Qualitative studies of PDEs: a dynamical systems approach

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Abstract

These notes are the support of lectures given in the Master 2 research program at the Université Paul Sabatier during the winter 2018/19. The aim of these lectures is to provide a fairly self-contained introduction to some topics related to the existence and stability of some special solutions of a class of Partial Differential Equations of parabolic type.

In Chapter 2, we present the main ideas in the finite dimensional case. More precisely, we study the nonlinear asymptotic stability of equilibrium points of autonomous ordinary differential equations in $\mathbb{R}^n$. We first recall well known results about linear (in)stability implies nonlinear (in)stability in the case of hyperbolic equilibrium points; that is when the Jacobian matrix at the equilibrium point does not possess eigenvalues on the imaginary axis. Then, in a second step, we present new results in the case of non hyperbolic equilibrium points and develop the concepts of center manifolds. The center manifold theory is based on the lectures by A. Vanderbauwhede [8]. For this chapter, only basic notions ODEs and linear algebra are required.

In Chapter 3, we give a theoretical framework to study the spectrum of closed linear operators. Closed linear operators naturally appear when linearizing a partial differential equation around a special solution (a traveling wave for example). These are the infinite-dimensional version of the Jacobian matrix found in the ODE setting. At the end of this chapter, one should be able for a given closed linear operator to fully characterize its spectrum. There are almost no prerequisites for this chapter as all notions will be introduced along the way. Most of the material covered in this chapter can be found in the excellent book of Kapitula & Promilsow [4].

Chapter 4 is a direct continuation of Chapter 3. The idea is to identify the relationship between the spectrum of a given closed operator $\mathcal{L}$ and the dynamics of the linear equation $\partial_t u = \mathcal{L}u$ generated by this operator. We will introduce the notion of a semigroup, and we will present a key result that relates the spectrum of a semigroup to the spectrum of its generator. As a key application, we demonstrate the nonlinear asymptotic stability of traveling fronts solutions for scalar bistable reaction-diffusion equations. This chapter is slightly more technical than the previous ones, but part of the results on semigroup theory should have been covered in the lectures on "Elliptic PDEs and evolution problems". I refer to the books [2, 6] for further readings on semigroup theory.

The last Chapter 5 is devoted to center manifold theorems in infinite dimensions. This is the natural generalization of the results presented in Chapter 2 in the finite dimensional case for ODEs. The formalism is directly taken from the book of Haragus & Iooss [3], and we also refer to the report [10] different proofs of the main results.

If you find any errors or typos, please signal it to me by sending an email to:

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Chapter 1

General introduction

The aim of these lectures is to provide a fairly self-contained introduction to some topics related to the existence and stability of some special solutions of a class of Partial Differential Equations of parabolic type. Two canonical examples will be used throughout these lectures. The first example is the following scalar reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where the scalar unknown $u(t,x) \in \mathbb{R}$ typically represents a population density and f a smooth reaction term. And, the second example, is the cubic Swift-Hohenberg equation

$$\partial_t u = -\left(\partial_x^2 + 1\right)^2 u + \mu u - u^3, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.2}$$

for some parameter $\mu \in \mathbb{R}$. More generally, we will consider PDEs of the form

$$\partial_t u = \mathcal{F}(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.3}$$

and look for special solutions $u_*(x)$ of (1.3) that satisfy

$$\mathcal{F}(u_*(x)) = 0, \quad \forall x \in \mathbb{R}. \tag{1.4}$$

Half of these lectures will provide some techniques to study (1.4) using spatial dynamics (see below). Once the existence of such solutions are established, we will be interested in their stability which motivates the other half of the lectures. To give a general idea of the types of problems that we will encounter, let us look for solutions of (1.3) that can be decomposed as

$$u(t,x) = u_*(x) + p(t,x),$$

with $p(t=0,x) = p_0(x)$ small (in some appropriate norm). Inserting this Ansatz into (1.3), we find

$$\begin{aligned}
\partial_t p = \partial_t u &= \mathcal{F}(u_* + p) \\
&= \mathcal{F}(u_*) + D\mathcal{F}(u_*)p + [\mathcal{F}(u_* + p) - \mathcal{F}(u_*) - D\mathcal{F}(u_*)p] \\
&= D\mathcal{F}(u_*)p + \mathcal{N}(p),
\end{aligned}$$

where $\mathcal{N}(p) := \mathcal{F}(u_* + p) - D\mathcal{F}(u_*)p$ contains terms that are at least quadratic in p assuming that $\mathcal{F}$ is smooth in some appropriate sense. Note that we explicitly used that $\mathcal{F}(u_*) = 0$ as u_* is a stationary solution. Upon denoting $\mathcal{L} := D\mathcal{F}(u_*)$, the evolution of the perturbation is governed by

$$\begin{cases} \partial_t p = \underbrace{\mathcal{L}p}_{\text{linear}} + \underbrace{\mathcal{N}(p)}_{\text{nonlinear}}, & t > 0, \quad x \in \mathbb{R}, \\ p(t=0,\cdot) = p_0, & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where we recall that p_0 is supposed to be small in some appropriate norm. Intuitively, for short time at least, we thus expect the linear equation $\partial_t p = \mathcal{L}p$ to give insights into whether perturbation will grow or decay. As a consequence, we will dedicate part of our time trying to study the properties of the solutions to the linearized equation

$$\begin{cases} \partial_t p = \mathcal{L}p, & t > 0, \quad x \in \mathbb{R}, \\ p(t=0,\cdot) = p_0, & x \in \mathbb{R}. \end{cases} \tag{1.6}$$

It turns out that the strategy is very analogous with the way one studies the stability of an equilibrium point of an ODE:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(v), \quad t > 0, \qquad F : \mathbb{R}^n \to \mathbb{R}^n.$$

One linearizes the ODE around the equilibrium point $v_* \in \mathbb{R}^n$ (solution to $F(v_*) = 0$) to get the Jacobian matrix $A := DF(v_*) \in \mathscr{M}_n(\mathbb{R})$, and then the eigenvalues of A are used to determine the stability of the equilibrium point. For ODEs, the associated linearized problem

$$\begin{cases} \frac{\mathrm{d}w}{\mathrm{d}t} = Aw, & t > 0, \\ w(t=0) = w_0, \end{cases} \tag{1.7}$$

has unique global solutions $w(t) = e^{At}w_0$ such that growth or decay of $w(t)$ are readily obtained via the properties of the eigenvalues of A. For PDEs, the situation is much more complex, and we will need to analyze:

1. the spectrum of $\mathcal{L}$, denoted $\sigma(\mathcal{L})$, that do not consist of eigenvalues only in general;
2. the solution of $\partial_t p = \mathcal{L}p$ (via the theory of semigroups);
3. relate the properties of the spectrum of $\mathcal{L}$ to the growing or decaying properties of the solutions of $\partial_t p = \mathcal{L}p$ (via spectral mapping theorems);
4. deduce the nonlinear (in)stability of the perturbations of (1.5).

Remark #1. The class of PDEs that we are studying are invariant by spatial translation. Typically, if $u_*(x)$ is a solution to (1.4) then $u_*(x+\tau)$, $\tau \in \mathbb{R}$, is also a solution, that is

$$\mathcal{F}(u_*(x+\tau)) = 0, \quad \forall x \in \mathbb{R}, \forall \tau \in \mathbb{R}.$$

Upon assuming some smoothness on both u_* and $\mathcal{F}$, we can differentiate the above equation with respect to τ to get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}(u_*(x+\tau))_{|\tau=0} = 0, x \in \mathbb{R},$$

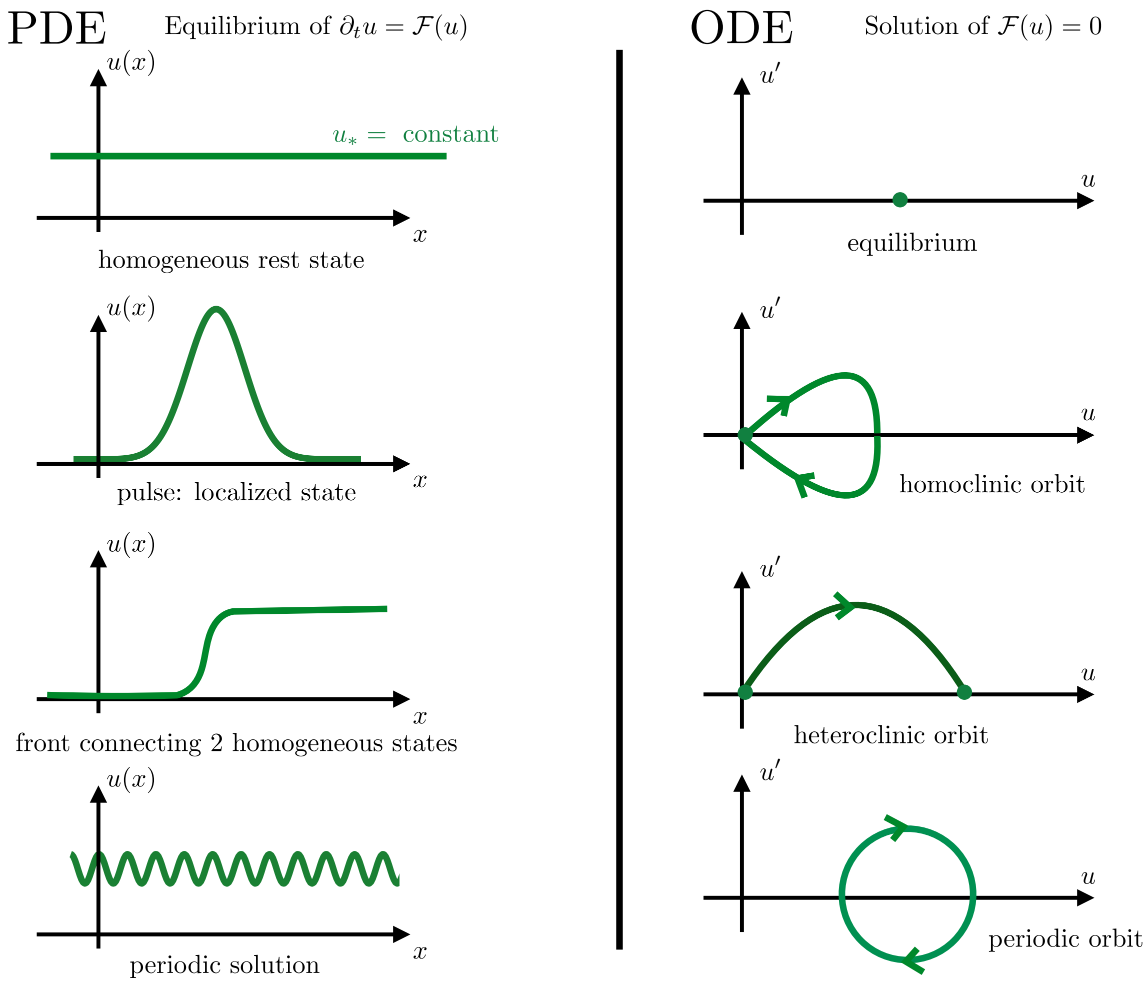

Figure 1.1: *Illustration of the correspondance between the stationary solutions of the PDE* (1.1) *and the associated ODE* (1.8) *for the reaction-diffusion equation.*

that is

$$\mathcal{L}u_*'(x) = D\mathcal{F}(u_*(x))u_*'(x) = 0,$$

which implies that $0 \in \sigma(\mathcal{L})$. It is important to realize that it does not necessarily mean that 0 is an eigenvalue with associated eigenvector u_*', there are many subtleties regarding the spectrum of operators in infinite dimension. Nevertheless, we will always be in the case where $0 \in \sigma(\mathcal{L})$ for which even the linear stability is not clear a priori.

Remark #2 – Spatial dynamics. Stationary solutions u_* of (1.4) are often realized as the solutions of high order ODE. For example, for the reaction-diffusion equation (1.1), we have

$$0 = u_*'' + f(u_*),$$

such that (u_*, u_*') are solutions of the first order system of ODEs

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} v \\ -f(u) \end{pmatrix} \tag{1.8}$$

where the space variable becomes the "time" variable for (1.8). Similarly, for the stationary solutions of the Swift-Hohenberg equation (1.2), we get

$$0 = -u_*'''' - 2u_*'' - u_* + \mu u_* - u_*^3$$

such that $(u_*, u_*', u_*'', u_*''')$ are solution of

$$\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu - 1 & 0 & -2 & 0 \end{pmatrix}}_{:=A_\mu} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u^3 \end{pmatrix}. \tag{1.9}$$

Remark that $u_* = 0$ is a solution of the PDE (1.2) and $(0, 0, 0, 0)$ is the corresponding equilibrium for the ODE (1.9). Linearizing the ODE around this equilibrium point, we obtain the linear ODE system $U' = A_\mu U$ with matrix A_μ being defined in (1.9). The eigenvalues of A_μ are those $\lambda \in \mathbb{C}$ such that

$$0 = -(\lambda^2 + 1)^2 + \mu,$$

where $\mu \in \mathbb{R}$ is some parameter. We can see that the eigenvalues are as follows:

- for $\mu < 0$: four complex conjugates eigenvalues with real part of order $\mathcal{O}\left(\pm\frac{\sqrt{-\mu}}{2}\right)$ for $|\mu|$ small;
- for $\mu = 0$: eigenvalues are $\pm\mathbf{i}$ (double);
- for $\mu > 0$: $\lambda = \pm\mathbf{i}\sqrt{1 \pm \sqrt{\mu}}$.

Once again, we are encountering situations where eigenvalues of the linearized operator sit on the imaginary axis.

Remark #3 – Traveling waves. We will be interested in traveling wave solutions $u(t, x) = u_*(x - ct)$, with constant velocity $c \in \mathbb{R}$ and fixed profile u_*, see Figure 1.2 for an illustration. As a consequence, it is natural to make the following change of variable $u(t, x) = \tilde{u}(t, x - ct)$ and set $y = x - ct$, such that (1.3) reads

$$\partial_t \tilde{u} = c\partial_y \tilde{u} + \mathcal{F}(\tilde{u}) := \widetilde{\mathcal{F}}_c(\tilde{u}),$$

so that traveling wave solutions of (1.3) are now stationary solutions of $\widetilde{\mathcal{F}}_c(u_*) = 0$. For example, the reaction-diffusion equation (1.1) becomes (dropping the tildes)

$$\partial_t u = \partial_y^2 u + c\partial_y u + f(u),$$

with traveling waves solutions of the ODE

$$0 = u'' + cu' + f(u).$$

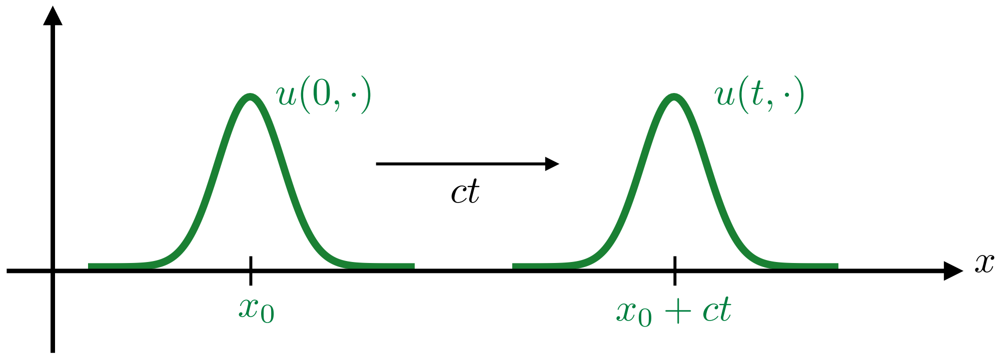

Figure 1.2: *Illustration of a traveling wave for* (1.3).

In the special case where the nonlinearity is given by $f(u) = u(1-u)(u-a)$ for $a \in (0,1)$, we look for traveling fronts

$$\begin{cases} 0 = u'' + cu' + f(u), \\ u(-\infty) = 1, \text{ and } u(+\infty) = 0, \text{ with } 0 < u < 1 \text{ on } \mathbb{R}. \end{cases} \tag{1.10}$$

One can construct explicit solutions of the form

$$u_{\mathrm{tw}}(y) = \frac{1}{1+e^{\sqrt{2}y}}, \quad c = \sqrt{2}\left(\frac{1}{2} - a\right).$$

Note that $u_{\mathrm{tw}}' < 0$. In fact such a result holds for general bistable nonlinearity f. We say that f is **bistable** if it satisfies

$$\begin{cases} f \in \mathscr{C}^1([0,1]), \quad f(0) = f(1) = f(a) = 0 \text{ for some } a \in (0,1), \\ f'(0) < 0, f'(1) < 0, \text{ and } f'(a) > 0, \\ f < 0 \text{ in } (0,a), \quad f > 0 \text{ in } (a,1). \end{cases} \tag{1.11}$$

In such a setting, a typical result is the following.

Theorem 1.1 (Fife & McLeod 77)**.** *Assume that f is bistable.*

(i) *There exists a unique (up to translation) traveling front solution (u_{tw}, c) of* (1.10) *which is monotone. The sign of the corresponding wave speed c is given by the sign of $\int_0^1 f(u)\mathrm{d}u$.*

(ii) *If $u_0 = u(0,\cdot) \neq 0$ is an initial condition satisfying $0 \leq u_0 \leq 1$ and*

$$\limsup_{x\to-\infty} u_0(x) > a, \quad \liminf_{x\to+\infty} u_0(x) < a,$$

then there exists $x_0 \in \mathbb{R}$ so that the corresponding solution of the Cauchy problem associated with (1.1) *and u_0 satisfies $u(t,x) \to u_{\mathrm{tw}}(x - ct + x_0)$ uniformly in x as $t \to +\infty$, with an exponential convergence rate.*

The proof of (i) relies on a phase-plane analysis (1.10), while the second point (ii) is established via comparison principle techniques which crucially rely on monotony properties of the equation. In the sequel, we will present an alternative proof of (ii) with stronger hypotheses on the initial datum (namely, it will be an $H^1(\mathbb{R})$ perturbation of the traveling front) leading to a local result. Compared to the global result of Theorem 1.1 it can be questionnable to present such a proof. The main reason is that the proof does not presuppose any structure on the considered system and can be applied in many other contexts, whereas comparison principle techniques will typically break down in general situations.

Chapter 2

Finite dimensional case – Old & New

2.1 Introduction

The aim is to study differential equations of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u), \tag{2.1}$$

for $u \in \mathbb{R}^n$, and $F : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathscr{C}^k$ for some $k \geq 2$. We assume that $u_* \in \mathbb{R}^n$ is an equilibrium point of the above ODE, that is $F(u_*) = 0$. We see that we can always rewrite the equation as

$$\frac{\mathrm{d}(u - u_*)}{\mathrm{d}t} = Df(u_*)(u - u_*) + (F(u) - F(u_*) - Df(u_*)(u - u_*)),$$

with $Df(u_*) \in \mathscr{M}_n(\mathbb{R})$and upon setting $v = u - u_*$, we get

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \underbrace{Df(u_*)}_{:=A} v + \underbrace{(F(v + u_*) - Df(u_*)v)}_{:=f(v)}.$$

As a consequence, we are let to study ODEs of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + f(u), \tag{2.2}$$

with $A \in \mathscr{M}_n(\mathbb{R})$, $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathscr{C}^k$ for some $k \geq 2$, $f(0) = 0_{\mathbb{R}^n}$ and $Df(0) = 0_{\mathscr{M}_n(\mathbb{R})}$. For each $u_0 \in \mathbb{R}^n$, we denote $t \mapsto \varphi(t, u_0)$ the unique maximal solution of (2.2) satisfying $u(0) = u_0$, defined on some interval $I(u_0)$.

Definition 2.1. *We say that $u_* = 0$ is a* ***stable*** *equilibrium point of (2.2) if*

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \textit{such that if } \|u_0\| < \delta,\ \textit{then } \|\varphi(t, u_0)\| < \epsilon \quad t \geq 0.$$

It is **asymptotically stable** *if*

$$u_* = 0 \textit{ is stable and } \exists \rho > 0,\ \textit{such that if } \|u_0\| < \rho,\ \textit{then } \|\varphi(t, u_0)\| \underset{t \to +\infty}{\longrightarrow} 0.$$

Finally, we say that $u_* = 0$ is **unstable** *if it is not stable.*

To determine the stability of $u_* = 0$ in (2.2), the very first step is to consider the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au. \tag{2.3}$$

The behavior of the solutions $\varphi_\ell(t, u_0) = e^{tA}u_0$ of (2.3) is completely determined by the spectrum of the matrix A. Here e^{tA} is given by definition by

$$e^{tA} := \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k, \quad t \in \mathbb{R}.$$

Let first denote by $\sigma = \sigma(A) := \{\lambda \in \mathbb{C} \mid \det(A - \lambda I_n) = 0\} \subset \mathbb{C}$ the set of eigenvalues of A. The algebraic multiplicity, $m_a(\lambda)$ of $\lambda \in \sigma(A)$ is the order of the zero of the characteristic polynomial $\det(A - \lambda I_n) = 0$. The geometric multiplicity, $m_g(\lambda)$ of $\lambda \in \sigma(A)$ is the dimension of $\ker(A - \lambda I_n)$. An eigenvalue is simple if $m_g(\lambda) = m_a(\lambda) = 1$, and semi-simple if $m_g(\lambda) = m_a(\lambda)$. Let us recall that if $\lambda \in \sigma(A)$ is such that $m_g(\lambda) < m_a(\lambda)$, then there will be at least one vector $u \neq 0$ such that $(A - \lambda I_n)^2 u = 0$ with $(A - \lambda I_n)u \neq 0$. It is clearly the case that $\ker(A - \lambda I_n)^k \subset \ker(A - \lambda I_n)^{k+1}$, and that these kernels grow until they reach full rank at $k = m_a(\lambda)$, that is

$$\begin{aligned} \dim\left[\ker(A - \lambda I_n)^{k-1}\right] < \dim\left[\ker(A - \lambda I_n)^{k}\right], \quad k < m_a(\lambda), \\ \dim\left[\ker(A - \lambda I_n)^{k}\right] = m_a(\lambda), \quad k \geq m_a(\lambda). \end{aligned}$$

For $\lambda \in \sigma(A)$, the generalized eigenspace, $\mathbb{E}_\lambda$ is given by

$$\mathbb{E}_\lambda = \mathrm{gker}(A - \lambda I_n) := \ker\left[(A - \lambda I_n)^{m_a(\lambda)}\right].$$

The spectrum $\sigma(A)$ is the disjoint union of the *stable spectrum* σ_s, the *unstable spectrum* σ_u and the *center spectrum* σ_c, where

$$\begin{aligned} \sigma_s &:= \{\lambda \in \sigma \mid \mathrm{Re}(\lambda) < 0\}, \\ \sigma_u &:= \{\lambda \in \sigma \mid \mathrm{Re}(\lambda) > 0\}, \\ \sigma_c &:= \{\lambda \in \sigma \mid \mathrm{Re}(\lambda) = 0\}. \end{aligned}$$

Let $\mathbb{E}_s$ be the subspace of $\mathbb{R}^n$ spanned by the generalized eigenvectors of A corresponding to eigenvalues $\lambda \in \sigma_s$; in a similar way we define $\mathbb{E}_u$ and $\mathbb{E}_c$. We use the decomposition

$$\mathbb{R}^n = \mathbb{E}_s \oplus \mathbb{E}_c \oplus \mathbb{E}_u.$$

Corresponding to this splitting, there are projections

$$\pi_s : \mathbb{R}^n \to \mathbb{E}_s, \quad \pi_c : \mathbb{R}^n \to \mathbb{E}_c, \text{ and } \pi_u : \mathbb{R}^n \to \mathbb{E}_u.$$

For future reference, we also denote $\pi_h := \pi_s + \pi_u$, the hyperbolic projection associated to the subspace $\mathbb{E}_h := \mathbb{E}_s \oplus \mathbb{E}_u = \ker(\pi_c)$. We also define numbers $\beta_\pm$ and β by

$$\begin{aligned} \beta_+ &:= \min\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma_u\} > 0, \\ \beta_- &:= \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma_s\} < 0, \\ \beta &:= \min\{\beta_+, -\beta_-\}, \end{aligned}$$

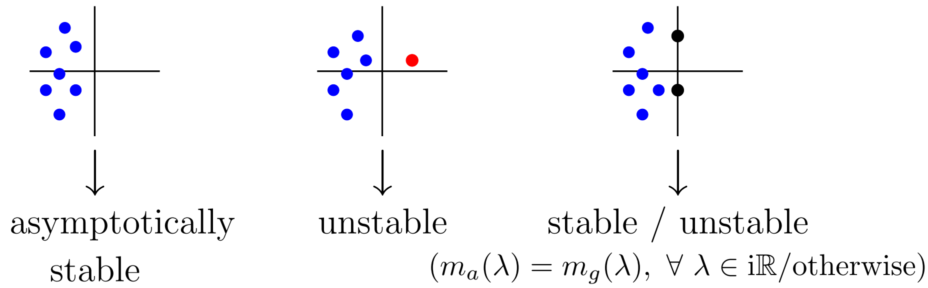

Figure 2.1: *Different stability results of* 0 *for the linear system* (2.3) *depending on the spectrum* $\sigma(A)$*. Left: if all eigenvalues are strictly contained to the half plane to the left of the imaginary axis, then* 0 *is asymptotically stable. Middle: if there is one eigenvalue with positive real part, then* 0 *is unstable. Right: all eigenvalues are strictly contained to the half plane to the left of the imaginary axis with some eigenvalues lying on the imaginary, if for each such eigenvalues the geometric multiplicity equals the algebraic multiplicity then* 0 *is stable, if not then* 0 *is unstable.*

with $\beta_+ = +\infty$ if $\sigma_u = \varnothing$, and $\beta_- = -\infty$ if $\sigma_s = \varnothing$. Then, we have the following estimates, which will be crucial for the study in the next sections.

Lemma 2.2. *For any* $\epsilon > 0$*, there exists some constant* $C(\epsilon) > 0$ *such that*

$$\left\|e^{tA}\pi_c\right\| \leq C(\epsilon)e^{\epsilon|t|}, \quad \forall t \in \mathbb{R}, \tag{2.4a}$$

$$\left\|e^{tA}\pi_u\right\| \leq C(\epsilon)e^{(\beta_+-\epsilon)t}, \quad \forall t \leq 0, \tag{2.4b}$$

$$\left\|e^{tA}\pi_s\right\| \leq C(\epsilon)e^{(\beta_-+\epsilon)t}, \quad \forall t \geq 0. \tag{2.4c}$$

Proof. Let $\mathbb{E}_\lambda := \ker{(A - \lambda I_n)}^{m_a(\lambda)}$ be a generalized eigenspace. On such a subspace, we have for $z \in \mathbb{E}_\lambda$,

$$e^{tA}z = e^{t\lambda}e^{t(A-\lambda I_n)}z = e^{t\lambda}\left(\sum_{p=0}^{m_a(\lambda)-1}\frac{t^p}{p!}(A-\lambda I_n)^p\right)z,$$

so that we get

$$\left\|e^{tA}z\right\| \lesssim (1+|t|^{m_a(\lambda)-1})e^{t\mathrm{Re}(\lambda)}\|z\|.$$

And the estimates in (2.4) easily follow from the definition of $\mathbb{E}_c$ and $\beta_\pm$. ∎

As a conclusion, we can get the following stability results for the linear system (2.3) that are illustrated in Figure 2.1.

Lemma 2.3. *For the linear system* $u' = Au$*, we have the following scenario regarding the stability of the equilibrium* $u_* = 0$*.*

(i) If for all $\lambda \in \sigma(A)$ *one has* $\Re(\lambda) < -\alpha < 0$ *for some* $\alpha > 0$*, then* 0 *is asymptotically stable.*

(ii) If there exists $\lambda \in \sigma(A)$ *with* $\Re(\lambda) > 0$*, then* 0 *is unstable.*

(iii) If for all $\lambda \in \sigma(A)$ *one has* $\Re(\lambda) \leq 0$*, then if for all* $\lambda \in \mathbb{E}_c$ *one has* $m_a(\lambda) = m_g(\lambda)$ *then* 0 *is stable, otherwise it is unstable.*

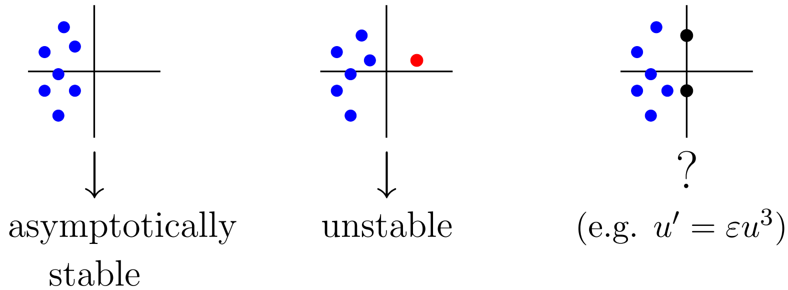

Figure 2.2: *Illustration of Theorem 2.1 and the different stability results of u_* for the nonlinear system (2.1) depending on the spectrum $\sigma(DF(u_*))$. Left: if for all $\lambda \in \sigma(DF(u_*))$ one has $\Re(\lambda) < -\alpha < 0$ for some $\alpha > 0$, then u_* is asymptotically stable. Middle: if there exists $\lambda \in \sigma(DF(u_*))$ with $\Re(\lambda) > 0$, then u_* is unstable. Right: if for all $\lambda \in \sigma(DF(u_*))$ one has $\Re(\lambda) \leq 0$, then we can not conclude about the (in)stability of u_*.*

We now return to the nonlinear equation (2.2). We have the following classical result and we refer to Figure 2.2 for an illustration.

Theorem 2.1. *Let $u_* \in \mathbb{R}^n$ be an equilibrium point of (2.1) and denote $A = DF(u_*)$.*

(i) If for all $\lambda \in \sigma(A)$ one has $\Re(\lambda) < -\alpha < 0$ for some $\alpha > 0$, then u_* is asymptotically stable.

(ii) If there exists $\lambda \in \sigma(A)$ with $\Re(\lambda) > 0$, then u_* is unstable.

(iii) If for all $\lambda \in \sigma(A)$ one has $\Re(\lambda) \leq 0$, then we can not conclude about the (in)stability of u_*.

We will prove (i)-(ii) in the next section, while in the following sections we will develop a theory that will allow to conclude in the third case (iii). We will often use the following terminology:

- $\sigma(A) \cap \mathbf{i}\mathbb{R} = \emptyset$, we say that 0 is an hyperbolic equilibrium point;
- $\sigma(A) \cap \mathbf{i}\mathbb{R} \neq \emptyset$, we say that 0 is not an hyperbolic equilibrium point.

2.2 Nonlinear (in)stability of hyperbolic equilibrium

It is obviously equivalent to prove Theorem 2.1 by considering the asymptotic (in)stability of $u_* = 0$ in (2.2).

2.2.1 Case (i) – Linear asymptotic stability implies nonlinear asymptotic stability

First of all, from the estimates 2.4 and the fact that for all $\lambda \in \sigma(A)$ one has $\Re(\lambda) < -\alpha < 0$ for some $\alpha > 0$, one gets the existence of a constant $c_1 > 1$ such that

$$\left\| e^{At} \right\| \leq c_1 e^{-\alpha t}, \quad t \geq 0.$$

Furthermore, using that $f \in \mathscr{C}^k$ for $k \geq 2$ and that $f(0) = 0_{\mathbb{R}^n}$ and $Df(0) = 0_{\mathscr{M}_n(\mathbb{R})}$, there exist $\rho_0 > 0$ and $c_2 > 0$ such that

$$\|f(u)\| \leq c_2 \|u\|^2, \text{ for } \|u\| < \rho_0.$$

The variation of constant formula yields

$$\varphi(t,u_0) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(\varphi(s,u_0))\mathrm{d}s, \quad t \in I(u_0).$$

Let $\epsilon > 0$ and $\delta > 0$ such that

$$\delta < \min\left\{\frac{\epsilon}{2c_1}, \frac{\rho_0}{2c_1}, \frac{\alpha}{4c_1^2 c_2}\right\}.$$

Finally consider initial conditions $u_0 \in \mathbb{R}^n$ such that $\|u_0\| < \delta$. We introduce the following time

$$T := \sup\left\{t \geq 0 \mid \|\varphi(s,u_0)\| < 2c_1\delta e^{-\alpha s}, \quad 0 \leq s \leq t\right\},$$

which is well defined and we also have $T > 0$ by assumption on the initial condition and the fact that $c_1 > 1$. We are going to show that $T = +\infty$ which will prove (i) of Theorem 2.1. First note that whenever $t \in [0,T)$ then

$$\|\varphi(t,u_0)\| < 2c_1\delta e^{-\alpha t} \leq 2c_1\delta < \rho_0,$$

from our condition on δ. Thus, for all $t \in [0,T)$, we have

$$\begin{aligned}\|\varphi(t,u_0)\| &\leq c_1\delta e^{-\alpha t} + 4c_1^3c_2\delta^2 \underbrace{\int_0^t e^{-\alpha(t-s)}e^{-2\alpha s}\mathrm{d}s}_{\leq \frac{1}{\alpha}e^{-\alpha t}}, \\ &\leq e^{-\alpha t}c_1\delta\left(1 + \frac{4}{\alpha}c_1^2c_2\delta\right) \\ &< 2c_1\delta e^{-\alpha t}.\end{aligned}$$

This implies that $T = +\infty$ and that for all $t \geq 0$

$$\|\varphi(t,u_0)\| \leq \epsilon e^{-\alpha t} \underset{t\to+\infty}{\longrightarrow} 0.$$

2.2.2 Case (ii) – Linear instability implies nonlinear instability

It will be slightly more difficult to prove case (ii) regarding nonlinear instability. Let $\lambda \in \sigma(A)$ be the eigenvalue with largest positive real part and denote $\alpha = \Re(\lambda) > 0$. For simplicity, we will only consider the case where λ is actually a real eigenvalue, and let the case where it is complex as an exercise. In that case, there exists $c_1 > 0$ such that

$$\left\|e^{At}\right\| \leq c_1 e^{\frac{3}{2}\alpha t}, \quad t \geq 0.$$

Once again, we assume that there exist $\rho_0 > 0$ and $c_2 > 0$ such that

$$\|f(u)\| \leq c_2\|u\|^2, \text{ for } \|u\| < \rho_0.$$

We define $\ell > 0$ as the following constant that depends only on α and $c_{1,2}$

$$\ell := 4e^{2\alpha}\frac{c_1c_2}{\alpha}\left(1 + \frac{e^{-\alpha}}{2}\right)^2.$$

It will become clear later on in the proof how this quantity will appear in the computations. Finally, let $\epsilon_0 > 0$ be such that

$$\epsilon_0 := \min\left\{\frac{\rho_0}{2}, \frac{1}{2\ell}\right\}.$$

Finally, let $u_\alpha \in \mathbb{R}^n$ be a unit eigenvector associated to $\lambda = \alpha$ such that $Au_\alpha = \alpha u_\alpha$ with $\|u_\alpha\| = 1$. For all $\delta > 0$ small enough such that

$$\delta < \min\left\{\frac{1}{\ell}, \frac{\rho_0}{2}, 1\right\},$$

we define $v := \delta u_\alpha \in \mathbb{R}^n$ such that $Av = \alpha v$ and $\|v\| = \delta$. Let $T_* > 0$ be such that

$$\frac{1}{\ell} < \delta e^{T_* \alpha} \leq \frac{e^\alpha}{\ell},$$

that is

$$T_* \in \left(\frac{1}{\alpha}\ln\left(\frac{1}{\delta\ell}\right), 1 + \frac{1}{\alpha}\ln\left(\frac{1}{\delta\ell}\right)\right].$$

We have the following facts:

- $\left\|e^{AT_*}v\right\| = \delta e^{\alpha T_*} > \frac{1}{\ell}$;
- $\left\|e^{At}v\right\| = \delta e^{\alpha t}$ for all $t \geq 0$.

Using the variation of constant formula from $u_0 = v$ yields

$$\varphi(t, v) = e^{tA}v + \int_0^t e^{(t-s)A} f(\varphi(s, v))\mathrm{d}s, \quad t \in I(v).$$

As a consequence, as long as $\|\varphi(t, v)\| < \rho_0$ we get

$$\left\|\varphi(t, v) - e^{At}v\right\| \leq c_1 c_2 \int_0^t e^{\frac{3}{2}(t-s)} \|\varphi(s, v)\|^2 \,\mathrm{d}s.$$

Let T be the following time

$$T := \sup\left\{t \mid \left\|\varphi(s, v) - e^{As}v\right\| \leq \frac{1}{2e^\alpha}\delta e^{\alpha s} \text{ and } \|\varphi(s, v)\| < \frac{\rho_0}{2}, \quad 0 \leq s \leq t\right\}.$$

Clearly $T > 0$ and for all $t \leq \min\{T, T_*\}$, we have the estimates

$$\begin{aligned}
\left\|\varphi(t, v) - e^{At}v\right\| &\leq c_1 c_2 \int_0^t e^{\frac{3}{2}(t-s)} \left(\left\|e^{sA}v\right\| + \left\|\varphi(s, v) - e^{As}v\right\|\right)^2 \mathrm{d}s \\
&\leq c_1 c_2 \left(1 + \frac{e^{-\alpha}}{2}\right)^2 \delta^2 \int_0^t e^{\frac{3}{2}(t-s)} e^{2\alpha s}\mathrm{d}s \\
&< \frac{2}{\alpha} c_1 c_2 \left(1 + \frac{e^{-\alpha}}{2}\right)^2 \delta^2 e^{2\alpha t} \\
&= \frac{\ell}{2e^{2\alpha}} \left(\delta e^{\alpha t}\right)^2.
\end{aligned}$$

We claim that either $T_* < T$ or $\|\varphi(T,v)\| = \frac{\rho_0}{2}$. Suppose that it is not the case, that is $T_* \geq T$ and $\|\varphi(T,v)\| < \frac{\rho_0}{2}$. Then, from the above estimate, we deduce

$$\frac{1}{2e^{\alpha}}\delta e^{\alpha T} = \left\|\varphi(T,v) - e^{AT}v\right\| < \frac{\ell}{2e^{2\alpha}}\left(\delta e^{\alpha T}\right)^2,$$

which gives

$$\delta e^{\alpha T} > \frac{e^{\alpha}}{\ell} \geq \delta e^{\alpha T_*},$$

and thus $T_* < T$ which is a contradiction. If $\|\varphi(T,v)\| = \frac{\rho_0}{2}$ we are done. So, let us suppose that $T_* < T$, then

$$\left\|\varphi(T_*,v) - e^{AT_*}v\right\| < \frac{\ell}{2e^{2\alpha}}\left(\delta e^{\alpha T_*}\right)^2 \leq \frac{1}{2\ell}.$$

As a consequence, we deduce that

$$\|\varphi(T_*,v)\| \geq \left\|e^{AT_*}v\right\| - \left\|\varphi(T_*,v) - e^{AT_*}v\right\| > \frac{1}{\ell} - \frac{1}{2\ell} = \frac{1}{2\ell}.$$

It follows that in any case, there is a time (either T or T_*) at which the solution $\|\varphi(t,v)\| \geq \min\left\{\frac{\rho_0}{2}, \frac{1}{2\ell}\right\} = \epsilon_0$, which concludes the proof.

Remark 2.4. *The proof shows that there exist $\epsilon_0 >$ and $C > 0$ such that for all $\delta > 0$ sufficiently small there is a solution $\varphi(\cdot,u_0)$ with $\|u_0\| < \delta$ such that $\sup\limits_{0\leq t\leq C|\ln(\delta)|}\|\varphi(t,u_0)\| \geq \epsilon_0$. The escape time of a given neighborhood is logarithmic.*

2.3 Stability of non hyperbolic equilibrium – Center manifold theory

Throughout this section, we will consider equation (2.2)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + f(u),$$

in the case where $u_* = 0$ is a non hyperbolic equilibrium that is $\mathbb{E}_c = \sigma(A)\cap \mathbf{i}\mathbb{R} \neq \emptyset$. We recall that $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathscr{C}^k$ for some $k \geq 1$, $f(0) = 0_{\mathbb{R}^n}$ and $Df(0) = 0_{\mathscr{M}_n(\mathbb{R})}$. More precisely, we would like to determine the asymptotic behavior of solutions $\varphi(t,u_0)$ of (2.2) starting from some initial condition $u_0 \in \mathbb{R}^n$ in a neighborhood of the origin.

2.3.1 Existence and uniqueness of global center manifolds

To get an intuition on the type of results that one can hope to prove in that non hyperbolic case, it is insightful to consider the linear system (2.3). From the estimates in (2.4) one sees that for any $\eta \in (0,\beta)$, then solutions of (2.3) in $\mathbb{E}_c$ are bounded by $Ce^{\eta|t|}$ for all $t\in\mathbb{R}$, while non-zero solutions in $\mathbb{E}_h$ blow up faster than $Ce^{\eta|t|}$ for $t\to+\infty$, for $t\to-\infty$, or both. This observation is formulated in the following lemma.

Lemma 2.5. *We have*

$$\mathbb{E}_c = \left\{u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}}\|\pi_h\varphi_\ell(t,u_0)\| < +\infty\right\}, \tag{2.5}$$

and for each $\eta \in (0,\beta)$:

$$\mathbb{E}_c = \left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} e^{-\eta|t|} \|\varphi_\ell(t,u_0)\| < +\infty \right\}. \tag{2.6}$$

Proof. If $u_0 \in \mathbb{E}_c$, then $\pi_h\varphi_\ell(t,u_0) = \pi_h e^{tA}u_0 = e^{tA}\pi_h u_0 = 0$ for all $t \in \mathbb{R}$, and hence $\pi_h\varphi_\ell(t,u_0)$ stays bounded for all t. That is

$$\mathbb{E}_c \subset \left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} \|\pi_h\varphi_\ell(t,u_0)\| < +\infty \right\}.$$

Next, let $u_0 \in \mathbb{R}^n$ be such that $\|\pi_h\varphi_\ell(t,u_0)\| < C < +\infty$ for all $t \in \mathbb{R}$, then if $\eta \in (0,\beta)$, we have

$$\begin{aligned}\|\varphi_\ell(t,u_0)\| &\le \|\pi_h\varphi_\ell(t,u_0)\| + \|\pi_c\varphi_\ell(t,u_0)\| \\ &\le C + C(\eta)e^{\eta|t|}\|u_0\| \le C_1 e^{\eta|t|}.\end{aligned}$$

This implies that

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|} \|\varphi_\ell(t,u_0)\| < +\infty,$$

and thus

$$\left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} \|\pi_h\varphi_\ell(t,u_0)\| < +\infty \right\} \subset \left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} e^{-\eta|t|} \|\varphi_\ell(t,u_0)\| < +\infty \right\}.$$

Finally, let $u_0 \in \mathbb{R}^n$ be such that $\|\pi_h\varphi_\ell(t,u_0)\| < Ce^{\mu|t|}$ for all $t \in \mathbb{R}$. Then, we have that for all $t \le 0$ and $\epsilon > 0$,

$$\|\pi_u u_0\| = \left\|e^{tA}\pi_u e^{-tA}u_0\right\| \le C(\epsilon)e^{(\beta_+-\epsilon)t}Ce^{-\eta t}.$$

Taking $\epsilon < \beta - \eta \le \beta_+ - \eta$ and letting $t \to -\infty$, we see that $\pi_u u_0 = 0$. Repeating the same argument one can prove $\pi_s u_0 = 0$, and hence $u_0 = \pi_c u_0 \in \mathbb{E}_c$, that is

$$\left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} e^{-\eta|t|} \|\varphi_\ell(t,u_0)\| < +\infty \right\} \subset \mathbb{E}_c.$$

■

It is important to note that the above Lemma is global, and we can expect a similar result for (2.2) only in a neighborhood of $u = 0$. The following construction will allow us to work out the theory in a global setting, and later on we will return to the local situation. To do so, we fix a smooth cut-off function $\chi : \mathbb{R}^n \to \mathbb{R}$, with the following properties:

(i) $0 \le \chi(u) \le 1$, for all $u \in \mathbb{R}^n$;

(ii) $\chi(u) = 1$ if $\|u\| \le 1$;

(iii) $\chi(u) = 0$ if $\|u\| \ge 2$.

Then for each $\rho > 0$, we define $f_\rho : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_\rho(u) := f(u)\chi(\rho^{-1}u), \quad \forall u \in \mathbb{R}^n. \tag{2.7}$$

The flow of (2.2) in the ball centered in $u = 0$ of radius ρ will coincide with the flow of the modified equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + f_\rho(u). \tag{2.8}$$

We define $\mathscr{C}_b^k(\mathbb{R}^n) = \mathscr{C}_b^k(\mathbb{R}^n, \mathbb{R}^n)$ as the Banach space of all mappings $w \in \mathscr{C}^k(\mathbb{R}^n, \mathbb{R}^n)$ and such that

$$|w|_j := \sup_{u\in\mathbb{R}^n} \|D^j w(u)\| < \infty, \qquad \text{for } 0 \leq j \leq k.$$

Lemma 2.6. *Let f be of class $\mathscr{C}^k$ for some $k \geq 1$, and define f_ρ by (2.7). Then $f_\rho \in \mathscr{C}_b^k(\mathbb{R}^n)$ for each $\rho > 0$, and*

$$\lim_{\rho\to 0} |f_\rho|_1 = 0.$$

Proof. Clearly f_ρ is of class $\mathscr{C}^k$ and has compact support given by the closed ball of radius 2ρ, hence we have $f_\rho \in \mathscr{C}_b^k(\mathbb{R}^n)$ for each $\rho > 0$. In order, to establish the limit of $|f_\rho|_1$ as $\rho \to 0$, we first note that

$$Df_\rho(u) = Df(u)\chi(\rho^{-1}u) + \rho^{-1} f(u) D\chi(\rho^{-1}u),$$

and since $f(0) = 0$, we have $\|f(u)\| \leq \|u\| \sup_{s\in[0,1]} \|Df(su)\|$. It then follows that

$$\begin{aligned}
|f_\rho|_1 &= \sup_{\|u\|\leq 2\rho} \|Df(u)\chi(\rho^{-1}u) + \rho^{-1} f(u) D\chi(\rho^{-1}u)\| \\
&\leq \sup_{\|u\|\leq 2\rho} \|Df(u)\| + \rho^{-1}|\chi|_1 \sup_{\|u\|\leq 2\rho} \sup_{s\in[0,1]} \|u\|\|Df(su)\| \\
&\leq (1 + 2|\chi|_1) \sup_{\|u\|\leq 2\rho} \|Df(u)\|.
\end{aligned}$$

Since $Df(0) = 0$, we have that

$$\lim_{\rho\to 0} \sup_{\|u\|\leq 2\rho} \|Df(u)\| = \|Df(0)\| = 0,$$

and the conclusion follows. ∎

From now on, we consider differential equations of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + g(u), \tag{2.9}$$

for $u \in \mathbb{R}^n$, $A \in \mathscr{M}_n(\mathbb{R})$, $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ for some $k \geq 1$, $g(0) = Dg(0) = 0$. Note that solutions $t \mapsto \varphi_g(t, u_0)$ of (2.9) are unique and global in time for each initial condition $u(0) = u_0 \in \mathbb{R}^n$. The starting point of the analysis will be a generalization of Lemma 2.5 to equation (2.9).

Lemma 2.7. *Let $g \in \mathscr{C}_b^k(\mathbb{R}^n)$, $k \geq 1$, and $\eta \in (0, \beta)$. Then we have*

$$\mathcal{M}_c := \left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} \|\pi_h \varphi_g(t, u_0)\| < +\infty \right\} = \left\{ u_0 \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} e^{-\eta|t|} \|\varphi_g(t, u_0)\| < +\infty \right\}.$$

Proof. The proof is based on the variation of constant formula

$$\varphi_g(t,u_0) = e^{(t-t_0)A}\varphi_g(t_0,u_0) + \int_{t_0}^{t} e^{(t-\tau)A} g(\varphi_g(\tau,u_0))\mathrm{d}\tau, \tag{2.10}$$

valid for any $t, t_0 \in \mathbb{R}$.

If $u_0 \in \mathcal{M}_c$ then, by definition, $\pi_h\varphi_g(t,u_0)$ is bounded for all time, and therefore

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|}\,\|\pi_h\varphi_g(t,u_0)\| < +\infty.$$

Taking $t_0 = 0$ in (2.10) and applying π_c gives

$$\pi_c\varphi_g(t,u_0) = e^{tA}\pi_c u_0 + \int_0^t e^{(t-\tau)A}\pi_c g(\varphi_g(\tau,u_0))\mathrm{d}\tau.$$

As a consequence, we obtain that

$$\begin{aligned}\|\pi_c\varphi_g(t,u_0)\| &\le C(\eta)e^{\eta|t|}\|u_0\| + C(\eta)|g|_0\int_0^t |e^{\eta|t-\tau|}|\mathrm{d}\tau \\ &\le C(\eta)e^{\eta|t|}\left(\|u_0\| + \eta^{-1}|g|_0\right),\end{aligned}$$

which implies that

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|}\,\|\pi_c\varphi_g(t,u_0)\| < +\infty,$$

and together with the previous estimate this gives

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|}\,\|\varphi_g(t,u_0)\| < +\infty.$$

Conversely, let $u_0 \in \mathbb{R}^n$ be such that

$$\|\varphi_g(t,u_0)\| \le Ce^{\eta|t|}$$

for all $t \in \mathbb{R}$ and some $C > 0$. We can use equation (2.10) and project it onto $\mathbb{E}_u$, this gives

$$\pi_u\varphi_g(t,u_0) = e^{(t-t_0)A}\pi_u\varphi_g(t_0,u_0) + \int_{t_0}^{t} e^{(t-\tau)A}\pi_u g(\varphi_g(\tau,u_0))\mathrm{d}\tau.$$

We fix $t \in \mathbb{R}$ and let $t_0 \ge \max(t,0)$, and let $\epsilon \in (0, \beta-\eta)$. Then, we have

$$\left\|e^{(t-t_0)A}\pi_u\varphi_g(t_0,u_0)\right\| \le C(\epsilon)e^{(\beta-\epsilon)(t-t_0)}Ce^{\eta t_0} = C(\epsilon)Ce^{(\beta-\epsilon)t}e^{-(\beta-\eta-\epsilon)t_0},$$

and the last term goes to zero as $t_0 \to +\infty$. And as a consequence, we obtain

$$\pi_u\varphi_g(t,u_0) = -\int_t^{+\infty} e^{(t-\tau)A}\pi_u g(\varphi_g(\tau,u_0))\mathrm{d}\tau, \quad \forall t\in\mathbb{R}.$$

Repeating this argument with π_s, we obtain

$$\pi_s\varphi_g(t,u_0) = \int_{-\infty}^{t} e^{(t-\tau)A}\pi_s g(\varphi_g(\tau,u_0))\mathrm{d}\tau, \quad \forall t\in\mathbb{R}.$$

As a consequence, for any $\epsilon \in (0,\beta)$, we have

$$\|\pi_h\varphi_g(t,u_0)\| \le 2(\beta-\epsilon)^{-1}C(\epsilon)|g|_0,$$

which implies that $u_0 \in \mathcal{M}_c$ and concludes the proof of the lemma. ■

Remark 2.8. *It is interesting to note that we have obtained the following useful formula*

$$\varphi_g(t,u_0) = e^{tA}\pi_c u_0 + \int_0^t e^{(t-\tau)A}\pi_c g(\varphi_g(\tau,u_0))\mathrm{d}\tau$$
$$+ \int_{-\infty}^t e^{(t-\tau)A}\pi_s g(\varphi_g(\tau,u_0))\mathrm{d}\tau - \int_t^{+\infty} e^{(t-\tau)A}\pi_u g(\varphi_g(\tau,u_0))\mathrm{d}\tau, \tag{2.11}$$

valid for all $t \in \mathbb{R}$ *and any* $u_0 \in \mathcal{M}_c$.

The main result of this section is the following.

Theorem 2.2. *Let* $A \in \mathscr{M}_n(\mathbb{R})$ *be given. Then, there exists* $\delta > 0$ *such that for each* $g \in \mathscr{C}_b^1(\mathbb{R}^n)$ *with* $|g|_1 < \delta$ *the following holds:*

(i) Invariance : *the set* $\mathcal{M}_c$ *is invariant under* (2.9)*;*

(ii) Structure : $\mathcal{M}_c$ *is a Lipschitz* $\mathscr{C}^0-$*submanifold of* $\mathbb{R}^n$*. More precisely, there exists some Lipschitz map* $\Psi \in \mathscr{C}_b^0(\mathbb{E}_c, \mathbb{E}_h)$*,* $\Psi(0) = 0$*, such that*

$$\mathcal{M}_c = \{u_0 + \Psi(u_0) \mid u_0 \in \mathbb{E}_c\};$$

(iii) Uniqueness : *if* $\widetilde{\Psi} \in \mathscr{C}_b^0(\mathbb{E}_c, \mathbb{E}_h)$ *is such that the manifold*

$$\mathcal{W}_c = \left\{u_0 + \widetilde{\Psi}(u_0) \mid u_0 \in \mathbb{E}_c\right\}$$

is invariant under the flow of (2.9)*, then* $\mathcal{W}_c = \mathcal{M}_c$ *and* $\widetilde{\Psi} = \Psi$*;*

(iv) Regularity and Tangency : *if* $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ *for* $k \geq 1$*, then* $\Psi \in \mathscr{C}_b^k(\mathbb{E}_c, \mathbb{E}_h)$ *and* $D\Psi(0) = 0$*.*

Under the setting of Theorem 2.2, we call $\mathcal{M}_c$ the unique global *center manifold* of (2.9). Note also, that in the trivial case $g = 0$, we have that $\mathcal{M}_c = \mathbb{E}_c$.

Remark 2.9. *Notice that* $\mathcal{M}_c$ *has the same dimension as* $\mathbb{E}_c$ *and that it is tangent to* $\mathbb{E}_c$ *at* $u = 0$*.*

2.3.2 Proof of Theorem 2.2

We first prove (i) and (iii). If $u \in \mathcal{M}_c$ and $t_0 \in \mathbb{R}$, then we have for all $t \in \mathbb{R}$ that

$$\pi_h \varphi_g(t, \varphi_g(t_0, u)) = \pi_h \varphi_g(t + t_0, u),$$

such that $\pi_h \varphi_g(t, \varphi_g(t_0, u))$ stays bounded for all $t \in \mathbb{R}$ and $\varphi_g(t_0, u) \in \mathcal{M}_c$ which proves the invariance. As for the uniqueness part; if $\widetilde{\Psi} \in \mathscr{C}_b^0(\mathbb{E}_c, \mathbb{E}_h)$ is such a function then for any $u_0 \in \mathbb{E}_c$ we have $\varphi_g(t, u_0 + \widetilde{\Psi}(u_0)) \in \mathcal{W}_c$ by invariance which means that

$$\pi_h \varphi_g(t, u_0 + \widetilde{\Psi}(u_0)) = \widetilde{\Psi}(\pi_c \varphi_g(t, u_0 + \widetilde{\Psi}(u_0))).$$

Since $\widetilde{\Psi}$ is bounded on $\mathbb{E}_c$, it follows that

$$\sup_{t \in \mathbb{R}} \|\pi_h \varphi_g(t, u_0 + \widetilde{\Psi}(u_0))\| < \infty,$$

and hence $u_0+\widetilde{\Psi}(u_0)\in\mathcal{M}_c$ for each $u_0\in\mathbb{E}_c$, which implies that $u_0+\widetilde{\Psi}(u_0)=y_0+\Psi(y_0)$ for some $y_0\in\mathbb{E}_c$. And thus, $u_0-y_0=\Psi(y_0)-\widetilde{\Psi}(u_0)\in\mathbb{E}_c\cap\mathbb{E}_h=\{0\}$ and the conclusion follows.

The remaining of this section is dedicated to the proof of (ii) which will be decomposed in several steps.

Step #1: parametrization of the center manifold. We first introduce the Banach space Y_η for $\eta\geq 0$ defined by

$$Y_\eta:=\left\{y\in\mathscr{C}^0(\mathbb{R},\mathbb{R}^n)\mid \|y\|_\eta:=\sup_{t\in\mathbb{R}}e^{-\eta|t|}\|y(t)\|<+\infty\right\}.$$

It is straightforward to check that for any $0\leq\eta\leq\zeta$ we have $Y_\eta\subset Y_\zeta$ and the embedding is continuous, namely

$$\|y\|_\zeta\leq\|y\|_\eta,\quad\forall y\in Y_\eta.$$

We can then rewrite the definition of $\mathcal{M}_c$ as

$$\mathcal{M}_c=\{u_0\in\mathbb{R}^n\mid\varphi_g(\cdot,u_0)\in Y_\eta\},$$

which holds for any $\eta\in(0,\beta)$. We can also equivalently see $\mathcal{M}_c$ as

$$\mathcal{M}_c=\{y(0)\in\mathbb{R}^n\mid y\in Y_\eta\text{ and }y\text{ solves (2.9)}\},$$

and this is precisely the approach we will follow.

Let us recall that a point $u_0\in\mathbb{R}^n$ belongs to $\mathcal{M}_c$ if and only if

$$\varphi_g(t,u_0)=e^{tA}\pi_cu_0+\int_0^te^{(t-\tau)A}\pi_cg(\varphi_g(\tau,u_0))\mathrm{d}\tau+\int_{-\infty}^{+\infty}B(t-\tau)g(\varphi_g(\tau,u_0))\mathrm{d}\tau,\quad\forall t\in\mathbb{R},$$

where $B:\mathbb{R}\to\mathscr{L}(\mathbb{R}^n)$ is defined as follows

$$B(t)=\begin{cases}-e^{tA}\pi_u & \text{if } t<0,\\ e^{tA}\pi_s & \text{if } t\geq 0.\end{cases}$$

And for any $\epsilon>0$, we have that

$$\|B(t)\|\leq C(\epsilon)e^{-(\beta-\epsilon)|t|},\quad\forall t\in\mathbb{R}.$$

Lemma 2.10. *Let $g\in\mathscr{C}^1_b(\mathbb{R}^n)$, $\eta\in(0,\beta)$ and $y\in Y_\eta$. Then y is a solution of (2.9) if and only if there exists some $u_0\in\mathbb{E}_c$ such that*

$$y(t)=e^{tA}u_0+\int_0^te^{(t-\tau)A}\pi_cg(y(\tau))\mathrm{d}\tau+\int_{-\infty}^{+\infty}B(t-\tau)g(y(\tau))\mathrm{d}\tau,\quad\forall t\in\mathbb{R}.\tag{2.12}$$

Proof. If y is a solution then $y(t)=\varphi_g(t,y(0))$ and $y(0)\in\mathcal{M}_c$, and the conclusion follows by taking $u_0=\pi_cy(0)$.

Conversely, suppose that $y\in Y_\eta$ satisfies

$$y(t)=e^{tA}u_0+\int_0^te^{(t-\tau)A}\pi_cg(y(\tau))\mathrm{d}\tau+\int_{-\infty}^{+\infty}B(t-\tau)g(y(\tau))\mathrm{d}\tau,\quad\forall t\in\mathbb{R},$$

for some $u_0 \in \mathbb{E}_c$. Then we have by definition of B

$$\begin{aligned} y(t) &= e^{tA}\left[u_0 + \int_{-\infty}^{0} e^{-\tau A}\pi_s g(y(\tau))\mathrm{d}\tau - \int_{0}^{+\infty} e^{-\tau A}\pi_u g(y(\tau))\mathrm{d}\tau\right] + \int_0^t e^{(t-\tau)A} g(y(\tau))\mathrm{d}\tau \\ &= e^{tA}y(0) + \int_0^t e^{(t-\tau)A} g(y(\tau))\mathrm{d}\tau, \end{aligned}$$

hence y is a solution of (2.9). ∎

Let Σ be the set of all $(u_0, y) \in \mathbb{E}_c \times Y_\eta$ such that (2.12) holds, then we have

$$\mathcal{M}_c = \{y(0) \in \mathbb{R}^n \mid (u_0, y) \in \Sigma\} = \{u_0 + \pi_h y(0) \mid (u_0, y) \in \Sigma\},$$

since $\pi_c y(0) = u_0$ for $(u_0, y) \in \Sigma$.

Step #2: fixed point argument. The next step is to determine the solution set Σ of (2.12). To do so we rewrite (2.12) as an equation in Y_η of the form

$$y = Su_0 + KG(y). \tag{2.13}$$

Here, we have used the following definitions:

- for each $u_0 \in \mathbb{E}_c$, we define $Su_0 : \mathbb{R} \to \mathbb{R}^n$ by

$$(Su_0)(t) := e^{tA}u_0, \quad \forall t \in \mathbb{R};$$

- for each function $y : \mathbb{R} \to \mathbb{R}^n$ we define $G(y) : \mathbb{R} \to \mathbb{R}^n$ by

$$G(y)(t) := g(y(t)), \quad \forall t \in \mathbb{R};$$

- for those functions $y : \mathbb{R} \to \mathbb{R}^n$ for which the integrals make sense we define $Ky : \mathbb{R} \to \mathbb{R}^n$ by

$$Ky(t) := \int_0^t e^{(t-\tau)A}\pi_c y(\tau)\mathrm{d}\tau + \int_{-\infty}^{+\infty} B(t-\tau)y(\tau)\mathrm{d}\tau, \quad \forall t \in \mathbb{R}.$$

Lemma 2.11. *S is a bounded linear operator from $\mathbb{E}_c$ into Y_η for each $\eta > 0$.*

Proof. We have for each $\eta > 0$

$$\|Su_0\|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|e^{tA}u_0\| \le C(\eta)\|u_0\|, \quad \forall u_0 \in \mathbb{E}_c.$$

∎

Lemma 2.12. *If $g \in \mathscr{C}_b^0(\mathbb{R}^n)$ then G maps $\mathscr{C}^0(\mathbb{R}, \mathbb{R}^n)$ into $\mathscr{C}_b^0(\mathbb{R}, \mathbb{R}^n)$; in particular G maps each Y_η into itself for any $\eta \ge 0$.*

If $g \in \mathscr{C}_b^1(\mathbb{R}^n)$ then we have for each $\eta > 0$

$$\|G(y_1) - G(y_2)\|_\eta \le |g|_1\|y_1 - y_2\|_\eta, \quad \forall y_1, y_2 \in Y_\eta.$$

Proof. The first part of the proof is obvious and the second part uses the fact that

$$|g(y_1(t)) - g(y_2(t))| \leq |g|_1 |y_1(t) - y_2(t)|,$$

by the mean value theorem. ∎

Lemma 2.13. *For each $\eta \in (0, \beta)$, K is a bounded linear operator on Y_η and there exists a continuous function $\gamma_0 : (0, \beta) \to \mathbb{R}_+$ such that*

$$\|K\|_\eta \leq \gamma_0(\eta), \quad \forall \eta \in (0, \beta).$$

Proof. Let $\eta \in (0, \beta)$ and $y \in Y_\eta$, then it follows from the definition of K that

$$\begin{aligned}
e^{-\eta|t|} \|Ky(t)\| &\leq \|y\|_\eta \sup_{t \in \mathbb{R}} e^{-\eta|t|} \left[\left| \int_0^t \left\| e^{(t-\tau)A} \pi_c \right\| e^{\eta|\tau|} \mathrm{d}\tau \right| + \int_{-\infty}^{+\infty} \|B(t-\tau)\| \, e^{\eta|\tau|} \mathrm{d}\tau \right] \\
&\leq \|y\|_\eta \sup_{t \in \mathbb{R}} \left[\left| \int_0^t \left\| e^{(t-\tau)A} \pi_c \right\| e^{-\eta|t-\tau|} \mathrm{d}\tau \right| + \int_{-\infty}^{+\infty} \|B(t-\tau)\| \, e^{-\eta|t-\tau|} \mathrm{d}\tau \right] \\
&= \|y\|_\eta \left[\max \left(\int_0^\infty \left\| e^{\tau A} \pi_c \right\| e^{-\eta\tau} \mathrm{d}\tau, \int_{-\infty}^0 \left\| e^{\tau A} \pi_c \right\| e^{\eta\tau} \mathrm{d}\tau \right) + \int_{-\infty}^{+\infty} \|B(\tau)\| \, e^{\eta\tau} \mathrm{d}\tau \right] \\
&\leq \|y\|_\eta C(\epsilon) \left[(\eta - \epsilon)^{-1} + 2(\beta - \eta - \epsilon)^{-1} \right],
\end{aligned}$$

for $\epsilon \in (0, \min(\eta, (\beta - \eta)))$. This shows that $K \in \mathscr{L}(Y_\eta)$ and we define

$$\gamma_0(\eta) := \max \left(\int_0^\infty \left\| e^{\tau A} \pi_c \right\| e^{-\eta\tau} \mathrm{d}\tau, \int_{-\infty}^0 \left\| e^{\tau A} \pi_c \right\| e^{\eta\tau} \mathrm{d}\tau \right) + \int_{-\infty}^{+\infty} \|B(\tau)\| \, e^{\eta\tau} \mathrm{d}\tau,$$

which is continuous on $(0, \beta)$ by the dominated convergence theorem. ∎

Lemma 2.14. *Let $\eta \in (0, \beta)$ and $g \in \mathscr{C}_b^1(\mathbb{R}^n)$ be such that*

$$\kappa := \|K\|_\eta |g|_1 < 1,$$

then $(\mathrm{Id} - KG)$ is a homeomorphism on Y_η, with inverse $\Phi : Y_\eta \to Y_\eta$, and the solution set of (2.12) is given by

$$\Sigma = \{(u_0, \Phi(Su_0)), \mid u_0 \in \mathbb{E}_c\}.$$

Proof. The two previous Lemma show that KG maps Y_η onto itself, and is globally Lipschitzian with Lipschitz constant κ and thus, under the condition $\kappa < 1$, the map $(\mathrm{Id} - KG) : Y_\eta \to Y_\eta$ is invertible, with an inverse $\Phi : Y_\eta \to Y_\eta$ which is itself Lipschitzian and the conclusion follows. ∎

Step #3: Lipschitz regularity of the center manifolds. We can now conclude the proof of Theorem 2.2. We define

$$\delta_0 := \sup_{\eta \in (0, \beta)} \gamma_0(\eta)^{-1},$$

and if $g \in \mathscr{C}_b^1(\mathbb{R}^n)$ and $|g|_1 < \delta_0$, then there exists some $\eta \in (0, \beta)$ such that $|g|_1 \gamma_0(\eta) < 1$, and thus the result of the previous Lemma applies. We can define $\Psi : \mathbb{E}_c \to \mathbb{E}_h$ by

$$\Psi(u_0) := \pi_h \Phi(Su_0)(0), \quad \forall u_0 \in \mathbb{E}_c.$$

Since Φ is continuous, also Ψ is continuous and by definition of Φ, we have

$$\Phi(Su_0) = Su_0 + KG(\Phi(Su_0)),$$

and thus

$$\Psi(u_0) = \int_{-\infty}^{+\infty} B(-\tau)g(\Phi(Su_0)(\tau))\mathrm{d}\tau, \quad \forall u_0 \in \mathbb{E}_c,$$

which gives the bound

$$\|\Psi(u_0)\| \leq 2C(\epsilon)|g|_0(\beta-\epsilon)^{-1}, \quad \forall u_0 \in \mathbb{E}_c, \quad \forall \epsilon(0,\beta).$$

And thus Ψ is bounded and globally Lipschitzian because Φ is.

2.3.3 Smoothness of the unique center manifold

The purpose of this section is to give a key result yielding to the proof of (iv) of Theorem 2.2. Since, $\Psi(u_0) = \pi_h \Phi(Su_0)(0)$ it is sufficient to show that the mapping $\Phi : Y_\eta \to Y_\eta$ is of class $\mathscr{C}^k$. The main difficulty here is that in general $G : Y_\eta \to Y_\eta$ is not differentiable. In fact, one can prove the following result.

Lemma 2.15. *Let $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ for some $k \geq 1$; let $\eta, \zeta \in (0,\beta)$ be such that $\zeta > k\eta$, and suppose that*

$$\kappa := \sup_{\xi \in [\eta,\zeta]} \|K\|_\xi |g|_1 < 1.$$

Then the mapping $\Phi : Y_\eta \to Y_\zeta$ is of class $\mathscr{C}^k$.

And the regularity of the center manifold directly follows from the above Lemma. There exists several proofs of the above result [8, 9] that use similar ideas on contractions on embedded Banach spaces (here Y_η). The proofs are quite technical and do not really fit within the scope of these lectures, and we refer the interested reader to the above references for more details.

2.3.4 Local center manifolds

In this section, we return to equation (2.2) with $f \in \mathscr{C}^k(\mathbb{R}^n)$ for some $k \geq 1$, and $f(0) = Df(0) = 0$. We denote simply by $\varphi(t, u_0)$ the flow of (2.2) which is defined for $u_0 \in \mathbb{R}^n$ on $I(u_0)$, the maximal interval of existence of solution. If $\Omega \subset \mathbb{R}^n$ is open and $u \in \mathbb{R}^n$, then we denote by $I_\Omega(u_0)$, the maximal time of existence with respect to Ω.

Theorem 2.3. *Let $A \in \mathscr{M}_n(\mathbb{R})$ be given and $f \in \mathscr{C}^k(\mathbb{R}^n)$ for some $k \geq 1$, with $f(0) = Df(0) = 0$. Then, there exists some $\Psi \in \mathscr{C}_b^k(\mathbb{E}_c, \mathbb{E}_h)$ and an open neighborhood Ω of the origin in $\mathbb{R}^n$ such that:*

(i) Tangency : $\Psi(0) = D\Psi(0) = 0$;

(ii) Invariance : *the manifold*

$$\mathcal{M}_c = \{u_0 + \Psi(u_0) \mid u_0 \in \mathbb{E}_c\};$$

is locally invariant for (2.2), i.e., we have

$$\varphi(t, u_0) \in \mathcal{M}_c, \quad \forall u_0 \in \mathcal{M}_c \cap \Omega, \quad \forall t \in I_\Omega(u_0);$$

(iii) $\mathcal{M}_c$ *contains the set of bounded solutions of* (2.2) *staying in* Ω *for all* $t \in \mathbb{R}$*; i.e., if* u *is a solution of* (2.2) *satisfying* $u(t) \in \Omega$ *for all* $t \in \mathbb{R}$*, then* $u(0) \in \mathcal{M}_c$*.*

Proof. Let $\delta_k > 0$, then one can find $\rho > 0$ such that $f_\rho \in \mathscr{C}_b^k(\mathbb{R}^n)$ is such that $|f_\rho|_1 < \delta_k$. Then the equation (2.8) coincides with (2.2) in $\Omega = \{u \in \mathbb{R}^n \mid \|u\| < \rho\}$, hence we have

$$\varphi(t, u_0) = \varphi_\rho(t, u_0), \quad \forall u_0 \in \mathbb{R}^n, \quad \forall t \in I_\Omega(u_0),$$

where $\varphi_\rho(t, u_0)$ denotes the flow of (2.8). Then, we can use the previous results to get the existence of a unique global center manifold for (2.8) given through the map $\Psi \in \mathscr{C}_b^k(\mathbb{E}_c, \mathbb{E}_h)$. Then we have $\Psi(0) = D\Psi(0) = 0$, the local invariance follows from the global invariance for the flow of (2.8). The characterization of bounded solutions also follows from the boundedness of the flow on the global center manifold. ∎

Remark 2.16. *Local center manifolds are in general not unique. This is due to the occurence in the proof of the smooth cut-off function which is not unique.*

Corollary 2.17. *Under the assumptions of Theorem 2.3, consider a solution* $u(t)$ *of* (2.2) *which belongs to* $\mathcal{M}_c$ *for* $t \in I$*, for some open interval* $I \subset \mathbb{R}$*. Then* $u(t) = u_0(t) + \Psi(u_0(t))$*, and* $u_0(t) \in \mathbb{E}_c$ *satisfies*

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = A_c u_0 + \pi_c f(u_0 + \Psi_c(u_0)), \quad t \in I, \tag{2.14}$$

where A_c *is the reduction of* A *to* $\mathbb{E}_s$*.*

This corollary shows that solutions on the center manifold are described by system of ODEs, also called *reduced system*, which has the same dimension as $\mathbb{E}_c$. It is not difficult to check that the reduction function ψ satisfies the equality:

$$D\Psi(u_0)\left(A_c u_0 + \pi_c f(u_0 + \Psi_c(u_0))\right) = A_h \Psi(u_0) + \pi_h f(u_0 + \Psi_c(u_0)), \quad \forall u_0 \in \mathbb{E}_c. \tag{2.15}$$

In practice, it is important to compute the reduced vector field in (2.31) (see Theorem 2.8), and more precisely its Taylor expansion.

Center-manifold reductions have become a central tool to the analysis of dynamical systems near non hyperbolique equilibrium points. The very first results on center manifolds go back to the pioneering works of Pliss [7] and Kelley [5] in the finite-dimensional setting. Our proof closely follows the one that can be found in [8].

2.3.5 Application on a simple example

Let us consider the following fourth order ODE

$$u'''' - u'' - u^2 = 0. \tag{2.16}$$

The first step is to write (2.16) as a first-order system. We set $U = (u_1, u_2, u_3, u_4)$ with $u = u_1$, $u' = u_2$, $u'' = u_3$ and $u''' = u_4$, and then the equation is equivalent with the system

$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU + f(U), \tag{2.17}$$

in which

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad f(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1^2 \end{pmatrix}.$$

Next, we need to compute the eigenvalues of A. A direct computations gives $\sigma(A) = \{0, \pm 1\}$. Note ± 1 are simple eigenvalues, while 0 is geometrically simple and algebraically double. As f is a smooth (polynomial) function from $\mathbb{R}^4$ to $\mathbb{R}^4$, we can apply the local center manifold theorem Theorem 2.3, and conclude of the existence of a local center manifold of class $\mathscr{C}^k$ for any $k \geq 1$. Since 0 is an algebraically double eigenvalue, the center space $\mathbb{E}_s$ is two-dimensional, so that the corresponding center manifold is also two-dimensional. Our purpose is to compute the Taylor expansion, up to order 2, of the vector field in the reduced equation.

Step #1: basis of $\mathbb{E}_c$ and projection onto $\mathbb{E}_c$. We start by computing a basis of $\mathbb{E}_c$, which is the two-dimensional generalized kernel of A. Solving for $A\zeta_0 = 0$, and then $A\zeta_1 = \zeta_0$, we find a basis $\{\zeta_0, \zeta_1\}$ for $\mathbb{E}_c$ given by

$$\zeta_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then, we compte the spectral projection π_c onto $\mathbb{E}_c$. One way is to compute π_c with the help of the adjoint matrix

$$A^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We claim that the spectral projection π_c is given by

$$\pi_c U = \langle U, \zeta_0^* \rangle \zeta_0 + \langle U, \zeta_1^* \rangle \zeta_1$$

where $\{\zeta_0^*, \zeta_1^*\}$ is a dual basis sastisfying

$$A^* \zeta_0^* = \zeta_1^*, \quad A^* \zeta_1^* = 0, \quad \langle \zeta_j, \zeta_j^* \rangle = \delta_{ij} \text{ for all } i, j \in \{0, 1\}.$$

We find that

$$\zeta_0^* = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \zeta_1^* = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Step #2: the reduced equation. According to the center manifold center, solutions on the center manifold are of the form

$$U(t) = U_0(t) + \Psi(U_0(t)),$$

in which $\Psi(0) = 0$, $D\Psi(0) = 0$, and $U_0(t) \in \mathbb{E}_s$, so that

$$U_0(t) = a(t)\zeta_0 + b(t)\zeta_1,$$

where a and b are real-valued functions. The reduced system is an ODE for $U_0 = (a, b)$ which can be obtained as follows. We compute both sides of (2.17)

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}t}\zeta_0 + \frac{\mathrm{d}b}{\mathrm{d}t}\zeta_1 + D\Psi(a\zeta_0 + b\zeta_1)\left(\frac{\mathrm{d}a}{\mathrm{d}t}\zeta_0 + \frac{\mathrm{d}b}{\mathrm{d}t}\zeta_1\right)$$

and

$$A(a\zeta_0 + b\zeta_1) + A\Psi(a\zeta_0 + b\zeta_1) = b\zeta_0 + A\Psi(a\zeta_0 + b\zeta_1)$$

together with

$$f(a\zeta_0 + b\zeta_1 + \Psi(a\zeta_0 + b\zeta_1)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a^2 + \mathcal{O}\left((|a| + |b|)^3\right) \end{pmatrix},$$

as $\Psi(U_0) = \mathcal{O}(\|U_0\|)^2$. Projecting along ζ_0 and ζ_1 one gets

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= b, \\ \frac{\mathrm{d}b}{\mathrm{d}t} &= -a^2 + \mathcal{O}\left((|a| + |b|)^3\right). \end{aligned}$$

2.3.6 Asymptotic behavior

In this section, we are going to study the behavior of solutions of (2.2) near any of its local center manifolds. Actually, we will only present the results for global problems and let as an exercise to pass from global to local using the cut-off function χ and the associated modified f_ρ from (2.7). The first step is the introduction of the concepts of a *center-stable* and *center-unstable manifold*. The theory concerning these manifolds completely parallels the center manifold theory given previously in the preceding sections, therefore we will give here an outline of the theory for the case of the center-unstable manifold.

We start with the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + g(u), \tag{2.18}$$

for $u \in \mathbb{R}^n$, $A \in \mathscr{M}_n(\mathbb{R})$, $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ for some $k \geq 1$, $g(0) = Dg(0) = 0$. We define the set

$$\mathcal{M}_{cu} := \left\{u_0 \in \mathbb{R}^n \mid \sup_{t \leq 0} \|\pi_s \varphi_g(t, u_0)\| < +\infty\right\}.$$

Thus, this manifold consists of $u_0 \in \mathbb{R}^n$ for which $\varphi_g(t, u_0)$ does not tend exponentially to 0 as $t \to +\infty$. We do not know exactly what $\varphi_g(t, u_0)$ does as $t \to +\infty$, it may or may not tend to 0 but it does not so exponentially.

From its definition, the set $\mathcal{M}_{cu}$ is clearly invariant by the flow of (2.18), and we have the following analogue of Lemma 2.7.

Lemma 2.18. *Let $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ for some $k \geq 1$ and $\eta \in (0, |\beta_-|)$, then we have*

$$\mathcal{M}_{cu} = \left\{u_0 \in \mathbb{R}^n \mid \sup_{t \leq 0} e^{\eta t} \|\varphi_g(t, u_0)\| < +\infty\right\}.$$

This naturally motivates us to introduce for each $\eta > 0$ the Banach space

$$Y_\eta^- := \left\{ y \in \mathscr{C}^0(\mathbb{R}_-, \mathbb{R}^n) \mid \|y\|_\eta := \sup_{t\le 0} e^{\eta t} \|y(t)\| < +\infty \right\}.$$

There is also an analogue of Lemma 2.10, which here takes the following form.

Lemma 2.19. *Let $g \in \mathscr{C}_b^1(\mathbb{R}^n)$, $\eta \in (0, |\beta_-|)$ and $y \in Y_\eta^-$. Then y is a solution of (2.18) if and only if there exists some $u_0 \in \mathbb{E}_{cu} := \mathbb{E}_c \oplus \mathbb{E}_u = \ker(\mathbb{E}_s)$ such that*

$$y(t) = e^{tA}u_0 + \int_0^t e^{(t-\tau)A}\pi_{cu} g(y(\tau))\mathrm{d}\tau + \int_{-\infty}^t e^{(t-\tau)A}\pi_s g(y(\tau))\mathrm{d}\tau, \quad \forall t \ge 0. \tag{2.19}$$

The combination of the above two lemma shows that

$$\mathcal{M}_{cu} = \{u_0 + \pi_s y(0) \mid (u_0, y) \in \Sigma_{cu}\},$$

where Σ_{cu} is the set of all $(u_0, y) \in \mathbb{E}_{cu} \times Y_\eta^-$ which satisfy (2.19). Therefore, we rewrite equation (2.19) in the form

$$y = Su_0 + K_{cu}G(y), \tag{2.20}$$

with S and G defined in the previous section, and

$$(K_{cu}y)(t) := \int_0^t e^{(t-\tau)A}\pi_{cu} y(\tau)\mathrm{d}\tau + \int_{-\infty}^t e^{(t-\tau)A}\pi_s y(\tau)\mathrm{d}\tau.$$

We have the following properties for these mappings:

- $S \in \mathscr{L}(\mathbb{E}_{cu}, Y_\eta^-)$ for each $\eta > 0$;
- $G : Y_\eta^- \to Y_\eta^-$ is globally Lipschitz for each $\eta > 0$, with Lipschitz constant $|g|_1$;
- $K_{cu} \in \mathscr{L}(Y_\eta^-)$ for each $\eta \in (0, |\beta_-|)$ and

  $$\|K_{cu}\|_\eta \le \gamma_{cu}(\eta), \quad \forall \eta \in (0, |\beta_-|),$$

  with $\gamma_{cu} : (0, |\beta_-|) \to \mathbb{R}_+$ a continuous function.

Let us note that γ_{cu} is given by

$$\gamma_{cu} := \int_{-\infty}^0 \left\|e^{\tau A}\pi_{cu}\right\| e^{\eta\tau}\mathrm{d}\tau + \int_0^{+\infty} \left\|e^{\tau A}\pi_s\right\| e^{\eta\tau}\mathrm{d}\tau.$$

Under the appropriate contraction condition the equation (2.20) has for each $u_0 \in \mathbb{E}_{cu}$ a unique solution $y \in Y_\eta^-$ which leads to the following result.

Theorem 2.4. *Let $\eta \in (0, |\beta_-|)$ and $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ with $k \ge 1$ be such that*

$$|g|_1 < \delta_{cu}^k,$$

for some $\delta_{cu}^k > 0$ small enough. Then there exists a mapping $\Psi_{cu} \in \mathscr{C}_b^k(\mathbb{E}_{cu}, \mathbb{E}_s)$ such that

$$\mathcal{M}_{cu} = \{u_0 + \Psi_{cu}(u_0) \mid u_0 \in \mathbb{E}_{cu}\}.$$

Moreover, if $\widetilde{\Psi} \in \mathscr{C}_b^k(\mathbb{E}_{cu}, \mathbb{E}_s)$ *is such that the manifold*

$$\mathcal{W}_{cu} = \left\{ u_0 + \widetilde{\Psi}_{cu}(u_0) \mid u_0 \in \mathbb{E}_{cu} \right\}$$

is invariant under the flow of (2.18)*, then* $\widetilde{\Psi} = \Psi_{cu}$ *and* $\mathcal{W}_{cu} = \mathcal{M}_{cu}$*.*

Now that we have a complete characterization of $\mathcal{M}_{cu}$, we consider the asymptotic behavior of the solutions of (2.18) which do not lie on the center-unstable manifold. The next theorem shows that each such solution converges exponentially for $t \to +\infty$ to a uniquely determined solution on the center-unstable manifold.

Theorem 2.5. *Under the conditions of Theorem 2.4, there exists a continuous mapping* $\mathcal{H}_{cu} : \mathbb{R}^n \to \mathcal{M}_{cu}$ *such that for each* $(u_0, \tilde{u}_0) \in \mathbb{R}^n \times \mathcal{M}_{cu}$ *we have*

$$\sup_{t \geq 0} e^{\eta t} \left\| \varphi_g(t, u_0) - \varphi_g(t, \tilde{u}_0) \right\| < +\infty \tag{2.21}$$

if and only if $\tilde{u}_0 = \mathcal{H}_{cu}(u_0)$*.*

We will prove this theorem in several steps.

Step #1: parametrization of the problem. We start with a definition and a lemma which outlines the strategy of the proof. For each $\eta \geq 0$ we define the Banach space Z_η by

$$Z_\eta := \left\{ z \in \mathscr{C}^0(\mathbb{R}, \mathbb{R}^n) \mid |||z|||_\eta := \sup_{t \in \mathbb{R}} e^{\eta t} \left\| z(t) \right\| < +\infty \right\}.$$

Observe that the restriction to $\mathbb{R}_-$ of a function $z \in Z_\eta$ belongs to Y_η^-, and that $\|z\|_\eta \leq |||z|||_\eta$. Using this definition, we have the following result.

Lemma 2.20. *Let* $\widehat{\varphi}_g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping such that*

- *(i)* $\widehat{\varphi}_g(t, u_0) = \varphi_g(t, u_0)$ *for* $t \geq 0$ *and* $u_0 \in \mathbb{R}^n$*;*
- *(ii)* $\widehat{\varphi}_g(\cdot, u_0) \in Y_\eta^-$ *for each* $u_0 \in \mathbb{R}^n$*.*

Let $u_0, \tilde{u}_0 \in \mathbb{R}^n$*. Then the following statements are equivalent:*

- *(1)* $\tilde{u}_0 \in \mathcal{M}_{cu}$ *and* (2.21) *holds;*
- *(2) there exists some* $z \in Z_\eta$ *such that* $\widehat{\varphi}_g(\cdot, u_0) + z$ *is a solution of* (2.18) *and* $\tilde{u}_0 = u_0 + z(0)$*.*

Proof.

($\Rightarrow$) Suppose that (1) holds, and let $z := \varphi(\cdot, \tilde{u}_0) - \widehat{\varphi}_g(\cdot, u_0)$. As $\tilde{u}_0 \in \mathcal{M}_{cu}$, we have from Lemma 2.18 that $\varphi(\cdot, \tilde{u}_0) \in Y_\eta^-$. From hypothesis (ii), this shows that $z \in Y_\eta^-$. Moreover hypothesis (i) and (2.21) show that also

$$\sup_{t \geq 0} e^{\eta t} \left\| z(t) \right\| < +\infty,$$

hence we have $z \in Z_\eta$. From the definition of z it is clear that $\tilde{u}_0 = u_0 + z(0)$.

($\Leftarrow$) Conversely, suppose that (2) holds. Then $\varphi(t, \tilde{u}_0) = \widehat{\varphi}_g(t, u_0) + z(t)$, then estimate (2.21) follows from (i) and the fact that $z \in Z_\eta$. Moreover, we have $z \in Y_\eta^-$ and $\widehat{\varphi}_g(\cdot, u_0) \in Y_\eta^-$, such that also $\varphi(\cdot, \tilde{u}_0) \in Y_\eta^-$. By Lemma 2.18, this implies $\tilde{u}_0 \in \mathcal{M}_{cu}$.

∎

The idea is to use Lemma 2.20. For that, we need to fix an appropriate choice for $\widehat{\varphi}_g(t, u_0)$ such that both conditions (i) and (ii) are satisfied. We cannot directly take $\widehat{\varphi}_g(t, u_0) = \varphi_g(t, u_0)$, since $\varphi_g(\cdot, u_0) \in Y_\eta^-$ only for $u_0 \in \mathcal{M}_{cu}$. But if we denote by $\varphi_g^{cu}(\cdot, u_0)$ the flow of the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A\pi_{cu}u + \pi_{cu}g(u),$$

then one can easily see that $\varphi_g^{cu}(\cdot, u_0) \in Y_\eta^-$ from the variation of constants formula and Lemma 2.18. Therefore, a judicious definition for $\widehat{\varphi}_g(t, u_0)$ is as follows

$$\widehat{\varphi}_g(t, u_0) = \begin{cases} \varphi_g(t, u_0) & \text{for } t \geq 0, \\ \varphi_g^{cu}(t, u_0) & \text{for } t \leq 0. \end{cases}$$

Then both conditions (i) and (ii) hold and moreover

$$\pi_{cu}\widehat{\varphi}_g(t, u_0) = e^{(t-t_0)}\pi_{cu}\widehat{\varphi}_g(t_0, u_0) + \int_{t_0}^{t} e^{(t-\tau)A}\pi_{cu}g(\widehat{\varphi}_g(\tau, u_0))\mathrm{d}\tau, \quad \forall t, t_0 \in \mathbb{R}. \tag{2.22}$$

Step #2: condition on z. Next, we find a condition for $z \in Z_\eta$ to be such that $\widehat{\varphi}_g(\cdot, u_0) + z$ is a solution of (2.18).

Lemma 2.21. *Let $u_0 \in \mathbb{R}^n$, $\eta \in (0, |\beta_-|)$, and $z \in Z_\eta$. Then $\widehat{\varphi}_g(\cdot, u_0) + z$ is a solution of (2.18) if and only if*

$$\begin{aligned} z(t) = -\pi_s\widehat{\varphi}_g(t, u_0) &+ \int_{-\infty}^{t} e^{(t-\tau)A}\pi_s g(\widehat{\varphi}_g(\tau, u_0) + z(\tau))\mathrm{d}\tau \\ &- \int_{t}^{+\infty} e^{(t-\tau)A}\pi_{cu}\left[g(\widehat{\varphi}_g(\tau, u_0) + z(\tau)) - g(\widehat{\varphi}_g(\tau, u_0))\right]\mathrm{d}\tau, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{2.23}$$

Proof. Suppose that $\widehat{\varphi}_g(\cdot, u_0) + z$ is a solution of (2.18). The variation of constants formula together with (2.22) gives

$$\begin{aligned} z(t) = &-\pi_s\widehat{\varphi}_g(t, u_0) + e^{(t-t_0)A}\left(\pi_s\widehat{\varphi}_g(t_0, u_0) + z(t_0)\right) \\ &+ \int_{t_0}^{t} e^{(t-\tau)A}\pi_{cu}\left[g(\widehat{\varphi}_g(\tau, u_0) + z(\tau)) - g(\widehat{\varphi}_g(\tau, u_0))\right]\mathrm{d}\tau \\ &+ \int_{t_0}^{t} e^{(t-\tau)A}\pi_s g(\widehat{\varphi}_g(\tau, u_0) + z(\tau))\mathrm{d}\tau, \quad \forall t, t_0 \in \mathbb{R}. \end{aligned}$$

Using estimates (2.4) in combination with $\|z(t)\| \leq e^{-\eta t}\|\|z\|\|_\eta$ and the fact that $\pi_s\widehat{\varphi}_g(t, u_0) = \pi_s u_0$ for $t \geq 0$ on then gets that for fixed $t \in \mathbb{R}$ one has

$$\lim_{t_0 \to -\infty} e^{(t-t_0)A}\left(\pi_s\widehat{\varphi}_g(t_0, u_0) + \pi_s z(t_0)\right) = 0,$$

and
$$\lim_{t_0 \to +\infty} e^{(t-t_0)A}\pi_{cu} z(t_0) = 0.$$
Therefore applying π_s and π_{cu} to $z(t)$ in the previous equation and taking the limit for $t_0 \to -\infty$ (respectively $t_0 \to +\infty$), one obtains (2.23).

Conversely, suppose that (2.23) holds. Recombining the terms and using (2.22) with $t_0 = 0$ one then easily finds
$$\widehat{\varphi}_g(t,u_0) + z(t) = e^{tA}(u_0 + z(0)) + \int_0^t e^{(t-\tau)A} g(\widehat{\varphi}_g(\tau,u_0) + z(\tau))\mathrm{d}\tau, \quad \forall t \in \mathbb{R},$$
which proves that $\widehat{\varphi}_g(\cdot,u_0) + z$ is a solution of (2.18). ∎

Step #3: solving (2.23). First, we rewrite (2.23) in the following form
$$\begin{aligned} z(t) = -\pi_s\widehat{\varphi}_g(t,u_0) &+ \int_{-\infty}^t e^{(t-\tau)A}\pi_s g(\widehat{\varphi}_g(\tau,u_0))\mathrm{d}\tau \\ &+ \int_{-\infty}^t e^{(t-\tau)A}\pi_s \hat{g}(\tau,u_0,z(\tau))\mathrm{d}\tau - \int_t^{+\infty} e^{(t-\tau)A}\pi_{cu}\hat{g}(\tau,u_0,z(\tau))\mathrm{d}\tau, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{2.24}$$
where $\hat{g} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by
$$\hat{g}(t,u_0,z) := g(\widehat{\varphi}_g(t,u_0) + z) - g(\widehat{\varphi}_g(t,u_0)).$$
One has
$$\|\hat{g}(t,u_0,z)\| \leq 2|g|_0 \text{ and } \|\hat{g}(t,u_0,z)\| \leq |g|_1 \|z\|.$$
It is not so difficult to check that for each $(u_0,z) \in \mathbb{R}^n \times Z_\eta$ each of the terms on the right-hand side of (2.24) represents a function belonging to Z_η. Therefore, we can write (2.24) in the form
$$z = z_0(u_0) + \widehat{K}\widehat{G}(u_0,z), \tag{2.25}$$
where $z_0 : \mathbb{R}^n \to \mathbb{R}^n$, $\widehat{K} : Z_\eta \to Z_\eta$ and $\widehat{G} : \mathbb{R}^n \times Z_\eta \to Z_\eta$ are defined by
$$\begin{aligned} z_0(u_0)(t) &:= -\pi_s\widehat{\varphi}_g(t,u_0) + \int_{-\infty}^t e^{(t-\tau)A}\pi_s g(\widehat{\varphi}_g(\tau,u_0))\mathrm{d}\tau, \\ (\widehat{K}z)(t) &:= \int_{-\infty}^t e^{(t-\tau)A}\pi_s z(\tau)\mathrm{d}\tau - \int_t^{+\infty} e^{(t-\tau)A}\pi_{cu} z(\tau)\mathrm{d}\tau, \end{aligned}$$
and
$$\widehat{G}(u_0,z)(t) := \hat{g}(t,u_0,z(t)).$$
One easily shows that:

- $\widehat{K} \in \mathscr{L}(Z_\eta)$ with $\left\|\widehat{K}\right\|_\eta \leq \gamma_{cu}(\eta)$;
- $\left\|\left\|\widehat{G}(u_0,z_1) - \widehat{G}(u_0,z_2)\right\|\right\|_\eta \leq |g|_1 \|\|z_1 - z_2\|\|_\eta$.

Therefore, we have the following result.

Lemma 2.22. *Assume that the conditions of Theorem 2.4 are satisfied. Then the equation (2.25) has for each $u_0 \in \mathbb{R}^n$ a unique solution $z = \hat{z}(u_0) \in Z_\eta$, and the conclusion of Theorem 2.5 holds with*

$$\mathcal{H}_{cu}(u_0) := u_0 + \hat{z}(u_0)(0), \quad \forall u_0 \in \mathbb{R}^n.$$

Proof. Fix $u_0 \in \mathbb{R}^n$, and suppose that $\tilde{u}_0 \in \mathcal{M}_{cu}$ is such that (2.21) holds. By Lemma 2.20 there exists some $z \in Z_\eta$ such that $\tilde{u}_0 = u_0 + z(0)$ while $\widehat{\varphi}_g(\cdot, u_0) + z$ is a solution of (2.18). By Lemma 2.21 this means that (2.25) holds; hence $z = \hat{z}(u_0)$ by uniqueness and $\tilde{u}_0 = u_0 + \hat{z}(u_0)(0) = \mathcal{H}_{cu}(u_0)$.

Conversely, if $\tilde{u}_0 = \mathcal{H}_{cu}(u_0)$ then $\tilde{u}_0 = u_0 + z(0)$ where $z = \hat{z} \in Z_\eta$ is such that $\widehat{\varphi}_g(\cdot, u_0) + z$ is a solution of (2.18). It then follows from Lemma 2.20 that $\tilde{u}_0 \in \mathcal{M}_{cu}$ and that (2.21) holds.

We conclude that $\mathcal{H}_{cu}$ is a mapping from $\mathbb{R}^n$ to $\mathcal{M}_{cu}$ satisfying the conclusion of Theorem 2.5. ■

Step #4: regularity of $\mathcal{H}_{cu}$. It only remains to show that $\mathcal{H}_{cu}$ is continuous which will directly follow from the continuity of the mapping $\hat{z} : \mathbb{R}^n \to Z_\eta$ as given by the next lemma.

Lemma 2.23. *Assume that the conditions of Theorem 2.4 are satisfied. Then the mapping $\hat{z} : \mathbb{R}^n \to Z_\eta$ is continuous.*

Proof. We have obtained $\hat{z}$ has a fixed point in Z_η of (2.25). To prove the continuity of $\hat{z}$ it is thus enough to show that the right-hand side of (2.25) depends continuously upon $u_0 \in \mathbb{R}^n$, that is z_0 and $\widehat{G}(\cdot, z)$ are both continuous.

- $z_0 : \mathbb{R}^n \to Z_\eta$ is continuous. From its definition we have for all $u_0, v_0 \in \mathbb{R}^n$ that

  $$\sup_{t \le 0} e^{\eta t} \|z_0(u_0)(t) - z_0(v_0)(t)\| \le \|\pi_s(u_0 - v_0)\| + M(|\beta_-| - \eta) \int_{-\infty}^{0} e^{\eta\tau} \|g(\widehat{\varphi}_g(\tau, u_0)) - g(\widehat{\varphi}_g(\tau, v_0))\| \,\mathrm{d}\tau.$$

  Since g is bounded and $g(\widehat{\varphi}_g(\tau, u_0)) \to g(\widehat{\varphi}_g(\tau, v_0))$ as $u_0 \to v_0$, it follows from the dominated convergence theorem that the right-hand side of the above equation converges to zero as $u_0 \to v_0$. Now, for $t \ge 0$ we use the fact that $\widehat{\varphi}_g(t, u_0) = \varphi_g(t, u_0)$ to rewrite $z_0(u_0)(t)$ as

  $$z_0(u_0)(t) = e^{tA} \pi_s z_0(u_0)(0), \quad t \ge 0, \quad \forall u_0 \in \mathbb{R}^n.$$

  It follows that

  $$\sup_{t \ge 0} e^{\eta t} \|z_0(u_0)(t) - z_0(v_0)(t)\| \le M(|\beta_-| - \eta) \|z_0(u_0)(0) - z_0(v_0)(0)\| \to 0 \text{ as } u_0 \to v_0.$$

- To gain some regularity for $\widehat{G}$ the idea is to remark that in fact $z \in Z_\eta \cap Z_\zeta$ for all $\zeta \in (\eta, |\beta_-|)$. So we will prove that $\widehat{G}(\cdot, z) : \mathbb{R}^n \to Z_\eta$ is continuous for each $z \in Z_\eta \cap Z_\zeta$. For each $u_0, v_0 \in \mathbb{R}^n$ and for each $R > 0$ we have the following estimate

  $$\begin{aligned} \left\| \widehat{G}(u_0, z) - \widehat{G}(v_0, z) \right\|_\eta &= \sup_{t \in \mathbb{R}} e^{\eta t} \|\hat{g}(t, u_0, z(t)) - \hat{g}(t, v_0, z(t))\| \\ &\le \max\left(4|g|_0 e^{-\eta R}, 2|g|_1 \|z\|_\zeta e^{(\eta - \zeta)R}, \sup_{|t| \le R} e^{\eta R} \|\hat{g}(t, u_0, z(t)) - \hat{g}(t, v_0, z(t))\| \right). \end{aligned}$$

  By first choosing $R > 0$ large enough and then v_0 sufficiently close to u_0, the above expression can be made arbitrarily small.

■

Corollary 2.24. *Assume the conditions of Theorem 2.4. Then there exists for each $\tilde{u}_0 \in \mathcal{M}_{cu}$ and for each $\epsilon > 0$ some $\delta > 0$ such that*

$$\|\varphi_g(t, u_0) - \varphi_g(t, \mathcal{H}_{cu}(u_0))\| \leq \epsilon e^{-\eta t}, \quad \forall t \geq 0,$$

for all $u_0 \in \mathbb{R}^n$ with $\|u_0 - \tilde{u}_0\| < \delta$.

Proof. First, we remark that

$$\hat{z}(u_0)(t) = \varphi_g(t, \mathcal{H}_{cu}(u_0)) - \varphi_g(t, u_0), \quad \forall t \geq 0, \quad \forall u_0 \in \mathbb{R}^n.$$

We also have by definition that $\mathcal{H}_{cu}(u_0) = u_0$ whenever $u_0 \in \mathcal{M}_{cu}$, and hence

$$\hat{z}(u_0)(t) = 0, \quad \forall t \geq 0, \quad \forall u_0 \in \mathcal{M}_{cu}.$$

Now let $\tilde{u} \in \mathcal{M}_{cu}$ and $\epsilon > 0$, by continuity of $\hat{z}$ we can find some $\delta > 0$ such that $|||\hat{z}(u_0) - \hat{z}(\tilde{u}_0)|||_\eta \leq \epsilon$ if $\|u_0 - \tilde{u}_0\| < \delta$. As a consequence we have

$$\sup_{t \geq 0} e^{\eta t} \|\hat{z}(u_0)(t)\| = \sup_{t \geq 0} e^{\eta t} \|\hat{z}(u_0)(t) - \hat{z}(\tilde{u}_0)(t)\| \leq |||\hat{z}(u_0) - \hat{z}(\tilde{u}_0)|||_\eta \leq \epsilon,$$

which concludes the proof. ■

If in the statement of Theorem 2.5 we do not impose the condition $\tilde{u}_0 \in \mathcal{M}_{cu}$ then the approach simplifies considerably, leading to the following result.

Theorem 2.6. *Under the conditions of Theorem 2.4, there exists a continuous mapping $\mathcal{J}_s : \mathbb{R}^n \times \mathbb{E}_s \to \mathbb{E}_{cu}$ such that for all $u_0 \in \mathbb{R}^n$ we have*

$$\left\{ \tilde{u}_0 \in \mathbb{R}^n \mid \sup_{t \geq 0} e^{\eta t} \|\varphi_g(t, \tilde{u}_0) - \varphi_g(t, u_0)\| < +\infty \right\} = \{u_s + \mathcal{J}_s(u_0, u_s) \mid u_s \in \mathbb{E}_s\}.$$

Proof. The argument should now be standard. Fix some $x \in \mathbb{R}^n$. If $\tilde{u}_0 \in \mathbb{R}^n$ is such that estimate (2.21) is satisfied then the function z defined by $z(t) := \varphi_g(t, \tilde{u}_0) - \varphi_g(t, u_0)$ belongs to the Banach space

$$Z_\eta^+ := \left\{ z \in \mathscr{C}^0(\mathbb{R}_+, \mathbb{R}^n) \mid |||z|||_{\eta,+} := \sup_{t \geq 0} e^{\eta t} \|z(t)| < +\infty \right\}.$$

Conversely, if $z \in Z_\eta^+$ is such that $\varphi_g(\cdot, u_0) + z$ is a solution of (2.18), then $z(t) = \varphi_g(t, \tilde{u}_0) - \varphi_g(t, u_0)$, where $\tilde{u}_0 = u_0 + z(0)$, hence (2.21) is satisfied. Therefore, we need to determine all $z \in Z_\eta^+$ such that $\varphi_g(\cdot, u_0) + z$ is a solution of (2.18). This is equivalent to finding all solutions $z \in Z_\eta^+$ of the equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + \tilde{g}(t, u_0, z), \tag{2.26}$$

where $\tilde{g} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\tilde{g}(t, u_0, z) := g(\varphi_g(t, u_0) + z) - g(\varphi_g(t, u_0)),$$

satisfies

$$\|\tilde{g}(t,u_0,z)\| \le 2|g|_0, \qquad \text{and} \qquad \|\tilde{g}(t,u_0,z)\| \le |g|_1 \, \|z\| \,.$$

It is easy to check that $z \in Z_\eta^+$ is a solution of (2.26) if and only if there exists some $u_s \in \mathbb{E}_s$ such that

$$z = Su_s + K_s \widetilde{G}(u_0, z), \tag{2.27}$$

where $S \in \mathscr{L}(\mathbb{E}_s, Z_\eta^+)$, $K_s \in \mathscr{L}(Z_\eta^+)$ and $\widetilde{G} : \mathbb{R}^n \times Z_\eta^+ \to Z_\eta^+$ are defined by

$$\begin{aligned} (Su_s)(t) &:= e^{tA}u_s, \\ (K_s z)(t) &:= \int_0^t e^{(t-\tau)A}\pi_s z(\tau)\mathrm{d}\tau - \int_t^{+\infty} e^{(t-\tau)A}\pi_{cu} z(\tau)\mathrm{d}\tau, \\ \widetilde{G}(u_0,z)(t) &:= \tilde{g}(t,u_0,z(t)). \end{aligned}$$

One easily verifies that

- $\left\|\left|\widetilde{G}(u_0,z_1) - \widetilde{G}(u_0,z_2)\right|\right\|_{\eta,+} \le |g|_1 \|\|z_1 - z_2\|\|_{\eta,+}$;
- $\|K_s\|_{\mathscr{L}(Z_\eta^+)} \le \gamma_{cu}(\eta)$ for all $\eta \in (0, |\beta_-|)$.

Therefore, provided that $\gamma_{cu}(\eta)|g|_1 < 1$, (2.27) has for each $(u_0, u_s) \in \mathbb{R}^n \times \mathbb{E}_s$ a unique solution $z = \tilde{z}(u_0, u_s) \in Z_\eta^+$, and we have, for each $u_0 \in \mathbb{R}^n$

$$\left\{\tilde{u}_0 \in \mathbb{R}^n \mid \varphi_g(\cdot, \tilde{u}_0) - \varphi_g(\cdot, u_0) \in Z_\eta^+\right\} = \left\{u_0 + \tilde{z}(u_0,u_s)(0) \mid u_s \in \mathbb{E}_s\right\}.$$

As a consequence, we define $\mathcal{J}_s : \mathbb{R}^n \times \mathbb{E}_s \to \mathbb{E}_{cu}$ by

$$\mathcal{J}_s(u_0, u_s) := \pi_{cu}u_0 + \pi_{cu}\tilde{z}(u_0, u_s - \pi_s u_0)(0),$$

as $\pi_s(u_0 + \tilde{z}(u_0,u_s)(0)) = \pi_s u_0 + u_s$. To prove that $\mathcal{J}_s$ is continuous, we prove that $\tilde{z} : \mathbb{R}^n \times \mathbb{E}_s \to Z_\eta^+$ is continuous, which boils down to prove that $u_0 \mapsto \widetilde{G}(u_0, z)$ is continuous from $\mathbb{R}^n$ to Z_η^+ for each fixed $z \in Z_\eta^+ \cap Z_\zeta^+$ with $\zeta \in (\eta, |\beta_-|)$. The argument is similar to the case $\widehat{G}$, studied previously, and thus let as an exercise. ∎

We make the following remarks.

1. If $g \in \mathscr{C}_b^k$ for some $k \ge 1$, then it is possible to show that the mapping $u_s \mapsto \mathcal{J}_s(u_0, u_s)$ is for each $u_0 \in \mathbb{R}^n$ of class $\mathscr{C}^k$ from $\mathbb{E}_s$ to $\mathbb{E}_{cu}$.

2. We see that for each $u_0 \in \mathbb{R}^n$ we have shown that

$$\mathcal{M}_{cu} \cap \{u_s + \mathcal{J}_s(u_0,u_s) \mid u_s \in \mathbb{E}_s\} = \{\mathcal{H}_{cu}(u_0)\}.$$

3. For each $u_0 \in \mathbb{R}^n$ there is a manifold

$$\mathcal{M}_s(u_0) := \{u_s + \mathcal{J}_s(u_0,u_s) \mid u_s \in \mathbb{E}_s\},$$

such that all solutions starting at $t = 0$ on $\mathcal{M}_s(u_0)$ converge exponentially to the solution $\varphi_g(t, u_0)$ as $t \to +\infty$. We call $\mathcal{M}_s(u_0)$ the *stable manifold* of $u_0 \in \mathbb{R}^n$, which is of class $\mathscr{C}^k$ if $g \in \mathscr{C}_b^k$ for some

$k \geq 1$. Each such stable manifold has a unique intersection point with $\mathcal{M}_{cu}$ (given by $\mathcal{H}_{cu}(u_0)$). It follows that we have the following foliation of $\mathbb{R}^n$:

$$\mathbb{R}^n = \bigcup_{\tilde{u}_0 \in \mathcal{M}_{cu}} \mathcal{M}_s(\tilde{u}_0).$$

This foliation is continuous, a the leaves $\mathcal{M}_s(\tilde{u}_0)$ are of class $\mathscr{C}^k$ if $g \in \mathscr{C}_b^k$ for some $k \geq 1$. Moreover, the flow of (2.18) leaves this foliation invariant.

4. One can also prove that there exists a homeomorphism $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ such that for each $u_0 \in \mathbb{R}^n$ the mapping $t \mapsto \Theta(\varphi(t, u_0))$ is a solution of the equation

$$\begin{aligned} \frac{\mathrm{d}u_s}{\mathrm{d}t} &= Au_s, \\ \frac{\mathrm{d}u_{cu}}{\mathrm{d}t} &= Au_{cu} + \pi_{cu} g(u_{cu} + \Psi_{cu}(u_{cu})). \end{aligned}$$

The second equation describes the flow on the center-unstable manifold.

It is obvious that there is a completely analogous set of results on the *center-stable manifold*, defined as

$$\mathcal{M}_{cs} := \left\{ u_0 \in \mathbb{R}^n \mid \sup_{t \geq 0} \|\pi_u \varphi_g(t, u_0)\| < +\infty \right\},$$

that we all summarize in the following theorem.

Theorem 2.7. *There exists some $\delta_{cs} > 0$ such that if $g \in \mathscr{C}_b^1(\mathbb{R}^n)$ and $|g|_1 < \delta_{cs}$ the there holds:*

(i) $\mathcal{M}_{cs} = \{u_0 + \Psi_{cs}(u_0) \mid u_0 \in \mathbb{E}_{cs}\}$ *with* $\Psi \in \mathscr{C}_b^1(\mathbb{E}_{cs}, \mathbb{E}_u)$*;*

(ii) there exists some $\eta \in (0, \beta_+)$ and continuous mappings $\mathcal{J}_u : \mathbb{R}^n \times \mathbb{E}_u \to \mathbb{E}_{cs}$ and $\mathcal{H}_{cs} : \mathbb{R}^n \to \mathcal{M}_{cs}$ such that for each $u_0 \in \mathbb{R}^n$ one has

$$\mathcal{M}_u(u_0) := \left\{ \tilde{u}_0 \in \mathbb{R}^n \mid \sup_{t \leq 0} e^{-\eta t} \|\varphi_g(t, \tilde{u}_0) - \varphi_g(t, u_0)\| < +\infty \right\} = \{w_u + \mathcal{J}_u(u_0, w_u) \mid w_u \in \mathbb{E}_u\},$$

and $\mathcal{M}_u(u_0) \cap \mathcal{M}_{cs} = \{\mathcal{H}_{cs}(u_0)\}$;

- *if $g \in \mathscr{C}_b^k$ for some $k \geq 1$, then i the mapping $w \mapsto \mathcal{J}_u(u_0, w)$ is for each $u_0 \in \mathbb{R}^n$ of class $\mathscr{C}^k$ from $\mathbb{E}_u$ to $\mathbb{E}_{cs}$;*
- *for each $k \geq 1$, there exists $\delta_{cs}^k > 0$ such that $\Psi \in \mathscr{C}_b^k(\mathbb{E}_{cs}, \mathbb{E}_u)$ if $g \in \mathscr{C}_b^k$ and $|g|_1 < \delta_{cs}^k$.*

For each $u_0 \in \mathbb{R}^n$, one calls $\mathcal{M}_u(u_0)$ the *unstable manifold* of u_0 which is of class $\mathscr{C}^k$ if $g \in \mathscr{C}_b^k$. Taking $\eta \in (0, \beta)$, it follows that

$$\mathcal{M}_c = \mathcal{M}_{cs} \cap \mathcal{M}_{cu}.$$

Moreover, if $\sigma_u = \emptyset$, then $\mathcal{M}_c = \mathcal{M}_{cu}$.

It is clear that by using the cut-off technique each of forgoing global results leads to a corresponding local result for the equation (2.2). Furthermore, we have the following result which allows to conclude in the case *(iii)* of Theorem 2.1.

Theorem 2.8. *Assume that $\sigma_u = \emptyset$ and that $f \in \mathscr{C}^k(\mathbb{R}^n)$ with $k \geq 2$. Let $\mathcal{M}_c^{loc}$ be a local center manifold for (2.2) as given by Theorem 2.3. Let Ω be some neighborhood of the origin. Let $u_0 \in \mathcal{M}_c^{loc} \cap \Omega$ be such that $\overline{\{\varphi(t, u_0) \mid t \geq 0\}} \subset \Omega$. Let $u_c(t) := \pi_c\varphi(t, u_0)$ for $t \geq 0$. Then:*

(i) $u_c(t)$ is a solution of the equation

$$\frac{\mathrm{d}u_c}{\mathrm{d}t} = Au_c + \pi_c f(u_c + \Psi_c(u_c)); \tag{2.28}$$

(ii) if $u_c(t)$ is stable (asymptotically stable, unstable) as a solution of (2.28), then $\varphi(t, u_0)$ is stable (asymptotically stable, unstable) as a solution of (2.2).

Proof. Only (ii) needs to be proved. Suppose that $u_c(t)$ is stable as a solution of (2.28). Using the cut-off function argument, one gets the existence of a unique global center manifold $\mathcal{M}_c^{\mathrm{glob}}$ with $g(u) = f(u)$ for all $u \in \Omega$ and $g \in \mathscr{C}_b^k(\mathbb{R}^n)$ such that $\mathcal{M}_c^{\mathrm{glob}} \cap \Omega = \mathcal{M}_c^{\mathrm{loc}} \cap \Omega$. As $\sigma_u = \emptyset$, then $\mathcal{M}_c^{\mathrm{glob}}$ is also the unique global center-unstable manifold. Then we have $\varphi(t, u_0) = \varphi_g(t, u_0)$ for all $t \geq 0$ and $u_c(t) = \pi_c\varphi_g(t, u_0)$ is a solution of the equation

$$\frac{\mathrm{d}u_c}{\mathrm{d}t} = Au_c + \pi_c g(u_c + \Psi_c(u_c)); \tag{2.29}$$

which in Ω coincides with (2.28). Hence $u_c(t)$ is also stable as a solution of (2.29).

Let $v_0 \in \mathcal{M}_c^{\mathrm{glob}}$, then $\varphi_g(t, v_0) = \pi_c\varphi_g(t, v_0) + \Psi(\pi_c\varphi_g(t, v_0))$ with $\pi_c\varphi_g(t, v_0)$ is a solution of (2.29), hence

$$\|\varphi_g(t, v_0) - \varphi_g(t, u_0)\| \leq (1 + |\Psi|_1) \|\pi_c\varphi_g(t, v_0) - u_c(t)\|.$$

As $u_c(t)$ is a stable solution of (2.29), it follows that for each $\epsilon > 0$, one can find $\delta > 0$ such that $v_0 \in \mathcal{M}_c^{\mathrm{glob}}$ and $\|u_0 - v_0\| < \delta$ imply $\|\varphi_g(t, v_0) - \varphi_g(t, u_0)\| \leq \epsilon$ for all $t \geq 0$.

Now consider $\mathcal{H}_{cu} : \mathbb{R}^n \to \mathcal{M}_c^{\mathrm{glob}}$ which is continuous and $\mathcal{H}_{cu}(u_0) = u_0$ as $u_0 \in \mathcal{M}_c^{\mathrm{loc}} \cap \Omega$. Hence, there exists some δ_1 such that $\|\mathcal{H}_{cu}(v_0) - u_0\| < \delta$ if $v_0 \in \mathbb{R}^n$ and $\|v_0 - u_0\| < \delta_1$. Moreover, we can take δ_1 sufficiently small that

$$\|\varphi_g(t, v_0) - \varphi_g(t, \mathcal{H}_{cu}(v_0))\| \leq \epsilon e^{-\eta t}, \quad \forall t \geq 0.$$

It follows that for each $v_0 \in \mathbb{R}^n$ with $\|v_0 - u_0\| < \delta_1$ we have

$$\|\varphi_g(t, v_0) - \varphi_g(t, u_0)\| \leq \|\varphi_g(t, v_0) - \varphi_g(t, \mathcal{H}_{cu}(v_0))\| + \|\varphi_g(t, \mathcal{H}_{cu}(v_0)) - \varphi_g(t, u_0)\| \leq \epsilon e^{-\eta t} + \epsilon,$$

for all $t \geq 0$. This proves that $\varphi_g(t, u_0)$ is a stable solution of (2.18) which coincides with (2.2) in the neighborhood Ω, which shows that $\varphi(t, u_0)$ is a stable solution of (2.2) for each $u_0 \in \mathcal{M}_c^{\mathrm{loc}} \cap \Omega$. ∎

2.3.7 Natural extensions

Parameter-dependent center-manifolds. In many applications, ODEs of the form (2.2) have parameters. It is then natural to extend our center-manifold results to this setting. Let us consider the *parameter-dependent* differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + f(u, \mu), \tag{2.30}$$

in the case where $u_* = 0$ is a non hyperbolic equilibrium that is $\mathbb{E}_c = \sigma(A) \cap \mathbf{i}\mathbb{R} \neq \emptyset$. We assume that the nonlinearity $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is defined in a neighborhood of $(0,0) \in \mathbb{R}^n \times \mathbb{R}^d$. Here $\mu \in \mathbb{R}^d$ is a parameter that we assume to be small. More precisely, we suppose that there exists neighborhoods $0 \in \mathcal{O}_u \in \mathbb{R}^n$ and $0 \in \mathcal{O}_\mu \in \mathbb{R}^d$ such that $f \in \mathscr{C}^k(\mathcal{O}_u \times \mathcal{O}_\mu, \mathbb{R}^n)$ for some $k \geq 1$, with $f(0,0) = 0_{\mathbb{R}^n}$ and $D_u f(0,0) = 0_{\mathscr{M}_n(\mathbb{R})}$.

Theorem 2.9. *Let $A \in \mathscr{M}_n(\mathbb{R})$ be given and $f \in \mathscr{C}^k(\mathcal{O}_u \times \mathcal{O}_\mu, \mathbb{R}^n)$ for some $k \geq 1$, with $f(0,0) = 0_{\mathbb{R}^n}$ and $D_u f(0,0) = 0_{\mathscr{M}_n(\mathbb{R})}$. Then, there exists some $\Psi \in \mathscr{C}^k(\mathbb{E}_c \times \mathbb{R}^d, \mathbb{E}_h)$ and an open neighborhood $\Omega_u \times \Omega_\mu$ of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^d$ such that:*

(i) Tangency : *$\Psi(0,0) = 0$, and $D_u\Psi(0,0) = 0$;*

(ii) Invariance : *for all $\mu \in \Omega_\mu$, the manifold*

$$\mathcal{M}_c(\mu) = \{u_0 + \Psi(u_0, \mu) \mid u_0 \in \mathbb{E}_c\};$$

is locally invariant for (2.30), i.e., if $\varphi(t,u_0)$ is a solution of (2.30) satisfying $u_0 \in \mathcal{M}_c(\mu) \cap \Omega_u$ and $\varphi(t,u_0) \in \Omega_u$ for all $t \in I_{\Omega_u}(u_0)$, then $\varphi(t,u_0) \in \mathcal{M}_c(\mu)$ for all $t \in I_{\Omega_u}(u_0)$;

(iii) $\mathcal{M}_c(\mu)$ contains the set of bounded solutions of (2.30) staying in Ω_u for all $t \in \mathbb{R}$; i.e., if u is a solution of (2.30) satisfying $u(t) \in \Omega_u$ for all $t \in \mathbb{R}$, then $u(0) \in \mathcal{M}_c(\mu)$.

The main idea for the proof of Theorem 2.9 is to augment (2.30) with the equation

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0,$$

and then apply the same type of arguments. In this parameter-dependent setting, it is possible to get an analogue of the reduced equation (2.31).

Corollary 2.25. *Under the assumptions of Theorem 2.9, consider a solution $u(t)$ of (2.30) which belongs to $\mathcal{M}_c(\mu)$ for $t \in I$ and for all $\mu \in \Omega_\mu$, for some open interval $I \subset \mathbb{R}$. Then $u(t) = u_0(t) + \Psi(u_0(t), \mu)$, and $u_0(t) \in \mathbb{E}_c$ satisfies*

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = A_c u_0 + \pi_c f(u_0 + \Psi_c(u_0, \mu), \mu), \quad t \in I, \tag{2.31}$$

where A_c is the reduction of A to $\mathbb{E}_s$.

Let us consider once again the example (2.16), now introducing a small parameter $\mu \in \mathbb{R}$ into the equation

$$u'''' - u'' - \mu u - u^2 = 0. \tag{2.32}$$

The first step is to write (2.32) as a first-order system. We set $U = (u_1, u_2, u_3, u_4)$ with $u = u_1$, $u' = u_2$, $u'' = u_3$ and $u''' = u_4$, and then the equation is equivalent with the system

$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU + f(U, \mu), \tag{2.33}$$

in which

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad f(U,\mu) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu u_1 + u_1^2 \end{pmatrix}.$$

We can apply the parameter-dependent center manifold theorem 2.9, and conclude that the solutions on the center-manifold are of the form

$$U(t) = U_0(t) + \Psi(U_0(t), \mu), \quad U_0(t) = a(t)\zeta_0 + b(t)\zeta_1,$$

with $\Psi(0,\mu) = 0$ (because $f(0,\mu) = 0$ for all μ), $D_u\Psi(0,0) = 0$ with $\Psi(U_0,\mu) = \mathcal{O}(\|U_0\| (|\mu| + \|U_0\|)$ near $(0,0)$. Finally, we easily check

$$\pi_c f(U_0 + \Psi(U_0,\mu)) = \left(-\mu a - a^2 + \mathcal{O}\left((|a| + |b|)(|\mu|^2 + |a|^2 + |b|^2)\right)\right)\zeta_1.$$

Projecting along ζ_0 and ζ_1 one gets

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= b, \\ \frac{\mathrm{d}b}{\mathrm{d}t} &= -\mu a - a^2 + \mathcal{O}\left((|a| + |b|)(|\mu|^2 + |a|^2 + |b|^2)\right). \end{aligned}$$

Symmetries. We consider the case of an equation that is equivariant under the action of a linear transformation. More precisely, we make the following assumptions: there exists $\mathbf{T} \in \mathscr{M}_n(\mathbb{R})$ which commutes with vector field in equation (2.2), that is

$$\mathbf{T}Au = A\mathbf{T}u, \quad \mathbf{T}f(u) = f(\mathbf{T}u), \forall u \in \mathbb{R}^n.$$

We further assume that the restriction $\mathbf{T}_c$ of $\mathbf{T}$ to the subspace $\mathbb{E}_c$ is an isometry.

Theorem 2.10. *Under the assumptions of Theorem 2.3 and the above equivariance condition, then one can find Ψ in Theorem 2.3 which commutes with $\mathbf{T}$, that is*

$$\mathbf{T}\Psi(u_0) = \Psi(\mathbf{T}u_0) \text{ for all } u_0 \in \mathbb{E}_c.$$

and such that the vector field in the reduced equation (2.31) commutes with $\mathbf{T}_c$.

In the proof, one needs to be careful when choosing the cut-off function χ, and it should satisfy

$$\chi(\mathbf{T}_c u_0) = \chi(u_0), \quad \forall u_0 \in \mathbb{E}_c.$$

As $\mathbf{T}_c$ is assumed to be an isometry, it is possible to choose χ to be a smooth function of $\|u_0\|^2$.

Reversibility. Next, we consider the case of reversible equations, when the vector field (2.2) anticommutes with a symmetry $\mathbf{S}$. More precisely, we make the following assumptions: there exists $\mathbf{S} \in \mathscr{M}_n(\mathbb{R})$ which commutes with vector field in equation (2.2), that is

$$\mathbf{S}Au = -A\mathbf{S}u, \quad \mathbf{S}f(u) = -f(\mathbf{S}u), \forall u \in \mathbb{R}^n.$$

Notice that in that case, if $t \mapsto u(t)$ is a solution of (2.2), then $t \mapsto \mathbf{S}u(-t)$ is also a solution of (2.2). Moreover, the spectrum $\sigma(A)$ is necessarily symmetric with respect to the origin in the complex plane. We also point out that if $\lambda \in \sigma(A)$ with associated eigenvector ζ, then $-\lambda$ is an eigenvalue with the associated eigenvector $\mathbf{S}\zeta$.

Theorem 2.11. *Under the assumptions of Theorem 2.3 and the above reversibility condition, then one can find* Ψ *in Theorem 2.3 which commutes with* $\mathbf{S}$*, that is*

$$\mathbf{S}\Psi(u_0) = \Psi(\mathbf{S}_c u_0) \text{ for all } u_0 \in \mathbb{E}_c.$$

where $\mathbf{S}_c$ *is the restriction of* $\mathbf{S}$ *to* $\mathbb{E}_c$*, and such that the vector field in the reduced equation (2.31) anti-commutes with* $\mathbf{T}_c$*.*

Coming back, once again, to example (2.32), we notice that the system (2.33) possesses a reversibility, *i.e.* A and $f(\cdot, \mu)$ anticommute with

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This symmetry is a consequence of the fact the equation (2.32) is invariant under the reflection $t \mapsto -t$. We notice that

$$\mathbf{S}\zeta_0 = \zeta_0, \quad \mathbf{S}\zeta_1 = -\zeta_1.$$

Furthermore, the map Ψ from the center-manifold theorem is such that $\mathbf{S}\Psi(U_0, \mu) = \Psi(\mathbf{S}_c U_0, \mu)$. We also have that the reduced system

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= b, \\ \frac{\mathrm{d}b}{\mathrm{d}t} &= -\mu a - a^2 + \mathcal{O}\left((|a| + |b|)(|\mu|^2 + |a|^2 + |b|^2)\right). \end{aligned}$$

anticcommutes with $\mathbf{S}_c$, which implies that the right-hand side in the second equation is even in b, so that the higher order terms in the expansion are in fact of order

$$\mathcal{O}\left((|a| + |b|^2)(|\mu|^2 + |a|^2 + |b|^4)\right).$$

2.3.8 Exercice (From last year's final exam)

We consider the Lorenz system

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 + x_2, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - x_2 - x_1 x_3, \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = -3x_3 + x_1 x_2, \end{cases} \tag{2.34}$$

where $x_{1,2,3}(t)$ are real valued functions. The aim is to study bounded solutions near the equilibrium point $(x_1, x_2, x_3) = (0, 0, 0)$. We denote $X = (x_1, x_2, x_3)$ and write the above Lorenz system as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X),$$

where $F(X)$ is defined by the right-hand side of (2.34). Finally, we denote by $L = DF(0) \in \mathscr{M}_3(\mathbb{R})$.

1. Compute the spectrum of L.

2. Show that 0 is a simple eigenvalue of L. Calculate a corresponding eigenvector ξ. Compute a spectral projection π_c onto $\mathbb{E}_c = \mathrm{Span}(\xi)$ using the adjoint matrix L^*.

3. The Lorenz system is equivariant under a certain symmetry $S \in \mathscr{M}_3(\mathbb{R})$, where $S^2 = I_3$ and $SF(X) = F(SX)$ for all $X \in \mathbb{R}^3$. Establish this symmetry, and show that $S\xi = -\xi$. (**Hint:** look at a reflectional symmetry leaving one axis invariant.)

4. Establish the existence of a one-dimensional center manifold $\mathcal{M}_c$ that can be parametrized as follows
$$\mathcal{M}_c = \{z\xi + \Psi(z) \mid z \in \mathbb{R}\},$$
where $\Psi : \mathbb{R} \to \mathbb{R}^3$ is of class $\mathscr{C}^k$ for any $k \geq 1$, $\Psi(z) \in \mathbb{E}_h$ for all $z \in \mathbb{R}$, and such that $\Psi(-z) = S\Psi(z)$ for all $z \in \mathbb{R}$.

5. The reduced equation governing the amplitude z is
$$\frac{\mathrm{d}z}{\mathrm{d}t} = f(z), \quad z \in \mathbb{R}. \tag{2.35}$$
Establish that $f : \mathbb{R} \to \mathbb{R}$ is such that $f(-z) = -f(z)$ for all $z \in \mathbb{R}$, and of class $\mathscr{C}^k$ for any $k \geq 1$.

6. Expand $\Psi : \mathbb{R} \to \mathbb{R}^3$ and $f : \mathbb{R} \to \mathbb{R}$ in Taylor series around $z = 0$, that is,
$$\Psi(z) = \sum_{p=0}^{3} \Psi_p z^p + \mathcal{O}(|z|^4), \quad f(z) = \sum_{p=0}^{3} f_p z^p + \mathcal{O}(|z|^4),$$
where $\Psi_p \in \mathbb{R}^3$ and $f_p \in \mathbb{R}$. Compte Ψ_p and f_p for $0 \leq p \leq 3$. (**Hint:** use the tangency of the center manifold and the symmetry S.)

7. Study the dynamics of the reduced equation (2.35) near $z = 0$.

8. Sketch the phase portrait of (2.34) in the (x_1, x_2, x_3)-space in the neighborhood of the origin $X = (x_1, x_2, x_3) = (0, 0, 0)$.

We introduce a parameter $\mu \in \mathbb{R}$ and consider the following perturbation of the Lorenz system
$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 + x_2, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - x_2 + \mu x_1 - x_1 x_3, \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = -3x_3 + x_1 x_2, \end{cases} \tag{2.36}$$
which can be written
$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X, \mu), \quad X = (x_1, x_2, x_3),$$
where $F(X, \mu)$ is defined by the right-hand side of (2.36). Note that when $\mu = 0$ we recover the previous system (2.34). Finally, we denote by $L(\mu) = DF(0, \mu) \in \mathscr{M}_3(\mathbb{R})$.

9. Show that $X = (x_1, x_2, x_3) = (0, 0, 0)$ is stable for $\mu < 0$ and unstable for $\mu > 0$.

10. Establish the existence of a local parameter dependent center manifold near $(X, \mu) = (0_{\mathbb{R}^3}, 0)$ that can be parametrized as follows

$$\mathcal{M}_c(\mu) = \{z\xi + \Psi(z, \mu) \mid z \in \mathbb{R}\}.$$

11. Show that the reduced equation governing the amplitude z

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f(z, \mu), \quad z \in \mathbb{R}, \tag{2.37}$$

has the following expansion near $(z, \mu) = (0, 0)$

$$f(z, \mu) = az\mu + bz^3 + \mathcal{O}\left(|z||\mu|^2 + |z|^5\right),$$

for some a and b in $\mathbb{R}$. Compute a and b. (**Hint:** b has already been computed in question 6.)

12. For $\mu > 0$ small, sketch the phase portrait of (2.36) in the (x_1, x_2, x_3)-space in the neighborhood of the origin $(x_1, x_2, x_3) = (0, 0, 0)$.

Chapter 3

Spectrum of closed operators and application to traveling fronts and pulses of reaction-diffusion equation

The aim of this chapter is to give a theoretical framework to study the spectrum of closed operators and apply this framework to analyze the spectrum of operators obtained by linearizing a reaction-diffusion equation around a traveling wave solution.

3.1 Elements of functional analysis

3.1.1 Bounded and closed operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces with respective norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, and assume that $\mathcal{Y} \subset \mathcal{X}$ is dense. Typical example will be $\mathcal{X} = L^2(\mathbb{R})$ and $\mathcal{Y} = H^k(\mathbb{R})$ for any $k \geq 1$. Consider linear operator $\mathcal{L}$, with $\mathcal{Y} = \mathcal{D}(\mathcal{L})$, the domain of $\mathcal{L}$, dense in $\mathcal{X}$ and $\mathcal{L} : \mathcal{Y} \to \mathcal{X}$. The kernel of $\mathcal{L}$ is given by

$$\ker(\mathcal{L}) := \{u \in \mathcal{Y} \mid \mathcal{L}u = 0\}.$$

The range of $\mathcal{L}$ is given by

$$\mathrm{rg}(\mathcal{L}) := \{\mathcal{L}u \in \mathcal{X} \mid u \in \mathcal{Y}\} \subset \mathcal{X}.$$

We say that a linear operator is **closed** if for any sequence $(u_j) \in \mathcal{Y}$ with

$$\lim_{j\to+\infty} \|u_j - u\|_{\mathcal{X}} = 0, \text{ and } \lim_{j\to+\infty} \|\mathcal{L}u_j - v\|_{\mathcal{X}} = 0,$$

then we have $u \in \mathcal{Y}$ and $\mathcal{L}u = v$. This is equivalent to saying that the domain $\mathcal{D}(\mathcal{L})$ is complete under the graph nom of $\mathcal{L}$,

$$\|u\|_{\mathcal{D}(\mathcal{L})} := \|u\|_{\mathcal{X}} + \|\mathcal{L}u\|_{\mathcal{X}}.$$

We say that the operator is **bounded** from $\mathcal{Y}$ to $\mathcal{X}$ if

$$\sup\left\{\|\mathcal{L}u\|_{\mathcal{X}} \mid u \in \mathcal{Y}, \|u\|_{\mathcal{Y}} = 1\right\} < +\infty.$$

We denote the space of bounded linear operators from $\mathcal{Y}$ to $\mathcal{X}$ by $\mathscr{L}(\mathcal{Y},\mathcal{X})$, with the induced norm of $\mathcal{L}$ given by

$$\|\mathcal{L}\|_{\mathscr{L}(\mathcal{Y},\mathcal{X})} := \sup_{\|u\|_{\mathcal{Y}}\neq 0} \frac{\|\mathcal{L}u\|_{\mathcal{X}}}{\|u\|_{\mathcal{Y}}}.$$

If $\mathcal{X}=\mathcal{Y}$, we simply denote $\mathscr{L}(\mathcal{X})$. Let us make some remarks.

- the sum of a closed operator and a bounded operator si a closed operator;
- if $\mathcal{L}$ is a closed operator with $\mathcal{X}=\mathcal{Y}$, then $\mathcal{L}$ is a bounded operator;
- if for each bounded sequence $(u_j)\in\mathcal{Y}$ the sequence $(\mathcal{L}u_j)\in\mathcal{X}$ has a convergent subsequence, then the operator is said to be **compact**;
- a compact operator is bounded.

3.1.2 Resolvent and spectrum

Definition 3.1 (Resolvent)**.** *The* **resolvent set** *of $\mathcal{L}$, denoted $\rho(\mathcal{L})$, is the set of complex numbers $\lambda\in\mathbb{C}$ such that*

(i) $\lambda\mathrm{Id}-\mathcal{L}$ *is invertible,*

(ii) $(\lambda\mathrm{Id}-\mathcal{L})^{-1}$ *is a bounded linear operator.*

Here $\mathrm{Id}:\mathcal{X}\to\mathcal{X}$ *is the identity operator. For $\lambda\in\rho(\mathcal{L})$, the operator $(\lambda\mathrm{Id}-\mathcal{L})^{-1}$ is called the* **resolvent operator** *of $\mathcal{L}$.*

Definition 3.2 (Spectrum)**.** *The* **spectrum** *of $\mathcal{L}$ is the complement of the resolvent set, i.e.,*

$$\sigma(\mathcal{L})=\mathbb{C}\backslash\rho(\mathcal{L}).$$

Definition 3.3 (Eigenvalue)**.** *A complex number $\lambda\in\sigma(\mathcal{L})$ is called an* **eigenvalue** *if* $\ker(\lambda\mathrm{Id}-\mathcal{L})\neq\{0\}$*. In other words, $\lambda\in\sigma(\mathcal{L})$ is an eigenvalue if there exists $u\in\mathcal{Y}$, $u\neq 0$, such that $\mathcal{L}u=\lambda u$. Equivalently, $\lambda\mathrm{Id}-\mathcal{L}$ is not injective.*

Remark 3.4. *If $\mathcal{L}$ is a closed operator, then $\sigma(\mathcal{L})$ is a closed set. If $\mathcal{L}$ is a bounded operator, then $\sigma(\mathcal{L})$ is closed, bounded and a nonempty set.*

In finite dimensions, the only way for $\lambda\mathrm{Id}-\mathcal{L}$ to fail to have a bounded inverse on all $\mathbb{R}^n$ is if λ is an eigenvalue. Hence, the spectrum of a matrix is exactly the set of its eigenvalues. In infinite dimensions, however, there are more ways for $\lambda\mathrm{Id}-\mathcal{L}$ to fail to have a bounded inverse. For example, its range could fail to be dense and/or fail to be closed. One can actually check that the range not being closed is equivalent to there existing a sequence of so-called **approximate eigenvalues**: $(\lambda-\mathcal{L})u_j\to 0$ as $j\to+\infty$, where $u_j\in\mathcal{D}(\mathcal{L})$ for all j, $\|u_j\|_{\mathcal{X}}=1$, but the sequence (u_j) does not have a limit in $\mathcal{X}$.

As an example, consider the Laplacian in one dimension: $\mathcal{L} = \partial_x^2$ and $\mathcal{X} = L^2(\mathbb{R})$ with $\mathcal{Y} = \mathcal{D}(\mathcal{L}) = H^2(\mathbb{R})$. For a given $\lambda \in \mathbb{C}$, we can try to solve the equation $(\lambda\mathrm{Id} - \mathcal{L})u = v$ for a given $v \in \mathcal{X}$ via $u = (\lambda\mathrm{Id} - \mathcal{L})^{-1}v$. Denoting the Fourier transform by $\mathcal{F}(u) = \hat{u}$, we find

$$u = (\lambda\mathrm{Id} - \mathcal{L})^{-1}v, \quad \hat{u}(\ell) = \frac{1}{\ell^2 + \lambda}\hat{v}(\ell), \quad u(x) = \mathcal{F}^{-1}\left(\frac{1}{(\cdot)^2 + \lambda}\hat{v}(\cdot)\right)(x), \quad \ell \in \mathbb{R}.$$

Since $\ell \in \mathbb{R}$, for any $\lambda \in (-\infty, 0]$ one can find a $v \in L^2(\mathbb{R})$ such that $\|u\|_{L^2(\mathbb{R})}$ is not finite. Hence, $(\lambda\mathrm{Id} - \mathcal{L})^{-1}$ is not well-defined on all $L^2(\mathbb{R})$ for any such λ. In addition, using the above formulation, we have that for $\lambda \notin (-\infty, 0]$ the resolvent operator is well-defined and bounded on all $L^2(\mathbb{R})$. Hence $\sigma(\mathcal{L}) = (-\infty, 0]$. On the other hand, if we fix $\lambda \in (-\infty, 0]$ and look for eigenvalues, we find $\partial_x^2 u = \lambda u$. This implies that $u(x, \lambda) = e^{\mathbf{i}\sqrt{|\lambda|}x}$, which is not in $L^2(\mathbb{R})$. One can check that for each such λ there is a sequence u_j of approximate eigenvalues. For example, one can use mollifiers of the form $e^{-\epsilon x^2}$ and set $u_j(x, \lambda) = C_j e^{\mathbf{i}\sqrt{|\lambda|}x} e^{-\epsilon_j x^2}$, where $\epsilon_j \to 0$ as $j \to +\infty$ and the constant C_j is chosen such that $\|u_j\|_{L^2(\mathbb{R})} = 1$.

Remark 3.5. *Because, there are multiple ways for the resolvent operator to fail to be bounded on all of $\mathcal{X}$, it will be useful to define subsets of the spectrum in terms of way(s) in which $(\lambda\mathrm{Id} - \mathcal{L})$ fails to have a bounded inverse.*

Suppose that $\lambda \in \sigma(\mathcal{L})$ is an eigenvalue.

- $\dim\left[\ker(\lambda\mathrm{Id} - \mathcal{L})\right]$ is called the **geometric multiplicity** of the eigenvalue, and is denoted by $m_g(\lambda)$.
- If $m_g(\lambda) = 1$, then the eigenvalue is called **geometrically simple**.
- If the eigenvalue is isolated, then the **algebraic multiplicity** of the eigenvalue, denoted $m_a(\lambda)$, is the dimension of the largest subspace $\mathcal{Y}_\lambda \subset \mathcal{Y}$ which
  1. is invariant under the action of $\mathcal{L}$, that is $u_\lambda \in \mathcal{Y}_\lambda \;\Rightarrow\; \mathcal{L}u_\lambda \in \mathcal{Y}_\lambda$,
  2. satisfies the property $\sigma\left(\mathcal{L}_{|\mathcal{Y}_\lambda}\right) = \{\lambda\}$.
- If $m_a(\lambda) = 1$, then the eigenvalue is called **algebraically simple**.
- It is true that $m_a(\lambda) \geq m_g(\lambda)$.
- An eigenvalue is called **semi-simple** if $m_a(\lambda) = m_g(\lambda)$.
- If $\mathcal{L}$ is a compact operator whose domain is $\mathcal{Y}$ is separable, *i.e.* has a countably infinite dense subset, then the following hold:
  1. $0 \in \sigma(\mathcal{L})$,
  2. if $\lambda \neq 0$, then λ is isolated and $m_a(\lambda) < +\infty$,
  3. $\sigma(\mathcal{L})$ is a countable set, the only possible accumulation point is $\lambda = 0$.
- If λ is an isolated eigenvalue, let $\Gamma \subset \mathbb{C}$ be a simple closed positively oriented curve surrounding λ that does not intersect the spectrum of $\mathcal{L}$ and whose interior contains no other points in $\sigma(\mathcal{L})$. The spectral projection $\mathcal{P}(\lambda) : \mathcal{X} \to \mathcal{Y}_\lambda$ is given by the Dunford integral formula

$$\mathcal{P}(\lambda) := \frac{1}{2\pi\mathbf{i}}\oint_\Gamma (\zeta\mathrm{Id} - \mathcal{L})^{-1}\mathrm{d}\zeta.$$

The operator $\mathcal{P}(\lambda)$ commutes with $\mathcal{L}$ and $\mathcal{P}(\lambda)\mathcal{P}(\lambda) = \mathcal{P}(\lambda)$. The range of $\mathcal{P}(\lambda)$ is the $\mathcal{L}$-invariant subspace $\mathcal{Y}_\lambda$ and $\mathcal{L}_{|\mathcal{Y}_\lambda} = \mathcal{L}\mathcal{P}(\lambda)$.

- A linear operator $\mathcal{L}$ has **compact resolvent** if
  1. $\rho(\mathcal{L}) \neq \emptyset$,
  2. for $\lambda \in \rho(\mathcal{L})$ the resolvent operator $(\lambda \mathrm{Id} - \mathcal{L})^{-1} : \mathcal{X} \to \mathcal{Y}$ is compact in $\mathscr{L}(\mathcal{X})$.
- If $(\lambda \mathrm{Id} - \mathcal{L})^{-1}$ is a compact operator for one $\lambda \in \rho(\mathcal{L})$, then it is compact for all $\lambda \in \rho(\mathcal{L})$.
- If $\mathcal{L}$ is an operator with compact resolvent, then $\sigma(\mathcal{L})$ is a countable set of isolated eigenvalues with finite algebraic multiplicity for which the only possible accumulation point is $\lambda = \infty$.

3.1.3 Adjoint and Fredholm operators

Assume that $\mathcal{X}$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and that $\mathcal{L}$ is a closed operator with a dense domain. The domain, $\mathcal{D}(\mathcal{L}^*)$, of the adjoint operator, $\mathcal{L}^*$, is the set of all $v \in \mathcal{X}$ for which the linear functional

$$u \mapsto \langle \mathcal{L}u, v \rangle,$$

is continuous in the Hilbert norm on $\mathcal{X}$. From the Riesz representation theorem, we deduce the existence of $w \in \mathcal{X}$ for which

$$\langle \mathcal{L}u, v \rangle = \langle u, w \rangle.$$

For such $v \in \mathcal{D}(\mathcal{L}^*)$ the adjoint operator is defined by the map $w = \mathcal{L}^* v$ that is, the adjoint operator is the unique operator that satisfies

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}^* v \rangle, \quad \forall u \in \mathcal{X}, \quad v \in \mathcal{D}(\mathcal{L}^*).$$

The adjoint operator is also closed, and its domain is also dense in $\mathcal{X}$. For example, consider the second-order differential operator

$$\mathcal{L} = \partial_x^2 + a_1(x)\partial_x + a_0(x) : H^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

the coefficients $a_{0,1}$ being smooth and uniformly bounded. Integration by parts shows that the adjoint operator is given by

$$\mathcal{L}^* = \partial_x^2 - \partial_x \left(a_1(x) \cdot \right) + a_0(x) : H^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

The resolvent set and spectrum of an operator and its adjoint are related by

$$\rho(\mathcal{L}^*) = \overline{\rho(\mathcal{L})}, \quad \sigma(\mathcal{L}^*) = \overline{\sigma(\mathcal{L})},$$

and the resolvent operators are related through

$$\left(\overline{\lambda} \mathrm{Id} - \mathcal{L}^*\right)^{-1} = \overline{(\lambda \mathrm{Id} - \mathcal{L})^{-1}}.$$

Definition 3.6 (Self-adjoint operator)**.** *An operator is said to be* **self-adjoint** *if $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}^*)$ and $\mathcal{L}u = \mathcal{L}^* u$ for all $u \in \mathcal{D}(\mathcal{L})$.*

Remark 3.7. *For self-adjoint operators, we have that $\sigma(\mathcal{L}) \subset \mathbb{R}$, and that all eigenvalues are semi-simple.*

Definition 3.8 (Fredhlom operator)**.** *The operator $\mathcal{L}$ is a* **Fredholm operator** *if*

(i) $\ker(\mathcal{L})$ *is finite-dimensional;*

(ii) $\mathrm{rg}(\mathcal{L})$ *is closed with finite codimension.*

The **Fredholm index** *of a Fredholm operator is defined by*

$$\mathrm{ind}(\mathcal{L}) = \dim\left[\ker(\mathcal{L})\right] - \mathrm{codim}\left[\mathrm{rg}(\mathcal{L})\right].$$

An operator is Fredholm if and only if its adjoint $\mathcal{L}^*$ is, and the indices are related via

$$\mathrm{ind}(\mathcal{L}) = -\mathrm{ind}(\mathcal{L}^*).$$

If $\lambda \in \sigma(\mathcal{L})$ is an isolated eigenvalue with $m_a(\lambda) < +\infty$, then $\lambda\mathrm{Id} - \mathcal{L}$ is a Fredholm operator with index 0.

It is easy to see that the range of $\mathcal{L}$ must be orthogonal to the kernel of $\mathcal{L}^*$. Indeed, if $v \in \ker(\mathcal{L}^*)$ and $\mathcal{L}u = f$, then

$$\langle f, v\rangle = \langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle = 0.$$

The sufficiency of this condition often goes by the name of the Fredholm alternative and reads.

Theorem 3.1 (Fredholm alternative)**.** *Suppose that $\mathcal{X}$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{L} : \mathcal{D}(\mathcal{L}) = \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}$ is a closed Fredholm operator with dense domain. For $f \in \mathcal{X}$ the nonhomogeneous problem $\mathcal{L}u = f$ has a solution $u \in \mathcal{D}(\mathcal{L})$ if and only if $f \in \ker(\mathcal{L}^*)^{\perp}$. In other words,*

$$\mathrm{rg}(\mathcal{L}) = \ker(\mathcal{L}^*)^{\perp}.$$

Moreover, the Fredholm index counts the dimension mismatch between the kernels of $\mathcal{L}$ and $\mathcal{L}^*$,

$$\mathrm{ind}(\mathcal{L}) = \dim\left[\ker(\mathcal{L})\right] - \dim\left[\ker(\mathcal{L}^*)\right].$$

As a consequence, for any Fredholm operator the space $\mathcal{X}$ can be decomposed as

$$\mathcal{X} = \mathrm{rg}(\mathcal{L}) \oplus \ker(\mathcal{L}^*).$$

If in addition $\mathrm{ind}(\mathcal{L}) = 0$, then $\dim\left[\ker(\mathcal{L})\right] = \dim\left[\ker(\mathcal{L}^*)\right]$ and:

- either $\mathcal{L}$ has a kernel,
- or the operator is one-to-one (no kernel) and onto (as $\mathrm{rg}(\mathcal{L}) = \mathcal{X}$).

However, if $\mathrm{ind}(\mathcal{L}) \neq 0$ then:

- either $\mathcal{L}$ has a kernel in which case $\mathcal{L}$ cannot be one-to-one,
- or $\mathcal{L}^*$ has a kernel in which case $\mathcal{L}$ cannot be onto.

In both of these cases $\mathcal{L}$ cannot be invertible.

3.1.4 Essential and point spectrum

From the previous discussion, we observe that if the Fredholm index is not zero, then invertibility is hopeless, while if the Fredholm index is zero, then invertibility follows if $\mathcal{L}$ has no kernel. This observation motivates the following classification of the spectral sets of operators.

Definition 3.9. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ be a closed linear operator with dense domain $\mathcal{D}(\mathcal{L})$ in $\mathcal{X}$. The spectrum of $\mathcal{L}$ is decomposed into the following two sets:*

(i) The **essential spectrum** *of a densely defined closed linear operator $\mathcal{L}$, denoted $\sigma_{\text{ess}}(\mathcal{L})$, is the set of all $\lambda \in \mathbb{C}$ such that either*

- $\lambda\mathrm{Id} - \mathcal{L}$ *is not Fredholm; or*
- $\lambda\mathrm{Id} - \mathcal{L}$ *is Fredholm, but* $\mathrm{ind}(\lambda\mathrm{Id} - \mathcal{L}) \neq 0$.

(ii) The **point spectrum** *of a densely defined closed linear operator $\mathcal{L}$ is the set defined by*

$$\sigma_{\text{pt}}(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \lambda\mathrm{Id} - \mathcal{L} \textit{ is Fredholm with } \mathrm{ind}(\lambda\mathrm{Id} - \mathcal{L}) = 0, \textit{ but } \lambda\mathrm{Id} - \mathcal{L} \textit{ is not invertible}\}.$$

Although this definition is very convenient as it makes the essential spectrum a large set, in this formulation, the point spectrum is not equivalent to the set of eigenvalues. The reason is that eigenvalues can be embedded in the essential spectrum.

Locating the essential spectrum requires the computation of the Fredholm index of an operator. Two techniques are commonly employed to achieve this. The first technique is to perturb a known operator.

Definition 3.10. *The operator $\mathcal{L}$ is a* **relatively compact perturbation** *of $\mathcal{L}_0$ if*

$$(\mathcal{L}_0 - \mathcal{L})(\lambda\mathrm{Id} - \mathcal{L}_0)^{-1} : \mathcal{X} \to \mathcal{X}$$

is compact for some $\lambda \in \rho(\mathcal{L}_0)$.

Theorem 3.2 (Weyl essential spectrum theorem)**.** *Let $\mathcal{L}$ and $\mathcal{L}_0$ be closed linear operators in a Banach space $\mathcal{X}$. If $\mathcal{L}$ is a relatively compact perturbation of $\mathcal{L}_0$, then the following properties hold:*

(i) the operator $\lambda\mathrm{Id} - \mathcal{L}$ is Fredholm if and only if $\lambda\mathrm{Id} - \mathcal{L}_0$ is Fredholm,

(ii) $\mathrm{ind}(\lambda\mathrm{Id} - \mathcal{L}) = \mathrm{ind}(\lambda\mathrm{Id} - \mathcal{L}_0)$,

(iii) $\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}_0)$.

The second result deals with operators with compact resolvent. In that case, the operator cannot have any essential spectrum.

Theorem 3.3. *If $\mathcal{X}$ is a Banach, $\mathcal{Y} \subset \mathcal{X}$ is dense and $\mathcal{L} : \mathcal{Y} \to \mathcal{X}$ is a closed Fredholm operator with compact resolvent, then* $\mathrm{ind}(\mathcal{L}) = 0$.

Example #1: If we come back to $\mathcal{L} = \partial_x^2$ on $\mathcal{X} = L^2(\mathbb{R})$ with domain $\mathcal{Y} = H^2(\mathbb{R})$. We have already seen that $\sigma(\mathcal{L}) = (-\infty, 0]$, and more specifically that the range of $\lambda\mathrm{Id} - \mathcal{L}$ is not closed by exhibiting approximate eigenvalues for $\lambda \in (-\infty, 0]$. So $\lambda\mathrm{Id} - \mathcal{L}$ is not Fredholm for $\lambda \in (-\infty, 0]$, and thus $\sigma_{\text{ess}}(\partial_x^2) = (-\infty, 0]$.

Example #2: We look at the following reaction-diffusion equation[1]

$$\partial_t u = \partial_x^2 u - u + u^3, \quad t > 0, \quad x \in \mathbb{R},$$

which has explicit stationary solution given by $u_*(x) = \sqrt{2}\mathrm{sech}(x)$. The linearized operator around u_* takes the form

$$\mathcal{L} = \partial_x^2 + (6\mathrm{sech}^2(x) - 1) : H^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Solving for $\lambda u = \mathcal{L}u$, we get for each value of λ two independent solutions

$$\begin{aligned} u_1(x,\lambda) &= e^{\sqrt{1+\lambda}x}\left(1 + \frac{\lambda}{3} - \sqrt{1+\lambda}\tanh(x) - \mathrm{sech}^2(x)\right), \\ u_2(x,\lambda) &= e^{-\sqrt{1+\lambda}x}\left(1 + \frac{\lambda}{3} + \sqrt{1+\lambda}\tanh(x) - \mathrm{sech}^2(x)\right). \end{aligned}$$

In order to determine the spectrum of $\mathcal{L}$, we need to investigate when the resolvent operator is well defined. Suppose that we are given a function $w \in L^2(\mathbb{R})$ and we seek a function u such that $(\lambda\mathrm{Id} - \mathcal{L})u = w$. We will use the method of variation of the constant, and assume that u as the form

$$u(x,\lambda) = v_1(x,\lambda)u_1(x,\lambda) + v_2(x,\lambda)u_2(x,\lambda),$$

and solve for the functions $v_{1,2}$ in terms of w. To do this, as usual, we impose that $v_1'u_1 + v_2'u_2 = 0$, and insert the above Ansatz into the equation $(\lambda\mathrm{Id} - \mathcal{L})u = w$. These two equations can be written

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \frac{1}{u_1u_2' - u_1'u_2}\begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix}\begin{pmatrix} 0 \\ -w \end{pmatrix} = -\frac{9}{2\lambda(3-\lambda)\sqrt{1+\lambda}}\begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix}\begin{pmatrix} 0 \\ -w \end{pmatrix}.$$

We immediately see that there will be problems if $\lambda \in \{0,3\} \cup (-\infty,-1]$. Suppose that $\lambda \notin \{0,3\} \cup (-\infty,-1]$, and let us continue our computations to get expressions for $v_{1,2}$ as

$$v_1'(x,\lambda) = -\frac{9}{2\lambda(3-\lambda)\sqrt{1+\lambda}}u_2(x,\lambda)w(x), \quad v_2'(x,\lambda) = \frac{9}{2\lambda(3-\lambda)\sqrt{1+\lambda}}u_1(x,\lambda)w(x).$$

We note that u_1 is well behaved at $-\infty$, while u_2 is well behaved at $+\infty$. Hence, we define

$$v_1(x,\lambda) = \frac{9}{2\lambda(3-\lambda)\sqrt{1+\lambda}}\int_x^{+\infty} u_2(y,\lambda)w(y)\mathrm{d}y, \quad v_2(x,\lambda) = \frac{9}{2\lambda(3-\lambda)\sqrt{1+\lambda}}\int_{-\infty}^x u_1(y,\lambda)w(y)\mathrm{d}y.$$

Inserting these formulas into the expression for u, one finds that u can be written

$$\begin{aligned} u(x) &= \frac{9}{2\lambda(3-\lambda)\sqrt{1+\lambda}}\int_{\mathbb{R}} \left(u_1(x,\lambda)u_2(y,\lambda)H(y-x) + u_2(x,\lambda)u_1(y,\lambda)H(x-y)\right)w(y)\mathrm{d}y \\ &=: \int_{\mathbb{R}} \mathbf{G}(x,y,\lambda)w(y)\mathrm{d}y. \end{aligned}$$

As a consequence, the action of the resolvent operator can be expressed through the integral kernel $\mathbf{G}(x,y,\lambda)$. One can now investigate for which values of λ this operator is well-defined and bounded on all $L^2(\mathbb{R})$ and obtain that

$$\sigma_{\mathrm{pt}}(\mathcal{L}) = \{0,3\}, \text{ and } \sigma_{\mathrm{ess}}(\mathcal{L}) = (-\infty,-1].$$

[1]This example is taken from Björn Sandstede's webpage: example of an explicit Evans function and the lecture notes of Margaret Beck: minicourse on stability.

More precisely, if $\lambda \in (-\infty, -1]$, then the functions $u_{1,2}$ are bounded and can be used to construct approximate eigenvalues showing that the range of $\lambda\mathrm{Id}-\mathcal{L}$ is not closed, so that $\lambda\mathrm{Id}-\mathcal{L}$ is not Fredholm, and thus $(-\infty, -1] \subset \sigma_{\mathrm{ess}}(\mathcal{L})$. If $\lambda \in \{0, 3\}$, then the kernel of $\lambda\mathrm{Id}-\mathcal{L}$ is one-dimensional as we have $-u_1(x, 0)+u_2(x, 0) = -\mathrm{sech}(x)\mathrm{tanh}(x)$ and $u_1(x, 3) + u_2(x, 3) = \mathrm{sech}^2(x)$ which are the corresponding eigenfunctions. In fact, by translation invariance of the equation, we already know that $u_*'(x) = -\sqrt{2}\mathrm{sech}(x)\mathrm{tanh}(x) \in H^2(\mathbb{R})$ is such that $\mathcal{L}u_*' = 0$ and thus an eigenvector associated to the eigenvalue $\lambda = 0$. Let ϕ be the eigenfunction associated to $\lambda \in \{0, 3\}$. Since $L^2(\mathbb{R})$ is a Hilbert space, we can set $\mathcal{E} = \mathrm{span}\{\phi\}$ and let $\mathcal{E}^\perp$ denote its orthogonal complement, which is closed. Since $\mathcal{L}$ is self-adjoint, $\mathcal{E}^\perp$ is the range of $\lambda\mathrm{Id} - \mathcal{L}$ (see [1]), and we see that the Fredholm index is zero. For all other values of λ, the explicit form of $\mathbf{G}$ can be used to show that the above resolvent operator is bounded on all of $L^2(\mathbb{R})$.

3.1.5 Point spectrum and Sturm-Liouville theory

The point spectrum does not behave well with respect to perturbations. And in practice, it is often much more delicate to establish the location of the point spectrum than the essential spectrum. However, for many scalar second-order operators, there are important results.

Sturm-Liouville operators on a bounded interval. A Sturm-Liouville operator $\mathcal{L}$ takes the form

$$\mathcal{L}p := \partial_x^2 p + a_1(x)\partial_x p + a_0(x)p.$$

Here, we will consider $\mathcal{L}$ to be defined on the bounded interval $[0, 1]$, subject to boundary conditions

$$b_1^l p(0) + b_2^l \partial_x p(0) = 0, \quad b_1^r p(1) + b_2^r \partial_x p(1) = 0.$$

And we assume that the above coefficients satisfy

$$\left(b_1^{l,r}\right)^2 + \left(b_2^{l,r}\right)^2 > 0,$$

and the coefficients $a_{0,1}(x)$ in $\mathcal{L}$ are $\mathscr{C}^1([0, 1])$ and real-valued. The spectral problem is naturally posed on $H^2_{bc}(]0, 1[)$ where

$$H^2_{bc}(]0, 1[) := \left\{u \in H^2(]0, 1[) \mid b_1^l u(0) + b_2^l \partial_x u(0) = 0, \quad b_1^r u(1) + b_2^r \partial_x u(1) = 0\right\}.$$

The operator $\mathcal{L}$ is self-adjoint in the weighted inner product

$$\langle u, v\rangle_\rho = \int_0^1 u(x)\overline{v(x)}\rho(x)\mathrm{d}x,$$

with associated norm $\|\cdot\|_\rho$, where the weight function is

$$\rho(x) := e^{\int_0^x a_1(y)\mathrm{d}y} > 0.$$

The associated eigenvalue problem is

$$\left\{\begin{array}{l} \mathcal{L}p = \lambda p, \\ b_1^l p(0) + b_2^l \partial_x p(0) = 0, \quad b_1^r p(1) + b_2^r \partial_x p(1) = 0. \end{array}\right. \tag{3.1}$$

Theorem 3.4. *Consider the eigenvalue problem (3.1) on $H^2_{bc}(]0,1[)$. Then all the eigenvalues are real-valued and simple, and can be enumerated in a strictly descending order*

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots, \qquad \lim_{n\to+\infty} \lambda_n = -\infty.$$

The eigenfunction $p_j(x)$ associated with the eigenvalue λ_j can be normalized so that

(i) p_j has j simple zeros in the open interval $(0,1)$;

(ii) the eigenfunctions are orthonormal in the ρ-weighted inner product;

(iii) the eigenfunctions form a complete orthonormal basis of $L^2(]0,1[)$ in the ρ-weighted inner product;

(iv) the largest eigenvalue can be characterized as the supremum of the bilinear form associated to $\mathcal{L}$

$$\lambda_0 = \sup_{\|u\|_\rho=1} \langle \mathcal{L}u, u\rangle_\rho,$$

moreover the supremum is achieved at $u = p_0$ which no zero on $(0,1)$.

Proof. We refer to [1, Chapter 8.6 Theorem 8.22]. It relies on the spectral decomposition of self-adjoint compact operators. ∎

Sturm-Liouville operator on the real line. We consider the same Sturm-Liouville operator $\mathcal{L}$ but now acting on $H^2(\mathbb{R})$ with smooth coefficients which decay exponentially to constants at $x \pm \infty$

$$\lim_{x\to\pm\infty} e^{\nu|x|}|a_1(x) - a_1^\pm| = 0, \qquad \lim_{x\to\pm\infty} e^{\nu|x|}|a_0(x) - a_0^\pm| = 0,$$

for some $\nu > 0$ and constants $a^\pm_{0,1} \in \mathbb{R}$. The operator is self-adjoint in the ρ-weighted inner product, where the weight has the finite asymptotic values

$$\rho_\pm := \lim_{x\to\pm\infty} e^{-a_1^\pm x}\rho(x).$$

Theorem 3.5. *Consider the eigenvalue problem $\mathcal{L}p = \lambda p$ on $H^2(\mathbb{R})$, where the coefficients satisfy the above conditions. The point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ consists of a finite number, possibly zero, of simple eigenvalues, which can be enumerated in a a strictly descending order*

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_N > b := \max\left\{a_0^-, a_0^+\right\}.$$

For $j = 0, \dots, N$ the eigenfunction p_j associated to the eigenvalue λ_j can be normalized so that:

(i) p_j has j simple zeros;

(ii) the eigenfunctions are orthonormal in the ρ-weighted inner product;

(iii) the largest eigenvalue, if it exists, can be characterized as the supremum of the bilinear form associated to $\mathcal{L}$

$$\lambda_0 = \sup_{\|u\|_\rho=1} \langle \mathcal{L}u, u\rangle_\rho,$$

moreover the supremum is achieved at $u = p_0$ which no zero.

The proof of the above theorem is beyond the scope of these lectures, but we will see how to use it in a couple of examples.

Example #1: stationary pulses of reaction-diffusion equations. We consider the following scalar bistable reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u),$$

where $f(u)$ satisfies the conditions (1.11) together with $\int_0^1 f(u)\mathrm{d}u > 0$. We look for stationary solutions $u_*(x)$ satisfying

$$0 = u_*'' + f(u_*),$$

together with the condition that

$$\lim_{x\to\pm\infty} u_*(x) = 0.$$

A phase plane analysis, together with the fact that

$$E(u,u') = \frac{1}{2}(u')^2 - \int_0^u f(v)\mathrm{d}v,$$

is an energy for the system, shows that there exists a unique solution which is symmetric about $x = 0$ and such that $u_*'(x) < 0$ for $x > 0$ and $u_*'(x) > 0$ for $x < 0$. Furthermore, as $u_*(x)$ is realized as homoclinc orbit to the fixed point $(0,0)$ which is a saddle, thus we have that

$$\lim_{x\to\pm\infty} e^{\eta|x|}u_*(x) = 0,$$

for some $\eta > 0$. Linearizing the reaction-diffusion around this stationary solution, one obtains the operator

$$\mathcal{L} = \partial_x^2 + f'(u_*(x)).$$

We have that the coefficient $a_1 = 0$ and $a_0(x) = f'(u_*(x))$. As $u_*(x)$ converges to 0 at an exponential rate, so his $a_0(x)$ with $a_0^\pm = f'(0) < 0$. As a consequence, we can apply Theorem 3.5 to $\mathcal{L}$. We have already seen that the translation invariance of the equation implies that

$$\mathcal{L}u_*'(x) = 0,$$

with $u_*' \in H^2(\mathbb{R})$ since $u_*' \to 0$ exponentially fast. We conclude that u_*' is an eigenfunction of $\mathcal{L}$ with corresponding eigenvalue $\lambda = 0$. Moreover, u_*' has precisely one zero at $x = 0$. Theorem 3.5 ensures that there must exist one positive eigenvalue $\lambda_0 > 0$ with an associated eigenfunction p_0 with no zero. All other nonzero eigenvalues must be negative. In conclusion, $\mathcal{L}$ has unstable point spectrum.

Example #2: traveling fronts of reaction-diffusion equations. Let us recall from the introduction that there exists a unique (up to translation) traveling front solution (u_{tw}, c_*) solution of

$$\begin{cases} 0 = u'' + cu' + f(u), \\ u(-\infty) = 1, \text{ and } u(+\infty) = 0, \text{ with } 0 < u < 1 \text{ on } \mathbb{R}. \end{cases}$$

that is a stationary solution of the PDE

$$\partial_t u = \partial_y^2 u + c\partial_y u + f(u),$$

with $f(u)$ bistable under the conditions (1.11). The eigenvalue problem associated to the linearization about $u = u_{\mathrm{tw}}$ takes the form

$$\mathcal{L}p = \lambda p, \quad \mathcal{L} = \partial_y^2 + c_*\partial_y + f'(u_{\mathrm{tw}}(y)).$$

As u_{tw} is realized as a heteroclinic orbit, we have that u_{tw} approaches exponentially 1 at $-\infty$ and 0 at $+\infty$. Furthermore, $u'_{\text{tw}} < 0$ also converges exponentially towards 0 as $y \to \pm\infty$. As a consequence, we have that

$$\mathcal{L}u'_{\text{tw}} = 0, \quad u'_{\text{tw}} \in H^2(\mathbb{R}), \quad u'_{\text{tw}} < 0,$$

such that $\lambda = 0$ is the largest eigenvalue of $\mathcal{L}$ with associated eigenvector u'_{tw} by application of Theorem 3.5. Also we deduce that the point spectrum is bounded from below by

$$b = \max\left\{f'(0), f'(1)\right\} < 0.$$

All other eigenvalues lie in the interval $(b, 0)$.

3.2 Essential spectrum

In this section, we will investigate the essential spectrum and Fredholm indices for differential operators on unbounded domain that typically arise as the linearization of a nonlinear PDE about a heteroclinic (front) or homoclinic (pulse) solution. We will only consider second-order differential operators to simplify the presentation, but all the theory carries out for nth order differential operators. Throughout this section, we will consider differential operators of the form

$$\mathcal{L} := \partial_x^2 + a_1(x)\partial_x + a_0(x), \quad x \in \mathbb{R}. \tag{3.2}$$

The domain $\mathcal{Y} = H^2(\mathbb{R})$ of $\mathcal{L}$ is dense in $\mathcal{X} = L^2(\mathbb{R})$.

We have the following first result.

Lemma 3.11. *Assume that the coefficients $a_j \in W^{1,\infty}(\mathbb{R})$ for $j = 0, 1$, then the operator $\mathcal{L} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is closed.*

Proof. Assume that $(u_k)_{k\geq 0} \in H^2(\mathbb{R})$ converges to u in $\|\cdot\|_2$, and that $v_k := \mathcal{L}u_k$ converges to v in $\|\cdot\|_2$. We must show that $u \in H^2(\mathbb{R})$ and $\mathcal{L}u = v$. For all $\omega > 0$, the operator $\partial_x^2 - \mathbf{i}\omega$ is invertible with

$$\left\|(\partial_x^2 - \mathbf{i}\omega)^{-1}u\right\|_{H^{s+1}} \leq C\left\|u\right\|_{H^s},$$

for any $s \geq 0$ and $C > 0$ independent of u. Furthermore, its resolvent satisfies the limit

$$\lim_{\omega\to+\infty}\left\|(\partial_x^2 - \mathbf{i}\omega)^{-1}\right\|_{\mathscr{L}(H^s,H^{s+1})} = 0.$$

As a consequence, we have that for all $\omega > 0$

$$v_k - \mathbf{i}\omega u_k = \mathcal{L}u_k - \mathbf{i}\omega u_k = \left(\partial_x^2 - \mathbf{i}\omega\right)u_k + (\mathcal{L} - \partial_x^2)u_k,$$

and inverting by $\partial_x^2 - \mathbf{i}\omega$ we get

$$\left(\partial_x^2 - \mathbf{i}\omega\right)^{-1}\left(v_k - \mathbf{i}\omega u_k\right) = u_k + (\partial_x^2 - \mathbf{i}\omega)^{-1}(\mathcal{L} - \partial_x^2)u_k,$$

where $\mathcal{L} - \partial_x^2 = a_1(x)\partial_x + a_0(x)$ is of order 1 and thus $\mathcal{L} - \partial_x^2 \in \mathscr{L}(H^2, H^1)$. Consequently the operator

$$\mathcal{B} = (\partial_x^2 - \mathbf{i}\omega)^{-1}(\mathcal{L} - \partial_x^2) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$$

is bounded and its norm tends to zero as $\omega \to +\infty$. Then, for fixed $\omega > 0$ sufficiently large, the operator $\mathrm{Id} + \mathcal{B} : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is invertible so that we have

$$u_k = (\mathrm{Id} + \mathcal{B})^{-1} (\partial_x^2 - \mathbf{i}\omega)^{-1} (v_k - \mathbf{i}\omega u_k).$$

Thus $(u_k)_{k\geq 0}$ is Cauchy in $H^2(\mathbb{R})$ (here we also used that $(\partial_x^2 - \mathbf{i}\omega)^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$), and passing to the limit $k \to +\infty$ in the above equation yields

$$u = (\mathrm{Id} + \mathcal{B})^{-1} (\partial_x^2 - \mathbf{i}\omega)^{-1} (v - \mathbf{i}\omega u),$$

and thus $\mathcal{L}u = v$. ■

Definition 3.12. *The operator $\mathcal{L}$ is said to be* **exponentially asymptotic** *if the coefficients a_0 and a_1 are smooth, real-valued functions that are asymptotically constant, that is, there exists $\nu > 0$ such that*

$$\lim_{x\to\pm\infty} e^{\nu|x|}|a_j(x) - a_j^{\pm}| = 0, \quad j = 0, 1.$$

Definition 3.13. *For an exponentially asymptotic operator $\mathcal{L}$, we define its asymptotic operator $\mathcal{L}_\infty$ as*

$$\mathcal{L}_\infty := \partial_x^2 + a_1^\infty(x)\partial_x + a_0^\infty(x), \tag{3.3}$$

where the coefficients a_j^∞ are piecewise constant functions obtained by replacing a_j with its limiting value on each half-line. Specifically, we define a_j^∞ by

$$a_j^\infty(x) = \begin{cases} a_j^-, & x < 0, \\ a_j^+, & x > 0. \end{cases}$$

We would like to characterize those values $\lambda \in \mathbb{C}$ for which the nonhomogeneous problem

$$(\mathcal{L}_\infty - \lambda)p = f, \tag{3.4}$$

fails to be boundedly invertible from $L^2(\mathbb{R})$ into $H^2(\mathbb{R})$. We will proceed into two steps.

Step #1: We construct solutions p which may not lie in $H^2(\mathbb{R})$, by rewriting the spectral problem as an initial value problem for a first-order system of ODEs, and we anticipe that the solutions will generically grow exponentially as $x \to \pm\infty$.

Step #2: We determine in terms of λ and the coefficients a_j^∞ if there exists a choice of initial data for which the corresponding solution p decays exponentially a $x \to \pm\infty$, yielding $p \in H^2(\mathbb{R})$.

We introduce $Y = (p, \partial_x p)^T$ and $F = (0, f)^T$, so that p solves (3.4) if and only if Y solves the first order system

$$\partial_x Y = A_\infty(x,\lambda)Y + F, \tag{3.5}$$

where the matrix $A_\infty(x,\lambda) \in \mathscr{M}_2(\mathbb{C})$ is piecewise constant in x, and is defined via the two asymptotic matrices

$$A_\infty(x,\lambda) = \begin{cases} A_-(\lambda), & x < 0 \\ A_+(\lambda), & x \geq 0, \end{cases} \qquad A_\pm(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - a_0^\pm & -a_1^\pm \end{pmatrix}.$$

3.2.1 Non hyperbolic asymptotic matrices.

Non hyperbolic asymptotic matrices always produces essential spectrum. We have the following result which is a generalization of our example on the Laplacian ∂_x^2.

Lemma 3.14. *Fix $\lambda \in \mathbb{C}$. If either of the asymptotic matrices $A_\pm(\lambda)$ is not hyperbolic, then the range of the operator $\lambda \mathrm{Id} - \mathcal{L}_\infty : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is not closed, and the operator is not Fredholm. In particular $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L}_\infty)$.*

Proof. The idea is to construct approximate eigenvalues for each $\lambda \in \mathbb{C}$ such that either of the asymptotic matrices $A_\pm(\lambda)$ is not hyperbolic. To simplify the presentation, we only consider the case where the asymptotic matrices are equal: $A_0(\lambda) = A_\pm(\lambda)$. The fact that $A_0(\lambda)$ is not hyperbolic is equivalent to the fact that $\mu(\lambda), \overline{\mu(\lambda)}$ the two eigenvalues of $A_0(\lambda)$ belong to $\mathbf{i}\mathbb{R}$. As a consequence $u(x,\lambda) = e^{\mu(\lambda)x}$ is not in $L^2(\mathbb{R})$. We can then use mollifiers and set

$$u_j(x,\lambda) = c_j e^{\mu(\lambda)x} e^{-\epsilon_j x^2}, \quad \epsilon_j \to 0 \text{ as } j \to +\infty,$$

with the constant $c_j > 0$ chosen such that $\|u_j(\cdot,\lambda)\|_{L^2(\mathbb{R})} = 1$, that is $c_j = \sqrt{\epsilon_j/\pi}$. With such a definition for $(u_j)_{j\geq 0}$ we that $u_j \in H^2(\mathbb{R})$ and

$$(\mathcal{L}_\infty - \lambda)u_j(x,\lambda) = -4\epsilon_j u_j(x,\lambda) + 4\epsilon_j^2 x^2 u_j(x,\lambda) - 2a_1^\infty \epsilon_j x u_j(x,\lambda),$$

and it is then a simple computation to check that

$$\|(\mathcal{L}_\infty - \lambda)u_j\|_{L^2(\mathbb{R})} \longrightarrow 0 \text{ as } j \to +\infty.$$

This concludes that $(u_j)_{j\geq 0}$ is a sequence of approximate eigenvalue and thus that $\mathrm{rg}(\mathcal{L}_\infty - \lambda \mathrm{Id})$ is not closed, which in turn implies that $\mathcal{L}_\infty - \lambda \mathrm{Id}$ is not Fredholm and $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L}_\infty)$. ∎

3.2.2 Hyperbolic asymptotic matrices.

We first show that if the asymptotic matrices are hyperbolic, then the asymptotic operator $\mathcal{L}_\infty - \lambda$ is Fredholm. In fact, our construction will allow to explicitly characterize the Fredholm index in terms of the dimensions of the unstable subspaces of $A_\pm(\lambda)$. We will need some notations. When $A_\pm(\lambda)$ are hyperbolic, their stable and unstable eigenspaces yield the decomposition

$$\mathbb{C}^2 = \mathbb{E}_s^- \oplus \mathbb{E}_u^- = \mathbb{E}_s^+ \oplus \mathbb{E}_u^+,$$

with corresponding $\pi_{s,u}^\pm$ spectral projections. We recall the following properties

- for all $Y \in \mathbb{C}^2$, we have $Y = \pi_u^\pm Y + \pi_s^\pm Y$, and if $Y \in \mathbb{E}_{s,u}^\pm$ then $\pi_{s,u}^\pm Y = Y$;
- for all $Y \in \mathbb{C}^2$, we have $e^{A_\pm(\lambda)x}\pi_{s,u}^\pm Y = \pi_{s,u}^\pm Y e^{A_\pm(\lambda)x} Y$;
- there are constants $C, \alpha > 0$ such that

$$\left\| e^{A_\pm(\lambda)x}\pi_u^\pm Y \right\| \leq Ce^{\alpha x}\|Y\|, x<0; \quad \left\| e^{A_\pm(\lambda)x}\pi_s^\pm Y \right\| \leq Ce^{-\alpha x}\|Y\|, \quad x>0;$$
$$\left\| e^{A_\pm(\lambda)x}\pi_s^\pm Y \right\| \geq Ce^{-\alpha x}\|Y\|, x<0; \quad \left\| e^{A_\pm(\lambda)x}\pi_u^\pm Y \right\| \geq Ce^{\alpha x}\|Y\|, \quad x>0.$$

The above bounds imply that $e^{A_\pm(\lambda)x}Y$ decays exponentially fast in norm as $x \to -\infty$ only if $Y \in \mathbb{E}_u^\pm$, and grows exponentially fast otherwise. Similarly, $e^{A_\pm(\lambda)x}Y$ decays exponentially fast in norm as $x \to +\infty$ only if $Y \in \mathbb{E}_s^\pm$, and grows exponentially fast otherwise. This **exponential dichotomy** drives the following analysis.

Strategy: For any given $F \in L^2(\mathbb{R})^2$ (each component is in $L^2(\mathbb{R})$), and for every initial condition Y_0 the system (3.5) has a unique solution defined on $\mathbb{R}$. The question is, does there exist any initial condition for which the corresponding solution decays in norm as $x \to \pm\infty$? Furthermore, is that initial condition unique? Since the choice of the initial condition allows 2 degrees of freedom, we expect the existence and uniqueness will result when the decay condition imposes 2 linearly independent constraints. To characterize these constraints, we start with an arbitrary $Y_0 \in \mathbb{C}^2$, and solve the inhomogeneous system (3.5) for $x < 0$. We get

$$Y(x) = e^{A_-(\lambda)x}Y_0 + \int_0^x e^{A_-(\lambda)(x-y)}F(y)\mathrm{d}y.$$

Using the projections $\pi_{s,u}^-$, we may rewrite the above equation as

$$Y(x) = e^{A_-(\lambda)x}\pi_u^- Y_0 + e^{A_-(\lambda)x}\pi_s^- Y_0 + \int_0^x e^{A_-(\lambda)(x-y)}\pi_u^- F(y)\mathrm{d}y + \int_0^x e^{A_-(\lambda)(x-y)}\pi_s^- F(y)\mathrm{d}y.$$

It is also more convenient to change the lower limit of integration on the inhomogeneous term with the stable projection, pushing it back to $-\infty$. In this case, we have

$$Y(x) = e^{A_-(\lambda)x}\pi_u^- Y_0^- + e^{A_-(\lambda)x}\pi_s^- Y_0^- - \int_x^0 e^{A_-(\lambda)(x-y)}\pi_u^- F(y)\mathrm{d}y + \int_{-\infty}^x e^{A_-(\lambda)(x-y)}\pi_s^- F(y)\mathrm{d}y,$$

where we have introduced the vector

$$Y_0^- = Y_0 - \int_{-\infty}^0 e^{-A_-(\lambda)y}\pi_s^- F(y)\mathrm{d}y$$

Introducing the Green's matrix function for the ODE system $A_-(\lambda)$

$$\mathbf{G}_-(x,\lambda) = \begin{cases} -e^{A_-(\lambda)z}\pi_u^-, & x < 0, \\ e^{A_-(\lambda)x}\pi_s^-, & x > 0, \end{cases}$$

we can write

$$Y(x) = e^{A_-(\lambda)x}\pi_u^- Y_0^- + e^{A_-(\lambda)x}\pi_s^- Y_0^- + (\mathbf{G}_- * \mathbf{F}_-)(x), \quad x \leq 0,$$

where

$$\mathbf{F}_-(x) = \begin{cases} F(x), & x \leq 0, \\ 0, & x > 0. \end{cases}$$

As F is assumed to be in $L^2(\mathbb{R})^2$, we see that $\mathbf{G}_- * \mathbf{F}_-$ is bounded in $L^q(\mathbb{R})$ for all $q \in [2, +\infty]$. Since the term $e^{A_-(\lambda)x}\pi_s^- Y_0^-$ grows exponentially in nom as $x \to -\infty$, and $e^{A_-(\lambda)x}\pi_u^- Y_0^-$ decays exponentially in nom as $x \to -\infty$, we see that

$$Y \in L^2(\mathbb{R}_-)^2 \Longleftrightarrow Y_0^- \in \mathbb{E}_u^-.$$

A similar analysis for $x \geq 0$ leads to an expression of the solution as

$$Y(x) = e^{A_+(\lambda)x}\pi_u^+ Y_0^+ + e^{A_+(\lambda)x}\pi_s^+ Y_0^+ + (\mathbf{G}_+ * \mathbf{F}_+)(x), \quad x \geq 0,$$

with corresponding definitions for Y_0^+, $\mathbf{G}_+$ and $\mathbf{F}_+$, from which we conclude that

$$Y \in L^2(\mathbb{R}_+)^2 \Longleftrightarrow Y_0^+ \in \mathbb{E}_s^+.$$

As a consequence, the composite function

$$Y(x) = \begin{cases} e^{A_-(\lambda)x} Y_0^- + (\mathbf{G}_- * \mathbf{F}_-)(x), & x < 0, \\ e^{A_+(\lambda)x} Y_0^+ + (\mathbf{G}_+ * \mathbf{F}_+)(x), & x > 0, \end{cases}$$

solves (3.5) on the disjoint intervals $(-\infty, 0)$ and $(0, +\infty)$ and decays exponentially in norm as $x \to \pm\infty$ if and only if both $Y_0^- \in \mathbb{E}_u^-$ and $Y_0^+ \in \mathbb{E}_s^+$. However, the solution Y solves (3.5) on the whole line if and only if it is continuous at $x = 0$, that is if and only of $Y(0^+) = Y(0^-)$. This continuity condition can be expressed as

$$Y_0^- - Y_0^+ = (\mathbf{G}_+ * \mathbf{F}_+ - \mathbf{G}_- * \mathbf{F}_-)(0) := \mathcal{G}(f), \tag{3.6}$$

subject to the constraints $Y_0^- \in \mathbb{E}_u^-$ and $Y_0^+ \in \mathbb{E}_s^+$. For a fixed $f \in L^2(\mathbb{R})$ the term $\mathcal{G}(f) \in \mathbb{C}$ is a known vector. Setting

$$n_u^- = \dim \mathbb{E}_u^- \text{ and } n_s^+ = \dim \mathbb{E}_s^+,$$

and $(y_1^u, \cdots, y_{n_u^-}^u)$ denote a basis of $\mathbb{E}_u^-$ and $(y_1^s, \cdots, y_{n_s^+}^s)$ denote a basis of $\mathbb{E}_s^+$. Forming the matrix

$$\mathbf{M}(\lambda) = \left(y_1^u, \cdots, y_{n_u^-}^u, y_1^s, \cdots, y_{n_s^+}^s \right), \tag{3.7}$$

the system (3.6) can be written in terms of some $\mathbf{Y} \in \mathbb{C}^{n_u^- + n_s^+}$ as

$$\mathbf{M}(\lambda)\mathbf{Y} = \mathcal{G}(f).$$

If the matrix $\mathbf{M}(\lambda)$ is square, that is $n_u^- + n_s^+ = 2$, and invertible, that is $\det \mathbf{M}(\lambda) \neq 0$ then there exists a unique solution to (3.6), and in this case

$$\|\mathbf{Y}\| \leq C(\lambda) \|f\|_{L^2(\mathbb{R})},$$

such that the composite function $Y(x)$ found previously satisfies

$$\|Y\|_{L^2(\mathbb{R})^2} \leq C_0(\lambda) \|f\|_{L^2(\mathbb{R})},$$

for some constant $C_0(\lambda) > 0$. Returning to the original variable p, the continuity condition makes both p and $\partial_x p$ continuous at $x = 0$, and the above estimate translates to

$$\|p\|_{H^2(\mathbb{R})} \leq C_0(\lambda) \|f\|_{L^2(\mathbb{R})},$$

which is precisely the invertibility of $\mathcal{L}_\infty - \lambda$.

The invertibility of the matrix $\mathbf{M}(\lambda)$ is equivalent to the following two conditions

$$\dim \mathbb{E}_u^- + \dim \mathbb{E}_s^+ = 2, \quad \dim \left[\mathbb{E}_u^- \cap \mathbb{E}_s^+\right] = 0,$$

which can be stated as

$$\mathbb{E}_u^- \oplus \mathbb{E}_s^+ = \mathbb{C}^2.$$

Results. So far, we have almost proved the following lemma.

Lemma 3.15. *Fix $\lambda \in \mathbb{C}$. If the asymptotic matrices $A_\pm(\lambda)$ are hyperbolic, then $\operatorname{rg}(\mathcal{L}_\infty - \lambda)$ is closed. Furthermore,*

$$\dim\left[\ker(\mathcal{L}_\infty - \lambda)\right] = \dim\left[\ker(\mathbf{M}(\lambda))\right], \quad \operatorname{codim}\left[\operatorname{rg}(\mathcal{L}_\infty - \lambda)\right] = \operatorname{codim}\left[\operatorname{rg}(\mathbf{M}(\lambda))\right]$$

where the matrix $\mathbf{M}(\lambda)$ is the matrix defined in (3.7). Moreover, $\mathcal{L}_\infty - \lambda$ is Fredholm with index

$$\operatorname{ind}(\mathcal{L}_\infty - \lambda) = n_u^-(\lambda) + n_s^+(\lambda) - 2,$$

where $n_{s,u}^\pm(\lambda) = \dim\left[\mathbb{E}_{s,u}^\pm(\lambda)\right]$ are the dimensions of the stable and unstable subspaces of $A_\pm(\lambda)$.

Proof. We have that $f \in \operatorname{rg}(\mathcal{L}_\infty - \lambda)$ if and only if $\mathcal{G}(f) \in \operatorname{rg}(\mathbf{M}(\lambda))$. Since $\operatorname{rg}(\mathbf{M}(\lambda))$ is closed and $f \mapsto \mathcal{G}(f)$ is continuous in $L^2(\mathbb{R})$, it follows that $\operatorname{rg}(\mathcal{L}_\infty - \lambda)$ is closed. It is also clear that each $p \in \ker(\mathcal{L}_\infty - \lambda)$ is in one-to-one correspondance with an $\mathbf{Y} \in \ker \mathbf{M}(\lambda)$. To demonstrate the equality of the codimensions, we consider only the case $\operatorname{codim}\left[\operatorname{rg}(\mathbf{M}(\lambda))\right] = 1$, in which case $\ker(\mathbf{M}^*(\lambda)) = \operatorname{span}(w)$ for some vector with $\|w\| = 1$. We know that $\mathcal{G}(f) \in \operatorname{rg}(\mathbf{M}(\lambda))$ if and only if $w \perp \mathcal{G}(f)$. Let $\xi \in L^2(\mathbb{R})$ be any function that satisfies $\mathcal{G}(\xi) = w$. The codimension one projection onto $\operatorname{rg}(\mathcal{L}_\infty - \lambda)$ is then given by

$$\pi_\infty f := f - \langle w, \mathcal{G}(f)\rangle \xi.$$

Every $f \in L^2(\mathbb{R})$ can be decomposed as $f = \pi_\infty f + \beta\xi$ where $\beta := \langle w, \mathcal{G}(f)\rangle \in \mathbb{C}$. Since $\pi_\infty \in \operatorname{rg}(\mathcal{L}_\infty - \lambda)$ we have the decomposition

$$L^2(\mathbb{R}) = \operatorname{rg}(\mathcal{L}_\infty - \lambda) \oplus \operatorname{span}(w),$$

which is precisely the meaning of $\operatorname{codim}\left[\operatorname{rg}(\mathcal{L}_\infty - \lambda)\right] = 1 = \operatorname{codim}\left[\operatorname{rg}(\mathbf{M}(\lambda))\right]$. ■

Definition 3.16. *The* **Morse index** *of a constant matrix A, denoted by $\mathbf{i}(A)$, is the dimension of the unstable subspace associated to A:*

$$\mathbf{i}(A) = \dim \mathbb{E}_u(A).$$

Combining the previous two lemmas, we have the following complete characterization.

Proposition 3.17. *For $\lambda \in \mathbb{C}$, the asymptotic operator $\mathcal{L}_\infty - \lambda$ is Fredholm if and only if the asymptotic matrices $A_\pm(\lambda)$ are hyperbolic. The resolvent set of $\mathcal{L}_\infty$ is comprised precisely of those $\lambda \in \mathbb{C}$ for which $\mathcal{L}_\infty - \lambda$ is Fredholm and $\mathbb{C}^2 = \mathbb{E}_u^-(\lambda) \oplus \mathbb{E}_s^+(\lambda)$, where $\mathbb{E}_{s,u}^\pm(\lambda)$ are the stable and unstable eigenspaces of the asymptotic matrices. Moreover, for λ in the resolvent set there exists $C = C(\lambda) > 0$ such that*

$$\left\|(\mathcal{L}_\infty - \lambda)^{-1}\right\|_{H^2(\mathbb{R})} \le C(\lambda)\,\|f\|_{L^2(\mathbb{R})}.$$

For those $\lambda \in \mathbb{C}$ for which the operator is Fredholm, the Fredholm index equals the difference of the Morse indices of the asymptotic matrices $\mathbf{i}_\pm(\lambda) := \mathbf{i}(A_\pm(\lambda))$

$$\operatorname{ind}(\mathcal{L}_\infty - \lambda) = \mathbf{i}_-(\lambda) - \mathbf{i}_+(\lambda).$$

In particular, we can characterize the essential spectrum of $\mathcal{L}_\infty$ as

$$\sigma_{\mathrm{ess}}(\mathcal{L}_\infty) = \{\lambda \in \mathbb{C} \mid \mathbf{i}_-(\lambda) \neq \mathbf{i}_+(\lambda)\} \cup \{\lambda \in \mathbb{C} \mid \dim \mathbb{E}_c(A_\pm(\lambda)) \neq 0\}.$$

3.2.3 Essential spectrum of exponentially asymptotic operators

We have the following theorem which tells us that the information on the essential spectrum of an exponentially asymptotic operator is completely contained in its asymptotic operator.

Theorem 3.6. *Assume that the operator given (3.2) is exponentially asymptotic with $H^1(\mathbb{R})$ coefficients. Then $\mathcal{L}$ is a relatively compact perturbation of the asymptotic operator $\mathcal{L}_\infty$ given in (3.3). In particular,*

$$\sigma_{\text{ess}}(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \mathbf{i}_-(\lambda) \neq \mathbf{i}_+(\lambda)\} \cup \{\lambda \in \mathbb{C} \mid \dim \mathbb{E}_c(A_\pm(\lambda)) \neq 0\}.$$

Moreover, for each $\lambda \neq \sigma_{\text{ess}}(\mathcal{L})$, either $\dim(\ker(\mathcal{L} - \lambda)) \neq 0$ *or there exists $C > 0$ such that*

$$\left\|(\mathcal{L} - \lambda)^{-1} f\right\|_{H^2(\mathbb{R})} \leq C \left\|f\right\|_{L^2(\mathbb{R})}.$$

Proof. We only consider the case where the coefficients $a_{0,1}(x)$ are constants except on a common compact interval $I \subset \mathbb{R}$. We fix $\lambda \in \rho(\mathcal{L}_\infty)$ so that $(\mathcal{L}_\infty - \lambda)^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is continuous. We can view $\mathcal{L}_\infty - \mathcal{L}$ as a piecewise map from $H^2(\mathbb{R})$ into $H^1(\mathbb{R}_+)$ and $H^1(\mathbb{R}_-)$ such that the composite map

$$(\mathcal{L}_\infty - \mathcal{L})(\mathcal{L}_\infty - \lambda)^{-1} : L^2(\mathbb{R}) \to H^1(\mathbb{R}_+) \oplus H^1(\mathbb{R}_-)$$

is continuous. Since the coefficients of $\mathcal{L}$ are constant off $I \subset \mathbb{R}$, we have

$$(\mathcal{L}_\infty - \mathcal{L})(\mathcal{L}_\infty - \lambda)^{-1} : L^2(\mathbb{R}) \to H^1(I_+) \oplus H^1(I_-)$$

with $I_+ = I \cap (0, +\infty)$ and $I_- = I \cap (-\infty, 0)$. In particular the map takes bounded sets into bounded sets. As bounded sets in $H^1(I_\pm)$ are equicontinuous and $I_\pm$ are compact, we deduce from the arzela-Ascoli theorem that the operator $(\mathcal{L}_\infty - \mathcal{L})(\mathcal{L}_\infty - \lambda)^{-1}$ maps bounded sets of $L^2(\mathbb{R})$ into precompact sets, and hence is compact. The Weyl essential spectrum theorem ensures that $\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}_\infty)$ and the first part of the theorem is proved. The final statement of the theorem follows from the Fredholm alternative. ∎

3.2.4 Boundaries of the essential spectrum: Fredholm borders

Now that the essential spectrum of an exponentially asymptotic operator has been completely characterized, we wish to study its boundary. The boundary of the essential spectrum is precisely described by those values of λ for which the asymptotic matrices lose their hyperbolicity: that is when a matrix eigenvalue becomes purely imaginary. The matrix eigenvalues $\mu_\pm(\lambda)$ are the zeros of the characteristic polynomials of $A_\pm(\lambda)$:

$$d_\pm(\mu, \lambda) := \det(A_\pm(\lambda) - \mu I_2) = \mu^2 + a_1^\pm \mu + a_0^\pm - \lambda.$$

The Fredholm border(s), denoted $\sigma_F(\mathcal{L})$ are those curves in the complex λ-plane for which there exists a purely imaginary matrix eigenvalue, that is

$$\sigma_F(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid d_\pm(\mathbf{i}\ell, \lambda) = 0, \text{ some } \ell \in \mathbb{R}\}.$$

In other words, the Fredholm boorder(s) can be conveniently parametrized by the real-valued parameter ℓ,

$$\lambda_\pm(\ell) := -\ell^2 + \mathbf{i}a_1^\pm \ell + a_0^\pm, \quad \ell \in \mathbb{R},$$

and thus

$$\sigma_F(\mathcal{L}) = \left\{-\ell^2 + \mathbf{i}a_1^\pm \ell + a_0^\pm \mid \ell \in \mathbb{R}\right\}.$$

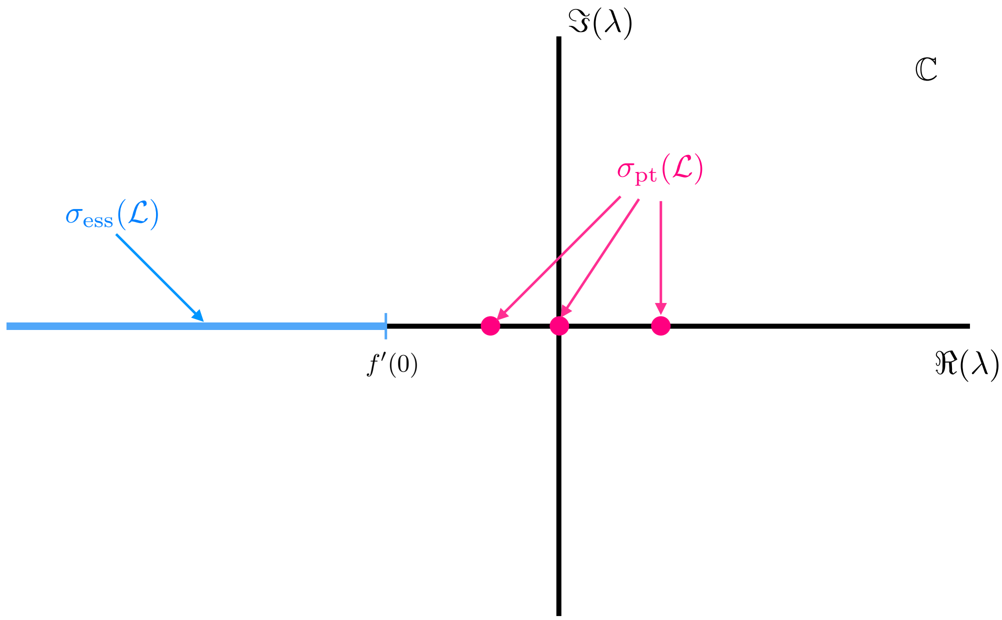

Figure 3.1: *Illustration of the spectrum of* $\mathcal{L} = \partial_x^2 + f'(u_*(x))$ *around a stationary pulse solutions* u_*.

Theorem 3.7. *Fix an exponentially asymptotic operator* $\mathcal{L}$ *as in* (3.2) *and let* $\sigma_F(\mathcal{L})$ *denote its Fredholm borders. There exists a finite number* N *of open, disjoint, connected sets* $S_j \subset \mathbb{C}$ *such that*

$$\mathbb{C} \backslash \sigma_F(\mathcal{L}) = \bigcup_{j=1}^{N} S_j.$$

For each j *the Fredholm index of* $\mathcal{L} - \lambda$ *is independent of* $\lambda \in S_j$. *Each set is either entirely within* $\sigma_{\mathrm{ess}}(\mathcal{L})$ *or is contained with* $\sigma_{\mathrm{pt}}(\mathcal{L}) \cup \rho(\mathcal{L})$. *If the asymptotic matrices are equal* $A_-(\lambda) = A_+(\lambda)$, *then the Fredholm borders coincide and comprise the entire essential spectrum* $\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_F(\mathcal{L})$.

We denote by $\sigma_F(\mathcal{L}_\pm)$ the two solutions curves $\lambda_\pm(\ell)$ which are oriented curves in the complex plane and parametrized by $\ell \in \mathbb{R}$, with the orientation coinciding with the direction of increasing ℓ. As λ crosses a Fredholm border separating a domain S_j from a domain S_k, the Fredholm operator $\mathcal{L} - \lambda$ may change. Indeed, it can be shown that $\mathrm{ind}(\mathcal{L} - \lambda)$:

- increases by one upon crossing the curve $\sigma_F(\mathcal{L}_+)$ from right to left with respect to its orientation;
- decreases by one upon crossing the curve $\sigma_F(\mathcal{L}_-)$ from right to left with respect to its orientation.

3.2.5 Application to pulses and fronts of reaction-diffusion equation

We continue our study of the scalar reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u).$$

Example #1: pulses. We have already established the existence (and uniqueness up to translation) of stationary pulse solutions u_* satisfying

$$0 = u_*'' + f(u_*), \text{ and } \lim_{x\to\pm\infty} e^{\eta|x|}u_*(x) = 0,$$

for some $\eta > 0$, and when $f(u)$ is bistable with $\int_0^1 f(u)\mathrm{d}u > 0$. We have characterized the point spectrum of the associated linearized operator

$$\mathcal{L} = \partial_x^2 + f'(u_*(x)),$$

and demonstrated that $\lambda = 0$ is an eigenvalue with corresponding eigenvector u_*' that vanishes once on the real line, so that Sturm-Liouville theory implied the existence of $\lambda_0 > 0$ in the point spectrum of $\mathcal{L}$. Lets us continue the characterization of $\sigma(\mathcal{L})$ by investigating its essential spectrum. First, $\mathcal{L}$ is an exponentially asymptotic operator with asymptotic operator

$$\mathcal{L}_\infty = \partial_x^2 + f'(0).$$

As a consequence, the essential spectrum is given by its Fredholm border parametrized by $\lambda(\ell) = -\ell^2 + f'(0)$. As a consequence, $\sigma_{\mathrm{ess}}(\mathcal{L})$ is the half line $(-\infty, f'(0)]$, with $f'(0) < 0$ by definition of bistability of f. To be exhaustive, the matrix eigenvalues are

$$\mu(\lambda) = \pm\sqrt{\lambda - f'(0)},$$

and the Morse index is $\mathbf{i}_0(\lambda) = 1$ for all $\lambda \in \mathbb{C}\backslash(-\infty, f'(0)]$.

Example #2: traveling fronts. We have already established the existence (and uniqueness up to translation) of traveling front solutions (u_{tw}, c_*) satisfying

$$0 = u_{\mathrm{tw}}'' + c_* u_{\mathrm{tw}} + f(u_{\mathrm{tw}}), \quad u_{\mathrm{tw}}(-\infty) = 1, \quad u_{\mathrm{tw}}(+\infty) = 0 \text{ and } \lim_{x\to\pm\infty} e^{\eta|x|}u_{\mathrm{tw}}'(x) = 0,$$

for some $\eta > 0$, and when $f(u)$ is bistable. We have characterized the point spectrum of the associated linearized operator

$$\mathcal{L} = \partial_x^2 + c_*\partial_x + f'(u_{\mathrm{tw}}(x)),$$

and demonstrated that $\lambda = 0$ is an eigenvalue with corresponding eigenvector $u_*' < 0$, so that by Sturm-Liouville theory all remaining point spectrum lies within the interval $(b, 0)$ where

$$b = \max\left\{f'(0), f'(1)\right\}.$$

Lets us continue the characterization of $\sigma(\mathcal{L})$ by investigating its essential spectrum. First, $\mathcal{L}$ is an exponentially asymptotic operator with asymptotic operator

$$\mathcal{L}_\infty = \partial_x^2 + c_*\partial_x + a_0^\pm, \quad a_0^- = f'(1) \text{ and } a_0^+ = f'(0).$$

The Fredholm borders are given by the two curves

$$\sigma_F(\mathcal{L}_-) = \left\{-\ell^2 + \mathbf{i}c_*\ell + f'(1) \mid \ell \in \mathbb{R}\right\}, \quad \sigma_F(\mathcal{L}_+) = \left\{-\ell^2 + \mathbf{i}c_*\ell + f'(0) \mid \ell \in \mathbb{R}\right\}.$$

Without loss of generality, let us assume that $f'(0) < f'(1) < 0$, then $\sigma_F(\mathcal{L}_+)$ will be to the left of $\sigma_F(\mathcal{L}_-)$. The complex plane is thus divided into three regions denoted $S_{1,2,3}$, with S_1 the region to the right of

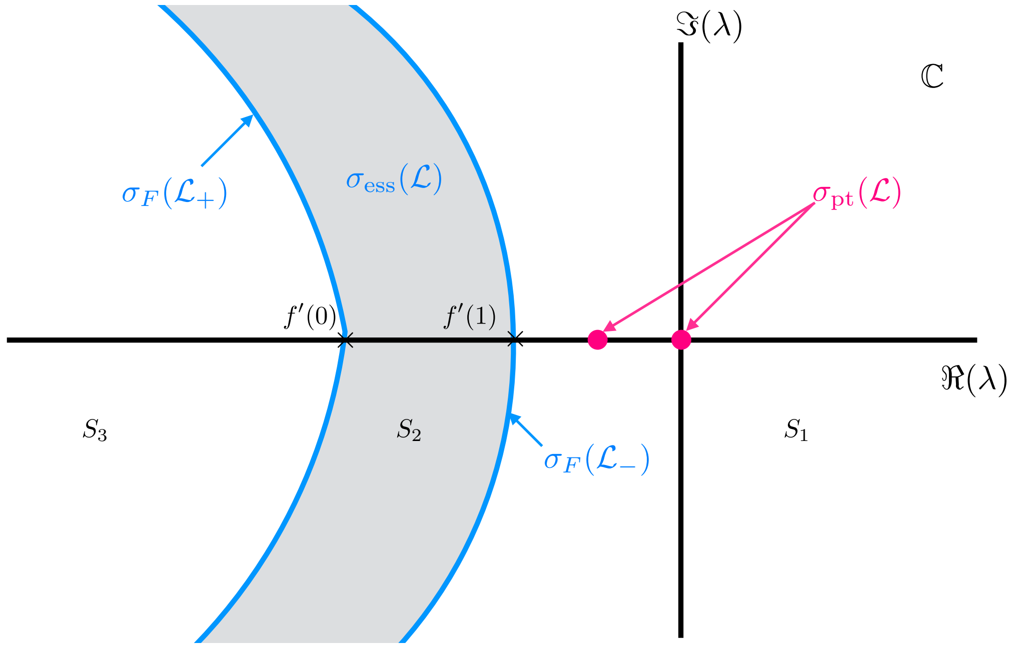

Figure 3.2: *Illustration of the spectrum of* $\mathcal{L} = \partial_x^2 + c_*\partial_x + f'(u_{\text{tw}}(x))$ *around a traveling front solution* u_{tw}.

$\sigma_F(\mathcal{L}_-)$, S_2 the region in between the two curves, and S_3 to the left of $\sigma_F(\mathcal{L}_+)$. The very first to remark is that for $\lambda \in \rho(\mathcal{L})$ for $\Re(\lambda)$ very large, such that $(L, +\infty) \subset S_1 \cap \rho(\mathcal{L})$ for some $L > 0$ large enough. As a consequence, $\mathcal{L} - \lambda$ has Fredholm index 0 on S_1. Then the Fredholm index of $\mathcal{L} - \lambda$ is -1 in S_2 and 0 in S_3. As a consequence, we have

$$\sigma_{\text{ess}}(\mathcal{L}) = S_2 \cup \sigma_F(\mathcal{L}_-) \cup \sigma_F(\mathcal{L}_+) = \overline{S_2}.$$

Chapter 4

Linear stability in infinite dimension

Let $\mathcal{L} : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}$ be a closed linear operator with dense domain $\mathcal{D}(\mathcal{L}) = \mathcal{Y}$, with $\mathcal{X}$, $\mathcal{Y}$ two Banach spaces. In the previous chapter, we have defined and characterized the spectrum of the operator $\mathcal{L}$. We now consider the following Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \mathcal{L}u(t,x), & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{4.1}$$

We wish to identify the relationship between the spectrum of $\mathcal{L}$ and the dynamics of (4.1). The impact of point spectrum is relatively easy to understand. Let $\lambda_0 \in \sigma_{\mathrm{pt}}(\mathcal{L})$ and $\psi_0 \in \mathcal{Y}$ be an associated eigenvector. Then $u(t,x) = e^{\lambda_0 t}\psi_0(x)$ is a solution to (4.1) with initial condition $u_0 = \psi_0$. Noting that $\|u\|_{\mathcal{Y}} = e^{\Re(\lambda_0)t}\|\psi_0\|_{\mathcal{Y}}$, we have the following characterization

- $\Re(\lambda_0) > 0$ implies the existence of exponentially growing solutions;
- $\Re(\lambda_0) < 0$ implies the existence of exponentially decaying solutions;
- $\Re(\lambda_0) = 0$ implies the existence of bounded, nondecaying solutions.

Moreover, if λ_0 is not algebraically simple, then $\Re(\lambda_0) = 0$ may imply the existence of solutions that grow polynomial in time. The impact of essential spectrum is more subtle. To motivate our approach, let us consider that the operator $\mathcal{L}$ has constant coefficients and is given by

$$\mathcal{L} = \partial_x^2 + a_1\partial_x + a_0 : H^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

For an initial condition of the form $u_0(x) = e^{\mathbf{i}\ell x}$ (which does not belong to $\mathcal{D}(\mathcal{L})$), the solution to (4.1) is given by

$$u(t,x) = e^{\lambda(\ell)t + \mathbf{i}\ell x},$$

where $\lambda(\ell) = -\ell^2 + \mathbf{i}a_1\ell + a_0$. Such $\lambda(\ell)$ form the Fredholm border, $\sigma_F(\mathcal{L})$, which comprises the rightmost boundary of the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$. While such solutions generically do not belong to $\mathcal{Y} = H^2(\mathbb{R})$ (since they are not localized in space), it is clear that $\Re(\lambda(\ell)) > 0$ (< 0) implies the existence of exponentially growing (decaying) solutions in $L^\infty(\mathbb{R})$, whereas $\Re(\lambda(\ell)) = 0$ gives the existence of solutions that are temporally bounded in $L^\infty(\mathbb{R})$ but that do not decay in time. In particular if the Fredholm boundary lies uniformly in the left-half plane (*i.e.* $a_0 < 0$), then so does the essential spectrum, and all solutions of the form $e^{\lambda(\ell)t+\mathbf{i}\ell x}$ decay exponentially in time.

We may also test the impact of $\sigma_{\mathrm{ess}}(\mathcal{L})$ on initial conditions in $L^2(\mathbb{R})$ by using the Fourier transform. System (4.1) becomes

$$\partial_t \widehat{u}(t,\ell) = \lambda(\ell)\widehat{u}(t,\ell), \quad \widehat{u}(0,\ell) = \widehat{u}_0(\ell),$$

from which we get that

$$\widehat{u}(t,\ell) = e^{\lambda(\ell)t}\widehat{u}_0(\ell).$$

And if $\sigma_F(\mathcal{L}) \subset \{\lambda \mid \Re(\lambda) \leq -\alpha < 0\}$, for some $\alpha > 0$, then we get

$$|\widehat{u}(t,\ell)| = e^{-\alpha t}|\widehat{u}_0(\ell)|,$$

and with Plancherel's theorem, we can conclude that

$$\|u\|_{L^2(\mathbb{R})} \leq e^{-\alpha t} \|u_0\|_{L^2(\mathbb{R})}.$$

How to proceed when the coefficients of $\mathcal{L}$ are not constant? The idea is to use Laplace transform. We define

$$\widetilde{u}(\lambda) = \int_0^{+\infty} e^{-\lambda t}u(t)\mathrm{d}t,$$

for $\Re(\lambda)$ large enough for the above integral to be convergent. We apply the Laplace transform to (4.1)

$$\lambda\widetilde{u}(\lambda,x) - u_0(x) = \mathcal{L}\widetilde{u}(\lambda,x),$$

that we can rewrite

$$(\lambda\mathrm{Id} - \mathcal{L})\widetilde{u}(\lambda,x) = u_0(x).$$

As a consequence, for all $\lambda \in \rho(\mathcal{L})$, we can invert the above equality to get

$$\widetilde{u}(\lambda,x) = \underbrace{(\lambda\mathrm{Id} - \mathcal{L})^{-1}}_{:=R(\lambda,\mathcal{L})} u_0(x).$$

Finally, we can inverse the Laplace transform by choosing a contour Γ, that is a positively oriented curve that approaches ∞ at either end, and that lies to the right of $\sigma(\mathcal{L})$, and obtain

$$u(t,x) = \mathcal{S}_{\mathcal{L}}(t) \cdot u_0(x)$$

where we have set

$$\mathcal{S}_{\mathcal{L}}(t) := \frac{1}{2\pi\mathbf{i}} \int_\Gamma e^{\lambda t}R(\lambda,\mathcal{L})\mathrm{d}\lambda. \tag{4.2}$$

Most of the time, the operator $\mathcal{S}_{\mathcal{L}}(t)$ is denoted $e^{\mathcal{L}t}$, but the notation is slightly confusing and we shall avoid it in these lectures. The resolvent operator $R(\lambda,\mathcal{L})$ is an analytic function of $\lambda \in \rho(\mathcal{L})$, so that the contour Γ can be continuously deformed without affecting the value of the integral, so long as the deformation does not push Γ across any spectrum of $\mathcal{L}$.

4.1 Semigroup theory

The above construction makes $\mathcal{S}_{\mathcal{L}}(t)$ what is called a semigroup, that is

$$\mathcal{S}_{\mathcal{L}}(0) = \mathrm{Id}, \quad \mathcal{S}_{\mathcal{L}}(t+s) = \mathcal{S}_{\mathcal{L}}(t)\mathcal{S}_{\mathcal{L}}(s) \text{ for all } t, s \geq 0.$$

Actually, we will need a stronger notion of semigroup that we now define.

Definition 4.1 (Strongly continuous semigroup)**.** *A family of bounded operators $S(t) \in \mathscr{L}(\mathcal{X})$, $t \geq 0$, on a Banach space is called a* **strongly continuous semigroup** *or $\mathscr{C}^0$ semigroup if*

- $S(0) = \mathrm{Id}$*;*
- $S(t+s) = S(t)S(s)$ *for all* $t, s \geq 0$*;*
- $\|S(t)u - u\|_{\mathcal{X}} \longrightarrow 0$ *as* $t \to 0^+$ *for all* $u \in \mathcal{X}$*.*

The term strongly continuous is used because this type of continuity is exactly continuity with respect to the strong operator topology. It is weaker than requiring continuity with respect to the uniform topology, which would require

$$|||S(t_k) - S(t)||| = \sup_{\|u\|_{\mathcal{X}}=1} \|S(t_k)u - S(t)u\|_{\mathcal{X}} \longrightarrow 0, \text{ as } t_k \to t,$$

for any $t \geq 0$.

Lemma 4.2. *Let $S(t)$ be a $\mathscr{C}^0$ semigroup, then there exists $\eta > 0$ and $M \geq 1$ such that*

$$|||S(t)||| \leq M e^{\eta t}.$$

Proof. The strong continuity of a semigroup $S(t)$ implies that there exists $\delta > 0$ and $M \geq 1$ such that $|||S(t)||| \leq M$ for all $0 \leq t \leq \delta$. Then for any $t \geq 0$, write $t = s + \delta n$ for $s \in [0, \delta]$, then we have

$$|||S(t)||| = |||S(s+\delta n)||| = |||S(s)S(\delta)\cdots S(\delta)||| \leq |||S(s)||| |||S(\delta)|||^n \leq M^{n+1} = M e^{n \log M} \leq M e^{\eta t},$$

for $\eta := (\log M)/\delta$, and this holds for all $t \geq 0$. ■

Definition 4.3. *The* **generator** *$\mathcal{L}$ of a strongly continuous semigroup $S(t)$ on a Banach space is the operator defined by*

$$\mathcal{L}u := \lim_{h \to 0^+} \frac{S(h)u - u}{h},$$

and the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ is precisely all $u \in \mathcal{X}$ for which the above limit exists, that is

$$\mathcal{D}(\mathcal{L}) := \left\{ u \in \mathcal{X} \mid \lim_{h \to 0^+} \frac{S(h)u - u}{h} \text{ exists in } \mathcal{X} \right\}.$$

We give the following result whose proof is let as an exercise.

Lemma 4.4. *If $\mathcal{L}$ is the generator of a $\mathscr{C}^0$ semigroup $S(t)$ then $\mathcal{D}(\mathcal{L})$ the domain of $\mathcal{L}$ is denses in $\mathcal{X}$ and $\mathcal{L}$ is a closed linear operator.*

If $S(t)$ is a strongly continuous semigroup with generator $\mathcal{L}$, then $S(t)u_0$ is the unique classical solution of the Cauchy problem (4.1) with $u_0 \in \mathcal{D}(\mathcal{L})$. And from our estimate

$$|||S(t)||| \leq Me^{\eta t},$$

we see that the solution cannot blow up in finite time. Conversely, when solving the Cauchy problem (4.1), one would like to know under what conditions the operator $\mathcal{L}$ is the generator of a strongly continuous semigroup. We have the following characterization (see Theorem 5.3 of Pazy [6]).

Theorem 4.1. *A linear operator $\mathcal{L}$ is the infinitesimal generator of a $\mathscr{C}^0$ semigroup $S(t)$ satisfying $|||S(t)||| \leq Me^{\eta t}$ if and only if*

(i) $\mathcal{L}$ is closed and $\mathcal{D}(\mathcal{L})$ is closed in $\mathcal{X}$;

(ii) the resolvent set $\rho(\mathcal{L})$ of $\mathcal{L}$ contains the ray $]\eta, +\infty[$ and

$$|||R(\lambda, \mathcal{L})^n||| \leq \frac{M}{(\lambda - \eta)^n}, \quad \text{for } \lambda > \eta, \quad n = 1, 2, \cdots.$$

We conclude this section with an important sufficient condition for an operator $\mathcal{L}$ to be the infinitesimal generator of a $\mathscr{C}^0$ semigroup.

Theorem 4.2. *Let $\mathcal{L}$ be a densely defined operator in $\mathcal{X}$ satisfying the following conditions:*

(i) for some $0 < \delta < \pi/2$, the sector $\Sigma_\delta = \{\lambda \in \mathbb{C} \mid |\arg\lambda| < \pi/2 + \delta\} \setminus \{0\}$ is contained in the resolvent set $\rho(\mathcal{L})$;

(ii) there exists a constant $M \geq 0$ such that

$$|||R(\lambda, \mathcal{L})||| \leq \frac{M}{|\lambda|}, \text{ for } \lambda \in \Sigma_\delta.$$

Then $\mathcal{L}$ is the infinitesimal generator of a $\mathscr{C}^0$ semigroup which is given by

$$S(t) = \frac{1}{2\pi \mathbf{i}} \int_\Gamma e^{\lambda t} R(\lambda, \mathcal{L}) \mathrm{d}\lambda, \quad t > 0,$$

where Γ is a smooth curve in Σ_δ running from $\infty e^{-\mathbf{i}\theta}$ to $\infty e^{\mathbf{i}\theta}$ for $\pi/2 < \theta < \pi/2 + \delta$.

Such operators $\mathcal{L}$ are called **sectorial** and generate **analytic semigroups** (see [6]).

Example: Let us show that $\mathcal{L} = \partial_x^2$ on $\mathcal{X} = L^2(\mathbb{R})$ satisfies the conditions of the above theorem. First, we recall that $\sigma(\mathcal{L}) =]-\infty, 0]$ which certainly lies in a sector. As a consequence, we just need to obtain the bound on the resolvent. By solving the equation $(\lambda - \partial_x^2)u = v$ using the variation of parameters, one finds that

$$u(x) = \int_\mathbb{R} \mathbf{G}_\lambda(x-y) v(y) \mathrm{d}y, \quad \mathbf{G}_\lambda(z) = \frac{1}{2\sqrt{\lambda}} \left(H(z) e^{-\sqrt{\lambda} z} + H(-z) e^{\sqrt{\lambda} z} \right).$$

Therefore, the above integral is a convolution: $u = \mathbf{G}_\lambda * v$. We find that $\|u\|_{L^2(\mathbb{R})} \leq C \|\mathbf{G}_\lambda\|_{L^1(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}$ for some constant $C > 0$. If we take $0 < \delta < \pi/2$, then for all $\lambda \in \Sigma_\delta$ we have $\Re(\sqrt{\lambda}) > 0$. Hence, if $\lambda = re^{\mathbf{i}\theta}$,

$$\|\mathbf{G}_\lambda\|_{L^1(\mathbb{R})} \leq \frac{1}{|2\sqrt{\lambda}|} \int_\mathbb{R} e^{-\Re(\sqrt{\lambda})|z|} \mathrm{d}z = \frac{1}{\sqrt{\lambda}\Re(\sqrt{\lambda})} \leq \frac{1}{|\lambda|} \frac{1}{\cos(\theta/2)} \leq \frac{M}{|\lambda|}.$$

We are going to show that the Laplacian being sectorial will in turn imply that

$$\mathcal{L} = \partial_x^2 + a_1(x)\partial_x + a_0(x) : H^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

is also sectorial provided that $a_{1,0}(x)$ are exponentially asymptotic and smooth enough ($H^1(\mathbb{R})$). For that we will consider $\mathcal{L}$ as a perturbation of ∂_x^2.

Theorem 4.3. *Let $\mathcal{A}$ be the infinitesimal generator of an analytic semigroup. Let $\mathcal{B}$ be a closed linear operator satisfying $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$ and*

$$\|\mathcal{B}u\|_{\mathcal{X}} \leq a \|\mathcal{A}u\|_{\mathcal{X}} + b \|u\|_{\mathcal{X}}, \quad u \in \mathcal{D}(\mathcal{A}).$$

There exists a positive number $\delta > 0$ such that if $0 \leq a \leq \delta$ then $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of an analytic semigroup.

We apply the above theorem to $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\partial_x^2 = \mathcal{A}$. First, as $a_0 \in H^1(\mathbb{R})$ and exponentially asymptotic, it is bounded and thus

$$\|a_0 u\|_{L^2(\mathbb{R})} \leq C \|u\|_{L^2(\mathbb{R})}.$$

Finally, we have that

$$\|\partial_x u\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\ell|^2 |\widehat{u}(\ell)|^2 \mathrm{d}\ell \leq \sqrt{\|u\|_{L^2(\mathbb{R})}} \sqrt{\|\partial_x^2 u\|_{L^2(\mathbb{R})}} \leq \frac{1}{2\epsilon} \|u\|_{L^2(\mathbb{R})} + \frac{\epsilon}{2} \left\|\partial_x^2 u\right\|_{L^2(\mathbb{R})}.$$

We conclude that

$$\|a_1 \partial_x u + a_0 u\| \leq \frac{\epsilon C}{2} \left\|\partial_x^2 u\right\|_{L^2(\mathbb{R})} + \frac{C}{\epsilon} \|u\|_{L^2(\mathbb{R})},$$

and the theorem applies.

4.2 Spectral mapping theorem

Let $S(t)$ be a $\mathscr{C}^0$ semigroup on a Banach space $\mathcal{X}$ and let $\mathcal{L}$ be its infinitesimal generator. In this section, we will be interested in the relations between the spectrum of $\mathcal{L}$ and the spectrum of each one of the operators $S(t)$ for $t \geq 0$. From a purely formal point of view (it is true in finite dimensions), one would expect the relation

$$\sigma(S(t)) \backslash \{0\} = e^{t\sigma(\mathcal{L})} := \left\{ e^{t\lambda} \mid \lambda \in \sigma(\mathcal{L}) \right\}. \tag{4.3}$$

This however not true in general. It turns out that one inclusion is always satisfied as is stated in the following lemma (see Theorem 2.3 [6]).

Lemma 4.5. *Let $S(t)$ be a $\mathscr{C}^0$ semigroup on a Banach space $\mathcal{X}$ and let $\mathcal{L}$ be its infinitesimal generator. Then,*

$$e^{t\sigma(\mathcal{L})} \subset \sigma(S(t)), \quad t \geq 0.$$

Definition 4.6. *Let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ be a closed operator. We call* **the spectral bound** *of $\mathcal{L}$ the following quantity:*

$$s(\mathcal{L}) := \sup \{\Re(\lambda) \mid \lambda \in \sigma(\mathcal{L})\}.$$

For the generator $\mathcal{L}$ of a $\mathscr{C}^0$ semigroup $S(t)$, we define the **growth bound** *as*

$$\eta_0 := \inf \left\{ \eta \in \mathbb{R} \mid \text{ there exists } M_\eta \geq 1 \text{ such that } \|\!|S(t)|\!\| \leq M_\eta e^{\eta t} \text{ for all } t \geq 0 \right\}.$$

For a $\mathscr{C}^0$ semigroup $S(t)$ on a Banach space $\mathcal{X}$ and its infinitesimal generator $\mathcal{L}$, one always has the relation

$$-\infty \leq s(\mathcal{L}) \leq \eta_0 < +\infty.$$

We have the following key result (see [2] Corollary 3.12).

Theorem 4.4 (Spectral mapping theorem). *For sectorial operators $\mathcal{L}$ generating analytic semigroup $S(t)$, one has*

$$\sigma(S(t)) \backslash \{0\} = e^{t\sigma(\mathcal{L})} := \left\{ e^{t\lambda} \mid \lambda \in \sigma(\mathcal{L}) \right\},$$

and the spectral bound equals the growth bound:

$$s(\mathcal{L}) = \eta_0.$$

4.3 Applications to traveling fronts of reaction-diffusion equations

Let us consider our running example of traveling fronts for the reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u),$$

with $f(u)$ bistable under the conditions (1.11). So far, we have proved the following results regarding the linearization around a traveling front solution (u_{tw}, c_*):

$$\mathcal{L} = \partial_y^2 + c_* \partial_y + f'(u_{\mathrm{tw}}(y)),$$

- $\mathcal{L} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a closed linear operator;
- $\mathcal{L}$ is exponentially asymptotic with some rate $\nu > 0$;
- $\sigma_{\mathrm{ess}}(\mathcal{L}) \subset \{\lambda \in \mathbb{C} \mid \Re(\lambda) \leq b\}$ with $b = \max\{f'(0), f'(1)\}$;
- $\lambda = 0$ is a simple isolated eigenvalue with corresponding eigenvector u_{tw}', *i.e.* $\mathcal{E}_0 := \ker \mathcal{L} = \operatorname{span}\{u_{\mathrm{tw}}'\}$, all remaining point spectrum (which is finite) lies within the interval $(b, 0)$;
- $\mathcal{L}$ is a sectorial operator and thus generates an analytic semigroup.

We define the spectral projection $\mathcal{P}_0$ onto $\mathcal{E}_0$ via the Dunford integral formula

$$\mathcal{P}_0 = \frac{1}{2\pi \mathbf{i}} \oint_{\Gamma_0} R(\lambda, \mathcal{L}) \mathrm{d}\lambda$$

where Γ_0 is a positively oriented, simple, closed curve that encircles $\lambda = 0$ in $\mathbb{C}$. We recall that it is the unique bounded operator that enjoys the following properties

1. $\mathcal{P}_0 : \mathcal{X} \to \mathcal{E}_0$, with $\mathcal{P}_{0|_{\mathcal{E}_0}} = \mathrm{Id}$;
2. $\mathcal{P}_0^2 = \mathcal{P}_0 \mathcal{P}_0 = \mathcal{P}_0$;
3. $\mathcal{P}_0 \mathcal{L} = \mathcal{L} \mathcal{P}_0$;

4. if $\mathcal{P}_{\lambda_1}$ is another spectral projection associated to a point eigenvalue $\lambda_1 \neq 0$, then $\mathcal{P}_0\mathcal{P}_{\lambda_1} = \mathcal{P}_{\lambda_1}\mathcal{P}_0 = 0$.

We can define the complementary spectral projection $\Pi_0 := \mathrm{Id} - \mathcal{P}_0$ which satisfies

$$\Pi_0 : \mathcal{X} \to (\mathcal{E}_0^a)^\perp, \quad \mathcal{E}_0^a = \mathrm{span}\left\{e^{c_* y} u_{\mathrm{tw}}'\right\},$$

and enjoys the properties

$$\ker \Pi_0 = \mathcal{E}_0, \quad \mathcal{L}\Pi_0 = \Pi_0\mathcal{L}, \quad \sigma\left(\Pi_0 \mathcal{L}_{|_{(\mathcal{E}_0^a)^\perp}}\right) = \sigma(\mathcal{L}) \backslash \{0\}.$$

Then, we can chose $\omega > 0$ so that all the point spectrum of $\mathcal{L}$ (except for the simple eigenvalue at $\lambda = 0$) lies to the left of the contour $\Gamma = \{\Re(\lambda) = -\omega\}$. Finally, we can apply the spectral mapping theorem to $\mathcal{L}\Pi_0$ and we deduce that there exists a constant $C > 0$ such that

$$\|\!|\mathcal{S}_{\mathcal{L}}(t)\Pi_0\|\!| \leq C e^{-\omega t}, \quad t \geq 0.$$

As a consequence, the traveling front solution (u_{tw}, c_*) is linearly asymptotically stable with respect to the projection Π_0. It is also not difficult to see that one has

$$\|\mathcal{S}_{\mathcal{L}}(t)\Pi_0 u\|_{H^1(\mathbb{R})} \leq C e^{-\omega t} \|u\|_{H^1(\mathbb{R})}, \quad t \geq 0,$$

for all $u \in H^1(\mathbb{R})$.

4.4 Nonlinear asymptotic stability of traveling fronts of reaction-diffusion equations

We are now in position to prove one of the main theorem of these lectures. Let us consider the reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{4.4}$$

and let us introduce the change of coordinate $y = x - c_* t$ such that the above equation transforms to

$$\partial_t u = \partial_y^2 u + c_* \partial_y u + f(u), \quad t > 0, \quad y \in \mathbb{R}. \tag{4.5}$$

Upon denoting by $\mathcal{L}$ the linearization around a traveling front solution (u_{tw}, c_*), we write

$$\partial_t v = \mathcal{L}v + \mathcal{N}(v), \quad t > 0, \quad y \in \mathbb{R},$$

where

$$\mathcal{L} = \partial_y^2 + c_* \partial_y + f'(u_{\mathrm{tw}}(y)), \quad \mathcal{N}(v) = f(u_{\mathrm{tw}} + v) - \mathcal{L}v,$$

that is we have set $u = u_{\mathrm{tw}} + v$.

We assume that $\mathcal{L}$ has dense domain $\mathcal{Z} = H^2(\mathbb{R}) \subset \mathcal{X} = L^2(\mathbb{R})$, and we denote $\mathcal{Y} = H^1(\mathbb{R})$.

Note that here $f(u)$ is a bistable nonlinearity satisfying (1.11). We further assume the following conditions for f.

Hypothesis 4.7. *Let f be a bistable nonlinearity satisfying* (1.11)*. We suppose that:*

(i) $D_u f$ *is locally Lipschitz on bounded sets, that is, for each* $R > 0$ *there exists* $M > 0$ *such that*

$$\|(D_u f(u) - D_u f(v))w\|_{\mathcal{Y}} \leq M \|u - v\|_{\mathcal{Y}} \|w\|_{\mathcal{Y}};$$

(ii) the nonlinearity $\mathcal{N}$ *is quadratic in* $\|\cdot\|_{\mathcal{Y}}$ *near zero, that is there exists* $R, M > 0$ *such that*

$$\|\mathcal{N}(v)\|_{\mathcal{Y}} \leq M \|v\|_{\mathcal{Y}}^2,$$

for all $\|v\|_{\mathcal{Y}} \leq R$.

The translation invariance of the reaction-diffusion (4.4) causes some difficulties. Indeed, there exists a one-dimensional manifold of traveling front solution of (4.5)

$$\mathcal{M}_0 := \{\gamma \cdot u_{\text{tw}} \mid \gamma \in \mathbb{R}\},$$

where $(\gamma \cdot u)(y) = u(y + \gamma)$ for all $y \in \mathbb{R}$. As a consequence, we cannot expect u_{tw} to be asymptotically stable when it is part of a manifold equilibria. The more reasonable expectation is that the manifold itself is stable under the flow. This approach, which leads to the idea of orbital stability, requires to introduce a better suited system of perturbations. More specifically, we have assumed that each solution can be decomposed as

$$u = u_{\text{tw}} + v.$$

In fact, for any $v \in \mathcal{Y}$ sufficiently small, we can uniquely write the sum $u_{\text{tw}} + v$ as a point on $\mathcal{M}_0$ and a normal vector. That is

$$u = u_{\text{tw}} + v = \gamma \cdot u_{\text{tw}} + w, \quad \mathcal{P}_0 w = 0,$$

where $\mathcal{P}_0$ is the spectral projection onto $\ker(\mathcal{L}^a)$ and the orthogonal projection onto the tangent space of the manifold $\mathcal{M}_0$ at u_{tw}.

Lemma 4.8. *There exists* $\delta > 0$ *and smooth functions* $(\gamma, w) : \mathcal{Y} \to \mathbb{R} \times \mathcal{Y}$ *satisfying* $\gamma(0) = 0$ *and* $w(0) = 0$ *such that for all* $\phi \in \mathcal{M}_0$ *and all* $\|v\|_{\mathcal{Y}} \leq \delta$

$$\phi + v = \gamma(v) \cdot \phi + w(v),$$

where $w(v) \in (\ker(\mathcal{L}^a))^\perp$.

Proof. Using a Taylor expansion and the regularity of u_{tw},

$$\gamma \cdot \phi = \phi + \gamma \partial_y \phi + T(\gamma, \phi),$$

with $\|T(\gamma, \phi)\|_{\mathcal{Y}} \leq C|\gamma|^2$ since $\phi \in \mathcal{M}_0$ is uniformly bounded in $H^2(\mathbb{R})$. As a consequence, we get that

$$w = v - \gamma \partial_y \phi - T(\gamma, \phi).$$

The constraint $\mathcal{P}_0 w = 0$ is equivalent to the equation

$$0 = g(\gamma, v) := \langle v, \psi^a \rangle - \gamma \langle \partial_y \phi, \psi^a \rangle - \langle T(\gamma, \phi), \psi^a \rangle,$$

where ψ^a is such that $\ker(\mathcal{L}^a) = \text{span}(\psi^a)$. We have that

$$0 = g(0, 0), \text{ and } \partial_\gamma g(0, 0) = -\langle \partial_y \phi, \psi^a \rangle \neq 0,$$

such that the implicit function theorem gives the existence of a neighborhood of $(0, 0)$ and a unique function $\gamma(v)$ such that $g(\gamma(v), v) = 0$. The dependence of w on v then immediately follows. ■

Definition 4.9. *For the flow generated by (4.4) we say that the manifold $\mathcal{M}_0$ of equilibria is* **asymptotically orbitally stable** *in $\|\cdot\|_{\mathcal{Y}}$ with exponential rate $\sigma > 0$ if there exists $C, \delta > 0$ such that $\|u_0 - \gamma_0 \cdot u_{\mathrm{tw}}\|_{\mathcal{Y}} \leq \delta$ for some $\gamma_0 \in \mathbb{R}$ implies there exists unique $\gamma_\infty = \gamma_\infty(u_0)$ such that*

$$\|u(t) - \gamma_\infty \cdot u_{\mathrm{tw}}\|_{\mathcal{Y}} \leq Ce^{-\sigma t} \|u_0 - \gamma_0 \cdot u_{\mathrm{tw}}\|_{\mathcal{Y}}, \quad t \geq 0.$$

Theorem 4.5. *Consider the reaction-diffusion equation (4.5) with bistable nonlinearity verifying Hypothesis 4.7. Then, the manifold $\mathcal{M}_0$ of equilibria is asymptotically orbitally stable in $H^1(\mathbb{R})$ with rate $0 < \sigma < \omega$.*

Proof. Let $u_0 \in H^1(\mathbb{R})$ be such that $\|u_0 - u_{\mathrm{tw}}\|_{H^1(\mathbb{R})}$ is sufficiently small. The flow is locally well-posed in $\mathcal{Y} = H^1(\mathbb{R})$, that is there exists $T = T(\|u_0\|_{\mathcal{Y}}) > 0$ for which there exists a unique solution $u(t) \in \mathcal{Y}$ for $t \in [0, T)$ of the Cauchy problem associated to (4.5) with $u(0) = u_0$. Assuming that $u(t)$ the solution of (4.5) stays close to u_{tw} for all times in $t \in [0, T)$, we can use the previous lemma to write

$$u(t, y) = \gamma(t) \cdot u_{\mathrm{tw}}(y) + w(t, y), \quad 0 = \mathcal{P}_0 w.$$

Here, without loss of generality, we assume that $\gamma(0) = 0$. Plugging this ansatz into (4.5), we obtain that

$$\partial_t w + (\gamma(t) \cdot u_{\mathrm{tw}}'(y))\gamma'(t) = \mathcal{L}w + \underbrace{f(\gamma \cdot u_{\mathrm{tw}} + w) - \mathcal{L}_\gamma w + (\mathcal{L}_\gamma - \mathcal{L})w}_{:=\mathcal{N}(\gamma, w)},$$

where $\mathcal{L}_\gamma w = D_u f(\gamma \cdot u_{\mathrm{tw}})$. With our assumptions, we have that

$$\|(\mathcal{L}_\gamma - \mathcal{L})w\|_{H^1(\mathbb{R})} \leq C|\gamma| \|w\|_{H^1(\mathbb{R})} \text{ and } \|f(\gamma \cdot u_{\mathrm{tw}} + w) - \mathcal{L}_\gamma w\|_{H^1(\mathbb{R})} \leq M \|w\|_{H^1(\mathbb{R})}^2$$

for $|\gamma|$ and $\|w\|_{H^1(\mathbb{R})}$ small enough. Furthermore, we can write

$$\gamma(t) \cdot u_{\mathrm{tw}}'(y) = u_{\mathrm{tw}}'(y) + \tau(\gamma), \text{ with } \|\tau(\gamma)\|_{H^1(\mathbb{R})} \leq C|\gamma|.$$

As a consequence, equation (4.5) can be written as

$$\partial_t w + \left(u_{\mathrm{tw}}'(y) + \tau(\gamma)\right) \gamma'(t) = \mathcal{L}w + \mathcal{N}(\gamma, w), \quad 0 = \mathcal{P}_0 w, \tag{4.6}$$

with

$$\|\tau(\gamma)\|_{H^1(\mathbb{R})} = \mathcal{O}(|\gamma|), \quad \|\mathcal{N}(\gamma, w)\|_{H^1(\mathbb{R})} = \mathcal{O}\left(|\gamma| \|w\|_{H^1(\mathbb{R})} + \|w\|_{H^1(\mathbb{R})}^2\right).$$

By construction, $w \in (\mathcal{E}_0^a)^\perp$ for all $t \in [0, T)$ which is equivalent to $\langle w, \psi^a \rangle = 0$. Here, we will assume that $\psi^a \in \mathcal{E}_0^a$ is normalized such that

$$\langle u_{\mathrm{tw}}', \psi^a \rangle = 1.$$

We can then take the inner product of the above equation (4.6) to obtain

$$(1 + \langle \tau(\gamma), \psi^a \rangle)\gamma'(t) = \mathbf{n}(\gamma, w) := \langle \mathcal{N}(\gamma, w), \psi^a \rangle.$$

Then, for small enough γ, one gets

$$\gamma'(t) = \frac{\mathbf{n}(\gamma, w)}{1 + \langle \tau(\gamma), \psi^a \rangle} = \mathcal{O}\left(|\gamma| \|w\|_{H^1(\mathbb{R})} + \|w\|_{H^1(\mathbb{R})}^2\right). \tag{4.7}$$

Similarly, using the complementary projection Π_0, we get

$$\partial_t w + (\Pi_0 \tau(\gamma))\, \gamma'(t) = \mathcal{L} w + \Pi_0 \mathcal{N}(\gamma, w),$$

which we rewrites

$$\partial_t w = \mathcal{L} w + \mathcal{N}_f(\gamma, w), \tag{4.8}$$

where the nonlinearity

$$\mathcal{N}_f(\gamma, w) := \Pi_0 \mathcal{N}(\gamma, w) - (\Pi_0 \tau(\gamma))\, \gamma'(t)$$

satisfies the estimate

$$\|\mathcal{N}_f(\gamma, w)\|_{H^1(\mathbb{R})} = \mathcal{O}\left(|\gamma|\, \|w\|_{H^1(\mathbb{R})} + \|w\|^2_{H^1(\mathbb{R})}\right).$$

We fix $\sigma \in (0, \omega)$ and introduce

$$M_w(t) := \sup_{0 \le s \le t} \left(e^{\sigma s} \|w(s)\|_{H^1(\mathbb{R})}\right), \quad M_\gamma(t) := \sup_{0 \le s \le t} |\gamma(s)|.$$

Using the variation of constant formula to (4.8) we obtain the solution

$$w(t) = \mathcal{S}_{\mathcal{L}}(t) w_0 + \int_0^t \mathcal{S}_{\mathcal{L}}(t-s) \mathcal{N}_f(\gamma(s), w(s)) \mathrm{d}s.$$

We use our result on the exponential decay of the semigroup $\mathcal{S}_{\mathcal{L}}(t)$ to get

$$\|w(t)\|_{H^1(\mathbb{R})} \le C e^{-\omega t} \|w_0\|_{H^1(\mathbb{R})} + C \int_0^t e^{-\omega(t-s)} \|\mathcal{N}_f(\gamma(s), w(s))\|_{H^1(\mathbb{R})} \mathrm{d}s,$$

which gives

$$\|w(t)\|_{H^1(\mathbb{R})} \le C e^{-\omega t} \|w_0\|_{H^1(\mathbb{R})} + C \int_0^t e^{-\omega(t-s)} \left(|\gamma(s)|\, \|w(s)\|_{H^1(\mathbb{R})} + \|w(s)\|^2_{H^1(\mathbb{R})}\right) \mathrm{d}s.$$

As a consequence, we get

$$\|w(t)\|_{H^1(\mathbb{R})} \le C e^{-\omega t} \|w_0\|_{H^1(\mathbb{R})} + C e^{-\omega t} \int_0^t \left(e^{(\omega-\sigma)s} M_\gamma(t) M_w(t) + e^{(\omega-2\sigma)s} M_w(t)^2\right) \mathrm{d}s.$$

Thus for $\sigma \in (\omega/2, \omega)$, we get

$$\|w(t)\|_{H^1(\mathbb{R})} \le C \left(e^{-\sigma t} \|w_0\|_{H^1(\mathbb{R})} + e^{-\sigma t} M_\gamma(t) M_w(t) + e^{-2\sigma t} M_w(t)^2\right),$$

such that

$$M_w(t) \le C \left(\|w_0\|_{H^1(\mathbb{R})} + M_\gamma(t) M_w(t) + M_w(t)^2\right).$$

Similarly, we obtain

$$M_\gamma(t) \le C_1 \left(M_\gamma(t) M_w(t) + M_w(t)^2\right).$$

Now assume that $\|w_0\|_{H^1(\mathbb{R})}$ and $T > 0$ are such that $M_v(t) \le 1/(2C_1)$ for all $t \in [0, T)$. We get that

$$M_\gamma(t) \le \frac{1}{2} M_\gamma(t) + C_1 M_w(t)^2,$$

which gives

$$M_\gamma(t) \leq 2C_1 M_w(t)^2, \quad t \in [0,T).$$

With this bound we get

$$M_w(t) \leq C_2 \left(\|w_0\|_{H^1(\mathbb{R})} + M_w(t)^2 + M_w(t)^3 \right), \quad C_2 > 1.$$

By continuity of $M_w(t)$ in t and the fact that $M_w(0) = \|w_0\|_{H^1(\mathbb{R})}$ can be chosen as small as we want, and in particular smaller than the first positive root $r_1 > 0$ of the polynomial

$$p(r) = C_2 \|w_0\|_{H^1(\mathbb{R})} - r + C_2 r^2 + C_2 r^3,$$

it follows that

$$M_w(t) \leq r_1 = \mathcal{O}(\|w_0\|_{H^1(\mathbb{R})}), \quad t \in [0,T).$$

As a consequence if $\|w_0\|_{H^1(\mathbb{R})}$ is sufficiently small such that $r_1 < 1/(2C_1)$, we can extend this process and get that $T = +\infty$. This gives that

$$\|w(t)\|_{H^1(\mathbb{R})} \leq C_3 e^{-\sigma t} \|w_0\|_{H^1(\mathbb{R})}, \quad t \geq 0.$$

Returning to (4.7) and integrating between $t_1 < t_2$ we have

$$|\gamma(t_1) - \gamma(t_2)| \leq C_4 e^{-\sigma t_1} \|w_0\|^2_{H^1(\mathbb{R})}.$$

This shows that the sequence $(\gamma(t))_{t\geq 0}$ is Cauchy, and thus converges at an exponential rate to a limit γ_∞.

■

Chapter 5

Center manifold theorems in infinite dimensions

Throughout this chapter, we will consider three Banach spaces with continuous embeddings

$$\mathcal{Z} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X},$$

and differential equation in $\mathcal{X}$ of the form

$$\partial_t u = \mathcal{L}u + \mathcal{R}(u), \tag{5.1}$$

in which we assume the linear part $\mathcal{L}$ and the nonlinear part $\mathcal{R}$ are such that the following holds.

Hypothesis 5.1 (Structure & Regularity)**.** *We assume that $\mathcal{L}$ and $\mathcal{R}$ in (5.1) have the following properties:*

(i) $\mathcal{L} \in \mathscr{L}(\mathcal{Z}, \mathcal{X})$*;*

(ii) for some $k \geq 2$, there exists a neighborhood $\mathcal{V} \subset \mathcal{Z}$ of 0 such that $\mathcal{R} \in \mathscr{C}^k(\mathcal{V}, \mathcal{Y})$ and

$$\mathcal{R}(0) = 0, \quad D_u\mathcal{R}(0) = 0.$$

The condition $\mathcal{R}(0) = 0$ means that $u = 0$ is a solution of (5.1), and the requirement $D_u\mathcal{R}(0) = 0$ ensures that $\mathcal{L}$ is the linearization of the vector field about $u = 0$, so that $\mathcal{R}$ represents the nonlinear terms which are $\mathcal{O}(\|u\|_{\mathcal{Z}}^2)$.

Definition 5.2. *A solution of the differential equation (5.1) is a function $u : I \to \mathcal{Z}$ defined on an interval $I \subset \mathbb{R}$ with the following properties:*

1. *the map $u : I \to \mathcal{Z}$ is continuous;*
2. *the map $u : I \to \mathcal{X}$ is continuously differentiable;*
3. *the equation (5.1) holds in $\mathcal{X}$ for all $t \in I$.*

Besides the fact that $\mathcal{L}$ is a bounded linear operator from $\mathcal{Z}$ to $\mathcal{X}$, we make two further assumptions regarding its spectrum.

Hypothesis 5.3 (Spectral decomposition)**.** *Consider the spectrum $\sigma(\mathcal{L})$ of the linear operator $\mathcal{L}$, and write:*

$$\sigma(\mathcal{L}) = \sigma_+(\mathcal{L}) \cup \sigma_0(\mathcal{L}) \cup \sigma_-(\mathcal{L}),$$

in which

$$\sigma_+(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) \mid \Re\lambda > 0\}, \quad \sigma_0(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) \mid \Re\lambda = 0\}, \quad \sigma_-(\mathcal{L}) = \{\lambda(\mathcal{L}) \in \sigma \mid \Re\lambda < 0\}.$$

We assume that:

(i) there exists a positive constant γ such that

$$\inf_{\lambda\in\sigma_+(\mathcal{L})} (\Re\lambda) > \gamma, \qquad \sup_{\lambda\in\sigma_-(\mathcal{L})} (\Re\lambda) < -\gamma;$$

(ii) the set $\sigma_0(\mathcal{L})$ consists of a finite number of eigenvalues with finite algebraic multiplicities.

As a consequence of Hypothesis 5.3(ii), we can define the spectral projection $\mathcal{P}_0 \in \mathscr{L}(\mathcal{X})$, corresponding to $\sigma_0(\mathcal{L})$, by the Dunford integral formula

$$\mathcal{P}_0 := \frac{1}{2\pi\mathbf{i}} \int_\gamma (\lambda\mathrm{Id} - \mathcal{L})^{-1} \mathrm{d}\lambda,$$

where Γ is a simple, oriented counterclockwise, closed curve surrounding $\sigma_0(\mathcal{L})$ and lying entirely in $\{\lambda \in \mathbb{C} \mid |\Re(\lambda)| < \gamma\}$. The range of $\mathcal{P}_0$ is finite-dimensional. We define the complementary projection

$$\mathcal{P}_h = \mathrm{Id} - \mathcal{P}_0 \in \mathscr{L}(\mathcal{X}).$$

Next, we consider the spectral subspaces associated with these projections

$$\mathcal{E}_0 := \mathrm{rg}\mathcal{P}_0 = \ker \mathcal{P}_h \subset \mathcal{Z}, \quad \mathcal{X}_h := \mathrm{rg}\mathcal{P}_h = \ker \mathcal{P}_0 \subset \mathcal{X},$$

which provides a decomposition of $\mathcal{X}$ into invariant subspaces

$$\mathcal{X} = \mathcal{E}_0 \oplus \mathcal{X}_h.$$

We also set $\mathcal{Z}_h = \mathcal{P}_h\mathcal{Z} \subset \mathcal{Z}$ and $\mathcal{Y}_h = \mathcal{P}_h\mathcal{Y} \subset \mathcal{Y}$ and we denote by $\mathcal{L}_0$ and $\mathcal{L}_h$ the restriction of $\mathcal{L}$ to $\mathcal{E}_0$ and $\mathcal{Z}_h$ respectively,

$$\mathcal{L}_0 \in \mathscr{L}(\mathcal{E}_0), \quad \mathcal{L}_h \in \mathscr{L}(\mathcal{Z}_h, \mathcal{X}_h).$$

As already noticed, the space $\mathcal{E}_0$ is finite-dimensional, then $\mathcal{L}_0$ acts in a finite-dimensional space, such that we can explicitly solve the linear ordinary differential equation

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \mathcal{L}_0 u_0 + f(t)$$

via the variation of constant formula

$$u_0(t) = e^{\mathcal{L}_0 t} u(0) + \int_0^t e^{\mathcal{L}_0(t-s)} f(s) \mathrm{d}s.$$

Our next hypothesis concerns the analogue of this linear problem for the infinite-dimensional operator $\mathcal{L}_h$.

Hypothesis 5.4 (Linear hyperbolic equation)**.** *For any $\eta \in [0,\gamma]$ and any $f \in \mathscr{C}_\eta(\mathbb{R},\mathcal{Y}_h)$ the linear problem*

$$\partial_t u_h = \mathcal{L}_h u_h + f(t),$$

has a unique solution $u_h = \mathcal{K}_h f \in \mathscr{C}_\eta(\mathbb{R},\mathcal{Z}_h)$. Furthermore, the linear map $\mathcal{K}_h$ belongs to $\mathscr{L}(\mathscr{C}_\eta(\mathbb{R},\mathcal{Y}_h),\mathscr{C}_\eta(\mathbb{R},\mathcal{Z}_h))$, and there exists a continuous map $C:[0,\gamma]\to\mathbb{R}_+$ such that

$$|||\mathcal{K}_h|||_{\mathscr{L}(\mathscr{C}_\eta(\mathbb{R},\mathcal{Y}_h),\mathscr{C}_\eta(\mathbb{R},\mathcal{Z}_h))} \leq C(\eta).$$

Note that for a given Banach space $\mathcal{X}$, we have used the following definition for $\mathscr{C}_\eta(\mathbb{R},\mathcal{X})$

$$\mathscr{C}_\eta(\mathbb{R},\mathcal{X}) := \left\{ u \in \mathscr{C}^0(\mathbb{R},\mathcal{X}) \mid \|u\|_{\mathscr{C}_\eta(\mathbb{R},\mathcal{X})} := \sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \|u(t)\|_{\mathcal{X}} \right) < +\infty \right\}.$$

Theorem 5.1 (Center manifold theorem)**.** *Assume that hypotheses 5.1, 5.3 and 5.4 hold. Then there exists a map $\Psi \in \mathscr{C}^k(\mathcal{E}_0,\mathcal{Z}_h)$, with*

$$\Psi(0) = 0, \quad D_u\Psi(0) = 0,$$

and a neighborhood $\mathscr{O}$ of 0 in $\mathcal{Z}$ such that the manifold:

$$\mathcal{M}_0 = \{u_0 + \Psi(u_0) \mid u_0 \in \mathcal{E}_0\} \subset \mathcal{Z}$$

has the following properties:

(i) $\mathcal{M}_0$ is locally invariant: if u is a solution of equation (5.1) satisfying $u(0) \in \mathcal{M}_0 \cap \mathscr{O}$ and $u(t) \in \mathscr{O}$ for all $t \in [0,T]$, then $u(t) \in \mathcal{M}_0$ for all $t \in [0,T]$.

(ii) $\mathcal{M}_0$ contains the set of bounded solutions of (5.1) staying in $\mathscr{O}$ for all $t \in \mathbb{R}$.

The manifold $\mathcal{M}_0$ is called a **local center manifold** of (5.1) and the map Ψ is referred to as the **reduction function**. Let u be a solution of (5.1) which belongs to $\mathcal{M}_0$ for $t \in I$, for some open interval $I \subset \mathbb{R}$. Then $u = u_0 + \Psi(u_0)$ and u_0 satisfies:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \mathcal{L}_0 u_0 + \mathcal{P}_0\mathcal{R}(u_0 + \Psi(u_0)). \tag{5.2}$$

The reduction function Ψ satisfies:

$$D_u\Psi(u_0)(\mathcal{L}_0 u_0 + \mathcal{P}_0\mathcal{R}(u_0+\Psi(u_0))) = \mathcal{L}_h\Psi(u_0) + \mathcal{P}_h\mathcal{R}(u_0+\Psi(u_0)), \quad \forall u_0 \in \mathcal{E}_0.$$

5.1 Proof of Theorem 5.1

Consider the differential equation (5.1), and assume that hypotheses 5.1, 5.3 and 5.4 hold. For any $u \in \mathcal{Z}$, we set

$$u = u_0 + u_h \in \mathcal{Z}, \quad u_0 = \mathcal{P}_0 u \in \mathcal{E}_0, \quad u_h = \mathcal{P}_h u \in \mathcal{Z}_h.$$

We can rewrite (5.1) as

$$\begin{cases} \dfrac{\mathrm{d}u_0}{\mathrm{d}t} = \mathcal{L}_0 u_0 + \mathcal{P}_0\mathcal{R}(u), \\ \partial_t u_h = \mathcal{L}_h u_h + \mathcal{P}_h\mathcal{R}(u). \end{cases} \tag{5.3}$$

We introduce a cut-off function $\chi : \mathcal{E}_0 \to \mathbb{R}$ of class $\mathscr{C}^\infty$ such that

$$\chi(u_0) = \begin{cases} 1 & \text{for} \quad \|u_0\|_{\mathcal{E}_0} \leq 1 \\ 0 & \text{for} \quad \|u_0\|_{\mathcal{E}_0} \geq 2 \end{cases}, \quad \chi(u_0) \in [0,1] \text{ for all } u_0 \in \mathcal{E}_0.$$

We set

$$\mathcal{R}^\epsilon(u) = \chi\left(\frac{u_0}{\epsilon}\right)\mathcal{R}(u) \text{ for all } \epsilon \in (0, \epsilon_0),$$

where $\epsilon_0 > 0$ is chosen such that

$$\left\{u = u_0 + u_h \mid \|u_0\|_{\mathcal{E}_0} \leq 2\epsilon_0, \quad \|u_h\|_{\mathcal{Z}_h} \leq \epsilon_0\right\} \subset \mathcal{V},$$

with $\mathcal{V}$ being the neighborhood of the origin given in Hypothesis 5.1. Then $\mathcal{R}^\epsilon$ is well defined in the closed set

$$\mathscr{O}^\epsilon = \mathcal{E}_0 \times \mathcal{B}_\epsilon(\mathcal{Z}_h), \quad \mathcal{B}_\epsilon(\mathcal{Z}_h) := \left\{u_h \in \mathcal{Z}_h \mid \|u_h\|_{\mathcal{Z}_h} \leq \epsilon\right\},$$

and satisfies

$$\mathcal{R}^\epsilon(u) = \mathcal{R}(u) \text{ for all } u \in \mathscr{O}^\epsilon, \quad \|u_0\|_{\mathcal{E}_0} \leq \epsilon.$$

We consider the modified system

$$\begin{cases} \dfrac{\mathrm{d}u_0}{\mathrm{d}t} = \mathcal{L}_0 u_0 + \mathcal{P}_0 \mathcal{R}^\epsilon(u), \\ \partial_t u_h = \mathcal{L}_h u_h + \mathcal{P}_h \mathcal{R}^\epsilon(u). \end{cases} \tag{5.4}$$

The nonlinear terms in the above system verify

$$\begin{aligned} \delta_0(\epsilon) &:= \sup_{u \in \mathscr{O}^\epsilon} \left(\|\mathcal{P}_0 \mathcal{R}^\epsilon(u)\|_{\mathcal{E}_0}, \|\mathcal{P}_h \mathcal{R}^\epsilon(u)\|_{\mathcal{Y}_h}\right) = \mathcal{O}(\epsilon^2), \\ \delta_1(\epsilon) &:= \sup_{u \in \mathscr{O}^\epsilon} \left(\|D_u \mathcal{P}_0 \mathcal{R}^\epsilon(u)\|_{\mathscr{L}(\mathcal{Z}, \mathcal{E}_0)}, \|D_u \mathcal{P}_h \mathcal{R}^\epsilon(u)\|_{\mathscr{L}(\mathcal{Z}, \mathcal{Y}_h)}\right) = \mathcal{O}(\epsilon). \end{aligned}$$

We replace system (5.4) by its integral reformulation

$$\begin{cases} u_0(t) = \mathcal{S}_{0,\epsilon}(u, t, u_0(0)) := e^{\mathcal{L}_0 t} u_0(0) + \int_0^t e^{\mathcal{L}_0(t-s)} \mathcal{P}_0 \mathcal{R}^\epsilon(u(s)) \mathrm{d}s, \\ u_h(t) = \mathcal{S}_{h,\epsilon}(u) := \mathcal{K}_h \mathcal{P}_h \mathcal{R}^\epsilon(u). \end{cases} \tag{5.5}$$

Here, $u_0(0) \in \mathcal{E}_0$ is arbitrary, and the exponential $e^{\mathcal{L}_0 t}$ is well defined as $\mathcal{E}_0$ is finite-dimensional. The second equation in (5.5) is obtained by using Hypothesis 5.4 with $f = \mathcal{P}_h \mathcal{R}^\epsilon(u) \in \mathscr{C}_0(\mathbb{R}, \mathcal{Y}_h)$. This integral system is thus equivalent to (5.4) whenever

$$u = (u_0, u_h) \in \mathcal{N}_{\eta,\epsilon} := \mathscr{C}_\eta(\mathbb{R}, \mathcal{E}_0) \times \mathscr{C}_0(\mathbb{R}, \mathcal{B}_\epsilon(\mathcal{Z}_h)),$$

with $0 < \eta \leq \gamma$ and $\epsilon \in (0, \epsilon_0)$. Notice that $\mathcal{N}_{\eta,\epsilon}$ is a closed subspace of $\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})$, so that it is complete when equipped with the norm of $\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})$.

Our aim is to show that (5.5) has a unique solution $u = (u_0, u_h) \in \mathcal{N}_{\eta,\epsilon}$ for any $u_0(0) \in \mathcal{E}_0$. For this, we use a fixed point argument to the map

$$\mathcal{S}_\epsilon(u, u_0(0)) := \left(\mathcal{S}_{0,\epsilon}(u, t, u_0(0)), \mathcal{S}_{h,\epsilon}(u)\right), \quad \mathcal{S}_\epsilon(\cdot, u_0(0)) : \mathcal{N}_{\eta,\epsilon} \to \mathcal{N}_{\eta,\epsilon}.$$

Step #1: The map $\mathcal{S}_\epsilon(\cdot, u_0(0))$ is well-defined. First, our assumption on the spectrum of $\sigma(\mathcal{L})$ implies that for any $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$\left|\left|\left| e^{\mathcal{L}_0 t} \right|\right|\right|_{\mathscr{L}(\mathcal{E}_0)} \leq c_\delta e^{\delta|t|}, \quad \forall t \in \mathbb{R}. \tag{5.6}$$

Using this inequality with $\delta = \eta$, we get

$$\sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \left\| e^{\mathcal{L}_0 t} u_0(0) \right\|_{\mathcal{E}_0} \right) \leq c_\eta \left\| u_0(0) \right\|_{\mathcal{E}_0},$$

which shows that the first term in $\mathcal{S}_{0,\epsilon}(u, \cdot, u_0(0))$ belongs to $\mathscr{C}_\eta(\mathbb{R}, \mathcal{E}_0)$ for any $\eta > 0$. Next, for any $u \in \mathcal{N}_{\eta,\epsilon}$, we have the estimates

$$\|\mathcal{P}_0 \mathcal{R}^\epsilon(u)\|_{\mathcal{E}_0} \leq \delta_0(\epsilon), \quad \|\mathcal{P}_h \mathcal{R}^\epsilon(u)\|_{\mathcal{Y}_h} \leq \delta_0(\epsilon),$$

which gives that

$$\sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \left\| \int_0^t e^{\mathcal{L}_0(t-s)} \mathcal{P}_0 \mathcal{R}^\epsilon(u(s)) \mathrm{d}s \right\|_{\mathcal{E}_0} \right) \leq c_{\eta/2} \delta_0(\epsilon) \sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \int_0^t e^{|t-s|\frac{\eta}{2}} \mathrm{d}s \right) \leq \frac{2c_{\eta/2}\delta_0(\epsilon)}{\eta},$$

and

$$\|\mathcal{K}_h \mathcal{P}_h \mathcal{R}^\epsilon(u)\|_{\mathscr{C}_0(\mathbb{R}, \mathcal{B}_\epsilon(\mathcal{Z}_h))} \leq C(0)\delta_0(\epsilon).$$

This shows that $\mathcal{S}_\epsilon(u, u_0(0)) \in \mathcal{N}_{\eta,\epsilon}$ provided that $C(0)\delta_0(\epsilon) \leq \epsilon$, which holds for ϵ sufficiently small as $\delta_0(\epsilon) = \mathcal{O}(\epsilon^2)$.

Step #2: The map $\mathcal{S}_\epsilon(\cdot, u_0(0))$ is a contraction. First, we find that

$$\begin{aligned}
\|\mathcal{R}^\epsilon(u_1) - \mathcal{R}^\epsilon(u_2)\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Y})} &= \sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \left\| \mathcal{R}^\epsilon(u_1(t)) - \mathcal{R}^\epsilon(u_2(t)) \right\|_{\mathcal{Y}} \right) \\
&\leq \delta_1(\epsilon) \sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \left\| u_1(t) - u_2(t) \right\|_{\mathcal{Z}} \right) \\
&\leq \delta_1(\epsilon) \left\| u_1 - u_2 \right\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})}
\end{aligned}$$

for any $u_1, u_2 \in \mathcal{N}_{\eta,\epsilon}$. Now, using estimate (5.6) with $\delta = \eta/2$ we obtain

$$\begin{aligned}
\|\mathcal{S}_{0,\epsilon}(u_1, \cdot, u_0(0)) - \mathcal{S}_{0,\epsilon}(u_2, \cdot, u_0(0))\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{E}_0)} &\leq c_{\eta/2}\delta_1(\epsilon) \sup_{t\in\mathbb{R}} \left(e^{-\eta|t|} \left| \int_0^t e^{\eta|s| + |t-s|\frac{\eta}{2}} \mathrm{d}s \right| \right) \|u_1 - u_2\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})} \\
&\leq \frac{2c_{\eta/2}\delta_0(\epsilon)}{\eta} \|u_1 - u_2\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})}.
\end{aligned}$$

Furthermore, we also have from Hypothesis 5.4 that

$$\|\mathcal{S}_{h,\epsilon}(u_1) - \mathcal{S}_{h,\epsilon}(u_2)\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z}_h)} \leq C(\eta)\delta_1(\epsilon) \|u_1 - u_2\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})}.$$

Since $\delta_1(\epsilon) = \mathcal{O}(\epsilon)$ for any $\eta \in (0, \gamma]$, we can choose ϵ small enough such that

$$\|\mathcal{S}_\epsilon(u_1, u_0(0)) - \mathcal{S}_\epsilon(u_2, u_0(0))\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})} \leq \frac{1}{2} \|u_1 - u_2\|_{\mathscr{C}_\eta(\mathbb{R}, \mathcal{Z})}.$$

Step #3: Fixed point theorem and first consequences. We can apply a fixed point theorem to $\mathcal{S}_\epsilon(\cdot, u_0(0))$ in $\mathcal{N}_{\eta,\epsilon}$ to get the existence of a unique solution of (5.5)

$$u := \Phi(u_0(0)) \in \mathcal{N}_{\eta,\epsilon},$$

for any $u_0(0) \in \mathcal{E}_0$, for any $\eta \in (0, \gamma]$, and ϵ sufficiently small. Clearly this is also a solution of (5.4). Next, notice that the map

$$u_0(0) \mapsto \mathcal{S}_{0,\epsilon}(u, \cdot, u_0(0))$$

is Lipschitz from $\mathcal{E}_0$ into $\mathscr{C}_\eta(\mathbb{R}, \mathcal{E}_0)$, so that the map

$$u_0(0) \mapsto \mathcal{S}_\epsilon(u, u_0(0))$$

is also Lipschitz. As a consequence, Φ is a Lipschitz map. In addition, the uniqueness of the fixed point implies that

$$\Phi(0) = 0.$$

Step #4: Construction of Ψ and its properties. We define now the map $\Psi : \mathcal{E}_0 \to \mathcal{Z}_h$ through

$$(u_0(0), \Psi(u_0(0))) := \Phi(u_0(0))(0), \quad \forall u_0(0) \in \mathcal{E}_0,$$

that is by taking the component in $\mathcal{Z}_h$ of the fixed point $\Phi(u_0(0))$ at $t = 0$. Since Φ is a Lipschitz map, so is Ψ. And since $\Phi(0) = 0$, we also have that

$$\Psi(0) = 0.$$

We now prove that Ψ has the properties (i) and (ii) of the theorem. First, we show that the manifold

$$\mathcal{M}_{\eta,\epsilon} = \{(u_0, \Psi(u_0)) \mid u_0 \in \mathcal{E}_0\}$$

is a global invariant manifold for the flow defined by (5.4). We define the shift operator ϕ_s through

$$(\phi_s \cdot u)(t) = u(t + s) \text{ for all } t, s \in \mathbb{R}.$$

Since system (5.4) is autonomous, it is equivariant under the action of ϕ_s for any $s \in \mathbb{R}$, so that if u is a solution of (5.4), then $\phi_s \cdot u$ is also a solution of (5.4). Moreover, $\phi_s \cdot u \in \mathcal{N}_{\eta,\epsilon}$ when $u \in \mathcal{N}_{\eta,\epsilon}$. Consider a solution of (5.4) with $u(0) = (u_0(0), \Psi(u_0(0)))$ for some $u_0(0) \in \mathcal{E}_0$. Then $u = \Phi(u_0(0)) \in \mathcal{N}_{\eta,\epsilon}$, and since $\phi_s \cdot u \in \mathcal{N}_{\eta,\epsilon}$ is also a solution, from the uniqueness of the fixed point we conclude that

$$\phi_s \cdot u = \Phi(u_0(s)) \quad \forall s \in \mathbb{R}.$$

Consequently,

$$u(s) = (u_0(s), \Psi(u_0(s))) \quad \forall s \in \mathbb{R},$$

which shows that $\mathcal{M}_{\eta,\epsilon}$ is globally invariant under the flow of (5.4). Since equation (5.3) coincides with (5.4) in $\mathscr{O}^\epsilon$, this proves part (i) of the theorem with $\mathcal{M}_0 = \mathcal{M}_{\eta,\epsilon}$ and $\mathscr{O} = \mathscr{O}^\epsilon$.

Consider now a solution of (5.3) which belongs to $\mathscr{O} = \mathscr{O}^\epsilon$ for all $t \in \mathbb{R}$. Then $u \in \mathcal{N}_{\eta,\epsilon}$ and it is also a solution of (5.4). As a consequence, $u = \Phi(u_0(0))$, so that $u(0) \in \mathcal{M}_{\eta,\epsilon} = \mathcal{M}_0$, which proves part (ii) of the theorem.

Step #5: Regularity of Ψ. So far we have proved that Ψ is a Lipschitz map using only the fact that $\mathcal{R}$ is of class $\mathscr{C}^1$. It remains to show that Ψ is $\mathscr{C}^k$ when $\mathcal{R}$ is of class $\mathscr{C}^k$. The proof of such a result is beyond the scope of these lectures, and once again we refer to [8, 9]. This concludes the proof of the theorem.

5.2 Checking Hypothesis 5.4

From the proof of Theorem 5.1, we see that Hypothesis 5.4 was precisely what was needed in order to solve the hyperbolic part of the equation. In practice, verifying Hypothesis 5.4 can be very challenging, this is why we present sufficient conditions on the resolvent of $\mathcal{L}$ for such an Hypothesis to be true. Throughout this section, we assume that Hypotheses 5.1 and 5.3 are satisfied. Furthermore, we assume that $\mathcal{Y} \subsetneq \mathcal{X}$, that is we are in the semilinear case.

Hypothesis 5.5 (Resolvent estimates)**.** *Assume that there exist positive constants $\omega_0 > 0$, $c > 0$, and $\alpha \in [0,1)$ such that for all $\omega \in \mathbb{R}$, with $|\omega| > \omega_0$, we have that $\mathbf{i}\omega$ belongs to the resolvent set of $\mathcal{L}$, and*

$$\left\|(\mathbf{i}\omega - \mathcal{L})^{-1}\right\|_{\mathscr{L}(\mathcal{X})} \leq \frac{c}{|\omega|}, \tag{5.7}$$

$$\left\|(\mathbf{i}\omega - \mathcal{L})^{-1}\right\|_{\mathscr{L}(\mathcal{Y},\mathcal{Z})} \leq \frac{c}{|\omega|^{1-\alpha}}. \tag{5.8}$$

Theorem 5.2 (Center manifold in the semilinear case)**.** *Assume that Hypotheses 5.1, 5.3 and 5.5 hold. Then*

- *Hypothesis 5.4 is satisfied;*
- *the result in Theorem 5.1 holds.*

Remark 5.6. *When $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert spaces the second estimate (5.8) is not necessary.*

Remark 5.7. *When $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert spaces, and when $\mathcal{L}$ is a sectorial operator then Hypothesis 5.5 is satisfied.*

The proof of the fact that Hypothesis 5.5 implies Hypothesis 5.4 can be found in [3] Appendix B2.

5.3 Extensions

5.3.1 Parameter-dependent center manifold

In the same frame as above, we consider parameter-dependent evolution equations of the form

$$\partial_t u = \mathcal{L}u + \mathcal{R}(u,\mu), \tag{5.9}$$

where the linear operator $\mathcal{L}$ is defined as previously and the nonlinear term $\mathcal{R}$ now depends on the parameter $\mu \in \mathbb{R}^m$ which we assume to be small. We modify Hypothesis 5.1 accordingly.

Hypothesis 5.8 (Structure & Regularity)**.** *We assume that $\mathcal{L}$ and $\mathcal{R}$ in (5.9) have the following properties:*

(i) $\mathcal{L} \in \mathscr{L}(\mathcal{Z},\mathcal{X})$;

(ii) for some $k \geq 2$, there exists a neighborhood $\mathcal{V}_u \subset \mathcal{Z}$ of 0 and $\mathcal{V}_\mu \subset \mathbb{R}^m$ of 0 such that $\mathcal{R} \in \mathscr{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathcal{Y})$ and

$$\mathcal{R}(0,0) = 0, \quad D_u\mathcal{R}(0,0) = 0.$$

Theorem 5.3 (Parameter-dependent center manifold theorem)**.** *Assume that hypotheses 5.8, 5.3 and 5.4 hold. Then there exists a map* $\Psi \in \mathscr{C}^k(\mathcal{E}_0 \times \mathbb{R}^m, \mathcal{Z}_h)$*, with*

$$\Psi(0,0) = 0, \quad D_u\Psi(0,0) = 0,$$

and a neighborhood $\mathscr{O}_u$ *of* 0 *in* $\mathcal{Z}$ *and* $\mathscr{O}_\mu$ *of* 0 *in* $\mathbb{R}^m$ *such that for* $\mu \in \mathscr{O}_\mu$ *the manifold:*

$$\mathcal{M}_0(\mu) = \{u_0 + \Psi(u_0, \mu) \mid u_0 \in \mathcal{E}_0\} \subset \mathcal{Z}$$

has the following properties:

(i) $\mathcal{M}_0(\mu)$ *is locally invariant: if* u *is a solution of equation* (5.9) *satisfying* $u(0) \in \mathcal{M}_0(\mu) \cap \mathscr{O}_u$ *and* $u(t) \in \mathscr{O}_u$ *for all* $t \in [0,T]$*, then* $u(t) \in \mathcal{M}_0(\mu)$ *for all* $t \in [0,T]$*.*

(ii) $\mathcal{M}_0(\mu)$ *contains the set of bounded solutions of* (5.9) *staying in* $\mathscr{O}_u$ *for all* $t \in \mathbb{R}$*.*

5.3.2 Symmetries and reversibility

It is not difficult to extend Theorems 2.10 and 2.11 to the infinite-dimensional setting.

Hypothesis 5.9 (Equivariant equation)**.** *We assume that there exists a linear operator* $\mathbf{T} \in \mathscr{L}(\mathcal{X}) \cap \mathscr{L}(\mathcal{Z})$*, which commutes with the vector field equation* (5.1)

$$\mathbf{T}\mathcal{L}u = \mathcal{L}\mathbf{T}u, \quad \mathbf{T}\mathcal{R}(u) = \mathcal{R}(\mathbf{T}u).$$

We further assume that the restriction $\mathbf{T}_0$ *of* $\mathbf{T}$ *to* $\mathcal{E}_0$ *is an isometry.*

Theorem 5.4. *Under the assumptions of Theorem 5.1 and the above equivariance hypothesis 5.9, then one can find* Ψ *in Theorem 5.1 which commutes with* $\mathbf{T}$*, that is*

$$\mathbf{T}\Psi(u_0) = \Psi(\mathbf{T}_0 u_0) \text{ for all } u_0 \in \mathcal{E}_0,$$

and such that the vector field in the reduced equation (5.2) *commutes with* $\mathbf{T}_0$*.*

Hypothesis 5.10 (Reversible equation)**.** *We assume that there exists a linear symmetry* $\mathbf{S} \in \mathscr{L}(\mathcal{X}) \cap \mathscr{L}(\mathcal{Z})$*, with*

$$\mathbf{S}^2 = \mathrm{Id}, \quad \mathbf{S} \neq \mathrm{Id},$$

and which anticommutes with the vector field equation (5.1)

$$\mathbf{T}\mathcal{L}u = -\mathcal{L}\mathbf{T}u, \quad \mathbf{T}\mathcal{R}(u) = -\mathcal{R}(\mathbf{T}u).$$

Theorem 5.5. *Under the assumptions of Theorem 5.1 and the above reversible hypothesis 5.10, then one can find* Ψ *in Theorem 5.1 which commutes with* $\mathbf{S}$*, that is*

$$\mathbf{S}\Psi(u_0) = \Psi(\mathbf{S}_0 u_0) \text{ for all } u_0 \in \mathcal{E}_0,$$

where $\mathbf{S}_0$ *is the restriction of* $\mathbf{S}$ *to* $\mathcal{E}_0$*, and such that the vector field in the reduced equation* (5.2) *is reversible, that is it anticommutes with* $\mathbf{S}_0$*.*

5.4 Application to the Swift-Hohenberg equation

We consider the cubic Swift-Hohenberg equation (SHE) given in the introduction

$$\partial_t u = -\left(1+\partial_x^2\right)^2 u + \mu u - u^3, \quad t>0, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ is a parameter. Notice that $u=0$ is a solution of (SHE) and that the equation is invariant under spatial translation, reflections $x \mapsto -x$ and $u \mapsto -u$.

Linear stability analysis of $u=0$. We first analyze the linear stability of the trivial solution $u=0$ by looking for solutions of the form

$$u(t,x) = e^{\mathbf{i}kx+\lambda t},$$

where k is a real number and $\lambda \in \mathbb{C}$, of the linearized (SHE)

$$\partial_t u = -\left(1+\partial_x^2\right)^2 u + \mu u.$$

We obtain the linear dispersion

$$\lambda(k,\mu) = -(1-k^2)^2 + \mu.$$

The trivial solution is linearly stable (resp. unstable) with respect to the mode $e^{\mathbf{i}kx}$ if $\Re(\lambda(k,\mu)) < 0$ (resp. $\Re(\lambda(k,\mu)) > 0$). The dispersion relation shows that $\lambda(k,\mu)$ is real for all k and μ. The modes $e^{\mathbf{i}kx}$ for which $\mu = (1-k^2)^2$ are the critical modes at the threshold of instability. Upon increasing μ, the first critical modes, $k \pm 1$ occur at $\mu = 0$. These modes correspond to 2π-periodic solutions $e^{\pm\mathbf{i}x}$ of the linearized equation, at the threshold of linear instability. We therefore expect that 2π-periodic solutions to play a particular role in the dynamics of the equation, and we will restrict ourselves to this type of solutions in our analysis.

Center manifold reduction. We write (SHE) with the operator $\mathcal{L}_\mu$ depending upon the parameter μ by setting

$$\mathcal{L}_\mu := -\left(1+\partial_x^2\right)^2 + \mu, \quad \mathcal{R}(u) = -u^3.$$

We choose the spaces

$$\mathcal{X} = L^2_{per}(0,2\pi), \quad \mathcal{Y} = \mathcal{Z} = H^4_{per}(0,2\pi).$$

$\mathcal{L}_\mu$ is a closed operator on $\mathcal{X}$ with dense domain $\mathcal{Z}$. We also note that

$$\|\mathcal{R}(u)\|_{\mathcal{Z}} \leq C \|u\|^3_{\mathcal{Z}},$$

and that $\mathcal{R} \in \mathscr{C}^k(\mathcal{Z})$ for all $k \geq 2$.

Next, we compute the spectrum of $\mathcal{L}$. We remark that the domain $\mathcal{Z}$ is compactly embedded in $\mathcal{X}$ such that $\mathcal{L}$ has a compact resolvent. Consequently the spectrum of $\mathcal{L}$ consists of only isolated eigenvalues with finite multiplicities. Since we work in spaces of 2π-periodic functions, we can use Fourier series to solve the eigenvalue problem and we easily conclude that

$$\sigma(\mathcal{L}_\mu) = \left\{\lambda_n := -(1-n^2)^2 + \mu \mid n \in \mathbb{N}\right\}.$$

All these eigenvalues are real and there is a sequence of values of μ, given by $\mu_n = (1-n^2)^2$, for which $\lambda = 0$ is an eigenvalue. The smallest value is $\mu_1 = 0$, and it is the one where $u = 0$ looses its stability when μ is increased. As a consequence, we are going to apply the parameter-dependent center manifold theorem for values of μ close to 0.

First, we rewrite (SHE) as

$$\partial_t u = \mathcal{L}_0 u + \underbrace{\mathcal{L}_\mu - \mathcal{L}_0 + \mathcal{R}(u)}_{:=\mathcal{R}(u,\mu)=\mu u - u^3}.$$

From the above discussion, we see that $\mathcal{L}_0$ and $\mathcal{R}(u,\mu)$ satisfy Hypothesis 5.8. Furthermore, Hypothesis 5.3 holds with

$$\sigma_0(\mathcal{L}_0) = \{0\}, \quad \sigma_+(\mathcal{L}_0) = \emptyset, \text{ and } \sigma_-(\mathcal{L}_0) = \left\{-(1-n^2) \mid n \in \mathbb{N}^*\right\}.$$

Furthermore, $\lambda = 0$ is an eigenvalue with geometric multiplicity two with corresponding eigenvectors $e^{\pm \mathbf{i}x}$. Let us verify that the algebraic multiplicity is also two. Let us assume that there exists $u \in \mathcal{Z}$ such that

$$-(1+\partial_x^2)^2 u = e^{\mathbf{i}x}.$$

Multiplying the equation by $e^{-\mathbf{i}x}$ and integrating between 0 and 2π, one gets on the one side

$$\begin{aligned} -\int_0^{2\pi} (1+\partial_x^2)^2 u(x) e^{-\mathbf{i}x} \mathrm{d}x &= -\int_0^{2\pi} u(x) e^{-\mathbf{i}x} \mathrm{d}x - 2\int_0^{2\pi} \partial_x^2 u(x) e^{-\mathbf{i}x} \mathrm{d}x + \int_0^{2\pi} \partial_x^4 u(x) e^{-\mathbf{i}x} \mathrm{d}x \\ &= -\int_0^{2\pi} u(x) e^{-\mathbf{i}x} \mathrm{d}x - 2(-\mathbf{i})^2 \int_0^{2\pi} u(x) e^{-\mathbf{i}x} \mathrm{d}x - (-\mathbf{i})^4 \int_0^{2\pi} u(x) e^{-\mathbf{i}x} \mathrm{d}x = 0, \end{aligned}$$

while on the right-hand side we have

$$\int_0^{2\pi} e^{\mathbf{i}x} e^{-\mathbf{i}x} \mathrm{d}x = 2\pi.$$

Finally, it only remains to check Hypothesis 5.4. As $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert spaces, we only need to check estimate (5.7). For that, let $\omega > 0$, then $\mathbf{i}\omega \in \rho(\mathcal{L}_0)$ and thus we have that for any $f \in \mathcal{X}$ there exists a unique $u \in \mathcal{Z}$ solution of

$$\mathbf{i}\omega u - \mathcal{L}_0 u = f.$$

Multiplying both side of the equation by $\bar{u}$ and integrating over 0 to 2π, we get

$$(\mathbf{i}\omega + 1)\|u\|_{\mathcal{X}}^2 - 2\|\partial_x u\|_{\mathcal{X}}^2 + \left\|\partial_x^2 u\right\|_{\mathcal{X}}^2 = \int_0^{2\pi} f(x)\bar{u}(x)\mathrm{d}x.$$

Taking the imaginary part of both side, we get that

$$\omega \|u\|_{\mathcal{X}}^2 = \Im\left(\int_0^{2\pi} f(x)\bar{u}(x)\mathrm{d}x\right),$$

from which we deduce that

$$\|u\|_{\mathcal{X}} \leq \frac{1}{|\omega|}\|f\|_{\mathcal{X}},$$

which is precisely estimate (5.7). As a consequence, we get the existence of a two-dimensional parameter-dependent center manifold $\mathcal{M}_0(\mu)$ for all μ small enough. Here, the center eigenspace is given by

$$\mathcal{E}_0 = \operatorname{span}\left\{e^{\mathbf{i}x}, e^{-\mathbf{i}x}\right\}.$$

Any solution on $\mathcal{M}_0(\mu)$ can be written

$$u(t,x) = A(t)e^{\mathbf{i}x} + \overline{A(t)}e^{-\mathbf{i}x} + \underbrace{\Psi\left(A(t), \overline{A(t)}, \mu\right)}_{\in \mathcal{Z}_h},$$

with $A : \mathbb{R} \to \mathbb{C}$. The reduced equation reads

$$\frac{\mathrm{d}A}{\mathrm{d}t} = g(A, \overline{A}, \mu),$$

together with the complex conjugate equation for $\overline{A}$. One can check (let as an exercise) that the action translation and the two reflections imply that the map $g \in \mathscr{C}^k$ can be written

$$g(A, \overline{A}, \mu) = Ah(|A|^2, \mu),$$

for some $\mathscr{C}^{k-1}$ map h. Considering the Taylor expansion of h, we get that

$$g(A, \overline{A}, \mu) = a\mu A + bA|A|^2 + \mathcal{O}\left(|A|(|\mu|^2 + |A|^4)\right),$$

for some complex coefficients a and b to be determined. It is a straightforward computation to see that necessarily we have

$$a = 1.$$

To compute b, we set $\mu = 0$, and we Taylor expand the map $\Psi(A, \overline{A}, 0)$ as

$$\Psi(A, \overline{A}, 0) = \sum_{p,q} \Psi_{pq} A^p \overline{A}^q, \quad \Psi_{pq} \in \mathcal{Z}_h.$$

Here we further have that

$$\Psi qp = \overline{\Psi_{pq}}, \text{ and } \Psi_{00} = \Psi_{01} = \Psi_{10} = 0.$$

Furthermore, as a consequence of the equivariance with respect to $u \mapsto -u$ of (SHE) we have that

$$\Psi(-A, -\overline{A}, 0) = -\Psi(A, \overline{A}, 0),$$

for all A and thus $\Psi_{pq} = 0$ when $p + q$ is even. Summarizing, we find the leading order expansion

$$\Psi(A, \overline{A}, 0) = \Psi_{30}A^3 + \Psi_{03}\overline{A}^3 + \Psi_{12}A\overline{A}^2 + \Psi_{21}A^2\overline{A} + \mathcal{O}(|A|^5).$$

Substituting the ansatz $u = A\zeta_0 + \overline{A\zeta_0} + \Psi(A, \overline{A}, 0)$ in (SHE) at $\mu = 0$, with $\zeta_0(x) = e^{\mathbf{i}x}$, one gets that

$$\begin{aligned}\mathcal{L}_0\Psi(A, \overline{A}, 0) + \mathcal{R}(A\zeta_0 + \overline{A\zeta_0} + \Psi(A, \overline{A}, 0), 0) =& (\mathcal{L}_0\Psi_{30} - e^{3\mathbf{i}x})A^3 + (\mathcal{L}_0\Psi_{03} - e^{-3\mathbf{i}x})\overline{A}^3 \\ &+ (\mathcal{L}_0\Psi_{12} - 3e^{-\mathbf{i}x})A\overline{A}^2 + (\mathcal{L}_0\Psi_{21} - 3e^{\mathbf{i}x})A^2\overline{A} + \mathcal{O}(|A|^5).\end{aligned}$$

Identifying terms of order A^3 and $A^2\overline{A}$, we get that

$$\begin{aligned}\mathcal{L}_0\Psi_{30} - e^{3\mathbf{i}x} &= 0, \\ \mathcal{L}_0\Psi_{21} - 3e^{\mathbf{i}x} &= be^{\mathbf{i}x}.\end{aligned}$$

Taking the inner product of the second equation with $e^{-\mathbf{i}x}$, we directly get that necessarily

$$b = -3,$$

as $\mathcal{L}_0^* e^{-\mathbf{i}x} = 0$. As a consequence, we conclude that the flow on the center-manifold is described by the set of ordinary differential equations

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mu A - 3A|A|^2 + \mathcal{O}\left(|A|(|\mu|^2 + |A|^4)\right),$$

and the complex conjugate equation. This tells us that there exists a family $(A_\alpha)_{\alpha\in\mathbb{R}\setminus 2\pi\mathbb{Z}}$ of stable equilibria emerging from 0 as μ crosses 0 and given by

$$A_\alpha = \sqrt{\frac{\mu}{3}} e^{\mathbf{i}\alpha} + \mathcal{O}\left(|\mu|^{3/2}\right), \quad \mu > 0,$$

and the corresponding family of 2π-periodic solutions of (SHE)

$$u_\alpha(x) = 2\sqrt{\frac{\mu}{3}} \cos(x+\alpha) + \mathcal{O}\left(|\mu|^{3/2}\right).$$

These steady 2π-periodic solutions of the (SHE) are called *roll* solutions. Actually, such solutions exist for a range of period close to 2π, for any sufficiently small μ.

Remark 5.11. *As $b = -3$, we have that $\mathcal{L}_0\Psi_{21} = 0$ and thus $\Psi_{21} \in \mathcal{E}_0 \cap \mathcal{Z}_h$, i.e. $\Psi_{21} = 0$. On the other hand $\mathcal{L}_0\Psi_{30} = e^{3\mathbf{i}x}$ gives that*

$$\Psi_{30}(x) = -\frac{1}{64} e^{3\mathbf{i}x} + \zeta(x), \quad \zeta \in \mathcal{E}_0.$$

As $\Psi_{30} \in \mathcal{Z}_h$, we have that $\langle \Psi_{30}, \zeta_0 \rangle = \langle \Psi_{30}, \overline{\zeta_0} \rangle = 0$ which implies that $\zeta = 0$.

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