Selected topics in statistics
Spatial Statistics
End-term exam
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## Specifications

- No documents, no calculators.
- You can use the results of the lectures without reproving them.
- You must prove your answers
- Clarity and readability will be taken into account in the final mark
- Tentative notation: 15/15/10.

## Reminder of the Gaussian conditioning theorem

We write $\mathcal{N}(m, S)$ as a shorthand for the multidimensional Gaussian distribution with mean vector $m$ and covariance matrix $S$.

Let

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right),$$

where $Y_1$ and $m_1$ are of size $n_1 \times 1$, $Y_2$ and $m_2$ are of size $n_2 \times 1$, $\Sigma_{11}$ is of size $n_1 \times n_1$, $\Sigma_{12}$ is of size $n_1 \times n_2$, $\Sigma_{21} = \Sigma_{12}^t$ and $\Sigma_{22}$ is of size $n_2 \times n_2$. Then, conditionally to $Y_1 = y_1$, we have

$$\mathcal{L}(Y_2 | Y_1 = y_1) = \mathcal{N}\left( m_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - m_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \right).$$

## Exercise 1

a) Is the following function $K : \mathbb{R}^2 \to \mathbb{R}$ defined by $K(x, y) = e^{-|e^x - e^y|}$ SNND?

b) Let us define, for $(a, b) \in \mathbb{R}^2$, the function $K_{a,b} : \mathbb{R}^2 \to \mathbb{R}$ defined by $K_{a,b}(x, y) = ax^2y^2 + bxy$. Find the set of the $(a, b) \in \mathbb{R}^2$ so that $K_{a,b}$ is SNND.

## Exercise 2

a) Let $Y$ be a Gaussian process on $\mathbb{R}$ with mean function 0 and covariance function $K(x, y) = \mathbf{1}_{|x-y|\leq 1}(1 - |x - y|)$. Then the vector

$$\begin{pmatrix} Y(-1/2) \\ Y(1/2) \\ Y(0) \end{pmatrix}$$

is a Gaussian vector. Find its mean vector and covariance matrix.

b) Find the distribution of $Y(0)$ conditionally to $(Y(-1/2) = -1, Y(1/2) = 1)$.

## Exercise 3

a) Consider a Gaussian process $Y$ on $\mathbb{R}$ with mean function known to be zero. Consider the parametric covariance function family $\{K_{\sigma^2}(x,y); \sigma^2 > 0\}$ with $K_{\sigma^2}(x,y) = \sigma^2 \mathbf{1}_{|x-y|\leq 1}(1-|x-y|)xy$. Consider $x_1, ..., x_n$ so that $x_i = i$. Let $y_1, ..., y_n \in \mathbb{R}$. Assume that one observes $Y(x_1) = y_1, ..., Y(x_n) = y_n$. Prove that the Maximum Likelihood estimator of $\sigma^2$ is

$$\hat{\sigma}^2_{ML} = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i^2}{i^2}.$$