Selected topics in statistics
Spatial Statistics
Homework 5

Lecturer: François Bachoc, PhD

Suggested programming language: R. Implement the following functions for Gaussian process modeling, and write a report answering the questions that are asked. The code will not be evaluated. The evaluation will be made on the written report answering the questions.

We model the deterministic function $f : [0,1] \to \mathbf{R}$, with $f(x) = \cos(4x) + x + x^2 - \exp(x)$, as the realization of a Gaussian process Y, with zero mean function, and Matérn $\frac{3}{2}$ covariance function, with variance parameter σ^2 and correlation-length parameter ℓ.

1)

Create a R function with

- Inputs
  - A vector x of size n_x
  - A vector y of size n_y
  - $\sigma^2 > 0$
  - $\ell > 0$
- Output
  - The $n_x \times n_y$ covariance matrix between x and y, with the Matérn $\frac{3}{2}$ covariance function, with variance parameter σ^2 and correlation-length parameter ℓ.

Suggested R commands: *outer(.,.,"-")*

2)

Write the results of your function for $x = y = (0.2, 0.4, 0.6, 0.8)$, $\sigma^2 = 0.3$, $\ell = 0.7$. Write the 4 eigen-values of this matrix. Give a few comments on the matrix and the eigenvalues (max 5 lines)

3)

Create a R function with

- Inputs
  - A vector x of size n_x
- Output
  - A vector v of size n_x with $v_i = f(x_i)$.

4)

Evaluate the function f on a vector of 1000 points that are equally spaced on $[0,1]$. Plot the graph of the function on $[0,1]$. Suggested R commands: *plot(.,.,type="l")*

5)

Evaluate f on the vector $(0, 0.3, 0.6, 1)$. Add to the figure these four value points. Suggested R commands: *points(.,.,col="red")*

6)

Create a R function with

- Inputs
  - A vector x of size n_x
  - A vector x_{obs} of size n_{obs}
  - A vector y_{obs} of size n_{obs}
  - $\sigma^2 > 0$
  - $\ell > 0$
- Output
  - The conditional mean vector of $(Y(x_1), ..., Y(x_{n_x}))$, conditionally to $(Y(x_{obs,1}) = f(x_1), ..., Y(x_{obs,n_{obs}}) = f(x_{obs,n_{obs}}))$, where Y is a Gaussian process with mean function 0 and Matérn $\frac{3}{2}$ covariance function, with variance parameter σ^2 and correlation-length parameter ℓ.

Suggested R commands: *matrix(nrow=.,ncol=1,data=.)*, *t(.)*, *solve(.)*, *%*%*.

7)

Write the result of the function of 6) at $x = 0.4$, $x_{obs} = (0, 0.3, 0.6, 1)$, $y_{obs} = (f(0), f(0.3), f(0.6), f(1))$, $\sigma^2 = 0.3$, $\ell = 0.7$. Write $f(0.4)$

8)

Calculate the result of the function of 6), with x a vector of 1000 equally-spaced points on $[0, 1]$, $x_{obs} = (0, 0.3, 0.6, 1)$, $y_{obs} = (f(0), f(0.3), f(0.6), f(1))$, $\sigma^2 = 0.3$, $\ell = 0.7$. Add to the figure this result. Suggested R commands: *lines(.,.,col="blue")*

9)

We call $\hat{v}$ the result of 8). We call v the result of 3). Compute

$$\frac{\sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(v_i - \hat{v}_i)^2}}{\sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(v_i - \overline{v})^2}},$$

where $\overline{v} = \frac{1}{1000}\sum_{i=1}^{1000} v_i$.

10)

Create a R function with

- Inputs
  - A vector x of size n_x
  - A vector x_{obs} of size n_{obs}
  - A vector y_{obs} of size n_{obs}
  - $\sigma^2 > 0$
  - $\ell > 0$
- Output
  - The diagonal vector of the covariance matrix of $(Y(x_1), ..., Y(x_{n_x}))$, conditionally to $(Y(x_{obs,1}) = f(x_1), ..., Y(x_{obs,n_{obs}}) = f(x_{obs,n_{obs}}))$, where Y is a Gaussian process with mean function 0 and Matérn $\frac{3}{2}$ covariance function, with variance parameter σ^2 and correlation-length parameter ℓ.

Suggested R commands: *matrix(nrow=.,ncol=1,data=.)*, *t(.)*, *solve(.)*, *%*%*, *diag*.

11)

Write the result of the function of 10), at $x = 0.4$, $x_{obs} = (0, 0.3, 0.6, 1)$, $y_{obs} = (f(0), f(0.3), f(0.6), f(1))$, $\sigma^2 = 0.3$, $\ell = 0.7$.

12)

Let x be a vector of 1000 equally-spaced points on $[0, 1]$. Let Y be a Gaussian process with mean function 0 and Matérn $\frac{3}{2}$ covariance function, with variance parameter $\sigma^2 = 0.3$ and correlation-length parameter $\ell = 0.7$. Compute the vector w so that, for $1 \leq i \leq 1000$,

$$P(|Y(x_i)| \leq w_i | Y(0) = f(0), Y(0.3) = f(0.3), Y(0.6) = f(0.6), Y(1) = f(1)) = 0.95.$$

Suggested R commands: *qnorm*. Add the graph of w to your figure. Suggested R commands: *lines(.,.,col="green")*