AUTOUR DES ENLACEMENTS

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TABLE DES MATIÈRES

1. INTRODUCTION

L'objet de ce texte est de présenter une partie des travaux de l'auteur, centrée sur les enlacements et les invariants de 3-variétés. Étant donné un anneau commutatif intègre R, un enlacement algébrique est une forme bilinéaire symétrique ou alternée

$M \times M \to F(R)/R$ où M est un R-module de torsion et $F(R)$ le corps des fractions de R. Une forme d'intersection algébrique est une forme bilinéaire symétrique ou alternée $M \times M \to R$ où M est un R-module projectif. Du point de vue topologique, dans le cas le plus simple ($R = \mathbb{Z}$), ces formes apparaissent naturellement comme sous-produits de la dualité de Poincaré à coefficients entiers. Ainsi toute variété orientée X de dimension paire $2n$ admet une forme d'intersection, donnée par le produit cohomologique $\cup$ évalué sur la classe fondamentale :

$$H^n(X) \times H^n(X) \to \mathbb{Z}, \ (a, b) \mapsto (a \cup b)([X]).$$

La symétrie ou l'anti-symétrie du produit $\cup$, qui est donnée par la parité de n, induit celle de la forme d'intersection. Une variété orientée de dimension impaire $2n+1$ possède quant à elle un enlacement sur le groupe de torsion $\mathrm{Tors}\, H^{n+1}(X)$ via le produit

$$H^{n+1}(X) \times H^n(X; \mathbb{Q}/\mathbb{Z}) \to H^{2n+1}(X; \mathbb{Q}/\mathbb{Z}).$$

Dans ce cas, l'enlacement est symétrique si n est impair, alterné si n est pair. La dualité de Poincaré sur les variétés compactes orientées sans bord assure que dans ce cadre, ces formes sont non-dégénérées.

Dans la théorie classique, ces formes conduisent à des invariants de cobordismes via les groupes de Witt qui leur sont associés. Une partie du présent travail s'interpréte comme une généralisation de ces invariants dans le cas $\dim X = -1 \bmod 4$.

Une fonction quadratique est une application q sur un module telle que $b_q : (x, y) \mapsto q(x+y) - q(x) - q(y)$ soit bilinéaire. La forme bilinéaire b_q est dite associée à q. Étant donné une forme bilinéaire b et une fonction quadratique q, on dira que q est un raffinement quadratique de b si $b_q = b$. Une forme d'intersection peut admettre zéro, un ou plusieurs raffinements quadratiques. Un enlacement algébrique admet en général plusieurs raffinements quadratiques.

Un enlacement (resp. une forme d'intersection réduite modulo N) admet, moyennant le choix d'une trivialisation stable du fibré tangent de la variété, un unique raffinement quadratique. Un exemple (qui est utilisé implicitement dans §5 dans le cadre des surfaces) se présente quand $\dim X = 2 \bmod 4$: le raffinement de la forme d'intersection (réduite modulo 2) est métabolique et l'invariant d'Arf associé donne un invariant de la classe de cobordisme trivialisé de la variété munie de la trivialisation (Kervaire, 1960). Dans le cas où $\dim X = -1 \bmod 4$, le raffinement quadratique de l'enlacement est à valeurs dans $\mathbb{Q}/\mathbb{Z}$. Il ne donne plus un invariant de cobordisme trivialisé, mais un invariant de cobordisme spin.

Nos travaux s'organisent autour de la notion de raffinement quadratique d'enlacement. Une observation essentielle est que tout raffinement quadratique q sur M obtenu à partir d'une trivialisation est homogène, c'est-à-dire qu'il vérifie la propriété $q(nx) = n^2 q(x)$, pour tous $n \in \mathbb{Z}$ et $x \in M$. Les raffinements quadratiques qui apparaissent dans nos travaux ne seront pas toujours déterminés par une trivialisation et seront donc éventuellement non-homogènes. Ils seront aussi éventuellement dégénérés. Une autre différence importante avec le point de vue traditionnel est que nous utilisons généralement la classe d'isomorphisme des raffinements quadratiques (ou des enlacements), et non simplement leur classe de Witt associée. Ce choix est en fait dicté par les constructions topologiques des parties §4 et §5.

La **première partie** pose un cadre combinatoire (la construction du discriminant) qui mime algébriquement la relation entre les invariants d'une variété de dimension $4n$ et ceux de son bord. Nous décrivons, à l'aide d'un théorème de plongement, les raffinements quadratiques en termes d'un quotient de formes caractéristiques d'un réseau. Le second résultat classifie à isomorphisme près les raffinements quadratiques provenant de la construction du discriminant. Cette question est liée à la classification stable des formes d'intersection sur les réseaux munies de formes caractéristiques. Pour lever l'indétermination du raffinement quadratique par rapport à son enlacement associé, nous réduisons la classification des raffinements quadratiques à celle d'une classe particulière d'enlacements pointés, c'est-à-dire d'enlacements munis d'éléments distingués. Ces résultats généralisent des résultats classiques de Wall, Nikulin, Durfee et Looijenga–Wahl. Enfin, nous résolvons la question du comportement de la construction du discriminant relativement au produit tensoriel de deux réseaux dans le cadre du groupe de Witt des raffinements quadratiques d'enlacements. En conséquence, en appliquant l'homomorphisme fourni par la somme de Gauss, nous obtenons une formule de réciprocité. Cette formule de réciprocité permet d'expliciter le lien entre les données combinatoires de la théorie topologique quantique des champs de §4 et les données topologiques. Elle s'applique au calcul de la plupart des invariants quantiques (voir par exemple [17], [21], [20]).

La classification des enlacements sur les groupes abéliens finis à partir d'invariants a été résolue sur les p-groupes avec p impair par Wall dans les années 60 et sur les 2-groupes (le cas le plus difficile) en 1980 par Kawauchi et Kojima. Dans la **deuxième partie**, nous proposons une nouvelle solution plus explicite à ce problème, qui consiste à décrire de façon combinatoire l'image d'un système complets d'invariants. En application, nous résolvons le problème de détecter un facteur orthogonal d'un enlacement donné. Des exemples d'applications topologiques sont l'existence d'applications de degré un sur un lenticulaire, y compris avec des raffinements spin et spin complexe.

La **troisième partie** considère les enlacements pointés sur les groupes abéliens finis. Le résultat principal est que les enlacements pointés sont classifiés par des sommes de Gauss généralisées. Il répond à une question soulevée par C. Gille. Le point le plus délicat est la classification sur les 2-groupes.

Dans la **quatrième partie**, nous exposons la construction d'une théorie topologique quantique des champs en dimension 3 à partir d'un enlacement. Cette théorie est « abélienne », ce qui permet de l'expliciter complètement en termes d'invariants topologiques classiques. Les travaux liés à cette théorie ont motivé dans une large mesure l'introduction des outils algébriques décrits dans les parties précédentes. Une conséquence de nos formules explicites est que la théorie obtenue se généralise aux dimensions supérieures (pourvu que la dimension soit congrue à -1 modulo 4).

La **cinquième partie** présente la construction d'une théorie dite à la Goussarov–Habiro des invariants de type fini pour les structures complexes spin de 3-variétés. Les résultats obtenus semblent spécifiques à la dimension 3. Un des résultats clés est le fait que l'ensemble des structures spin complexes sur une 3-variété X orientée compacte (qui possède une structure affine sur $H^2(M)$) se plonge de façon naturelle dans l'ensemble des raffinements quadratiques de l'enlacement sur M. On montre que le raffinement quadratique coïncide avec celui déterminé par la torsion de Reidemeister–Turaev.

La dernière partie présente quelques perspectives actuelles de nos travaux.

2. Le discriminant

Un *réseau* est un groupe abélien M libre de type fini muni d'une forme bilinéaire symétrique $f : M \times M \to \mathbb{Z}$. Un adjoint d'une forme bilinéaire symétrique sera toujours noté par $\widehat{\cdot}$. Ainsi $\widehat{f}$ désigne l'application adjointe $M \to M^* = \mathrm{Hom}(M, \mathbb{Z})$. Un réseau donne lieu, par extension des scalaires, à une forme bilinéaire $f_{\mathbb{Q}} : V \times V \to \mathbb{Q}$ où $V = M \otimes \mathbb{Q}$. Le réseau *dual* est $M^\sharp = \{x \in V \mid f_{\mathbb{Q}}(x, M) \subset \mathbb{Z}\}$. Un réseau est *unimodulaire* si $M^\sharp = M$.

Une *forme caractéristique* pour un réseau est un élément $c \in M^*$ tel que

$$f(x,x) - c(x) \in 2\mathbb{Z}, \quad \text{pourtout } x \in M.$$

L'ensemble $\mathrm{Char}(f)$ des formes caractéristiques d'un réseau est non vide et est un espace affine sur $\mathrm{Hom}(M, 2\mathbb{Z})$.

Étant donné un réseau comme ci-dessus, $G_f = M^\sharp/M$ est un groupe de torsion. On définit dessus l'enlacement $L_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$ par

$$L_f([x],[y]) = f_{\mathbb{Q}}(x,y) \bmod \mathbb{Z}, \quad x, y \in M^\sharp.$$

Si le réseau est muni d'une forme caractéristique c, cet enlacement admet un raffinement quadratique $\phi_{f,c}$. Soit $c_{\mathbb{Q}} : V \to \mathbb{Q}$ l'extension linéaire de c. On définit alors

$$\phi_{f,c}([x]) = \frac{1}{2}(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x)) \bmod \mathbb{Z}, \quad x \in M^\sharp.$$

Observons que le défaut d'homogénéité, défini par $d : x \mapsto q(x) - q(-x)$, est donné par

$$d(x) = -c_{\mathbb{Q}}(x) \bmod \mathbb{Z}.$$

Remarque. Le signe $-$ dans la formule ci-dessus se justifie par une convention usuelle d'orientation en topologie. Voir la remarque après le théorème 1.

Wall a montré que l'application $[f] \mapsto [L_f]$ est surjective sur le monoïde des (classes d'isomorphismes d') enlacements sur les groupes abéliens finis. Nous allons voir en quoi l'énoncé analogue en remplaçant enlacement par raffinement quadratique tombe en défaut.

2.1. **Le théorème de plongement.**

Soit $\mathrm{Quad}(L_f)$ l'ensemble des raffinements quadratiques de l'enlacement L_f. C'est un espace affine modelé sur $\mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$.

On dispose d'une application bilinéaire naturelle $M^* \times M^\sharp \to \mathbb{Q}$ définie par $(\alpha, x) \mapsto \alpha_{\mathbb{Q}}(x)$, où $\alpha_{\mathbb{Q}}$ désigne l'extension de α en tensorisant par $\mathbb{Q}$. Cette application induit une application bilinéaire

$$\langle -, - \rangle : \mathrm{Coker}\widehat{f} \times G_f \to \mathbb{Q}/\mathbb{Z}$$

dont on peut montrer qu'elle est toujours non-dégénérée à gauche et à droite, et même toujours non-singulière à droite. Notons Φ_f : Coker$\widehat{f} \to \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$ l'adjoint à gauche.

On montre que tout homomorphisme Ker$\widehat{f} \to \mathbb{Z}$ induit par produit tensoriel avec $\mathbb{Q}/\mathbb{Z}$ un homomorphisme Ker$\widehat{L}_f \to \mathbb{Q}/\mathbb{Z}$. Soit $j_f : \mathrm{Hom}(\mathrm{Ker}\widehat{f}, \mathbb{Z}) \to \mathrm{Hom}(\mathrm{Ker}\widehat{L}_f, \mathbb{Q}/\mathbb{Z})$ le morphisme correspondant.

Théorème 1 (Deloup–Massuyeau, [14])**.** *L'application $c \mapsto \phi_{f,c}$ induit un plongement affine*

$$\phi_f : \mathrm{Char}(f)/2\widehat{f}(M) \hookrightarrow \mathrm{Quad}(L_f)$$

au-dessus du morphisme de groupe $-\Phi_f$: Coker$\widehat{f} \to \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$; de plus,

$$\mathrm{Coker}\phi_f = \mathrm{Coker} j_f$$

et étant donné $q \in \mathrm{Quad}(L_f)$, $q \in \mathrm{Im}\phi_f$ si et seulement si $q|_{\mathrm{Ker}\widehat{L}_f} \in \mathrm{Im} j_f$.

En particulier, l'application ϕ_f est surjective si et seulement si f est non-dégénérée. Cette construction apparaît en topologie quand le réseau s'identifie au groupe libre $H_{2n}(X)$/Tors $H_{2n}(X)$ d'une $4n$-variété orientée compacte munie de sa forme d'intersection. Supposons par exemple que X soit munie d'une structure presque complexe ; La forme caractéristique peut s'interpréter comme la donnée d'une classe $c_1(X)$ de Chern. Dans ce cas, l'enlacement produit par la construction du discriminant s'identifie comme suit : soit I l'image de l'application naturelle $\partial|_{\mathrm{Tors}}$: Tors $H_{2n}(X, \partial X) \to$ Tors $H_{2n-1}(\partial X)$. Alors L_f est l'opposé de l'enlacement de ∂X (avec la convention usuelle d'orientation) défini sur $I^{\perp}/I$ (qui est un sous-quotient de Tors $H_{2n-1}(\partial X)$) et $\phi_{f,c}$ est un raffinement de cet enlacement. Dans le cas plus spécifique où ∂X est de dimension 3, Char$(f)/2\widehat{f}(M)$ s'identifie avec l'espace affine des structures spin complexes sur ∂X. Voir à ce sujet §5.1.

Le théorème 1 a été généralisé dans un travail en cours par l'auteur à des enlacements sur des R-modules de torsion autres que $R = \mathbb{Z}$. Essentiellement une telle généralisation existe dès qu'on peut définir une notion raisonnable de forme caractéristique.

2.2. **Le théorème de classification stable.** Le résultat que nous présentons ici généralise un résultat dû à Wall et Durfee, ainsi qu'à Looijenga–Wahl. L'idée consiste à utiliser la construction du discriminant pour résoudre le problème de classifier les enlacements, et plus généralement les raffinements quadratiques, à isomorphisme près.

Il existe deux notions naturelles d'équivalences sur les réseaux munis de formes caractéristiques. Deux réseaux (M, f, c) et (M', f', c') munis de formes caractéristiques sont *équivalents* s'il existe un isomorphisme $\psi : M \to M'$ tel que $\psi^* f' = f$ et $\psi^* c' = c \mod 2\widehat{f}(M)$. Deux tels réseaux sont dits *stablement équivalents* s'ils deviennent équivalents après stabilisations avec des réseaux unimodulaires munis de formes caractéristiques. Les réseaux unimodulaires induisent des enlacements triviaux par la construction du discriminant ; aussi une équivalence stable entre réseaux induit-elle un isomorphisme entre les raffinements quadratiques correspondants. Qu'en est-il de la réciproque ? L'adaptation du résultat de Wall pour les enlacements permet de répondre positivement dans le cas où les réseaux sont non-dégénérés. L'inclusion du cas où les réseaux sont potentiellement dégénérés est motivé par l'étude des invariants de type fini des structures spin complexes sur les 3-variétés, voir §5.

Rappelons que l'adjoint à droite de l'application bilinéaire $\langle -, - \rangle : \mathrm{Coker}\widehat{f} \times G_f \to \mathbb{Q}/\mathbb{Z}$ est toujours un isomorphisme $G_f \to \mathrm{Hom}(\mathrm{Coker}\widehat{f}, \mathbb{Q}/\mathbb{Z})$. Par conséquent, tout isomorphisme $\psi : \mathrm{Coker}\widehat{f} \to \mathrm{Coker}\widehat{f'}$ induit un isomorphisme $\psi^\sharp : G_{f'} \to G_f$. Ceci donne lieu à une application injective

$$\psi \mapsto \psi^\sharp, \quad \mathrm{Iso}(\mathrm{Coker}\widehat{f}, \mathrm{Coker}\widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f).$$

Théorème 2 (Deloup–Massuyeau, [14])**.** *Deux réseaux (M, f, c) et (M', f', c') munis de formes caractéristiques sont stablement équivalents si et seulement s'il existe un élément $\psi^\sharp$ tel que les raffinements quadratiques $(G_f, \phi_{f,c})$ et $(G_{f'}, \phi_{f',c'})$ soient isomorphes via $\psi^\sharp$. De plus, un tel isomorphisme se relève en une équivalence stable entre (M, f, c) et (M', f', c').*

Dans le cas où les réseaux sont non-dégénérés, l'application $\psi \mapsto \psi^\sharp$ est bijective, donc l'existence de $\psi^\sharp$ équivaut à celle de ψ (qui se relève toujours), de sorte que l'équivalence stable sur les réseaux est équivalente à l'isomorphisme des raffinements quadratiques correspondants.

On note ± 1 les deux formes sur $\mathbb{Z}$ définie par $(1, 1) \mapsto \pm 1$, munies toutes deux de la forme caractéristique $1 \mapsto 1$.

Corollaire 2.1. *Soient (M, f, c) et (M', f', c') deux réseaux munis de formes caractéristiques. Nous avons $(G_f, \varphi_{f,c}) \simeq (G_{f'}, \varphi_{f',c'})$ si et seulement si les réseaux deviennent équivalents par stabilisation avec des formes comme ci-dessus.*

Remarque. Ce résultat est bien connu pour les réseaux sans formes caractéristiques.

3. Combinatoire des enlacements

Les enlacements considérés dans cette section sont tous sur des groupes abéliens finis. Certains résultats de cette section admettent des généralisations aux modules de torsion de type fini sur des domaines de Dedekind [6].

3.1. **Le monoïde des enlacements.**

Comme indiqué dans l'introduction, la classification des enlacements a été considérée comme achevée avec la classification des enlacement sur les 2-groupes, par Kawauchi et Kojima en 1980. Ils construisent, par générateurs et relations, le monoïde $\mathfrak{M}$ des classes d'isomorphismes (l'opération étant la somme orthogonale).

Nous proposons une construction différente du même objet. Outre le fait de simplifier les démonstrations, notre construction dévoile un objet combinatoire naturel et permet de répondre concrètement à la question de déterminer si un enlacement possède un facteur orthogonal donné.

L'idée de base est la suivante. Soit p un entier. Les p-groupes abéliens d'ordre p^n sont classés, à isomorphisme près, par un objet combinatoire qui est une partition $(l_1, \ldots, l_r)$ de n. Nous nous proposons de déterminer l'objet combinatoire qui va classifier les enlacements sur les p-groupes d'ordre p^n. Dans le cas où p est impair, il suffit essentiellement de remplacer partition par partition signée. Dans le cas où $p = 2$, la combinatoire est plus compliquée.

Pour la décrire, on va décrire l'image d'un système d'invariants complets des enlacements sur les 2-groupes. Étant donné une fonction quadratique $q : G \to \mathbb{Q}/\mathbb{Z}$, la somme de Gauss non normalisée associée est $\Gamma(q) = \sum_{x\in G} \exp(2\pi i q(x))$. Il sera en fait agréable par la suite de considérer la somme de Gauss normalisée

$$(1) \qquad \gamma(q) = \begin{cases} \frac{\Gamma(q)}{|\Gamma(q)|} & \text{si } \Gamma(q) \neq 0; \\ 0 & \text{si } \Gamma(q) = 0. \end{cases}$$

Il est bien connu que si $\gamma(q) \neq 0$ alors $\gamma(q)$ est une racine 8-ème de l'unité. Soit maintenant $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ un enlacement. Posons alors $q_k(x) = 2^{k-1}\lambda(x,x)$ pour tout $k \geq 1$. On définit

$$(2) \qquad \sigma_k(\lambda) = \begin{cases} \frac{1}{2\pi}\text{Arg } \gamma(q_k) \in \mathbb{Z}/8\mathbb{Z} & \text{if } \gamma(q_k) \neq 0 \\ \infty & \text{if } \gamma(q_k) = 0 \end{cases}$$

Soit $\overline{\mathbb{Z}}_8 = \mathbb{Z}/8\mathbb{Z} \cup \{\infty\}$. Il s'agit du monoïde obtenu en adjoignant au groupe cyclique à 8 éléments un élément supplémentaire noté ∞, avec la règle $\infty + a = a + \infty = \infty = \infty + \infty$ pour tout $a \in \mathbb{Z}/8\mathbb{Z}$. Il résulte de ce qui précède que σ définit une application $\mathbb{N}^\times \to \overline{\mathbb{Z}}_8$.

Par ailleurs, étant donné un 2-groupe G, on dispose des invariants de rang des sous-groupes homogènes. Si la partition associée à G est $(l_1, \dots, l_r)$, on définit, pour chaque $k \geq 1$,

$$(3) \qquad \rho_k(\lambda) = |\{j \in \mathbb{N} \mid l_j = k\}| \in \mathbb{N}.$$

Naturellement, cet invariant ne dépend que du groupe et non de l'enlacement défini dessus. On peut alors regrouper les invariants (rang, signature) ρ et σ sous la forme d'une seule application $(\rho, \sigma) : \mathbb{N}^\times \to \mathbb{N} \times \overline{\mathbb{Z}}_8$. E. Burger [5] à l'aide de la théorie de Minkowski, puis A. Kawauchi et S. Kojima [24] ont montré que cette application classifie l'enlacement.

Soit $\mathcal{M}$ un monoïde additif et I un suite d'entiers consécutifs. Un tableau est une application $I \to \mathcal{M}$, qu'il sera pratique de considérer comme un diagramme de la forme

I	k	$k+1$	$\dots$	l
$\mathcal{M}$	m_k	m_{k+1}	$\dots$	m_l

Afin de simplifier la notation, les notations de l'intervalle ainsi que du monoïde seront omises des tableaux suivants. La longueur d'un tableau T est l'entier $1 + \sup_{(m,n)\in I\times I} |m - n| \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Un tableau T' est un prolongement d'un tableau T si T' prolonge T en tant qu'application. Dans ce cas, T est un tableau extrait de T'. Étant donné un tableau $T : I \to \mathcal{M}$ quelconque, on peut toujours le prolonger trivialement sur $\mathbb{N}$ entier en définissant $\tilde{T}(n) = 0$ pour $n \in \mathbb{N} - I$. En pratique, on confondra un tableau T et son prolongement trivial $\tilde{T}$ à $\mathbb{N}$ ainsi défini. Ainsi on dira qu'un tableau T est *fini* s'il est de longueur finie ou s'il est le prolongement trivial $\tilde{T}'$ d'un tableau T' de longueur finie. (C'est la définition habituelle de support fini.) Comme $\mathcal{M}$ est un monoïde, l'addition de tableaux est bien définie. La somme de deux tableaux T_1 et T_2 est définie sur $\mathbb{N}$ par $T_1 + T_2 = \tilde{T}_1 + \tilde{T}_2$ où $\tilde{T}_i$, $i = 1, 2$, désigne le prolongement trivial à $\mathbb{N}$. L'ensemble des tableaux

$\mathbb{N} \to \mathcal{M}$ forme un monoïde. L'élément neutre 0 est le tableau envoyant $\mathbb{N}$ sur 0. Le délimiteur à gauche (resp. à droite) d'un tableau $T : I \to \mathcal{M}$ est l'élément $-1 \leq \operatorname{Inf} I - 1 < \infty$ (resp. l'élément $0 \leq \operatorname{Sup} I + 1 \leq \infty$).

Considérons à présent les tableaux à valeur dans le monoïde $\mathcal{M} = \mathbb{N} \times \overline{\mathbb{Z}}_8$, que nous noterons $T : m \mapsto (r(m), s(m))$, avec $r(m) \in \mathbb{N}$ (rang formel) and $s(m) \in \overline{\mathbb{Z}}_8$ (signature formelle). Nous dirons qu'un tableau est *admissible* s'il existe un enlacement (G, λ) sur un 2-groupe tel que $r(m) = \rho_m(\lambda)$ et $s(m) = \sigma_m(\lambda)$ pour tout $m \in I$.

Un entier $m \in I$ sera dit *régulier* pour un tableau T si $r(m) = 0$ ou $s(m) \neq \infty$. On note $I_{\text{reg}} \subseteq I$ l'ensemble des éléments réguliers de T. Présentons quatre types particuliers distincts de tableaux :

- Type T_0. Tout tableau de longueur impaire de la forme $T = (0, s(m))_{m \in I}$.
- Type T_1. Tout tableau de la forme $\begin{array}{|c|}\hline m \\ \hline 1 \\ \hline \infty \\ \hline \end{array}$ pour un entier non nul m.
- Type T_2. Tout tableau de la forme $\begin{array}{|c|}\hline m \\ \hline 2 \\ \hline \infty \\ \hline \end{array}$ pour un entier non nul m.
- Type T_3. Tout tableau de longueur impaire tel que $I = I_{\text{reg}}$.

Le résultat principal est un critère nécessaire et suffisant pour qu'un tableau soit admissible.

Théorème 3 (Deloup, [12]). *Un tableau fini* $T : \mathbb{N}^{\times} \to \mathbb{N} \times \overline{\mathbb{Z}}_8,\ m \mapsto (r(m), s(m))$ *est admissible si et seulement si les conditions suivantes sont satisfaites :*

(1) $r(I_{\text{reg}}) \subseteq 2\mathbb{N}$.

(2) $s(m) = \sum_{k \geq m+1} r(k) \pmod 2$ *pour tout* $m \in I_{\text{reg}}$.

(3) $s(m) + s(m+1) = 2 \sum_{k \geq m+2} r(k) \pmod 4$ *pour tout* $\{m, m+1\} \subseteq I_{\text{reg}}$.

(4) *Pour tout tableau* T_{ext} *extrait de* T *et pour toute paire de délimiteurs* m, n *de* T_{ext} *dans* I_{reg}*, les valeurs possibles de* $s(m) - s(n)$ *sont déterminées comme ci-dessous :*

Type de T_{ext}	T_0	T_1	T_2	T_3
$s(m) - s(n)$	0	± 1	$0, \pm 2$	$0, 4$

Compte-tenu du fait que le groupe d'un enlacement est fini, il est aisé d'observer sur le rang et la signature que tout tableau admissible est fini. Ceci garantit en particulier que les sommes intervenant dans les conditions (2) et (3) sont finies. (En particulier, la condition (2) implique que $s(m) \neq \infty$ dès que $r(m) = 0$: les entiers réguliers m de T sont exactement les entiers m tels que $s(m) \neq \infty$.) De manière générale, la nécessité des conditions énoncées dans le Théorème 3 est une conséquence de calculs d'enlacements et de sommes de Gauss. La preuve de la suffisance est constructive et se fait par récurrence sur la longueur de l'intervalle I, voir [12].

Notons $\mathfrak{T}$ le monoïde constitué des tableaux $T : \mathbb{N}^{\times} \to \mathbb{N} \times \overline{\mathbb{Z}}_8$. On déduit du Théorème 3 que la somme de deux tableaux admissibles est encore un tableau admissible, de sorte que le sous-ensemble des tableaux admissibles constitue un sous-monoïde $\mathfrak{T}^{\mathrm{adm}}$ de $\mathfrak{T}$. Puisque ρ, σ sont des invariants complets du monoïde $\mathfrak{M}$ des classes d'isomorphismes d'enlacements sur les 2-groupes, l'application (ρ, σ) : $\mathfrak{M} \to \mathfrak{T}$ est injective. Il en résulte la description combinatoire de $\mathfrak{M}$ ci-dessous.

Corollaire 3.1. *Le monoïde $\mathfrak{M}$ des classes d'isomorphismes d'enlacements sur les 2-groupes est isomorphe au sous-monoïde $\mathfrak{T}^{\mathrm{adm}}$ des tableaux admissibles.*

Le théorème 3 se généralise aussi aux fonctions quadratiques homogènes, voir [12]. Il est raisonnable de conjecturer qu'il se généralise aussi aux fonctions quadratiques : l'idée est d'utiliser le théorème 4 afin de réduire cette question aux enlacements pointés (voir la section §3.2).

Considérons à présent la question de reconnaître si un enlacement λ' est un facteur orthogonal d'un enlacement λ, c'est-à-dire s'il existe un enlacement λ" tel que

$$\lambda = \lambda' \oplus \lambda".$$

Décrivons tout d'abord des conditions nécessaires simples pour qu'une telle décomposition orthogonale existe. Il est clairement nécessaire que

$$\rho_k(\lambda) \geq \rho_k(\lambda') \text{ pour tout } k \geq 1. \tag{4}$$

Une seconde condition nécessaire, résultant de l'additivité de σ sur les sommes orthogonales, dit que

$$\sigma_k(\lambda') = \infty \implies \sigma_k(\lambda) = \infty, \quad \text{pour tout } k \geq 1. \tag{5}$$

Supposons à présent ces conditions (4) et (5) vérifiées. Nous allons associer à (λ, λ') un ensemble

$$S_{\lambda,\lambda'} = \{T_\alpha\}_{\alpha \in \overline{\mathbb{Z}}_8}$$

de tableaux. Pour $\alpha \in \overline{\mathbb{Z}}_8$, nous définissons le tableau $T_\alpha = (r_\alpha, s_\alpha) : \mathbb{N}^\times \to \mathbb{N} \times \overline{\mathbb{Z}}_8$ par

$$\begin{array}{rcl} r_\alpha(k) & = & \rho_k(\lambda) - \rho_k(\lambda') \\ s_\alpha(k) & = & \left\{ \begin{array}{cl} \alpha & \text{si } \sigma_k(\lambda') = \infty \\ \sigma_k(\lambda) - \sigma_k(\lambda') & \text{si } \sigma_k(\lambda') \neq \infty. \end{array} \right. \end{array} \qquad k \in \mathbb{N}^\times. \tag{6}$$

Le tableau T_α est bien défini grâce à la condition (4) et au fait que ∞ est le seul élément non inversible dans $\overline{\mathbb{Z}}_8$.

Corollaire 3.2. *Un enlacement λ' est un facteur orthogonal d'un enlacement λ si et seulement si les conditions* (4) *et* (5) *ci-dessus sont vérifiées et s'il existe un tableau admissible $T \in S_{\lambda,\lambda'}$.*

Une première série d'applications est fournie par les applications de degré 1 sur les espaces lenticulaires $L(m, p)$ (ou plus généralement de degré n où est premier avec m), voir [12]. On peut également déterminer les lenticulaires proscrits ou prescrits en fonction de l'existence ou non d'une application de degré 1 à partir d'une variété de dimension trois donnée. Une autre application concerne la torsion de Reidemeister, raffinée par Turaev [12].

3.2. **La classification des enlacements pointés.** Soit M un R-module de torsion. Un enlacement pointé est un enlacement $b : M \times M \to F/R$ muni d'un nombre fini éléments $x_1, \ldots, x_n \in M$. La notion d'isomorphisme entre enlacements s'étend immédiatement aux enlacements pointés : $(M, b, (x_1, \ldots, x_n))$ et $(M', b', (x'_1, \ldots, x'_n))$ sont isomorphes s'il existe un isomorphisme $\psi : M \to M'$ tel que $\psi^* b' = b$ et $\psi(x_j) = x'_j$, $j = 1, \ldots, n$. Il est parfois utile de relaxer la notion en remplaçant dans la définition $\psi^{\otimes n}(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)$ par $\psi(\{x_1, \ldots, x_n\}) = \{x'_1, \ldots, x'_n\}$.

Contrairement à la théorie des formes pointés (ou des isométries) sur les modules *libres*, dont l'outil essentiel est le théorème de Witt[1] [23], la classification des enlacements pointés est beaucoup plus délicate, même sur les modules de torsion de type fini les plus simples, à savoir les groupes abéliens finis.

Une motivation supplémentaire pour s'intéresser aux enlacements pointés est le fait que les classifications du paragraphe précédent ne semblent pas mener à une classification concrète en pratique. Or les fonctions quadratiques s'interprètent presque comme des enlacements pointés. On se restreint à présent aux enlacements sur les groupes abéliens finis.

Théorème 4 (Deloup–Massuyeau et Deloup, [14] [6])**.**

1. *Deux raffinements quadratiques q, q' d'enlacements b, b' respectivement, sont isomorphes si et seulement si $\gamma(q) = \gamma(q')$, s'il existe un isomorphisme ψ tel que $\psi^* b' = b$ et $\psi^* d_{q'} = d_q$.*
2. *Le monoïde des fonctions quadratiques se surjecte sur le monoïde des enlacements pointés de la forme (G, b, x) où G est un groupe abélien fini, b un enlacement sur G et $x \in 2G$.*

En d'autres termes, la classification des fonctions quadratiques se réduit à celle d'une classe particulière d'enlacements pointés, modulo l'invariant de Gauss (qui est un invariant de Witt). Remarquons qu'il s'agit ici de la classification des classes d'*isomorphisme* et non de la classification, au sens plus *faible*, des classes de Witt.

La première affirmation généralise un théorème dû à Nikulin qui correspond au cas particulier $d_q = d_{q'} = 0$ (raffinements quadratiques homogènes). La démonstration est très différente de celle de Nikulin. Nikulin démontre le résultat par récurrence sur le nombre de générateurs, en utilisant la classification des enlacements sur l'anneau p-adique $\mathbb{Z}_p$. Nous démontrons le résultat directement en utilisant l'action de $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ sur l'invariant de Gauss $\gamma(q)$.

La formule suivante due à van der Blij et à Milgram calcule la somme de Gauss $\gamma(G_f, \varphi_{f,v})$ via la signature $\sigma(f) \in \mathbb{Z}$.

Lemme 1 (Van der Blij, [4])**.** $\gamma(G_f, \varphi_{f,v}) = e^{\frac{\pi i}{4}(\sigma(f) - f_{\mathbb{Q}}(v,v))}$.

[1]Le théorème de Witt affirme que toute isométrie entre deux sous-espaces V et W d'un k-espace vectoriel quadratique U s'étend en une isométrie de (U, q). Deux sous-espaces V, W sont donc isométriques si et seulement si $q|_V$ et $q|_W$ sont isomorphes.

Pour décrire la classification des enlacements pointés sur les groupes abéliens finis, on introduit des sommes de Gauss généralisées. On se donne un triplet

$$(V, f, s)$$

où $f : V \to \mathbb{Z}$ est un raffinement quadratique d'un réseau et $s = (s_1, \ldots, s_n) \in (V^*)^n$. Étant donné un enlacement pointé

$$(G, \lambda, a)$$

où $a \in G^n$, on associe une nouvelle fonction quadratique sur $V \otimes G$ définie par

$$f \otimes \lambda + (\mathrm{id}_{V^*} \otimes \hat{\lambda})(s \otimes a)$$

où $s \otimes a = \sum_j s_j \otimes a_j \in V^* \otimes G$. On définit $\gamma_{f,s}(\lambda, a)$ la somme de Gauss associée à cette fonction quadratique.

Théorème 5 (Deloup, [6])**.** *Deux enlacements pointés (G, λ, a) et (G, λ', a') sont isomorphes si et seulement si $\gamma_{f,s}(\lambda, a) = \gamma_{f,s}(\lambda', a')$ pour tous les triplets (V, f, s).*

Il suffit bien sûr d'un nombre fini de tels triplets pour déterminer la classe d'isomorphisme. Les réseaux (V, f, s) de dimension un sont insuffisants, même pour les enlacements pointés avec un seul élément distingué.

Problème. Trouver un système complet *minimal* d'invariants (ce qui serait bien utile pour généraliser les modèles combinatoires de la section précédente).

Le théorème 5 est l'outil algébrique sous-jacent à la construction de l'invariant τ des entrelacs colorés exposé dans la section §4.2.

3.3. **La formule de réciprocité.** La construction du discriminant $f \mapsto L_f$ préserve les sommes orthogonales. En revanche, son comportement relativement au produit tensoriel de deux réseaux est plus subtil. L'étude va nous permettre de généraliser la formule de Van der Blij présentée dans le paragraphe précédent.

Le monoïde $\mathfrak{M}$ des classes d'isomorphismes d'enlacements sur les groupes abéliens finis possède une infinité de générateurs et de relations, voir [24]. Il est traditionnel de considérer un objet plus simple à manipuler : le groupe de Witt $\mathfrak{W}$. Rappelons sa définition. Un enlacement $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ est dit métabolique s'il existe un sous-groupe H tel que

$$H = H^{\perp} = \{x \in \ \mid b(x, H) = 0\}.$$

On ne demande pas a priori que ce sous-groupe H soit un facteur orthogonal. Le groupe de Witt $\mathfrak{W}$ s'obtient à partir de $\mathfrak{M}$ en décrétant que les formes métaboliques sont nulles. Le groupe de Witt $\mathfrak{WQ}$ des fonctions quadratiques se définit de manière analogue, une fonction quadratique $q : G \to \mathbb{Q}/\mathbb{Z}$ étant métabolique si elle vérifie $q(N) = 0$ et $N = N^{\perp}$ (orthogonalité relative à l'enlacement associé). Le calcul de ces groupes de Witt est classique, voir [31]. La somme de Gauss γ introduite au §3 est invariant du groupe de Witt $\mathfrak{WQ}$: elle se factorise en effet en un homomorphisme $\mathfrak{WQ} \to S^1$.

Le problème de base est le suivant. Soit $f : V \times V \to \mathbb{Z}$ et $g : W \times W \to \mathbb{R}$ deux réseaux non-dégénérés. Le produit tensoriel $f \otimes g : (V \otimes W) \times (V \otimes W) \to \mathbb{Z}$ est le réseau défini par

$$(f \otimes g)(x \otimes y, x' \otimes y') = f(x, x')g(y, y'), \quad x, x' \in V, \ y, y' \in W.$$

Peut-on relier, dans $\mathfrak{M}$, les (classes d'isomorphismes d') enlacements L_f, L_g et $L_{f\otimes g}$? Cette question paraît difficile en général. En revanche, on peut répondre à cette question dans le cadre des groupes de Witt.

On dispose d'un homomorphisme naturel $j_f : G_f \otimes W \to G_{f\otimes g}$ défini par

$$j_f([x] \otimes y) = [x \otimes y], \quad x \in V^\sharp, y \in W.$$

De manière similaire, on définit $j_g : V \otimes G_g \to G_{f\otimes g}$ par

$$j_g(x \otimes [y]) = [x \otimes y], \quad x \in V, y \in W^\sharp.$$

Soient A et B les images dans $G_{f\otimes g}$ de j_f et j_g respectivement. On peut alors considérer les restrictions $\phi_f = \phi_{f\otimes g,c}|_A$ et $\phi_g = \phi_{f\otimes g,c}|_B$. Ce sont des raffinements quadratiques des enlacements $L_f \otimes g$ et de $f \otimes L_g$ respectivement. On peut montrer que $A = B^\perp$ et par suite, que ces raffinements quadratiques induisent des fonctions quadratiques $\bar{\phi}_f$ et $\bar{\phi}_g$ sur les quotients $A/(A \cap A^\perp)$ et $A^\perp/(A \cap A^\perp)$ respectivement.

Théorème 6 (Deloup, [6])**.** *L'égalité suivante est vérifiée dans* $\mathfrak{WQ}$ *:*

$$[G_{f\otimes g}, \phi_{f\otimes g,c}] = [A/(A \cap A^\perp), \bar{\phi}_f] + [A^\perp/(A \cap A^\perp), \bar{\phi}_g]. \tag{7}$$

Remarque. Le théorème 6 reste essentiellement[2] valide sur tout module de torsion de type fini sur un domaine de Dedekind R.

Revenons au cas où $R = \mathbb{Z}$. En appliquant γ à l'égalité (7), nous obtenons la formule de réciprocité ci-dessous pour les sommes de Gauss. Cette formule généralise la formule de Van der Blij (Lemme 1). On note $\sigma(\cdot)$ la signature d'un réseau (tensorisé par $\mathbb{R}$), $\tilde{c} \in (V \otimes W) \otimes \mathbb{Q}$ l'élément uniquement déterminé par la propriété $c_{\mathbb{Q}}(\cdot) = f_{\mathbb{Q}}(\tilde{c}, \cdot)$ et par une barre la conjugaison canonique sur $\mathbb{C}$.

Corollaire 6.1 (Turaev, [33])**.**

$$\gamma(\varphi_{f\otimes g,c} \circ j_f) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - (f\otimes g)_{\mathbb{Q}}(\tilde{c},\tilde{c}))} \ \overline{\gamma(\varphi_{f\otimes g,c} \circ j_g)}.$$

Cette formule est elle-même une généralisation de la formule de ma thèse.

Corollaire 6.2 (Deloup, [7])**.** *Supposons f, g munis de formes caractéristiques $v = f_{\mathbb{Q}}(\tilde{v}, \cdot)$ et $w = g_{\mathbb{Q}}(\tilde{w}, \cdot)$ respectivement. Alors*

$$\gamma(\varphi_{f,v} \otimes g) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - (f(\tilde{v},\tilde{v})g(\tilde{w},\tilde{w}))} \ \overline{\gamma(f \otimes \varphi_{g,w})}.$$

Les formules de réciprocité sont utilisées dans le calcul d'invariants quantiques. Elles sont un ingrédient technique important dans la (re)construction explicite de la TTQC exposée dans la section suivante §4.

[2]Il n'y a rien à modifier si la multiplication par 2 est un isomorphisme ou si $r^2 - r \in 2R$ pour tout r ; dans le cas général, on peut manquer de formes caractéristiques ; le théorème reste vrai pour les réseaux pairs ($c = 0$).

4. Théorie topologique quantique des champs abélienne

Cette section expose la motivation proprement topologique de nos travaux. Les résultats font largement appel aux techniques algébriques exposées dans la première partie (discriminant, réciprocité, classification stable des enlacements).

4.1. **Invariant associé à l'enlacement.** Soit M une variété orientée compacte de dimension 3. On sait que M admet une présentation par chirurgie sur un entrelacs L parallélisé dans S^3. La topologie quantique explicite, à partir de la théorie des catégories monoïdales tressées, les données combinatoires sur un diagramme D_L d'entrelacs (plus généralement d'un écheveau) pour construire un invariant de la variété $M = S^3_L$ obtenue par chirurgie le long de L. Soit $e : G \times G \to \mathbb{Q}/\mathbb{Z}$ un enlacement sur un groupe abélien fini et $f : G \to \mathbb{Q}/\mathbb{Z}$ un homomorphisme tel que $2f(x) = 0$ pour tout $x \in G$. Ces données sont équivalentes à un raffinement quadratique homogène $q : G \to \mathbb{Q}/\mathbb{Z}$ de e.

On formalise ces données combinatoires en un foncteur $\mathcal{F}$ entre une catégorie des écheveaux enrubannés $\mathcal{R}_C$ et une catégorie enrubannée C ([32, Chap. I] pour les définitions générales). Soit $S = \mathbb{Q}/\mathbb{Z} \cup \{\infty\}$ le monoïde additif où ∞ est l'unique élément non inversible et $\infty + r = r + \infty = \infty$ pour tout $r \in \mathbb{Q}/\mathbb{Z}$. Les objets de C sont les éléments de G. Les morphismes sont définis par $\mathrm{Hom}(g, g) = S$ et $\mathrm{Hom}(g, h) = \{\infty\}$ pour $g \neq h$. La composition et le produit tensoriel de deux morphismes sont donnés par la somme correspondante dans $\mathbb{Q}/\mathbb{Z}$. Le tressage $\tau_{g,h} \in \mathrm{Hom}(g \otimes h, h \otimes g)$ est $e(g, h) \in \mathbb{Q}/\mathbb{Z}$. La volte $\theta_g \in \mathrm{Hom}(g, g)$ est $q(g) \in \mathbb{Q}/\mathbb{Z}$. L'objet dual à $g \in G$ est $-g \in G$. On vérifie que ces données définissent une catégorie enrubannée C.

Fixons une orientation de $\mathbb{R}^2 \times [0, 1]$. Un écheveau est une surface compacte orientée dans $\mathbb{R}^2 \times [0, 1]$ formée d'une réunion disjointe de bandes (copies homéomorphes du carré $[0, 1] \times [0, 1]$) et d'anneaux (copies homéomorphes du cylindre $S^1 \times [0, 1]$). Les bandes sont munies d'une base : c'est l'image de $[0, 1] \times 0$ et de $[0, 1] \times 1$. On requiert que ces images soient respectivement sur $0 \times \mathbb{R} \times 0$ et $0 \times \mathbb{R} \times 1$ respectivement. De plus, l'âme (l'image de $1/2 \times [0, 1]$) de chaque bande est orientée. La figure 1 donne un exemple d'écheveau. Pour une définition plus complète, voir par exemple [22, Chap. XII]. Un entrelacs est un écheveau sans bandes.

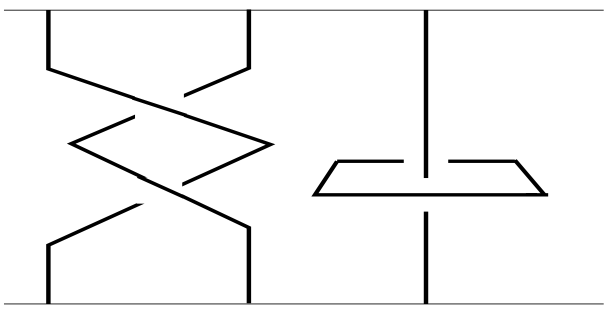

Fig. 1. Un écheveau.

Un C-écheveau est un écheveau muni d'un morphisme de C de la façon suivante : chaque base d'une bande est munie d'une paire d'objets $g, h \in \mathrm{Obj}(C) = G$, où g est associé à la composante de la base sur $0\times\mathbb{R}\times 0$ et g est associé à la composante de la base sur $0\times\mathbb{R}\times 1$. Chaque objet est muni d'un signe ± 1, selon que l'orientation de l'âme de la bande coïncide avec l'orientation de $0 \times 0 \times [0,1]$ ou non. Classons les objets $g_1, \dots, g_n$ (resp. $h_1, \dots, h_n$) dans l'ordre (induit par l'orientation de $\mathbb{R}$ dans $0 \times \mathbb{R} \times 0$ et $0 \times \mathbb{R} \times 1$) dans lequel l'écheveau rencontre $0 \times \mathbb{R} \times 0$ (resp. $0 \times \mathbb{R} \times 1$). Chaque g_i (resp. h_i) est muni d'un signe ε_i (resp. ε'_i). On peut alors munir l'écheveau d'un morphisme dans

$$\mathrm{Hom}(g_1^{\varepsilon_1} \otimes \cdots \otimes g_n^{\varepsilon_n}, h_1^{\varepsilon'_1} \otimes \cdots \otimes h_n^{\varepsilon'_n}) = \mathrm{Hom}\Big(\sum_{1\le j\le n} \varepsilon_j g_j, \sum_{1\le j\le n} \varepsilon'_j h_j\Big).$$

Inversement, à tout C-écheveau correspond une paire ordonnée de suites d'objets signés. Les morphismes d'identité sont représentés par des écheveaux sans anneaux qui consistent en des bandes verticales non enlacées. Notons que l'écheveau vide est aussi un C-échéveau.

Soit maintenant la catégorie $\mathcal{R}_C$ des écheveaux sur C : les objets de $\mathcal{R}_C$ sont les suites finies $(g_1, \varepsilon_1), \dots, (g_n, \varepsilon_n)$ où $g_1, \dots, g_n \in G$ et $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$. Un morphisme $\alpha \to \beta$ dans $\mathcal{R}_C$ est une classe d'isotopie de C-écheveaux dont α, β est la paire ordonnée de suites d'objets signés associée. Cette catégorie est elle-même monoïdale, le produit tensoriel d'objets étant leur juxtaposition.

Lemme 2. *Il existe un unique foncteur tensoriel covariant $\mathcal{F} : \mathcal{R}_C \to C$ tel que $\mathcal{F}(g, \pm 1) = \pm g$ et dont les valeurs sur les C-écheveaux élémentaires soient données par la figure* 2.

Ce résultat est un cas particulier du théorème général de représentation des C-écheveaux, voir [32] et [22]. L'idée est qu'il suffit d'expliciter les valeurs de $\mathcal{F}$ sur des écheveaux élémentaires qui engendrent la catégorie des écheveaux : tout écheveau s'écrit à l'aide de compositions et de produits tensoriels (= juxtaposition) d'écheveaux élémentaires et d'endomorphismes d'identité.

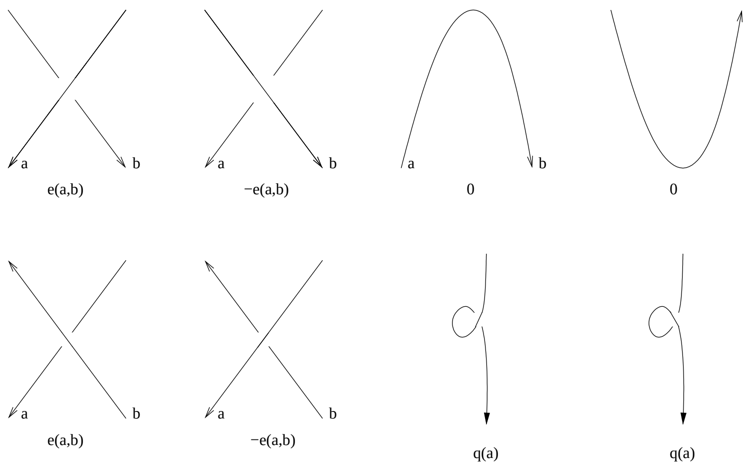

FIG. 2. Le codage des écheveaux élémentaires.

En particulier, un entrelacs muni d'un élément $g \in G$ s'identifie à un endomorphisme de l'objet vide. La valeur prise par le foncteur sur un tel entrelacs s'identifie donc à un élément de S. Il résulte des définitions et du lemme ci-dessus que si un entrelacs L s'écrit comme la juxtaposition $E \otimes E'$ (E' est juxtaposé à droite de E) de deux C-écheveaux E, E', alors $\mathcal{F}(L) = \mathcal{F}(E \otimes E') = \mathcal{F}(E) + \mathcal{F}(E')$. De même, $\mathcal{F}(E \circ E') = \mathcal{F}(E) + \mathcal{F}(E')$.

À partir de ces données, nous construisons un invariant $\tau(M, q)$ d'une 3-variété M qui généralise l'invariant de Murakami-Okada-Ohtsuki [26]. Cet invariant s'obtient également par une autre construction due à Viro à partir d'un 3-cocyle abélien au sens d'Eilenberg-McLane [27].

Soit M une 3-variété orientée compacte présentée par chirurgie sur un entrelacs $L = L_1 \cup \cdots L_n$ orienté parallélisé dans S^3. On note A la matrice d'enlacement de L, c'est-à-dire $A_{ij} = lk(L_i, L_j)$ où lk désigne le coefficient entier d'enlacement des composantes L_i et L_j dans S^3 (si $i = j$, $lk(L_i, L_i) = lk(L_i, L_i')$ où L_i' est un parallèle de L_i désigné à homotopie près par la parallélisation de L). Nous montrons dans [10] que les données ci-dessus déterminent $\tau(M, q)$ par la formule globale suivante :

$$\tau(M, q) = \gamma(q)^{-\sigma(A)} \gamma(q \otimes A) |H^1(M; G)|^{\frac{1}{2}}. \tag{8}$$

Remarques. Une condition nécessaire pour $\tau(M, q)$ soit nulle est que $q \otimes A$ soit dégénérée. Même si q et A sont non-dégénérés, le produit tensoriel $q \otimes A$ peut l'être.

Théorème 7 (Deloup, [10]). *$\tau(M, q)$ est nul si et seulement si λ_M possède un facteur orthogonal cyclique d'ordre une puissance de 2.*

La nullité de $\tau(M, q)$ intervient de façon cruciale dans l'extension de τ aux cobordismes. On ne peut donc pas régulariser $q \otimes A$ de façon à renormaliser la somme de Gauss dans $\tau(M, q)$ (et à assurer qu'elle soit non nulle).

À l'aide de la formule de réciprocité, nous explicitons $\tau(M, q)$ en termes de l'enlacement λ_M qui est intrinsèque à la 3-variété M :

Théorème 8 (Deloup, [10]). *Soit Q un raffinement quadratique homogène de l'enlacement λ_M sur Tors $H_1(M)$. Soit f un réseau muni d'une forme caractéristique $v = f(\tilde{v}, \cdot)$ présentant q via la construction du discriminant, i.e., tel que $\varphi_{f,c} = q$. Alors*

$$\tau(M, q) = \gamma(Q)^{-f(\tilde{v}, \tilde{v})} \gamma(Q \otimes f) |H^1(M; G)|^{\frac{1}{2}}. \tag{9}$$

Corollaire 8.1. *L'invariant τ se généralise aux $(4n-1)$-variétés (c'est-à-dire aux variétés closes dont l'enlacement est symétrique).*

À partir de la théorie de Minkowski–Burger, on déduit du Th. 8 :

Corollaire 8.2 (Deloup–Gille, [13]). *Deux $(4n-1)$-variétés closes orientées M et M' ont même premier nombre de Betti et enlacements isomorphes si et seulement si $\tau(M, q) = \tau(M', q)$ pour toutes les formes quadratiques $q : G \to \mathbb{Q}/\mathbb{Z}$.*

On peut remarquer que l'enlacement λ_M et la fonction quadratique q jouent des rôles symétriques du point de vue algébrique. Cette remarque peut s'exprimer de façon rigoureuse de la façon suivante. Notons $\mathfrak{M}$ le monoïde des enlacements et $\mathfrak{MQ}$ le monoïde des raffinements quadratiques. L'invariant τ peut s'interpréter comme essentiellement une forme bilinéaire $\langle\cdot,\cdot\rangle$ sur $\mathfrak{M} \times \mathfrak{MQ}$. Dans ce cadre, l'extension de l'invariant τ aux variétés munies de structures spinorielles est immédiate. En utilisant le fait qu'une structure spin σ sur une 3-variété définit un raffinement quadratique q_σ de l'enlacement (voir §5.1), si l'on note $\tau^{\mathrm{spin}}(M,\sigma)$ l'extension spinorielle, on démontre

Corollaire 8.3 (Deloup, [9])**.** *Soient (M,σ) et (M',σ') deux 3-variétés closes orientées munies de structures spin et ayant même nombre de Betti. Alors les raffinements quadratiques associées q_σ et $q_{\sigma'}$ sont isomorphes si et seulement si $\tau^{\mathrm{spin}}(M,\sigma) = \tau^{\mathrm{spin}}(M',\sigma')$.*

La généralisation la plus immédiate au cadre spin complexe ne donne pas le résultat escompté : l'invariant ne détermine alors plus le raffinement quadratique associé à isomorphisme près (voir [9]). L'explication tient au fait que le raffinement est non homogène en général. Il est en fait possible de définir une autre généralisation de τ pour les variétés munies d'un 1-cycle (voir aussi la section suivante). Cette généralisation-là classifie les (classes d'isomorphismes de) raffinements quadratiques associées aux structures spin complexes, [6].

4.2. **L'extension aux cobordismes.** Un cobordisme orienté est une variété X^{n+1} orientée telle que $\partial X = -A \coprod B$. Ainsi A et B sont deux n-variétés sans bord, spécifiées, appelées bases du cobordisme. On note le cobordisme $X = (X, A, B)$. Étant donnés deux cobordismes (X, A, B) et (Y, B, C), on sait les recoller le long de B, on obtient ainsi un cobordisme $(X \cup_B Y, A, C)$. L'idée conduit naturellement à la notion de catégorie de cobordisme, dont les objets sont les variétés orientées de dimension n (éventuellement munies de structures supplémentaires) et les morphismes sont les classes d'équivalence de cobordismes. On demande que l'équivalence soit donnée par des isomorphismes de cobordismes qui soient l'identité sur les bases. La composition est induite par le recollement. Le cylindre $A \times [0,1]$ représente l'identité. La catégorie possède une involution (renversement d'orientation) et les sommes finies (unions disjointes).

L'invariant τ admet une extension aux 3-cobordismes. Cette extension est très particulière : elle entre dans le cadre des théories topologiques quantiques des champs (TTQCs)[3]. Une TTQC est grosso modo un foncteur τ de la catégorie des cobordismes vers la catégories des morphismes d'espaces vectoriels (de dimension finie) avec certaines propriétés essentielles relatives à son comportement sur le recollement et l'union disjointe. Ce foncteur présuppose l'existence $\mathcal{T}$ d'un foncteur modulaire sur la catégorie des variétés closes vers la catégorie des espaces vectoriels de dimension finie, avec notamment la propriété caractéristique suivante :

$$\mathcal{T}(A \coprod B) = A \otimes B.$$

Cette propriété est à contraster avec l'additivité des foncteurs de nature homologique ou cohomologique. Le foncteur τ envoie un cobordisme sur un morphisme

[3]connues sous l'acronyme TQFT en anglais

d'espace vectoriel. La propriété clé est la multiplicativité projective de τ sur les compositions de cobordismes. Si $X \circ Y$ désigne la composition du cobordisme Y suivi de X, alors

$$\tau(X \circ Y) = \mu \cdot \tau(X) \circ \tau(Y), \text{pour } \mu \in \mathbb{C} - \{0\}.$$

L'élément μ est appelé l'anomalie de la TTQC.

Il résulte des travaux de Witten [35], ainsi qu'il est explicité dans [3], qu'il existe une procédure universelle pour étendre les invariants quantiques à des invariants de 3-cobordismes. Néanmoins cette construction n'est pas complètement explicite. De plus, même si elle peut être explicitée en dimension 3, elle requiert une combinatoire a priori extrinsèque à la variété, voir [8].

Dans un premier temps, on peut expliciter l'invariant obtenu en réalisant une présentation combinatoire du 3-cobordisme à l'aide d'écheveaux rubannés. Soit $M = (M, \Sigma_-, \Sigma_+)$ un 3-cobordisme orienté. Pour simplifier, supposons ici que les bases sont connexes. Pour chaque entier $g \geq 0$, on fixe un corps en anses orienté non noué de genre g, le corps en anses standard $\mathcal{H}_g$. On note g_+ et g_- les genres respectifs de Σ_+ et Σ_-. On choisit des paramétrisations (= homéomorphismes préservant l'orientation) $f_\pm : \partial\mathcal{H}_{g_\pm} \to \Sigma_\pm$. L'idée est la suivante : étant donné un tel cobordisme M, on peut « remplir » les bases de M, à savoir, considérer la 3-variété orientée *fermée* $\tilde{M} = \mathcal{H}_{g_-} \cup_{f_-} M \cup_{f_+} \mathcal{H}_{g_+}$, à l'aide des paramétrisations $f_\pm$.

Le cobordisme paramétré se décrit sur $\tilde{M}$ par un entrelacs orienté et parallélisé $L = L_+ \cup L_-$ où $L_\pm$ a $g_\pm$ composantes $L_1^\pm, \ldots, L_{g_\pm}^\pm$. Les composantes sont les images par $f_\pm$ des longitudes de $\partial\mathcal{H}_{g_\pm}$.

Les *couleurs* sont les éléments de $G^{g_\pm}$. Soit $\mathcal{T}(\Sigma_\pm)$ un espace vectoriel sur $\mathbb{C}$ librement engendré par $G^{g_\pm}$. Par définition, $\mathcal{T}(\Sigma_\pm)$ est de dimension $|G|^{g_\pm}$. À $(c_1^\pm, \ldots, c_{g_\pm}^\pm) \in G^{g_\pm}$, on associe un 1-cycle dans $\Sigma_\pm$ à coefficients dans G en définissant

$$\hat{\theta}_\pm = \sum_k c_k^\pm \otimes L_k^\pm.$$

Nous ferons l'abus de notation consistant à utiliser la même notation pour désigner l'image de ce 1-cycle dans $\tilde{M}$. On pose

$$\hat{\theta} = \hat{\theta}_+ - \hat{\theta}_-.$$

C'est un 1-cycle dans $\tilde{M}$ à coefficients dans G. Par la suite, quand nous aurons besoin de préciser que $\hat{\theta}$ dépend du choix des couleurs, on écrira $\hat{\theta}_{c+,c^-}$ au lieu de $\hat{\theta}$.

On considère à présent le couple $(\tilde{M}, L)$. Un isomorphisme entre deux couples $(\tilde{M}, L), (\tilde{M}', L')$ est un homeomorphisme $\phi : \tilde{M} \to \tilde{M}'$ tel que $\phi(L) = L'$. On définit alors un invariant de $(\tilde{M}, L)$ de la manière suivante. On présente $(\tilde{M}, L)$ par chirurgie, c'est-à-dire qu'on considère une paire ordonnée (J, J') d'entrelacs disjoints orientés parallélisés dans S^3 tels que $J = J_1 \cup \cdots \cup J_m$ est l'entrelacs de chirurgie : M s'obtient de $S^3 - J$ en recollant m tores solides, envoyant chaque méridien sur la courbe de chirurgie déterminée par la parallélisation sur chaque composante J_k de J, tandis que J' est un entrelacs à $g_+ + g_-$ composantes qui survit à la chirurgie et donne l'entrelacs L après que la chirurgie sur J a eu lieu.

On note $\sigma(J)$ la signature de la matrice d'enlacement de J dans S^3 et A la matrice d'enlacement de $J \cup J'$ dans S^3.

$$\text{(10)} \quad \tau(\tilde{M}, L, q, c_+, c_-) = \gamma(q)^{-\sigma(J)}|G|^{-m/2} \sum_{(x_1,\ldots,x_m)\in G^m} \exp(2\pi\sqrt{-1})\ (q \otimes A)(x_1, \ldots, x_m, c_1^+, \ldots, c_{g_+}^+, c_1^-, \ldots, c_{g_-}^-).$$

Il est démontré dans [8] que ce nombre est un invariant topologique du couple $(\tilde{M}, \hat{\theta}_{c^-,c^+})$. On peut donc le noter $\tau(\tilde{M}, \hat{\theta}_{c^-,c^+}, q)$. Chaque choix de couleurs c^-, c^+ donne lieu à un 1-cycle $\hat{\theta}_{c^-,c^+}$ (parallélisé). On est alors en mesure de définir $\tau(M) = \tau(M, \Sigma_+, \Sigma_-) : \mathcal{T}(\Sigma_-) \to \pm_+$ sous forme matricielle relativement aux bases données par les couleurs. Soit $\tau(M) = (\tau_{c^-,c^+})_{\substack{c^-\in G^{g_-}\\ c^+\in G^{g_+}}}$. Alors

$$\tau_{c^-,c^+} = |G|^{-g_+/2}\tau(\tilde{M}, \hat{\theta}_{c^-,c^+}, q).$$

Théorème 9 (Deloup, [8]). *La règle $\tau : (M, \Sigma_+, \Sigma_-) \mapsto \tau(M)$ définit une TTQC en dimension 3 dont l'anomalie est fonction de l'indice de Leray–Maslov.*

Cette présentation est spécifique à la dimension trois et n'est pas intrinsèque : elle fait appel à la variété auxilaire $\tilde{M}$ pour définir l'opérateur $\tau(M)$.

On va décrire ici comment construire $\tau(M)$ de façon plus explicite. Rappelons que chaque base est naturellement munie de sa forme d'intersection

$$H_1(\Sigma_\pm) \times H_1(\Sigma_\pm) \to \mathbb{Z}$$

qui est anti-symétrique. Les 1-cycles $L_j^\pm$ construits précédemment sur les bases de M déterminent des Lagrangiens $\Lambda_\pm$ dans $H_1(\Sigma_\pm)$. On se donne aussi des Lagrangiens supplémentaires $\Lambda'_\pm$ de sorte que

$$\Lambda_- \oplus \Lambda'_- = H_1(\Sigma_-), \quad \Lambda_+ \oplus \Lambda'_+ = H_1(\Sigma_+).$$

On remarque que

$$\ell = \Lambda_- \oplus \Lambda_+, \quad \ell' = \Lambda'_- \oplus \Lambda'_+$$

sont des lagrangiens supplémentaires de $H_1(\partial M) = H_1(\Sigma_-) \oplus H_1(\Sigma_+)$.

Soit A^-, resp. A^+, le groupe abélien libre engendré par $L_1^-, \ldots, L_{g_-}^-$, resp. par $L_1^+, \ldots, L_{g_+}^+$. On définit $\mathcal{T}(\Sigma_\pm) = \mathbb{C}[G \otimes A^\pm]$. Dans ce qui suit, on notera (abusivement) i_* tous les homomorphismes induits en homologie par les inclusions. Au lieu de considérer l'enlacement de la variété M, on considère l'enlacement induit sur Tors $H_1(M)/i_*(\ell')$. On le note encore λ_M.

Il est expliqué dans [11] que l'enlacement $e : G \times G \to \mathbb{Q}/\mathbb{Z}$ associé à q et l'enlacement λ_M déterminent un élément caractéristique χ d'ordre 2. Cet élément est nul si l'un des deux groupes G ou Tors $H_1(M)/i_*(\ell')$ est d'ordre impair. On peut alors décrire $\tau(M) : \mathbb{C}[G \otimes A^-] \to \mathbb{C}[G \otimes A^+]$ de la manière suivante. Il suffit de décrire l'action de $\tau(M)$ sur les éléments de la base, c'est-à-dire sur les éléments de $G \otimes A^-$. Par définition, un élément $x \in G \otimes A^-$ s'écrit $x = \sum_k g_k \otimes L_k^-$. On définit

$$\mathcal{H}(x) = \{y \in G \otimes A^+ \mid i_*(y) - i_*(x) = \chi\}.$$

En d'autres termes, un 1-cycle $\hat{y} \in G \otimes A^+$ est considéré comme un élément de $\mathcal{H}(x)$ si et seulement si lorsqu'une fois déformés dans M, les cycles x et y forment un élément homologue à l'élément caractéristique $\chi \in G \otimes H_1(M)/i_*(\ell')$. Soit $\hat{\chi}_{xy} = i_*(x) - i_*(y)$, c'est un 1-cycle à coefficients dans G représentant χ.

Soit maintenant (V, f, c) une présentation de q, i.e., $\varphi_{f,c} = q$ (construction du discriminant). On peut supposer que f est non-dégénérée, de sorte que l'extension de l'adjoint $\widehat{f_{\mathbb{Q}}} : V \otimes \mathbb{Q} \to \mathrm{Hom}(V \otimes \mathbb{Q}, \mathbb{Q})$ soit bijective. En particulier, $\widehat{f_{\mathbb{Q}}}$ envoie $V^\sharp$ sur $V^* = \mathrm{Hom}(V, \mathbb{Z})$. Par définition de la construction, le raffinement quadratique $\varphi_{f,c} : G \to \mathbb{Q}/\mathbb{Z}$ à se relève en une fonction quadratique $V^\sharp \to \mathbb{Q}$ que nous continuons de noter $\varphi_{f,c}$. L'application

$$\Phi_{f,c} = \varphi_{f,c} \circ \widehat{f_{\mathbb{Q}}}^{-1} : V^* \to \mathbb{Q}$$

est une fonction quadratique.

La présentation (V, f) détermine une résolution de G, i.e., on dispose de la suite exacte $0 \to V \xrightarrow{\widehat{f}} V^* \to G \to 0$. On peut alors relever à V^* les coefficients des cycles et des classes d'homologie à coefficients dans G. On conservera la même notation.

Soit maintenant un raffinement quadratique homogène q_M de l'enlacement λ_M. On note enfin $\mathrm{lk} = \mathrm{lk}_{\ell'}$ l'enlacement rationnel des *cycles* représentant des éléments de torsion dans $H_1(M)/i_*(\ell')$.

On définit le poids suivant :

$$C(M, \ell', q) = \gamma(f \otimes q_M + (\mathrm{id}_{V^*} \otimes \widehat{\lambda}_M)(\chi)) \cdot \left(\frac{|G \otimes \mathrm{Tors}\ (H_1(M)/i_*(\ell'))|}{|G \otimes \Lambda'_+|} \right)^{1/2}.$$

Dans le membre de droite ci-dessus, $f \otimes q_M : V \otimes \mathrm{Tors}\ (H_1(M)/i_*(\ell')) \to \mathbb{Q}/\mathbb{Z}$ est un raffinement quadratique de l'enlacement $f \otimes \lambda_M$. L'application $(\mathrm{id}_{V^*} \otimes \widehat{\lambda}_M)(\chi)$ est un homomorphisme $V \otimes \mathrm{Tors}\ (H_1(M)/i_*(\ell')) \to \mathbb{Q}/\mathbb{Z}$. Donc l'application $f \otimes q_M + (\mathrm{id}_{V^*} \otimes \widehat{\lambda}_M)(\chi)$ est bien un raffinement quadratique de $f \otimes \lambda_M$. Le fait que χ soit un élément caractéristique pour le couple (e, λ_M) assure que la somme de Gauss ci-dessus est toujours non nulle.

On définit alors

$$\tau(M)x = C(M, \ell', q) \cdot \sum_{y \in \mathcal{H}(x)} \exp\left(2\pi i (\Phi_{f,c} \otimes \mathrm{lk})(\hat{\chi}_{xy})\right) y \ \in \mathbb{C}[G \otimes A^+]$$

Théorème 10 (Deloup, [8])**.** *Les propriétés suivantes sont vérifiées par l'opérateur* $\tau(M)$ *:*

1. *L'opérateur* $\tau(M)$ *est indépendant du choix du raffinement quadratique de* q_M *sur* λ_M *et de la présentation de* (G, q)*. Il ne dépend que* ℓ'*,* λ_M *et* q*.*
2. *La règle* $(M, \Sigma_-, \Sigma_+) \mapsto \tau(M)$ *définit une TTQC en dimension 3. Avec le choix des lagrangiens induits par les paramétrisations des bases,* τ *coïncide avec la TTQC décrite précédemment.*

Du point de vue topologique, la TTQC ne dépend que de l'enlacement de la variété, modulo le choix des lagrangiens. Une conséquence est le

Corollaire 10.1. *La TTQC se généralise aux cobordismes de dimension* $4n-1$ *avec bases de dimension* $4n-2$*.*

En écrivant les éléments du groupe de difféotopie (mapping class group) $\mathfrak{M}_g$ de surfaces fermées comme des cylindres paramétrés, on montre que la représentation de $\mathfrak{M}_g$ (qui est toujours projective) est une version de la représentation de Shale-Weil [25]. Elle se factorise en particulier par le groupe métaplectique $\mathrm{Mp}(n, \mathbb{R})$ qui est le revêtement double du groupe symplectique $\mathrm{Sp}(n, \mathbb{R})$.

5. La théorie complexe spin des invariants de type fini

Un invariant de variété de dimension trois est de type fini s'il se comporte comme un polynôme relativement à une opération élémentaire fixée. Pour donner un sens précis à cette idée, commençons par rappeler une définition équivalente à la définition usuelle de polynôme. Soit T_u l'opérateur translation $P(X) \mapsto P(X+u)$ sur l'algèbre $\mathbb{R}[X]$ des polynômes. Une application $f : \mathbb{R} \to \mathbb{R}$ est polynômiale de degré $< n$ si et seulement si

$$\sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|} \left(\prod_{j \in J} T_{x_j} \right) f = 0 \quad \text{pour tout } (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

L'objet de cette section est d'étendre une théorie d'invariants de type fini, la théorie de Goussarov–Habiro, aux 3-variétés munies de structures spin complexes. Les structures spin complexes sont des structures de type homotopique apparaissant dans différents contextes, dont l'intérêt a été renouvelé en petites dimensions par la torsion de Reidemeister–Turaev (en dimension 3, avec l'identification des structures d'Euler avec les structures spin complexes) et les invariants de Seiberg-Witten (de variétés simplement connexes en dimension 4).

Soit M une variété close et orientée de dimension trois. Un *trèfle* G dans M est un plogement $G : F \to M$ d'une surface F obtenue comme épaississement d'un graphe en forme de Y auquel on a attaché à chaque sommet univalent une copie de S^1. La surface F est donc de genre 0 et possède quatre composantes de bord. Les cercles épaissis sont les *feuilles* de G et les sommets trivalents les *nœuds* de G. La trivialisation $j : H_3 \to M$ d'un voisinage régulier H_3 de G dans M, est un plongement, essentiellement unique, d'un corps en anses de genre 3 dans l'intérieur de M. Dans un corps en anses de genre 3 standard, soit L l'entrelacs parallélisé à six composantes représenté dans la figure suivante.

Notons $(H_3)_L$ le résultat d'une chirurgie sur L avec la convention usuelle de parallélisation. On pose

$$M_G = M - \mathrm{Int}(\mathrm{Im}\ j) \cup_{j|_{\partial H_3}} (H_3)_L.$$

Une telle chirurgie est appelée chirurgie *borroméenne*. La relation d'équivalence borroméenne est la relation d'équivalence sur les 3-variétés compactes orientées engendrées par les chirurgies borroméennes et les difféomorphismes préservant l'orientation.

Lemme 3 (Deloup-Massuyeau, [16]). *Il existe une bijection équivariante canonique* $\mathrm{Spin}^c(M) \to \mathrm{Spin}^c(M_G)$ *permettant d'étendre la chirurgie borroméenne aux 3-variétés compactes orientées munies de structures spin complexes.*

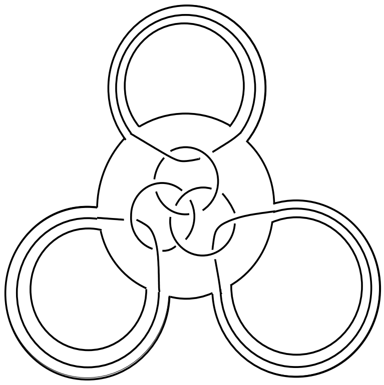

FIG. 3. L'entrelacs L. La parallélisation est induite par la convention que le premier vecteur rencontre l'oeil du lecteur.

En d'autres termes, étant donné une structure complexe spin $\alpha \in \mathrm{Spin}^c(M)$ et un trèfle G, on peut lui associer de façon non ambigue une variété M_G munie d'une structure complexe spin $\alpha_G \in \mathrm{Spin}^c(M_G)$. Il est géométriquement clair qu'étant donnés deux trèfles disjoints G_1, G_2 dans M, les variétés spin complexes $((M_{G_1})_{G_2}, (\alpha_{G_1})_{G_2})$ et $((M_{G_2})_{G_1}, (\alpha_{G_2})_{G_1})$ sont Spin^c-difféomorphes. Il y a donc un sens à définir $(M_{\mathcal{F}}, \alpha_{\mathcal{F}})$ pour une famille finie quelconque de trèfles deux à deux disjoints dans M. On est alors en mesure d'étendre la définition d'invariant de type fini aux structures spin complexes.

Soit f un invariant de structures spin complexes de 3-variétés, à valeurs dans un groupe abélien A. On dit que f est de degré au plus d si pour toute variété complexe spin (M, σ) et pour toute famille $\mathcal{F}$ d'au moins $d+1$ trèfles deux à deux disjoints dans M,

$$\sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|} f(M_{\mathcal{F}'}, \sigma_{\mathcal{F}'}) = 0.$$

En particulier, les invariants de degré 0 sont donc les invariants qui ne distinguent pas la chirurgie borroméenne. Ils sont explicités dans [16]. Le point clé est un plongement de $\mathrm{Spin}^c(M)$ dans un espace de fonctions quadratiques.

5.1. **Le plongement des structures spin complexes en dimension 3.** Soit M une variété orientée close de dimension trois. Une structure spin complexe σ sur M induit un raffinement quadratique φ_σ sur son enlacement $\lambda_M : H^1(M; \mathbb{Q}/\mathbb{Z}) \times H^1(M; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. [4] Notons $\mathrm{Spin}^c(M)$ l'ensemble des structures spin complexes de M et $\mathrm{Quad}(\lambda_M)$ l'ensemble des raffinements quadratiques de λ_M.

Théorème 11 (Deloup–Massuyeau, [16]). *Il existe un plongement canonique*

$$\mathrm{Spin}^c(M) \to \mathrm{Quad}(\lambda_M), \ \sigma \mapsto \varphi_\sigma.$$

[4] C'est un enlacement légèrement modifié par rapport à la définition usuelle (en particulier, $H^1(M; \mathbb{Q}/\mathbb{Z})$ est infini ssi $b_1(M) \geq 1$) ; il coincide avec l'enlacement précédemment défini dans l'introduction si M est une sphère d'homologie rationnelle.

Ce plongement est affine au-dessus de l'application

$$H^2(M) \to \mathrm{Hom}(H^1(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}), a \mapsto (a \cup \cdot)([M]).$$

De plus :

1. *L'image φ_σ d'une structure σ est homogène si et seulement si σ est induite (en général pas de façon injective) par une structure spin.*
2. *L'application φ est bijective si et seulement si M est une sphère d'homologie rationnelle.*

Ce plongement, ainsi que son conoyau, est explicitement décrit dans [16]. Il généralise un résultat du à C. Gille [19] et à l'auteur [11], tous deux obtenus par des méthodes sensiblement différentes. Il est le pendant géométrique du théorème 1. Il montre que les structures spin complexes sont *déterminées* par les raffinements quadratiques correspondants, contrairement à ce qui se passe pour les structures spin.

Corollaire 11.1. *Le raffinement quadratique φ_σ est dégénéré dès que $b_1(M) \geq 1$.*

À l'aide du théorème 1, on caractérise les invariants de degré 0 dans la théorie de Goussarov-Habiro pour les structures spin complexes.

Théorème 12 (Deloup-Massuyeau, [16])**.** *Tout invariant de degré* 0 *se déduit de la classe d'isomorphisme du raffinement quadratique associé à la structure spin complexe. De façon équivalente, deux structures spin complexes σ, σ' sur des 3-variétés M, M' sont équivalentes au sens borroméen si et seulement φ_σ et $\varphi_{\sigma'}$ sont isomorphes.*

5.2. **Applications à la torsion de Reidemeister–Turaev.** La torsion de - Reidemeister–Turaev est un invariant fondamental des structures spin complexes en dimension 3 [34] [28]. On se limite ici au cadre des 3-sphères d'homologie rationnelle orientées. Soit donc M^3 orientée, compacte, connexe telle que $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. Alors $H = H_1(M)$ est un groupe fini (traditionnellement noté multiplicativement). On suppose M munie d'une structure spin complexe σ. Dans ce cas, la torsion τ de Reidemeister–Turaev est un élément $\tau(M) \in \mathbb{Q}[H]$ qu'on peut écrire

$$\tau(M,\sigma) = \sum_{h \in H} \tau_\sigma(h) h \in \mathbb{Q}[H].$$

Rappelons que H agit transitivement et librement, via la dualité de Poincaré, sur $\mathrm{Spin}^c(M)$. Une propriété importante de τ est l'équivariance relativement à l'action de H :

$$h \cdot \tau(M,\sigma) = \tau(M, h \cdot \sigma), \quad h \in H. \tag{11}$$

Une autre propriété démontrée par Turaev est que τ détermine l'enlacement de M par la relation

$$\tau_\sigma(h_1 h_2) - \tau_\sigma(h_1) - \tau_\sigma(h_2) + \tau_\sigma(1) = -\lambda_M(h_1, h_2) \bmod \mathbb{Z}$$

pour tous $h_1, h_2 \in H$. En particulier, $\tau(M,\sigma)$ détermine également un raffinement quadratique q_σ de l'enlacement en posant $q_\sigma(h) = \tau_\sigma(1) - \tau_\sigma(h^{-1})$.

Il est naturel de comparer ce raffinement quadratique avec celui obtenu par le théorème 11.

Théorème 13 (Deloup–Massuyeau, [15])**.** *Pour toute sphère d'homologie rationnelle M munie d'une structure spin complexe, $q_\sigma = \varphi_\sigma$.*

Dans sa monographie [28], L. Nicolaescu a démontré le même résultat par une méthode analytique. Notre démonstration est purement topologique et combinatoire. Une conséquence immédiate du théorème 13 est le

Corollaire 13.1. *Le raffinement quadratique φ_σ est déterminé par $\tau(M,\sigma)$ mod 1.*

Mentionnons une conjecture qui a résisté jusqu'ici à nos efforts et qui permettrait de préciser le lien entre raffinement quadratique associé à une structure spin et l'invariant de Casson–Walker. Pour la motiver, commençons par observer que la réciproque du corollaire 13.1 n'est pas vérifiée. Soit maintenant

$$c_\sigma = \tau_\sigma(1) \bmod 1.$$

La torsion de Reidemeister–Turaev ayant la propriété d'équivariance $h \cdot \tau(M,\sigma) = \tau(M, h\cdot\sigma)$ pour tout $h \in H$, on en déduit que

$$c_{h\cdot\sigma} = c_\sigma - \varphi_\sigma(h).$$

Soit maintenant $d_\sigma = \frac{1}{2\pi}\arg\gamma(\varphi_\sigma)$. À l'aide de ce qui précède, on montre que le nombre $C(M) = d_\sigma - c_\sigma$ mod 1 est un invariant topologique de M, indépendant de la structure σ, additif sur les sommes connexes, s'annule sur les sphères d'homologie *entière* et change de signe sous le renversement de l'orientation.

Question. Est-ce que $C(M)$ coïncide avec $\frac{1}{|H|}\lambda_{CW}(M)$ modulo 1 où λ_{CW} désigne l'invariant de Casson-Walker (avec la normalisation de Lescop) des sphères d'homologie rationnelle ?

6. Perspectives

Nous réunissons dans cette section quelques thèmes de recherche autour des enlacements et des intersections que nous abordons dans nos travaux en cours (voir [6]).

6.1. **Classifications combinatoires.** Nous avons présenté dans §3 le monoïde des enlacements comme un sous-monoïde de tableaux admissibles. Si l'on se restreint aux p-groupes d'ordre fixé, un tableau admissible généralise une partition d'un entier fixé. Dans le cas impair, on obtient une partition signée (et le problème qui suit admet une solution triviale). Dans le cas $p = 2$, on définit l'application oubli qui à un tableau admissible associe sa partition. On peut aussi définir une application oubli – qui se factorise à travers la précédente – en associant à un tableau admissible T son « profil de rangs », c'est-à-dire les entiers k tels que $r_k(T) \neq 0$. Il est raisonnable d'espérer classifier le monoïde $\mathfrak{M}$ à partir des fibres génériques de l'une ou l'autre de ces applications.

Présentons à présent une question un peu plus spéculative. Il est bien connu que l'ensemble Λ_n des partitions (ordonnées) d'un entier n est en correspondance bijective avec l'ensemble des classes de conjugaison du groupe symétrique S_n qui sont elles-mêmes en correspondance bijective avec les représentations irréductibles de S_n. Soit

$\mathfrak{T}_n^{\mathrm{adm}}$ l'ensemble des tableaux admissibles $T = (r, s)$ tels que $n = \sum_k r(k)$. Cet objet s'interprète-t-il comme l'ensemble des représentations d'un groupe généralisant le groupe symétrique ? Les représentations irréductibles s'identifieraient alors aux tableaux admissibles. Question : généraliser la combinatoire des tableaux de Young aux tableaux admissibles.

6.2. **Généralisations.** On peut proposer une généralisation non abélienne de la théorie topologique quantiques des champs présentée dans §4. En principe, on remplace le groupe abélien G de la théorie par un groupe non abélien. Par quoi généraliser l'enlacement sur G ? Dans [30], il est essentiellement remplacé par un 3-cocycle « quasi-abélien » sur G. Ce 3-cocycle s'interprète (Street, Turaev) dans le cadre d'une généralisation des catégories monoïdales tressées, autorisant les isomorphismes d'associativité. M. Sokolov a proposé une interprétation en calcul d'écheveau (skein calculus) de l'invariant τ, qui admet elle aussi une généralisation naturelle.

Une généralisation dans une autre direction consiste à considérer des revêtements de variétés et d'y étudier les raffinements de l'enlacement. L'idée est de prendre en compte une action de groupe π. La symétrie de l'enlacement est alors à entendre comme le fait d'être hermitien relativement à l'involution induite par $G \to G,\ g \mapsto g^{-1}$.

6.3. **Structures spin complexes.** Le théorème de plongement affirme que toute structure spin complexe σ sur une 3-variété M compacte orientée s'interprète comme un raffinement quadratique φ_σ de l'enlacement λ_M. Supposons $b_1(M) \geq 1$. Si l'on munit $\mathrm{Quad}(\lambda_M)$ de la topologie naturelle, modelée sur celle de $H^1(M; \mathbb{Q}/\mathbb{Z})^* = \mathrm{Hom}(H^1(M; \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, alors l'image de φ_σ est *dense* dans $\mathrm{Quad}(\lambda_M)$. L'invariant $\tau(M, \sigma)$, ou plus généralement les invariants de type fini décrits dans la section précédente, s'interprètent donc comme étant associés à un raffinement dans l'image de φ. Il est raisonnable de les généraliser à un raffinement quadratique *quelconque* de λ_M. Cela suggère d'introduire une notion raisonnable de limite d'invariants de type fini.

6.4. **Application aux 4-cobordismes.** Il s'agit de décrire les intersections positives de 4-variétés lisses bordées par des 3-sphères d'homologie rationnelle. Le point de départ est une interprétation topologique simple du corollaire 2.1 :

Théorème 14. *Soit X une 3-sphère d'homologie rationnelle et soit f une présentation sur réseau de son enlacement. Alors il existe une 4-variété lisse simplement connexe Y telle que $\partial Y = X$ et dont la forme d'intersection est $f \oplus g$ où g est unimodulaire.*

Un résultat profond de Ozsváth-Szabó affirme que l'invariant $\gamma(\varphi_\sigma)$ associé à une structure spin^c d'une 3-sphère d'homologie rationnelle admet une *extension rationnelle* :

Théorème 15. *Soit M une 3-sphère d'homologie rationnelle et $\sigma \in \mathrm{Spin}^c(M)$. Il existe un invariant $d_\sigma \in \mathbb{Q}$ de Spin^c 3-cobordisme de (M, σ) tel que*

$$d_\sigma = \frac{1}{2\pi} \mathrm{Arg}(\gamma(\varphi_\sigma)) \bmod \mathbb{Z}.$$

De plus, si X est une 4-variété lisse simplement connexe définie positive munie d'une structure spin^c θ telle que $\theta|_M = \sigma$, alors

$$d_\sigma \leq \frac{c_1(\theta)^2 - \mathrm{rk}(H^2(X))}{4},$$

où $c_1(\theta)$ est la classe de Chern associée à θ (elle correspond à la forme caractéristique introduite en §2).

On utilise ici une normalisation différente de la normalisation originale.

En l'associant au théorème 14 et à ses raffinements, ce résultat permet d'obtenir des obstructions sur la forme d'intersection d'une 4-variété lisse simplement connexe bordée par une 3-sphère d'homologie dont l'enlacement est fixé. De façon un peu plus précise, nous pouvons énoncer la conjecture suivante.

Conjecture. Soit X une 4-variété lisse simplement connexe compacte de bord M une 3-sphère d'homologie rationnelle. Notons i_M l'intersection algébrique sur $H_2(M)$ et λ_M l'enlacement sur Tors $H_1(M)$. Supposons i_M définie positive. Alors il existe un nombre fini de rationnels $r_1 < \ldots < r_n$ et de formes d'intersections algébriques $f_1, \ldots, f_n$ au-dessus de λ_M (au sens du discriminant, c'est-à-dire que $\lambda_M = L_{f_i}$ pour tout $1 \leq i \leq n$) tels que

1. $\min\limits_\sigma d_\sigma \leq r_n$ avec égalité si et seulement si i_M est isomorphe à f_n ;
2. De plus, si $r_j < \min\limits_\sigma d_\sigma < r_{j+1}$ $(1 \leq j \leq n-1)$, alors il existe un nombre fini de classes d'isomorphisme pour i_M ;
3. Si $\min\limits_\sigma d_\sigma = r_j$ $(1 \leq j \leq n-1)$, alors i_M est isomorphe à f_j.

Le minimum ci-dessus est pris sur toutes les structures spin complexe.

La conjecture ci-dessus peut sembler a priori contraire à l'intuition, dans la mesure où l'on ne s'attend pas normalement à ce qu'il existe des formes d'intersection privilégiées au-dessus d'une forme d'enlacement fixée. Mais l'invariant d_σ encrypte des restrictions très fortes, de nature arithmétique.

Nous ne préciserons pas ici les questions arithmétiques que cette conjecture soulève ; cependant, cette conjecture généraliserait le théorème de Donaldson qui affirme qu'une 4-variété lisse compacte sans bord a ou bien une forme d'intersection non définie ou bien une forme d'intersection définie diagonalisable. (Il correspond au cas où $n = 1$ et $r_1 = 0$ ci-dessus.)

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An explicit construction of an Abelian topological quantum field theory in dimension 3

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Abstract

A geometrical description of an elementary topological quantum field theory based on the linking pairing is given, based on a talk given at the PIMS/Summer 1999 Workshop on invariants of 3-manifolds and topological quantum field theories at Nakoda Lodge, Alberta, Canada.
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1. Introduction

The goal of this note is to define a topological quantum field theory (tqft) in dimension 3 and to describe it explicitly in terms of familiar topological invariants. Tqfts were introduced in 1988 by Witten [12], using Feynman path integrals, who discovered an intimate relationship between the Jones polynomial and gauge theory. Roughly speaking, a tqft can be described as a functor from the category of oriented cobordisms to the category of vector spaces. Here the tqft in question is elementary: it depends on a quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$ on a finite Abelian group and is topologically determined by homotopical invariants (Betti numbers and certain linking pairings). Precisely because of its elementary nature, the tqft can be expressed intrinsically.

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This work is the natural continuation of our study of the invariant τ in [3,4,6]. A certain degree of familiarity with those papers will be helpful, notably in Section 3 and Section 4 and the proof of Theorem 2, but most of (the ideas in) this note can be read independently.

Plan of the paper. First, we give a reasonably self-contained account of the definition of topological quantum field theories (tqft) (Section 2.1) and discuss the extension of quantum invariants to tqfts (Section 2.2). Second, we proceed in Section 3 to a preliminary description of the tqft associated to the invariant τ studied in [3]. Finally, we give an explicit description of the tqft in terms of linking pairings and Lagrangians (Section 4).

2. Construction of tqft: Generalities

In this section, we give a brief definition of tqft, avoiding technicalities, which will be sufficient for the purpose of this paper. However, we point out to the reader that these technicalities are necessary for the coherence of the general theory (to encompass the most general situations that tqft's deal with). For further details, the reader is referred to [1,9,2].

2.1. Axioms

Following [9, III], we define a modular functor on oriented closed d-manifolds as a covariant functor $\mathcal{T}$ from the category of oriented closed d-manifolds (possibly endowed with some extra structure) to the category of finite-dimensional vector spaces, defined as follows. To each (structure preserving) homeomorphism $f:\Sigma \to \Sigma'$ between oriented closed d-manifolds, is assigned a $\mathbb{C}$-linear isomorphism $f_\#:\mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ between finite-dimensional vector spaces over $\mathbb{C}$. The $\mathbb{C}$-vector space $\mathcal{T}(\Sigma)$ is called the space of states of Σ. The functor $\mathcal{T}$ is required to be multiplicative with respect to disjoint union: if $f:\Sigma_0 \to \Sigma_0'$ and $g:\Sigma_1 \to \Sigma_1'$ are two (structure preserving) homeomorphisms, the isomorphisms $(f \cup g)_\#:\mathcal{T}(\Sigma_0 \cup \Sigma_1) \to \mathcal{T}(\Sigma_0' \cup \Sigma_1')$ and $f_\# \otimes g_\#:\mathcal{T}(\Sigma_0) \otimes \mathcal{T}(\Sigma_1) \to \mathcal{T}(\Sigma_0') \otimes \mathcal{T}(\Sigma_1')$ obtained by applying the functor $\mathcal{T}$ are identified. It is required that for any oriented closed d-manifold Σ, there is a non-degenerate bilinear pairing $d_\Sigma:\mathcal{T}(\Sigma) \times \mathcal{T}(-\Sigma) \to \mathbb{C}$. The family of pairings $\{d_\Sigma\}_\Sigma$ is natural with respect to disjoint union and (structure preserving) homeomorphisms. Furthermore, $\mathcal{T}$ satisfies a number of natural conditions related to disjoint union. Finally, one usually finds convenient to add a normalization condition: $\mathcal{T}(\emptyset) = \mathbb{C}$.

Any oriented $(n+1)$-manifold M with boundary decomposed as $\partial M = -A \coprod B$ is called an oriented cobordism from A to B and is denoted (M, A, B). Here A and B are oriented n-manifolds and $-A$ denotes A with reversed orientation. Given two cobordisms (M, A, B) and (N, B, C), one can glue them together along B to obtain a cobordism from A to C. This naturally leads to the notion of cobordism category, whose objects are oriented n-manifolds (possibly endowed with some extra structure) and whose morphisms are equivalent classes of cobordisms. (Here the equivalence is given by isomorphisms of cobordism, being the identity on the bases.) Composition is defined by gluing. The cylinder $A \times [0,1]$ represents the identity. This category has an involution (orientation reversal) and finite sums (disjoint unions).

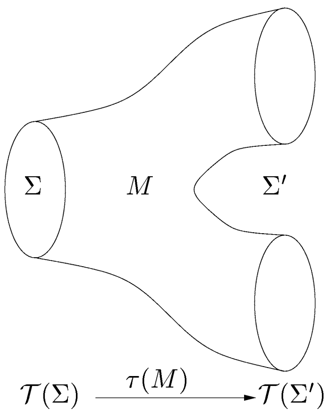

Fig. 1. Typical picture of a tqft (here in dimension $1 + 1$).

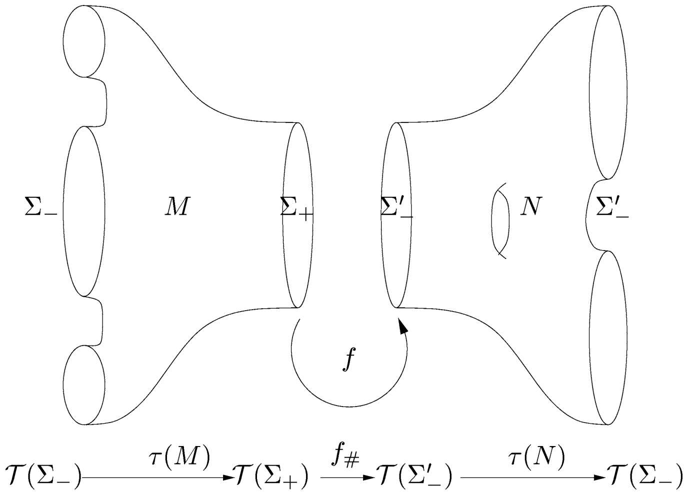

Fig. 2. The gluing axiom.

A tqft in dimension $(n + 1)$ with underlying modular functor $\mathcal{T}$ in dimension n is a map τ which associates to each cobordism (M, Σ, Σ') a $\mathbb{C}$-linear map $\tau(M) : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$, subject to the following four axioms. The first axiom describes how τ behaves on disjoint unions of two cobordisms M and N:

(i) Multiplicativity on disjoint union: $\tau(M \cup N) = \tau(M) \otimes \tau(N)$.

This axiom should be constrasted to the behaviour of homology functors. The second axiom describes the behaviour of τ under the gluing of two cobordisms (M, Σ_-, Σ_+) and $(N, \Sigma'_-, \Sigma'_+)$ along a homeomorphism $f : \Sigma_+ \to \Sigma'_-$ (preserving the extra structures). Denote by $N_f M$ the resulting cobordism.

(ii) Gluing axiom: there exists $k \in \mathbb{C} \setminus \{0\}$ such that $\tau(N_f M) = k \cdot \tau(N) \circ f_\# \circ \tau(M)$.

The number k depends on M, N and f. It is called the gluing anomaly of (M, N, f). The map $f_\# : \mathcal{T}(\Sigma_+) \to \mathcal{T}(\Sigma'_-)$ is induced by f (using the modular functor τ which is an integral part of the definition of τ). If k can always be taken to be 1, then the tqft is said to be anomaly free.

The third axiom describes how τ behaves on a (structure preserving) homeomorphism $f:(M,\Sigma_-,\Sigma_+)\to(N,\Sigma'_-,\Sigma'_+)$ of cobordisms.

(iii) Naturality axiom: the square

$$\begin{array}{ccc} \mathcal{T}(\Sigma_-) & \xrightarrow{\tau(M)} & \mathcal{T}(\Sigma_+) \\ \downarrow{\scriptstyle (f|_{\Sigma_-})_\#} & & \downarrow{\scriptstyle (f|_{\Sigma_+})_\#} \\ \mathcal{T}(\Sigma'_-) & \xrightarrow[\tau(N)]{} & \mathcal{T}(\Sigma'_+) \end{array}$$

is commutative.

It is convenient to add a fourth axiom which fixes the value of τ on the cylinder:

(iv) Normalization axiom: for any closed oriented d-manifold Σ, $\tau(\Sigma\times[0,1],\Sigma\times 0,\Sigma\times 1)=\mathrm{Id}_{\mathcal{T}(\Sigma)}$.

By definition, a tqft yields a topological invariant of $(d+1)$-cobordisms (under homeomorphisms (preserving structures) which are the identity on the bases). If M is any oriented $(d+1)$-manifold, M can be seen as the cobordism $(M,\emptyset,\partial M)$, so $\tau(M):\mathcal{T}(\emptyset)\to\mathcal{T}(\partial M)$ is determined by its value on $1\in\mathcal{T}(\emptyset)=\mathbb{C}$. So identifying $\tau(M)=\tau(M)(1)$, we have: $\tau(M)\in\mathcal{T}(\partial M)$. It is easy to see from the axioms that this identification is natural with respect to homeomorphisms and disjoint union. It is also a consequence of the axioms that $\mathcal{T}(-\Sigma)=\mathcal{T}(\Sigma)^*=\mathrm{Hom}(\mathcal{T}(\Sigma),\mathbb{C})$ for any closed oriented d-manifold. If the tqft τ is anomaly free, then τ is a functor from the category of cobordisms to the category of vector spaces (the gluing axiom reduces to functoriality). We recover in this fashion Atiyah's original axiomatic presentation of tqft's.

2.2. Extension of quantum invariants to tqfts

Given a $\mathbb{C}$-valued topological invariant τ_0 of closed d-manifolds, can it be extended to a tqft in dimension d? Under two easy-to-check conditions (described below), the answer is almost affirmative. Let us say that a weak tqft τ is the same object as a tqft, except that τ does not necessarily satisfy the first axiom (i). A $\mathbb{C}$-valued invariant τ_0 of closed manifolds is involutive if $\tau_0(-M)=\overline{\tau_0(M)}$ (bar denotes complex conjugation here). Assume that τ_0 is involutive and multiplicative under disjoint union (some authors take this to be the definition of a quantum invariant, while others require various additional conditions). Then there exists a unique extension τ of τ_0 to a weak (anomaly free) tqft. The construction is described in [2]. Very briefly: for a closed $(d-1)$-manifold Σ, define $T(\Sigma)$ to be the $\mathbb{C}$-vector space whose basis consists of all d-manifolds M whose boundary is homeomorphic to Σ (the homeomorphism should preserve the additional structures on Σ). Define a bilinear pairing $T(\Sigma)\otimes T(-\Sigma)\to\mathbb{C}$ by

$$\langle M,N\rangle=\tau_0(M\cup_\Sigma N)\in\mathbb{C}.$$

Then $\mathcal{T}(\Sigma)$ is the quotient of $T(\Sigma)$ by the left annihilator of $\langle,\rangle$ which induces a non-degenerate bilinear pairing $d_\Sigma=\langle,\rangle_\Sigma$. Now if M is a cobordism between Σ_1 and Σ_2, then $\tau(M):\mathcal{T}(\Sigma_1) \to \mathcal{T}(\Sigma_2)$ is defined as follows. Let $[N]$ be a generator of $\mathcal{T}(\Sigma_1)$. Then

$$\tau(M)[N] = [N \cup_{\Sigma_1} M] \in \mathcal{T}(\Sigma_2).$$

It is easy to check that τ satisfies all the axioms of a weak anomaly free tqft. However, in general, one may require axiom (i) to hold as well as finite-dimensionality of the spaces of states $\mathcal{T}(\Sigma)$. To this end, an additional sufficient condition is required in [9, III §4]: each closed d-manifold Σ has a so-called splitting system with respect to τ_0. Roughly speaking, if a closed d-manifold M is obtained by gluing two manifolds N_1 and N_2 along their common boundary Σ, $\tau_0(M)$ can be evaluated using a finite number of closed d-manifolds $V_k \cup_\Sigma N_2$, $N_1 \cup_\Sigma W_k$. Here V_k and W_k are d-manifolds with boundary Σ and $-\Sigma$ respectively (they form the splitting system associated to Σ). This condition seems hard to check in practice on a candidate invariant τ_0.

As a conclusion, even if there exists a universal procedure to extend quantum invariants to tqfts, we note that the procedure is not a completely explicit one. It is difficult in general to describe in geometrical terms what the tqft obtained from a quantum invariant looks like.

Therefore, to describe explicitly the extension of the topological invariant of closed 3-manifolds introduced in [3] to a full tqft, we have to resort to other techniques. We do this in Section 3. In the next section, we give a preliminary construction, which is rather specific to dimension 3.

3. Preliminary construction of a tqft τ in dimension 3

We will work in this section in the category of oriented 3-cobordisms with parametrized boundaries (bases). From the Abelian quantum invariant τ described in [4], we construct a tqft in dimension $2+1$. In dimensions 2 and 3, there is a general procedure, for constructing a tqft from a quantum topological invariant. This is done by gluing "standard" manifolds $\mathcal{H}_-, \mathcal{H}_+$ to a cobordism (M, Σ_-, Σ_+) to obtain a closed manifold $\widetilde{M}$ (in general equipped with extra structure) on which we can evaluate the invariant. However, one needs to keep track of the parametrizations used to perform the gluings. The reason why this procedure works is that in low dimensions, we can choose canonical manifolds with prescribed boundaries. See Fig. 3.

Let $M = (M, \Sigma_-, \Sigma_+)$ be an oriented 3-cobordism. First, we treat the case when the bases Σ_- and Σ_+ are connected. For each non-negative integer g, we fix an unknotted oriented handlebody $\mathcal{H}_g$ of genus g, called the standard handlebody of genus g. Denote by g_- (respectively g_+) the genus of Σ_- (respectively Σ_+). We say M has parametrized boundaries if it is equipped with (orientation preserving) homeomorphisms $f_-:\partial\mathcal{H}_{g_-} \to \Sigma_-$ and $f_+:\partial\mathcal{H}_{g_+} \to \Sigma_+$.

Given such a 3-cobordism M, we can "fill in" the bases of M, i.e., form the closed oriented 3-manifold $\widetilde{M} = \mathcal{H}_{g_-} \cup_{f_-} M \cup_{f_+} \mathcal{H}_{g_+}$ using the parametrization maps f_- and f_+ as gluing maps.

Each surface $\partial\mathcal{H}_g$ comes equipped with a set of (isotopy classes of) longitudes $l_1, \ldots, l_g$ and a set of (isotopy classes of) meridians $m_1, \ldots, m_g$. Orient them in way compatible

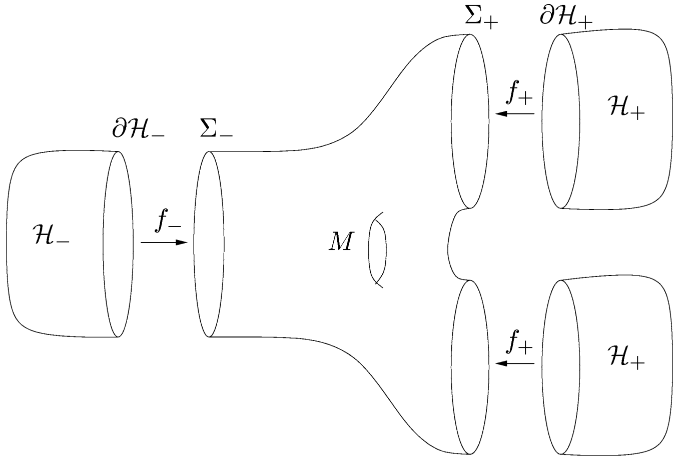

Fig. 3. Filling in a cobordism (here in dimension $1+1$): $\widetilde{M} = \mathcal{H}_- \cup_{f_-} M \cup_{f_+} \mathcal{H}_+$.

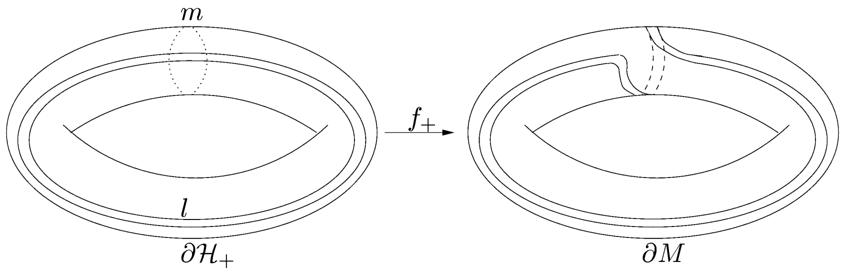

Fig. 4. The parametrization induces a framing on longitudes.

with the orientation of $\partial\mathcal{H}_g$. That is, at the unique point x of (transversal) intersection of m_k and l_k, the orientation of the direct sum of $T_x m_k$ and $T_x l_k$ is the same as that of $T_x\partial\mathcal{H}_g$. Denote by $L_1^-, \ldots, L_{g_-}^-$ (respectively by $L_1^+, \ldots, L_{g_+}^+$) the images $f_-(l_1), \ldots, f_-(l_{g_-})$ in Σ_- (respectively the images $f_+(l_1), \ldots, f_+(l_{g_+})$ in Σ_+). The parametrizations f_- and f_+ induce a framing on each L_j^- $(1 \leqslant j \leqslant g_-)$ and L_k^+ $(1 \leqslant k \leqslant g_+)$, respectively. (Just take the image of a parallel.) By a framing on a longitude, we mean a parallel of it, or equivalently, a non-vanishing vector field on it. See Fig. 4 for an example when $\partial M = \Sigma_+$ is an unknotted torus and the homeomorphism f_+ is isotopic to a Dehn twist about the meridian m.

Therefore, $\widetilde{M}$ comes equipped with an oriented and framed link $L = L^- \cup L^+$ where $L^- = \bigcup_{k=1}^{g_-} L_k^-$ and $L^+ = \bigcup_{k=1}^{g_+} L_k^+$. Thus L has $g_- + g_+$ components. Note that all components may be linked with each other.

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic form on a finite Abelian group G. We denote by $\gamma(G, q)$ the following Gauss sum

$$\gamma(G,q) = \left|\ker \hat{b}_q\right|^{-1/2} |G|^{-1/2} \sum_{x\in G} e^{2\pi i q(x)} \in \{0\} \cup \left\{z \in \mathbb{C},\ z^8 = 1\right\}. \tag{1}$$

Let $\mathcal{T}(\Sigma_-)$ (respectively $\mathcal{T}(\Sigma_+)$) be a vector space V_{g_-} (respectively V_{g_+}) over $\mathbb{C}$ of dimension $|G|^{g_-}$ (respectively $|G|^{g_+}$). Fix a base of $\mathcal{T}(\Sigma_-)$ (respectively $\mathcal{T}(\Sigma_+)$) indexed by elements of G^{g_-} (respectively of G^{g_+}). These elements are called colors. For $c^- = (c_1^-, \dots, c_{g_-}^-) \in G^{g_-}$ (respectively for $c^+ = (c_1^+, \dots, c_{g_+}^+) \in G^{g_+}$, we define a 1-cycle in Σ_- (respectively in Σ_+) with coefficients in G by $\hat{\theta}_- = \sum_k c_k^- \otimes L_k^-$ (respectively by $\hat{\theta}_+ = \sum_k c_k^+ \otimes L_k^+$). We use the same notation for those 1-cycles viewed in $\widetilde{M}$. We set $\hat{\theta} = \hat{\theta}_+ - \hat{\theta}_-$ (a 1-cycle in $\widetilde{M}$). If we need to emphasize that $\hat{\theta}$ depends on the colors chosen, we write $\hat{\theta}_{c^-,c^+}$ instead of $\hat{\theta}$.

We now recall the quantum topological invariant τ of links in oriented closed 3-manifolds introduced in [4]. By a surgery presentation of $(\widetilde{M}, L)$, we mean a pair (J, J') of disjoint oriented and framed links in S^3 such that

- $J = J_1 \cup \cdots \cup J_m$ is the surgery link: M is obtained from $S^3 \setminus J$ by gluing m solid tori, sending each meridian to the surgery curve determined by the framing on each component J_k of J.
- $J' = J_{m+1} \cup \cdots \cup J_{m+n}$ $(n = g_+ + g_-)$ yields the link L in M after the surgery on J is performed.

We denote by $A = (a_{jk})_{1 \leqslant j,k \leqslant m+n}$ the linking matrix of $J \cup J'$, that is, the $(m+n) \times (m+n)$ matrix of integers defined as follows: $a_{jk} = \mathrm{lk}(J_j, J_k)$ is the linking number of J_j and J_k in S^3 if $j \neq k$ and a_{jj} is the framing number of J_j. Denote by A_J the $m \times m$ submatrix of A given by $A_J = (a_{jk})_{1 \leqslant j,k \leqslant m}$. Let $\sigma(J)$ denote the signature of (the real symmetric bilinear form induced by) A_J.

Assume that $\gamma(G, q) \neq 0$ (a condition always satisfied if q is non-degenerate). Let $c = (c^+, -c^-) \in G^{g_+ + g_-} = G^n$. We define

$$\tau(\widetilde{M}, L; G, q, c) = \gamma(G, q)^{-\sigma(L')}\big(|G|\big|\ker \hat{b}_q\big|\big)^{-m/2} \times \sum_{(x_1,\dots,x_m) \in G^m} \mathrm{e}^{2\pi \mathrm{i}(q \otimes A)(x_1,\dots,x_m,c_1^+,\dots,c_{g_+}^+,-c_1^-,\dots,-c_{g_-})}. \tag{2}$$

(The tensor product $q \otimes A$ is defined in [4].) In this setting, Theorem 1 in [4] reads:

Theorem 1. *The number $\tau(\widetilde{M}, L; G, q, c)$ is a topological invariant of the pair $(\widetilde{M}, \hat{\theta}_{c^-,c^+})$.*

Therefore we denote it by $\tau(\widetilde{M}, \hat{\theta}_{c^-,c^+}; q)$ in the sequel. Each choice of colors c^+, c^- gives rise to a framed 1-cycle $\hat{\theta}_{c^-,c^+}$. We are now ready to define the $\mathbb{C}$-linear map $\tau(M) = \tau(M, \Sigma_-, \Sigma_+) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$ in matrix form, i.e., $\tau(M) = (\tau_{c^-,c^+})_{\substack{c^- \in G^{g_-} \\ c^+ \in G^{g_+}}}$ as follows

$$\tau_{c^-,c^+} = |G|^{-g_+/2} \cdot \tau\big(\widetilde{M}, \hat{\theta}_{c^-,c^+}; q\big). \tag{3}$$

Now we treat the case when M has possibly non-connected bases. Instead of having one genus for the bottom base and one genus for the top base, we have multiple genera $g_1^-, \dots, g_{r_-}^-$ for the bottom base Σ_- and $g_1^+, \dots, g_{r_+}^+$ for the top base Σ_+. Here r_-, r_+ denote the number of components of Σ_-, Σ_+ respectively. We set $\mathcal{T}(\Sigma_-) = \bigotimes_{k=1}^{r_-} V_{g_k^-}$,

$\mathcal{T}(\Sigma_+) = \bigotimes_{k=1}^{r_+} V_{g_k^+}$. As above, we present τ as a matrix τ_{c^-,c^+}. Now $c^- \in G^{g_1^-} \times \cdots \times G^{g_{r_-}^-}$, i.e., runs over sequences $c_1^- \in G^{g_1^-}, \ldots, c_{r_-}^- \in G^{g_{r_-}^-}$ and $c^+ \in G^{g_1^+} \times \cdots \times G^{g_{r_+}^+}$, i.e., runs over sequences $c_1^- \in G^{g_1^+}, \ldots, c_{r_+}^+ \in G^{g_{r_+}^+}$. We fill in the bases of M in the same way as above, using the parametrizations for each connected component of Σ_- and Σ_+, respectively. Call $\widetilde{M}$ the resulting closed oriented 3-manifold. As before, $\widetilde{M}$ comes equipped with an oriented and framed link $L = L^- \cup L^+$, the only difference being that now L^- is a link with $\sum_{1\leqslant j\leqslant r_-} g_j^-$ components and L^+ is a link with $\sum_{1\leqslant j\leqslant r_-} g_j^-$ components. Fix indices c^-, c^+. Each component of Σ_- (respectively of Σ_+) will contribute a 1-cycle in $\widetilde{M}$ with coefficients in G as above; call $\hat{\theta}_-$ (respectively $\hat{\theta}_+$) the sum of these 1-cycles. As before, set $\hat{\theta} = \hat{\theta}_+ - \hat{\theta}_-$. Then we define

$$\tau_{c^-,c^+} = |G|^{-\frac{1}{2}\sum_{1\leqslant j\leqslant r_+} g_j^+} \cdot \tau\big(\widetilde{M}, \hat{\theta}_{c^-,c^+}; q\big). \tag{4}$$

Theorem 2. *The assignment $\tau : (M, \Sigma_-, \Sigma_+) \mapsto \tau(M)$ defines a tqft in dimension $2+1$.*

Proof (*Sketch*)**.** Verification of the axioms (i), (iii) and (iv) is straightforward from the definition of τ in [4]. The only contribution of τ which does not behave multiplicatively (with respect to the gluing axiom (iii)) is a finite Gauss sum $\gamma(G, q)^{-\sigma(L)}$ where $\sigma(L)$ is the signature of the linking matrix of the surgery link J in S^3 as above. On the other hand, $-\sigma(L)$ can be seen as the signature of a smooth simply-connected 4-manifold X bounded by $\widetilde{M}$ (it is the 4-manifold obtained from the 4-ball B^4 by attaching 2-handles along the components of the surgery link $J \subset S^3 = \partial B^4$); therefore the gluing axiom (iii) can be checked using Wall's signature formula [11]. (The anomaly can be expressed as a certain Leray–Maslov index modulo 8.) □

This presentation, specific to dimension 3, is not completely intrinsic, in the sense that we had to resort to $\widetilde{M}$ in order to define $\tau(M)$. In other words, $\tau(M)$ is computed inside the closed 3-manifold $\widetilde{M}$ rather than inside M.

4. An explicit construction of τ

Let $M = (M, \Sigma_-, \Sigma_+)$ be a connected compact oriented 3-cobordism. As Σ_- is a closed 2-manifold, $H_1(\Sigma_-)$ carries an antisymmetric bilinear pairing, which can be defined using Poincaré duality and the cup product in cohomology:

$$H^1(\Sigma_-) \times H^1(\Sigma_-) \to \mathbb{Z}.$$

Similarly for Σ_+. Therefore, there is a well defined notion of isotropic subgroups and Lagrangians in $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$, respectively. For example, recall the definition of a Lagrangian in $H_1(\Sigma_-)$. Let A be a subgroup of $H_1(\Sigma_-)$. First define the orthogonal $A^\perp$ of A by $A^\perp = \{h \in H_1(\Sigma_-), h \cdot A = 0\}$ where dot denotes the antisymmetric pairing on $H_1(\Sigma_-)$. We say that A is *totally isotropic* if $A \subset A^\perp$; we say that A is a *Lagrangian* of $H_1(\Sigma_-)$ if A is a maximal totally isotropic subgroup, that is, if $A = A^\perp$. Clearly, if A

is Lagrangian then A contains $H_1(\Sigma_-)^\perp$ which may be non-trivial as the pairing may be degenerate.

We shall assume that M is equipped with distinguished framed 1-cycles $L_1^-, \ldots, L_{g_-}^- \subset \Sigma_-$ and $L_1^+, \ldots, L_{g_+}^+ \subset \Sigma_+$, respectively, generating Lagrangians λ_- and λ_+ in $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$. Furthermore, we fix Lagrangians $\lambda'_- \subset H_1(\Sigma_-)$ and $\lambda'_+ \subset H_1(\Sigma_+)$ such that

$$\lambda_- \oplus \lambda'_- = H_1(\Sigma_-), \qquad \lambda_+ \oplus \lambda'_+ = H_1(\Sigma_+).$$

We say that the framed cycles and Lagrangians above form a *parametrized Lagrangian structure* of M. Note that, as Σ_- and Σ_+ are disjoint,

$$\lambda = \lambda_- \oplus \lambda_+ \quad \text{and} \quad \lambda' = \lambda'_- \oplus \lambda'_+ \tag{5}$$

are complimentary Lagrangians of $H_1(\partial M) = H_1(-\Sigma_-) \oplus H_1(\Sigma_+)$.

Lemma 1. *Let A be an Abelian group equipped with an antisymmetric bilinear pairing and let $\lambda \subset A$ be a Lagrangian. Then there exists a Lagrangian $\lambda' \subset A$ such that $\lambda \oplus \lambda' = A$.*

Example. If $A = H_1(\Sigma)$ where Σ is a *parametrized* closed connected surface in ∂M such that λ is generated by the images of longitudes, then we can simply take λ' to be the subgroup generated by images of meridians.

Remark. The pairing is not assumed to be non-degenerate in Lemma 1.

Proof. Choose a totally isotropic subgroup λ', maximal among those with the property that $\lambda \cap \lambda' = 0$. Then $\lambda + \lambda'^\perp = \lambda^\perp + \lambda'^\perp = (\lambda \cap \lambda')^\perp = A$. By definition, $\lambda' \subset \lambda'^\perp$. To show the reverse inclusion, assume that there is $x \in \lambda'^\perp$ such that $x \notin \lambda'$. Then the subgroup generated by λ' and x is totally isotropic (because $\lambda'^\perp \subset \lambda'^{\perp\perp}$) and does not intersect λ non-trivially. Thus λ' is not maximal. □

Lemma 2. *Any connected oriented 3-cobordism (M, Σ_-, Σ_+) can be equipped with a parametrized Lagrangian structure.*

Proof. The existence of Lagrangians is clear from the discussion above and Lemma 1. Choose systems of generators for the Lagrangians and then 1-cycle representatives. Choosing a framing for a cycle $c \in \Sigma_+$ amounts to choosing a section of the normal bundle to c in Σ_+, which is always possible in this dimension (1). □

As in the previous section, we have at our disposal a finite Abelian group G (equipped with a quadratic form q). Denote by A^- (respectively A^+) the free Abelian group generated by $L_1^-, \ldots, L_{g_-}^-$ (respectively by $L_1^+, \ldots, L_{g_+}^+$). We define $\mathcal{T}(\Sigma_-) = \mathbb{C}[G \otimes A^-]$ and $\mathcal{T}(\Sigma_+) = \mathbb{C}[G \otimes A^+]$. Simply put, elements in $\mathcal{T}(\Sigma_-)$ are formal combinations with coefficients in $\mathbb{C}$ of elements in $G \otimes A^-$. As vector spaces over $\mathbb{C}$, $\mathcal{T}(\Sigma_-)$ and $\mathcal{T}(\Sigma_+)$ have dimension $|G|^{g_-}$ and $|G|^{g_+}$, respectively. (The algebra structure will not be used.)

In the remainder of this section, we denote by i_* inclusion homomorphisms. The context should make clear which inclusion we are referring to. Recall that any closed oriented 3-manifold M carries a non-degenerate symmetric bilinear pairing $\mathcal{L}_M : \operatorname{Tors} H_1(M) \times$

$\operatorname{Tors} H_1(M) \to \mathbb{Q}/\mathbb{Z}$ defined by $\mathcal{L}_M(x, y) = \tilde{x} \cdot y \in \mathbb{Q}/\mathbb{Z}$, where $\tilde{x} \in H_1(M; \mathbb{Q}/\mathbb{Z})$ is a lift of x. If M is not closed, one can still define a symmetric bilinear pairing as follows. Because λ' is a Lagrangian of $H_1(\partial M)$, the torsion subgroup of $H_1(M)/i_*(\lambda')$ carries a non-degenerate symmetric bilinear pairing, which we denote also $\mathcal{L}_M$. For example, see [8, §6] for a proof of this well-known fact.

The quadratic form q on G gives rise to a symmetric bilinear pairing $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$ by the formula $b_q(x, y) = q(x + y) - q(x) - q(y),\ x, y \in G$. To the two non-degenerate pairings $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\mathcal{L}_M : \operatorname{Tors}(H_1(M)/i_*(\lambda')) \times \operatorname{Tors}(H_1(M)/i_*(\lambda')) \to \mathbb{Q}/\mathbb{Z}$, we can associate a characteristic element $\chi \in G \otimes \operatorname{Tors}(H_1(M)/i_*(\lambda'))$ by [4, §1.3]. χ is of order dividing 2 and is zero if the order of G or $\operatorname{Tors}(H_1(M)/i_*(\lambda'))$ is odd. See [4] for details.

We now proceed to the definition of $\tau(M) : \mathbb{C}[G \otimes A^-] \to \mathbb{C}[G \otimes A^+]$. Since $\tau(M)$ is linear, it suffices to describe the action of $\tau(M)$ on the elements of the basis, that is, on elements in $G \otimes A^-$. By definition, an element $\hat{x} \in G \otimes A^-$ can be written as $\hat{x} = \sum_k g_k \otimes L_k^-$. We define

$$\mathcal{H}(x) = \left\{ \hat{y} \in G \otimes A^+,\ i_*(y) - i_*(x) = \chi \right\}.$$

In other words, a 1-cycle $\hat{y} \in G \otimes A^+$ can be considered to be an element in $\mathcal{H}(x)$ if and only if when it is deformed so that it lies in M, the cycles $\hat{y}$ and $\hat{x}$ are homologous to the characteristic element $\chi \in G \otimes H_1(M)/i_*(\lambda')$. If $\hat{y} \in \mathcal{H}(x)$, we denote by $\hat{\chi}_{xy}$ the 1-cycle $i_*\hat{y} - i_*\hat{x}$ in M (with coefficients in G), which is a cycle representating χ.

An even presentation (V, f) of (G, q) (cf. [4]) consists of a resolution of G

$$0 \longrightarrow V \xrightarrow{\hat{f}} V^* \longrightarrow G \longrightarrow 0$$

with the following properties:

(1) V is a finitely generated free Abelian group and $V^* = \operatorname{Hom}(V, \mathbb{Z})$.
(2) $\hat{f} : V \to V^*$ is the adjoint map of a symmetric bilinear pairing $f : V \times V \to \mathbb{Z}$ satisfying $f(v, v) = 0 \bmod 2$ for all $v \in V$. (Such a pairing is called an even pairing.)

To describe the last property, let $f_{\mathbb{Q}} : (V \otimes \mathbb{Q})^2 \to \mathbb{Q}$ be the rational extension of f and let $\hat{f}_{\mathbb{Q}} : V \otimes \mathbb{Q} \to \operatorname{Hom}(V \otimes \mathbb{Q}, \mathbb{Q})$ be the adjoint map. Define a symmetric bilinear pairing $b^f : V^* \times V^* \to \mathbb{Q}$ by $b^f(x, y) = f_{\mathbb{Q}}(\hat{f}_{\mathbb{Q}}^{-1}(x), \hat{f}_{\mathbb{Q}}^{-1}(y))$. Since $\hat{f}$ is injective, $\hat{f}_{\mathbb{Q}}$ is bijective and hence b^f is well-defined. We now state the third property:

(3) The pairing b^f induces a map $G \times G \to \mathbb{Q}/\mathbb{Z}$ such that $q(x) = \frac{1}{2} b^f(x, x) \bmod 1$.

In fact, (2) is equivalent to (3) for the right-hand side of the equality in (3) is well defined if and only if f is even. Hence, if (V, f) is an even presentation of (G, q), q can be reconstructed from f by the formula above. Conversely, any (homogeneous) quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$ has such an even presentation [10].

Once an even presentation (V, f) for (G, q) is fixed, we find it convenient in the sequel to adopt the following notation. A tilde will always denote a fixed lift with respect to the projection $V^* \to G$ (i.e., a set-theoretic section). Hence $\widetilde{\chi} \in V^* \otimes H_1(M)/\lambda'$ is obtained from χ by lifting the coefficients that lie in G to V^*. Similarly, $\widetilde{\chi}_{xy}$ is the 1-cycle obtained from χ_{xy} by lifting the coefficients that lie G to V^* of the cycle $\widehat{\chi}_{xy}$ (using the same lift). Let q_M be any quadratic form over $\mathcal{L}_M$. Set

$$C\big(M, \lambda'; \widetilde{\chi}\big) = \gamma\big(V \otimes \operatorname{Tors}\big(H_1(M)/i_*(\lambda')\big), f \otimes q_M + \big(1_{V^*} \otimes \widehat{\mathcal{L}}_M\big)(\widetilde{\chi})\big) \times \left(\frac{|G \otimes (H_1(M)/i_*(\lambda'))|}{|G \otimes \lambda'_+|}\right)^{1/2}.$$

Recall that lk denotes linking number (for cycles representing torsion elements in $H_1(M)$). Define

$$\tau(M)\hat{x} = C\big(M, \lambda'; \widetilde{\chi}\big) \sum_{\hat{y} \in \mathcal{H}(x)} \exp\big(\mathrm{i}\pi \big(b^f \otimes \mathrm{lk}\big)\big(\widetilde{\chi}_{xy}\big)\big)\hat{y}. \tag{6}$$

Lemma 3. *The map $\tau(M)$ defined by the formula above is independent of the choice of q_M over $\mathcal{L}_M$ and independent of the choice of the lift over G. In particular, $\tau(M)$ does not depend on the even presentation (V, f) of (G, q).*

Proof. First, since $(f \otimes q_M)(u \otimes v) = \frac{1}{2} f(u,u) \cdot \mathcal{L}_M(v,v)$ (because f is even), the formula above only depends on $\mathcal{L}_M$, rather than q_M. Secondly, if $\widetilde{\chi}$ and $\widetilde{\chi}'$ are two lifts of χ, then $\widetilde{\chi} - \widetilde{\chi}' \in \ker(1_{V^*} \otimes \widetilde{\mathcal{L}}_M)$. Thus $C(M, \lambda'; \widetilde{\chi}) = C(M, \lambda'; \widetilde{\chi}')$. Similarly a change of lift only affects the coefficients of $\widetilde{\chi}_{xy}$ by adding elements in the image of $\hat{f} : V \to V^*$. Hence, by definition of b^f and since f is even, a change of lift does not affect $(b^f \otimes lk)(\widetilde{\chi}_{xy})$ modulo 2. Thus $\tau(M)$ is invariant under a change of lift, which is the desired result. □

We now state our main result.

Theorem 3. *The assignment $(M, \Sigma_-, \Sigma_+) \mapsto \tau(M)$ defines a tqft in dimension 3. If the choice of Lagrangians is induced by parametrizations of the boundaries, then τ coincides with the tqft described in Theorem 2, Section 3.*

Proof. First, we work in the category of oriented 3-cobordisms with parametrized boundaries. For simplicity, we shall assume that all boundaries are connected, leaving to the reader the modifications for the general case. Theorem 1 provides us with a tqft τ, which to a 3-cobordism (M, Σ_-, Σ_+) associates a $\mathbb{C}$-linear map $\tau(M) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$. The vector space $\mathcal{T}(\Sigma_-)$ can be identified to $\mathbb{C}[G \otimes A^-]$ once its basis is chosen to consist of elements of $G \otimes A^-$. Similarly for $\mathcal{T}(\Sigma_+)$. We now have to identify the map itself. First, we identify $H_1(\widetilde{M})$. Recall that $\widetilde{M}$ is the closed 3-manifold obtained by gluing to M two handlebodies of genus g_- and g_+ respectively whose boundaries coincide with Σ_- and Σ_+ respectively. Let λ'_- be the Lagrangian of $H_1(\Sigma_-)$ generated in homology by $f_-(m_1), \ldots, f_-(m_{g_-})$ (the images in Σ_- of the meridians of the standard handlebody of genus g_- under the parametrization f_-). Similarly, let λ'_+ be the Lagrangian of $H_1(\Sigma_+)$ generated in homology by $f_+(m_1), \ldots, f_+(m_{g_+})$ (the images in Σ_+ of the meridians of the standard handlebody of genus g_+ under the parametrization f_+). Then $\lambda' = \lambda'_- \oplus \lambda'_+$ is a Lagrangian in $H_1(\partial M) = H_1(-\Sigma_-) \oplus H_1(\Sigma_+)$. A Mayer–Vietoris argument shows that $H_1(\widetilde{M}) = H_1(M)/i_*(\lambda')$. So the linking pairing on the torsion subgroup of $H_1(\widetilde{M})$ can be considered to be a pairing on the torsion subgroup of $H_1(M)/i_*(\lambda')$. For the same reason, the 1-cycle θ in $\widetilde{M}$ with coefficients in G can be viewed as a 1-cycle in M such that $\theta \in G \otimes H_1(M)/i_*(\lambda')$. Consider formula (3). We apply [4, Corollary 3.2] and find that

$$\begin{aligned}\tau\big(\widetilde{M}, \hat{\theta}; q\big) &= \mathrm{e}^{\mathrm{i}\pi(b^f \otimes lk)(\hat{\eta})} \cdot \gamma\big(V \otimes \operatorname{Tors} H_1\big(\widetilde{M}\big), \tfrac{1}{2} f \otimes \mathcal{L}_{\widetilde{M}} \\ &\qquad + \big(1_{V^*} \otimes \widehat{\mathcal{L}}_{\widetilde{M}}\big)(\eta)\big) \cdot \big|H^1(M; G)\big|^{1/2} \\ &= \mathrm{e}^{\mathrm{i}\pi(b^f \otimes lk)(\hat{\eta})} \cdot \gamma\big(V \otimes \operatorname{Tors} H_1(M)/i_*(\lambda'), \tfrac{1}{2} f \otimes \mathcal{L}_M \\ &\qquad + \big(1_{V^*} \otimes \widehat{\mathcal{L}}_M\big)(\eta)\big) \cdot \big|G \otimes H_1(M)/i_*(\lambda')\big|^{1/2}.\end{aligned}$$

Here $\hat{\eta}$ is a 1-cycle obtained by lifting the coefficients of $\hat{\theta}$ to V^*. According to [4, Theorem 4], the quantity above is nonzero if and only if θ is the characteristic element in $G \otimes H_1(\widetilde{M}) = G \otimes H_1(M)/i_*(\lambda')$ (associated to b_q and $\mathcal{L}_M$), which we denote by χ. Recall now that θ depends on the colors and therefore on both elements chosen in $G \otimes A^-$ and in $G \otimes A^+$ respectively, namely $\theta = \theta_+ - \theta_-$. Viewing $\hat{\theta}$ as a 1-cycle in M, we can rewrite $\hat{\theta} = i_*(\hat{y}) - i_*(\hat{x}) = \widehat{\chi}_{xy}$, with $\hat{x} \in G \otimes A^-$ and $\hat{y} \in G \otimes A^+$. The factor $|G \otimes \lambda'_+|^{-1/2}$ is clearly equal to $|G|^{-g_+/2}$. Thus, evaluating τ on an element $x \in G \otimes A^-$, we obtain exactly formula (6). At this stage, it is clear that our verification depends on the parametrizations of boundaries only to the extent of distinguishing framed cycles and Lagrangians, i.e., of inducing a Lagrangian structure. Conversely, any Lagrangian in $H_1(\Sigma_-)$ can be realized as the image $(f_-)_*(L)$ of a fixed Lagrangian L in $H_1(\Sigma)$ where Σ is a fixed surface parametrizing Σ_- (by f_-). Similarly for Lagrangians of $H_1(\Sigma_+)$. (Here we use the fact that the Lagrangians are integral.) This finishes the proof. □

Remark. In fact, a direct verification (without appealing to parametrizations) of the gluing axiom for the tqft τ seems possible and should lead to tqfts in higher dimensions. This is currently investigated in [5].

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Reidemeister–Turaev torsion modulo one of rational homology three–spheres

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Abstract

Given an oriented rational homology 3–sphere M, it is known how to associate to any Spin^c–structure σ on M two quadratic functions over the linking pairing. One quadratic function is derived from the reduction modulo 1 of the Reidemeister–Turaev torsion of (M, σ), while the other one can be defined using the intersection pairing of an appropriate compact oriented 4–manifold with boundary M.

In this paper, using surgery presentations of the manifold M, we prove that those two quadratic functions coincide. Our proof relies on the comparison between two distinct combinatorial descriptions of Spin^c–structures on M: Turaev's charges vs Chern vectors.

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1 Introduction and statement of the result

1.1 Introduction

Any closed oriented 3–manifold M can be equipped with a *complex spin structure*, or *Spin^c–structure*. While they seem to have been originally introduced in the '50s and '60s [5], in the framework of Dirac operators and K–theory [8], the revival of interest in Spin^c–structures over the last decade is certainly due to symplectic geometry and Seiberg–Witten invariants of 4–manifolds. For a general introduction to Spin^c–structures, the reader is referred to [8]. It was observed somewhat more recently [16] that, in dimension 3, Spin^c–structures have a simple and natural interpretation: any Spin^c–structure on a closed oriented 3–manifold M can be represented by a nowhere vanishing vector field on M. This enabled Turaev to reinterpret a topological invariant of Euler structures on 3–manifolds, which he had introduced earlier, as an invariant of Spin^c–structures. Since this invariant is a refinement of the Reidemeister torsion, we call this invariant the Reidemeister–Turaev torsion.

We will be interested in the restriction of this invariant to the class of rational homology 3–spheres. Our work is motivated by and based on two observations.

- On the one hand, there is the following special feature of the Reidemeister–Turaev torsion $\tau_{M,\sigma}$ of an oriented rational homology 3–sphere M with a Spin^c–structure σ: its reduction modulo 1 induces a quadratic function $q_{M,\sigma}$ over the linking pairing λ_M [19].
- On the other hand, there is a canonical bijective correspondence, denoted by $\sigma \mapsto \phi_{M,\sigma}$, between Spin^c–structures on M and quadratic functions over the linking pairing λ_M [10, 4, 2]. The quadratic function $\phi_{M,\sigma}$ can be defined, extrinsically, using the intersection pairing of a compact oriented 4–manifold with boundary M and first Betti number equal to zero.

Thus, the question naturally arises to compare the quadratic functions $q_{M,\sigma}$ and $\phi_{M,\sigma}$.

1.2 Statement of the result

Let us begin by developing the above two observations and fixing some notations.

The Reidemeister–Turaev torsion of a closed oriented 3–manifold equipped with a Spin^c–structure is a fundamental topological invariant. A concise and almost self-contained introduction is [14]. A broader introduction is [17], while the monographs [11, 19] contain the most recent developments. We give here a succinct presentation sufficient for our purpose.

Let M be a connected oriented 3–manifold, compact without boundary. All homology and cohomology groups will be with integral coefficients unless explicity stated otherwise. We set $H = H_1(M)$, the first homology group, written multiplicatively. Let $Q(H)$ denote the classical ring of fractions of the group ring $\mathbb{Z}[H]$. The *maximal Abelian Reidemeister torsion* $\tau(M)$ of M is an element in $Q(H)$ defined up to multiplication by an element of $\pm H \subset Q(H)$. This invariant, defined in [13], can be thought of as a generalization of the Alexander polynomial. Next, its indeterminacy in $\pm H$ can be disposed of by specifying two extra structures: a homology orientation of M and an Euler structure of M (see [15]). On the one hand, using the intersection pairing, the choosen orientation of M induces a canonical homology orientation. On the other hand, the Euler structures on M, defined as punctured homotopy classes of nowhere vanishing vector fields on M, are in canonical bijective correspondence with the Spin^c–structures on M [16]. Therefore, if (M, σ) is a connected closed Spin^c–manifold of dimension 3, one can define its *Reidemeister–Turaev torsion*

$$\tau(M, \sigma) \in Q(H).$$

It has the following equivariance property:

$$\forall h \in H, \quad h \cdot \tau(M, \sigma) = \tau(M, h \cdot \sigma) \in Q(H). \tag{1.1}$$

Here, the left hand side involves a multiplication in $Q(H)$ while, in the right hand side, $h \cdot \sigma$ involves the free and transitive action of $H^2(M)$ (or $H_1(M)$ via Poincaré duality) on the set $\text{Spin}^c(M)$: see, eg, [8].

Now and throughout the paper, **we assume that M is an oriented rational homology 3–sphere**, ie, we suppose that

$$H_*(M; \mathbb{Q}) = H_*\left(\mathbf{S}^3; \mathbb{Q}\right).$$

Then H is finite and $Q(H) = \mathbb{Q}[H]$. Hence $\tau(M, \sigma)$ determines a function $\tau_\sigma : H \to \mathbb{Q}$ such that

$$\tau(M, \sigma) = \sum_{h \in H} \tau_\sigma(h) \cdot h \in \mathbb{Q}[H].$$

It has been proved in [16, Theorem 4.3.1] that the modulo 1 reduction of the function τ_σ satisfies the property that

$$\forall h_1, h_2 \in H, \quad \tau_\sigma(h_1 h_2) - \tau_\sigma(h_1) - \tau_\sigma(h_2) + \tau_\sigma(1) = -\lambda_M(h_1, h_2) \mod 1. \tag{1.2}$$

Here, $\lambda_M : H \times H \to \mathbb{Q}/\mathbb{Z}$ denotes the *linking pairing* of M: this is a symmetric nondegenerate bilinear pairing, which gives partial information on the way knots are linked in the manifold M [12]. It immediately follows from (1.2) that

$$\forall h \in H, \quad \tau_\sigma(h) = \tau_\sigma(1) - q_{M,\sigma}\left(h^{-1}\right) \bmod 1,$$

where $q_{M,\sigma}$ is a *quadratic function over* the linking pairing λ_M, in the sense that it satisfies the following property:

$$\forall h, k \in H, \quad q_{M,\sigma}(hk) - q_{M,\sigma}(h) - q_{M,\sigma}(k) = \lambda_M(h,k).$$

It is also easily seen from (1.1) and (1.2) that

$$\forall h \in H, \quad q_{M,h\cdot\sigma} = q_{M,\sigma} + \lambda_M\left(h, -\right). \tag{1.3}$$

This equation suggests to define the following free transitive action of the group H on the set $\mathrm{Quad}(\lambda_M)$ of quadratic functions over λ_M:

$$H \times \mathrm{Quad}(\lambda_M) \to \mathrm{Quad}(\lambda_M), \ (h,q) \mapsto h \cdot q$$

where

$$\forall x \in H, \ (h \cdot q)(x) = q(x) + \lambda_M\left(h, x\right).$$

On the other hand, it is known [10, 4, 2] (see [3] for arbitrary closed oriented 3–manifolds) how to define another bijective H–equivariant correspondence

$$\mathrm{Spin}^c(M) \to \mathrm{Quad}(\lambda_M), \ \sigma \mapsto \phi_{M,\sigma}.$$

This map is defined combinatorially, starting from a surgery presentation of the manifold M and using its linking matrix. (The detailed construction will be recalled in subsection 2.4.)

Theorem *For any oriented rational homology 3–sphere M equipped with a Spin^c–structure σ, the quadratic functions $q_{M,\sigma}$ and $\phi_{M,\sigma}$ are equal.*

In his monograph [11], Nicolaescu has proved the same result, with an analytic proof based on the connection between the Reidemeister–Turaev torsion and the Seiberg–Witten invariant. Our proof is combinatorial and purely topological. A surgery presentation of M provides two combinatorial descriptions of Spin^c–structures on M. One description (called *charges*) is defined by Turaev in [18] in terms of the complement in $\mathbf{S}^3$ of the framed surgery link, and is used there to compute $\tau(M,\sigma)$. Another description (called *Chern vectors*) relies on the 4–manifold with boundary M associated to the surgery presentation, and is well suited for the computation of $\phi_{M,\sigma}$. Our main contribution consists in comparing those two descriptions of Spin^c–structures.

Before going into the proof of the Theorem, let us discuss the following immediate consequence.

Corollary *The quadratic function* $\phi_{M,\sigma}$ *is determined by* $\tau(M,\sigma)$ mod 1.

We claim that the converse of the Corollary does not hold. To justify this, define the "constant"

$$c_\sigma = \tau_\sigma(1) \bmod 1.$$

From (1.1), we obtain that

$$\forall h \in H, \quad c_{h\cdot\sigma} = c_\sigma - \phi_{M,\sigma}(h)\,. \tag{1.4}$$

Let also $d_\sigma \in \mathbb{R}/\mathbb{Z}$ be such that

$$\exp(2i\pi\ d_\sigma) = \frac{1}{\sqrt{|H|}} \cdot \sum_{x\in H} \exp(2i\pi\ \phi_{M,\sigma}(x)) \in \mathbb{C}.$$

Since $\phi_{M,\sigma}$ is nondegenerate, the Gauss sum on the right hand side is well-known to be a complex number of modulus 1. It can also be proved that $d_\sigma \in \mathbb{Q}/\mathbb{Z}$. Observe that

$$d_{h\cdot\sigma} = d_\sigma - \phi_{M,\sigma}(h)\,. \tag{1.5}$$

As an immediate consequence of (1.4) and (1.5), we obtain the following

Proposition *The number* $c(M) = c_\sigma - d_\sigma \in \mathbb{Q}/\mathbb{Z}$ *is a topological invariant of the oriented rational homology* 3*–sphere* M.

Explicit computations can be performed on the lens spaces. For instance, we find that $8c(L(7,1)) = 3/7 \neq 2/7 = 8c(L(7,2))$; since $L(7,1)$ and $L(7,2)$ have isomorphic linking pairings, we deduce that $c(M)$ can not be computed from $\phi_{M,\sigma}$.

It is not difficult to verify that $c(M)$ is additive under connected sums, vanishes if M is an integer homology 3–sphere and changes sign when the orientation of M is reversed. Let $\lambda(M) \in \mathbb{Q}$ denote the Casson-Walker invariant of M in Lescop's normalization [9]. We ask the following

Question Does the invariant $c(M) \in \mathbb{Q}/\mathbb{Z}$ coincide with $-\lambda(M)/|H|$ mod 1?

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2 Chern vectors and charges

This section contains preliminary material for the proof of the Theorem (Section 3). The heart of this section is devoted to the presentation of two equivalent, but distinct, combinatorial descriptions of complex spin structures on M. The proof of this equivalence will be given in Section 3. Even though we shall not need it, note that subsections 2.1, 2.2 and 2.3 are valid for *any* closed oriented connected 3–manifold (ie, with arbitrary first Betti number).

As a convention, boundaries of oriented manifolds will be always given orientation by the "outward normal vector first" rule.

2.1 Surgery presentation

In this paragraph and throughout Section 2, we fix an ordered oriented framed n–component link L in $\mathbf{S}^3$, such that the oriented 3–manifold V_L obtained from $\mathbf{S}^3$ by surgery along L is diffeomorphic to our oriented rational homology 3–sphere M.

Let $b_{ij} = \mathrm{lk}_{\mathbf{S}^3}(L_i, L_j)$ for all $1 \leq i \neq j \leq n$, and let b_{ii} be the framing number of L_i for all $1 \leq i \leq n$. We denote by $B_L = (b_{ij})_{i,j=1,\ldots,n}$ the linking matrix of L in $\mathbf{S}^3$. We also denote by W_L the *trace* of the surgery. In other words,

$$M = V_L = \partial W_L \quad \text{with} \quad W_L = \mathbf{D}^4 \cup \bigcup_{i=1}^{n} \left(\mathbf{D}^2 \times \mathbf{D}^2\right)_i,$$

where the 2–handle $\left(\mathbf{D}^2 \times \mathbf{D}^2\right)_i$ is attached by embedding $-\left(\mathbf{S}^1 \times \mathbf{D}^2\right)_i$ into $\mathbf{S}^3 = \partial \mathbf{D}^4$ in accordance with the specified framing and orientation of L_i. The group $H_2(W_L)$ is free Abelian of rank n. It is given the *preferred* basis $([S_1], \ldots, [S_n])$. Here, the closed surface S_i is taken to be

$$S_i = \left(\mathbf{D}^2 \times 0\right)_i \cup (-\Sigma_i),$$

where Σ_i is a Seifert surface for L_i in $\mathbf{S}^3$ which has been pushed into the interior of $\mathbf{D}^4$ as shown in Figure 2.1. Also, $H^2(W_L)$ will be identified with $\mathrm{Hom}(H_2(W_L), \mathbb{Z})$ by Kronecker evaluation, and will be given the dual basis. Note that the matrix of the intersection pairing $\bullet : H_2(W_L) \times H_2(W_L) \to \mathbb{Z}$ relatively to the preferred basis of $H_2(W_L)$ is B_L.

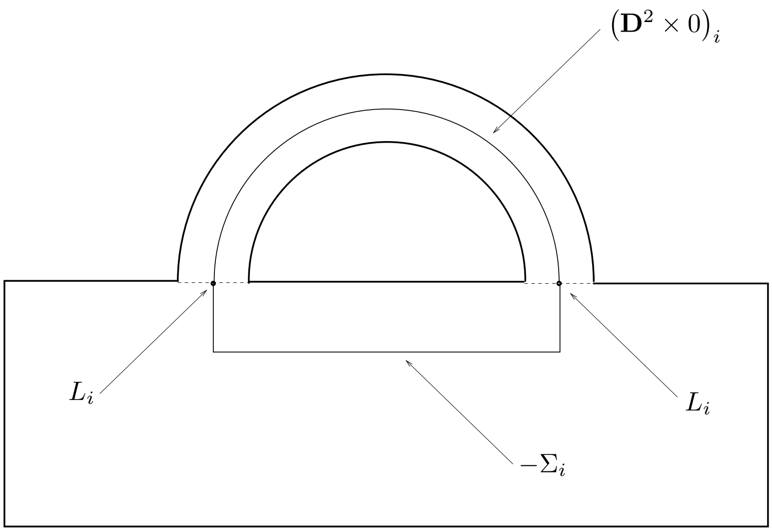

Figure 2.1: The preferred basis of $H_2(W_L)$

2.2 Chern vectors

We define the set of *Chern vectors* (associated to the link L) to be

$$\tilde{\mathcal{V}}_L = \{s = (s_i)_{i=1}^n \in \mathbb{Z}^n \ : \ \forall i = 1, \dots, n, \ s_i \equiv b_{ii} \bmod 2\}.$$

Set $\mathcal{V}_L = \dfrac{\tilde{\mathcal{V}}_L}{2 \cdot \mathrm{Im}\, B_L}$. A basic result of [3] (where the reader is referred to for full details) asserts that

$$\mathrm{Spin}^c(V_L) \simeq \mathcal{V}_L. \tag{2.1}$$

This is our first combinatorial description of Spin^c–structures on V_L, which we now recall briefly. Let $\sigma \in \mathrm{Spin}^c(V_L)$. Extend σ to a Spin^c–structure $\tilde{\sigma} \in \mathrm{Spin}^c(W_L)$. Thus the Chern class $c(\tilde{\sigma}) \in H^2(W_L) \simeq \mathrm{Hom}(H_2(W_L), \mathbb{Z})$ is given by an element in $\mathbb{Z}^n$ (according to the basis dual to the preferred basis). The isomorphism (2.1) is induced by the map $\sigma \mapsto c(\tilde{\sigma})$.

2.3 Charges

Charges were introduced by Turaev in [18], as a combinatorial description of Euler structures. We give a brief description.

The set of *charges* (associated to the link L) is defined to be

$$\tilde{\mathcal{C}}_L = \left\{ k = (k_i)_{i=1}^n \in \mathbb{Z}^n \ : \ \forall i = 1, \ldots, n, \ k_i \equiv 1 + \sum_{1 \leq j \leq n, \ j \neq i} b_{ij} \bmod 2 \right\}.$$

Set $\mathcal{C}_L = \dfrac{\tilde{\mathcal{C}}_L}{2 \cdot \mathrm{Im}\ B_L}$. We shall recall below that

$$\mathrm{Spin}^c(V_L) \simeq \mathcal{C}_L. \tag{2.2}$$

We can alternatively view V_L, without reference to W_L, as

$$V_L = \mathbf{E} \cup \bigcup_{i=1}^n Z_i,$$

where $\mathbf{E}$ denotes the exterior of a tubular neighborhood of L in $\mathbf{S}^3$ and Z_i is a (reglued) solid torus, homeomorphic to $\mathbf{S}^1 \times \mathbf{D}^2$. A solid torus Z is said to be *directed* when its core is oriented. We direct the solid torus Z_j in the following way: we denote by $m_j \subset \mathbf{E}$ the meridian of L_j which is oriented so that $\mathrm{lk}_{\mathbf{S}^3}(m_j, L_j) = +1$, and we require the oriented core of Z_j to be isotopic in V_L to m_j.

In general, let N be a compact oriented 3–manifold with boundary ∂N endowed, this time, with a Spin–structure σ. There is a well-defined set of *Spin^c–structures on N relative to σ*, denoted by $\mathrm{Spin}^c(N, \sigma)$. The Abelian group $H^2(N, \partial N)$ acts freely and transitively on $\mathrm{Spin}^c(N, \sigma)$. Also, there is a *Chern class map*

$$c : \mathrm{Spin}^c(N, \sigma) \to H^2(N, \partial N)$$

which is affine over the square map (where $H^2(N, \partial N)$ is written multiplicatively). For details about relative Spin^c–structures and their gluings, see [3].

The torus $\mathbf{S}^1 \times \mathbf{S}^1$ has a canonical Spin–structure σ^0, which is induced by its Lie group structure. Hence $\partial\mathbf{E}$ can be endowed with a distinguished Spin–structure, which is denoted by $\cup_{i=1}^n \sigma^0$. A directed solid torus Z has a *distinguished* Spin^c–structure relative to the canonical Spin–structure σ^0 on ∂Z: this is the one whose Chern class is Poincaré dual to the opposite of the oriented core of Z. Hence by gluing any Spin^c–structure on $\mathbf{E}$ relative to $\cup_{i=1}^n \sigma^0$ to the distinguished relative Spin^c–structures on the directed solid tori Z_j's, we define a map

$$g : \mathrm{Spin}^c\left(\mathbf{E}, \cup_{i=1}^n \sigma^0\right) \to \mathrm{Spin}^c(V_L).$$

This map g is affine, via the Poincaré duality isomorphisms $P : H_1(\mathbf{E}) \to H^2(\mathbf{E}, \partial\mathbf{E})$ and $P : H_1(V_L) \to H^2(V_L)$, over the natural inclusion homomorphism $H_1(\mathbf{E}) \to H_1(V_L)$. In particular, g is onto.

Another useful general fact is that the Chern class $c(\alpha)$ of a Spin^c–structure α relative to a Spin–structure on the boundary has a nice explicit expression modulo 2, which we briefly explain. Let S be a closed oriented surface. Denote by $\mathrm{Quad}(S)$ the set of quadratic functions over the mod 2 intersection pairing of S. Hence, an element $q \in \mathrm{Quad}(S)$ is a map $q : H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$ such that $q(x+y) - q(x) - q(y) = x \bullet y$ for all $x, y \in H_1(S;\mathbb{Z}_2)$, where $\bullet$ denotes the mod 2 intersection pairing. The Atiyah-Johnson correspondence [1, 6] is a bijective $H_1(S,\mathbb{Z}_2)$–equivariant map

$$J : \mathrm{Spin}(S) \to \mathrm{Quad}(S), \ \sigma \mapsto J_\sigma.$$

Here, the function J_σ is defined, for any simple oriented closed curve γ, by $J_\sigma([\gamma]) = 1$ or 0 according to whether $(\gamma, \sigma|_\gamma)$ is homotopic to $\mathbf{S}^1$ with the Spin–structure induced from the Lie group structure or not [7, pages 35–36].

Lemma 2.1 (See [3]) *Let N be a compact oriented 3–manifold with boundary, $\sigma \in \mathrm{Spin}(\partial N)$ and $\alpha \in \mathrm{Spin}^c(N,\sigma)$. Then*

$$\forall y \in H_2(N, \partial N), \quad \langle c(\alpha), y\rangle \equiv J_\sigma\left(\partial_*(y)\right) \bmod 2,$$

where $\langle \cdot, \cdot \rangle$ denotes Kronecker evaluation, and where $\partial_* : H_2(N,\partial N) \to H_1(\partial N)$ is the connecting homomorphism of the pair $(N, \partial N)$.

A canonical bijection between $\mathrm{Spin}^c\left(\mathbf{E}, \cup_{i=1}^n \sigma^0\right)$ and $\tilde{\mathcal{C}}_L$ can be defined in the following way: for any $\alpha \in \mathrm{Spin}^c\left(\mathbf{E}, \cup_{i=1}^n \sigma^0\right)$, calculate $P^{-1}c(\alpha) \in H_1(\mathbf{E})$ and identify $H_1(\mathbf{E})$ with $\mathbb{Z}^n$ taking the meridians $([m_1], \ldots, [m_n])$ as a basis; it is a consequence of Lemma 2.1 that the multi-integer we obtain is actually a charge on L. Thus, since g is surjective and since $\mathrm{Ker}\,(H_1(\mathbf{E}) \to H_1(V_L))$ is generated by the n characteristic curves of the surgery, it follows that the map g induces a bijection

$$\frac{\tilde{\mathcal{C}}_L}{2 \cdot \mathrm{Im}\ B_L} \to \mathrm{Spin}^c(V_L)$$

as claimed.

2.4 The quadratic function $\phi_{M,\sigma}$

In this paragraph, we recall how to compute the quadratic function $\phi_{M,\sigma}$ [10, 4, 2] from the surgery presentation L for M and a Chern vector $s \in \mathbb{Z}^n$ representing $\sigma \in \mathrm{Spin}^c(M)$. By the homology exact sequence associated to the pair (W_L, V_L), the choice of the preferred basis for $H_2(W_L)$ induces an identification

$$H \simeq \mathrm{Coker}\ B_L = \mathbb{Z}^n / \mathrm{Im}\ B_L. \tag{2.3}$$

Let $x \in H$ and let $X \in \mathbb{Z}^n$ be a representative of x by (2.3). We have

$$\phi_{M,\sigma}(x) = -\frac{1}{2}\left(X^{\mathrm{T}} \cdot {B_L}^{-1} \cdot X + X^{\mathrm{T}} \cdot {B_L}^{-1} \cdot s\right) \mod 1. \tag{2.4}$$

Example 2.2 Suppose that the surgery link L is algebraically split (ie, B_L is diagonal). As before, denote by m_i the meridian of L_i oriented so that $\mathrm{lk}_{\mathbf{S}^3}(L_i, m_i) = +1$ and let $[m_i] \in H$ be its homology class in M. It follows from (2.3) and the orientation convention that

$$\phi_{M,\sigma}([m_i]) = -\frac{1}{2b_{ii}}(1 - s_i) \bmod 1. \tag{2.5}$$

3 Proof of the Theorem

A technical difficulty lies in the computation of $q_{M,\sigma}$ from the torsion $\tau(M,\sigma)$. Fortunately, $\tau(M,\sigma)$ can be computed from a surgery presentation of M and a charge representing σ (see [18] or [19]). In the previous section, we computed $\phi_{M,\sigma}$ from a surgery presentation of M and a Chern vector representing σ. Thus, the proof consists in two steps: 1. compare charges to Chern vectors (there must be a bijective correspondence between them); 2. compare $q_{M,\sigma}$ to $\phi_{M,\sigma}$ using surgery presentations.

We shall use the notations of the previous section. In particular, we have fixed an ordered oriented framed n–component link L in $\mathbf{S}^3$, such that the oriented 3–manifold V_L obtained by surgery along L is diffeomorphic to our oriented rational homology 3–sphere M.

The comparison of the two combinatorial descriptions of $\mathrm{Spin}^c(V_L)$ is contained in the following

Claim 3.1 *If $\sigma \in \mathrm{Spin}^c(V_L)$ corresponds to $[k] \in \mathcal{C}_L$, then σ corresponds to $[s] \in \mathcal{V}_L$, where*

$$\forall j \in \{1, \ldots, n\}, \quad s_j = 1 - k_j + \sum_{i=1}^{n} b_{ij}. \tag{3.1}$$

Remark 3.2 Claim 3.1 is true for *any* closed oriented connected 3–manifold (ie, with arbitrary first Betti number).

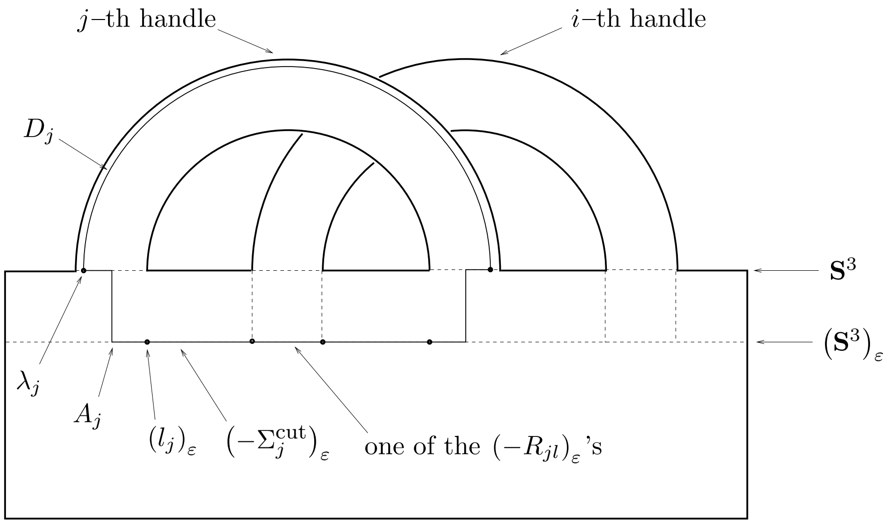

Figure 3.1: A decomposition of the surface S_j

Proof of the Claim 3.1 We denote by σ_2 the distinguished relative Spin^c–structure in $\mathrm{Spin}^c\left(\cup_{j=1}^n Z_j, \cup_{j=1}^n \sigma^0\right)$. Let also $\sigma_1 \in \mathrm{Spin}^c(\mathbf{E}, \cup_{j=1}^n \sigma^0)$ be such that

$$\sigma = \sigma_1 \cup \sigma_2 \in \mathrm{Spin}^c(V_L).$$

Pick an extension $\tilde{\sigma}$ of σ to W_L and let ξ be the isomorphism class of $U(1)$–principal bundles determined by $\tilde{\sigma} \in \mathrm{Spin}^c(W_L)$. On the one hand, the first Chern class $c_1(\xi)$ of ξ, when expressed in the preferred basis $([S_j]^*)_{j=1}^n$ of $H^2(W_L) \simeq \mathrm{Hom}(H_2(W_L), \mathbb{Z})$, gives a multi–integer $s \in \mathbb{Z}^n$; then $[s] \in \mathcal{V}_L$ corresponds to σ. On the other hand, the Poincaré dual to the relative Chern class of $\sigma_1 \in \mathrm{Spin}^c\left(\mathbf{E}, \cup_{j=1}^n \sigma^0\right)$, when expressed in the preferred basis $([m_j])_{j=1}^n$ of $H_1(\mathbf{E})$, gives a multi–integer $k \in \mathbb{Z}^n$; then $[k] \in \mathcal{C}_L$ corresponds to σ. Thus, proving that those specific integers k and s verify (3.1) modulo $2 \cdot \mathrm{Im}\, B_L$ will be enough.

In the sequel we denote by $\left(\mathbf{S}^3\right)_\varepsilon$ a collar push-off of $\mathbf{S}^3 = \partial \mathbf{D}^4$ in the interior of $\mathbf{D}^4$. The surface S_j can be decomposed (up to isotopy) in W_L as

$$S_j = D_j \cup A_j \cup \left(-\Sigma_j^{\mathrm{cut}}\right)_\varepsilon \cup \bigcup_l (-R_{jl})_\varepsilon$$

where the subsurfaces, illustrated on Figure 3.1, are defined as follows:

- D_j is a meridian disc of Z_j such that ∂D_j is the characteristic curve λ_j of the j–th surgery;

- A_j is the annulus of an isotopy of $-\lambda_j$ to L_j, union the annulus of an isotopy of $-L_j$ to $(L_j)_\varepsilon$, union the annulus of an isotopy of $(-L_j)_\varepsilon$ to $(l_j)_\varepsilon$, where l_j denotes the preferred parallel of L_j in $\mathbf{S}^3$ (ie, $\operatorname{lk}_{\mathbf{S}^3}(l_j, L_j) = 0$);
- Σ_j is a Seifert surface for l_j in $\mathbf{S}^3$ disjoint from L_j and in transverse position with the L_i's ($i \neq j$). For each intersection point x_l between Σ_j and a L_i, remove a small disc R_{jl} so that $\Sigma_j = \Sigma_j^{\mathrm{cut}} \cup \bigcup_l R_{jl}$.

By definition of s, we have $s_j = \langle c_1(\xi), [S_j]\rangle = \langle c_1(p|_{S_j}), [S_j]\rangle$ where p is representative for ξ and where $c_1(p|_{S_j}) \in H^2(S_j)$ is the obstruction to trivialize p over S_j. So $P^{-1}c_1(p|_{S_j}) = s_j \cdot [\mathrm{pt}] \in H_0(S_j)$. Let tr be a trivialization of p on $\partial\mathbf{E}$ and let tr_ε be the corresponding trivialization of p on $(\partial\mathbf{E})_\varepsilon$. A classical argument (calculus of obstructions in compact oriented manifolds by means of Poincaré dualities) leads to the equality

$$\begin{aligned} H_0(S_j) \ni P^{-1}c_1(p|_{S_j}) &= i_*P^{-1}c_1\left(p|_{D_j}, \mathrm{tr}|_{\lambda_j}\right) \\ &\quad + i_*P^{-1}c_1\left(p|_{A_j}, \mathrm{tr}|_{-\lambda_j} \cup \mathrm{tr}_\varepsilon|_{(l_j)_\varepsilon}\right) \\ &\quad - i_*P^{-1}c_1\left(p|_{\left(\Sigma_j^{\mathrm{cut}}\right)_\varepsilon}, \mathrm{tr}_\varepsilon|_{\left(\partial\Sigma_j^{\mathrm{cut}}\right)_\varepsilon}\right) \\ &\quad - \sum_l i_*P^{-1}c_1\left(p|_{\left(R_{jl}\right)_\varepsilon}, \mathrm{tr}_\varepsilon|_{\left(\partial R_{jl}\right)_\varepsilon}\right), \end{aligned} \tag{3.2}$$

where P denotes a Poincaré duality isomorphism for the appropriate surface (D_j, A_j, Σ_j^{cut} or R_{jl}). For an appropriate choice of p in the class ξ and for an appropriate choice of tr, we have

$$\begin{aligned} c_1\left(p|_{\mathbf{E}}, \mathrm{tr}\right) &= c(\sigma_1) \in H^2(\mathbf{E}, \partial\mathbf{E}) \\ c_1\left(p|_{\cup_j Z_j}, \mathrm{tr}\right) &= c(\sigma_2) \in H^2\left(\cup_j Z_j, \cup_j \partial Z_j\right) \\ c_1\left(p|_{\mathrm{N}(L)}, \mathrm{tr}\right) &= c(\sigma_3) \in H^2\left(\mathrm{N}(L), \partial\mathrm{N}(L)\right) \end{aligned}$$

where, in this last requirement, $\mathrm{N}(L)$ is a tubular neighborhood of L in $\mathbf{S}^3$ and σ_3 is an arbitrary element of $\mathrm{Spin}^c\left(\mathrm{N}(L), \cup_j \sigma^0\right)$. For such choices, we now compute separately each term of the right hand side of (3.2).

(1) The first term is of the form $d_j \cdot [\mathrm{pt}]$. Here

$$d_j = \langle c(\sigma_2), [D_j]\rangle = -\left(\text{oriented core of } Z_j\right) \bullet [D_j] = +1,$$

where the intersection is taken in Z_j. (Note that $Z_j = \left(\mathbf{D}^2 \times \mathbf{S}^1\right)_j$ if we denote by $\left(\mathbf{D}^2 \times \mathbf{D}^2\right)_j$ the 2–handle of W_L corresponding to L_j, and be careful of the fact that the above specified oriented core of Z_j is $-\left(0 \times \mathbf{S}^1\right)_j$.)

(2) The second term is of the form $a_j \cdot [\mathrm{pt}]$. Here $a_j = \langle c(\sigma_3), [A_j]\rangle$ where A_j is regarded as a relative 2–cycle in $(\mathrm{N}(L), \partial\mathrm{N}(L))$ once the collar has been squeezed. Since ∂A_j is $-\lambda_j \cup l_j$, $[A_j]$ is $-b_{jj}$ times the class of the meridian disc of L_j (oriented so that its oriented boundary is m_j) in $H_2(\mathrm{N}(L), \partial\mathrm{N}(L))$. Then, $a_j = -b_{jj} \cdot \rho_j$ where ρ_j is defined to be

$$\rho_j = \langle c(\sigma_3), [\text{meridian disc of } L_j]\rangle \in \mathbb{Z}.$$

Note that $\rho_j \equiv J_{\sigma^0}([m_j]) \equiv 1 \bmod 2$ (by the Atiyah-Johnson correspondence, see Lemma 2.1).

(3) The third term is $-g_j \cdot [\mathrm{pt}]$ where $g_j = \langle c(\sigma_1), [\Sigma_j^{\mathrm{cut}}]\rangle$. But, that integer is equal to

$$g_j = \left(P^{-1}c(\sigma_1)\right) \bullet [\Sigma_j^{\mathrm{cut}}] = \left(\sum_i k_i[m_i]\right) \bullet [\Sigma_j^{\mathrm{cut}}] = \sum_i k_i \delta_{ij} = k_j$$

where the intersection is taken in $\mathbf{E}$.

(4) The fourth term is given by $-\sum_l r_{jl} \cdot [\mathrm{pt}]$. Here $r_{jl} = \langle c(\sigma_3), [R_{jl}]\rangle$. For each index l, denote by $i(l)$ the integer i such that x_l is an intersection point of Σ_j with L_i, and denote by $\epsilon(l)$ the sign of the intersection point x_l. Then, from the definition of ρ_i (given for the second term), we have $r_{jl} = \epsilon(l) \cdot \rho_{i(l)}$. Hence

$$\sum_l r_{jl} = \sum_{\substack{i=1 \\ i\neq j}}^{n} b_{ij}\rho_i.$$

Putting those computations together, we obtain that (3.2) is equivalent to the identity

$$\begin{aligned} s_j &= d_j + a_j - g_j - \sum_l r_{jl} \\ &= 1 - b_{jj}\rho_j - k_j - \sum_{\substack{i=1 \\ i\neq j}}^{n} b_{ij}\rho_i \\ &= \left(1 - k_j + \sum_{i=1}^{n} b_{ij}\right) - \sum_{i=1}^{n} b_{ij}\left(\rho_i + 1\right). \end{aligned}$$

The claim now follows from the fact that $\rho_i \equiv 1 \bmod 2$ for all $i = 1, \ldots, n$. □

We are now able to prove the Theorem. Assume first that M is obtained by surgery along an algebraically split link L, and that σ is represented by a charge

k on L. Then, according to [19, Chapter X, Section 5.4], we have that

$$q_{M,\sigma}([m_j]) = \frac{1}{2} - \frac{k_j}{2b_{jj}} \bmod 1.$$

Substituting $k_j = 1 - s_j + \sum_i b_{ij}$, we find that this formula agrees with (2.5) of Example 2.2. This proves the Theorem in this particular case. Now consider the general case, when L is not necessarily algebraically split. We shall use the following observation due to Ohtsuki.

Lemma 3.3 *Let M be an oriented rational homology 3–sphere. There exist non-zero integers $n_1, \ldots, n_r$ such that $M\#L(n_1,1)\#\cdots\#L(n_r,1)$ can be presented by surgery along a framed link L algebraically split in $\mathbf{S}^3$.*

Here # denotes connected sum and $L(n,1)$ is the 3–dimensional lens space obtained by surgery along a trivial knot with framing $n \neq 0$ in $\mathbf{S}^3$. Apply that lemma to the oriented rational homology 3–sphere M we are working with, and consider the resulting manifold $M' = M\#L(n_1,1)\#\cdots\#L(n_r,1)$. Set $\sigma' = \sigma\#\sigma_1\#\cdots\#\sigma_r \in \mathrm{Spin}^c(M')$ where $\sigma_1, \ldots, \sigma_r$ denote arbitrary Spin^c–structures on the lens spaces. Then, we have $q_{M',\sigma'} = \phi_{M',\sigma'}$. By definition of #, there is a small 3–ball $B \subset M$ such that $M \setminus B \subset M'$. This inclusion induces a (injective) homomorphism $i_* : H_1(M) \to H_1(M')$. Since we can compute $\phi_{M',\sigma'}$ from a split surgery presentation of M' using the surgery formula (2.4), we have that $\phi_{M,\sigma} = \phi_{M',\sigma'} \circ i_*$. It follows from [19, Chapter XII, Section 1.2] (which describes the behaviour of the Reidemeister–Turaev torsion under #) that, similarly, $q_{M,\sigma} = q_{M',\sigma'} \circ i_*$. We deduce that $q_{M,\sigma} = \phi_{M,\sigma}$ and we are done.

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Quadratic functions and complex spin structures on three-manifolds

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Abstract

We show how the space of complex spin structures of a closed oriented three-manifold embeds naturally into a space of quadratic functions associated to its linking pairing. Besides, we extend the Goussarov–Habiro theory of finite type invariants to the realm of compact oriented three-manifolds equipped with a complex spin structure. Our main result states that two closed oriented three-manifolds endowed with a complex spin structure are undistinguishable by complex spin invariants of degree zero if, and only if, their associated quadratic functions are isomorphic.
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Complex spin structures, or Spin^c-structures, are additional structures with which manifolds may be equipped. They are needed to define the Seiberg–Witten invariants of 4-manifolds, as well as the

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Heegaard–Floer homologies of 3-manifolds by Ozsváth and Szabó. Any closed oriented 3-manifold M can be endowed with a Spin^c-structure and, in that case, Spin^c-structures are in canonical correspondence with Euler structures. The latter are classes of nonsingular vector fields on M which have been introduced by Turaev in order to refine Reidemeister torsion.

In this paper, we investigate the rôle played by quadratic functions in the topology of closed oriented 3-manifolds equipped with a Spin^c-structure or, equivalently, an Euler structure.

Extending constructions from [18,19,24], we associate, to any closed oriented 3-manifold M with a Spin^c-structure σ, its *linking quadratic function*

$$H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}.$$

The function $\phi_{M,\sigma}$ is quadratic in the sense that the symmetric pairing defined by $(x,y)\mapsto \phi_{M,\sigma}(x+y)-\phi_{M,\sigma}(x)-\phi_{M,\sigma}(y)$ is bilinear. Moreover, this symmetric bilinear pairing coincides with $L_M := \lambda_M \circ (B\times B)$ where

$$\mathrm{Tors}H_1(M;\mathbb{Z}) \times \mathrm{Tors}H_1(M;\mathbb{Z}) \xrightarrow{\lambda_M} \mathbb{Q}/\mathbb{Z}$$

is the linking pairing of M and B denotes the Bockstein homomorphism associated to the short exact sequence of coefficients $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$. In contrast with $\phi_{M,\sigma}$, the bilinear pairing L_M does not depend on σ. Spin^c-structures on a given manifold M are determined by their corresponding quadratic functions.

Theorem 1. *Let M be a closed connected oriented* 3*-manifold. The map $\sigma\mapsto\phi_{M,\sigma}$ defines a canonical embedding*

$$\mathrm{Spin}^c(M) \overset{\phi_M}{\hookrightarrow} \mathrm{Quad}(L_M)$$

from the set of Spin^c*-structures on M to the set of quadratic functions with L_M as associated bilinear pairing*.

Via the map ϕ_M, topological notions can be put in correspondence with algebraic ones. For instance, the Chern class $c(\sigma)\in H^2(M)$ of the Spin^c-structure σ corresponds to the homogeneity defect $d_{\phi_{M,\sigma}}: H_2(M;\mathbb{Q}/\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$ of the quadratic function $\phi_{M,\sigma}$, which is defined by $d_{\phi_{M,\sigma}}(x)=\phi_{M,\sigma}(x)-\phi_{M,\sigma}(-x)$.

When the Chern class $c(\sigma)$ is torsion, $\phi_{M,\sigma}$ happens to factor through B to a quadratic function

$$\mathrm{Tors}H_1(M;\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}$$

with λ_M as associated bilinear pairing and is equivalent to the quadratic function constructed by Looijenga and Wahl [19] (see also [4,9]). In particular, the Spin^c-structure may arise from a classical spin structure, or Spin-structure. In that case, which is detected by the vanishing of $c(\sigma)$, the quadratic function $\phi_{M,\sigma}$ is homogeneous and coincides with yet earlier constructions due to Lannes and Latour [18], as well as Morgan and Sullivan [24] (see also [17,27]).

The linking quadratic function is used here to solve a problem related to the theory of finite type invariants by Goussarov and Habiro. Their theory [8,11,12] deals with compact oriented 3-manifolds and is based on an elementary move called Y-surgery. The Y-equivalence, which is defined to be the equivalence relation among such manifolds generated by this move, has been characterized by Matveev in the closed case [22]. This characterization amounts to recognize the degree 0 invariants of the theory. His result, anterior to the work of Goussarov and Habiro, can be re-stated as follows: two closed oriented 3-manifolds M and M' are Y-equivalent if and only if they have isomorphic pairs (homology, linking pairing). A Spin-refinement of the Goussarov–Habiro theory (the possibility of which was announced in [11] and [12]) has also been considered in [21], where Matveev's theorem is extended to closed oriented 3-manifolds equipped with a Spin-structure.

We show that the Y-surgery move makes sense for closed oriented 3-manifolds equipped with a Spin^c-structure as well. The equivalence relation generated by this move among such manifolds is called, here, *Y^c-equivalence*. It follows that there exists a Spin^c-refinement of the Goussarov–Habiro theory. Our main result is a characterization of the Y^c-equivalence relation in terms of the linking quadratic function. In order to state this more precisely, let us fix a few notations.

Given an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$, the dual isomorphism to ψ by the intersection pairings is denoted by $\psi^\sharp : H_2(M';\mathbb{Q}/\mathbb{Z}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$:

$$\forall x \in H_1(M;\mathbb{Z}),\ \forall y' \in H_2(M';\mathbb{Q}/\mathbb{Z}),\ x \bullet \psi^\sharp(y') = \psi(x) \bullet y' \in \mathbb{Q}/\mathbb{Z}.$$

Also, given sections s and s' of the surjections $B : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \operatorname{Tors} H_1(M;\mathbb{Z})$ and $B : H_2(M';\mathbb{Q}/\mathbb{Z}) \to \operatorname{Tors} H_1(M';\mathbb{Z})$ respectively, we say that s and s' are ψ-compatible if the diagram

$$\begin{array}{ccc} H_2(M';\mathbb{Q}/\mathbb{Z}) & \xleftarrow{\ s'\ } & \operatorname{Tors} H_1(M';\mathbb{Z}) \\ {\scriptstyle \psi^\sharp}\big\downarrow \simeq & & {\scriptstyle \psi|}\big\uparrow \simeq \\ H_2(M;\mathbb{Q}/\mathbb{Z}) & \xleftarrow[\ s\ ]{} & \operatorname{Tors} H_1(M;\mathbb{Z}) \end{array}$$

commutes. We denote by P a Poincaré isomorphism and we recall that the Gauss sum of a quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$, defined on a finite Abelian group G, is the complex number $\sum_{x\in G} \exp(2i\pi q(x))$.

Theorem 2. *Let (M,σ) and (M',σ') be two closed connected oriented 3-manifolds with Spin^c-structure. The following assertions are equivalent*:

(1) *The Spin^c-manifolds (M,σ) and (M',σ') are Y^c-equivalent.*
(2) *There is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \psi^\sharp$.*
(3) *There is an isomorphism $\psi : H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ such that*

- $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$,
- $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$,
- *forsome ψ-compatible sections s and s' of the Bockstein homomorphisms, $\phi_{M,\sigma} \circ s$ and $\phi_{M',\sigma'} \circ s'$ have identical Gauss sums.*

Two special cases deserve to be singled out. First, consider manifolds whose first homology group is torsion free. The following result is deduced from Theorem 2.

Corollary 1. *Let (M,σ) and (M',σ') be two closed connected oriented 3-manifolds with Spin^c-structure, such that $H_1(M;\mathbb{Z})$ and $H_1(M';\mathbb{Z})$ are torsion free. The following assertions are* *equivalent*:

(1) *The* Spin^c*-manifolds* (M, σ) *and* (M', σ') *are* Y^c*-equivalent.*
(2) *There is an isomorphism* $\psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z})$ *such that* $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$.

Second, consider the case of rational homology 3-spheres. According to what has been said above, if M is an oriented rational homology 3-sphere, then $\phi_{M,\sigma}$ can be regarded as a quadratic function $H_1(M; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ with λ_M as associated bilinear pairing. In that case, Theorem 2 specializes to the following corollary.

Corollary 2. *Let* (M, σ) *and* (M', σ') *be two oriented rational homology* 3*-spheres with* Spin^c*-structure. The following assertions are equivalent*:

(1) *The* Spin^c*-manifolds* (M, σ) *and* (M', σ') *are* Y^c*-equivalent.*
(2) *There is an isomorphism* $\psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z})$ *such that* $\phi_{M,\sigma} = \phi_{M',\sigma'} \circ \psi$.
(3) *There is an isomorphism* $\psi : H_1(M; \mathbb{Z}) \to H_1(M'; \mathbb{Z})$ *such that*
   - $\lambda_M = \lambda_{M'} \circ (\psi \times \psi)$,
   - $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$,
   - $\phi_{M,\sigma}$ *and* $\phi_{M',\sigma'}$ *have identical Gauss sums.*

The paper is organized as follows. In Section 1, we briefly review Spin^c-structures from a general viewpoint. Next, we restrict ourselves to the dimension 3, in which case one can work with Euler structures as well. At the end of the section, the technical problem of gluing Spin^c-structures is considered. This is needed to define the *Y*-surgery move in the setting of manifolds equipped with a Spin^c-structure, since this move is defined as a “cut and paste” operation. Our gluing lemma involves Spin^c-structures, on a compact oriented 3-manifold with boundary, which are relative to a fixed Spin-structure on the boundary.

Section 2 is devoted to the construction and study of the linking quadratic function. First, we give a combinatorial description of the Spin^c-structures of a given closed oriented 3-manifold presented by surgery along a link in $\mathbf{S}^3$. This leads to a Spin^c-refinement of Kirby’s theorem. Next, we define the quadratic function $\phi_{M,\sigma}$ associated to a closed 3-dimensional Spin^c-manifold (M, σ): this is done essentially by defining a cobordism invariant of singular 3-dimensional Spin^c-manifolds over $K(\mathbb{Q}/\mathbb{Z}, 1)$. The quadratic function $\phi_{M,\sigma}$ can be computed combinatorially as soon as (M, σ) is presented by surgery along a link in $\mathbf{S}^3$. We prove Theorem 1 and some other basic properties of the map ϕ_M. Lastly, regarding σ as an Euler structure, we give for $\phi_{M,\sigma}$ an intrinsic formula that does not make reference to the dimension 4 anymore. This is obtained by presenting, à la Sullivan, elements of $H_2(M; \mathbb{Q}/\mathbb{Z})$ as immersed surfaces with *n*-fold boundary.

In Section 3, the Y^c-surgery move is defined using the above-mentioned gluing lemma. Next, Theorem 2 is proved working with surgery presentations of Spin^c-manifolds. We use the material of the previous section and a result due to Matveev, Murakami and Nakanishi [22,25] on ordered oriented framed links having the same linking matrix. Some algebraic ingredients about quadratic functions on torsion Abelian groups are needed as well. Those results, some of them well-known in the case of finite Abelian groups, have been proved aside in [5]. We conclude this paper by giving some applications of Theorem 2 and stating some problems.

1. Complex spin structures on three-manifolds

In this section, we review Spin^c-structures and other related structures, with special emphasis on the dimension 3. We also give a gluing lemma for Spin^c-structures.

1.1. Some conventions

In this paper, any manifold M is assumed to be compact, smooth and oriented. We denote by $-M$ the manifold obtained from M by reversing its orientation. If M has non-empty boundary, ∂M has the orientation given by the "outward normal vector first" rule. The oriented tangent bundle of M is denoted by TM.

Vector bundles will be stabilized from the left side. A section of a vector bundle is said to be *nonsingular* if it does not vanish at any point.

If G is an Abelian group, a *G-affine space A* is a set A on which G acts freely and transitively. The affine action is denoted additively; thus, for $a, a' \in A$, the unique element $g \in G$ satisfying $a' = a + g$ will be written $a' - a$.

Unless otherwise specified, all (co)homology groups are assumed to be computed with integer coefficients.

1.2. Complex spin structures

In this subsection, we consider a n-manifold M. We recall basic facts about Spin^c-structures on M, adopting a viewpoint which is analogous to that used in [3] for Spin-structures.

1.2.1. From Spin^c *onto* SO

Let $n \geqslant 1$ be an integer. The group Spin(n) is the 2-fold covering of the special orthogonal group SO(n):

$$1 \longrightarrow \mathbb{Z}_2 \to \mathrm{Spin}(n) \longrightarrow \mathrm{SO(n)} \longrightarrow 1.$$

The group Spin$^c(n)$ is defined by

$$\mathrm{Spin}^c(n) = \frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ is generated by $[(-1,-1)]$, hence the following short exact sequence of groups:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(n) \stackrel{\pi}{\longrightarrow} \mathrm{SO(n)} \longrightarrow 1,$$

where the first map sends z to $[(1, z)]$ and where π is induced by the projection of Spin(n) onto SO(n).

The inclusion of SO(n) into SO$(n+1)$, defined by $A \mapsto (1) \oplus A$, induces a monomorphism Spin$^c(n) \rightarrowtail$ Spin$^c(n+1)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spin}^c(n) & \rightarrowtail & \mathrm{Spin}^c(n+1) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathrm{SO}(n) & \rightarrowtail & \mathrm{SO}(n+1). \end{array}$$

commutes, hence a diagram at the level of classifying spaces:

$$\begin{array}{ccc} \mathrm{BSpin}^c(n) & \longrightarrow & \mathrm{BSpin}^c(n+1) \\ \mathrm{B}\pi \downarrow & & \downarrow \mathrm{B}\pi \\ \mathrm{BSO}(n) & \longrightarrow & \mathrm{BSO}(n+1). \end{array} \tag{1.1}$$

Here, we take $\mathrm{BSO}(n)$ to be the Grassman manifold of oriented n-planes in $\mathbb{R}^\infty$ and the map $\mathrm{BSO}(n) \to \mathrm{BSO}(n+1)$ to be the usual one. We fix the classifying spaces $\mathrm{BSpin}^c(n)$ (in their homotopy equivalence classes) and, next, we fix the maps $\mathrm{B}\pi : \mathrm{BSpin}^c(n) \to \mathrm{BSO}(n)$ (in their homotopy classes) to be fibrations. Then, the map from $\mathrm{BSpin}^c(n)$ to $\mathrm{BSpin}^c(n+1)$ is choosen (in its homotopy class) to make diagram (1.1) *strictly* commute.

We denote by $\gamma_{\mathrm{SO}(n)}$ the universal n-dimensional oriented vector bundle over $\mathrm{BSO}(n)$. Let $\gamma_{\mathrm{Spin}^c(n)}$ be the pull-back of $\gamma_{\mathrm{SO}(n)}$ by $\mathrm{B}\pi$. Thanks to (1.1), there is a well-defined morphism between $(n+1)$-dimensional oriented vector bundles $\mathbb{R} \oplus \gamma_{\mathrm{Spin}^c(\mathrm{n})} \to \gamma_{\mathrm{Spin}^c(\mathrm{n+1})}$ induced by the usual one $\mathbb{R} \oplus \gamma_{\mathrm{SO}(n)} \to \gamma_{\mathrm{SO}(n+1)}$.

1.2.2. Rigid Spin^c*-structures*

Recall that M is a n-manifold to which some conventions, stated in Section 1.1, apply.

Definition 1.1. A *rigid* Spin^c*-structure* on M is a morphism $\mathrm{T}M \to \gamma_{\mathrm{Spin}^c(n)}$ between n-dimensional oriented vector bundles. A Spin^c*-structure* (or *complex spin structure*) on M is a homotopy class of rigid Spin^c-structures on M. We denote by $\mathrm{Spin}^c_r(M)$ the set of rigid Spin^c-structures on M, and by $\mathrm{Spin}^c(M)$ the set of its Spin^c-structures.

Obviously, a different choice of the classifying space $\mathrm{BSpin}^c(n)$ (in its homotopy type) or a different choice of the map $\mathrm{B}\pi$ (in its homotopy class) would lead to a different notion of rigid Spin^c-structure, but would not affect the definition of a Spin^c-structure. Rigid structures will be used later to define gluing maps.

Let β be the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

The fibration $\mathrm{B}\pi : \mathrm{BSpin}^c(n) \to \mathrm{BSO}(n)$ has fiber $\mathrm{BU}(1) \simeq K(\mathbb{Z}, 2)$ and, indeed, is a principal fibration with characteristic class $w := \beta w_2 \in H^3(\mathrm{BSO}(n))$, where w_2 is the second Stiefel–Whitney class. Then, by obstruction theory, we obtain the following well-known fact about existence and parametrization of Spin^c-structures.

Proposition 1.1. *The manifold M can be given a* Spin^c*-structure if and only if the cohomology class* $\beta w_2(M) \in H^3(M)$ *vanishes. In that case,* $\mathrm{Spin}^c(M)$ *is a* $H^2(M)$*-affine space.*

One may easily verify that the homotopy-theoretical definition of a Spin^c-structure, which we have adopted here, agrees with the usual one.

Lemma 1.1. *Suppose that M is equipped with a Riemannian metric and denote by* $\mathrm{SO}(\mathrm{T}M)$ *the bundle of its oriented orthonormal frames. A* Spin^c*-structure on M is equivalent to an isomorphism class of pairs*

(η, H), where η is a principal $\mathrm{Spin}^c(n)$*-bundle over M and where* $H : \eta/\mathrm{U}(1) \to \mathrm{SO}(\mathrm{T}M)$ *is a principal* $\mathrm{SO}(n)$*-bundle isomorphism*.

To go to the point, we have only defined (rigid) Spin^c-structures on the manifold M. Nevertheless, the notion of a (rigid) Spin^c-structure obviously extends to any oriented vector bundle over any base space.

Remark 1.1. Thanks to the map $\mathbb{R} \oplus \gamma_{\mathrm{Spin}^c(\mathrm{n})} \to \gamma_{\mathrm{Spin}^c(\mathrm{n+1})}$ constructed at the end of Section 1.2.1, a rigid Spin^c-structure on $\mathrm{T}M$ gives rise to one on $\mathbb{R} \oplus \mathrm{T}M$. This induces a canonical map

$$\mathrm{Spin}^c(M) = \mathrm{Spin}^c(\mathrm{T}M) \to \mathrm{Spin}^c(\mathbb{R} \oplus \mathrm{T}M),$$

which is $H^2(M)$-equivariant and, so, bijective. Thus, a Spin^c-structure on M is equivalent to a Spin^c-structure on its stable oriented tangent bundle.

1.2.3. Orientation reversal

The *time-reversing* map is the orientation-reversing automorphism of $\mathbb{R} \oplus \mathrm{T}M$ defined by $(t, v) \mapsto (-t, v)$. Composition with that map transforms a rigid Spin^c-structure on $\mathbb{R} \oplus \mathrm{T}M$ to one on $\mathbb{R} \oplus \mathrm{T}(-M)$. So, by Remark 1.1, we get a canonical $H^2(M)$-equivariant map

$$\mathrm{Spin}^c(\mathrm{M}) \xrightarrow{\ -\ } \mathrm{Spin}^c(-M).$$

1.2.4. Relative Spin^c*-structures*

Suppose that M has some boundary and fix a rigid structure $s \in \mathrm{Spin}^c_r(\mathrm{T}M|_{\partial M})$ over ∂M.

Definition 1.2. A Spin^c*-structure on M relative to s* is a homotopy class rel ∂M of rigid Spin^c-structures on M that extend s. We denote by $\mathrm{Spin}^c(M, s)$ the set of such structures.

The following relative version of Proposition 1.1 is also proved by obstruction theory applied to the fibration $\mathrm{B}\pi$.

Proposition 1.2. *There exists a rigid* Spin^c*-structure on M that extends s if and only if a certain cohomology class*

$$w(M, s) \in H^3(M, \partial M)$$

vanishes. In that case, $\mathrm{Spin}^c(M, s)$ *is a* $H^2(M, \partial M)$*-affine space.*

1.2.5. Restriction to the boundary

Suppose that M has some boundary. Observe that there is a well-defined homotopy class of isomorphisms between the oriented vector bundles $\mathbb{R} \oplus \mathrm{T}\partial M$ and $\mathrm{T}M|_{\partial M}$, which is defined by any section of $\mathrm{T}M|_{\partial M}$ transverse to ∂M and directed outwards.

In particular, a Spin^c-structure on $\mathrm{T}M|_{\partial M}$ can be identified without ambiguity to a Spin^c-structure on ∂M. Thus, we get a canonical *restriction* map

$$\mathrm{Spin}^c(M) \longrightarrow \mathrm{Spin}^c(\partial M),$$

which is affine over the homomorphism $H^2(M) \to H^2(\partial M)$ induced by inclusion.

1.2.6. From Spin *to* Spin^c

Proceeding as in Section 1.2.2, we define the set $\mathrm{Spin}_r(M)$ of *rigid* Spin-*structures* on M and the set $\mathrm{Spin}(M)$ of Spin-*structures* on M. The latter is a $H^2(M;\mathbb{Z}_2)$-affine space as soon as $w_2(M)$ vanishes. The reader is referred to [3] for details.[1] The group homomorphism

$$\mathrm{Spin}(n) \stackrel{\beta}{\longrightarrow} \mathrm{Spin}^c(n)$$

defined by $\beta(x) = [(x,1)]$, makes the two projections onto $\mathrm{SO}(n)$ agree. This allows us to define a morphism $\gamma_{\mathrm{Spin(n)}} \to \gamma_{\mathrm{Spin}^c\mathrm{(n)}}$ between oriented n-dimensional vector bundles, the composition with which transforms a rigid Spin-structure u to a rigid Spin^c-structure denoted by $\beta(u)$. Thus, we get a canonical map

$$\mathrm{Spin}(M) \stackrel{\beta}{\longrightarrow} \mathrm{Spin}^c(M)$$

which is affine over the Bockstein homomorphism $\beta: H^1(M;\mathbb{Z}_2) \to H^2(M)$.

If M has some boundary, we define *relative* Spin-*structures* on M as well. Their construction goes as in Section 1.2.4. Thus, for a fixed $s \in \mathrm{Spin}_r(\mathrm{T}M|_{\partial M})$, we get a map

$$\mathrm{Spin}(M,s) \stackrel{\beta}{\longrightarrow} \mathrm{Spin}^c(M,\beta s),$$

which is affine over the Bockstein homomorphism $\beta: H^1(M,\partial M;\mathbb{Z}_2) \to H^2(M,\partial M)$.

1.2.7. From U *to* Spin^c

Let m be an integer such that $n \leqslant 2m$. We take $\mathrm{BU}(m)$ to be the Grassman manifold of complex m-planes in $\mathbb{C}^\infty$. The map $\mathrm{BU}(m) \to \mathrm{BSO}(2m)$, which consists in forgetting the complex structure on a complex m-plane, represents the usual inclusion of $\mathrm{U}(m)$ into $\mathrm{SO}(2m)$. We define $\gamma_{\mathrm{U(m)}}$ to be the pull-back of $\gamma_{\mathrm{SO}(2m)}$ by this map $\mathrm{BU}(m) \to \mathrm{BSO}(2m)$, which can be identified with the $2m$-dimensional oriented vector bundle underlying the universal m-dimensional complex vector bundle. Then, as we did in the Spin and Spin^c cases, we could define a "rigid U-structure" on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ to be a morphism $\mathbb{R}^{2m-n} \oplus \mathrm{T}M \to \gamma_{\mathrm{U(m)}}$ between $2m$-dimensional oriented vector bundles. Such a morphism induces a complex structure on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ by pulling back the canonical one on $\gamma_{\mathrm{U(m)}}$ and, conversely, any complex structure on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ inducing the given orientation arises that way. Then, a "U-structure" on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ is equivalent to a homotopy class of complex structures on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ compatible with the given orientation.

There is a canonical way to embed $\mathrm{U}(m)$ into $\mathrm{Spin}^c(2m)$: see, for instance, [10, Proposition D.50]. This inclusion

$$\mathrm{U}(m) \stackrel{\omega}{\rightarrowtail} \mathrm{Spin}^c(2m)$$

makes the two maps to $\mathrm{SO}(2m)$ commute. This allows us to define a morphism $\gamma_{\mathrm{U}(m)} \to \gamma_{\mathrm{Spin}^c(2m)}$ between oriented $2m$-dimensional vector bundles, the composition with which transforms a "rigid U-structure" on $\mathbb{R}^{2m-n} \oplus \mathrm{T}M$ to a rigid Spin^c-structure on it. As a consequence of Remark 1.1, we get a

[1] In [3], rigid Spin-structures are called "w_2-structures" and are defined on the stable oriented tangent bundle. An observation similar to that given in Remark 1.1 for Spin^c-structures applies to Spin-structures.

canonical map

$$\mathrm{U}^s(M) \stackrel{\omega}{\longrightarrow} \mathrm{Spin}^c(M)$$

from the set of stable complex structures on TM compatible with the orientation to the set of Spin^c-structures on M. (See [10, Proposition D.57] for a construction of ω involving the usual definition of a Spin^c-structure.)

1.2.8. Chern class

A Spin^c-structure α on M induces an isomorphism class of principal Spin$^c(n)$-bundles over M and, so, an isomorphism class of principal U(1)-bundles thanks to the homomorphism $\mathrm{Spin}^c(n) \to \mathrm{U}(1)$ defined by $[(x, y)] \mapsto y^2$. The first Chern class of the latter is denoted by $c(\alpha)$. We get a *Chern class* map

$$\mathrm{Spin}^c(M) \stackrel{c}{\longrightarrow} H^2(M)$$

which is affine over the doubling map defined by $x \mapsto 2x$. When $c(\alpha)$ belongs to Tors $H^2(M)$, the Spin^c-structure α is said to be *torsion*.

1.3. Complex spin structures in dimension 3

In this subsection, we turn to 3-manifolds which, by Section 1.1, are assumed to be compact smooth and oriented. The preliminary remark is that any 3-manifold M can be endowed with a Spin^c-structure, since $w_2(M)$ is well-known to vanish.

We start by removing the rigidity of relative Spin^c-structures which is still remaining along the boundary. Next, we recall Turaev's observation that Spin^c-structures can be regarded as classes of vector fields. This holds true in the relative case as well.

1.3.1. Relative Spin^c*-structures*

Let M be a 3-manifold with boundary and let σ be a Spin-structure on ∂M. We define Spin^c-structures on M which are relative to σ. Note that, thanks to the observation initiating Section 1.2.5, one can identify $\sigma \in \mathrm{Spin}(\partial M)$ to a Spin-structure on $\mathrm{T}M|_{\partial M}$.

Lemma 1.2. *For any rigid* Spin*-structure s on* $\mathrm{T}M|_{\partial M}$ *representing σ (which we denote by $s \in \sigma$), the rigid* Spin^c*-structure $\beta(s)$ can be extended to M. Moreover, for any $s, s' \in \sigma$, there exists a canonical $H^2(M, \partial M)$-equivariant bijection*

$$\mathrm{Spin}^c\,(M, \beta s) \stackrel{\rho_{s,s'}}{\longrightarrow} \mathrm{Spin}^c(M, \beta s').$$

Lastly, for any $s, s', s'' \in \sigma$, we have that $\rho_{s',s''} \circ \rho_{s,s'} = \rho_{s,s''}$.

Definition 1.3. A Spin^c*-structure on M relative to* σ is a pair (u, s) where $s \in \sigma$ and $u \in \mathrm{Spin}^c(M, \beta s)$, two such pairs (u, s) and (u', s') being considered as equivalent when $u' = \rho_{s,s'}(u)$. The set of such structures is denoted by $\mathrm{Spin}^c(M, \sigma)$ and can naturally be given the structure of a $H^2(M, \partial M)$-affine space.

Remark 1.2. There is an analogue to Lemma 1.1 that formulates what a Spin^c-structure on M relative to σ is in terms of principal bundles.

Example 1.1. Suppose that ∂M is a disjoint union of tori. The 2-torus has a distinguished Spin-structure σ^0 that is induced by its Lie group structure. Using the previous remark, it can be verified that a Spin^c-structure on M relative to the union of copies of σ^0 is equivalent to a relative Spin^c-structure in the sense of Turaev [30, Section 1.2].

Proof of Lemma 1.2. Let $w_2(M, s) \in H^2(M, \partial M; \mathbb{Z}_2)$ denote the obstruction to extend s to a rigid Spin-structure on M. We have that

$$\beta(w_2(M, s)) = w(M, \beta s) \in H^3(M, \partial M).$$

Thus, $w(M, \beta s)$ is of order at most 2 and, so, vanishes.

We now prove the second statement. Let $\varphi : [-1, 0] \times \partial M \hookrightarrow M$ be a collar neighborhood of ∂M. In particular, φ induces a specific isomorphism between $\mathbb{R} \oplus \mathrm{T}\partial M$ and $\mathrm{T}M|_{\partial M}$: the rigid Spin-structures on $\mathbb{R} \oplus \mathrm{T}\partial M$ corresponding to s and s' are denoted by s_0 and s_1, respectively. By assumption, s_0 and s_1 are homotopic: let $S = (s_t)_{t\in[0,1]}$ be such a homotopy. This defines a rigid Spin-structure S on $[0, 1] \times \partial M$ by identifying, at each time t, $\mathbb{R} \oplus \mathrm{T}\partial M$ with the restriction of $\mathrm{T}([0, 1] \times \partial M)$ to $t \times \partial M$. The same collar neighborhood allows us to define a smooth gluing $M \cup ([0, 1] \times \partial M)$, as well as a positive diffeomorphism $\tilde{\varphi} : M \to M \cup ([0, 1] \times \partial M)$ (based on the affine identification between $[-1, 0]$ and $[-1, 1]$). Consider the map

$$\mathrm{Spin}^c(M, \beta s) \xrightarrow{\rho_S} \mathrm{Spin}^c(M, \beta s')$$

defined, for any $u \in \mathrm{Spin}^c_r(M)$ extending $\beta(s)$, by $\rho_S([u]) = [(u \cup \beta(S)) \circ \mathrm{T}\tilde{\varphi}]$.

The map ρ_S is $H^2(M, \partial M)$-equivariant and is independent of the choice of φ. So, we are left to prove that ρ_S does not depend on the choice of the homotopy S between s_0 and s_1, which will allow us to set $\rho_{s,s'} = \rho_S$. To see that, consider the map β constructed in Section 1.2.6 from $\mathrm{Spin}([0, 1] \times \partial M, 0 \times (-s_0) \cup 1 \times s_1)$ to $\mathrm{Spin}^c([0, 1] \times \partial M, 0 \times (-\beta s_0) \cup 1 \times (\beta s_1))$, where $-s_0 \in \mathrm{Spin}_r(\mathbb{R} \oplus \mathrm{T}(-\partial M))$ is obtained from s_0 by time-reversing. The Bockstein homomorphism β from $H^1([0, 1] \times \partial M, \partial[0, 1] \times \partial M; \mathbb{Z}_2)$ to $H^2([0, 1] \times \partial M, \partial[0, 1] \times \partial M)$ is trivial, since its codomain is isomorphic to the free Abelian group $H_1(\partial M)$. It follows that the former map β collapses, and the conclusion follows. □

Remark 1.3. The set of Spin-*structures on M relative to* σ is defined to be

$$\mathrm{Spin}(M, \sigma) = \{\alpha \in \mathrm{Spin}(M) \ : \ \alpha|_{\partial M} = \sigma\},$$

which may be empty. One can construct a canonical map

$$\mathrm{Spin}(M, \sigma) \xrightarrow{\beta} \mathrm{Spin}^c(M, \sigma)$$

by means of a rigid Spin-structure s on $\mathrm{T}M|_{\partial M}$ representing σ and the map β defined in Section 1.2.6 from $\mathrm{Spin}(M, s)$ to $\mathrm{Spin}^c(M, \beta s)$.

1.3.2. Spin^c*-structures as vector fields: the closed case*

Let M be a closed 3-manifold. We recall Turaev's definition [28] of an Euler structure on M, and how this corresponds to a Spin^c-structure on M.

Lemma 1.3. *The group* $\mathrm{Spin}^c(3)$ *can be identified with* $\mathrm{U}(2)$ *in such a way that the diagram*

$$\begin{array}{ccccc} \mathrm{SO}(2) & \xrightarrow{\simeq} & \mathrm{U}(1) & \rightarrowtail & \mathrm{U}(2) \\ \downarrow & & & & \downarrow \simeq \\ \mathrm{SO}(3) & & \xleftarrow[\pi]{} & & \mathrm{Spin}^c(3) \end{array}$$

commutes. Here, π *is the canonical projection,* $\mathrm{SO}(2)$ *is identified with* $\mathrm{U}(1)$ *in the usual way and is embedded into* $\mathrm{SO}(3)$ *by* $A \mapsto (1) \oplus A$, *whereas* $\mathrm{U}(1)$ *is embedded into* $\mathrm{U}(2)$ *by* $A \mapsto A \oplus (1)$.

Proof. There is a well-known way to construct a 2-fold covering from SU(2) onto SO(3), which consists in identifying SU(2) with the group of unit quaternions, $\mathbb{R}^3$ with the space of pure quaternions and making the former act on the latter by conjugation. Thus, we get a unique group isomorphism $\mathrm{SU}(2) \xrightarrow{\simeq} \mathrm{Spin}(3)$ which makes the two projections onto SO(3) commute. Then, the isomorphism

$$\frac{\mathrm{SU}(2) \times \mathrm{U}(1)}{\mathbb{Z}_2} \xrightarrow{\simeq} \mathrm{U}(2)$$

sending $[(A, z)]$ to zA induces a group isomorphism $\mathrm{U}(2) \xrightarrow{\simeq} \mathrm{Spin}^c(3)$. The reader may easily verify the commutativity of the above diagram. □

Definition 1.4. An *Euler structure* on M is a punctured homotopy class of nonsingular vector fields on M. Precisely, two nonsingular vector fields v and v' on M are considered as equivalent, when there exists a point $x \in M$ such that the restrictions of v and v' to $M \setminus x$ are homotopic among nonsingular vector fields on $M \setminus x$. The set of Euler structures on M is denoted by $\mathrm{Eul}(M)$.

If a cellular decomposition of M is given, punctured homotopy coincides with homotopy on the 2-skeleton of M. Then, obstruction theory applied to the bundle of non-zero vectors tangent to M says that Euler structures do exist (Poincaré–Hopf theorem: $\chi(M) = 0$) and that they form a $H^2(M; \pi_2(\mathrm{T}_y M \setminus 0))$-affine space (where $y \in M$). Since M has come with an orientation, $\mathrm{Eul}(M)$ is naturally a $H^2(M)$-affine space.

Lemma 1.4 (*Turaev [29]*). *There exists a canonical* $H^2(M)$*-equivariant bijection*

$$\mathrm{Eul}(M) \xrightarrow{\mu} \mathrm{Spin}^c(M).$$

Proof. Let v be a nonsingular vector field on M. We are going to associate to v a Spin^c-structure in the usual sense (see Lemma 1.1) and, for this, we need to endow M with a metric. Orient $\langle v \rangle^{\perp}$, the orthogonal complement of $\langle v \rangle$ in TM, with the "right hand" rule (v being taken as right thumb). Then, $\mathrm{SO}(\langle v \rangle^{\perp})$ is a reduction of $\mathrm{SO}(\mathrm{T}M)$ with respect to the inclusion of SO(2) into SO(3) defined by $A \mapsto (1) \oplus A$. The bundle $\mathrm{SO}(\langle v \rangle^{\perp})$, together with the homomorphism $\mathrm{SO}(2) \simeq \mathrm{U}(1) \to \mathrm{U}(2)$ defined by $A \mapsto A \oplus (1)$, induces a principal U(2)-bundle η. According to Lemma 1.3, η can be declared to be a principal $\mathrm{Spin}^c(3)$-bundle and can be accompanied with an isomorphism $H : \eta/\mathrm{U}(1) \to \mathrm{SO}(\mathrm{T}M)$. The Spin^c-structure $[(\eta, H)]$ on M only depends on the punctured homotopy class of v, and we set $\mu([v]) = [(\eta, H)]$. The map μ can be verified to be $H^2(M)$-equivariant. □

Remark 1.4. Let $[v]$ be an Euler structure on M. The isomorphism class of principal $\mathrm{U}(1)$-bundles induced by the Spin^c-structure $\mu([v])$ in Section 1.2.8 is represented by $\mathrm{SO}(\langle v^{\perp}\rangle)$, since the homomorphism $\mathrm{Spin}^c(3) \to \mathrm{U}(1)$ used there corresponds to the determinant map through the isomorphism $\mathrm{Spin}^c(3) \simeq \mathrm{U}(2)$ of Lemma 1.3. Consequently, the Chern class of $\mu([v])$ is the Euler class $e(\mathrm{T}M/\langle v\rangle)$, i.e., the obstruction to find a nonsingular vector field on M transverse to v.

According to the previous remark, Spin^c-structures arising from Spin-structures correspond to nonsingular vector fields on M which can be completed.

More precisely, let a *parallelization* of M be a punctured homotopy class of trivializations $t=(t_1, t_2, t_3)$ of the oriented vector bundle TM, and denote the set of such structures by $\mathrm{Parall}(M)$. Obstruction theory applied to the bundle of oriented frames of M says that parallelizations do exist (Stiefel theorem: $w_2(M)=0$) and that they form a $H^1(M;\mathbb{Z}_2)$-affine space. (In the case of trivializations of TM, homotopy on the 2-skeleton coincides with homotopy on the 1-skeleton since $\pi_2(\mathrm{GL}_+(3)) = 0$.) Thus, one obtains the following well-known fact [16,23].

Lemma 1.5. *There exists a canonical $H^1(M;\mathbb{Z}_2)$-equivariant bijection*

$$\mathrm{Parall}(M) \xrightarrow{\mu} \mathrm{Spin}(M).$$

Define a map $\beta : \mathrm{Parall}(M) \to \mathrm{Eul}(M)$ by $\beta([t]) = [t_1]$ for any trivialization $t=(t_1, t_2, t_3)$ of TM. The next lemma follows from the definitions.

Lemma 1.6. *The following diagram is commutative*:

$$\begin{array}{ccc} \mathrm{Parall}(M) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}(M) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \mathrm{Eul}(M) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}^c(M). \end{array}$$

1.3.3. Spin^c*-structures as vector fields: the boundary case*

Let M be a 3-manifold with boundary. We define Euler structures on M which are relative to a homotopy class of trivializations of $\mathbb{R} \oplus \mathrm{T}\partial M$. We start with a preliminary observation.

What has been done in Section 1.3.2 for the oriented tangent bundle of a closed 3-manifold works for any 3-dimensional oriented vector bundle. In particular, if S is a closed surface, Section 1.3.2 can be repeated for $\mathbb{R} \oplus \mathrm{T}S$. This repetition ends with the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Parall}\,(\mathbb{R}\oplus \mathrm{T}S) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}(\mathbb{R}\oplus \mathrm{T}S) & \xleftarrow{\simeq} & \mathrm{Spin}(S) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \\ \mathrm{Eul}\,(\mathbb{R}\oplus \mathrm{T}S) & \xrightarrow[\simeq]{\mu} & \mathrm{Spin}^c(\mathbb{R}\oplus \mathrm{T}S) & \xleftarrow{\simeq} & \mathrm{Spin}^c(S). \end{array}$$

The only change is that, because the base space S is now 2-dimensional, homotopies are not punctured anymore. An *Euler structure* on $\mathbb{R} \oplus \mathrm{T}S$ is defined to be a homotopy class of nonsingular sections of this vector bundle and, similarly, a *parallelization* on $\mathbb{R} \oplus \mathrm{T}S$ is a homotopy class of trivializations of this oriented vector bundle.

Example 1.2. Thus, the section $v = (1, 0)$ of $\mathbb{R} \oplus \mathrm{T}S$ determines a Spin^c-structure $\mu([v])$ on the surface S. By Remark 1.4, the Chern class of $\mu([v])$ coincides with the Euler class $e(\mathrm{T}S)$ of the surface S.

In the sequel, we fix a parallelization τ on $\mathbb{R} \oplus \mathrm{T}\partial M$. The observation at the beginning of Section 1.2.5 allows us to identify τ with a homotopy class of trivializations of $\mathrm{T}M|_{\partial M}$.

Fix, in this paragraph, a nonsingular section s of $\mathrm{T}M|_{\partial M}$. An *Euler structure on M relative to s* is a punctured homotopy class rel ∂M of nonsingular vector fields on M that extend s. We denote by $\mathrm{Eul}(M, s)$ the set of such structures. Obstruction theory says that there is an obstruction $w(M, s) \in H^3(M, \partial M)$ to the existence of such structures and, when the latter happens to vanish, that the set $\mathrm{Eul}(M, s)$ is naturally a $H^2(M, \partial M)$-affine space. (Here, again, we use the given orientation of M to make $\mathbb{Z}$ the coefficients group.) As an application of the Poincaré–Hopf theorem and obstruction calculi on the double $M \cup_{\mathrm{Id}} (-M)$, one obtains that

$$2 \cdot \langle w(M, s), [M, \partial M]\rangle = \langle e(\mathrm{T}M|_{\partial M}/\langle s\rangle), [\partial M]\rangle \in \mathbb{Z}. \tag{1.2}$$

The following lemma can be proved formally the same way as Lemma 1.2. The first statement is also a direct consequence of (1.2).

Lemma 1.7. *For any trivialization $t = (t_1, t_2, t_3)$ of $\mathrm{T}M|_{\partial M}$ representing τ (which we denote by $t \in \tau$), the nonsingular vector field t_1 can be extended to M. Moreover, for any $t, t' \in \tau$, there exists a canonical $H^2(M, \partial M)$-equivariant bijection*

$$\mathrm{Eul}\,(M, t_1) \xrightarrow{\rho_{t,t'}} \mathrm{Eul}(M, t'_1).$$

Lastly, for any $t, t', t'' \in \tau$, we have that $\rho_{t',t''} \circ \rho_{t,t'} = \rho_{t,t''}$.

Definition 1.5. An *Euler structure on M relative to* τ is a pair (v, t) where $t \in \tau$ and $v \in \mathrm{Eul}(M, t_1)$, two such pairs (v, t) and (v', t') being considered as equivalent when $v' = \rho_{t,t'}(v)$. The set of such structures is denoted by $\mathrm{Eul}(M, \tau)$ and can naturally be given the structure of a $H^2(M, \partial M)$-affine space.

Remark 1.5. Following Turaev, one can describe concretely how a $x \in H^2(M, \partial M)$ acts on a $[(v, t)] \in \mathrm{Eul}(M, \tau)$. Let $P^{-1}x \in H_1(M)$ be represented by a smooth oriented knot $K \subset \mathrm{int}(M)$, and let v' be the vector field obtained from v by "Reeb turbulentization" along K (see [28, Section 5.2]). Then, (v', t) represents $[(v, t)] + x$.

The following relative version of Lemma 1.4 can be proved similarly.

Lemma 1.8. *There exists a canonical $H^2(M, \partial M)$-equivariant bijection*

$$\mathrm{Eul}(M, \tau) \xrightarrow{\mu} \mathrm{Spin}^c\,(M, \mu(\tau))\,.$$

1.3.4. Relative Chern classes

Let M be a 3-manifold with boundary and let σ be a Spin-structure on ∂M. In the relative case too, there is a *Chern class* map

$$\mathrm{Spin}^c(M, \sigma) \xrightarrow{c} H^2(M, \partial M),$$

which is affine over the doubling map. It can be defined directly (using Remark 1.2), or undirectly regarding relative Spin^c-structures as classes of vector fields (Section 1.3.3). This is done in the next paragraph.

Let τ be the parallelization on $\mathbb{R} \oplus \mathrm{T}\partial M$ corresponding to σ by μ. For any trivialization t of $\mathrm{T}M|_{\partial M}$ representing τ and for any nonsingular vector field v on M extending t_1, we can consider the relative Euler class

$$e(\mathrm{T}M/\langle v\rangle, t_2) \in H^2(M, \partial M),$$

i.e., the obstruction to extend the nonsingular section t_2 of $\mathrm{T}M/\langle v\rangle$ from ∂M to the whole of M. Clearly, this only depends on the equivalence class $[(v,t)]$ of (v,t) in the sense of Definition 1.5. Thus, we get a canonical map

$$\mathrm{Eul}(M,\tau) \longrightarrow H^2(M, \partial M)$$

which can be verified to be affine over the doubling map thanks to Remark 1.5. Its composition with μ^{-1} is defined to be c. (Compare with Remark 1.4.)

Remark 1.6. For any $\alpha \in \mathrm{Spin}^c(M,\sigma)$, the Chern class $c(\alpha)$ vanishes if and only if α comes from the set $\mathrm{Spin}(M,\sigma)$ defined in Remark 1.3.

We now compute the modulo 2 reduction of a relative Chern class. First, recall that the cobordism group Ω_1^{Spin} is isomorphic to $\mathbb{Z}_2$ [16,23]. For a closed surface S, there is the Atiyah–Johnson correspondence

$$\mathrm{Spin}(S) \underset{\simeq}{\xrightarrow{q}} \mathrm{Quad}(S)$$

between spin structures on S and quadratic functions with the modulo 2 intersection pairing of S as associated bilinear pairing [1,13]. The quadratic function $q_\sigma : H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$ corresponding to $\sigma \in \mathrm{Spin}(S)$ is defined by

$$q_\sigma([\gamma]) = [(\gamma, \sigma|_\gamma)] \in \Omega_1^{\mathrm{Spin}} \simeq \mathbb{Z}_2$$

for any oriented simple closed curve γ on S.

Lemma 1.9. *The following identity holds for any* $\alpha \in \mathrm{Spin}^c(M,\sigma)$:

$$\forall y \in H_2(M,\partial M),\ \langle c(\alpha), y\rangle \bmod 2 = q_\sigma(\partial_*(y)).$$

Here, $\partial_* : H_2(M,\partial M) \to H_1(\partial M)$ *denotes the connecting homomorphism of the pair* $(M,\partial M)$ *and is followed by the modulo* 2 *reduction.*

Proof. The modulo 2 reduction of $c(\alpha)$ is

$$w_2(M,\sigma) \in H^2(M,\partial M;\mathbb{Z}_2),$$

i.e., the obstruction to extend σ to the whole manifold M. Let Σ be a connected immersed surface in M such that $\partial\Sigma$ is $\partial M \cap \Sigma$, $\partial\Sigma$ has no singularity and Σ represents the modulo 2 reduction of y. Then, $\langle c(\alpha), y\rangle \bmod 2 = \langle w_2(M,\sigma), [\Sigma]\rangle$ is equal to $\langle w_2(\Sigma, \sigma|_{\partial\Sigma}), [\Sigma]\rangle$ and so is the obstruction to extend the

Spin-structure $\sigma|_{\partial\Sigma}$ to the whole surface Σ. Since Σ is connected, this is the class of $(\partial\Sigma, \sigma|_{\partial\Sigma})$ in Ω_1^{Spin}. Thus, we have that $\langle c(\alpha), y\rangle \bmod 2 = q_\sigma([\partial\Sigma]) = q_\sigma(\partial_*(y))$. □

Example 1.3. Suppose that ∂M is a disjoint union of tori. Let τ^0 be the distinguished parallelization corresponding to the distinguished Spin-structure σ^0 on the 2-torus (see Example 1.1). An Euler structure on M relative to the union of copies of τ^0 is equivalent to a relative Euler structure in the sense of Turaev [28, Section 5.1]. Lemma 1.9 is a generalization of [31, Lemma 1.3].

1.3.5. Spin^c*-structures as stable complex structures*

We conclude this subsection devoted to the dimension 3 by recalling that, in this case, a Spin^c-structure is equivalent to a stable complex structure on the oriented tangent bundle.

Lemma 1.10. *If M is a closed 3-manifold, then the canonical map*

$$\mathrm{U}^s(M) \xrightarrow{\omega} \mathrm{Spin}^c(M)$$

introduced in Section 1.2.7 is bijective.

Proof. Endow M with a Riemannian metric and consider a nonsingular vector field v on M. Then, $\mathbb{R}\oplus\mathrm{T}M$ splits as $(\mathbb{R}\oplus\langle v\rangle)\oplus\langle v\rangle^\perp$, which is the sum of two oriented 2-dimensional vector bundles. So, via the inclusion of $\mathrm{U}(1)\times\mathrm{U}(1)$ into $\mathrm{U}(2)$ defined by $(A, B)\mapsto (A)\oplus(B)$, v defines a complex structure J_v on $\mathbb{R}\oplus\mathrm{T}M$. Thus, we get a map from $\mathrm{Eul}(M)$ to the set of stable complex structures on $\mathrm{T}M$ up to punctured homotopy. By obstruction theory applied to the fibration $\mathrm{BU}\to\mathrm{BSO}$ with fiber type SO/U, the latter set is a $H^2(M)$-affine space and that map is $H^2(M)$-equivariant. Thus, since $\pi_3(\mathrm{SO}/\mathrm{U})$ is zero, we get a bijective map

$$\mathrm{Eul}(M) \underset{\simeq}{\xrightarrow{J}} \mathrm{U}^s(M).$$

It can be verified that $\omega\circ J$ is the map μ from Lemma 1.4. (This verification amounts to checking that some two group homomorphisms from $\mathrm{U}(1)$ to $\mathrm{Spin}^c(4)$ coincide.) □

1.4. Gluing of complex spin structures

In this subsection, we deal with the technical problem of gluing Spin^c-structures. We formulate the gluing in terms of (rigid) Spin^c-structures, but the reader may easily translate the statement and the proof in terms of vector fields and Euler structures.

Let M be a closed n-manifold obtained by gluing two n-manifolds M_1 and M_2 along their boundaries:

$$M = M_1 \cup_f M_2.$$

This involves a positive diffeomorphism $f : -\partial M_2 \to \partial M_1$ as well as a collar neighborhood of ∂M_i in M_i. The inclusion $M_i \hookrightarrow M$ will be denoted by j_i.

Lemma 1.11. *For $i = 1, 2$, let s_i be a rigid* Spin^c*-structure on $\mathrm{T}M_i|_{\partial M_i}$. Having identified $\mathbb{R}\oplus\mathrm{T}\partial M_i$ with $\mathrm{T}M_i|_{\partial M_i}$ thanks to the collar, we assume that $s_1\circ(-\mathrm{Id}\oplus\mathrm{T}f) = s_2$. If the relative obstructions*

$w(M_i, s_i)$'s vanish, then the absolute obstruction $w(M)$ does too and there is a canonical gluing *map*

$$\mathrm{Spin}^c(M_1, s_1) \times \mathrm{Spin}^c(M_2, s_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M)$$

which is affine over

$$\begin{array}{ccc} H^2(M_1, \partial M_1) \oplus H^2(M_2, \partial M_2) & \dashrightarrow & H^2(M) \\ {\scriptstyle P^{-1}\times P^{-1}} \downarrow & & \uparrow {\scriptstyle P} \\ H_{n-2}(M_1) \oplus H_{n-2}(M_2) & \xrightarrow{j_{1,*}\oplus j_{2,*}} & H_{n-2}(M). \end{array}$$

Proof. For $i = 1, 2$, let $\alpha_i \in \mathrm{Spin}^c(M_i, s_i)$ be represented by a rigid structure a_i. The structures a_1 and a_2 can be glued together by means of Tf: we obtain a rigid Spin^c-structure on M whose homotopy class does not depend on the choices of a_1 and a_2 in α_1 and α_2, respectively. We denote it by $\alpha_1 \cup_f \alpha_2 \in \mathrm{Spin}^c(M)$.

Let us prove that this map $\cup_f$ is affine. For $i = 1, 2$, let $\mathscr{C}_i$ be a smooth triangulation of M_i such that $\mathscr{C}_1|_{\partial M_1}$ corresponds to $\mathscr{C}_2|_{\partial M_2}$ by f. We denote by $\mathscr{C}_i^*$ the cellular decomposition of M_i dual to the triangulation $\mathscr{C}_i$.

On the one hand, we consider the *union* $\mathscr{C}$ of the triangulations $\mathscr{C}_1$ and $\mathscr{C}_2$: a simplex of $\mathscr{C}$ is a simplex of $\mathscr{C}_i$ for $i = 1$ or 2, and simplices of ∂M_1 are identified with simplices of ∂M_2 by f. On the other hand, we consider the *gluing* $\mathscr{C}^*$ of the cellular decompositions $\mathscr{C}_1^*$ and $\mathscr{C}_2^*$: a cell of $\mathscr{C}^*$ either is a cell of $\mathscr{C}_i^*$ which does not intersect ∂M_i, either is the gluing by f of a cell belonging to $\mathscr{C}_1^*$ with a cell of $\mathscr{C}_2^*$ along a face lying in $\partial M_1 \cong -\partial M_2$. Then, $\mathscr{C}$ is a smooth triangulation of M and $\mathscr{C}^*$ is its dual cellular decomposition. Cohomology will be calculated with $\mathscr{C}$ while homology will be computed with $\mathscr{C}^*$.

For $i = 1, 2$, consider some $\alpha_i, \alpha_i' \in \mathrm{Spin}^c(M_i, s_i)$ and set $\alpha = \alpha_1 \cup_f \alpha_2$ and $\alpha' = \alpha_1' \cup_f \alpha_2'$. We want to prove the following equality:

$$j_{1,*}P^{-1}(\alpha_1 - \alpha_1') + j_{2,*}P^{-1}(\alpha_2 - \alpha_2') = P^{-1}(\alpha - \alpha') \in H_{n-2}(M). \tag{1.3}$$

For $i = 1, 2$, let $a_i, a_i' \in \mathrm{Spin}_r^c(M_i)$ represent α_i and α_i' respectively and coincide on the 1-skeleton of $\mathscr{C}_i$ (and, of course, on ∂M_i). Suppose that we have fixed a morphism of oriented vector bundles $\mathrm{T}M_i \to \gamma_{\mathrm{SO}(n)}$: then, the rigid structures a_i and a_i' can be identified with lifts $M_i \to \mathrm{BSpin}^c(n)$ by Bπ of the base maps $M_i \to \mathrm{BSO}(n)$. The obstruction $\alpha_i - \alpha_i' \in H^2(M_i, \partial M_i)$ is the class of the 2-cocycle which assigns to each 2-simplex e_k^i of $\mathscr{C}_i$ outside ∂M_i, this element z_k^i of $\pi_2(\mathrm{BU}(1)) \simeq \pi_2(K(\mathbb{Z}, 2)) \simeq \mathbb{Z}$ obtained by gluing $a_i|_{e_k^i}$ and $a_i'|_{e_k^i}$ along ∂e_k^i. So, we have that $P^{-1}(\alpha_i - \alpha_i') = [\sum_k z_k^i \cdot e_k^{*,i}]$ if $e_k^{*,i}$ denotes the $(n-2)$-cell dual to e_k^i.

Moreover, $a := a_1 \cup_f a_2$ and $a' := a_1' \cup_f a_2'$ represent α and α' respectively. Using these rigid structures, we can describe explicitely a 2-cocycle representing $\alpha - \alpha'$ as well. This 2-cocycle sends any 2-simplex of $\mathscr{C}_1 \cup_f \mathscr{C}_2$ contained in $\partial M_1 \cong -\partial M_2$ to $0 \in \mathbb{Z}$ so that $P^{-1}(\alpha - \alpha')$ is represented by $\sum_k z_k^1 \cdot e_k^{*,1} + \sum_k z_k^2 \cdot e_k^{*,2}$. □

Suppose now that the manifolds have dimension $n = 3$. This is the gluing lemma that we will use in the next sections.

Lemma 1.12. *Let* $\sigma_1 \in \mathrm{Spin}(\partial M_1)$ *and* $\sigma_2 \in \mathrm{Spin}(\partial M_2)$ *be such that* $f^*(\sigma_1) = -\sigma_2$. *Then, there is a canonical* gluing *map*

$$\mathrm{Spin}^c(M_1, \sigma_1) \times \mathrm{Spin}^c(M_2, \sigma_2) \xrightarrow{\cup_f} \mathrm{Spin}^c(M)$$

which is affine over

$$\begin{array}{ccc} H^2(M_1, \partial M_1) \oplus H^2(M_2, \partial M_2) & \dashrightarrow & H^2(M) \\ \Big\downarrow {\scriptstyle P^{-1}\times P^{-1}} & & \Big\uparrow {\scriptstyle P} \\ H_1(M_1) \oplus H_1(M_2) & \xrightarrow{j_{1,*}\oplus j_{2,*}} & H_1(M). \end{array}$$

Moreover, for any $\alpha_1 \in \mathrm{Spin}^c(M_1, \sigma_1)$ *and* $\alpha_2 \in \mathrm{Spin}^c(M_2, \sigma_2)$, *the following identity between Chern classes holds*:

$$P^{-1}c\left(\alpha_1 \cup_f \alpha_2\right) = j_{1,*}P^{-1}c(\alpha_1) + j_{2,*}P^{-1}c(\alpha_2) \in H_1(M).$$

Proof. Choose a rigid Spin-structure s_1 on $\mathrm{T}M_1|_{\partial M_1}$ representing σ_1, which we denoted by $s_1 \in \sigma_1$. This induces a $s_2 \in \sigma_2$ by setting $s_2 = s_1 \circ (-\mathrm{Id} \oplus \mathrm{T}f)$. By Lemma 1.2, the obstructions $w(M_i, \beta s_i)$'s vanish and so, by Lemma 1.11, there is a gluing map with domain $\mathrm{Spin}^c(M_1, \beta s_1) \times \mathrm{Spin}^c(M_2, \beta s_2)$.

Another choice $s_1' \in \sigma_1$ would induce another $s_2' \in \sigma_2$ and would lead to another gluing map this time with domain $\mathrm{Spin}^c(M_1, \beta s_1') \times \mathrm{Spin}^c(M_2, \beta s_2')$. Nevertheless, using the "double collar" of $\partial M_1 \cong -\partial M_2$ in *M*, one easily sees that the identifications $\rho_{s_1,s_1'}$ and $\rho_{s_2,s_2'}$ from Lemma 1.2 make those two gluing maps agree.

The first assertion of the lemma then follows. The second one is proved with arguments similar to those used in the proof of Lemma 1.11 (gluing of obstructions in compact oriented manifolds using Poincaré duality). □

Remark 1.7. If *M* is obtained by gluing M_1 and M_2 along only part of their boundaries (so that $\partial M \neq \emptyset$), Lemma 1.12 can easily be generalized to produce Spin^c-structures on *M* relative to a fixed Spin-structure on its boundary.

2. Linking quadratic function of a three-manifold with complex spin structure

In this section, we define the quadratic function $\phi_{M,\sigma}$ associated to a closed oriented 3-manifold *M* equipped with a Spin^c-structure σ. We present its elementary properties and connect it to previously known constructions.

2.1. Quadratic functions on torsion Abelian groups

We fix some notations. If *A* and *B* are Abelian groups and if $b : A \times A \to B$ is a symmetric bilinear pairing, we denote by $\widehat{b} : A \to \mathrm{Hom}(A, B)$ the adjoint map. The pairing *b* is said to be *nondegenerate* (respectively *nonsingular*) if $\widehat{b}$ is injective (respectively bijective). We denote by A^* the group $\mathrm{Hom}(A, \mathbb{Z})$ when *A* is free, the group $\mathrm{Hom}(A, \mathbb{Q})$ when *A* is a $\mathbb{Q}$-vector space and the group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ when *A* is torsion. Lastly, application of the functor $- \otimes \mathbb{Q}$ is indicated by a subscript $\mathbb{Q}$.

2.1.1. Basic notions about quadratic functions

Let G be a torsion Abelian group.

A map $q : G \to \mathbb{Q}/\mathbb{Z}$ is said to be a *quadratic function* on G if

$$b_q(x, y) = q(x + y) - q(x) - q(y)$$

defines a (symmetric) bilinear pairing $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$. The quadratic function q is said to be *nondegenerate* if b_q is nondegenerate, and *homogeneous* if $q(-x) = q(x)$ for any $x \in G$. Apart from the bilinear pairing b_q, one can associate to q its *radical*

$$\mathrm{Ker}(q) = \mathrm{Ker}\,\widehat{b_q} \subset G,$$

its *homogeneity defect*

$$d_q : G \to \mathbb{Q}/\mathbb{Z}, \quad x \mapsto q(x) - q(-x)$$

and, in case when G happens to be finite, its *Gauss sum*

$$\gamma(q) = \sum_{x \in G} \exp(2i\pi q(x)) \in \mathbb{C}.$$

Given a symmetric bilinear pairing $b : G \times G \to \mathbb{Q}/\mathbb{Z}$, we say that $q : G \to \mathbb{Q}/\mathbb{Z}$ is a quadratic function *over* b if $b_q = b$. The group G^* acts freely and transitively on $\mathrm{Quad}(b)$, the set of quadratic functions over b, just as maps $G \to \mathbb{Q}/\mathbb{Z}$ add up. So, $\mathrm{Quad}(b)$ is a G^*-affine space.

There is a procedure to produce quadratic functions on torsion Abelian groups, known as the "discriminant" construction.

2.1.2. The discriminant construction

In the literature, the discriminant construction is usually restricted to nondegenerate bilinear lattices and produces quadratic functions on finite Abelian groups. The general case has been considered in [5], to which we refer for details and proofs. Here, we briefly review the construction.

A *lattice* H is a free finitely generated Abelian group. A *bilinear lattice* (H, f) is a symmetric bilinear pairing $f : H \times H \to \mathbb{Z}$ on a lattice H. Let also

$$H^\sharp = \{x \in H_\mathbb{Q} : f_\mathbb{Q}(x, H) \subset \mathbb{Z}\}$$

be the dual lattice. A *Wu class* for (H, f) is an element $w \in H$ such that

$$\forall x \in H, \ \ f(x, x) - f(w, x) \in 2\mathbb{Z}.$$

A *characteristic form* for (H, f) is an element $c \in H^* = \mathrm{Hom}(H, \mathbb{Z})$ satisfying

$$\forall x \in H, \ \ f(x, x) - c(x) \in 2\mathbb{Z}.$$

The sets of characteristic forms and Wu classes for (H, f) are denoted by $\mathrm{Char}(f)$ and $\mathrm{Wu}(f)$ respectively. Those sets are not empty and are related by the map $w \mapsto \widehat{f}(w)$, $\mathrm{Wu}(f) \to \mathrm{Char}(f)$.

Let (H, f) be a bilinear lattice. Consider the torsion Abelian group $G_f = H^\sharp/H$ and the map

$$L_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}, \ \ ([x], [y]) \mapsto f_\mathbb{Q}(x, y) \bmod 1.$$

The pairing L_f is symmetric and bilinear, with radical $\mathrm{Ker}\,\widehat{L_f} \simeq (\mathrm{Ker}\,\widehat{f}) \otimes \mathbb{Q}/\mathbb{Z}$.

Observe that the adjoint map $\widehat{f_{\mathbb{Q}}} : H_{\mathbb{Q}} \to H^*_{\mathbb{Q}}$ restricted to $H^\sharp$ induces an epimorphism $G_f \to$ Tors Coker $\widehat{f}$. Hence the short exact sequence

$$0 \to \text{Ker } \widehat{L_f} \to G_f \to \text{Tors Coker } \widehat{f} \to 0, \tag{2.1}$$

which can be verified to split (non-canonically). Therefore, G_f is the direct sum of a finite Abelian group with as many copies of $\mathbb{Q}/\mathbb{Z}$ as the rank of Ker $\widehat{f}$. It follows also from (2.1) that the pairing L_f factors to a nondegenerate symmetric bilinear pairing

$$\text{Tors Coker } \widehat{f} \times \text{Tors Coker } \widehat{f} \xrightarrow{\lambda_f} \mathbb{Q}/\mathbb{Z}.$$

The bilinear map $H^* \times H^\sharp \to \mathbb{Q}$ defined by $(\alpha, x) \mapsto \alpha_{\mathbb{Q}}(x)$ induces a bilinear pairing

$$\text{Coker } \widehat{f} \times G_f \xrightarrow{\langle -,- \rangle} \mathbb{Q}/\mathbb{Z} \tag{2.2}$$

which is left nondegenerate and right nonsingular. It is left nonsingular if and only if *f* is nondegenerate.

Let now (H, f, c) be a bilinear lattice equipped with a characteristic form $c \in H^*$. One can associate to this triple a quadratic function over L_f, namely

$$\phi_{f,c} : G_f \to \mathbb{Q}/\mathbb{Z}, \quad [x] \mapsto \frac{1}{2}(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x)) \text{ mod } 1.$$

Definition 2.1. The assignation $(H, f, c) \mapsto (G_f, \phi_{f,c})$ is called the *discriminant* construction.

Let us make a few observations about this construction. First, note that $\phi_{f,c}$ depends on *c* only mod $2\widehat{f}(H)$. Second, the Abelian group $H^*/\widehat{f}(H) = \text{Coker } \widehat{f}$ acts freely and transitively on $\text{Char}(f)/2\widehat{f}(H)$ by setting

$$\forall [\alpha] \in \text{Coker } \widehat{f}, \quad \forall [c] \in \text{Char}(f)/2\widehat{f}(H), \quad [c] + [\alpha] = [c + 2\alpha] \in \text{Char}(f)/2\widehat{f}(H).$$

Third, since Ker $\widehat{L_f}$ is canonically isomorphic to (Ker $\widehat{f}$) $\otimes \mathbb{Q}/\mathbb{Z}$, any form Ker $\widehat{f} \to \mathbb{Z}$ induces a homomorphism Ker $\widehat{L_f} \to \mathbb{Q}/\mathbb{Z}$. Thus, we get a homomorphism $j_f : (\text{Ker } \widehat{f})^* \to (\text{Ker } \widehat{L_f})^*$.

Theorem 2.1 ([5]). *The assignation* $c \mapsto \phi_{f,c}$ *induces an embedding*

$$\text{Char}(f)/2\widehat{f}(H) \overset{\phi_f}{\hookrightarrow} \text{Quad}(L_f)$$

which is affine over the opposite of the left adjoint of the pairing (2.2). *Moreover, a function* $q \in \text{Quad}(L_f)$ *belongs to* Im ϕ_f *if and only if* $q|_{\text{Ker } \widehat{L_f}}$ *belongs to* Im j_f.

Remark 2.1. The map ϕ_f is bijective if and only if *f* is nondegenerate.

We now use the algebraic notions above as combinatorial descriptions of topological notions.

2.2. Combinatorial descriptions associated to a surgery presentation

In this subsection, we fix an ordered oriented framed *n*-component link *L* in $\mathbf{S}^3$.

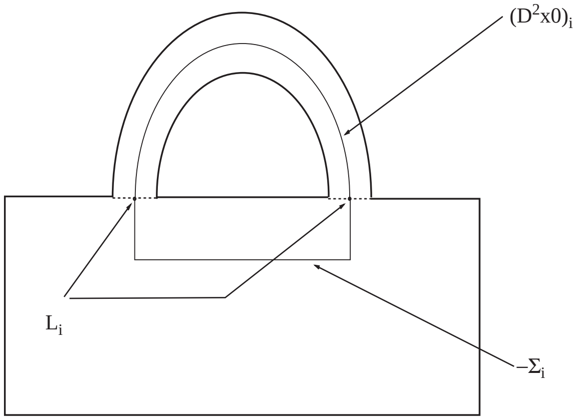

Fig. 1. The preferred basis of $H_2(W_L)$.

We call V_L the 3-manifold obtained from $\mathbf{S}^3$ by surgery along L and we denote by W_L the *trace* of the surgery:

$$V_L = \partial W_L \quad \text{with } W_L = \mathbf{D}^4 \cup \bigcup_{i=1}^{n} (\mathbf{D}^2 \times \mathbf{D}^2)_i,$$

where the 2-handle $(\mathbf{D}^2 \times \mathbf{D}^2)_i$ is attached by embedding $-(\mathbf{S}^1 \times \mathbf{D}^2)_i$ into $\mathbf{S}^3 = \partial \mathbf{D}^4$ in accordance with the specified framing and orientation of L_i.

The group $H_2(W_L)$ is free Abelian of rank n, and is given the *preferred* basis $([S_1], \ldots, [S_n])$ defined as follows. The closed surface S_i is taken to be $(\mathbf{D}^2 \times 0)_i \cup (-\Sigma_i)$, where Σ_i is a Seifert surface for L_i in $\mathbf{S}^3$ which has been pushed off into the interior of $\mathbf{D}^4$ as shown in Fig. 1. The group $H^2(W_L)$ is identified with $\mathrm{Hom}(H_2(W_L), \mathbb{Z})$ by Kronecker evaluation, and is given the dual basis. In the sequel, we simplify the notations by setting $H = H_2(W_L)$ (so that $H^2(W_L)$ is identified with H^*) and by denoting by $f : H \times H \to \mathbb{Z}$ the intersection pairing of W_L. The matrix of f relatively to the preferred basis of H is the linking matrix

$$B_L = (b_{ij})_{i,j=1}^{n}$$

of L. Since (H, f) is a bilinear lattice, the constructions of Section 2.1 apply.

2.2.1. Combinatorial description of Spin-structures

We recall a combinatorial description of $\mathrm{Spin}(V_L)$ due to Blanchet [2]. Define the set

$$\mathscr{S}_L = \left\{ [r] = ([r_i])_{i=1}^{n} \in (\mathbb{Z}_2)^n \ : \ \forall i = 1, \ldots, n, \ \sum_{j=1}^{n} b_{ij} r_j \equiv b_{ii} \bmod 2 \right\}.$$

The elements of $\mathscr{S}_L$ are called *characteristic solutions* of B_L.

Lemma 2.1. *There are canonical bijections*

$$\mathrm{Spin}(V_L) \underset{\simeq}{\longrightarrow} \mathrm{Wu}(f)/2H \underset{\simeq}{\longrightarrow} \mathscr{S}_L.$$

Thus, $\mathscr{S}_L$ shall be referred to as the *combinatorial description of* $\mathrm{Spin}(V_L)$. A refined Kirby's theorem dealing with surgery presentations of closed 3-dimensional Spin-manifolds can be derived from this lemma [2, Theorem (I.1)].

Proof of Lemma 2.1. The preferred basis of H induces an isomorphism $H/2H \simeq (\mathbb{Z}_2)^n$: the bijection between $\mathrm{Wu}(f)/2H$ and $\mathscr{S}_L$ is obtained this way. We now describe a bijection between $\mathrm{Spin}(V_L)$ and $\mathrm{Wu}(f)/2H$. Let σ be a Spin-structure on V_L. The obstruction $w_2(W_L, \sigma)$ to extend σ to W_L belongs to the group $H^2(W_L, V_L; \mathbb{Z}_2) \simeq H_2(W_L; \mathbb{Z}_2) \simeq H/2H$. Since $w_2(W_L, \sigma)$ is sent to $w_2(W_L)$ by the restriction map $H^2(W_L, V_L; \mathbb{Z}_2) \to H^2(W_L; \mathbb{Z}_2)$, a representative for $w_2(W_L, \sigma)$ in H has to be a Wu class for *f*. □

2.2.2. Combinatorial description of Spin^c*-structures*

Define the set

$$\mathscr{V}_L = \frac{\{s = (s_i)_{i=1}^n \in \mathbb{Z}^n \ : \ \forall i = 1, \ldots, n, \ s_i \equiv b_{ii} \bmod 2\}}{2 \cdot \mathrm{Im}\ B_L},$$

the elements of which are called *Chern vectors* of B_L. According to the following lemma, this set shall be referred to as the *combinatorial description of* $\mathrm{Spin}^c(V_L)$.

Lemma 2.2. *There are canonical bijections*

$$\mathrm{Spin}^c(V_L) \underset{\simeq}{\longrightarrow} \mathrm{Char}(f)/2\widehat{f}(H) \underset{\simeq}{\longrightarrow} \mathscr{V}_L.$$

Proof. The preferred basis of H defines an isomorphism $H^* \simeq \mathbb{Z}^n$, which induces a bijection between $\mathrm{Char}(f)/2\widehat{f}(H)$ and $\mathscr{V}_L$. The restriction map $\mathrm{Spin}^c(W_L) \to \mathrm{Spin}^c(V_L)$ is affine over the map $H^2(W_L) \to H^2(V_L)$ induced by inclusion. By exactness of the pair (W_L, V_L), the latter is surjective and its kernel coincides with the image of $\widehat{f} : H \to H^*$ (by Poincaré duality). Moreover, since $H^2(W_L)$ is free Abelian, a Spin^c-structure on W_L is determined by its Chern class in $H^2(W_L) \simeq H^*$. Such a class has to be a characteristic form for *f* since its modulo 2 reduction coincides with the second Stiefel–Whitney class $w_2(W_L) \in H^2(W_L; \mathbb{Z}_2) \simeq \mathrm{Hom}(H, \mathbb{Z}_2)$. Therefore, there is a bijection between $\mathrm{Spin}^c(V_L)$ and $\mathrm{Char}(f)/2\widehat{f}(H)$ defined by $\sigma \mapsto [c(\tilde{\sigma})]$ where $\tilde{\sigma}$ is an extension of σ to W_L. (This extension exists since $w(W_L, \sigma)$ lives in $H^3(W_L, V_L) = 0$, see Proposition 1.2.) □

If the Chern vector $[s]$ corresponds to the Spin^c-structure σ, we say that $(L, [s])$ is a *surgery presentation* of the closed 3-dimensional Spin^c-manifold (V_L, σ). On a diagram, we draw the framed link L using the blackboard framing convention, indicate its orientation and decorate each of its components L_i with the integer s_i.

Next, Kirby's theorem [15] can easily be extended to deal with surgery presentations of Spin^c-manifolds. This Spin^c version of Kirby's calculus will be used in the next section.

Theorem 2.2. *Let L and L' be ordered oriented framed links in* $\mathbf{S}^3$*. Equip them with Chern vectors* $[s]$ *and* $[s']$*, which correspond to* Spin^c*-structures* σ *and* σ' *on* V_L *and* $V_{L'}$ *respectively. Then, the* Spin^c*-manifolds* (V_L, σ) *and* $(V_{L'}, \sigma')$ *are* Spin^c*-diffeomorphic if and only if the pairs* $(L, [s])$ *and* $(L', [s'])$ *are, up to re-ordering and up to isotopy, related one to the other by a finite sequence of the moves drawn on Fig.* 2.

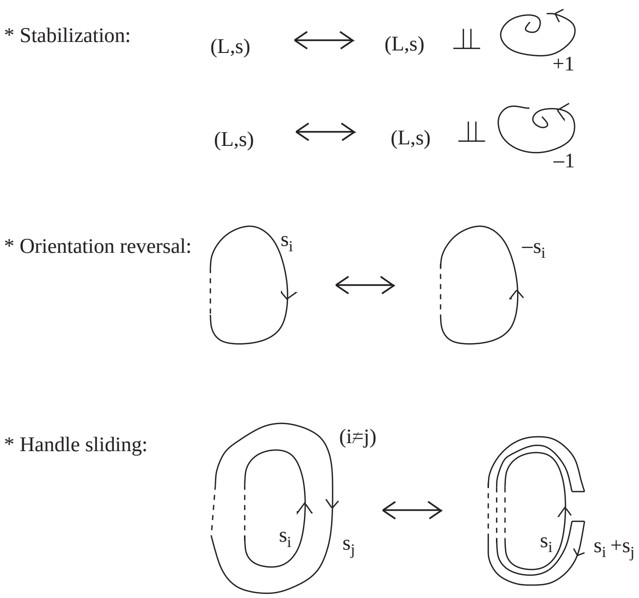

Fig. 2. Spin^c Kirby's moves. (Recall that the blackboard framing convention is used, and that labels refer to Chern vectors.)

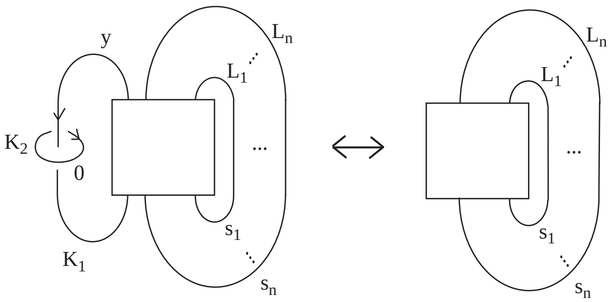

Fig. 3. Spin^c slam dunk move.

Proof. This follows from the usual Kirby's theorem. It suffices to show that, for each Kirby's move $L_1 \to L_2$, the corresponding canonical diffeomorphism $V_{L_1} \to V_{L_2}$ acts at the level of Spin^c-structures as combinatorially described on Fig. 2. This is a straightforward verification. □

Example 2.1. Look at the *slam dunk* move depicted in Fig. 3. Here, we are considering the ordered union $L \cup (K_1, K_2)$ of an n-component ordered oriented framed link L with an oriented framed knot K_1 together with its oriented meridian K_2. The move is

$$(L \cup (K_1, K_2), [(s_1, \dots, s_n, y, 0)]) \longleftrightarrow (L, [(s_1, \dots, s_n)]),$$

where y is the framing number of K_1. It relates two closed Spin^c-manifolds which are Spin^c-diffeomorphic, as can be shown by re-writing the proof of [7, Lemma 5] with Spin^c Kirby's calculi.

Remark 2.2. There exists a canonical isomorphism $\varrho : \mathrm{Coker}\ \widehat{f} \to H^2(V_L)$, as defined by the following commutative diagram:

$$\begin{array}{ccccccc} H^2(W_L, V_L) & \longrightarrow & H^2(W_L) & \longrightarrow & H^2(V_L) & \longrightarrow & 0 \\ \simeq \uparrow P & & \simeq \uparrow & & \simeq \uparrow \varrho & & \\ H & \xrightarrow{\widehat{f}} & H^* & \longrightarrow & \mathrm{Coker}\ \widehat{f} & \longrightarrow & 0. \end{array}$$

Then, the affine action of $H^2(V_L)$ on $\mathrm{Spin}^c(V_L)$ writes combinatorially:

$$\forall [x] \in \mathrm{Coker}\ \widehat{f},\ \forall [c] \in \mathrm{Char}(f)/2\widehat{f}(H),\ \ [c] + [x] = [c + 2x].$$

The Chern class map $c : \mathrm{Spin}^c(V_L) \to H^2(V_L)$ is combinatorially described by the map $c : \mathrm{Char}(f)/2\widehat{f}(H) \to \mathrm{Coker}\ \widehat{f},\ [c] \mapsto [c]$.

2.2.3. From Spin *to* Spin^c *in a combinatorial way*

We now relate the combinatorial description of $\mathrm{Spin}(V_L)$ to that of $\mathrm{Spin}^c(V_L)$.

Lemma 2.3. *The canonical map* $\beta : \mathrm{Spin}(V_L) \to \mathrm{Spin}^c(V_L)$ *corresponds to the map* $\beta : \mathrm{Wu}(f)/2H \to \mathrm{Char}(f)/2\widehat{f}(H)$ *defined by* $\beta([w]) = [\widehat{f}(w)]$ *or, equivalently, to the map* $\beta : \mathscr{S}_L \to \mathscr{V}_L$ *defined by* $\beta([r]) = [B_L \cdot r]$.

Proof. Take $\sigma \in \mathrm{Spin}(V_L)$ and let $r_\sigma \in H^2(W_L, V_L) \simeq \mathbb{Z}^n$ be an integral representative for the obstruction $w_2(W_L, \sigma) \in H^2(W_L, V_L; \mathbb{Z}_2) \simeq (\mathbb{Z}_2)^n$ to extend σ to W_L. Let also $\tilde{\sigma} \in \mathrm{Spin}^c(W_L)$ be an extension of $\beta(\sigma) \in \mathrm{Spin}^c(V_L)$. Then, the lemma will follow from the fact that r_σ goes to $c(\tilde{\sigma})$ by the natural map $H^2(W_L, V_L) \to H^2(W_L)$ provided $\tilde{\sigma}$ is appropriately choosen with respect to r_σ. This can be proved undirectly as follows. In case when σ can be extended to W_L, this is certainly true: indeed, we can take $r_\sigma = 0$ and choose as $\tilde{\sigma}$ the image by β of the unique extension of σ to W_L, so that $c(\tilde{\sigma})$ vanishes. The general case can be reduced to this particular one for the following two reasons. First, it is easily verified that for each Kirby's move $L_1 \to L_2$ between ordered oriented framed links, the induced bijections $\mathscr{S}_{L_1} \to \mathscr{S}_{L_2}$ and $\mathscr{V}_{L_1} \to \mathscr{V}_{L_2}$, which are respectively described in [2, Theorem (I.1)] and Theorem 2.2, are compatible with the maps $\beta : \mathscr{S}_{L_k} \to \mathscr{V}_{L_k}$ $(k = 1, 2)$ defined by $\beta([r]) = [B_{L_k} \cdot r]$. Second, according to a theorem of Kaplan [14], there exists an oriented framed link L' in $\mathbf{S}^3$ related to L by a finite sequence of Kirby's moves, and through which $\sigma \in \mathrm{Spin}(V_L)$ goes to $\sigma' \in \mathrm{Spin}(V_{L'})$ with the property that σ' can be extended to $W_{L'}$. □

2.2.4. A combinatorial description of $H_2(V_L; \mathbb{Q}/\mathbb{Z})$

We maintain the notations used in Section 2.1.

Lemma 2.4. *There exists a canonical isomorphism*

$$\frac{H^\sharp}{H} \underset{\simeq}{\xrightarrow{\kappa}} H_2(V_L; \mathbb{Q}/\mathbb{Z}).$$

Proof. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
0 & \longrightarrow & H_2(V_L;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H_2(W_L;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H_2(W_L,V_L;\mathbb{Q}/\mathbb{Z}) \\
 & & \uparrow & & \uparrow{\scriptstyle d} & & \uparrow \\
0 & \longrightarrow & H_2(V_L;\mathbb{Q}) & \longrightarrow & H_2(W_L;\mathbb{Q}) & \xrightarrow{c} & H_2(W_L,V_L;\mathbb{Q}) \\
 & & \uparrow & & \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle b} \\
0 & \longrightarrow & H_2(V_L;\mathbb{Z}) & \longrightarrow & H_2(W_L;\mathbb{Z}) & \longrightarrow & H_2(W_L,V_L;\mathbb{Z}) \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
\end{array}$$

The group $H^\sharp$ is the subgroup of $H \otimes \mathbb{Q} = H_2(W_L;\mathbb{Q})$ comprising those $x \in H_2(W_L;\mathbb{Q})$ such that $c(x) \in H_2(W_L, V_L;\mathbb{Q})$ satisfies $c(x) \bullet a(y) \in \mathbb{Z}$ for all $y \in H_2(W_L;\mathbb{Z})$, where $\bullet$ is the rational intersection pairing in W_L. So, we have that

$$H^\sharp = c^{-1} b(H_2(W_L, V_L;\mathbb{Z})).$$

Seeing $H_2(V_L;\mathbb{Q}/\mathbb{Z})$ as a subgroup of $H_2(W_L;\mathbb{Q}/\mathbb{Z})$, we deduce the announced isomorphism from the map d. □

Recall that the quotient group $H^\sharp/H$, which is denoted by G_f in Section 2.1, appears in the short exact sequence (2.1). We now interpret this sequence as an application of the universal coefficients theorem to V_L. We denote by B the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Lemma 2.5. *The following diagram is commutative*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} \widehat{L_f} & \longrightarrow & G_f & \longrightarrow & \operatorname{Tors} \operatorname{Coker} \widehat{f} & \longrightarrow & 0 \\
 & & {\scriptstyle \kappa|}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \kappa}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \varrho|}\downarrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & H_2(V_L) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_2\left(V_L;\mathbb{Q}/\mathbb{Z}\right) & \xrightarrow[-P\circ B]{} & \operatorname{Tors} H^2(V_L) & \longrightarrow & 0.
\end{array}$$

Proof. It is enough to prove the commutativity of the right square. Start with a class $m \in H_2(V_L;\mathbb{Q}/\mathbb{Z})$. It can be written as $m = [S \otimes [\frac{1}{n}]]$ where n is a positive integer, S is a 2-chain in V_L with boundary $\partial S = n \cdot X$ and X is a 1-cycle. Then, we have that $B(m) = x \in H_1(V_L)$ if x denotes $[X]$. Let also Y be a relative 2-cycle in (W_L, V_L) with boundary $\partial Y = X$ and set $y = [Y] \in H_2(W_L, V_L)$. Lastly, consider the 2-cycle $U = n \cdot Y - S$ in W_L and set $u = [U] \in H = H_2(W_L)$.

Note that $u \otimes \frac{1}{n} \in H \otimes \mathbb{Q}$ belongs to the dual lattice $H^\sharp$: indeed, $P^{-1}\widehat{f}(u) = i_*(u) \in H_2(W_L, V_L)$ equals $n \cdot y$ so that $\widehat{f}(u) = n \cdot P(y)$. This also shows that $\widehat{f_{\mathbb{Q}}}(u \otimes \frac{1}{n})|_H = P(y)$. So, the map $G_f \to \text{Tors Coker } \widehat{f}$ that is featured by the short exact sequence (2.1) sends $[u \otimes \frac{1}{n}]$ to $[P(y)]$.

The canonical map $H \otimes \mathbb{Q} \simeq H_2(W_L; \mathbb{Q}) \to H_2(W_L; \mathbb{Q}/\mathbb{Z})$ sends $u \otimes \frac{1}{n}$ to $[(n \cdot Y - S) \otimes [\frac{1}{n}]] = [-S \otimes [\frac{1}{n}]]$. Consequently, we get that $\kappa([u \otimes \frac{1}{n}]) = -m$.

The conclusion then follows from the commutativity of the diagram

$$\begin{array}{ccc} H_2(W_L, V_L) & \xrightarrow{\partial_*} & H_1(V_L) \\ \downarrow P & & \downarrow P \\ H^2(W_L) & \xrightarrow{i^*} & H^2(V_L), \end{array}$$

which implies that $\varrho([P(y)]) = P(x)$. □

Remark 2.3. Similarly, the pairing (2.2) can easily be interpreted as the intersection pairing of V_L

$$H_1(V_L) \times H_2(V_L; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\bullet} \mathbb{Q}/\mathbb{Z}$$

via the isomorphisms $P^{-1}\varrho : \text{Coker } \widehat{f} \to H_1(V_L)$ and $\kappa : G_f \to H_2(V_L; \mathbb{Q}/\mathbb{Z})$.

2.3. A 4-dimensional definition of the linking quadratic function

Let M be a closed connected oriented 3-manifold equipped with a Spin^c-structure σ. In this subsection, we construct the quadratic function $\phi_{M,\sigma}$ announced in the introduction.

Lemma 2.6. *Fix a homology class $m \in H_2(M; \mathbb{Q}/\mathbb{Z})$. Consider a quadruplet (W, ψ, α, w) formed by a compact oriented 4-manifold W, a positive diffeomorphism $\psi : \partial W \to M$, a* Spin^c*-structure α on W which restricts to $\psi^*(\sigma)$ on the boundary and a class $w \in H_2(W; \mathbb{Q})$, the reduction of which in $H_2(W; \mathbb{Q}/\mathbb{Z})$ coincides with the image of m. Then, the quantity*

$$\phi(M, \sigma, m) = \left[\frac{1}{2}\left(\langle c(\alpha), w\rangle - w \bullet w\right)\right] \in \mathbb{Q}/\mathbb{Z}$$

does not depend on the choice of such a quadruplet.

Remark 2.4. If W is a compact oriented 4-manifold such that $H_1(W) = 0$ and there exists a positive diffeomorphism $\psi : \partial W \to M$, then the pair (W, ψ) can be completed to a quadruplet (W, ψ, α, w) with the above property. In particular, such quadruplets do exist since M possesses surgery presentations.

Proof. Let (W', ψ', α', w') be another such quadruplet. We wish to compare the rational numbers $A := w \bullet w - \langle c(\alpha), w\rangle$ and $A' := w' \bullet w' - \langle c(\alpha'), w'\rangle$.

The homology class m of M can be written as $m = [S \otimes [\frac{1}{n}]]$, where n is a positive integer, S is a 2-chain with boundary $\partial S = n \cdot X$ and X is a 1-cycle. Then, we have that $B(m) = [X]$. Since the image of m in $H_2(W; \mathbb{Q}/\mathbb{Z})$ belongs to the image of $H_2(W; \mathbb{Q})$, the image of $[X] \in H_1(M)$ in $H_1(W)$ is zero. So, one can find a relative 2-cycle Y in $(W, \partial W)$ with boundary $\partial Y = \psi^{-1}(X)$. Consider the 2-cycle

$U = n \cdot Y - \psi^{-1}(S)$ in W. Then, by assumption, w can be written as $w = [-U \otimes \frac{1}{n}] + w_0 \in H_2(W;\mathbb{Q})$, where $w_0 \in H_2(W;\mathbb{Q})$ belongs to the image of $H_2(W;\mathbb{Z})$. We do the same for w' in W' (getting thus some Y', U', w'_0).

Next, we consider the closed oriented 4-manifold

$$\overline{W} := W \cup_{\psi^{-1}\circ\psi'} (-W').$$

Gluing rigid Spin^c-structures, it is easy to find a Spin^c-structure $\overline{\alpha}$ on $\overline{W}$ which restricts to α and $-\alpha'$ on W and $-W'$, respectively.

Set $\overline{Y} = i(Y) - i'(Y')$, where i and i' denote the inclusions of W and W' respectively. This is a 2-cycle in $\overline{W}$ with the property that the identity

$$[\overline{Y} \otimes 1] = [i(U) \otimes 1/n - i'(U') \otimes 1/n] = (-i_*(w) + i_*(w_0)) + (i'_*(w') - i'_*(w'_0))$$

holds in $H_2(\overline{W};\mathbb{Q})$. It follows from this identity that

$$\begin{aligned}[\overline{Y}] \bullet [\overline{Y}] = {} & (w \bullet w + w_0 \bullet w_0 - 2 \cdot w \bullet w_0) \\ & + (-w' \bullet w' - w'_0 \bullet w'_0 + 2 \cdot w' \bullet w'_0),\end{aligned} \tag{2.3}$$

and that

$$\langle c(\overline{\alpha}), [\overline{Y}]\rangle = (-\langle c(\alpha), w\rangle + \langle c(\alpha), w_0\rangle) + (\langle c(\alpha'), w'\rangle - \langle c(\alpha'), w'_0\rangle). \tag{2.4}$$

Recall that $w_0 \in H_2(W;\mathbb{Q})$ and $w'_0 \in H_2(W';\mathbb{Q})$ come from integral classes. Then, by the Wu formula and the fact that a Chern class reduces modulo 2 to the second Stiefel–Whitney class, the integers $[\overline{Y}]\bullet[\overline{Y}]$, $w_0 \bullet w_0$ and $w'_0 \bullet w'_0$ are congruent modulo 2 to $\langle c(\overline{\alpha}), [\overline{Y}]\rangle$, $\langle c(\alpha), w_0\rangle$ and $\langle c(\alpha'), w'_0\rangle$, respectively. Adding (2.3) to (2.4), we find that

$$A - A' - 2 \cdot w \bullet w_0 + 2 \cdot w' \bullet w'_0 \equiv 0 \bmod 2.$$

Because the image of $w \in H_2(W;\mathbb{Q})$ in $H_2(W;\mathbb{Q}/\mathbb{Z})$ comes from $H_2(M;\mathbb{Q}/\mathbb{Z})$ and because $w_0 \in H_2(W;\mathbb{Q})$ comes from $H_2(W;\mathbb{Z})$, the rational number $w \bullet w_0$ belongs to $\mathbb{Z}$. The same holds for $w' \bullet w'_0$. We conclude that the rational number $A - A'$ belongs to $2 \cdot \mathbb{Z}$. □

Remark 2.5. A universal class $u \in H^1(K(\mathbb{Q}/\mathbb{Z},1);\mathbb{Q}/\mathbb{Z})$ induces a homomorphism

$$\Omega_3^{\mathrm{Spin}^c}(K(\mathbb{Q}/\mathbb{Z},1)) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by $[(M,\sigma,f)] \mapsto \phi(M,\sigma,P^{-1}f^*(u))$. This follows from the definition of ϕ in Lemma 2.6.

Consider the linking pairing $\lambda_M : \mathrm{Tors}\, H_1(M) \times \mathrm{Tors}\, H_1(M) \to \mathbb{Q}/\mathbb{Z}$. Composing this with the Bockstein B, one gets a symmetric bilinear pairing

$$H_2(M;\mathbb{Q}/\mathbb{Z}) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{L_M} \mathbb{Q}/\mathbb{Z}$$

with radical $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$. Using a cobordism W as in Remark 2.4, one easily proves, for any $m, m' \in H_2(M;\mathbb{Q}/\mathbb{Z})$, the following identity:

$$\phi(M,\sigma,m+m') - \phi(M,\sigma,m) - \phi(M,\sigma,m') = m \bullet B(m') = L_M(m,m').$$

Definition 2.2. The *linking quadratic function* of the Spin^c-manifold (M, σ) is the map denoted by

$$H_2(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}$$

and defined by $m \mapsto \phi(M, \sigma, m)$.

The discriminant construction allows us to compute combinatorially the quadratic function $\phi_{M,\sigma}$, as soon as a surgery presentation of the Spin^c-manifold (M, σ) is given. Indeed, let L be an ordered oriented framed link in $\mathbf{S}^3$ together with a positive diffeomorphism $\psi : V_L \to M$. With the notations from Section 2.2, (H, f) still denotes the bilinear lattice $(H_2(W_L)$, intersection pairing of $W_L)$, to which the constructions from Section 2.1 apply. Let also $c \in \mathrm{Char}(f)$ represent $\psi^*(\sigma) \in \mathrm{Spin}^c(V_L)$ (in the sense of Lemma 2.2). Then, as can be verified from the definitions, the following diagram commutes:

$$\begin{array}{ccc} H_2(M;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\phi_{M,\sigma}} & \mathbb{Q}/\mathbb{Z} \\ \psi_* \uparrow \simeq & & \\ H_2(V_L;\mathbb{Q}/\mathbb{Z}) & \nearrow_{-\phi_{f,c}} & \\ \kappa \uparrow \simeq & & \\ G_f & & \end{array} \tag{2.5}$$

Note that, in this context, the pairings λ_f and L_f are topologically interpreted as $-\lambda_M$ and $-L_M$, respectively.

2.4. Properties of the linking quadratic function

In this subsection, we fix a closed connected oriented 3-manifold M and prove properties of the map $\phi_M : \mathrm{Spin}^c(M) \to \mathrm{Quad}(L_M)$ defined by $\sigma \mapsto \phi_{M,\sigma}$. Those properties are proved "combinatorially" using (2.5), but may also be proved directly from the very definition of $\phi_{M,\sigma}$.

Next lemma says that $\phi_{M,\sigma}$ is determined on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ by the Chern class $c(\sigma)$. Recall that the modulo 2 reduction of $c(\sigma)$ is $w_2(M) = 0$.

Lemma 2.7. *For any $\sigma \in \mathrm{Spin}^c(M)$, the function $\phi_{M,\sigma}$ is linear on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$:*

$$\forall x \otimes [r] \in H_2(M) \otimes \mathbb{Q}/\mathbb{Z}, \quad \phi_{M,\sigma}(x \otimes [r]) = \frac{\langle c(\sigma), x\rangle}{2} \cdot [r] \in \mathbb{Q}/\mathbb{Z}.$$

Proof. The first statement follows from the fact that Ker $\widehat{L_M} = H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$. As for the second statement, it suffices to prove it when $M = V_L$. Suppose that σ is represented by the characteristic form $c \in \mathrm{Char}(f)$ and that $x \in H_2(V_L)$ goes to y in $H = H_2(W_L)$. Then, $x \otimes [r]$ as an element of $H_2(V_L; \mathbb{Q}/\mathbb{Z})$ corresponds to $[y \otimes r]$ in $H^\sharp/H$. Consequently, we have that $\phi_{M,\sigma}(x \otimes [r]) = -\phi_{f,c}([y \otimes r]) = -\frac{1}{2}(r^2 f(y, y) - r \cdot c(y))$ mod 1. Since y belongs to Ker $\widehat{f}$, we obtain that $\phi_{M,\sigma}(x \otimes [r]) = \frac{1}{2} r \cdot c(y) \bmod 1 = \frac{1}{2} r \cdot \langle c(\sigma), x\rangle \bmod 1$, by Remark 2.2. □

Let us consider, for a while, the case when $\sigma \in \mathrm{Spin}^c(M)$ is torsion. Then, Lemma 2.7 implies that $\phi_{M,\sigma}$ vanishes on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$: Consequently, $\phi_{M,\sigma}$ factors to a quadratic function over λ_M. In this torsion case, our linking quadratic function is readily seen to agree with that of [4] and, up to a minus sign, with that of [9]. In the next subsection, it is also shown to coincide with that of [19].

In particular, σ may arise from a Spin-structure on M, which happens if and only if $c(\sigma)$ vanishes. Then, the factorization of $\phi_{M,\sigma}$ to Tors $H_1(M)$ coincides with the linking quadratic form defined in [18,24] or [27]. In [21], this quadratic form is used to classify degree 0 invariants in the Spin-refinement of the Goussarov–Habiro theory.

In the sequel, we will use the homomorphism

$$H^2(M) \xrightarrow{\mu_M} \mathrm{Hom}\,(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$$

defined by $\mu_M(y) = \langle y, - \rangle$.

Lemma 2.8. *For any $\sigma \in \mathrm{Spin}^c(M)$, the Chern class $c(\sigma)$ is sent by μ_M to the homogeneity defect $d_{\phi_{M,\sigma}} : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ of the quadratic function $\phi_{M,\sigma}$.*

Proof. Again suppose that $M = V_L$ and that σ is represented by $c \in \mathrm{Char}(f)$. Take $x \in H_2(V_L;\mathbb{Q}/\mathbb{Z})$ represented by $y \in H^\sharp$. One computes that $\phi_{M,\sigma}(x) - \phi_{M,\sigma}(-x) = -\phi_{f,c}([y]) + \phi_{f,c}(-[y]) = c_{\mathbb{Q}}(y) \bmod 1 = \langle c(\sigma), x\rangle$, by Remark 2.2. □

Recall that $\mathrm{Spin}^c(M)$ is an affine space over $H^2(M)$ and that $\mathrm{Quad}(L_M)$ is an affine space over $\mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$. Let

$$\mathrm{Hom}\,(H_2(M),\mathbb{Z}) \xrightarrow{j_M} \mathrm{Hom}\,(H_2(M)\otimes\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$$

be the homomorphism defined by $j_M(l) = l \otimes \mathbb{Q}/\mathbb{Z}$. Next result, which contains Theorem 1, is a direct application of Theorem 2.1 and Remark 2.3.

Theorem 2.3. *The map $\phi_M : \mathrm{Spin}^c(M) \to \mathrm{Quad}(L_M)$ is an affine embedding over the group monomorphism μ_M. Moreover, a function $q \in \mathrm{Quad}(L_M)$ belongs to $\mathrm{Im}\,\phi_M$ if and only if $q|_{H_2(M)\otimes\mathbb{Q}/\mathbb{Z}}$ belongs to $\mathrm{Im}\, j_M$.*

Remark 2.6. The map ϕ_M is bijective if and only if M is a rational homology 3-sphere.

2.5. An intrinsic definition of the linking quadratic function

Let M be a closed connected oriented 3-manifold equipped with a Spin^c-structure σ. In this subsection, we give for the quadratic function $\phi_{M,\sigma}$ an intrinsic formula which does not refer to 4-dimensional cobordisms.

Here is the idea. Take a $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$. It follows from Lemma 2.8 that

$$2 \cdot \phi_{M,\sigma}(x) = L_M(x,x) + \langle c(\sigma), x\rangle \in \mathbb{Q}/\mathbb{Z}.$$

For any $y \in \mathbb{Q}/\mathbb{Z}$, we denote by $\frac{1}{2}\cdot y$ the set of elements z of $\mathbb{Q}/\mathbb{Z}$ such that $z + z = y$. We are going to select, *correlatively*, an element z_1 in $\frac{1}{2}\cdot L_M(x,x)$ *and* an element z_2 in $\frac{1}{2}\cdot\langle c(\sigma), x\rangle$ such that $\phi_{M,\sigma}(x) = z_1 + z_2$.

Write $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$ as $x = [S \otimes [1/n]]$, where n is a positive integer and S is an oriented immersed surface in M with boundary $n \cdot K$, a bunch of n parallel copies of an oriented knot K in M. Apply now the following stepwise procedure:

- *Step* 1: Choose a nonsingular vector field v on M representing σ as an Euler structure, and which is transverse to K (we claim that it is possible to find such v).
- *Step* 2: Let V be a sufficiently small regular neighborhood of K in M and let K_v be the parallel of K, lying on ∂V, obtained by pushing K along the trajectories of v. By an isotopy, ensure that S is in transverse position with respect to K_v with boundary contained in the interior of V.
- *Step* 3: Define a Spin-structure α_v on $\partial(M\backslash\text{int}(V))$ by requiring its Atiyah–Johnson quadratic form q_{α_v} (Section 1.3.4) to be such that

$$q_{\alpha_v}([\text{meridian of } K]) = 0 \quad \text{and} \quad q_{\alpha_v}([K_v]) = 1.$$

- *Step* 4: Together with the vector field tangent to K_v, v represents a Spin^c-structure σ_v on $M\backslash\text{int}(V)$ relative to the Spin-structure α_v (we claim this). Consider the Chern class $c(\sigma_v) \in H^2(M\backslash\text{int}(V), \partial(M\backslash\text{int}(V)))$.

Proposition 2.1. *By applying the above procedure, we get*

$$\phi_{M,\sigma}(x) = \underbrace{\left[\frac{1}{2n} \cdot K_v \bullet S\right]}_{\in \frac{1}{2}\cdot L_M(x,x)} + \underbrace{\left[\frac{1}{2n} \cdot \langle c(\sigma_v), [S \cap (M\backslash\text{int}(V))]\rangle + \frac{1}{2}\right]}_{\in \frac{1}{2}\cdot\langle c(\sigma),x\rangle} \in \mathbb{Q}/\mathbb{Z}. \tag{2.6}$$

In [19], Looijenga and Wahl associate a quadratic function over λ_M to each pair $(M, \mathscr{J})$ formed by

- a closed connected oriented 3-manifold M,
- a homotopy class of complex structures $\mathscr{J}$ on $\mathbb{R} \oplus \mathrm{T}M$ whose first Chern class is torsion.

There is a Spin^c-structure $\omega(\mathscr{J})$ associated to $\mathscr{J}$ (see Section 1.2.7). By assumption, its Chern class is torsion so that $\phi_{M,\omega(\mathscr{J})}$ factors to a quadratic function over λ_M. One can verify, using the inverse of ω described in the proof of Lemma 1.10, that formula (2.6) is equivalent in this case to formula (3.4.1) in [19].

Proof of Proposition 2.1. First of all, we have to justify that the above procedure can actually be carried out.

We begin by proving the claim of Step 1. Let v be an arbitrary nonsingular vector field on M representing σ. It suffices to prove the following claim.

Claim 2.1. Let w be an arbitrary nonsingular vector field tangent to M defined on K. Then, v can be homotoped so as to coincide with w on K.

Proof. Choose a tubular neighborhood W of K, plus an identification $W = (2\mathbf{D}^2) \times \mathbf{S}^1$ such that K corresponds to $0 \times \mathbf{S}^1$. We denote by (e_1, e_2) the standard basis of $\mathbb{R}^2 \supset 2\mathbf{D}^2$. We define $\pi : W \to K$ to be the projection on the core. The solid torus W is parametrized by the cylindric coordinates

$$((r \in [0, 2], \theta \in \mathbb{R}/2\pi\mathbb{Z}), \phi \in \mathbb{R}/2\pi\mathbb{Z}).$$

If $p, q \in W$ are such that $\pi(p) = \pi(q)$ (i.e., they belong to the same meridional disk $2\mathbf{D}^2 \times *$), we define the transport map $t_{p,q} : \mathrm{T}_p W \to \mathrm{T}_q W$ as the unique linear map fixing the basis $(e_1, e_2, \frac{\partial}{\partial\phi})$. Deform the vector field v through the homotopy $(v^{(t)})_{t\in[0,1]}$ given at time t and point $p \in W$ by

$$v_p^{(t)} = \begin{cases} t_{\pi(p),p}(v_{\pi(p)}) & \text{if } r(p) \in [0,t] \\ t_{q(p,t),p}(v_{q(p,t)}) & \text{if } r(p) \in [t,2], \text{ with } q(p,t) = \left(\frac{r(p)-t}{1-t/2}, \theta(p), \phi(p)\right) \end{cases}$$

and at time t and point $p \notin W$ by $v_p^{(t)} = v_p$. After such a deformation, the vector field v satisfies the following property: $\forall p \in \mathbf{D}^2 \times \mathbf{S}^1,\ t_{p,\pi(p)}(v_p) = v_{\pi(p)}$. Now, since $\pi_1(\mathbf{S}^2)$ is trivial, $v|_K$ and w have to be homotopic; let $(w^{(t)})_{t\in[0,1]}$ be such a homotopy, beginning at $w^{(0)} = v|_K$ and ending at $w^{(1)} = w$. The homotopy given by

$$v_p^{(t)} = \begin{cases} t_{\pi(p),p}(w_{\pi(p)}^{(t-r(p))}) & \text{if } r(p) \in [0,t] \\ v_p & \text{if } r(p) \in [t,2] \end{cases}$$

if $p \in W$ and by $v_p^{(t)} = v_p$ if $p \notin W$, allows us to deform v to a nonsingular vector field which coincides with w on K. □

Since v is now transverse to K, we can find a regular neighborhood V of K in M plus an identification $V = \mathbf{D}^2 \times \mathbf{S}^1$, such that K corresponds to $0 \times \mathbf{S}^1$ and such that $v|_V$ corresponds to e_1 (recall that (e_1, e_2) denotes the standard basis of $\mathbb{R}^2 \supset \mathbf{D}^2$). We apply steps 2 and 3 (note that K_v then corresponds to $1 \times \mathbf{S}^1$) and we now prove the claim of Step 4. Let $\tau_v \in \mathrm{Spin}(V)$ be defined by the trivialization $(e_1, e_2, \frac{\partial}{\partial\phi})$ of $\mathrm{T}V$. Since $(\tau_v|_{\partial V})|_{1\times\mathbf{S}^1}$ is the non-bounding Spin-structure and since $(\tau_v|_{\partial V})|_{\partial\mathbf{D}^2\times 1}$ spin bounds, we have that $\tau_v|_{\partial V} = -\alpha_v$, i.e. τ_v belongs to $\mathrm{Spin}(V, -\alpha_v)$ with the notation of Remark 1.3. Thus, $v|_{M\setminus\mathrm{int}(V)}$ together with the trivialization $(e_1, e_2, \frac{\partial}{\partial\phi})|_{\partial V}$ of $\mathrm{T}(M\setminus\mathrm{int}(V))|_{\partial V}$ define a $\sigma_v \in \mathrm{Spin}^c(M\setminus\mathrm{int}(V), \alpha_v)$, as claimed in Step 4. For further use, note that σ is the gluing $\sigma_v \cup \beta(\tau_v)$, where $\beta : \mathrm{Spin}(V, -\alpha_v) \to \mathrm{Spin}^c(V, -\alpha_v)$ has been defined in Remark 1.3.

Set $z_1 = [1/2n \cdot K_v \bullet S] \in \mathbb{Q}/\mathbb{Z}$ and $z_2 = [1/2n \cdot \langle c(\sigma_v), [S']\rangle + 1/2] \in \mathbb{Q}/\mathbb{Z}$, where $S' = S \cap (M\setminus\mathrm{int}(V))$. We have that

$$2 \cdot z_1 = [1/n \cdot K_v \bullet S] = [\lambda_M(B(x), B(x))] = L_M(x, x).$$

Moreover, we have that

$$\begin{aligned} 2 \cdot z_2 &= [1/n \cdot \langle c(\sigma_v), [S']\rangle] \\ &= [1/n \cdot P^{-1}(c(\sigma_v)) \bullet [S']] \text{ (intersection in } M\setminus\mathrm{int}(V)) \\ &= P^{-1}(c(\sigma)) \bullet x \text{ (intersection in } M) \\ &= \langle c(\sigma), x\rangle \end{aligned}$$

where the third equality follows from the facts that $x = [S \otimes [1/n]]$, $P^{-1}(c(\sigma)) = i_* P^{-1}(c(\sigma_v)) + i_* P^{-1}(c(\beta(\tau_v))) \in H_1(M)$ (since $\sigma = \sigma_v \cup \beta(\tau_v)$) and $c(\beta(\tau_v)) = 0$ (by Remark 1.6).

We now prove formula (2.6), i.e., the equality $\phi_{M,\sigma}(x) = z_1 + z_2$. Let us work with surgery presentations (even if we could use more general cobordisms as well). Let M' be the 3-manifold obtained from M by doing surgery along the framed knot $(K, (e_1, e_2))$. Conversely, M is the result of the surgery on M' along the dual knot K' of K. Pick a surgery presentation $V_{L'}$ of M'; up to isotopy, the knot $K' \subset M'$ is in

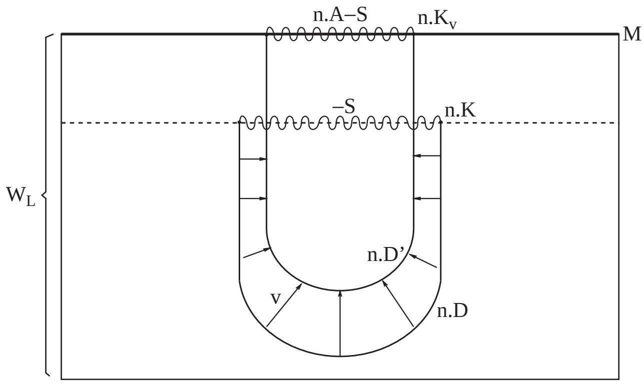

Fig. 4. Two representants of u in transverse position.

$\mathbf{S}^3\backslash L'$. We then find a surgery presentation V_L of M by setting L to be L' union K' with the appropriate framing. This surgery presentation of M has the following advantage: K bounds in the trace W_L of the surgery a disk D whose normal bundle is trivialized by some extension of the trivialization (e_1, e_2) of the normal bundle of K in M. We use the notations fixed in Section 2.2. In particular, $H = H_2(W_L)$ and $f : H \times H \to \mathbb{Z}$ is the intersection pairing of W_L. We define the 2-cycle $U = n \cdot D - S$ where $n \cdot D$ is a bunch of n parallel copies of the disk D with boundary $n \cdot K$; we also set $u = [U] \in H$. Then $u \otimes \frac{1}{n}$ belongs to $H^\sharp$ and the isomorphism $\kappa : H^\sharp/H \to H_2(M; \mathbb{Q}/\mathbb{Z})$ sends$[u \otimes \frac{1}{n}]$ to $-x = -[S \otimes [\frac{1}{n}]]$ (see the proof of Lemma 2.5). So, by diagram (2.5), we obtain that

$$\phi_{M,\sigma}(x) = -\phi_{f,c}\left(-\left[u \otimes \frac{1}{n}\right]\right) = -\frac{1}{2}\left(\frac{1}{n^2} f(u,u) + \frac{1}{n} c(u)\right) \bmod 1,$$

where c is a characteristic form representative for σ.

We calculate the quantity $f(u,u)$. The 2-cycle U is a representant of u. Let D' be a push-off of D by the extension of $e_1 = v|_V$ in such a way that $\partial D'$ is K_v. Let also A be the annulus of an isotopy from $-K_v$ to K in V (e.g., $A = -[0,1] \times \mathbf{S}^1$ in $V = \mathbf{D}^2 \times \mathbf{S}^1$). A second representative for u is $U' = n \cdot D' + n \cdot A - S$. By adding a collar to W_L and stretching the top of U', we can make U in transverse position with U' (see Fig. 4). So, we have that $f(u,u) = U \bullet U' = -nS \bullet K_v$ where the first intersection is calculated in W_L and the second one in M; we are led to

$$\phi_{M,\sigma}(x) = \frac{1}{2n} S \bullet K_v - \frac{1}{2n} c(u) \bmod 1. \tag{2.7}$$

We are now interested in the quantity $c(u)$. Let $\tilde{\sigma}$ be an extension of σ to the manifold W_L and let ξ be the isomorphism class of principal U(1)-bundles on W_L defined by $\tilde{\sigma}$; then c can be choosen to be $c_1(\xi)$. Let p be a representant of ξ and let tr be a trivialization of p on ∂V. Decompose the singular surface U' as $U' = U'_1 \cup U'_2 \cup U'_3$, where $U'_1 = n \cdot D'$, $U'_2 = n \cdot A \cup (-S \cap V)$ and $U'_3 = -S'$. By desingularizing U' so as to be reduced to a calculus of obstructions in an oriented manifold, we obtain that

$$c(u) = \langle c_1(p|_{U'}), [U'] \rangle = \sum_{i=1}^{3} \langle c_1(p|_{U'_i}, \mathrm{tr}\,|_{\partial U'_i}), [U'_i] \rangle \in \mathbb{Z}, \tag{2.8}$$

where $c_1(p|_{U_i'}, \mathrm{tr}\,|_{\partial U_i'}) \in H^2(U_i', \partial U_i')$ is the obstruction to extend the trivialization $\mathrm{tr}\,|_{\partial U_i'}$ of $p|_{U_i'}$ on $\partial U_i'$ to the whole of U_i'. Let $V' \subset W_L$ be the solid torus such that $M' = M\backslash \mathrm{int}(V) \cup V'$. For an appropriate choice of $\tilde{\sigma}$, there exists a Spin^c-structure $\sigma_1 \in \mathrm{Spin}^c(V', -\alpha_v)$ such that $\sigma_v \cup \sigma_1 = \tilde{\sigma}|_{M'}$. Also, for some appropriate choices of p in the class ξ and tr, we have

$$c_1(p|_{V'}, \mathrm{tr}) = c(\sigma_1) \in H^2(V', \partial V'),$$
$$c_1(p|_{V}, \mathrm{tr}) = c(\beta(\tau_v)) \in H^2(V, \partial V),$$
$$c_1(p|_{M\backslash \mathrm{int}(V)}, \mathrm{tr}) = c(\sigma_v) \in H^2(M\backslash \mathrm{int}(V), \partial(M\backslash \mathrm{int}(V))).$$

Then, Eq. (2.8) becomes

$$c(u) = n \cdot \langle c(\sigma_1), [D'] \rangle + \langle c(\beta(\tau_v)), [U_2'] \rangle - \langle c(\sigma_v), [S'] \rangle \in \mathbb{Z}.$$

From the fact that $c(\beta(\tau_v)) = 0$, we deduce that

$$\frac{1}{2n} \cdot c(u) = -\frac{1}{2n} \cdot \langle c(\sigma_v), [S'] \rangle + \frac{1}{2} \cdot \langle c(\sigma_1), [D'] \rangle \in \mathbb{Q}.$$

Then, showing that $\langle c(\sigma_1), [D'] \rangle$ is an odd integer together with (2.7) will end the proof of the proposition. Since $\langle c(\sigma_1), [D'] \rangle = q_{-\alpha_v}(\partial_*[D']) = q_{\alpha_v}([K_v]) = 1 \bmod 2$ (by Lemma 1.9), we are done. □

3. Goussarov–Habiro theory for three-manifolds with complex spin structure

In this section, we explain how the Goussarov–Habiro theory can be extended to the context of 3-manifolds equipped with a Spin^c-structure. Then, using the linking quadratic function, we prove Theorem 2 stated in the introduction. This amounts to identifying the degree 0 invariants in the generalized theory.

3.1. Review of the Y-equivalence relation

Recall that the Goussarov–Habiro theory is a theory of finite type invariants for compact oriented 3-manifolds [8,11,12] and is based on the *Y*-surgery as elementary move. In this subsection, we just recall how this surgery move is defined.

Suppose that M is a compact oriented 3-manifold. Let $j : H_3 \hookrightarrow M$ be a positive embedding of the genus 3 handlebody into the interior of M. Set

$$M_j = M\backslash \mathrm{int}(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}} (H_3)_B.$$

Here, $(H_3)_B$ is the surgered handlebody along the six-component framed link B shown in Fig. 5 with the blackboard framing convention.

Remark 3.1. Observe that there is a canonical inclusion $M\backslash \mathrm{int}(\mathrm{Im}(j)) \hookrightarrow M_j$. One can define a self-diffeomorphism h of ∂H_3 (explicitely, as the composition of 6 Dehn twists) such that there exists a diffeomorphism

$$M_j \cong M\backslash \mathrm{int}(\mathrm{Im}(j)) \cup_{j|_{\partial H_3} \circ h} H_3 \tag{3.1}$$

restricting to the identity on $M\backslash \mathrm{int}(\mathrm{Im}(j))$. Moreover, h can be verified to act trivially in homology.

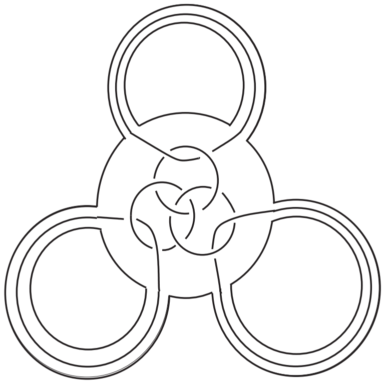

Fig. 5. The framed link B.

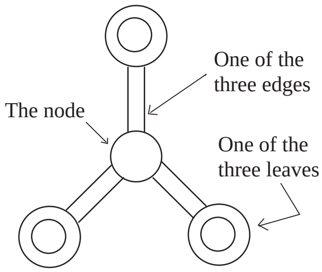

Fig. 6. A Y-graph.

A *Y-graph G* in M is an embedding of the surface drawn in Fig. 6 into the interior of M. This surface, of genus 0 with 4 boundary components, is decomposed between *leaves*, *edges* and *node*. Let $j : H_3 \hookrightarrow M$ be a trivialization of a regular neighborhood of G in M. The embedding j is unique, up to ambient isotopy.

Definition 3.1. The manifold obtained from M by *Y-surgery along G*, denoted by M_G, is the positive diffeomorphism class of the manifold M_j. The *Y-equivalence* is the equivalence relation among compact oriented 3-manifolds generated by Y-surgeries and positive diffeomorphisms.

Remark 3.2. The Y-surgery move has been introduced by Goussarov [11] and is equivalent to Habiro's "A_1-move" [12]. It is equivalent to Matveev's "Borromean surgery" as well, hence the Y-equivalence relation is characterized in [22].

3.2. *The Y^c-equivalence relation*

We define the Y^c-surgery move announced in the introduction, and we outline how this suffices to extend the Goussarov–Habiro theory to manifolds equipped with a Spin^c-structure.

3.2.1. Twist and Spin^c*-structures*

As in Section 1.4, we consider a closed oriented 3-manifold

$$M = M_1 \cup_f M_2$$

obtained by gluing two compact oriented 3-manifolds M_1 and M_2 with a positive diffeomorphism $f : -\partial M_2 \to \partial M_1$. We add the assumption that ∂M_2 *is connected.*

Let $h : \partial M_2 \to \partial M_2$ be a diffeomorphism *which acts trivially in homology* and consider the manifold

$$M' = M_1 \cup_{f \circ h} M_2.$$

The manifold M' is said to be obtained from M by a *twist*. By Remark 3.1, the Y-surgery move is an instance of a twist move.

By a Mayer–Vietoris argument, there is an isomorphism $\Phi : H_1(M) \to H_1(M')$ which is unambiguously defined by the commutative diagram

$$\begin{array}{ccccc} & & H_1(M) & & \\ & \nearrow^{j_{1,*}} & & \nwarrow^{j_{2,*}} & \\ H_1(M_1) & & \simeq \Big\downarrow \Phi & & H_1(M_2) \\ & \searrow_{j'_{1,*}} & & \swarrow_{j'_{2,*}} & \\ & & H_1(M') & & \end{array}$$

where j_1, j_2, j'_1 and j'_2 denote inclusions.

Proposition 3.1. *The twist from M to M' induces a canonical bijection*

$$\mathrm{Spin}^c(M) \xrightarrow[\simeq]{\Omega} \mathrm{Spin}^c(M')$$

which is affine over $P\Phi P^{-1} : H^2(M) \to H^2(M')$. *Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{Spin}^c(M) & \xrightarrow{\Omega} & \mathrm{Spin}^c(M') \\ {\scriptstyle c}\Big\downarrow & & \Big\downarrow{\scriptstyle c} \\ H^2(M) & \xrightarrow[P\Phi P^{-1}]{} & H^2(M') \end{array}$$

is commutative.

Proof. For any $\alpha \in \mathrm{Spin}^c(M)$, we define $\Omega(\alpha)$ as follows. *Choose* $\sigma_2 \in \mathrm{Spin}(\partial M_2)$ and set $\sigma_1 = f_*(-\sigma_2) \in \mathrm{Spin}(\partial M_1)$. Since $h_* : H_1(\partial M_2; \mathbb{Z}_2) \to H_1(\partial M_2; \mathbb{Z}_2)$ is the identity, h acts trivially on $\mathrm{Spin}(\partial M_2)$: this follows from the naturality of the Atiyah–Johnson correspondence $\mathrm{Spin}(\partial M_2) \to \mathrm{Quad}(\partial \mathrm{M}_2)$

(see Section 1.3.4). According to Lemma 1.12, there are two gluing maps

$$\mathrm{Spin}^c(M_1,\sigma_1)\times \mathrm{Spin}^c(M_2,\sigma_2)\xrightarrow{\cup_f}\mathrm{Spin}^c(M)$$

$$\mathrm{Spin}^c(M_1,\sigma_1)\times \mathrm{Spin}^c(M_2,\sigma_2)\xrightarrow{\cup_{f\circ h}}\mathrm{Spin}^c(M')$$

which are affine, via Poincaré duality, over $j_{1,*}\oplus j_{2,*}$ and $j'_{1,*}\oplus j'_{2,*}$ respectively. Since ∂M_2 is connected, the map $\cup_f$ is surjective. *Choose* $\alpha_1\in\mathrm{Spin}^c(M_1,\sigma_1)$ and $\alpha_2\in\mathrm{Spin}^c(M_2,\sigma_2)$ such that $\alpha=\alpha_1\cup_f\alpha_2$, next set

$$\alpha'=\alpha_1\cup_{f\circ h}\alpha_2\in\mathrm{Spin}^c(M')$$

and define $\Omega(\alpha)$ to be α'.

We have to verify that $\Omega(\alpha)$ is well-defined by that procedure. Assume other intermediate choices $\tilde\sigma_2$, $\tilde\alpha_1$ and $\tilde\alpha_2$ instead of σ_2, α_1 and α_2 respectively, leading to $\tilde\alpha':=\tilde\alpha_1\cup_{f\circ h}\tilde\alpha_2$. We claim that $\alpha'=\tilde\alpha'$.

Consider first the particular case when $\tilde\sigma_2=\sigma_2\in\mathrm{Spin}(\partial M_2)$. Since $\alpha_1\cup_f\alpha_2=\alpha=\tilde\alpha_1\cup_f\tilde\alpha_2$, we have that

$$j_{1,*}P^{-1}(\alpha_1-\tilde\alpha_1)+j_{2,*}P^{-1}(\alpha_2-\tilde\alpha_2)=P^{-1}(\alpha-\alpha)=0\in H_1(M).$$

Applying Φ to that identity, we obtain the equation

$$j'_{1,*}P^{-1}(\alpha_1-\tilde\alpha_1)+j'_{2,*}P^{-1}(\alpha_2-\tilde\alpha_2)=0\in H_1(M')$$

whose left term equals $P^{-1}(\alpha'-\tilde\alpha')$. We conclude that $\alpha'=\tilde\alpha'$.

We now turn to the general case. For this, choose an arbitrary element

$$\tau_2\in\mathrm{Spin}^c([0,1]\times\partial M_2,0\times(-\sigma_2)\cup 1\times\tilde\sigma_2).$$

Having set $\tilde\sigma_1=f_*(-\tilde\sigma_2)$, define

$$\tau_1=(\mathrm{Id}\times f)_*(-\tau_2)\in\mathrm{Spin}^c([0,1]\times\partial M_1,0\times(-\sigma_1)\cup 1\times\tilde\sigma_1).$$

Here, $-\tau_2\in\mathrm{Spin}^c(-[0,1]\times\partial M_2,0\times\sigma_2\cup 1\times(-\tilde\sigma_2))$ is obtained from τ_2 by time-reversing. For $i=1,2$, the collar of ∂M_i in M_i and Lemma 1.12 give a map

$$\mathrm{Spin}^c(M_i,\sigma_i)\times\mathrm{Spin}^c\left([0,1]\times\partial M_i,0\times(-\sigma_i)\cup 1\times\tilde\sigma_i\right)\xrightarrow{\cup_{\mathrm{col}}}\mathrm{Spin}^c(M_i,\tilde\sigma_i).$$

From the definition of the gluing map $\cup_f$ and by using the “double collar” of $\partial M_1\cong-\partial M_2$ in M, one sees that $\alpha=\alpha_1\cup_f\alpha_2$ may also be written as

$$\alpha=(\alpha_1\cup_{\mathrm{col}}\tau_1)\cup_f(\alpha_2\cup_{\mathrm{col}}\tau_2)\,.$$

It follows from the special case treated previously that, whatever the choices of $\tilde\alpha_1$ and $\tilde\alpha_2$ have been,

$$\tilde\alpha'=(\alpha_1\cup_{\mathrm{col}}\tau_1)\cup_{f\circ h}(\alpha_2\cup_{\mathrm{col}}\tau_2)\,.$$

On the other hand, having set

$$\tau'_1=(\mathrm{Id}\times(f\circ h))_*(-\tau_2)\in\mathrm{Spin}^c([0,1]\times\partial M_1,0\times(-\sigma_1)\cup 1\times\tilde\sigma_1),$$

one sees that $\alpha' = \alpha_1 \cup_{f\circ h} \alpha_2$ may also be written as

$$\alpha' = \left(\alpha_1 \cup_{\mathrm{col}} \tau_1'\right) \cup_{f\circ h} (\alpha_2 \cup_{\mathrm{col}} \tau_2)\,.$$

Consequently, it is enough to prove that

$$\tau_1 = \tau_1' \in \mathrm{Spin}^c([0,1]\times \partial M_1, 0\times(-\sigma_1)\cup 1\times\tilde{\sigma}_1). \tag{3.2}$$

The latter space of relative Spin^c-structures is classified by the Chern class map since $H^2([0,1]\times \partial M_1, \partial[0,1]\times\partial M_1)$ has no 2-torsion. Moreover, the naturality of the Chern class and the fact that h preserves the homology imply that

$$c(\tau_1) = (\mathrm{Id}\times f)_*(c(-\tau_2)) = (\mathrm{Id}\times (f\circ h))_*(c(-\tau_2)) = c(\tau_1').$$

We conclude that identity (3.2) holds and that the map Ω is well-defined.

The fact that Ω is affine and the last statement of the proposition are readily derived from the properties of the gluing maps $\cup_f$ and $\cup_{f\circ h}$ stated in Lemma 1.12, and from the definition of the isomorphism Φ. □

Remark 3.3. We could have considered as well the case when M_1 and M_2 have disconnected boundary, but are glued together along a connected component of their boundary to give M (so that $\partial M \cong \partial M' \neq \emptyset$). Then, in view of Remark 1.7, Proposition 3.1 can easily be generalized to involve Spin^c-structures on M and M' relative to a fixed Spin-structure on their identified boundaries.

3.2.2. Definition of the Y^c-surgery move

We explain how Y-surgery makes sense in the setting of Spin^c-manifolds. For simplicity, we consider only the case of a closed oriented 3-manifold M.

Let $j : H_3 \hookrightarrow M$ be an embedding. We denote by $\Phi_j : H_1(M) \to H_1(M_j)$ the isomorphism defined by the commutative diagram

$$\begin{array}{ccc} & & H_1(M) \\ & \overset{k_*}{\nearrow} & \\ H_1(M\setminus \mathrm{int}(\mathrm{Im}(j))) & & \simeq \Big\downarrow \Phi_j \\ & \underset{k'_*}{\searrow} & \\ & & H_1(M_j) \end{array}$$

where $k : M\setminus\mathrm{int}(\mathrm{Im}(j)) \hookrightarrow M$ and $k' : M\setminus\mathrm{int}(\mathrm{Im}(j)) \hookrightarrow M_j$ denote inclusions.

Lemma 3.1. *There exists a canonical bijection*

$$\mathrm{Spin}^c(M) \xrightarrow[\simeq]{\Omega_j} \mathrm{Spin}^c\left(M_j\right), \quad \alpha \longmapsto \alpha_j$$

which is affine over $P\Phi_j P^{-1}$. *Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{Spin}^c(M) & \xrightarrow{\Omega_j} & \mathrm{Spin}^c(M_j) \\ \downarrow c & & \downarrow c \\ H^2(M) & \xrightarrow[P\Phi_j P^{-1}]{} & H^2(M_j) \end{array}$$

is commutative.

Proof. By Remark 3.1, one can define a self-diffeomorphism h of ∂H_3 acting trivially in homology and such that there exists a diffeomorphism

$$M_j = M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}} (H_3)_B \xrightarrow[\cong]{f} M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}\circ h} H_3$$

which restricts to the identity on $M\backslash\mathrm{int}(\mathrm{Im}(j))$. This diffeomorphism induces a bijection

$$\mathrm{Spin}^c\left(M_j\right) \xrightarrow[\simeq]{f_*} \mathrm{Spin}^c\left(M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}\circ h} H_3\right).$$

Also, by Section 3.2.1, there is a canonical bijection

$$\mathrm{Spin}^c\,(M) \xrightarrow[\simeq]{\Omega} \mathrm{Spin}^c\left(M\backslash \mathrm{int}\,(\mathrm{Im}(j)) \cup_{j|_{\partial H_3}\circ h} H_3\right).$$

We define Ω_j to be the composite $f_*^{-1}\Omega$. This composite is easily verified to be independent of the pair (h, f) with the above property. □

Let G be a Y-graph in M. Let also $j : H_3 \hookrightarrow M$ and $j' : H_3 \hookrightarrow M$ be some trivializations of regular neighborhoods of G in M. There exists an ambiant isotopy $(q_t : M \to M)_{t\in[0,1]}$ between j and j': $q_0 = \mathrm{Id}_M$ and $q_1 \circ j = j'$. Let $q : M_j \to M_{j'}$ be the positive diffeomorphism induced by q_1 in the obvious way. One can verify that $q_* \circ \Omega_j = \Omega_{j'}$. Thus, for any Spin^c-structure α on M, the Spin^c-manifolds (M_j, α_j) and $(M_{j'}, \alpha_{j'})$ are Spin^c-diffeomorphic.

Definition 3.2. The Spin^c-manifold obtained from (M, α) by *Y^c-surgery along G*, denoted by (M_G, α_G), is the Spin^c-diffeomorphism class of the manifold (M_j, α_j). We call *Y^c-equivalence* the equivalence relation among closed 3-dimensional Spin^c-manifolds generated by Y^c-surgeries and Spin^c-diffeomorphisms.

In the sequel, the notation M_G will sometimes refer to a representative M_j obtained by fixing a trivialization j of a regular neighborhood of G in M. Similarly, α_G, Ω_G and Φ_G will stand for α_j, Ω_j and Φ_j, respectively.

Remark 3.4. In the case of compact oriented 3-manifolds with boundary, the Y^c-surgery move is defined similarly using Spin^c-structures relative to Spin-structures. (See Remark 3.3.)

It follows from the definition that, for any two disjoint Y-graphs G_1 and G_2 in M, the Spin^c-manifolds $((M_{G_1})_{G_2}, (\alpha_{G_1})_{G_2})$ and $((M_{G_2})_{G_1}, (\alpha_{G_2})_{G_1})$ are Spin^c-diffeomorphic. So, the Y^c-surgery along a family of disjoint Y-graphs makes sense.

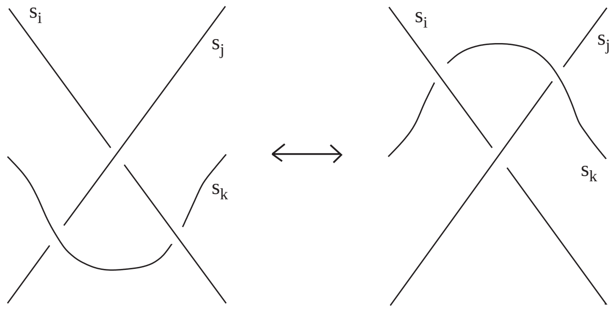

Fig. 7. A Δ^c-move.

Definition 3.3. Let I be an invariant of 3-dimensional Spin^c-manifolds with values in an Abelian group A. The invariant I is said to be *of degree at most d* if, for any 3-dimensional Spin^c-manifold (N, σ) and for any family S of at least $d+1$ pairwise disjoint Y-graphs in N, the identity

$$\sum_{S' \subset S} (-1)^{|S'|} \cdot I(N_{S'}, \sigma_{S'}) = 0 \in A \tag{3.3}$$

holds. Here, the sum is taken over all sub-families S' of S.

Thus, the Y^c-surgery move is the elementary move of a Spin^c-refinement of the Goussarov–Habiro theory of finite type invariants. In particular, two 3-dimensional Spin^c-manifolds are Y^c-equivalent if and only if they are not distinguished by degree 0 invariants. It can be shown that the "calculus of clovers" from [8], which is equivalent to the "calculus of claspers" from [12], extends to Spin^c-manifolds.

Remark 3.5. A Spin-refinement of the Goussarov–Habiro theory has been considered in [21]. In particular, it is shown that the Y-surgery along G induces a canonical bijection $\Theta_G : \text{Spin}(M) \to \text{Spin}(M_G)$. Both refinements of the theory are compatible, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \text{Spin}(M) & \xrightarrow[\simeq]{\Theta_G} & \text{Spin}(M_G) \\ \beta \downarrow & & \downarrow \beta \\ \text{Spin}^c(M) & \xrightarrow[\Omega_G]{\simeq} & \text{Spin}^c(M_G). \end{array}$$

3.2.3. A combinatorial description of the Y^c-equivalence relation

A given equivalence relation among closed oriented 3-manifolds can sometimes be derived from an unknotting operation via surgery presentations in $\mathbf{S}^3$. It is well-known that the Y-equivalence relation can be formulated that way with the *Δ-move* of [25] as unknotting operation. We refine this to the context of Spin^c-manifolds.

Lemma 3.2. *The Y^c-equivalence relation is generated by* Spin^c*-diffeomorphisms and Δ^c-moves, if the Δ^c-move is defined to be the move depicted on* Fig. 7 *between surgery presentations of closed* 3*-dimensional* Spin^c*-manifolds* (*see* Section 2.2.2).

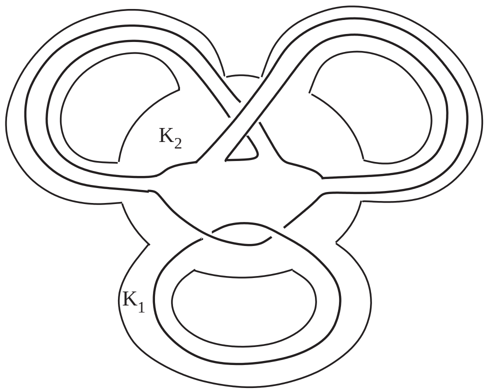

Fig. 8. *Y*-surgery as surgery along a 2-component link.

Proof. Let M be a closed connected oriented 3-manifold and let G be a Y-graph in M. Let $\psi : M \to V_L$ be a surgery presentation of M, where L is an n-component ordered oriented framed link in $\mathbf{S}^3$. Isotope G in M so that $\psi(G)$ becomes disjoint from the link dual to L, then $\psi(G)$ can be regarded as a subset of $\mathbf{S}^3 \setminus L$. In the image by ψ of the regular neighborhood of G in M, put the 2-component framed link K depicted on Fig. 8. The link K can be obtained from the link B of Fig. 5 by some slam dunks (see Example 2.1) and handle slidings in H_3. In particular, there is an obvious surgery presentation $\psi' : M_G \to V_{L\cup K}$ induced by ψ. With the viewpoint from Section 2.2.2, we want to identify the combinatorial analog of the bijection Ω_G. In other words, we look forthe map O_G making the diagram

$$\begin{array}{ccc}
\mathcal{V}_L & \overset{O_G}{\dashrightarrow} & \mathcal{V}_{L\cup K} \\
\simeq \uparrow & & \uparrow \simeq \\
\mathrm{Spin}^c(V_L) & & \mathrm{Spin}^c(V_{L\cup K}) \\
\simeq \uparrow \psi_* & & \psi'_* \uparrow \simeq \\
\mathrm{Spin}^c(M) & \underset{\Omega_G}{\longrightarrow} & \mathrm{Spin}^c(M_G)
\end{array}$$

commute. This is contained in the next claim, which will allow us to prove that the Δ^c-move and the Y^c-surgery move are equivalent.

Claim 3.1. Let B_L denote the linking matrix of L and let K be appropriately oriented so that the ordered union of ordered oriented framed links $L \cup K$ has its linking matrix of the form

$$B_{L\cup K} = \left(\begin{array}{ccc|cc}
 & & & x_1 & 0 \\
 & B_L & & \vdots & \vdots \\
 & & & x_l & 0 \\
\hline
x_1 & \cdots & x_l & x & 1 \\
0 & \cdots & 0 & 1 & 0
\end{array}\right).$$

Then, the map O_G sends a Chern vector $[s]$ to the Chern vector $[(s, x, 0)]$.

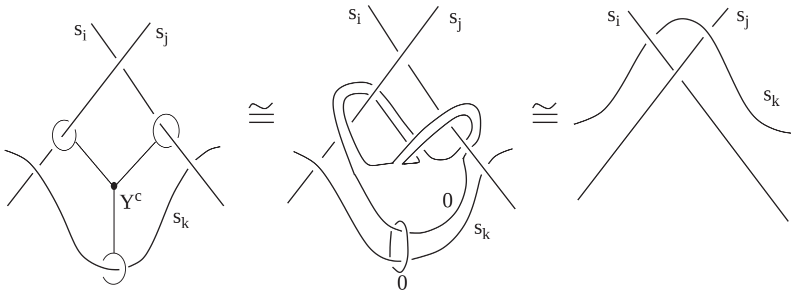

Fig. 9. A Δ^c-move can be realized by a Y^c-surgery.

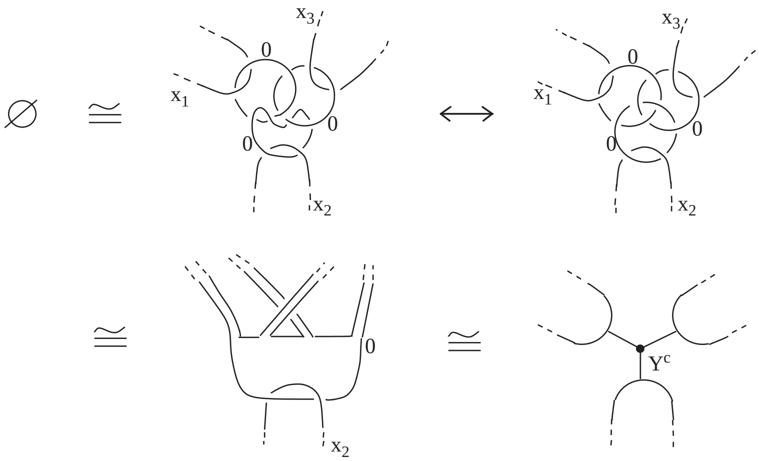

Fig. 10. A Y^c-surgery can be realized by a Δ^c-move.

Proof. As pointed out in Remark 3.5, a Y-surgery along G induces a bijection $\Theta_G : \mathrm{Spin}(M) \to \mathrm{Spin}(M_G)$, a combinatorial analog of which is given in [21]. Using the compatibility between Θ_G and Ω_G together with Section 2.2.3, we see that the claim holds at least for those Chern vectors that come from $\mathscr{S}_L$.

Denote by (H, f) the lattice corresponding to the intersection pairing of W_L, and by (H', f') that of $W_{L\cup K}$. Recall from Remark 2.2 that there are canonical isomorphisms $H^2(V_L) \simeq \mathrm{Coker}\,\widehat{f}$ and $H^2(V_{L\cup K}) \simeq \mathrm{Coker}\,\widehat{f'}$. The isomorphism $P\Phi_G P^{-1} : H^2(M) \to H^2(M_G)$ corresponds then to the isomorphism $\mathrm{Coker}\,\widehat{f} \to \mathrm{Coker}\,\widehat{f'}$ defined by $[y] \mapsto [(y, 0, 0)]$.

Take now $[s] \in \mathscr{V}_L$ arising from $\mathscr{S}_L$ and let $[y] \in \mathbb{Z}^n/\mathrm{Im}\,B_L \simeq \mathrm{Coker}\,\widehat{f}$. We aim to calculate $O_G([s]+[y]) \in \mathscr{V}_{L\cup K}$. The "+" here corresponds to the action of $H^2(V_L)$ on $\mathrm{Spin}^c(V_L)$ (see Remark 2.2). The map Ω_G being affine over $P\Phi_G P^{-1}$, we have that $O_G([s]+[y]) = O_G([s]) + [(y,0,0)] = [(s,x,0)] + [(y,0,0)] = [(s+2y,x,0)]$. Therefore, the claim also holds for $[s]+[y] = [s+2y]$. The transitivity of the action of $H^2(V_L)$ on $\mathrm{Spin}^c(V_L)$ allows us to conclude. □

Figs. 9 and 10 prove that, up to Spin^c-diffeomorphisms, a Δ^c-move can be realized by a Y^c-surgery and vice versa.

In Fig. 9, the first Spin^c-diffeomorphism is obtained by applying Claim 3.1, while the second one is obtained from one handle sliding and one slam dunk.

In Fig. 10, the first Spin^c-diffeomorphism is obtained from three slam dunks. Next, a Δ^c-move is applied. The second Spin^c-diffeomorphism is obtained by Spin^c Kirby's calculi (in particular, two slam dunks have been performed), and the last one is obtained from Claim 3.1. □

3.3. Proof of Theorem 2

In this subsection, we prove the characterization of the Y^c-equivalence relation, as announced in the introduction. We need two results concerning classification of quadratic functions up to isomorphism, proved in [5].

3.3.1. Isomorphism classes of quadratic functions

There is a natural notion of isomorphism among triples (H, f, c) defined by bilinear lattices with characteristic form (see Section 2.1): we say that two triples (H, f, c) and (H', f', c') are *isomorphic* if there is an isomorphism $\psi : H \to H'$ such that $f = f' \circ (\psi \times \psi)$ and $c = c' \circ \psi \bmod 2\widehat{f}(H)$. Such triples form a monoid for the orthogonal sum $\oplus$. Two triples (H, f, c) and (H', f', c') are said to be *stably equivalent* if they become isomorphic after stabilizations with some copies of $(\mathbb{Z}, \pm 1, \mathrm{Id})$, which denotes the bilinear lattice defined on $\mathbb{Z}$ by $(1, 1) \mapsto \pm 1$ and equipped with the characteristic form $\mathrm{Id} = \mathrm{Id}_{\mathbb{Z}}$. Note that, for any bilinear lattices (H, f) and (H', f'), there is a map

$$\psi \mapsto \psi^\sharp, \ \ \mathrm{Iso}(\mathrm{Coker}\, \widehat{f}, \ \mathrm{Coker}\, \widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f)$$

since the pairing (2.2) is right nonsingular.

Theorem 3.1 (*Deloup and Massuyeau [5]*). *Two bilinear lattices with characteristic form (H, f, c) and (H', f', c') are stably equivalent if, and only if, there exists an element*

$$\psi^\sharp \in \mathrm{Im}(\mathrm{Iso}(\mathrm{Coker}\, \widehat{f}, \mathrm{Coker}\, \widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f))$$

such that the associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ are isomorphic via $\psi^\sharp$. Furthermore, any such isomorphism between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ lifts to a stable equivalence between (H, f, c) and (H', f', c').

Remark 3.6. Let Ψ be an isomorphism between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ and suppose that f and f' are degenerate. Then, Ψ does not necessarily arise from an isomorphism $\psi : \mathrm{Coker}\, \widehat{f} \to \mathrm{Coker}\, \widehat{f'}$. In fact, it does if and only if $\Psi|_{\mathrm{Ker}\, \widehat{L_{f'}}} : \mathrm{Ker}\, \widehat{L_{f'}} \to \mathrm{Ker}\, \widehat{L_f}$ lifts to an isomorphism $\mathrm{Ker}\, \widehat{f'} \to \mathrm{Ker}\, \widehat{f}$. (See [5] for details.)

Let now $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on an Abelian group G. We shall say that q meets the *finiteness condition* if

- $G/\mathrm{Ker}\, \widehat{b_q}$ is finite,
- the extension G of $\mathrm{Ker}\, \widehat{b_q}$ by $G/\mathrm{Ker}\, \widehat{b_q}$ is split.

We shall also denote by r_q the homomorphism obtained by restricting q to $\mathrm{Ker}\, \widehat{b_q}$.

Theorem 3.2 (*Deloup and Massuyeau [5]*). *Two quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ satisfying the finiteness condition are isomorphic if, and only if, there is an isomorphism $\Psi : G' \to G$ such that $b_{q'} = b_q \circ (\Psi \times \Psi)$, $d_{q'} = d_q \circ \Psi$, $r_{q'} = r_q \circ \Psi|$ and $\gamma(q' \circ s') = \gamma(q \circ s)$ for some Ψ-compatible sections s and s' of the canonical epimorphisms $G \to G/\mathrm{Ker}\,\widehat{b_q}$ and $G' \to G'/\mathrm{Ker}\,\widehat{b_{q'}}$.*

Here, the Ψ-compatibility condition refers to the commutativity of the diagram

$$\begin{array}{ccc} G' & \xleftarrow{\ s'\ } & G'/\mathrm{Ker}\ \widehat{b_{q'}} \\ {\scriptstyle \Psi}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle [\Psi]} \\ G & \xleftarrow[\ s\ ]{} & G/\mathrm{Ker}\ \widehat{b_q} \end{array}$$

where $[\Psi]$ is the isomorphism induced by Ψ.

Remark 3.7. Theorem 3.2 does not claim that $q' = q \circ \Psi$ if the four conditions hold. Nevertheless, as follows from the proof in [5], it is true that there exists an isomorphism $\varphi : G' \to G$ such that $q' = q \circ \varphi$ and $\varphi|_{\mathrm{Ker}\,\widehat{b_{q'}}} = \Psi|_{\mathrm{Ker}\,\widehat{b_{q'}}}$.

We now go into the proof of Theorem 2. In the sequel, we consider two closed connected 3-dimensional Spin^c-manifolds, (M, σ) and (M', σ').

3.3.2. Proof of the equivalence (2) $\Longleftrightarrow$ (3) *of Theorem 2*

Next lemma is easily proved from the definitions.

Lemma 3.3. *Let $\psi : H_1(M) \to H_1(M')$ be an isomorphism, which induces a dual isomorphism $\psi^\sharp : H_2(M'; \mathbb{Q}/\mathbb{Z}) \to H_2(M; \mathbb{Q}/\mathbb{Z})$ with respect to the intersection pairings. The following assertions are equivalent*:

(a) $L_{M'} = L_M \circ (\psi^\sharp \times \psi^\sharp)$,
(b) $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$,
(c) *The following diagram is commutative*:

$$\begin{array}{ccc} H_2(M';\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\ B\ } & \mathrm{Tors}\ H_1(M') \\ {\scriptstyle \psi^\sharp}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \psi|}\uparrow{\scriptstyle \simeq} \\ H_2(M;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\ B\ } & \mathrm{Tors}\ H_1(M). \end{array}$$

Suppose that condition (2) of Theorem 2 is satisfied. This implies that $L_{M'} = L_M \circ (\psi^\sharp \times \psi^\sharp)$ and so that $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$ by Lemma 3.3.

Condition (2) also implies the relation $d_{\phi_{M',\sigma'}} = d_{\phi_{M,\sigma}} \circ \psi^\sharp$ between homogeneity defects of quadratic functions. So, by Lemma 2.8, we have $\langle c(\sigma'), x' \rangle = \langle c(\sigma), \psi^\sharp(x') \rangle$ for all $x' \in H_2(M'; \mathbb{Q}/\mathbb{Z})$. By left nondegeneracy of the pairing $\bullet : H_1(M') \times H_2(M'; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, we conclude that $P^{-1}c(\sigma') = \psi(P^{-1}c(\sigma))$.

Last, the quadratic function

$$\phi_{M,\sigma} \circ s = \phi_{M,\sigma} \circ \psi^{\sharp} \circ s' \circ \psi| = \phi_{M',\sigma'} \circ s' \circ \psi|$$

is isomorphic to $\phi_{M',\sigma'} \circ s'$: hence, these two quadratic functions have identical Gauss sums. Therefore condition (3) holds.

Conversely, suppose that the condition (3) of Theorem 2 is satisfied. The short exact sequence

$$0 \longrightarrow H_2(M) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H_2(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{B} \mathrm{Tors}\ H_1(M) \longrightarrow 0$$

is split, we have that $H_2(M) \otimes \mathbb{Q}/\mathbb{Z} = \mathrm{Ker}\ \widehat{L_M}$ and Tors $H_1(M)$ is finite: thus, $\phi_{M,\sigma}$ meets the finiteness condition of Section 3.3.1. Since $\lambda_M = \lambda_{M'} \circ (\psi| \times \psi|)$, we obtain by Lemma 3.3 that $L_{M'} = L_M \circ (\psi^{\sharp} \times \psi^{\sharp})$. Since $\psi(P^{-1}c(\sigma)) = P^{-1}c(\sigma')$, we deduce from Lemmas 2.7 and 2.8 that $r_{\phi_{M',\sigma'}} = r_{\phi_{M,\sigma}} \circ \psi^{\sharp}|$ and that $d_{\phi_{M',\sigma'}} = d_{\phi_{M,\sigma}} \circ \psi^{\sharp}$, respectively. Also, since $\psi| \circ B \circ \psi^{\sharp} = B$ (by Lemma 3.3),the ψ-compatibility condition between s and s' required by condition (3) of Theorem 2 coincides with the $\psi^{\sharp}$-compatibility in the sense of Section 3.3.1. Therefore, by Theorem 3.2, the quadratic functions $\phi_{M,\sigma}$ and $\phi_{M',\sigma'}$ are isomorphic. More precisely, according to Remark 3.7, there exists an isomorphism $\varphi : H_2(M'; \mathbb{Q}/\mathbb{Z}) \to H_2(M; \mathbb{Q}/\mathbb{Z})$ such that $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \varphi$ and $\varphi|_{H_2(M') \otimes \mathbb{Q}/\mathbb{Z}}$ coincides with $\psi^{\sharp}|_{H_2(M') \otimes \mathbb{Q}/\mathbb{Z}} = \psi^{\sharp} \otimes \mathbb{Q}/\mathbb{Z}$. This latter fact, together with Remark 3.6, allows us to precise that φ equals $\eta^{\sharp}$ for a certain isomorphism $\eta : H_1(M) \to H_1(M')$. Consequently, $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \eta^{\sharp}$.

3.3.3. *Proof of the equivalence* (1) $\Longleftrightarrow$ (2) *of Theorem 2*

We prove implication (1) $\Longrightarrow$ (2) first. By Lemma 3.2, it suffices to prove it when (M, σ) and (M', σ') are related by one Spin^c-diffeomorphism or, for some fixed surgery presentations, by one Δ^c-move. The first case follows immediately from the definition of the linking quadratic function. The second case is deduced from the combinatorial formula for the latter given at the end of Section 2.3, and from the fact that a Δ-move between ordered oriented framed links preserve the linking matrices.

Suppose now that condition (2) is satisfied. We can assume that $M = V_L$ and $M' = V_{L'}$, where L and L' are ordered oriented framed links in $\mathbf{S}^3$. As in Section 2.2, we denote by (H, f) and (H', f') the intersection pairings of W_L and $W_{L'}$, respectively. Let also $c \in \mathrm{Char}(f)$ and $c' \in \mathrm{Char}(f')$ represent σ and σ', respectively. By hypothesis, the quadratic functions $\phi_{f,c} : G_f \to \mathbb{Q}/\mathbb{Z}$ and $\phi_{f',c'} : G_{f'} \to \mathbb{Q}/\mathbb{Z}$ are isomorphic via an isomorphism which is induced by an isomorphism Coker $\widehat{f} \to$ Coker $\widehat{f'}$. So, by Theorem 3.1, the bilinear lattices with characteristic form (H, f, c) and (H', f', c') are stably equivalent.

An isomorphism of bilinear lattices with characteristic form can be topologically realized by a finite sequence of Spin^c Kirby's moves (see Theorem 2.2): handle slidings and reversings of orientation. Similarly, a stabilization by $(\mathbb{Z}, \pm 1, \mathrm{Id})$ corresponds to a stabilization by the unknot. Therefore, we can suppose, without loss of generality, that $(H, f, c) \simeq (H', f', c')$ through the isomorphism that identifies the preferred basis of H with that of H'. Concretely, this means that the linking matrices B_L and $B_{L'}$ are equal and that there is a multi-integer s such that the Chern vectors $[s] \in \mathscr{V}_L$ and $[s] \in \mathscr{V}_{L'}$ represent σ and σ', respectively.

A theorem[2] of Murakami and Nakanishi [25, Theorem 1.1] states that two ordered oriented framed links have identical linking matrices if, and only if, they are Δ-equivalent. Then, the “decorated” links

[2] In fact, the first reference is [22] but the proof there is not detailed.

(L, s) and (L', s) are Δ-equivalent: therefore, by Lemma 3.2, the Spin^c-manifolds (M, σ) and (M', σ') are Y^c-equivalent.

Remark 3.8. Observe that the present proof allows for a more precise statement of the equivalence (1) $\Longleftrightarrow$ (2) of Theorem 2. Any finite sequence of Spin^c-diffeomorphisms and Y^c-surgeries

$$(M, \sigma) = (M_0, \sigma_0) \rightsquigarrow (M_1, \sigma_1) \rightsquigarrow (M_2, \sigma_2) \rightsquigarrow \cdots \rightsquigarrow (M_n, \sigma_n) = (M', \sigma')$$

yields an isomorphism $\psi : H_1(M) \to H_1(M')$. This is the composite of the isomorphisms $H_1(M_i) \to H_1(M_{i+1})$, which is taken to be either g_* if the step $(M_i, \sigma_i) \rightsquigarrow (M_{i+1}, \sigma_{i+1})$ is a Spin^c-diffeomorphism g, either the isomorphism Φ_G if the step is the Y^c-surgery along a Y-graph $G \subset M_i$ (Section 3.2.2). This isomorphism ψ satisfies $\phi_{M',\sigma'} = \phi_{M,\sigma} \circ \psi^\sharp$. Conversely, given an isomorphism $\psi : H_1(M) \to H_1(M')$ with this property, one can find a finite sequence of Spin^c-diffeomorphisms and Y^c-surgeries from (M, σ) to (M', σ') inducing ψ at the level of $H_1(-)$. Here, we use the second statement of Theorem 3.1.

3.4. *Applications and problems*

We conclude this paper with some applications of our results illustrated by a few examples. We also state a few problems.

3.4.1. *The quotient set* Spin$^c(M)/Y^c$

Given a closed oriented 3-manifold M, one may consider the quotient set

$$\mathrm{Spin}^c(M)/Y^c$$

of Spin^c-structures on M modulo the Y^c-equivalence relation. Let us consider a few examples.

Example 3.1. Take $M = \mathbb{R}\mathrm{P}^3$. This manifold has two distinct Spin^c-structures σ_0 and σ_1, both arising from Spin-structures. The quadratic functions ϕ_{M,σ_0} and ϕ_{M,σ_1} have different Gauss sums (which are $\exp(2i\pi/8)$ and $\exp(-2i\pi/8) \in \mathbb{C}$). Therefore, by Corollary 2, σ_0 is not Y^c-equivalent to σ_1.

Example 3.2. Take M such that $H_1(M) \simeq \mathbb{Z}^n$. According to Corollary 1, the set $\mathrm{Spin}^c(M)/Y^c$ can be identified with $(2\mathbb{Z}^n)/\mathrm{GL}(n; \mathbb{Z})$ by the Chern class map.

In particular, if $M = \mathbf{S}^2 \times \mathbf{S}^1$ and if an isomorphism $H_1(M) \simeq \mathbb{Z}$ is fixed, we denote by α_k the unique element of Spin$^c(M)$ such that $c(\alpha_k) = 2k \in \mathbb{Z}$, with $k \in \mathbb{Z}$. Then, the Y^c-equivalence classes are $\{\alpha_0\}$ and $\{\alpha_k, \alpha_{-k}\}$ with $k > 0$. Observe from Theorem 2.2, that these classes coincide with the diffeomorphism classes.

Example 3.3. Take $M = (\mathbf{S}^2 \times \mathbf{S}^1)\sharp\mathbb{R}\mathrm{P}^3$. By applying equivalence (1) $\Leftrightarrow$ (2) of Theorem 2, the Y^c-equivalence classes are seen to be $\{\alpha_0\sharp\sigma_0\}$, $\{\alpha_0\sharp\sigma_1\}$, $\{\alpha_k\sharp\sigma_0,\ \alpha_k\sharp\sigma_1,\ \alpha_{-k}\sharp\sigma_0,\ \alpha_{-k}\sharp\sigma_1\}$ with $k > 0$ odd,$\{\alpha_k\sharp\sigma_0, \alpha_{-k}\sharp\sigma_0\}$ and $\{\alpha_k\sharp\sigma_1, \alpha_{-k}\sharp\sigma_1\}$ with $k > 0$ even. Again, observe from Theorem 2.2, that these classes coincide with the diffeomorphism classes.

In light of the previous examples, it is natural to ask whether the diffeomorphism classes of Spin^c-structures of a given closed oriented 3-manifold M coincide with the Y^c-equivalence classes. To answer this question by the negative, let us consider a class of manifolds for which the Spin^c-structures have been classified: the family of lens spaces. Let $p \geqslant 2$ be an integer, let q_1, q_2 be some invertible elements of $\mathbb{Z}_p$ and let $L(p; q_1, q_2)$ be the corresponding lens space with the orientation induced from the canonical orientation of $\mathbf{S}^3$.

Theorem 3.3 (*Turaev [28]*). *The number of orbits of* Spin^c*-structures under the action of the group of positive self-diffeomorphisms of* $L(p; q_1, q_2)$ *is*

- $[p/2] + 1$, *if* $q_1^2 \neq q_2^2$ *or* $q_1 = \pm q_2$,
- $p/2 - b(p; q_1, q_2)/4 + c(p; q_1, q_2)/2$, *if* $q_1^2 = q_2^2$*and* $q_1 \neq \pm q_2$.

Here, for $x \in \mathbb{Q}$, $[x]$ *denotes the greatest integer less or equal than x,* $b(p; q_1, q_2)$ *is the number of* $i \in \mathbb{Z}_p$ *for which i,* $q_1 + q_2 - i$ *and* $q_2 q_1^{-1} i$ *are pairwise different, and* $c(p; q_1, q_2)$ *is the number of* $i \in \mathbb{Z}_p$ *such that* $i = q_1 + q_2 - i = q_2 q_1^{-1} i$.

Proof. In [28, Section 9.2.1], the Euler structures on $L(p; q_1, q_2)$ are classified up to diffeomorphisms. The same kind of arguments can be used to classify these up to *positive* diffeomorphisms. Details are left to the reader. □

The classification of the Spin^c-structures on $L(p; q_1, q_2)$ up to Y^c-equivalence is easily obtained from Corollary 2. For instance, let us suppose that p is odd. Then, Spin$^c(L(p; q_1, q_2))/Y^c$ can be identified via the Chern class map with the quotient set $\mathbb{Z}_p/\sim$, where

$$\forall i, j \in \mathbb{Z}_p, \quad (i \sim j) \iff (\exists r \in \mathbb{Z}_p,\ r^2 = 1 \text{ and } j = ri).$$

Example 3.4. Let $k \geqslant 4$ be an even integer and let $p = k^2 - 1$. Then, there are some Spin^c-structures on $L(p; 1, 1)$ which are Y^c-equivalent but which are not diffeomorphic. Indeed, according to Theorem 3.3, Spin$^c(L(p; 1, 1))$ contains $(p-1)/2 + 1$ diffeomorphism classes. But, $k^2 = 1 \in \mathbb{Z}_p$ and $k \neq \pm 1 \in \mathbb{Z}_p$, so the cardinality of Spin$^c(L(p; 1, 1))/Y^c$ is strictly less than $(p-1)/2 + 1$.

3.4.2. Reidemeister–Turaev torsions

Let $\tau(M, \sigma)$ denote the maximal Abelian Reidemeister–Turaev torsion of a closed oriented 3-manifold M equipped with an Euler structure or, equivalently, a Spin^c-structure σ [32]. If M is a rational homology sphere, it turns out that $\phi_{M,\sigma}$ can be explicitely computed from $\tau(M, \sigma)$ [6,26]. Thus, according to Corollary 2, part of $\tau(M, \sigma)$ is of degree 0.

Problem 3.1. Derive from Reidemeister–Turaev torsions higher degree finite type invariants of closed 3-dimensional Spin^c-manifolds.

In the last chapter of [20], it is studied how Reidemeister–Turaev torsions vary under those twists defined in Section 3.2.1. This variation is difficult to control for a generic Y-graph. Nevertheless, this variation can be calculated explicitely in case of “looped clovers”. It is shown that Reidemeister–Turaev torsions satisfy a certain multiplicative degree 1 relation involving surgeries along looped clovers.

3.4.3. From the Spin-refinement of the theory to its Spin^c*-refinement*

According to Remark 3.5, any Spin^c-invariant of degree d in the Goussarov–Habiro theory induces a Spin-invariant of degree d. The converse is not true.

Example 3.5. The Rochlin invariant $R(M,\sigma)\in\mathbb{Z}_{16}$ of a closed Spin-manifold (M,σ) of dimension 3 is a finite type invariant of degree 1 [21]. But, it does not lift to an invariant of Spin^c-manifolds in general. Indeed, consider the torus $\mathbf{T}^3$ and its canonical Spin-structure σ^0 (induced by its Lie group structure), choose also σ' in Spin($\mathbf{T}^3$) different from σ^0. Then, $\beta(\sigma')$ and $\beta(\sigma^0)$ coincide, but $R(\mathbf{T}^3,\sigma^0)=8$ is not equal to $R(\mathbf{T}^3,\sigma')=0$.

On the contrary, we have in degree 0 the following consequence of both Theorem 2 and [21, Theorem 1].

Corollary 3.1. *Let* (M,σ) *and* (M',σ') *be closed* 3*-dimensional* Spin*-manifolds. Then,* (M,σ) *and* (M',σ') *are distinguished by degree* 0 Spin*-invariants if and only if* $(M,\beta(\sigma))$ *and* $(M',\beta(\sigma'))$ *are distinguished by degree* 0 Spin^c*-invariants.*

Problem 3.2. Compare in higher degrees the Spin^c-refinement of the Goussarov–Habiro theory with its Spin-refinement.

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Complexity invariance of real interpretations

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Abstract. In the field of implicit computational complexity, we are considering in this paper the fruitful branch of interpretation methods. In this area, the synthesis problem is solved by Tarski's decision procedure, and consequently interpretations are usually chosen over the reals rather than over the integers. Doing so, one cannot use anymore the (good) properties of the natural (well-) ordering of **N** employed to bound the complexity of programs. We show that, actually, polynomials over the reals benefit from some properties that allow their safe use for complexity. We illustrate this by two characterizations, one of PTIME and one of PSPACE.

To prove the termination of a rewrite system, it is natural to interpret terms into a well-founded ordering. For instance, Lankford describes interpretations as monotone Σ-algebras with domain of interpretation being the natural numbers with their usual ordering (c.f. [15, 14]).

However, in the late seventies, Dershowitz showed in a seminal paper [8] that the well-foundedness of the domain of interpretation is not necessary whenever the interpretations are chosen to be monotonic and to have the sub-term property. Thus, the domain of the Σ-algebra mentioned above can be the set of real numbers.

One of the main interesting points about choosing of real numbers rather than natural numbers is that we get (at least from a theoretical point of view) a procedure to verify the validity of an interpretation of a program by Tarski's decomposition procedure [25] and an algorithm to compute interpretations up to some fixed degree. Following Roy et al. [3], the complexity of these algorithms is exponential with respect to the size of the program.

A second good point is that the use of reals (as opposed to integers) enlarges the set of rewriting systems that are compatible with an interpretation, as shown recently by Lucas [17].

In the last years, the study of termination methods has been one of the major tools in implicit computational complexity. For instance, Moser et al. have characterized PTIME by means of POP* in [2], and context dependent interpretations in [22] after their introduction by Hofbauer [11]. One of our two characterizations, Theorem 6, use dependency pairs (c.f. [1]). In this vein, we mention here the work of Hirokawa and Moser [10], and, in the same spirit, Lucas and Peña in [19] made some investigation on the tools of rewriting to tackle the complexity of a first order functional programs.

But, the main concern of the present paper is to show that the structure of polynomials over the reals has an important role from the point of view of complexity. Our thesis is that, in the field of complexity, due to Stengle's Positivstellensatz [24], polynomials over the reals can safely replace polynomials over the integers. It is illustrated by two theorems, Theorem 4 and Theorem 6. We show that one may recover both derivational complexity (up to a polynomial) and size bounds (also, up to a polynomial) on terms as applications of Positivstellensatz. Moreover, this can be done in a constructive way. Our thesis But, let us draw briefly the roadmap of the key technical features of this work.

Given a strict interpretation for a term rewriting system, it follows immediately that for all rewriting steps $s \to t$, we have $(\!|s|\!) > (\!|t|\!)$. If one takes the interpretation on natural numbers (as they were introduced by Lankford [15]), this can be used to give a bound on the derivation height. Thus, Hofbauer and Lautemann have shown in [12] that the derivation height is bounded by a double exponential. However, their argument uses deeply the fact that the interpretation of a term is itself a bound on the derivation height:

$$\mathtt{dh}(t) \leq (\!|t|\!). \tag{1}$$

Suppose $t_0 \to t_1 \to \cdots \to t_n$, since $(\!|t_0|\!) > (\!|t_1|\!) > \cdots > (\!|t_n|\!)$, on natural numbers, this means that $n \leq (\!|t_0|\!)$. Such a proof does not hold with real numbers. The inequalities $(\!|t_i|\!) > (\!|t_{i+1}|\!)$ are due to a) for all rewrite rule $\ell \to r$, the inequality $(\!|\ell|\!) > (\!|r|\!)$ implies that

$$(\!|\ell|\!) \geq (\!|r|\!) + 1 \tag{2}$$

and b) that for all $x_i > y_i$:

$$(\!|f|\!)(x_1, \ldots, x_i, \ldots, x_n) - (\!|f|\!)(x_1, \ldots, y_i, \ldots, x_n) \geq x_i - y_i. \tag{3}$$

The two inequalities 2, 3 do not hold in general for real interpretations. To recover the good properties holding with natural numbers, people have enforced the inequalities on terms. For instance [17, 21] suppose the existence of some real $\delta > 0$ such that for any rule $\ell \to r : (\!|\ell|\!) \geq (\!|r|\!) + \delta$. We prove that even without the existence of such a δ, the derivation height of a term t is bounded by $(\!|t|\!)$ up to a polynomial.

To save some space, we have omitted some proofs. The reader will find them in the extended version of the paper, see [5].

1 Preliminaries

We suppose that the reader has familiarity with first order rewriting. We briefly recall some of the main notions of the theory, essentially to fix the notations. Dershowitz and Jouannaud's survey [9] is a good entry point.

Let $\mathcal{X}$ denote a (countable) set of *variables*. Given a *signature* Σ, the set of *terms* over Σ and $\mathcal{X}$ id denoted by $\mathcal{T}(\Sigma, \mathcal{X})$ and the set of *ground terms* as $\mathcal{T}(\Sigma)$. The size $|t|$ of a term t is defined as the number of symbols in t.

Given a signature Σ, a rule is an oriented equation $\ell \to r$ with $\ell, r \in \mathcal{T}(\Sigma, \mathcal{X})$ such that variables occurring in r also occur in ℓ. A Term Rewrite System (TRS) is a finite set of such rules. A TRS induces a rewriting relation denoted by $\to$. The relation $\xrightarrow{+}$ is the transitive closure of $\to$ and $\xrightarrow{*}$ is the reflexive and transitive closure of $\to$. Finally, we say that a term t is a *normal form* if there is no term u such that $t \to u$. Given two terms t and u, $t \xrightarrow{!} u$ denotes the fact that $t \xrightarrow{*} u$ and u is a normal form. We write $t_0 \to^n t_n$ the fact that $t_0 \to t_1 \cdots \to t_n$. One defines the derivation height for a term t as the maximal length of a derivation: $\mathtt{dh}(t) = \max\{n \in \mathbf{N} \mid \exists v : t \to^n v\}$.

A *context* is a term C with a particular variable $\Diamond$. If t is a term, $C[t]$ denotes the term C where the variable $\Diamond$ has been replaced by t. A substitution is a mapping from variables to terms. A substitution σ can be extended canonically to terms and we note $t\sigma$ the application of the substitution σ to the term t.

1.1 Syntax of programs

Definition 1. *A program is a 5-tuple $\mathtt{f} = \langle \mathcal{X}, \mathcal{C}, \mathcal{F}, \mathtt{main}, \mathcal{E} \rangle$ with:*

- *$\mathcal{C}$ is a (finite) signature of* constructor *symbols and $\mathcal{F}$ a (finite) signature of* function symbols*. $\mathtt{main} \in \mathcal{F}$ is the "main" function symbol*
- *$\mathcal{E}$ is a finite set of rules of the shape $\mathtt{f}(p_1, \cdots, p_n) \to r$ where $\mathtt{f} \in \mathcal{F}$ and $p_i \in \mathcal{T}(\mathcal{C}, \mathcal{X})$.*

Moreover, we suppose programs to be confluent. This is achieved by the following syntactic restriction due to Huet [13] (see also [23]): (i) Each rule $\mathtt{f}(p_1, \ldots, p_n) \to t$ is left-linear, that is a variable appears only once in $\mathtt{f}(p_1, \cdots, p_n)$, and (ii) there are no two left hand-sides which are overlapping.

The program $\langle \mathcal{X}, \mathcal{C}, \mathcal{F}, \mathtt{f}, \mathcal{E} \rangle$ computes the partial function $[\![\mathtt{f}]\!] : \mathcal{T}(\mathcal{C})^n \to \mathcal{T}(\mathcal{C})$ defined as follows. For every $u_1, \cdots, u_n \in \mathcal{T}(\mathcal{C}), [\![\mathtt{f}]\!](u_1, \cdots, u_n) = v$ iff $\mathtt{f}(u_1, \cdots, u_n) \xrightarrow{*} v$ and $v \in \mathcal{T}(\mathcal{C})$. Otherwise, it is undefined.

Example 1. The following program computes the membership in a list. The constructors of lists are **cons**, **nil**. Elements in the list are the tally natural numbers build from **0** and **s**.

$$\begin{array}{lll} \mathtt{not}(\mathbf{tt}) \to \mathbf{ff} & \mathtt{or}(\mathbf{tt}, y) \to \mathbf{tt} & \mathbf{0} = \mathbf{0} \quad \to \mathbf{tt} \\ \mathtt{not}(\mathbf{ff}) \to \mathbf{tt} & \mathtt{or}(x, \mathbf{tt}) \to \mathbf{tt} & \mathbf{0} = \mathbf{s}(y) \to \mathbf{ff} \\ & \mathtt{or}(\mathbf{ff}, \mathbf{ff}) \to \mathbf{ff} & \mathbf{s}(x) = \mathbf{0} \quad \to \mathbf{ff} \\ & & \mathbf{s}(x) = \mathbf{s}(y) \to x = y \end{array}$$

$$\begin{array}{r} \mathtt{in}(x, \mathbf{nil}) \to \mathbf{ff} \\ \mathtt{in}(x, \mathbf{cons}(a, l)) \to \mathtt{or}(x = a, \mathtt{in}(x, l)) \end{array}$$

Definition 2 (Call-tree). *Suppose we are given a program $\langle \mathcal{X}, \mathcal{C}, \mathcal{F}, \mathcal{E} \rangle$. Let $\rightsquigarrow$ be the relation*

$$(f, t_1, \ldots, t_n) \rightsquigarrow (g, u_1, \ldots, u_m) \Leftrightarrow f(t_1, \ldots, t_n) \to C[g(v_1, \ldots, v_m)] \xrightarrow{*} C[g(u_1, \ldots, u_m)]$$

where f and g are defined symbols, $t_1, \ldots, t_n, u_1, \ldots, u_m$ are ground constructor terms and $v_1, \ldots, v_m$ are arbitrary (ground) terms. Given a term $f(t_1, \ldots, t_n)$, the relation $\rightsquigarrow$ defines a tree whose root is $(f, t_1, \ldots, t_n)$ and η' is a daughter of η iff $\eta \rightsquigarrow \eta'$.

1.2 Interpretations of programs

Given a signature Σ, a Σ-algebra on the domain $\mathbf{R}^+$ is a mapping $(\!|-|\!)$ which associates to every n-ary symbol $f \in \Sigma$ an n-ary function $(\!|f|\!) : {\mathbf{R}^+}^n \rightarrow \mathbf{R}^+$. Such a Σ-algebra can be extended to terms by:

- $(\!|x|\!) = 1_{\mathbf{R}^+}$, that is the identity on $\mathbf{R}^+$, for $x \in \mathcal{X}$,
- $(\!|f(t_1, \ldots, t_m)|\!) = \mathrm{comp}((\!|f|\!), (\!|t_1|\!), \ldots, (\!|t_m|\!))$ where comp is the composition of functions.

Given a term t with n variables, $(\!|t|\!)$ is a function ${\mathbf{R}^+}^n \rightarrow \mathbf{R}^+$.

Definition 3. *Given a program $\langle \mathcal{X}, \mathcal{C}, \mathcal{F}, \mathfrak{f}, \mathcal{E} \rangle$, let us consider a $(\mathcal{C} \cup \mathcal{F})$-algebra $(\!|-|\!)$ on $\mathbf{R}^+$. It is said to:*

1. *be strictly monotonic if for any symbol f, the function $(\!|f|\!)$ is a strictly monotonic function, that is if $x_i > x'_i$, then*
$$(\!|f|\!)(x_1, \ldots, x_n) > (\!|f|\!)(x_1, \ldots, x'_i, \ldots, x_n),$$
2. *be weakly monotonic if for any symbol f, the function $(\!|f|\!)$ is a weakly monotonic function, that is if $x_i \geq x'_i$, then*
$$(\!|f|\!)(x_1, \ldots, x_n) \geq (\!|f|\!)(x_1, \ldots, x'_i, \ldots, x_n),$$
3. *have the strict sub-term property if for any symbol f, the function $(\!|f|\!)$ verifies $(\!|f|\!)(x_1, \ldots, x_n) > x_i$ with $i \in \{1, \ldots, n\}$,*
4. *to be strictly compatible (with the rewriting relation) if for all rules $\ell \rightarrow r$, $(\!|\ell|\!) > (\!|r|\!)$,*
5. *to be a sup-approximation if for all constructor terms $t_1, \ldots, t_n$, we have the inequality $(\!|f(t_1, \ldots, t_n)|\!) \geq (\!|[\![f]\!](t_1, \ldots, t_n)|\!)$.*

Definition 4. *Given a program $\langle \mathcal{X}, \mathcal{C}, \mathcal{F}, \mathfrak{f}, \mathcal{E} \rangle$, a $(\mathcal{C} \cup \mathcal{F})$-algebra on $\mathbf{R}^+$ is said to be a strict interpretation whenever it verifies (1), (3), (4). It is a sup-interpretation whenever it verifies (2) and (5).*

Sup-interpretation have been introduced by Marion and Pechoux in [20]. We gave here a slight variant of their definition. In [20], the last inequality refers to the size of normal forms. We preferred to have a more uniform definition. Clearly, a strict interpretation is a sup-interpretation.When we want to speak arbitrarily of one of those concepts, we use the generic word "interpretation". We also use this terminology to speak about the function $(\!|f|\!)$ given a symbol f.

Finally, by default, we restrict the interpretations of symbols to be *Max-Poly functions*, that is functions obtained by finite compositions of the constant functions, maximum, addition and multiplication.

Definition 5. *The interpretation of a symbol f is said to be additive if it has the shape $\sum_i x_i + c$ with $c > 0$. A program with an interpretation is said to be additive when its* constructors *are additive.*

Example 2. The program given in Example 1 has both an additive strict interpretation (left side, black) and an additive sup-interpretation (right side, blue):

$$\begin{aligned} (\!|\mathbf{tt}|\!) = (\!|\mathbf{ff}|\!) = (\!|\mathbf{0}|\!) = (\!|\mathbf{nil}|\!) &= 1 \\ (\!|\mathbf{s}|\!)(x) &= x+1 \\ (\!|\mathbf{cons}|\!)(x,y) &= x+y+3 \\ (\!|\mathtt{not}|\!)(x) &= x+1 \\ (\!|\mathtt{or}|\!)(x,y) = (\!|=|\!)(x,y) &= x+y+1 \\ (\!|\mathtt{in}|\!)(x,y) &= (x+1)(y+1) \end{aligned}$$

$$\begin{aligned} (\!|\mathbf{tt}|\!) = (\!|\mathbf{ff}|\!) = (\!|\mathbf{0}|\!) = (\!|\mathbf{nil}|\!) &= 1 \\ (\!|\mathbf{s}|\!)(x) &= x+1 \\ (\!|\mathbf{cons}|\!)(x,y) &= x+y+1 \\ (\!|\mathtt{not}|\!)(x) &= 1 \\ (\!|\mathtt{or}|\!)(x,y) = (\!|=|\!)(x,y) &= 1 \\ (\!|\mathtt{in}|\!)(x,y) &= 1 \end{aligned}$$

Example 3. The Quantified Boolean Formula (QBF) problem is well known to be PSPACE complete. It consists in determining the validity of a boolean formula with quantifiers over propositional variables. Without loss of generality, we restrict formulae to $\neg, \vee, \exists$. Variables are represented by tally numbers. The QBF problem is solved extending the preceding program with:

$$\begin{aligned} \mathtt{verify}(\mathbf{Var}(x), t) &\to \mathtt{in}(x, t) \\ \mathtt{verify}(\mathbf{Not}(\varphi), t) &\to \mathtt{not}(\mathtt{verify}(\varphi, t)) \\ \mathtt{verify}(\mathbf{Or}(\varphi_1, \varphi_2), t) &\to \mathtt{or}(\mathtt{verify}(\varphi_1, t), \mathtt{verify}(\varphi_2, t)) \\ \mathtt{verify}(\mathbf{Exists}(n, \varphi), t) &\to \mathtt{or}(\mathtt{verify}(\varphi, \mathbf{cons}(n, t)), \mathtt{verify}(\varphi, t)) \\ \mathtt{qbf}(\varphi) &\to \mathtt{verify}(\varphi, \varepsilon) \end{aligned}$$

It has a sup-interpretation but not a strict interpretation:

$$\begin{aligned} (\!|\mathbf{Not}|\!)(x) = (\!|\mathbf{Var}|\!)(x) &= x+1 \\ (\!|\mathbf{Or}|\!)(x,y) = (\!|\mathbf{Exists}|\!)(x,y) &= x+y+1 \\ (\!|\mathtt{verify}|\!)(x,y) = (\!|\mathtt{qbf}|\!)(x) &= 1 \end{aligned}$$

Actually, as Theorem 4 will prove it, unless PTIME = PSPACE, there is no program computing QBF with an additive strict interpretation.

2 Positivstellensatz and applications

In this section, we introduce a deep mathematical result, the Positivstellensatz. Then we give some applications to polynomial interpretations. They will be key points of the Theorems 4 and 6 in our analysis of the role of reals in complexity (§B).

Let $n > 0$. Denote by $\mathbf{R}[x_1, \dots, x_n]$ the $\mathbf{R}$-algebra of polynomials with real coefficients. Denote by $(\mathbf{R}^+)^n = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1, \dots, x_n > 0\}$ the first quadrant. Since we need to consider only the $\mathbf{R}$-algebra of polynomial functions $(\mathbf{R}^+)^n \to \mathbf{R}$, it will be convenient to identify the two spaces. In particular throughout this section, all polynomial functions are defined on $(\mathbf{R}^+)^n$.

Theorem 3 (Positivstellensatz, Stengle [24]). *Suppose that we are given polynomials $P_1, \ldots, P_m \in \mathbf{R}[x_1, \ldots, x_k]$, the following two assertions are equivalent:*

1. $\{x_1, \ldots, x_k : P_1(x_1, \ldots, x_k) \geq 0 \wedge \cdots \wedge P_m(x_1, \ldots, x_k) \geq 0\} = \emptyset$
2. $\exists Q_1, \ldots, Q_m : -1 = \sum_{i \leq m} Q_i P_i$ *where each Q_i is a sum of squares of polynomials (and so is positive and monotonic).*

Moreover, these polynomials $Q_1, \ldots, Q_m$ can effectively computed. We refer the reader to the work of Lombardi, Coste and Roy [6]. As a consequence, all the constructions given below can be actually (at least theoretically) computed.

It will be convenient to derive from the Positivstellensatz some propositions useful for our applications.

Proposition 2. *Suppose that a TRS (Σ, R) admits an interpretation $(\!| - |\!)$ over* Max-Poly *such that for all rules $\ell \to r$, we have $(\!| \ell |\!) > (\!| r |\!)$. There is a positive, monotonic polynomial P such that for any rule $\ell \to r$, we have $(\!| \ell |\!)(x_1, \ldots, x_k) - (\!| r |\!)(x_1, \ldots, x_k) \geq \dfrac{1}{P(x_1, \ldots, x_k)}$.*

The proof is direct when ℓ and r are polynomials. By a finite case analysis, one may cope with the max function.

Proposition 2 has an important consequence. Since, in a derivation all terms have an interpretation bounded by the interpretation of the first term, there is a minimal decay for each rule of the derivation.

Proposition 3. *Suppose that a TRS (Σ, R) admits a strict interpretation $(\!| - |\!)$ over* Max-Poly*. For all $A > 0$, the set of terms $\{t \in \mathcal{T}(\Sigma) \mid (\!| t |\!) < A\}$ is finite.*

Proposition 4. *Suppose that a TRS (Σ, R) admits a strict interpretation $(\!| - |\!)$ over Poly. There are a real $A > 0$ and a positive, monotonic polynomial P such that for all $x_1, \ldots, x_n \geq 0$, if $x_{i_1}, \ldots, x_{i_k} > A$, then for all symbols f, we have $(\!| f |\!)(x_1, \ldots, x_n) \geq x_{i_1} + \cdots + x_{i_k} + \dfrac{1}{P((\!| f |\!)(x_1, \ldots, x_n))}$.*

3 The role of reals in complexity

We have now all the tools to prove that reals can safely replace integers from a complexity point of view. This is illustrated by Theorem 4 and Theorem 6.

Theorem 4. *Functions computed by programs with an additive strict interpretation (over the reals) are exactly* PTIME *functions.*

The rest of the section is devoted to the proof of the Theorem. The main difficulty of the proof is that inequalities as given by the preceding section only hold for sufficiently large values. So, the main issue is to split "small" terms (and "small rewriting steps") from "large" ones. Positivstellensatz gives us the arguments for the large terms (Lemma 7), Lemmas 8,9 show that there are not too many small steps between two large steps. Lemma 11 describe how small steps and big steps alternate.

From now on, we suppose we are given a program with an additive strict interpretation over polynomials. The following Lemmas are direct applications of Proposition 2,4 and Corollary 2, they are the main steps to prove both Theorem 4 and Theorem 6. A full proof of the lemmas can be found in the technical report.

Lemma 7. *There is a polynomial P and a real $A > 0$ such that for all steps $\ell\sigma \to r\sigma$ with $(\!| r\sigma |\!) > A$, then, for all contexts C, we have $(\!| C[\ell\sigma] |\!) \geq (\!| C[r\sigma] |\!) + \dfrac{1}{P((\!| \ell\sigma |\!))}$.*

Definition 7. *Given a real $A > 0$, we say that the A-size of a closed term t is the number of subterms u of t (including itself) such that $(\!| u |\!) > A$. We note $|t|_A$ the A-size of t.*

Lemma 8. *There is a constant A such that for all $C > A$, there is a polynomial Q for which $|t|_C \leq Q((\!| t |\!))$ for all closed terms t.*

For $A > 0$, we say that $t = C[\ell\sigma] \to C[r\sigma] = u$ is an A-step whenever $(\!| r\sigma |\!) > A$. We note such a rewriting step $t \to_{>A} u$. Otherwise, it is an $\leq A$-step, and we note it $t \to_{\leq A} u$. We use the usual $*$ notation for transitive closure. In case we restrict the relation to the call by value strategy[1], we add "cbv" as a subscript. Take care that an $\to_{\leq A}$-normal form is not necessarily a normal form for $\to$.

Lemma 9. *There is a constant A such that for all $C > A$ there is a (monotonic) polynomial P such that for all terms t, any call by value derivation $t \to^*_{\leq C, cbv} u$ has length less than $P((\!| t |\!))$.*

Lemma 10. *For constructor terms, we have $(\!| t |\!) \leq \mathit{\Gamma} \times |t|$ for some constant $\mathit{\Gamma}$.*

Lemma 11. *Let us suppose we are given a program with an additive strict interpretation. There is a strategy such that for all function symbol f, for all constructor terms $t_1, \ldots, t_n$, any derivation following the strategy starting from $f(t_1, \ldots, t_n)$ has length bounded by $Q(\max(|t_1|, \ldots, |t_n|))$ where Q is a polynomial.*

Proof. Let us consider A as defined in Lemma 9, B and P_1 as defined in Lemma 7. We define $C = \max(A, B)$. Let P_0 be the polynomial thus induced from Lemma 9. Finally, let us consider the strategy as introduced above: rewrite as long as possible the according to $\to_{\leq C,\text{cbv}}$, and then, apply an C-step. That is, we have $t_1 \to^*_{\leq C,\text{cbv}} t'_1 \to_{>C,\text{cbv}} t_2 \to^*_{\leq C,\text{cbv}} t'_2 \to^*$. In Lemma 9, we have seen that there are at most $P_0((\!| t_i |\!))$ steps in the derivation $t_i \to^*_{\leq C,\text{cbv}} t'_i$. From Lemma 7, we can state that there are at most $(\!| t_1 |\!) \times P_1((\!| t_1 |\!))$ such C-steps. Consequently, the

[1] Innermost in the present context.

derivation length is bounded by $(\!|t_1|\!) \times P_1((\!|t_1|\!)) \times P_0((\!|t_1|\!))$ since for all $i \geq 1$, $(\!|t_i|\!) \leq (\!|t_1|\!)$.

Consider now a function symbol $f \in \mathcal{F}$, from Lemma 10, $(\!|f(t_1,\ldots,t_n)|\!) = (\!|f|\!)((\!|t_1|\!),\ldots,(\!|t_n|\!)) \leq (\!|f|\!)(\Gamma \max(|t_1|,\ldots,|t_n|),\ldots,\Gamma \max(|t_1|,\ldots,|t_n|))$. Then, the conclusion is immediate.

Proof. Of Theorem 4 With the strategy defined above, we have seen that the derivation length of a term $f(t_1,\ldots,t_n)$ is polynomial wrt to $\max(|t_1|,\ldots,|t_n|)$. The computation can be done in polynomial time due to dal Lago and Martini, see [7], together with the fact that the normal form has polynomial size (Lemma 10). For the converse part, we refer the reader to [4] where a proof that PTIME programs can be computed by functional programs with strict interpretations over the integers. This proof can be safely used in the present context.

3.1 Dependency Pairs with polynomial interpretation over the reals

Termination by Dependency Pairs is a general method introduced by Arts and Giesl [1]. It puts into light recursive calls. Suppose $f(t_1,\ldots,t_n) \to C[g(u_1,\ldots,u_m)]$ is a rule of the program. Then, $(F(t_1,\ldots,t_n), G(u_1,\ldots,u_m))$ is a dependency pair where F and G are new symbols associated to f and g respectively. $S(\mathcal{C},\mathcal{F},R)$ denotes the program thus obtained by adding these rules. The dependency graph links dependency pairs $(u,v) \to (u',v')$ if there is a substitution σ such that $\sigma(v) \xrightarrow{*} \sigma(u)$ and termination is obtained when there is no cycles in the graph. Since the definition of the graph involves the rewriting relation, its computation is undecidable. In practice, one gives an approximation of the graph which is bigger. Since this is not the issue here, we suppose that we have a procedure to compute this supergraph which we call the dependency graph.

Theorem 5. *[Arts,Giesl [1]] A TRS $(\mathcal{C},\mathcal{F},R)$ is terminating iff there exists a well-founded weakly monotonic quasi-ordering $\geq$, where both $\geq$ and $>$ are closed under substitution, such that*

- *$\ell \geq r$ for all rules $\ell \to r$,*
- *$s \geq t$ for all dependency pairs (s,t) on a cycle of the dependency graph and*
- *$s > t$ for at least one dependency pair on each cycle of the graph.*

It is natural to use sup interpretations for the quasi-ordering and the ordering of terms. However, the ordering $>$ is not well-founded on $\mathbf{R}$, so that system may not terminate. Here is such an example.

Example 4. Consider the non terminating system:

$$\begin{pmatrix} \mathtt{f}(\mathbf{0}) \to \mathbf{0} \\ \mathtt{f}(x) \to \mathtt{f}(\mathbf{s}(x)) \end{pmatrix}$$

Take $(\!|0|\!) = 1$, $(\!|\mathbf{s}|\!)(x) = x/2$. There is a unique dependency pair $F(x) \to F(\mathbf{s}(x))$. We define $(\!|F|\!)(x) = (\!|f|\!)(x) = x+1$.

One way to avoid these infinite descent is to force the inequalities over reals to be of the form $P(x_1, \ldots, x_n) \geq Q(x_1, \ldots, x_n) + \delta$ for some $\delta > 0$ (see for instance Lucas's work [18]). Doing so, one gets a well-founded ordering on reals. We propose an alternative approach to that problem, keeping the original ordering of $\mathbf{R}$.

Definition 8. *A* $\mathbf{R}$*-DP-interpretation for a program associates to each symbol* f *a monotonic function* $(\!| f |\!)$ *such that*

1. *constructors have additive interpretations,*
2. $(\!| \ell |\!) \geq (\!| r |\!)$ *for* $\ell \to r \in R$,
3. $(\!| s |\!) \geq (\!| r |\!)$ *for* $(s, r) \in DP(R)$,
4. *for each dependency pair* (s, t) *in a cycle,* $(\!| s |\!) > (\!| r |\!)$ *holds.*

Example 5. Let us come back to Example 3. The QBF problem can be given a $\mathbf{R}$-DP interpretation. Let us add the interpretations:

$$(\!|\mathrm{NOT}|\!)(x) = x$$
$$(\!|\mathrm{OR}|\!)(x, y) = (\!|\mathrm{EQ}|\!)(x, y) = \max(x, y)$$
$$(\!|\mathrm{IN}|\!)(x, y) = x + y$$
$$(\!|\mathrm{VERIFY}|\!)(x, y) = 2x + y + 1$$
$$(\!|\mathrm{QBF}|\!)(x) = 2x + 1$$

Theorem 6. *Functions computed by programs*

- *with additive* $\mathbf{R}$*-DP-interpretations*
- *the interpretation of any capital symbol* F *has the sub-term property*

are exactly Pspace *computable functions.*

Proof. The completeness comes from the example of the QBF, plus the compositionality of such interpretation.

In the other direction, the key argument is to prove that the call tree has a polynomial depth w.r.t. the size of arguments. The proof relies again on Lemmas 7, 8, 9, 11 adapted to dependency pairs (in cycles). Indeed, since capital symbol have the sub-term property, the lemmas are actually valid in the present context.[2] The rewriting steps of dependency pairs can be reinterpreted as depth-first traversal in the call-tree. Thus, we can state that the depth of the call tree is polynomial, as we stated in an analogous way that the derivation length was polynomial.

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A Positivstellensatz and applications

In this section, we introduce a deep mathematical result, the Positivstellensatz. Then we give some applications to polynomial interpretations. They will be key points of the Theorems 4 and 6 in our analysis of the role of reals in complexity (§B).

Let $n > 0$. Denote by $\mathbf{R}[x_1, \ldots, x_n]$ the $\mathbf{R}$-algebra of polynomials with real coefficients. Denote by $(\mathbf{R}^+)^n = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1, \ldots, x_n > 0\}$ the first quadrant. Since we need to consider only the $\mathbf{R}$-algebra of polynomial functions $(\mathbf{R}^+)^n \to \mathbf{R}$, it will be convenient to identify the two spaces. In particular throughout this section, all polynomial functions are defined on $(\mathbf{R}^+)^n$.

A.1 Preliminary results

Definition 6. *A polynomial $P \in \mathbf{R}[x_1, \ldots, x_n]$ is said* over-homogeneous of level d *if there exists $A, \alpha > 0$ such that for any $\lambda > \alpha$ and any $x \in (\mathbf{R}^+)^n$, $||x|| > A \implies P(\lambda x) \geq \lambda^d P(x)$.*

Remark. This notion is essentially independent of the chosen norm on $\mathbf{R}^n$. We say that P is (A, α)-over-homogeneous when we need to specify the bounds (A, α). In this case, the bounds depend on the choice of the norm $|| \cdot ||$ on $\mathbb{R}^n$.

Lemma 1. *Let $Q \in \mathbf{R}[x_1, \ldots, x_n]$ an over-homogeneous polynomial of level $d \geq 1$ such that there is some $r > A$ such that*

$$\inf_{||x||=r} Q(x_1, \ldots, x_n) > 0$$

For any $C > 0$, there is $D > 0$ such that $||x|| > D \implies Q(x) > C$.

Proof. Let us write $\alpha = \inf_{||x||=r} Q(x) > 0$. One observes that $Q(\lambda x) \geq \lambda^d Q(x) \geq \lambda^d \alpha > C$ for all $\lambda > (C/\alpha)^{1/d} = D$.

Lemma 2. *Let $Q \in \mathbf{R}[x_1, \ldots, x_n]$ an (A, α)-over-homogeneous polynomial of level $d \geq 1$.*

(i) If there is $x \in (\mathbf{R}^+)^n$ such that $P(x) \geq 0$ then $P(\lambda\ x) \geq 0$ for all $\lambda > \frac{\alpha A}{||x||}$; furthermore, the same statement holds with all inequalities replaced by strict inequalities.
(ii) If there is $x \in (\mathbf{R}^+)^n$ such that $P(x) < 0$ then $P(\lambda\ x) < 0$ for all $\lambda > 0$;

Proof. (i): $P(\lambda\ x) \geq \lambda^d\ P(x) \geq 0$. (ii): arguing by contradiction, suppose there is $\lambda_0 > 0$ such that $P(\lambda_0\ x) > 0$. Then by (i), $P(\lambda x) > 0$ for all $\lambda > 0$. Hence $P(x) > 0$, which is a contradiction.

Lemma 3. *Given $P \in \mathbf{R}[x_1, \ldots, x_n]$, a over-homogeneous polynomial of level $d \geq 2$ such that $\forall x \in (\mathbf{R}^+)^n, ||x|| > B \implies P(x) \geq 0$ for some constant C. Suppose that Q is over-homogeneous of level k with $1 \leq k < d$. Then, $P + Q$ is over-homogeneous of level k.*

Proof. Suppose that P is (A, α)-over-homogeneous and Q is (B, β)-over-homogeneous. Let $C = \max(A, B)$. For $||x|| > C$, $\lambda > \max(1, \alpha, \beta)$,

$$\begin{aligned} P(\lambda x) + Q(\lambda x) &\geq \lambda^d P(x) + \lambda^k Q(x) \\ &\geq \lambda^k P(x) + \lambda^k Q(x) \quad \text{since } P(x) \geq 0 \\ &= \lambda^k (P + Q)(x) \end{aligned}$$

Given a polynomial $P \in \mathbf{R}[x_1, \ldots, x_n]$, we can decompose it in homogeneous components $P = P_d + P_{d-1} + \cdots P_0$ with each P_i of degree i. Furthermore we note $P_{\geq k}$ the polynomial $\sum_{i=k}^{d} P_i$.

Lemma 4. *Let $P \in \mathbf{R}[x_1, \ldots, x_n]$ of total degree greater than 1. Suppose that there is some $C > 0$ such that for all $k \geq 0$, and all $x \in (\mathbf{R}^+)^n$ with $||x|| > C$, we have $P(x) > 0$. Then, $P_{\geq k}$ is over-homogeneous of level k for $1 \leq k \leq d$ and $P_{\geq k}(x) > 0$ with $||x|| > C'$ for some C'.*

Proof. By descending induction on $k \leq d$. For the base case, $P_{\geq d} = P_d$ is homogeneous, hence over-homogeneous, of level d. Moreover, P_d has the sign of P for $||x|| > C$ ([3]) hence P_d is nonnegative for $||x|| > C$. Since each P_i is homogeneous (and consequently over-homogeneous) of level i, the induction step is a direct consequence of Lemma 3.

Theorem 1. *Given a polynomial $P \in \mathbf{R}[x_1, \ldots, x_n]$ such that*

(i) $\forall x_1, \ldots, x_n \geq 0 : P(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n)$,
(ii) $\forall x'_i > x_i, x_1, \ldots, x_n \geq 0 : P(x_1, \ldots, x'_i, x_{i+1}, \ldots, x_n) > P(x_1, \ldots, x_n)$,

then, there exist $A \geq 0$ such that $P(x_1, \ldots, x_n) > x_1 + \cdots + x_n$ whenever $||x|| > A$.

[3] This comes from the equality $P(\lambda x) = \lambda^d P_d(x) + R(\lambda x)$ with $\deg_\lambda(R) < d$.

Proof. We propose a proof for $n = 2$. It can be extended to any n by induction. So, let us write $P(x, y) \in \mathbf{R}[x, y]$ such a polynomial having hypothesis (i) and (ii). Due to (i), we may suppose that P has degree wrt both x and y greater than 0. We make an analysis by case.

Let us suppose that P can be written $P(x, y) = P_0(y) + xP_1(y)$. Due to (ii), we have $P_0(y) = P(0, y) > y$.

First , we prove that $P_1(y) \geq 1$ for all $y \geq 0$. Ad absurdum, suppose that $P_1(y) < 1$. Then, take $x > \dfrac{P_0(y)}{1 - P_1(y)} > 0$. Due to (i), we have

$$P(x, y) - x = P_0(y) + x(P_1(y) - 1) > 0. \tag{4}$$

Since $P_1(y) - 1 < 0$ and $x > \dfrac{P_0(y)}{1 - P_1(y)} > 0$, we have

$$P_0(y) + x(P_1(y) - 1) < P_0(y) + \frac{P_0(y)}{1 - P_1(y)} \times (P_1(y) - 1) = 0$$

which contradicts 4.

Now, since $P_0(y) > y$ and $P_1(y) \geq 1$, we can state that $P(x, y) > x + y$ for all $x \geq 0, y \geq 0$.

The case $P(x, y) = Q_0(x) + yQ_1(x)$ is symmetric. So, the remaining case is when for both x and y we have components of degree strictly higher than 1. Let us write $P(x, y) = P_k(y)x^k + \cdots + P_1(y)x + P_0(y)$. Due to (i), for all y, $\lim_{x\to\infty} P(x, y) = +\infty$, we can say that $P_k(y) \geq 0$. Take y_0 such that $P_k(y_0) > 0$. Such a y_0 exists, otherwise $P_k = 0$.

Consider the polynomial $R(x) = P(x, y_0) = x^k P_k(y_0) + \cdots + P_0(y_0)$. Due to (i), $R(x) > x$. Then, we state that there is a bound A such that for all $x \geq A$, we have $R(x) > 2x$. So, we conclude that for all $\max(A, y_0) \leq y \leq x$, we have $x + y \leq 2x < R(x) \leq P(x, y)$. For the last inequality, we use (ii). Flipping x and y, we get an other bound, say B such that for all $\max(B, x_0) \leq x \leq y$ we have $x + y < P(x, y)$. We conclude taking $\max(A, B, y_0, x_0)$.

Lemma 5. *If P is positive, over-homogeneous of level $k \geq 2$ and satisfies the hypothesis (i), then for any $K > 0$, there exists $A = A_K > 0$ such that $P(x_1, \ldots, x_n) > K(x_1 + \cdots + x_n)$ whenever $||x|| > A$. In particular, the theorem holds.*

Proof of the Lemma. Let $x = (x_1, \ldots, x_n) \in (\mathbf{R}^+)^n$, let $\lambda \in \mathbf{R}^+$ and let $X = \lambda\, x$. We have

$$\begin{aligned} P(X) = P(\lambda\, x) \geq \lambda^k P(x) &\geq \lambda^k \max(x_1, \ldots, x_n) \\ &\geq \lambda^k\ \frac{x_1 + \cdots + x_n}{n} \\ &= \lambda^{k-1} \frac{X_1 + \cdots + X_n}{n}. \end{aligned}$$

Since $k \geq 2$, there exists $A \geq 0$ such that for any $\lambda > A$, $\frac{\lambda^{k-1}}{n} > K > 0$. This proves the lemma.

In particular the theorem holds if P has no homogeneous component P_1 of degree 1. We now complete the proof when $P_1(x) = a_1x_1 + \cdots + a_nx_n \neq 0$.

Lemma 6. *$P_{\geq 2} = P - P_1$ satisfies the hypothesis (i).*

Proof of the lemma. Argue by contradiction and suppose it does not. There exists then $x = (x_1, \ldots, x_n) \in (\mathbf{R}^+)^n$ such that $P_{\geq 2}(x) - x_i \leq 0$ for some fixed $1 \leq i \leq n$. Since $P_{\geq 2}(x)$ is over-homogeneous, Lemma 3 ensures that the polynomial $Q(x) = P_{\geq 2}(x) - x_i$ is over-homogeneous of level 1. By Lemma 2, $Q(\lambda x) = P_{\geq 2}(\lambda\ x) - \lambda\ x_i \leq 0$ for any $\lambda \geq 0$. Since $P_{\geq 2}$ is positive, the map $\lambda \mapsto P_{\geq 2}(\lambda x)$ is a positive polynomial of degree d. Thus

$$\lim_{\lambda \to +\infty} P_{\geq 2}(\lambda x) - \lambda x_i = +\infty,$$

which is a contradiction.

Apply Lemma 5 to $P_{\geq 2}$ with $K = 1 + \max(|a_1|, \ldots, |a_n|)$. We obtain $P_{\geq 2}(x) > K(x_1 + \cdots + x_n)$. Hence

$$P(x) = P_{\geq 2}(x) + P_1(x) > (K + a_1)x_1 + \cdots + (K + a_n)x_n > x_1 + \cdots + x_n.$$

This achieves the proof of the theorem.

Corollary 1. *Let $P(x_1, \ldots, x_n)$ be a polynomial with the hypotheses of Theorem 1. There is a constant A such that for any subset $I \subseteq \{1..n\}$, such that*

$$x_1, \ldots, x_n \geq 0,\ x_i \geq A,\ \text{for all } i \in I \implies P(x_1, \ldots, x_n) > \sum_{i \in I} x_i.$$

Proof. Use the norm $||x|| = \max(|x_1|, \ldots, |x_n|)$.

Theorem 2. *Given a polynomial $P \in \mathbf{R}[x_1, \ldots, x_n]$ such that*

(i) $\forall x_1 \geq 0, \ldots, x_n \geq 0 : P(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n)$,

(ii) $\forall x_1 \geq 0, \ldots, x_n \geq 0 : \frac{\partial P}{\partial x_i}(x_1, \ldots, x_n) > 0$ *for all* $i \leq n$,

then, there exist $A > 0$ such that for any $\Delta > 0$, we have $P(x_1, \ldots, x_i + \Delta, \ldots, x_n) > P(x_1, \ldots, x_n) + \Delta$ whenever $||x|| > A$.

Proof. For simplicity we take $n = 2$ (the reader will readily extend the argument to the general case) and we make an analysis by case. Consider the case where $P(x, y) = P_0(y) + xP_1(y)$. Let us consider $\frac{\partial P}{\partial x}(x, y) = P_1(y)$. First , we prove that $P_1(y) \geq 1$ for all $y \geq 0$. Ad absurdum, suppose that $P_1(y) < 1$. Then, take $x > \dfrac{P_0(y)}{1 - P_1(y)} > 0$. Due to (i), we have

$$P(x, y) - x = P_0(y) + x(P_1(y) - 1) > 0. \tag{5}$$

Since $P_1(y) - 1 < 0$ and $x > \dfrac{P_0(y)}{1 - P_1(y)} > 0$, we have

$$P_0(y) + x(P_1(y) - 1) < P_0(y) + \frac{P_0(y)}{1 - P_1(y)} \times (P_1(y) - 1) = 0$$

which contradicts (5).

In particular, for y large enough, $P_1(y) > 1$. The conclusion follows by the mean value inequality.

Consider now $\dfrac{\partial P}{\partial y}(x, y) = P_0'(y) + xP_1'(y)$. Suppose $P_1'(y_0) < 0$ for some $y_0 > 0$. Then

$$\lim_{x \to \infty} \frac{\partial P}{\partial y}(x, y_0) = \lim_{x \to \infty} P_0'(y_0) + xP_1(y_0) = -\infty.$$

This contradicts (ii). So, $P_1'(y) \geq 0$. Now, $P(0, y) = P_0(y) > y$ shows that $P_0'(y) - 1 \geq 0$ for y sufficiently large, say $y > A$. Then we have $\dfrac{\partial P}{\partial y}(x, y) = P_0'(y) + xP_1'(y) > 1$. We conclude as before.

The case $P(x, y) = Q_0(x, y) + yQ_1(x, y)$ is symmetric. So the remaining case is when $\deg_x(P) \geq 1$ and $\deg_y(P) \geq 1$. Thus $Q := \dfrac{\partial P}{\partial x}$ is positive and has total degree greater than 2. According to Lemma 4, $Q_{\geq 1}$ is positive over-homogeneous. Hence Lemma 1 applies to $Q_{\geq 1}$ with $C = 1 + \left|\dfrac{\partial P}{\partial x}(0, 0)\right|$. Hence there is some $D > 0$ such that for $||(x, y)|| > D$,

$$\frac{\partial P}{\partial x}(x, y) = Q_{\geq 1}(x, y) + Q(0, 0) > 1 + C + \frac{\partial P}{\partial x}(0, 0) > 1.$$

We conclude with the mean value inequality.

Corollary 2. *With the hypotheses of Theorem 2, there is a bound B such that for all $x_1, \ldots, x_n \geq 0$, if $x_i > B$, then, $P(x_1, \ldots, x_i + \Delta, \ldots, x_n) > P(x_1, \ldots, x_n) + \Delta$.*

Proof. Apply Theorem 2 with the norm $||x|| = \max(|x_1|, \ldots, |x_n|)$.

A.2 Positivstellensatz and applications

Theorem 3 (Positivstellensatz, Stengle [24]). *Suppose that we are given polynomials $P_1, \ldots, P_m \in \mathbf{R}[x_1, \ldots, x_k]$, the following two assertions are equivalent:*

1. $\{x_1, \ldots, x_k : P_1(x_1, \ldots, x_k) \geq 0 \wedge \cdots \wedge P_m(x_1, \ldots, x_k) \geq 0\} = \emptyset$
2. $\exists Q_1, \ldots, Q_m : -1 = \sum_{i \leq m} Q_i P_i$ *where each Q_i is a sum of squares of polynomials (and so is positive and monotonic).*

Moreover, these polynomials $Q_1, \ldots, Q_i$ can effectively computed. We refer the reader to the work of Lombardi, Coste and Roy [16, 6]. As a consequence, all the constructions given below can be actually (at least theoretically) computed.

It will be convenient to derive from the Positivstellensatz a proposition useful for our applications.

Proposition 1. *Let $(P_i)_{i \in I}$ and $(Q_j)_{j \in J}$ two finite families of polynomials in $\mathbf{R}[x_1, \ldots, x_n]$. Suppose that $\max_{i \in I} P_i(x) > \max_{j \in J} Q_j(x)$ for all $x \in (\mathbf{R}^+)^n$. Then there exists a polynomial $R \in \mathbf{R}[x_1, \ldots, x_n]$ such that*

$$R(x) > 0 \quad \textit{and} \quad \max_{i \in I} P_i(x) \geq \max_{j \in J} Q_j(x) + \frac{1}{R(x)}, \quad \textit{for all } x \in (\mathbf{R}^+)^n.$$

Proof. First we prove the result when J is a singleton. Suppose that $\max_{i \in I} P_i(x) > Q(x)$ for all $x \in (\mathbf{R}^+)^n$. Let $s \in I$. Define

$$D_s = \{x \in (\mathbf{R}^+)^n \ : \ P_s(x) \geq P_i(x) \text{ for all } i \neq s\}.$$

For $x \in D_s$, $P_s(x) = \max_{i \in I} P_i(x) > Q(x)$. Therefore the set

$$\left\{ \begin{array}{l} x_1 \geq 0, \ldots, x_n \geq 0, \\ P_s(x) - P_i(x) \geq 0, \text{for all} i \neq s, \\ Q(x) - P_s(x) \geq 0 \end{array} \right\}$$

is empty. The Positivstellensatz yields positive monotonic polynomials $(T_k)_{1 \leq k \leq n}$, $(U_i)_{i \neq s}$ and V_s such that

$$\sum_k U_k(x) x_k + \sum_{i \neq s} T_i(x)(P_s(x) - P_i(x)) + V_s(x)(Q(x) - P_s(x)) = -1.$$

Hence for $x \in D_s$,

$$(P_s(x) - Q(x)) V_s(x) = 1 + \sum_{i \neq s} T_i(x)(P_s(x) - P_i(x)) \geq 1.$$

Thus

$$\forall x \in D_s, \ \max_{i \in I} P_i(x) - Q(x) \geq \frac{1}{V_s(x)}.$$

Hence, setting $R(x) = \sum_{s \in I} V_s(x) > 0$, we have

$$\forall x \in (\mathbf{R}^+)^n,\ \max_{i \in I} P_i(x) - Q(x) \geq \frac{1}{R(x)}.$$

Next, we treat the general case. The previous argument yields for each $j \in J$ a positive monotonic polynomial R_j such that $\max_{i \in I} P_i(x) - Q_j(x) \geq \frac{1}{R_j(x)}$ for all $x \in (\mathbf{R}^+)^n$. Set $R(x) = \sum_{j \in J} R_j(x)$. For any $j \in J$,

$$\max_{i \in I} P_i(x) \geq Q_j(x) + \frac{1}{R(x)}.$$

Hence $\max_{i \in I} P_i(x) \geq \max_{j \in J} Q_j(x) + \frac{1}{R(x)}$.

We give a first application.

Proposition 2. *Suppose that a TRS (Σ, R) admits an interpretation $(\!|-|\!)$ over* Max-Poly *such that for all rules $\ell \to r$, we have $(\!|\ell|\!) > (\!|r|\!)$. There is a positive, monotonic polynomial P such that for any rule $\ell \to r$, we have $(\!|\ell|\!)(x_1, \ldots, x_k) - (\!|r|\!)(x_1, \ldots, x_k) \geq \dfrac{1}{P(x_1, \ldots, x_k)}$.*

Proof. For each symbol ℓ, there is a finite family of polynomials $(P_i)_{i \in I}$ such that $(\!|\ell|\!)(x_1, \ldots, x_n) = \max_{i \in I} P_i(x_1, \ldots, x_n)$. Therefore, if $(\!|\ell|\!) > (\!|r|\!)$, Proposition 1 applies: there is a positive monotonic polynomial $P_{\ell \to r}$ such that such that $(\!|\ell|\!)(x_1, \ldots, x_n) - (\!|r|\!)(x_1, \ldots, x_n) \geq \dfrac{1}{P_{\ell \to r}(x_1, \ldots, x_n)}$. Since there are only finitely many rules, we can take $P = \sum_{\ell \to r \in R} P_{\ell \to r}$.

Proposition 2 has an important consequence. Since, in a derivation all terms have an interpretation bounded by the interpretation of the first term, there is a minimal decay for each rule of the derivation.

Proposition 3. *Suppose that a TRS (Σ, R) admits a strict interpretation $(\!|-|\!)$ over* Max-Poly. *For all $A > 0$, the set of terms $\{t \in \mathcal{T}(\Sigma) \mid (\!|t|\!) < A\}$ is finite.*

Proof. For all symbols $f \in \Sigma$, we have $(\!|f|\!)(x_1, \ldots, x_n) > x_i$ for all i. By Proposition 2, there is a polynomial P such that $(\!|f|\!)(x_1, \ldots, x_n) \geq x_i + \dfrac{1}{P(x_1, \ldots, x_n)}$. Take a term $f(t_1, \ldots, t_n)$ such that $(\!|f(t_1, \ldots, t_n)|\!) < A$.

$$\begin{aligned} (\!|f(t_1, \ldots, t_n)|\!) &\geq (\!|t_i|\!) + \frac{1}{P((\!|t_1|\!), \ldots, (\!|t_n|\!))} \\ &\geq (\!|t_i|\!) + \frac{1}{P(A, \ldots, A)} \end{aligned}$$

where the second inequality is due to the sub-term property together with the monotonicity of P. By induction[4], one proves that $\mathrm{H}(t) \leq 1 + P(A, \ldots, A) \times (\!| t |\!)$ where $\mathrm{H}(t)$ denotes the height of the term t. Consequently, the height of a term t with $(\!| t |\!) < A$ is bounded by $1 + A \times P(A, \ldots, A)$. There are only finitely many such terms.

Proposition 4. *Suppose that a TRS (Σ, R) admits a strict interpretation $(\!| - |\!)$ over Poly. There is a real $A > 0$ and a positive, monotonic polynomial P such that for all $x_1, \ldots, x_n \geq 0$, if $x_{i_1}, \ldots, x_{i_k} > A$, then for all symbol f, we have $(\!| f |\!)(x_1, \ldots, x_n) \geq x_{i_1} + \cdots + x_{i_k} + \dfrac{1}{P((\!| f |\!)(x_1, \ldots, x_n))}$.*

Proof. Due to Corollary 1, there is a bound A such that for all $x_1, \ldots, x_n \geq 0$, if $x_{i_1}, \ldots, x_{i_k} > A$, then for all symbol f, we have $(\!| f |\!)(x_1, \ldots, x_n) > x_{i_1} + \cdots + x_{i_k}$. Applying Theorem 3, we get Q_f such that $(\!| f |\!)(x_1, \ldots, x_n) \geq x_{i_1} + \cdots + x_{i_k} + \dfrac{1}{Q_f(x_1, \ldots, x_n)} \geq x_{i_1} + \cdots + x_{i_k} + \dfrac{1}{Q_f((\!| f |\!)(x_1, \ldots, x_n), \ldots, (\!| f |\!)(x_1, \ldots, x_n))}$. It is then tedious but not difficult to get a uniform polynomial wrt to all symbols.

B The role of reals in complexity

We have now all the tools to prove that reals can safely replace integers from a complexity point of view. This is illustrated by Theorem 4 and Theorem 6.

Theorem 4. *Functions computed by programs with an additive interpretation (over the reals) are exactly* PTIME *functions.*

The rest of the section is devoted to the proof of the Theorem. The main difficulty of the proof is that inequalities as given by the preceding section only hold for sufficiently large values. So, the main issue is to split "small" terms (and "small rewriting steps") from "large" ones. Positivstellensatz gives us the arguments for the large terms (Lemma 7), Lemmas 8,9 show that there are not too many small steps between two large steps. Lemma 11 describe how small steps and big steps alternate.

From now on, we suppose we are given a program with an additive strict interpretation over polynomials.

Lemma 7. *There is a polynomial P and a real $A > 0$ such that for all step $\ell\sigma \to r\sigma$ with $(\!| r\sigma |\!) > A$, then, for all context C, we have $(\!| C[\ell\sigma] |\!) \geq (\!| C[r\sigma] |\!) + \dfrac{1}{P((\!| \ell\sigma |\!))}$.*

Proof. From Proposition 2, we have a polynomial P such that $(\!| \ell |\!) - (\!| r |\!) > \dfrac{1}{P}$. From Corollary 2, we have a bound A such that for all symbol f, $f(x_1, \ldots, x_i + \Delta, \ldots, x_n) \geq f(x_1, \ldots, x_n) + \Delta$ if $x_i > A$. The Lemma is then obtained by

[4] Restricted to terms such that $(\!| t |\!) < A$

induction on the context C. Suppose that $\ell = \mathtt{f}(p_1, \ldots, p_n)$ and $t_i = p_i\sigma$. For the base case, we have $(\!|\ell\sigma|\!) \geq (\!|r\sigma|\!) + \dfrac{1}{P((\!|t_1|\!), \ldots, (\!|t_n|\!))}$. By sub-term property, $(\!|t_i|\!) \leq (\!|\ell\sigma|\!)$, so that $(\!|\ell\sigma|\!) \geq (\!|r\sigma|\!) + \dfrac{1}{P((\!|\ell\sigma|\!), \ldots, (\!|\ell\sigma|\!))}$. Using A as defined above, the induction step is immediate (by the sub-term property).

Definition 7. *Given a real $A > 0$, we say that the A-size of a closed term t is the number of subterms u of t (including itself) such that $(\!|u|\!) > A$. We note $|t|_A$ the A-size of t.*

Lemma 8. *There is a constant A such that for all $C > A$, there is a polynomial Q for which $|t|_C \leq Q((\!|t|\!))$ for all closed terms t.*

Proof. Take A and P as given by Proposition 4. Consider one term t, we note $a = \dfrac{1}{P((\!|t|\!))}$. By induction on sub-terms u of t, we show that $|u|_A \leq (\!|u|\!)/a$. If $(\!|u|\!) \leq A$, then $|u|_A = 0 \leq (\!|t|\!)/a$. Otherwise, $(\!|u|\!) > A$. Let us write $u = f(u_1, \ldots, u_n)$, and say that $u_{i_1}, \ldots, u_{i_k}$ verify $(\!|u_{i_j}|\!) > A$. Then,

$$\begin{aligned}
|u|_A &= 1 + |u_{i_1}|_A + \cdots + |u_{i_k}|_A \\
&\leq (a + (\!|u_{i_1}|\!) + \cdots + (\!|u_{i_k}|\!))/a \text{ (by induction)} \\
&\leq (\frac{1}{P((\!|u|\!))} + (\!|u_{i_1}|\!) + \cdots + (\!|u_{i_k}|\!))/a \text{ (sub-term property)} \\
&\leq (\!|f(u_1, \ldots, u_n)|\!)/a \text{ (due to Proposition 4).}
\end{aligned}$$

As a consequence, $|t|_A \leq (\!|t|\!) \times P((\!|t|\!))$. The proof can be adapted for any $C > A$.

For $A > 0$, we say that $t = C[\ell\sigma] \to C[r\sigma] = u$ is an A-step whenever $(\!|r\sigma|\!) > A$. We note such a rewriting step $t \to_{>A} u$. Otherwise, it is an $\leq A$-step, and we note it $t \to_{\leq A} u$. We use the usual $*$ notation for transitive closure. In case we restrict the relation to the call by value strategy[5], we add "cbv" as a subscript. Take care that an $\to_{\leq A}$-normal form is not necessarily a normal form for $\to$.

Lemma 9. *There is a constant A and a polynomial P such that for all terms t, any call by value derivation $t \to^*_{\leq A,cbv} u$ has length less than $P((\!|t|\!))$.*

Proof. Take A and P as in the Lemma above. Let S be the finite set of terms with interpretation smaller than A. According to Proposition 3, the set S is finite. Then, we can define a to be the maximal derivation length of terms in S. Let d be the maximal arity of a term. By induction on terms, we show that the derivation length $t \to^*_{\leq A,\text{cbv}} u$ of $\leq A$-steps is bounded by $\max(|t|_A \times (d+1) \times (a+1), a)$.

[5] Innermost in the present context.

– If $(\!| t |\!) \leq A$, then, by definition of a, we have $\mathtt{dh}(t) \leq a = \max(|t|_A \times (d+1) \times (a+1), a)$.
– Otherwise, let us write $t = f(t_1, \dots, t_n)$. According to $\leq A$-call by value, consider the rewriting $f(t_1, \dots, t_n) \to^m_{\leq A,\mathrm{cbv}} f(u_1, \dots, u_n)$. Let us note $i_1, \dots, i_k$ indices for which $(\!| t_{i_j} |\!) > A$, we have by induction

$$
\begin{aligned}
m &\leq (n-k) \times a + (d+1) \times (a+1) \times \sum_{j=1}^{k} |t_{i_j}|_A \\
&\leq d \times a + (d+1) \times (a+1) \times \sum_{j=1}^{k} |t_{i_j}|_A .
\end{aligned}
$$

Consider $f(u_1, \dots, u_n)$, if $f(u_1, \dots, u_n)$ is not a $\leq A$-normal form, since the u_i are $\leq A$-normal form, a $\leq A$-step is of the form $f(u_1, \dots, u_n) \to u$ with $(\!| u |\!) \leq A$. In which case, the derivation length of u is bounded by a. To sum up, the derivation length is then bounded by $1 + a + d \times (a+1) + (d+1) \times (a+1) \times \sum_{j=1}^{k} |t_{i_j}|_A = (d+1) \times (a+1) \times |t|_A$.

Using the Lemma above, $|t|_A \leq P((\!| t |\!))$, and consequently, the derivation length is bounded by $(d+1) \times (a+1) \times P((\!| t |\!))$.

Lemma 10. *For constructor terms, we have $(\!| t |\!) \leq \Gamma \times |t|$ for some constant Γ.*

Proof. Take $\Gamma = \max\{\frac{1}{\gamma_c} \mid (\!| c |\!)(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i + \gamma_c\}$. By induction on terms.

Lemma 11. *Let us suppose we are given an additive program with interpretation in **Poly**. For a given function symbol f, there is a strategy such that for all constructor terms $t_1, \dots, t_n$, the derivation length of $f(t_1, \dots, t_n)$ is bounded by $Q(\max(|t_1|, \dots, |t_n|))$ where Q is a polynomial.*

Proof. Let us consider A and P_0 as defined in Lemma 9 and B and P_1 as defined in Lemma 7. We define $C = \max(A, B)$. Let us consider the strategy as introduced above: rewrite as long as possible the according to $\to_{\leq C,\mathrm{cbv}}$, and then, apply an C-step. That is, we have $t_1 \to^*_{\leq C,\mathrm{cbv}} t'_1 \to_{>C,\mathrm{cbv}} t_2 \to^*_{\leq C,\mathrm{cbv}} t'_2 \to^*$. In Lemma 9, we have seen that there are at most $(d+1) \times a \times P_0((\!| t_i |\!))$ steps in the derivation $t_i \to^*_{\leq C,\mathrm{cbv}} t'_i$. From Lemma 7, we can state that there are at most $(\!| t_1 |\!) \times P_1((\!| t_1 |\!))$ such C-steps. Consequently, the derivation length is bounded by $(\!| t_1 |\!) \times P_1((\!| t_1 |\!)) \times (1 + (d+1) \times a \times P_0((\!| t_1 |\!)))$ since $(\!| t_i |\!) < (\!| t_1 |\!)$.

Consider now a function symbol $f \in \mathcal{F}$, from Lemma 10, $(\!| f(t_1, \dots, t_n) |\!) = (\!| f |\!)((\!| t_1 |\!), \dots, (\!| t_n |\!)) \leq (\!| f |\!)(\Gamma \max(|t_1|, \dots, |t_n|), \dots, \Gamma \max(|t_1|, \dots, |t_n|))$. The conclusion is immediate.

Proof. Of Theorem 4 With the strategy defined above, we have seen that the derivation length of a term $f(t_1, \dots, t_n)$ is polynomial wrt to $\max(|t_1|, \dots, |t_n|)$. The computation can be done in polynomial time due to dal Lago and Martini, see [7], together with the fact that the normal form has polynomial size

(Lemma 10). For the converse part, we refer the reader to [4] where a proof that Ptime programs can be computed by functional programs with strict interpretations over the integers. This proof can be safely used in the present context.

B.1 Dependency Pairs with polynomial interpretation over the reals

Termination by Dependency Pairs is a general method introduced by Arts and Giesl [1]. It puts into light recursive calls.

Suppose $f(t_1, \ldots, t_n) \rightarrow C[g(u_1, \ldots, u_n)]$ is a rule of the program. Then, $(F(t_1, \ldots, t_n), G(u_1, \ldots, u_n))$ is a dependency pair where F and G are new symbols associated to f and g respectively. $S(\mathcal{C}, \mathcal{F}, R)$ denotes the program thus obtained by adding these rules. The dependency graph links dependency pairs $(u, v) \rightarrow (u', v')$ if there is a substitution σ such that $\sigma(v) \xrightarrow{*} \sigma(u)$ and termination is obtained when there is no cycles in the graph. Since the definition of the graph involves the rewriting relation, its computation is undecidable. In practice, one gives an approximation of the graph which is bigger. Since this is not the issue here, we suppose that we have a procedure to compute this supergraph which we call the dependency graph.

Theorem 5. *[Arts,Giesl [1]] A TRS $(\mathcal{C}, \mathcal{F}, R)$ is terminating iff there exists a well-founded weakly monotonic quasi-ordering $\geq$, where both $\geq$ and $>$ are closed under substitution, such that*

- *$\ell \geq r$ for all rules $\ell \rightarrow r$,*
- *$s \geq t$ for all dependency pairs (s, t) on a cycle of the dependency graph and*
- *$s > t$ for at least one dependency pair on each cycle of the graph.*

It is natural to use the polynomial orderings presented above for the quasi-ordering and the ordering of terms. However, the ordering $>$ is not well-founded on $\mathbf{R}$, so that system may not terminate. Here is such an example.

Example 4. Consider the non terminating system:

$$\begin{pmatrix} f(0) \rightarrow 0 \\ f(x) \rightarrow f(\mathbf{s}(x)) \end{pmatrix}$$

Take $(\!|0|\!) = 1$, $(\!|\mathbf{s}|\!)(x) = x/2$. The system has a unique dependency pair $F(x) \rightarrow F(\mathbf{s}(x))$ for which we can give the interpretation $(\!|F|\!)(x) = x+1$.[6] Take $(\!|f|\!)(x) = x$.

One way to avoid these infinite descent is to force the inequalities over reals to be of the form $P(x_1, \ldots, x_n) \geq Q(x_1, \ldots, x_n) + \delta$ for some $\delta > 0$ (see for instance Lucas's work [18]). Doing so, one gets a well-founded ordering on reals. We propose an alternative approach to that problem, keeping the original ordering of $\mathbf{R}$.

[6] The interpretation is correct since for all terms t, $(\!|t|\!) > 0$.

Definition 8. *A* **R***-DP-interpretation for a program associates to each symbol f a monotonic function $(\!| f |\!)$ such that*

1. *constructors have additive interpretations,*
2. $(\!| \ell |\!) \geq (\!| r |\!)$ *for* $\ell \to r \in R$,
3. $(\!| s |\!) \geq (\!| r |\!)$ *for* $(s, r) \in DP(R)$,
4. *for each dependency pair* (s, t) *in a cycle,* $(\!| s |\!) > (\!| r |\!)$ *holds.*

The main difference with say [21] is that we do not ask for the existence of some δ such that $(\!| s |\!) \geq (\!| r |\!) + \delta$ in the last equation.

Example 5. Let us come back to Example 3. The QBF problem can be given a **R**-DP interpretation. Let us add the interpretations:

$$\begin{aligned} (\!|\mathrm{NOT}|\!)(x) &= x \\ (\!|\mathrm{OR}|\!)(x, y) = (\!|\mathrm{EQ}|\!)(x, y) &= \max(x, y) \\ (\!|\mathrm{IN}|\!)(x, y) &= x + y \\ (\!|\mathrm{VERIFY}|\!)(x, y) &= 2 \times x + y + 1 \\ (\!|\mathrm{QBF}|\!)(x) &= 2x + 1 \end{aligned}$$

Theorem 6. *Functions computed by programs*

- *with additive* **R***-DP-interpretations*
- *the interpretation of any capital symbol F has the sub-term property*

are exactly PSPACE *computable functions.*

Proof. The completeness comes from the example of the QBF, plus the compositionality of such interpretation.

In the other direction, the key argument is to prove that the call tree has a polynomial depth w.r.t. the size of arguments. For the Theorem, let us consider two nodes of the call tree: $(f, t_1, \ldots, t_n) \rightsquigarrow (g, u_1, \ldots, u_m)$. Then, there is a context C such that $f(t_1, \ldots, t_n) \to C[g(v_1, \ldots, v_m)] \xrightarrow{*} C[g(u_1, \ldots, u_m)]$. One may note that $(\!| F(t_1, \ldots, t_n) |\!) > (\!| G(v_1, \ldots, v_m) |\!)$ for all F, G in a cycle. Let us restrict our attention to such cycles. It is then routine to reconstruct the argument for the entire call graph. We focus on nodes in the call graph with F and G in a cycle.

Lemma 7 remains valid for dependency pairs. Since $(\!| t |\!) \geq (\!| u |\!)$ whenever $t \xrightarrow{*} u$, then $(\!| G(u_1, \ldots, u_m) |\!) \leq (\!| G(v_1, \ldots, v_m) |\!)$, it is valid for two nodes $(f, t_1, \ldots, t_n) \rightsquigarrow (g, u_1, \ldots, u_m)$ in the call graph.

Lemma 8 is immediate for nodes $(f, t_1, \ldots, t_n) \rightsquigarrow (g, u_1, \ldots, u_m)$: take $A = 1$ and $Q(x) = x$.

Lemma 9 remains valid. Indeed, it depends only Proposition 3 which is valid and the fact that we use a call by value strategy.

Lemma 11 depends on the preceding lemmas, thus, it is still valid.

As a consequence, we conclude that the depth of the call tree is polynomial, as we stated in an analogous way that the derivation length was polynomial.

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The genus of regular languages

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Abstract. The article defines and studies the genus of finite state deterministic automata (FSA) and regular languages. Indeed, a FSA can be seen as a graph for which the notion of genus arises. At the same time, a FSA has a semantics via its underlying language. It is then natural to make a connection between the languages and the notion of genus. After we introduce and justify the the notion of the genus for regular languages, the following questions are addressed. First, depending on the size of the alphabet, we provide upper and lower bounds on the genus of regular languages : we show that under a relatively generic condition on the alphabet and the geometry of the automata, the genus grows at least linearly in terms of the size of the automata. Second, we show that the topological cost of the powerset determinization procedure is exponential. Third, we prove that the notion of minimization is orthogonal to the notion of genus. Fourth, we build regular languages of arbitrary large genus: the notion of genus defines a proper hierarchy of regular languages.

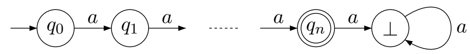
Beyond the set-theoretic description of graphs, there is the notion of an embedding of a graph in a surface. Intuitively speaking, an embedding of a graph in a surface is a drawing without edge-crossings. Planar graphs are drawn on the sphere S_0, the graphs K_5 and $K_{3,3}$ are drawn on the torus S_1 and more generally, any graph can be drawn on some closed orientable surface S_k, that is a sphere with k "handles". The genus of a graph G is the minimal index k such that G can be drawn on S_k.

The aim of this work is to explore standard notions of finite state automata (FSA) theory with this topological point of view. The novelty of this point of view lies in the fact that finite state automata are not only *graphs*, they are *machines*. These machines compute regular languages. The correspondence is onto: one language may be computed by infinitely many automata. It is then natural to define the genus of a regular language to be the minimal genus of its representing deterministic automata.

It should be noted that the word "deterministic" in the previous sentence is crucial: any regular language is recognized by some *planar* nondeterministic automaton. The earliest reference for this result we could find is [BoCh76]. The cost in terms in extra states and transitions is analyzed in [BP99]. By contrast, we show in this paper the existence of regular languages having *arbitrary high genus*.

The use of topology in the study of languages may come as a surprise at first. We suggest two motivations of very different nature. First, the question arises naturally if one wants to build physically the FSA. Think of boolean circuits, they also are graph-machines. There is an immense literature about their electronic implementation, that is about the layout of Very-Large Scale Integration (VLSI) (for instance [CKC83]). In particular, the problem of minimization of via is close to the current one. Many contributions suppose a fixed number of layers (holes), but some consider an arbitrary one [SHL90]. As we will show, a smaller number of states may not necessarily mean a smaller cost in terms of the electronic implementation.

There is a second and more fundamental reason why one should consider topology in general and the genus in particular in the study of regular languages. Low-dimensional topology is a natural tool in order to estimate the complexity of languages (or the complexity of the computation of languages). The main invariant of a regular language L is usually the number of states (the size) of the minimal automaton recognizing L. This invariant describes the size of the table data in which transitions are stored, that is the size of the machine's memory. However, simple counting costs memory without complexifying the internal structure of automata. As a simple example, the language $L_n = \{a^n\}$ is represented by an automaton of size $n + 2$ but with the simple shape of a line:



The genus, as a complexity measure, has been introduced for formal logical proofs by R. Statman [Sta74], and further studied by A. Carbone [Car09]. Cut-elimination is presented as a way of diminishing the complexity of proofs, that is of simplifying proofs. We are not aware of other use of low dimensional topology as a complexity measure besides this work. To the best of our knowledge, classical textbooks (e.g. [HMU06], [Sak09], [RS97]) about automata theory are devoted to the set-theoretic approach. Our long term objective is a topological study of the well known constructions such as minimization, determinization, union, concatenation, and so on. This paper is devoted to the notion of genus.

As a first step, we derive a closed formula for the genus of a deterministic finite automaton (Theorem 5). Then we show that under a rather mild hypotheses on the size of the alphabet (≥ 4) and on the geometry, the genus of a deterministic finite automaton at least increases linearly in terms of the number of states (Theorem 6). Since the hypotheses depend only on the abstract representing automaton and not on a particular embedding, we deduce an estimation of the genus of regular languages (Theorem 7).

Theorem 1. *Let $(L_n)_{n\geq 1}$ be a sequence of regular languages L_n of size n, with alphabet size $m \geq 4$. Assume that for any deterministic automaton recognizing L_n, the number of cycles of length 1 and 2 is negligible with respect to n. For any $\varepsilon > 0$, there is $N > 0$ such that for all $n \geq N$,*

$$1 + \left(\frac{m-3}{6m} - \varepsilon\right) mn \leq g(L_n) \leq mn.$$

We present several remarkable consequences of this result throughout this paper.

We mention two particular cases of interest. It is known that the size of the union of two automata increases linearly with the product of their respective size. We prove that the *genus* of the union of two automata `A` and `B` increases linearly with the product of the sizes of `A` and `B` (Corollary 4). We also provide an example of a nondeterministic automaton `A` such that the genus of the powerset-determinized form of a `A` is exponential up to a linear factor with respect to the size of `A` (Theorem 8).

In a second step, we study further the link between languages, their representation in terms of automata and their genus. The comparison with state minimization is instructive. Myhill-Nerode Theorem ensures that two deterministic automata with same minimal number of states that recognize the same laguage must be isomorphic. We show that this uniqueness property does not hold if we replace minimal number of states by minimal genus. There is no simple analog to Myhill-Nerode Theorem. As a consequence nonisomorphic automata representing the same language may have minimal genus. There may be even nonisomorphic automata of minimal size within the set of genus-minimal automata.

As a final step, we describe *explicit* languages having arbitrary high genus (Theorem 9). These results imply the existence of a nontrivial hierarchy of regular languages based on the genus and yields a far-reaching generalization of the results of [BoCh76, §4]. In particular, the genus yields a nontrivial measure of complexity of regular languages.

1 Finite State Automata

We briefly recall the main definitions of the theory of finite state automata and regular languages. An *alphabet* is a (finite) set of *letters*. A word on an alphabet A is a finite sequence of letters in the alphabet. Let A^* be the set of all words on A, ϵ is the empty word and the concatenation of two words w and w' is denoted by $w \cdot w'$. We define repetitions as follows. Given some word w, let $w^0 = \epsilon$ and $w^{n+1} = w^n \cdot w$.

A language on an alphabet A is a subset of A^*. Given two languages, let $L + L'$, $L \cdot L'$ and L^* denote respectively the union, the catenation and the star-operation on L (and L'). Rational languages are those languages build from finite sets and the three former operations.

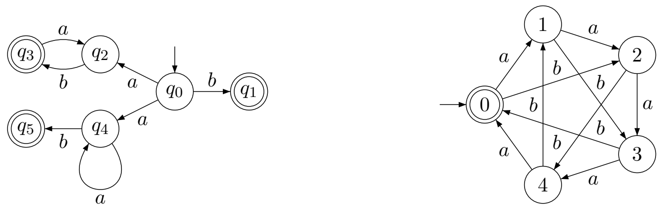
A (finite state) automaton is a 5-tuple $\mathtt{A} = \langle Q, A, q_0, F, \delta \rangle$ with Q, a finite set of *states* among which q_0 is the *initial* state, $F \subseteq Q$ is the set of *final* states, A is an alphabet and $\delta \subseteq Q \times A \times Q$ is the *transition relation*. The relation δ extends to words by setting $\delta(q, \epsilon, q)$ for all $q \in Q$ and by defining $\delta(q, a \cdot w, q')$ if and only if $\delta(q, a, q'')$ and $\delta(q'', w, q')$ for some state $q'' \in Q$. Such an automaton induces a language

$$\mathcal{L}_{\mathtt{A}} = \{ w \in A^* \mid \delta(q_0, w, q_f) \wedge q_f \in F \}.$$

The language $\mathcal{L}_{\mathtt{A}}$ is said to be recognized (or represented) by $\mathtt{A}$. A fundamental result is Kleene's theorem.

Theorem 2 (Kleene). *A language is regular if and only if it is recognized by some finite state automaton.*

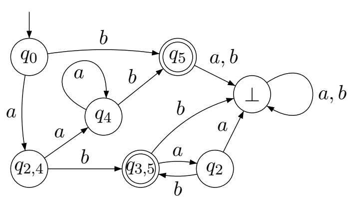
Example 1. On the alphabet $\{a, b\}$, let us define the automaton $\mathtt{F}$ on the left and $\mathtt{K}_5$ on the right:



The small arrows indicate the initial states and final states are doubly circled. The language recognized by $\mathtt{F}$ is $\mathcal{L}_{\mathtt{F}} = \{a^n \cdot b \mid n \in \mathbb{N}\} \cup \{(a \cdot b)^n \mid n > 0\} = a^* \cdot b + (a \cdot b)^*$. The language recognized by $\mathtt{K}_5$ is $\mathcal{L}_{\mathtt{K}_5}$ composed of the words of "weight" 0 modulo 5. The weight of a being 1 and the one of b being 2.

An automaton $\mathtt{A} = \langle Q, A, q_0, F, \delta\rangle$ is said to be *deterministic* (resp. *complete*) if for any state $q \in Q$ and any symbol $a \in A$, the cardinality of the set $\{q' \in Q \mid \delta(q, a, q')\}$ is at most one (resp. at least one). In the case when $\mathtt{A}$ is deterministic and complete, δ is actually a function $Q \times A \to Q$. In that case, $\delta(q, u) = q'$ stands for $\delta(q, u, q')$. It is well known that regular languages restrict to (complete) deterministic automata. All deterministic automata in this paper shall be finite and complete unless stated otherwise.

Example 2. The automaton $\mathtt{K}_5$ is deterministic, but $\mathtt{F}$ is not. Nevertheless, $\mathcal{L}_{\mathtt{F}}$ is recognized by the automaton $\mathtt{F}'$:



Note that the only function of the state symbolized by $\bot$ is to make the automaton $\mathtt{F}'$ complete. It is traditionally denoted the "trash state". Once this state is reached, the final states are unreachable.

Given a language L, a distinguishing extension of two words u and v is a word w such that $u \cdot w \in L$ and $v \cdot w \notin L$. Let R_L be the (equivalence) relation $u\ R_L\ v$ if and only if u and v have no distinguishing extension.

Theorem 3 (Myhill-Nerode [Myh57,Ner58]). *A language L is regular if and only if R_L has finitely many equivalence classes.*

Actually, the equivalence classes are the states of an automaton–called the minimal automaton– which, remarkably, is *the* smallest deterministic automaton recognizing L. By smallest, we mean the one with the minimal number of states. Thus, the notion of size of an automaton $\mathtt{A}$, denoted $|\mathtt{A}|$ in the sequel, is the number of states of $\mathtt{A}$. We emphasized the determinant 'the' in the first sentence of the paragraph to stress the fact that there is only one (up to isomorphism) automaton of minimal size representing L.

Example 3. The automaton $\mathtt{F}'$ is minimal.

The size of an automaton serves as an evaluation of its complexity (see for instance [Yu00]). Due to Myhill-Nerode, one may define the complexity of a regular language to be the size of its minimal automaton.

Example 4. There are regular languages of arbitrary large complexity. For instance, on the alphabet $A = \{a\}$, consider for all $n > 0$ the language $L_n = \{a^n\}$ that consists of all words on A of length n. The linear automaton depicted in the introduction is the minimal automaton representing L_n: it has size $n + 2$.

Proposition 1. *If two words u and v have a distinguishing extension for a regular language L, then, for any deterministic automaton $\mathtt{A}$ representing L, $\delta(q_0, u) \neq \delta(q_0, v)$.*

Proof. Ad absurdum. Suppose that $\delta(q_0, u) = q_1 = \delta(q_0, v)$. Then, $\delta(q_0, u \cdot w) = \delta(q_1, w) = \delta(q_0, v \cdot w)$. Since $u \cdot w \in L$, $\delta(q_1, w) \in F$. Thus, $v \cdot w \in L$, in contradiction with the hypothesis. ∎

2 The genus of a regular language

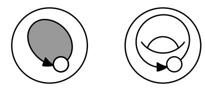
Let $\mathtt{A}$ be a finite automaton. In the constructions to follow, we regard $\mathtt{A}$ as a graph where the vertices are the states and the edges are the transitions[3]. We simply forget about the extra structures on it (namely, the orientation and the labels of the edges). We are interested in a class of embeddings of $\mathtt{A}$ into oriented surfaces. Recall that a 2-*cell* is a topological two-dimensional disc. An automaton is *planar* if it embeds into a 2-cell (or equivalently a sphere or a plane).

By means of elementary operations, one can show that $\mathtt{A}$ embeds into a closed oriented surface Σ. Among all embeddings that share that property, choose one such that the complement of the image of $\mathtt{A}$ in Σ is a

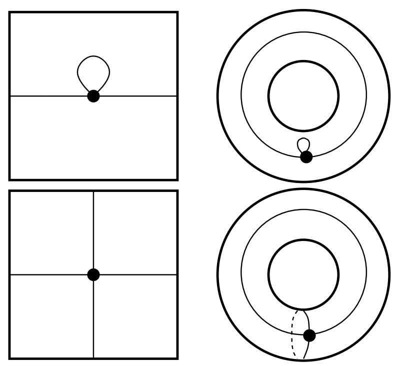
[3] In particular, two vertices may be joined by several edges.

disjoint union of a finite number of open 2-cells. Such an embedding will be called a *cellular embedding*. Again by elementary operations, one can show that there exists a cellular embedding of $\mathtt{A}$.

As a very simple example, the automaton $\mathtt{A}$ that consists of one state and one loop embeds in the obvious fashion into the 2-sphere. Note that the geometrical realization of $\mathtt{A}$ coincides with the loop. The embedding is cellular because the complement of the loop is the union of two 2-cells. The same automaton embeds also into the torus T as depicted. In this case the embedding is not cellular because $T \setminus \mathtt{A}$ is a cylinder and not a disjoint union of 2-cells.



Example 5. Another example is given by an automaton with one state and two loops. Of the two embeddings depicted into the torus T, the top one is noncellular and the bottom one is cellular. (One should identify the opposite sides of the square on the left side to obtain the embedding depicted on the right side.)



In this context, the following observation is a tautology:

Lemma 1. *A cellular embedding of an automaton* $\mathtt{A} \subset \Sigma$ *determines a finite CW-complex decomposition of the surface* Σ *in which the* 1*-skeleton* Σ^1 *of* Σ *is the image of* $\mathtt{A}$*.*

A CW-complex is a topological space made up of k-dimensional cells. Here we use 0-cells (points, corresponding to states), 1-cells (topological segments, corresponding to transitions) and 2-cells (topological discs). For the precise definition of a CW-complex decomposition, see for instance [Bre93, Chap. IV, §8]. For instance, the cellular embedding of $\mathtt{A}$ into the torus T of Example 5 induces one CW-complex decomposition of the torus consists of one 0-cell (induced by the unique state of $\mathtt{A}$), two 1-cells (induced by the two transitions of $\mathtt{A}$) and one 2-cell (thought of as the complement of $\mathtt{A}$ in T).

Recall that the *genus* of a closed oriented surface Σ is the integer $g = \frac{1}{2} \dim H_1(\Sigma; \mathbb{R})$. In our context, it is useful to note that the genus of Σ is the maximal number of disjoint cycles that can be removed from Σ such that the complement remains connected.

Definition 1. *A cellular embedding of* $\mathtt{A}$ *into* Σ *is* minimal *if the genus of* Σ *is minimal among all possible surfaces* Σ *into which* $\mathtt{A}$ *embeds cellularly.*

Example 6. The second embedding of Example 5 is cellular: the complement of $\mathtt{A}$ consists in one open 2-cell. It is not minimal. Indeed, the automaton embeds into the 2-sphere S^2: it is realized as the wedge of two circles  (whose complement in S^2 consists of three open 2-cells).

Definition 2. *The* genus $g(\mathtt{A})$ *of a finite deterministic automaton* $\mathtt{A}$ *is the genus of* Σ *where* Σ *is a closed oriented surface into which* $\mathtt{A}$ *embeds minimally.*

Example 7. The genus of the automaton that consists in one state and an arbitrary number of loops is zero because it embeds into the 2-sphere.

Let $g_{\mathtt{A}}$ be the smallest number $g_\Sigma \in \mathbb{N}$ where Σ is a closed oriented surface into which $\mathtt{A}$ can be embedded. Then $g_{\mathtt{A}} \leq g(\mathtt{A})$ (since all possible embeddings, included noncellular ones, are considered).

Theorem 4 (J.W.T. Youngs [You63]). *For any automaton* $\mathtt{A}$*,* $g_{\mathtt{A}} = g(\mathtt{A})$*. In other words, an embedding with minimal genus is cellular.*

We shall use Youngs' result throughout this paper.

Example 8. Consider the example of the graph K_5, the complete graph on five vertices. It is well known that K_5 is not planar. Embed it into the torus T as depicted in Fig. 1. Since the torus has genus 1, the embedding is minimal. One verifies that it is also cellular: the complement of K_5 in T consists of five disjoint open 2-cells.

We can now formally state the definition of the genus of a regular language.

Definition 3. *Let* L *be a regular language. The* genus $g(L)$ *of* L *is the minimal genus of a complete finite deterministic automaton recognizing* L*:*

$$g(L) = \min\{g(\mathtt{A}) \mid L = L_{\mathtt{A}},\ \mathtt{A} \text{ complete finite deterministic}\}.$$

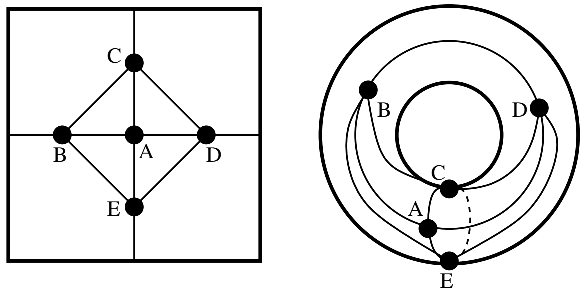


Fig. 1. A cellular embedding of the graph K_5 in the torus T.

There is a simple upper bound for the genus of a deterministic automaton.

Proposition 2. *Let* $\mathtt{A}$ *be a deterministic automaton with* m *letters and* n *states. Then*

$$g(\mathtt{A}) \leq mn.$$

Proof. This follows from Euler's formula (7). ■

Given some fixed alphabet, Prop. 2 shows that the genus of a regular language L is smaller than the size of a minimal automaton recognizing L up to some linear factor. Hence the main problem we face is to compute a lower bound for the genus (see Th. 7).

The next two results deal with the completeness of automata and reachable states. They are instrumental in nature: they say that the completion of a automaton of minimal genus and the suppression of its unreachable states do not modify the genus. These facts will be used in the sequel without further notice.

Proposition 3. *For any regular language* L *with genus* g*, there is a complete, deterministic automaton of genus* g *representing* L*.*

Proof. Let L be a regular language with genus g. Then, there is a deterministic automaton $\mathtt{A} = \langle Q, A, q_0, F, \delta \rangle$ representing L that embeds cellularly in a surface Σ of genus $g(\mathtt{A}) = g$. First, to any state q of $\mathtt{A}$ which would not be complete, add a new trash state $\bot_q$ with the transitions $\delta(q, a) = \bot_q$ for all letter a such that $\delta(q, a)$ is not defined. Second, to each of these new trash states, add loops $\delta(\bot_q, a) = \bot_q$ for all $a \in A$. Clearly, the new transitions embed into Σ and do not modify the genus of $\mathtt{A}$. ■

A state q of a (deterministic) automaton $\mathtt{A} = \langle Q, A, q_0, F, \delta \rangle$ is said to be *reachable* if there is a word w such that $\delta(q_0, w) = q$.

Proposition 4. *For any regular language L with genus g, there is a deterministic, complete automaton $\mathtt{A}$ of genus g representing L such that all states of $\mathtt{A}$ are reachable.*

Proof. Consider an automaton $\mathtt{A}$ of genus g representing L. Remove all unreachable states and the corresponding transitions from $\mathtt{A}$. The language recognized by the modified automaton is still L. Being a subgraph of $\mathtt{A}$, the new automaton has a genus smaller or equal to g. Since it represents the same language L, its genus must be equal to g. All its states are reachable. ■

2.1 Combinatorial cycles and faces

In this paragraph, we introduce cycles and faces. A cycle is a notion that depends only on the abstract graph, while a face depends on a cellular embedding of the graph. The notion of faces is crucial in the Genus Formula (Theorem 5) and instrumental in the Genus Growth Theorem (Theorem 6).

Definition 4. *Let $p \geq 1$. A* walk *in $\mathtt{A}$ is a finite alternating sequence of vertices (states) and edges $s_0, t_1, s_1, t_2 \ldots, t_p, s_p$ of $\mathtt{A}$ such that for each $j = 1, \ldots, p$, the states s_{j-1} and s_j are the endpoints of the edge t_j. The* length *of the walk is the number of edges (counting repetitions). An* internal vertex *of the walk is any vertex in the walk, distinct from the first vertex s_0 and the last vertex s_p. The walk is* closed *if the first vertex is the last vertex, $s_0 = s_p$.*

Recall that we regard $\mathtt{A}$ as an unoriented graph: one can walk along an edge opposite to the original orientation of the transition. The edge should be nonempty: there should be an actual transition in one direction or the other. In particular, if there is no transition from a state s to itself, then the vertex s cannot be repeated in the sequence defining a walk.

If the underlying graph is simple, then we suppress the notation of the edges: a walk is represented by a sequence of vertices $s_0, s_1, \ldots, s_p$ such that any two consecutive vertices are adjacent.

Definition 5. *Consider the set $W(p)$ of closed walks of length p in $\mathtt{A}$. The group of cyclic permutations of $\{1, \ldots, p\}$ acts on $W(p)$. A* combinatorial cycle *of length p, or simply a p-*cycle*, is an orbit of a closed walk of length p.*

In other words, two closed walks represent the same combinatorial cycle if there is a cyclic permutation that sends one onto the other. This definition is propped by the fact that we are interested in geometric cycles only and we do not want to count them with multiplicities with respect with the start of each node.

Remark 1. Our definition of a cycle departs from the traditional one in graph theory: repetitions of edges and internal vertices may occur. A combinatorial cycle in which no edge occurs more than one will be called a *simple* cycle. (We still allow repetition of an internal vertex that has several loops.)

We denote the set of all p-cycles in $\mathtt{A}$ by $Z_p(\mathtt{A})$. Since $\mathtt{A}$ is finite, $Z_p(\mathtt{A})$ is finite. We set $z_p = |Z_p(\mathtt{A})|$.

The definitions of walks and cycles are intrinsic to the graph: they do not depend on an embedding (or a geometric realization) of the graph. However, they are directly related to topology once an embedding is given. Let $\mathtt{A}$ be an automaton embedded in a surface Σ. Each combinatorial cycle determines a geometric 1-cycle (in the sense of singular homology) in Σ. Therefore combinatorial loops are thought of as combinatorial analogues of singular 1-cycles (in the sense of singular homology).

In what follows, consider a cellular embedding of $\mathtt{A}$ into a closed oriented surface Σ. By definition, the set $\pi_0(\Sigma - \mathtt{A})$ of connected components of $\Sigma - \mathtt{A}$ consists of a finite number of 2-cells. The image in Σ of the set $\mathtt{A}^1$ of edges of $\mathtt{A}$ is the 1-skeleton Σ^1 of Σ. With a slight abuse of notation, we shall denote by the same symbol Σ^1 the collection of embedded edges of $\mathtt{A}$. Consider an edge $e \in \Sigma^1$ and an open 2-cell $c \in \pi_0(\Sigma - \mathtt{A})$. It follows from definitions that if $\mathrm{Int}(e)$ and $\mathrm{Fr}(c)$ intersect nontrivially then $e \subset \mathrm{Fr}(c)$. Since Σ is a 2-manifold, there is at most one component c' of $\Sigma - \mathtt{A}$, $c' \neq c$, such that $e \subset \mathrm{Fr}(c)$.

Without loss of generality, we may assume that the embedded edge e is a smooth arc. Let x be a point in e. Define a small nonzero normal vector $\overrightarrow{n}$ at x. If $\overrightarrow{n}$ and $-\overrightarrow{n}$ point to distinct components c, c' of $\Sigma - \mathtt{A}$, then $e \subset \mathrm{Fr}(c) \cap \mathrm{Fr}(c')$: there are two distinct components separated by e. In this case we say that e is *bifacial*. If $\overrightarrow{n}$ and $-\overrightarrow{n}$ point to the same component c of $\Sigma - \mathtt{A}$, then c is the unique component of $\Sigma - \mathtt{A}$ such that $e \subset \mathrm{Fr}(c)$. In this case, one says that the edge e is *monofacial*.

We define a pairing $\langle -, - \rangle : \Sigma^1 \times \pi_0(\Sigma - \mathtt{A}) \to \{0, 1, 2\}$ by

$$\langle e, c\rangle = \begin{cases} 0 \text{ if } e \cap \mathrm{Fr}(c) = \varnothing \\ 1 \text{ if } e \subset \mathrm{Fr}(c) \text{ and } e \subset \mathrm{Fr}(c') \text{ for } c' \in \pi_0(\Sigma - \mathtt{A}) - \{c\} \\ 2 \text{ if } c \text{ is the unique component of } \Sigma - \mathtt{A} \text{ such that } e \subset \mathrm{Fr}(c). \end{cases}$$

From the discussion above, it follows that for any edge $e \in \Sigma^1$,

$$\sum_{c \in \pi_0(\Sigma - \mathtt{A})} \langle e, c\rangle = 2. \tag{1}$$

Definition 6. *Let $k \geq 1$. A component c of $\Sigma - \mathtt{A}$ is a k-*face *if*

$$\sum_{e \in \Sigma^1} \langle e, c\rangle = k.$$

The set of k-faces is denoted F_k. We let f_k denote the number of elements in F_k. A face *is a k-face for some $k \geq 1$. The set of faces is denoted F.*

The following properties hold:

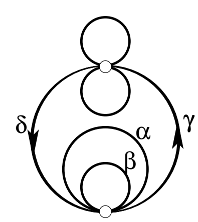
(1) The sets $F_k, k \geq 1$ are disjoint;
(2) All sets F_k but finitely many are empty;
(3) $F = \bigcup_{k \geq 1} F_k = \pi_0(\Sigma - \mathtt{A})$.

Definition 7. *Let $k \geq 1$. A* combinatorial k-gon *in Σ is a k-cycle of $\mathtt{A}$ that bounds a k-face of $\Sigma - \mathtt{A}$. A 2-gon will also be called a* bigon. *A cycle of length 1 will be called a* loop.

Lemma 2. *Any 1-gon has a bifacial edge.*

The proof follows from the more general fact that a contractible simple closed curve is separating. See §8.1 for a proof.

The automaton depicted opposite has two states; each state has three outgoing transitions. It is cellularly embedded into the plane. All edges are bifacial. The edge α is a loop contractible in Σ but is not a 1-gon; $\alpha\beta$ is a cycle of length 2 that is a bigon; $\gamma\delta$ is a cycle of length 2, contractible in Σ, that is not a bigon.



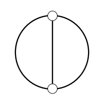
According to Lemma 2, a cycle of length 1 is monofacial if and only if it is not a 1-gon (if and only if it represents a nontrivial element in 1-homology). A bigon may have monofacial edges, even in a cellular embedding: the cellular embedding of Example 5 provides such an instance.

Remark 2. For any $k \geq 1$, $f_k \leq z_k$. Indeed, just as in homology, a combinatorial cycle does not necessarily bound a combinatorial face. For instance, in the K_5 embedded in the torus as in Example 8, the simple cycle $BCDB$ of length 3 is not a 3-gon. In fact, it does not bound any 2-cell.

Lemma 3 (1-gon lemma). *There exists a minimal embedding of* $\mathtt{A}$ *such that any cycle of length one is a* 1*-gon and in particular, is bifacial.*

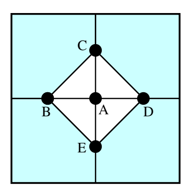
See §8 for a proof. In the sequel, we shall frequently use Lemma 3 without further notice.

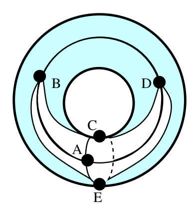
By contrast, there exists automata for which there is no embedding such that every cycle of length two is a bigon. A simple example can be constructed using the subgraph opposite.



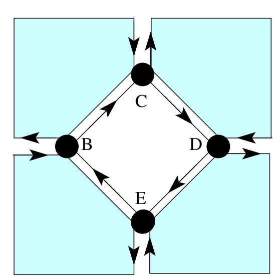
Our definition of a combinatorial cycle mimics that of a geometric 1-cycle c in the sense of a singular 1-chain such that $\partial c = 0$. We remark that in order to represent a singular 1-cycle by a combinatorial cycle, the combinatorial cycle in question may have repetitions of edges and internal vertices.

For instance, consider anew the cellular embedding of Example 8. As mentioned, the complement of K_5 in T has five components which are open 2-cells. Four of them are 3-faces bounded respectively by $ABCA$, $ACDA$, $ADEA$ and $AEBA$. The fifth component is an open 2-cell: removing the other four open 2-cells and the edges BD and CE yields a 2-cell. (Removing the four open 2-cells yields a punctured torus, which is a regular neighborhood of the wedge of a meridian and a longitude; removing the edges BD and CE amounts to cutting transversally the meridian and the longitude respectively, yielding a topological 2-cell.)





This fifth 2-cell is a bit more complicated to describe: it is not bounded by any simple cycle. It is bounded by the closed walk $BCEBDECDB$ which represents a combinatorial cycle of length 8. It therefore represents an 8-gon. Note that the monofacial edges CE and BD are travelled twice in opposite orientations.

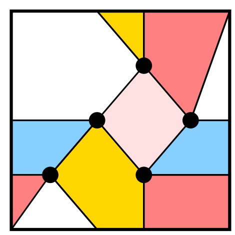


2.2 Digression: face embeddings and strong face embeddings

This paragraph is not necessary to understand our results and their proofs (and hence may be skipped on a first reading). Indeed they do not depend on the notions introduced here. In particular, they do not depend on whether the Strong Embedding Conjecture (or a related conjecture) is true or not. It is true however that for a graph that has a strong embedding in a surface of minimal genus, then the Genus Formula has a particularly simple form. (However, it is known in general that this needs not always be the case. There exist 2-connected graphs of genus 1 that have no strong cellular embedding in a torus, see [Xuo77].) We include this paragraph for clarification.

As we have seen, k-gons that appear in cellular embeddings need not be simple. We may request them to be, at the expense of a more restrictive definition.

Definition 8. *A* face embedding *of a graph* $\mathtt{A}$ *into a closed oriented surface* Σ *is a cellular embedding of* $\mathtt{A}$ *into* Σ *such that each* k*-face in* $\Sigma - \mathtt{A}$ *is bounded by a simple* k*-gon.*



Opposite is depicted another embedding of K_5 into the torus (the opposite sides in the square are identified as usual). This embedding is a face embedding of K_5: the complement of K_5 consists of five 4-faces. Hence this embedding is not equivalent to the cellular embedding of Example 8.

Note that a face embedding does not rule out the possibility that an edge be monofacial. Recall that e is bifacial if there is a component c in $\Sigma - \mathtt{A}$ such that $\langle e, c\rangle = 1$.

Definition 9. *Let* e *be an edge of embedded graph* $\mathtt{A}$ *into a closed oriented surface* Σ*. A* strong face embedding *of a graph* $\mathtt{A}$ *into a closed oriented surface* Σ *is a face embedding of* $\mathtt{A}$ *into* Σ *such that every* edge *is bifacial.*

For instance, in Example 8, all but the edges BD and CE are bifacial. The face embedding of K_5 above is strong. The second embedding of Example 5 is a face embedding that is not strong.

This definition is related to that of strong cellular embedding: a strong cellular embedding of $\mathtt{A}$ is an embedding in Σ such that the closure of each connected component $\Sigma - \mathtt{A}$ is a closed 2-cell. Equivalently, every k-face of $\Sigma - \mathtt{A}$ is bounded by a true simple cycle without repetition of an internal vertex. A strong cellular embedding is a strong face embedding. The converse does not hold in general.

The strong cellular embedding conjecture [Jae85] is that every 2-connected graph has a strong cellular embedding into some closed surface (orientable or not). Even though the question is theoretically simpler, we do not know whether every 2-connected graph has a strong face embedding into some closed surface (orientable or not).

3 Genus Formula

Our first main result is a closed formula for the genus of a regular language.

Theorem 5 (Genus formula). *Let* $\mathtt{A}$ *be a deterministic automaton with* m *letters. Then for any cellular embedding of* $\mathtt{A}$*,*

$$g(\mathtt{A}) \leq 1 - \frac{m+1}{4m} f_1 - \frac{1}{2m} f_2 + \frac{m-3}{4m} f_3 + \frac{2m-4}{4m} f_4 + \frac{3m-5}{4m} f_5 + \cdots \quad (2)$$

with equality if and only if the embedding is minimal.

The faces $f_1, f_2, \ldots$ are determined by the cellular embedding of $\mathtt{A}$. It follows from §2.1 that for each cellular embedding, there is some $M > 0$ such that $f_k = 0$ for all $k \geq M$. In particular, the sum $\sum_{k=1}^{\infty} \frac{k(m-1)-2m}{4m} f_k$ that appears on the right hand side of (2) is finite.

Remark 3. In the case when (2) is an equality, it is *not* claimed that the embedding is unique. Thus inequivalent minimal embeddings for $\mathtt{A}$ lead to distinct formulas for the genus of $\mathtt{A}$.

Remark 4. In the case when (2) is an equality, it is *not* claimed that the automaton $\mathtt{A}$ is the minimal state automaton (in the sense of Myhill-Nerode). Indeed, the automaton with the least number of states does not have necessarily minimal genus (see below §5).

The Genus Formula is proved in §9.

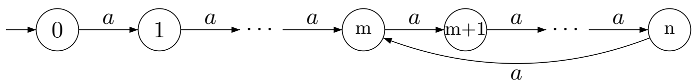
The following corollary is a straightforward consequence.

Corollary 1. *Let* L *be a regular language on* m *letters. For any minimal embedding of a deterministic automaton* $\mathtt{A}$ *representing* L*,*

$$g(L) \leq 1 + \sum_{k=1}^{\infty} \frac{k(m-1)-2m}{4m} f_k. \quad (3)$$

4 Genus growth

Let us begin with a simple example. Any language on a 1-letter alphabet is represented by a planar deterministic automaton. Indeed, these deterministic automata have a quasi-loop (planar) shape:



Actually, there is an indirect proof of the result. For a unary alphabet, $e_1 \leq e_0$. Since there is at least one face, Euler's relation states that $1 \leq 2 - 2g$, that is $g \leq 1/2$. And thus $g = 0$. The remark shows that to get graphs of higher genus, one should augment the number of edges. Then the relation $e_1 = me_0$ forces to augment the size of alphabets. Thus a general study of the genus of automata depends on the size of alphabet. Consider now the case of a 2-letter alphabet.

Proposition 5. *Let $(\mathtt{A}_n)_{n\in\mathbb{N}^\times}$ be a sequence of deterministic finite automata of size n on the same alphabet. Assume that the number m of letters is two. For any cellular embedding of $\mathtt{A}_n$,*

- *either there exists $M > 0$ such that $\sup_{n\geq 1} \big(f_1(n) + f_2(n) + f_3(n)\big) \leq M$*
- *or $\lim_{n\to+\infty} \sum_{k\geq 5} \frac{k-4}{8} f_k(n) = +\infty$.*

Proof. Suppose neither condition is satisfied. Then

$$f_1(n) \underset{n\to+\infty}{\to} +\infty \quad \text{or} \quad f_2(n) \underset{n\to+\infty}{\to} +\infty \quad \text{or} \quad f_3(n) \underset{n\to+\infty}{\to} +\infty$$

(since these are sequences of nonnegative integers) and $\sum_{k\geq 5} f_k(n)$ remains bounded. For $m = 2$, the second, third and fourth terms respectively in the genus inequality (2) are negative or zero. The fifth term is always zero for $m = 2$. It follows easily that $g(n)$ is negative for n large enough, which is a contradiction. ∎

Corollary 2. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of regular languages on two letters. If for each n, L_n is recognized by a deterministic automaton $\mathtt{A}_n$ of size n having a cellular embedding such that $\sum_{k\geq 5} \frac{k-4}{8} f_k(n)$ remains bounded as $n \to +\infty$, then the genus $g(L_n)$ of L_n also remains bounded.*

Proof. By hypothesis, the number $f_k(n)$ of k faces in a cellular embedding of $\mathtt{A}_n$ into a surface Σ_n verify $\sum_{k\geq 5} \frac{k-4}{8} f_k(n) < +\infty$. By Prop. 5, $f_1(n)$, $f_2(n)$ and $f_3(n)$ are bounded. The genus formula then shows that $g(\mathtt{A}_n)$ remains bounded. Since $g(L_n) \leq g(\mathtt{A}_n)$, the conclusion follows. ∎

Each $\mathtt{A}_n$ in general has several nonequivalent cellular embeddings. But for *any* cellular embedding, the alternative of Prop. 5 holds for the various numbers of faces $f_k(n)$ (determined by the embedding). Corollary 2 states a sufficient condition for the genus of a *language* to be bounded. The interest in this result lies in the fact that it discriminates between the respective contribution of the faces to the genus.

There is a similar result when the number of letters is three.

Proposition 6. *Let $(\mathtt{A}_n)_{n\in\mathbb{N}^\times}$ be a sequence of deterministic finite automata of size n on the same alphabet. Assume that the number m of letters is three. For any minimal embedding of $\mathtt{A}$,*

- *either there exists $M > 0$ such that $\sup_{n\geq 1} \big(f_1(n) + f_2(n)\big) \leq M$*
- *or $\lim_{n\to+\infty} \sum_{k\geq 4} \frac{k-3}{6} f_k(n) = +\infty$.*

Corollary 3. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of regular languages on two letters. If for each n, L_n is recognized by a deterministic automaton $\mathtt{A}_n$ of size n having a cellular embedding such that $\sum_{k\geq 4} \frac{k-3}{6} f_k(n)$ remains bounded as $n \to +\infty$, then the genus $g(L_n)$ of L_n also remains bounded.*

The proofs of Prop. 6 and Cor. 3 are similar to those of Proposition 5 and Corollary 2.

We state our main result on the genus growth of automata and languages.

Theorem 6 (Genus Growth). *Let $(\mathtt{A}_n)_{n\in\mathbb{N}^\times}$ be a sequence of deterministic finite automata with m letters and $n \geq 1$ states. Let $g(n)$ be the genus of $\mathtt{A}_n$. Assume*

(1) $m \geq 4$.
(2) *The numbers $z_k(n)$ of cycles of length 1 and 2 in $\mathtt{A}_n$ are negligible with respect to the size n of $\mathtt{A}_n$: $\lim_{n\to+\infty} \frac{z_1(n)}{n} = \lim_{n\to+\infty} \frac{z_2(n)}{n} = 0$.*

Then for any $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$,

$$g(n) \geq 1 + \left(\frac{m-3}{6m} - \varepsilon\right) mn.$$

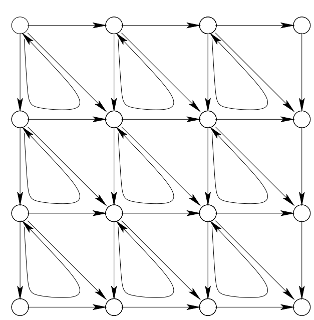
The Genus Growth Theorem is proved in §10.

We begin with examples showing that we cannot easily dispense with the hypotheses of Theorem 6.

Example 9 (Quasi-loop automaton). Any quasi-loop finite deterministic automaton with alphabet of cardinality $m \leq 1$ is planar. This of course does not contradict the Genus Growth Theorem because Hypothesis (1) does not hold.

Example 10 (Genus 1 automaton). Let $n \geq 3$. Define an automaton $\mathtt{A}_n$ as follows. Consider the set $\tilde{S}_n = \{(i,j) \mid 0 \leq i,j \leq n\}$ inside the square $C = [0,n] \times [0,n]$. The quotient T of C under the identifications $(0,t) = (1,t)$ and $(t,0) = (t,1)$ is a torus. The image S_n of $\tilde{S}_n$ in T is the set of states. Note that there are exactly n^2 states. For each $(i,j) \in (\mathbb{Z}/n\mathbb{Z})^2$, define two outgoing transitions $(i,j) \to (i+1,j) \pmod{n}$ and $(i,j) \to (i,j+1) \pmod{n}$. Choosing arbitrary initial and final states yields a finite deterministic complete automaton $\mathtt{A}_n$ with n^2 states. Clearly $g(\mathtt{A}_n) \leq 1$ for any $n \geq 3$. (This also follows from Cor. 2.) This does not contradict the Genus Growth Theorem because the alphabet has only two letters. The same example with an extra outgoing transition $(i,j) \to (i+1,j+1)$ (i.e., with an extra letter for the alphabet) still yields an automaton $\mathtt{B}_n$ with $g(\mathtt{B}_n) \leq 1$ for any $n \geq 3$. (This also follows from Cor. 3.) This still does not contradict the Genus Growth Theorem: the alphabet has only three letters.

Example 11 (Another genus 1 automaton). Start with the previous example $\mathtt{B}_n$. To each state (i,j), add an outgoing transition pointing to (i,j). This yields a deterministic complete automaton with n^2 states and an alphabet that consists now of 4 letters. There is an obvious cellular (minimal) embedding in the torus T as before. This does not contradict the Genus Growth Theorem because now the number of cycles of length one (loops) is n^2 (the number of states), so the second hypothesis is not satisfied.



A natural consequence of the Genus Growth Theorem for automata is an estimation of the genus of regular languages.

Theorem 7 (Genus Growth of Languages). *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of regular languages on m letters, $m \geq 4$. Suppose that for each n large enough, any automaton recognizing L_n has at least n states and that the number of its cycles of length 1 and 2 are negligible with respect to n. Then for any $\varepsilon > 0$, there is $N > 0$ such that for all $n \geq N$,*

$$1 + \left(\frac{m-3}{6m} - \varepsilon\right) mn \leq g(L_n) \leq mn$$

Proof. The upper bound for the genus follows from Prop. 2. The lower bound is a direct consequence of the Genus Growth Theorem. ■

In particular, under the hypothesis of Theorem 7, the genus $g(L_n)$ grows linearly in the size n of the minimal automaton $\mathtt{A}_n$ representing L_n.

We take up the question of explicitly constructing such a sequence of regular languages in §6. There we detail an explicit construction, that shows that there is a hierarchy of regular languages based on the genus (Th. 9).

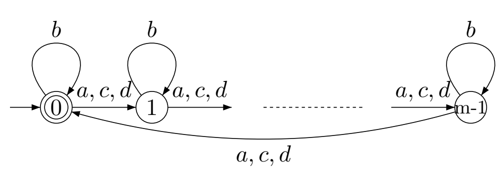
Another application of the Genus Growth Theorem is the estimation of the genus of product automata.

4.1 The genus of product automata

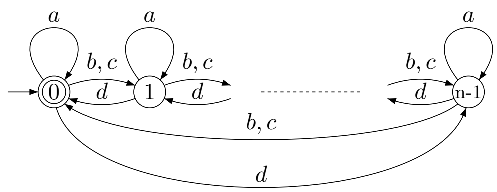
It is well know that the size of the product automaton corresponding to the union of two deterministic automata $\mathtt{A}$ and $\mathtt{B}$ is bounded by $m \times n$, the product of the size of $\mathtt{A}$ and the size of $\mathtt{B}$. This bound is actually a lower bound as presented by S. Yu in [Yu00]. By Prop. 2, up to a linear factor due to the size of the alphabet, $m \times n$ is also an upper bound on the genus of the product automaton. We prove that it is also a lower bound.

Corollary 4. *There is a family $(\mathtt{A}_m, \mathtt{B}_n)_{m\in\mathbb{N},n\in\mathbb{N}}$ of planar automata $\mathtt{A}_m$ and $\mathtt{B}_n$ of respective size m and n such that the deterministic minimal automata $\mathtt{A}_n \cup \mathtt{B}_m$ has genus $O(m \times n)$.*

Proof. Let $\mathtt{A}_m$ be the m-state automaton defined as follows.



Let $\mathtt{B}_n$ be the n-state automaton defined as follows.



The minimal automaton $\mathtt{A}_m \cup \mathtt{B}_n$ has size $m \times n$ and it contains neither loops, nor bigons. Thus, Theorem 6 applies and leads to the conclusion. ■

4.2 The exponential genus growth of determinization

We prove that determinization leads to an exponential growth, as this is the case for state-complexity (see for instance [GMRY12]). Consider the following family of automata $(\mathtt{A}_n)_{n\in\mathbb{N}^\times}$. The alphabet $\mathscr{A}_n = \{x_1, \ldots, x_n\}$ is a set of cardinality n. The states of $\mathtt{A}_n$ consist of one initial state s_0, n states (one state for each letter) $s_1, \ldots, s_n$ and one trash state. All states except the initial state and the trash state are final. The transitions of $\mathtt{A}_n$ are defined as follows:

- From the initial state s_0 to each state s_i $(1 \le i \le n)$, there are $n-1$ transitions whose labels lie in $\mathscr{A}_n - \{x_i\}$.
- From each state s_i $(1 \le i \le n)$ to itself, there are $n-1$ transitions whose labels lie in $\mathscr{A}_n - \{x_i\}$.
- From each state s_i $(1 \le i \le n)$ to the trash state, there is one transition whose label is x_i. (One can add n loops with labels $x_1, \ldots, x_n$ to the trash state so that the resulting automaton is complete.)

If follows from the definition that the language recognized by $\mathtt{A}_n$ is the set of words containing at most $n-1$ distinct letters. It is also clear from the definition that for any $n \ge 2$, $\mathtt{A}_n$ is planar and nondeterministic. (Note that the fact that we include or not the trash state with or with its loops is irrelevant.)

Theorem 8. *The determinization of $\mathtt{A}_n$ is minimal and has genus*

$$g_n \ge 1 + \left(\frac{n}{4} - 1\right) 2^{n-1}.$$

For instance, $g_4 \ge 1$ so the determinization $\mathtt{A}_4^{\det}$ of $\mathtt{A}_4$ is not planar. This can be seen by Kuratowski's theorem (as it is can be seen $\mathtt{A}_4^{\det}$ contains the utility graph $K_{3,3}$). It is not hard to embed $\mathtt{A}_3^{\det}$ into a plane so $g_3 = 0$. Of course the meaning of the theorem is that the genus of $\mathtt{A}_n^{\det}$ grows at least exponentially in n.

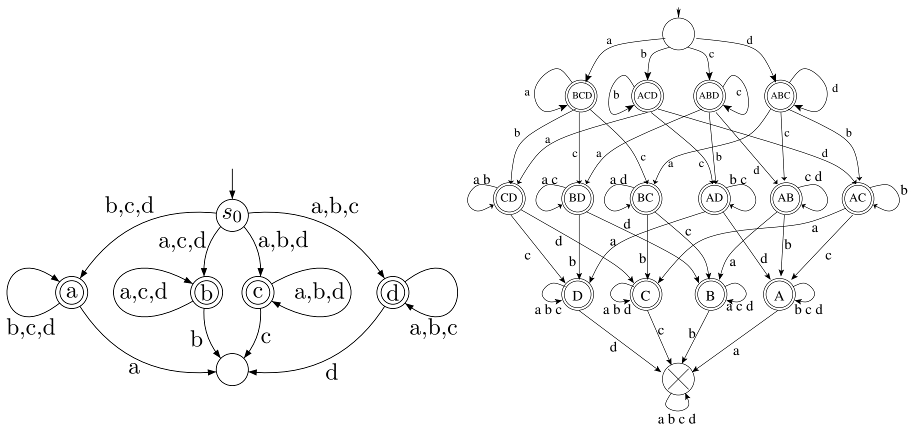


Fig. 2. The automaton $\mathtt{A}_4$ and its determinized form.

Proof. Let us describe an isomorphic variant $\mathtt{A}_n^{\mathrm{det}}$ of the determinized form of $\mathtt{A}$ by the powerset method. The states of $\mathtt{A}_n^{\mathrm{det}}$ consist of all subsets of Σ_n. The initial state of $\mathtt{A}_n^{\mathrm{det}}$ is $\mathscr{A}_n$ itself. The trash state is the empty set. Any state but the trash state and the initial state is a final state.

Therefore, the number e_0 of states of $\mathtt{A}_n^{\mathrm{det}}$ is 2^n. The transitions are described as follows. For each letter $x \in \mathscr{A}_n$, there is one transition from S to the state $S - \{x\}$. Minimality follows from definitions: there are no indistinguishable states.

Let us consider the number e_1^o of transitions of A^{det} that are loops. By definition, each state labelled by a subset of cardinality k contributes exactly $n - k$ loops. We conclude that

$$e_1^o = \sum_{k=0}^{n} \binom{n}{k} \, (n-k) = \sum_{k=0}^{n} \binom{n}{k} \, k = n \cdot 2^{n-1}. \tag{4}$$

It follows that exactly half of the transitions are loops:

$$e_1 = 2e_1^o. \tag{5}$$

Consider now a minimal embedding of $\mathtt{A}_n^{\mathrm{det}}$ into a closed oriented surface Σ. Consider one loop l in Σ^1. Since it is bifacial, it is the intersection of exactly two distinct adjacent closed 2-cells. Therefore removing the loop (while keeping the state) amounts to merging two 2-cells into one 2-cell. The union of states and transitions (minus l) still induces a CW-complex decomposition of Σ. Therefore, according to Euler's relation, the genus of Σ is unaffected. We can therefore remove all loops from Σ^1. Thus we can assume that $e_1 = e_1^o$ (from (5)) and $f_1 = 0$.

Lemma 4. *For the new graph minimally embedded in Σ, the following properties hold:*

- $f_2 = 0$*;*
- *For any* $k \geq 1$*,* $f_{2k+1} = 0$*;*
- *For any* $k \geq n$*,* $f_{2k} = 0$*.*

Proof. These observations are consequences of the particular structure of the original graph $\mathtt{A}_n^{\text{det}}$: they follow from the definition of A^{det} and are left to the reader. ■

We return to the proof of Theorem 8. We have

$$2e_1 = f_1 + 2f_2 + 3f_3 + 4f_4 \cdots = 4f_4 + 6f_6 + \cdots + (2n-2)f_{2n-2}.$$

The first equality is relation (9) and the second equality follows from Lemma 4. Since all numbers are nonnegative numbers, we have

$$2e_1 \geq 4(f_4 + f_6 + \cdots + f_{2n-2}) = 4\ e_2.$$

Thus $e_2 \leq \frac{1}{2}e_1$. From Euler's relation, we deduce that

$$2g = 2 - e_0 + e_1 - e_2 \geq 2 - e_0 + e_1 - \frac{1}{2}e_1 = 2 - e_0 + \frac{1}{2}e_1.$$

Substituting values for e_0 and e_1, we obtain

$$2g \geq 2 + \left(\frac{n}{4} - 1\right) 2^n.$$

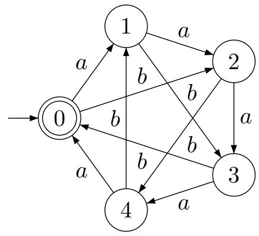
This is the desired result. ■

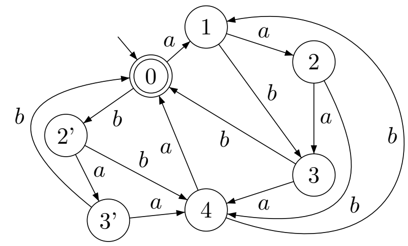
5 State-minimal automata versus genus-minimal automata

Minimal automata–as given by Myhill-Nerode Theorem–have the remarkable properties to be unique up to isomorphism, leading to a fruitful relation between rational languages and automata. In this section, we show that state-minimality is a notion orthogonal to genus-minimality. First, consider the following proposition:

Proposition 7. *There are deterministic automata with a genus strictly lower than the genus of their corresponding minimal automaton.*

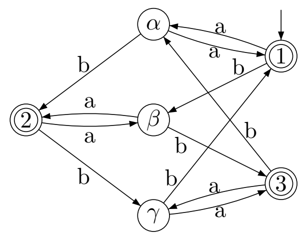
Proof. Let $\mathtt{K}_5$, $\mathtt{K}'$ be the automata:



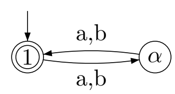


Clearly, $\mathtt{K}_5$ and $\mathtt{K}'$ represent the same language $\mathcal{L}$, $\mathtt{K}_5$ is minimal, $\mathtt{K}_5$ has genus 1 and $\mathtt{K}'$ is planar.

Example 12. Minimal automata have not necessarily the maximal genus. For instance, the following deterministic automaton

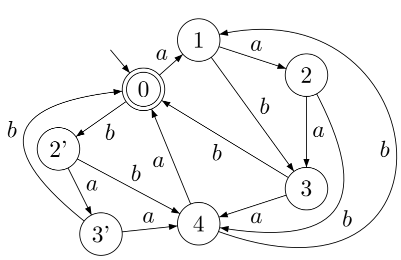


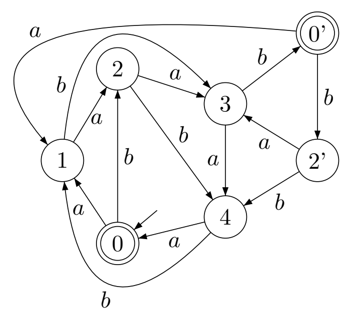
has genus 1, but its minimal corresponding automaton



has genus 0.

Contrarily to state minimization, there is no isomorphism between genus-minimal automata, even restricted to minimal state size. Indeed, the automaton $\mathtt{K}''$ below on the right represents $\mathcal{L}$, but it is not isomorphic to $\mathtt{K}'$ on the left (the vertex 4 of automaton $\mathtt{K}'$ has 6 adjacent edges, the automaton $\mathtt{K}''$ has no vertex with 6 adjacent edges):





To sum up, the two automata $\mathtt{K}'$ and $\mathtt{K}''$ (a) represent the same language, (b) have equal and minimal genus, (c) have minimal size given that genus, (d) have non isomorphic underlying graphs.

6 A hierarchy of regular languages

Is there always a planar deterministic representation of a regular language? For finite languages, the answer is positive. Indeed, finite languages are represented by trees (which are planar). In general, as evidenced by the Genus Growth Theorem for regular languages, the answer is a big "no". This section is devoted to an explicit constructive proof.

Theorem 9 (Genus-Based Hierarchy). *There are regular languages of arbitrarily large genus.*

Proof. Consider the alphabet $A = \{a, b, c, d\}$. For all $n > 0$, let U_n be the intersection of the languages $\mathcal{L}(\mathtt{A}_n)$ and $\mathcal{L}(\mathtt{B}_n)$ (see §4.1 for the definition). Consider an arbitrary deterministic automaton $\mathtt{C}_n = \langle Q, A, s, F, \delta \rangle$ representing U_n.

First, as justified by Proposition 4, we can suppose without loss of generality that any state of $\mathtt{C}_n$ is reachable.

Second, the automaton $\mathtt{C}_n$ is necessarily complete. Indeed, for any word w, there is an extension of w which belongs to the language (since the final state can be reached), so that any word must be read completely.

The automaton $\mathtt{C}_n$ has the following key properties:

(i) $\mathtt{C}_n$ has at least n^2 states,
(ii) $\mathtt{C}_n$ has no loops,
(iii) $\mathtt{C}_n$ has no bigons.

Since the size of the alphabet A is 4, by the genus formula (2) (Th. 5), we conclude from (i–iv) that $g(\mathtt{C}_n) \geq O(n^2)$ and the result follows. So, it remains to prove the four properties above.

Let us consider the function $\alpha : A \to \mathbb{N} \times \mathbb{N}$ with $\alpha(a) = (1, 0)$, $\alpha(b) = (0, 1)$, $\alpha(c) = (1, 1)$ and $\alpha(d) = (-1, 1)$. The function α extends to words in A^* by means of the equations $\alpha(\epsilon) = (0, 0)$ and $\alpha(e \cdot w) = \alpha(e) + \alpha(w) \mod (n, n)$ for $e \in A$, $w \in A^*$ and $(x, y) + (x', y') \mod (t, u) \triangleq (x + x' \mod u, y + y' \mod v)$. A crucial observation is that

$$w \in U_n \iff \alpha(w) = (0, 0). \tag{6}$$

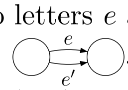
Remark that α induces a function–again denoted α–on states defined as follows. For the initial state s, let $\alpha(s) = (0, 0)$. Consider now a state

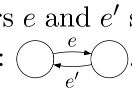
$q \neq s$. Since it is reachable, there is some word w such that $\delta(s, w) = q$. Define $\alpha(q) = \alpha(w)$. Note that the function is well-defined, that is, the definition does not depend on w. Indeed, suppose the existence of a word w' such that $\delta(s, w') = q$ and $\alpha(w') \neq \alpha(w)$. Let $\alpha(w) = (i, j)$. Consider $v = a^{n-i} \cdot b^{n-j}$. Then, $\alpha(w \cdot v) = (0, 0) \neq \alpha(w' \cdot v)$. And thus, v is a distinguishing extension of w and w'. This implies $\delta(s, w) \neq \delta(s, w')$ in contradiction with the hypothesis.

We come now to the three properties.

(i) Let $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$. Consider the word $v_{i,j} = a^i \cdot b^j$. Then, $\alpha(v_{i,j}) = (i, j)$. Thus, there are at least $n \times n$ states.

(ii) Suppose that at state q, there is a loop labelled by a letter $e \in A$. Since q is reachable, there is some word w such that $\delta(s, w) = q$. Since $\delta(s, w \cdot e) = \delta(s, w)$, we can state that $\alpha(w \cdot e) = \alpha(w)$, which itself implies $\alpha(e) = (0, 0)$. But there is no such letter e in A.

(iii) Similarly to (ii), since there are no two letters e and e' such that $\alpha(e) = \alpha(e')$, there are no bigons of the shape: 

. Since there are no two letters e and e' such that $\alpha(e) + \alpha(e') = (0, 0)$, there are no bigons of the shape: 

. Thus, there are no bigons. ■

7 Nondeterministic planar representation

The genus of a regular language L was defined in §2 as the minimal genus of a *deterministic* automaton recognizing L. In this section, we point out that that the word "deterministic" is crucial in the previous sentence. The following result is essentially proved by R.V. Book and A.K. Chandra [BoCh76, Th. 1a & 1b]. (See also [BP99].)

Theorem 10 (Planar Nondeterministic Representation). *For any regular language L, there exists a planar nondeterministic automaton* $\mathtt{A}$ *recognizing L.*

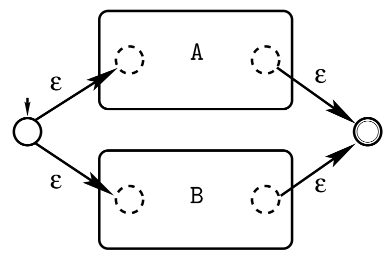
Proof. We include two proofs for the convenience of the reader. Both follow closely [BoCh76] with minor modifications. Let $L = L(R)$ be a regular language given by a regular expression R. We shall show that $L = L(\mathtt{A})$ for some planar nondeterministic automaton $\mathtt{A}$.

The proof follows the recursive definition of a regular expression. An expression that is not the empty string is regular if and only if it is constructed from a finite alphabet using the operations of union, concatenation and Kleene's $^+$-operation. Consider the class $\mathcal{C}$ of planar finite nondeterministic automata that have exactly one initial state, exactly one final state such that the initial state and the final state are distinct.

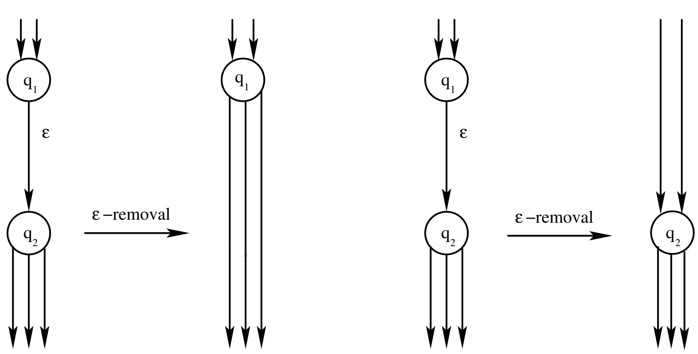
Clearly, $\mathcal{C}$ contains an automaton that recognizes the regular expressions $R = \varnothing$ (take `A` to be the automaton with two states, one initial, one final and no transition) and $R = a \in A$ (take the automaton with two states, one initial, one final and one a-labelled transition from the initial state to the final state).

Next, we show that the class $\mathcal{C}$ is closed under the three operations mentioned above. Suppose given two subexpressions R and S recognized by `A` and `B` in $\mathcal{C}$ respectively.

Consider union: first we construct an automaton `A`+`B` with ε-transitions that recognizes $R + S$.



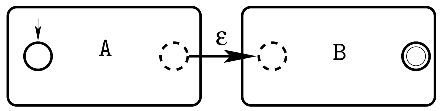
Define an ε-removal operation as follows. Consider an ε-transition that goes from state q_1 to state q_2. (We assume that $q_1 \neq q_2$.) We suppress the ε-transition and merge the two states q_1 and q_2 into one state q. Ascribe all incoming and outgoing transitions at q_1 and q_2 respectively, to the new state q. The ε-removal is best visualized by pulling the state q_2 back to the state q_1, or by pushing the state q_1 forward to the state q_2 before actually merging them[4].



[4] Note that the result of the ε-removal operation does not depend on the orientation of the transitions.

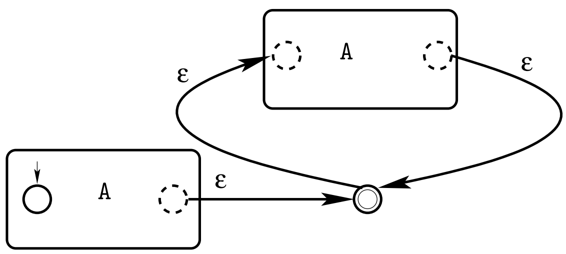
We apply this operation four times (in any order) to the automaton above. Clearly the result is an automaton that remains in $\mathcal{C}$.

Consider concatenation: the following planar automaton with one ε-transition recognizes the expression $R \cdot S$.



Next we remove the ε-transition by the ε-removal operation. This provides us with the desired automaton in $\mathcal{C}$.

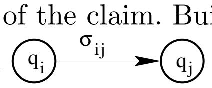
Finally consider Kleene's operation: suppose that the automaton A recognizes the expression R. The following planar automaton with three ε-transitions recognizes $R^+ = \cup_{k\geq 1} R^k$.



We remove the ε-transitions as before. This leaves us with the desired automaton in $\mathcal{C}$. This finishes the first proof.

The second proof is short but clever. Define $\mathtt{A}_n$ be the following deterministic finite automaton with set of states $[n] = \{1, \ldots, n\}$ and alphabet $A_n = \{\sigma_{ij} \mid 1 \leq i, j \leq n\}$. For $1 \leq i, j \leq n$, set a transition with symbol σ_{ij} from i to j. We take 1 to be the initial state and 2 to be the final state.

Claim 1. *The automaton $\mathtt{A}_n$ has the universal property that any nondeterministic n-state automaton $\mathtt{A} = ([n], A, 1, \delta, 2)$ can be recovered (up to equivalence) by "parallelization" of the transitions of $\mathtt{A}_n$.*

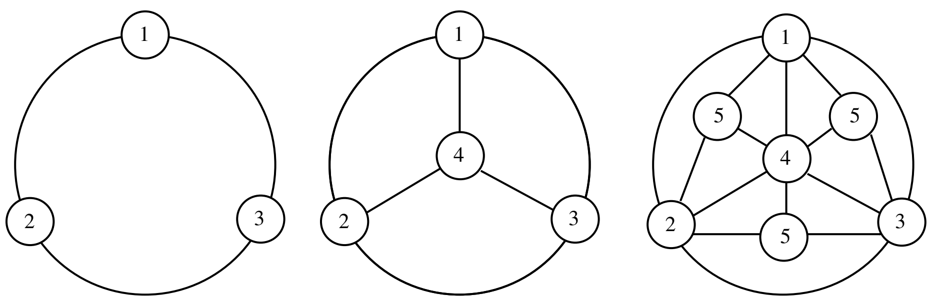
Proof of the claim. Build an n-state automaton $\mathtt{C}_n$ by replacing each transition 

in $\mathtt{A}_n$ by $T_{ij} = \{\sigma \in A \mid j \in \delta(i, \sigma)\}$. (If T_{ij} is empty, then we remove the transition σ_{ij}. Otherwise, we have $|T_{ij}|$ distinct "parallel" transitions from i to j.) The automaton $\mathtt{C}_n$ is equivalent to A. ■

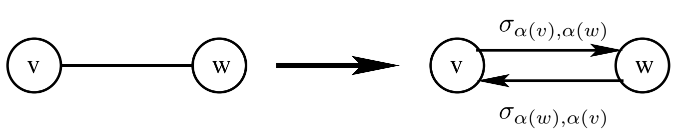
Claim 2. *If $\mathtt{A}_n$ has an equivalent planar automaton $\mathtt{B}_n$ then any nondeterministic n-state automaton $\mathtt{A} = ([n], A, 1, \delta, 2)$ has an equivalent planar automaton.*

Proof of the claim. We process the same proof as above with $\mathtt{B}_n$ instead of $\mathtt{A}_n$, observing that parallelization preserves planarity. ■

It remains to construct a planar automaton $\mathtt{B}_n$ equivalent to $\mathtt{A}_n$. The construction goes by induction on n. For $n = 3$, as a graph, $\mathtt{A}_3$ is the complete graph on three vertices, hence is planar. So we take $\mathtt{B}_3 = \mathtt{A}_3$. Suppose we have constructed a planar automaton $\mathtt{B}_n$, equivalent to $\mathtt{A}_n$, together with an embedding of $\mathtt{B}_n$ into $\mathbb{R}^2$ and a surjective map $\alpha : Q_n \to [n]$ from the set of states of $\mathtt{B}_n$ to the set of states of $\mathtt{A}_n$. We have to construct $\mathtt{B}_{n+1}$. Consider $\mathtt{B}_n \subset \mathbb{R}^2$. For any pair of distinct states q, q' of $\mathtt{B}_n$, merge the transitions from q to q' and from q' to q into one unoriented edge. (If there is no transition, we do not perform any operation.) Finally we remove loops at each state. We obtain in this fashion an undirected simple graph G_n whose vertices are exactly the states of $\mathtt{B}_n$. For each face f of $\mathbb{R}^2 - G_n$, place one vertex v inside f except for the exterior face (the unbounded component of $\mathbb{R}^2 - G_n$), and connect it to all vertices of the face f and itself. We obtain a new graph G_{n+1}. See the figure below for the recursive constructive of G_3, G_4 and G_5.



We extend α by setting $\alpha(v) = n+1$. We restore all previous (oriented) transitions between any pair of vertices, we label the new loop at v by the symbol $\sigma_{n+1,n+1}$ and we unfold each newly created edge from v to any other (old) vertex w into two transitions with opposite orientations with symbols $\sigma_{n+1,\alpha(w)}$ and $\sigma_{\alpha(w),n+1}$ respectively.



This yields a new automaton $\mathtt{B}_{n+1}$. It is clear that the recursive step does not affect the initial state and the final state of $\mathtt{B}_{n+1}$ (that were already constructed together with $\mathtt{B}_3$). The automaton $\mathtt{B}_{n+1}$ is planar since G_{n+1} is planar and the unfolding of the edges preserves planarity. It remains to see that $\mathtt{B}_{n+1}$ is equivalent to $\mathtt{A}_{n+1}$. It follows from the definition of $\mathtt{B}_n$ that for $q, q' \in Q_n$ and $\sigma_{ij} \in A_n$, there is a σ_{ij}-transition from q to q' if and only if $\alpha(q) = i$ and $\alpha(q') = j$. It follows that every word recognized by $\mathtt{B}_n$ is also recognized by $\mathtt{A}_n$. To prove the converse, one shows that for any sequence $x_1 = 1, x_2, \ldots, x_k = 2$ in $[n]$ (which is a word in the language recognized by $\mathtt{A}_n$), there is a path[5] $y_1, \ldots, y_k$ in $\mathtt{B}_n$ such that $\alpha(y_j) = j$ for each $1 \leq j \leq k$. This is proved by induction on $k \leq m$ by using the facts that it is true for $m = 3$ and that G_n contains isomorphic copies of G_{n-1}. ■

8 Proof of the 1-gon Lemma

8.1 Proof of Lemma 2

Geometrically, a bifacial embedded loop is nothing else than a separating simple closed curve with a basepoint. It suffices to prove that a contractible simple closed curve is separating. Consider an embedded loop α in Σ^1 based at $q \in \Sigma^0$. Assume that α is monofacial (nonseparating). Consider a small segment I transversal (say, normal) to α such that $I \cap \alpha = \{q\}$. Since α is monofacial, the endpoints of I lie in the same connected component of $\Sigma - \mathtt{A}$. Hence I extends to a loop β such that $\beta \cap (\Sigma - \mathtt{A}) = \beta \cap \alpha = \{q\}$. It follows that the algebraic 1-homology intersection $[\beta] \cdot [\alpha] = \pm 1$. In particular, $[\alpha] \neq 0$ in $H_1(\Sigma)$. Thus α is not contractible. ■

8.2 Proof of the 1-gon Lemma

Consider a state q of $\mathtt{A} \subset \Sigma$ that has at least one noncontractible loop. Consider a small enough open disc D in Σ centered in q such that the following properties hold: 1) $D \cap (\Sigma - \mathtt{A})$ is a disjoint union of open cells; 2) The intersection $D \cap \mathtt{A}$ is a wedge of semi-open arcs intersecting in their common endpoint q; 3) Each arc α is bifacial: there are exactly two adjacent cells $c, c' \in C = \{c_1, \ldots, c_r\}$ such that $\alpha \subset \mathrm{Fr}(c) \cap \mathrm{Fr}(c')$.

Let A be the set of arcs. The orientation of Σ induces a circular ordering $\alpha_1, c_1, \alpha_2, c_2, \ldots, \alpha_r, c_r$ of $A \cup D$ where the arcs and cells alternate and such that any two consecutive cells are adjacent.

We fix now an arc α_1 and perform successively the following operations on the arcs following the circular ordering. If the arc α_j does not belong

[5] A path is a walk such that no edge occurs more than once and no internal vertex is repeated.

to a loop (i.e. is part of a transition that is not a loop), we do not do anything. Otherwise there is another arc β belonging to the same loop. If the two arcs are enumerated consecutively in the circular ordering, we remove the whole loop inside Σ and replace it by a small 1-gon ℓ based at q such that $\ell - q$ lies entirely in the open cell c_j. At the end of the process, we have replaced all cycles of length 1 by contractible loops, hence by 1-gons. This does not change the surface hence it does not affect the genus of the embedding. In particular, if the embedding is minimal, the new embedding remains minimal (hence cellular), with the desired properties. Now by Lemma 2 each 1-gon consists of one bifacial edge. ■

9 Proof of the Genus Formula

9.1 Preliminary results

Consider a minimal embedding of an automaton $\mathtt{A}$ into a closed oriented surface Σ. We let e_0 denote the number of 0-cells (points, i.e. states), e_1 the number of 1-cells (open transitions) and e_2 the number of 2-cells (that is, the number of connected components of $\Sigma - \mathtt{A}$). The first classical result is Euler's formula ([Eul36], [Bre93, Chap. IV, §13]) that relates the genus to the CW-decomposition of Σ. In our context, since Σ is oriented and minimal, the formula takes the following form.

Lemma 5 (Euler's formula).

$$\chi(\Sigma) = 2 - 2g(\mathtt{A}) = e_0 - e_1 + e_2. \qquad (7)$$

Another useful observation is a consequence of the decomposition $\pi_0(\Sigma - \mathtt{A}) = \coprod_{k \geq 0} F_k$. Namely,

$$e_2 = f_1 + f_2 + f_3 + \ldots = \sum_{k \geq 0} f_k. \qquad (8)$$

The sum above is finite since the total number of 2-cells is finite. In particular, there is a maximal index $k \geq 0$ such that $f_k > 0$ and $f_l = 0$ for all $l > k$.

We need one more result that relates the number of 1-cells to the number of faces.

Lemma 6.

$$2e_1 = f_1 + 2f_2 + 3f_3 + \ldots = \sum_{k \geq 0} k \; f_k. \qquad (9)$$

Proof. We begin with the relation (1): $\sum_{c\in\pi_0(\Sigma-\mathtt{A})}\langle e,c\rangle = 2$. It follows that

$$\sum_{e\in\Sigma^1}\sum_{c\in\pi_0(\Sigma-\mathtt{A})}\langle e,c\rangle = 2|\Sigma^1| = 2e_1.$$

Now use the decomposition of the cells into k-faces: $\pi_0(\Sigma-\mathtt{A}) = \coprod_{k\geq 0} F_k$.

$$\begin{aligned}\sum_{e\in\Sigma^1}\sum_{c\in\pi_0(\Sigma-\mathtt{A})}\langle e,c\rangle = \sum_{c\in\pi_0(\Sigma-\mathtt{A})}\sum_{e\in\Sigma^1}\langle e,c\rangle &= \sum_{k\geq 0}\sum_{c\in F_k}\sum_{e\in\Sigma^1}\langle e,c\rangle\\ &= \sum_{k\geq 0}\sum_{c\in F_k} k\\ &= \sum_{k\geq 0} k\ f_k,\end{aligned}$$

where we used the relation $\sum_{e\in\Sigma^1}\langle e,c\rangle = k$ for a k-face c. This completes the proof. ∎

9.2 Proof of Theorem 5 (Genus formula)

Consider a cellular embedding of A into a closed oriented surface Σ. Euler's formula (7) for the genus of Σ gives $g_\Sigma = 1 - \frac{e_0-e_1+e_2}{2}$. Since the automaton is complete, each state has exactly m outgoing transitions. Therefore $e_0 = e_1/m$. Next use the relations (9) and (8) to express e_1 and e_2 in terms of the k-faces. This yields the formula

$$g_\Sigma = 1 + \sum_{k=1}^{+\infty}\frac{k(m-1)-2m}{4m} f_k.$$

Now $g(\mathtt{A}) \leq g_\Sigma$ with equality if and only if the embedding into Σ is minimal. This achieves the proof. ∎

10 Proof of the Genus Growth Theorem

It is convenient to introduce the following functions:

$$A(n) = \sum_{k\geq 3}\frac{k(m-1)-2m}{4m} f_k(n) \text{ and } B(n) = \sum_{k\geq 3} k\ f_k(n).$$

We begin with

Lemma 7. *There is a constant $\alpha > 0$ such that*

$$A(n) \geq \alpha B(n). \tag{10}$$

Proof. To prove the claim, we first find $\alpha > 0$ such that

$$\frac{(m-1)k - 2m}{4m} \geq \alpha\ k \quad \text{for all } k \geq 3.$$

It suffices, therefore, to choose α such that

$$\frac{m-1}{4m} - \frac{1}{2k} \geq \alpha \quad \text{for all } k \geq 3.$$

This condition is satisfied if we choose

$$\inf_{k \geq 3} \left(\frac{m-1}{4m} - \frac{1}{2k} \right) = \frac{m-3}{12m} = \alpha_0 \geq \alpha.$$

(Note that $\alpha_0 > 0$ for $m \geq 4$.) This proves the lemma. ∎

Lemma 8.

$$\lim_{n \to +\infty} \frac{f_j(n)}{B(n)} = 0 \quad \textit{for } j = 1, 2. \tag{11}$$

Proof. Since $f_k \leq z_k$, we have $\frac{f_k}{n} \leq \frac{z_k}{n} \underset{n \to +\infty}{\to} 0$ for $k = 1, 2$. Thus for any positive constants a, b,

$$\frac{n}{af_1 + bf_2} \underset{n \to +\infty}{\to} +\infty. \tag{12}$$

Observe that $2\ mn = e_1(n) = f_1(n) + 2f_2(n) + B(n)$. Hence

$$n = \frac{1}{2m}(f_1(n) + 2f_2(n) + B(n)).$$

Replacing n in (12) by this expression, with $a = 1/(2m)$ and $b = 1/m$, we find that

$$\frac{\frac{1}{2m}B(n)}{\frac{1}{2m}f_1(n) + \frac{1}{m}f_2(m)} = \frac{B(n)}{f_1(n) + 2f_2(n)} \underset{n \to +\infty}{\to} +\infty.$$

Then

$$\max\left(\frac{B(n)}{f_1(n)}, \frac{B(n)}{f_2(n)} \right) \geq \frac{B(n)}{f_1(n) + 2f_2(n)} \underset{n \to +\infty}{\to} \infty,$$

as desired. ∎

Let us come to the proof of the Genus Growth Theorem. Let $\alpha > \varepsilon > 0$ satisfying the condition of Lemma 7. Lemma 8 ensures there is $N > 0$ such that for any $n \geq N$,

$$\left(\alpha + \frac{m+1}{4m} \right) f_1(n) + \left(2\alpha + \frac{1}{2m} \right) f_2(n) \leq \varepsilon\ B(n).$$

Hence for $n \geq N$,

$$A(n) \geq (\alpha - \varepsilon)B(n) + \left(\alpha + \frac{m+1}{4m}\right) f_1(n) + \left(2\alpha + \frac{1}{2m}\right) f_2(n).$$

Thus

$$\begin{aligned} A(n) - \frac{m+1}{4m} f_1(n) - \frac{1}{2m} f_2(n) &\geq (\alpha - \varepsilon)(B(n) + f_1(n) + 2f_2(n)) \\ &= 2(\alpha - \varepsilon)e_1(n) \\ &= 2(\alpha - \varepsilon)mn. \end{aligned}$$

According to Theorem 5, $A(n) - \frac{m+1}{4m} f_1(n) - \frac{1}{2m} f_2(n) = g(n) - 1$. Thus

$$g(n) \geq 1 + 2(\alpha - \varepsilon)m\ n.$$

This achieves the proof of the theorem. ■

11 Conclusion

The topological tool we employ here, the genus as a complexity measure of the language, leads to a viewpoint that seems orthogonal to the standard one: it is not compatible with set-theoretic minimization (that is, state minimization). However, the genus does behave similarly to the state complexity with respect to operations such as determinization and union (up to a linear factor); furthermore, there is a hierarchy of regular languages based on the genus. This suggests a more systematic study of all operations : e.g. concatenation, star-operation, and composition of those. We take up this task in a sequel to this paper [BD13].

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