Topological Quantum Field Theory, Reciprocity and the Weil representation First version v1 – Jan 2006 This version v20 – Dec 2018

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Part 1

Algebra

CHAPTER 1

Linking pairings and finite quadratic functions

1. ε -symmetric bilinear pairings

Let S, T and U be three sets. A pairing between S and T with values in Uis a map $p: S \times T \to U$. The left and right adjoint map associated to p are respectively the maps $S \to U^T$, $s \mapsto p(s, -)$ and $T \to U^S$, $t \mapsto p(-, t)$. A pairing p is left (resp. right) nondegenerate if the left (resp. right) adjoint map is injective. A pairing is left (resp. right) nonsingular if its left (resp. right) adjoint map is bijective. A pairing $p: S \times S \to U$ is said symmetric if p(x, y) = p(y, x) for all $x, y \in S$. Assume that U is an abelian group. The pairing $p: S \times S \to U$ is said antisymmetric (resp. symplectic) if p(x, y) = -p(y, x) for all $x, y \in S$ (resp. if p(x, x) = 0 for all $x \in S$). Let $\varepsilon \in \{\pm 1\}$. A pairing is an ε -symmetric pairing if it is either symmetric $(\varepsilon = 1)$ or antisymmetric ($\varepsilon = -1$).

LEMMA 1.1. A symplectic pairing $p: S \times S \rightarrow U$ is antisymmetric.

PROOF. 0 = p(x + y, x + y) = p(x, x) + p(x, y) + p(y, x) + p(y, y) = p(x, y) + p(y, x).

Conversely if $p: S \times S \to U$ is antisymmetric, then 2p(x, x) = 0 for all $x \in S$. In particular, if U has no 2-torsion, then p is symplectic.

Clearly a ε -symmetric pairing is nondegenerate (resp. nonsingular) if and only if one of its adjoint maps is injective (resp. bijective). If p is symmetric then the left adjoint map coincides with the right adjoint map and we denote it $\hat{p}: S \to U^S$. If p is antisymmetric then the left adjoint map, still denoted \hat{p} , is the opposite to the right adjoint map.

Let $p: S \times S \to U$ be an ε -pairing. The orthogonal V^{\perp} of a subset $V \subseteq S$ is defined as the set

$$V^{\perp} = \{ s \in S \mid p(s, v) = 0 \text{ for all } v \in V \}.$$

For any subset $V, V \subseteq (V^{\perp})^{\perp}$. If $V \subseteq W \subset S$ then $W^{\perp} \subseteq V^{\perp}$. Two subsets V, W of S are *orthogonal* if p(v, w) = 0 for all $v \in V$ and $w \in W$. Equivalently $V \subseteq W^{\perp}$.

Suppose that S, T and U are abelian groups. A pairing $p: S \times T \to U$ is bilinear if p(s+s',t) = p(s,t) + p(s',t) and p(s,t+t') = p(s,t) + p(s,t') for all $s, s' \in S$ and $t, t' \in T$. It follows that the left (resp. right) adjoint map is a homomorphism $S \to \text{Hom}(T, U)$ (resp. $T \to \text{Hom}(S, U)$).

Two bilinear pairings $p: S \times S \to U$ and $q: T \times T \to V$ are *isomorphic* if there exists an isomorphism $\varphi: S \to T$ such that $q(\varphi(s), \varphi(s')) = p(s, s')$ for all $s, s' \in S$. We write: $\varphi^*q = p$.

If $p: S \times S \to U$ is an ε -symmetric bilinear pairing, then V^{\perp} is a subgroup of S for any subset $V \subseteq S$. It is also the orthogonal of the subgroup generated by V. For any subgroups V, W of S,

(1.1)
$$V^{\perp} \cap W^{\perp} = (V+W)^{\perp}.$$

Since $S^{\perp} = \text{Ker } \hat{p}$, an ε -symmetric bilinear pairing $p : S \times S \to U$ is nondegenerate if and only if $S^{\perp} = 0$. A subgroup V of S is said *isotropic* if $V \subseteq V^{\perp}$. A subgroup V of S is a *Lagrangian* if $V = V^{\perp}$. Any isotropic subgroup V induces a quotient ε -symmetric bilinear pairing $\bar{p} : V^{\perp}/V \times V^{\perp}/V \to U$ by $\bar{p}(s + V, t + V) = p(s, t), s \in V^{\perp}$.

LEMMA 1.2. Let V be a Lagrangian in S and let W be an isotropic subgroup in S. The following statements are equivalent:

- (1) W is a Lagrangian and $V \oplus W = S$;
- (2) $S = V \oplus W = V \oplus W^{\perp}$.

PROOF. (1) \implies (2) is clear. Conversely, we want to show that $W = W^{\perp}$. Let $x \in W^{\perp}$. There is a unique decomposition x = v + w with $v \in V$ and $w \in W$. Since $x \in W^{\perp}$, for any $y \in W$,

$$0 = p(x, y) = p(v + w, y) = p(v, y) + p(w, y) = p(v, y).$$

Thus $v \in W^{\perp}$. But v was in V so $v \in V \cap W^{\perp} = 0$. Hence $x = 0 + w = w \in W$.

The opposite of a bilinear pairing $p: S \times T \to U$ is the bilinear pairing $-p: S \times T \to U$ defined by (-p)(s,t) = -p(s,t).

Let $p: S \times S \to U$ and $p': S' \times S' \to U$ be two bilinear pairings, both symmetric (resp. both antisymmetric). The *orthogonal sum* of p and p' is the symmetric (resp. antisymmetric) bilinear pairing $p \oplus p': (S \oplus S') \times (S \oplus S') \to U$ defined by

$$(p \oplus p')(x + x', y + y') = p(x, y) + p'(x', y'), \quad x, y \in S, \ x', y' \in S'.$$

Clearly $S = S \oplus 0$ and $S' = 0 \oplus S$ are mutually orthogonal in $S \oplus S'$, i.e. $S^{\perp} = S'$ and $S'^{\perp} = S$. If the pairings on S and S' are implicitly understood, then we denote the orthogonal sum of the pairings (S, p) and (S', p') by $S \oplus S'$.

Conversely if $p'': S'' \times S'' \to U$ is an ε -symmetric pairing such that there exist subgroups S and S' such that $S^{\perp} = S'$ and $S'^{\perp} = S$, then p'' splits as an orthogonal sum

$$p'' = p|_{S \times S} \oplus p|_{S' \times S'}.$$

LEMMA 1.3. Let $p: S \times S \to U$ be a nonsingular ε -symmetric pairing. Let V be a subgroup of S. The following statements are equivalent:

- (1) $p|_{V \times V} : V \times V \rightarrow U$ is nonsingular;
- (2) $S = V \oplus V^{\perp}$ and $p|_{V^{\perp} \times V^{\perp}} : V^{\perp} \times V^{\perp} \to U$ is nonsingular.

A subgroup V satisfying one of the properties stated in Lemma 1.3 is an *orthogonal summand* of S.

PROOF. See [97, Lemma (1)].

Let $p: S \times S \to U$ and $q: T \times T \to V$ be two bilinear pairings. The respective adjoint maps $\hat{p}: S \to \text{Hom}(S, U)$ and $\hat{q}: T \to \text{Hom}(T, V)$ induce a homomorphism

(1.2)
$$S \otimes T \xrightarrow{\hat{p} \otimes \hat{q}} \operatorname{Hom}(S, U) \otimes \operatorname{Hom}(T, V) \xrightarrow{\otimes} \operatorname{Hom}(S \otimes T, U \otimes V).$$

DEFINITION 1.1. The *tensor product* of p and q is the bilinear pairing

$$p \otimes q : (S \otimes T) \times (S \otimes T) \to U \otimes V$$

whose left adjoint map is the homomorphism above.

Alternatively, the tensor product of p and q can be regarded as the bilinear pairing induced by the multilinear map $p \times q : (S \times T) \times (S \times T) \rightarrow U \otimes V$ defined by $(p \times q)(s,t;s',t') = p(s,s') \otimes q(t,t'), s,s' \in S, t,t' \in T$.

If p and q are both symmetric or both antisymmetric, then $p \otimes q$ is symmetric. ric. If p is symmetric (resp. antisymmetric) and q is antisymmetric (resp. symmetric), then $p \otimes q$ is antisymmetric.

The tensor product of two bilinear pairings take value in the tensor product $U \otimes V$ of the groups where the respective pairings take their values. Here are two examples.

EXAMPLE 1.1. The tensor product of an antisymmetric bilinear pairing $p: S \times S \to \mathbb{Z}$ on a free abelian group S and a symmetric bilinear pairing $q: T \times T \to \mathbb{Q}/\mathbb{Z}$ on a torsion group T is an antisymmetric bilinear pairing $p \otimes q: (S \otimes T) \times (S \otimes T) \to \mathbb{Q}/\mathbb{Z}$. The tensor product is induced by pointwise product $\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$:

$$(p \otimes q)(x \otimes y, x' \otimes y') = p(x, x') \cdot q(y, y').$$

EXAMPLE 1.2. Let r, s be positive integers and let t be their greatest common divisor. There is a canonical isomorphism

 $\mathbb{Z}/r\mathbb{Z} \otimes \mathbb{Z}/s\mathbb{Z} \simeq \mathbb{Z}/t\mathbb{Z}, \ (1 \mod r, \ 1 \mod s) \mapsto 1 \mod t.$

The tensor product of two symmetric bilinear pairings $p: S \times S \to \mathbb{Z}/r\mathbb{Z}$ and $q: T \times T \to \mathbb{Z}/s\mathbb{Z}$ on torsion groups S and T respectively is a symmetric bilinear pairing $p \otimes q: (S \otimes T) \times (S \otimes T) \to \mathbb{Z}/t\mathbb{Z}$. In the particular case r = s, the tensor product is induced by pointwise product.

REMARK 1.1. It is sometimes convenient to simplify the notation and write S for an ε -symmetric bilinear pairing $\lambda : S \times S \to U$ when the underlying pairing λ is implicitly understood. In this case, we write -S for the opposite pairing, $S \oplus T$ for orthogonal sum, etc.

2. ε -linking pairings on finite abelian groups

Let G be a finite abelian group. The dual group G^* of G is $\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Let $\varepsilon \in \{\pm 1\}$.

DEFINITION 1.2. An ε -linking pairing is an ε -symmetric bilinear pairing on a finite abelian group. A finite abelian together with a ε -linking pairing on it will be called a ε -linking group.

REMARK 1.2. The ε shall be dropped when the context is clear.

The definition forces the value group to be a finite subgroup of \mathbb{Q}/\mathbb{Z} . So a linking (resp. finite symplectic) pairing can be defined as a symmetric (resp. symplectic) bilinear pairing $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$. Alternatively, λ can be defined via its left adjoint map as a homomorphism $\hat{\lambda} : G \to G^*$.

It is sometimes convenient to take a smaller subgroup of values rather than the whole group \mathbb{Q}/\mathbb{Z} . For any integer *n*, the cyclic group $\mathbb{Z}/n\mathbb{Z}$ canonically embeds in \mathbb{Q}/\mathbb{Z} by the map

$$j_n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \ (1 \mod n) \mapsto \frac{1}{n} \mod 1.$$

For a finite group G, let $e_G \in \mathbb{N}^{\times}$ be the *period* of G, that is the smallest positive integer n such that $n \ x = 0$ for all $x \in G$.

LEMMA 2.1. Any ε -linking pairing $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ factors through an ε -linking pairing $\lambda' : G \times G \to \mathbb{Z}/e_G\mathbb{Z}$:



LEMMA 2.2. An ε -linking pairing is nonsingular if and only if it is nondegenerate.

PROOF. It suffices to see that nondegenerate implies nonsingular. The adjoint map $G \to G^*$ is injective. Since G is finite, the dual group G^* is also finite with $|G^*| = |G|$. Hence the adjoint map is bijective.

LEMMA 2.3. Let $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate ε -linking pairing. For any subgroup H of G,

(2.1)
$$|G| = |H| \cdot |H^{\perp}|$$
 and $(H^{\perp})^{\perp} = H$.

PROOF. There is a short exact sequence

$$0 \longrightarrow H^{\perp} \longrightarrow G \xrightarrow{\hat{\lambda}|_{H}} H^* \longrightarrow 0,$$

where by definition $\hat{\lambda}|_{H}(h) = \lambda(h, -) \in H^*$ for all $h \in H$. Hence $G/H^{\perp} \simeq H^*$. Therefore

$$|G| = |H| \cdot |H^{\perp}|.$$

This equality is true for any subgroup H of G. Applying this equality to the subgroup H^{\perp} , we obtain $|G| = |H^{\perp}| \cdot |(H^{\perp})^{\perp}|$. It follows that $|(H^{\perp})^{\perp}| = |H|$. Since $H \subseteq (H^{\perp})^{\perp}$, the equality follows.

We define a tensor product for ε -linking pairings. The general definition 1.22 does not apply here since $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$. However, since subgroups of \mathbb{Q}/\mathbb{Z} distinct from \mathbb{Q}/\mathbb{Z} are all cyclic, note the following observation.

LEMMA 2.4. Let A, B subgroups of \mathbb{Q}/\mathbb{Z} both Sdistinct from \mathbb{Q}/\mathbb{Z} . The following assertions are equivalent:

(1) $A \otimes B = 0;$ (2) $A \cap B = 0;$ (3) e_A and e_B are coprime.

This leaves nontrivial cases of nonzero tensor products and suggests to resort to the ideas of Lemma 2.1 and of Example 1.2.

Set $\hat{G} = \text{Hom}(G, \mathbb{Z}/e_G\mathbb{Z})$. Let $\lambda : G \to \hat{G}$ and $\lambda : G' \to \hat{G'}$ be two ε -linking pairings given by their left adjoint maps. They induce a homomorphism

(2.2)
$$\Phi: G \otimes G' \xrightarrow{\hat{\lambda} \otimes \hat{\lambda}'} \hat{G} \otimes \hat{G}' \xrightarrow{\otimes} \widehat{G \otimes G'}.$$

Observe that $e_{G \otimes G'}$ is the g.c.d. of e_G and e_G , so $\mathbb{Z}/e_G \mathbb{Z} \otimes \mathbb{Z}/e_{G'} \mathbb{Z} = \mathbb{Z}/e_{G \otimes G'} \mathbb{Z}$.

DEFINITION 1.3. The tensor product $\lambda \otimes \lambda'$ of two ε -linking pairings $\lambda : G \times G \to \mathbb{Z}/e_G\mathbb{Z}$ and $\lambda' : G' \times G' \to \mathbb{Z}/e_{G'}\mathbb{Z}$ is the bilinear pairing defined by

$$(\lambda \otimes \lambda')(x,y) = \Phi(x)(y), \quad x,y \in G \otimes G'.$$

Alternatively, $\lambda \otimes \lambda'$ is the linking pairing defined on $G \otimes G'$ defined by

 $(\lambda \otimes \lambda')(x \otimes x', y \otimes y') = \lambda(x, y) \otimes \lambda(x', y') \in \mathbb{Z}/e_{G \otimes G'}\mathbb{Z}.$

The tensor product is symmetric if both pairings are symmetric or both pairings are antisymmetric. The tensor product is antisymmetric (resp. symplectic) if one of the pairings is symmetric and the other one is antisymmetric (symplectic).

The natural map $\hat{G} \otimes \hat{G}' \to \widehat{G} \otimes \widehat{G}'$, $f \otimes f' \mapsto f(-) \otimes f'(-)$ is an isomorphism. As a consequence of this and functoriality, we record

LEMMA 2.5. If λ and λ' are nondegenerate, then $\lambda \otimes \lambda'$ is nonsingular.

3. Decomposition of ε -linking pairings

Let $\lambda : A \times A \to \mathbb{Q}/\mathbb{Z}$ be an ε -linking pairing on a finite abelian group A. We assume throughout this section that λ is nondegenerate. Recall that this is equivalent to $A^{\perp} = 0$.

LEMMA 3.1. For any subgroup $B \subseteq A$, $|B| \cdot |B^{\perp}| = |A|$ and $(B^{\perp})^{\perp} = B$.

PROOF. By definition, B^{\perp} is the kernel of the surjective map

$$A \mapsto \operatorname{Hom}(B, \mathbb{Q}/\mathbb{Z}), \ a \mapsto \lambda(a)|_B.$$

Thus $|A/B^{\perp}| = |\text{Hom}(B, \mathbb{Q}/\mathbb{Z})| = |B|$, the first equality follows. Applying the first equality to B and B^{\perp} respectively yields $|B||B^{\perp}| = |A| = |B^{\perp}||B^{\perp\perp}|$, so $|B| = |B^{\perp\perp}|$. Since $B \subseteq B^{\perp\perp}$, the second equality follows.

LEMMA 3.2. There exists a canonical orthogonal splitting

$$(A,\lambda) = \bigoplus_{p \text{ prime}} (A_p,\lambda_p)$$

where $A_p = \{x \in A \mid p^N x = 0 \text{ for some } N \in \mathbb{N}\}$. In particular, each orthogonal summand A_p is a p-group.

DEFINITION 1.4. Each linking pairing (A_p, λ_p) is the *p*-primary component of (A, λ) .

PROOF. Let A_p denote the subgroup of all elements in A of order a power of p. Clearly, $A = \bigoplus_{p \text{ prime}} A_p$. We claim that $\lambda(A_p, A_q) = 0$ for any two distinct primes p, q. Let $x \in A_p$ and $y \in A_q$. By definition, we have $0 = \lambda(p^k x, y) = p^k \lambda(x, y)$ for some integer k. Similarly, we have $0 = \lambda(x, q^l y) = q^l \lambda(x, y)$. Thus $\lambda(x, y)$ is annihilated in \mathbb{Q}/\mathbb{Z} by both p^k and q^l which are coprime. It follows that $\lambda(x, y) = 0$.

LEMMA 3.3. Let $x \in A$ and let B the subgroup generated by x. Let n be the order of B. The following assertions are equivalent:

- (1) $\lambda(x,x)$ has order n in \mathbb{Q}/\mathbb{Z} ;
- (2) $\lambda|_{B\times B}$ is nonsingular;
- (3) $A = B \oplus B^{\perp}$ and $\lambda|_{B^{\perp} \times B^{\perp}}$ is nonsingular.

PROOF. The equivalence $(2) \iff (3)$ follows from Lemma 1.3. Let us prove $(1) \iff (2)$. Suppose that $\lambda(x, x)$ has order n. Let $y = k \ x \in B$. The equation $0 = \lambda(x, y) = k \ \lambda(x, x)$ implies that k is a multiple of n hence y = 0. Thus $\lambda|_{B \times B}$ is nondegenerate, hence nonsingular. Conversely, let mbe the order of $\lambda(x, x)$. Since $n\lambda(x, x) = \lambda(nx, x) = 0$, m divides n. Now $0 = \lambda(x, mx)$ so $0 = \lambda(kx, mx) = 0$ for all $k \ge 0$. Thus $mx \in \text{Ker } \hat{\lambda}$. Since $\lambda|_{B \times B}$ is nonsingular, mx = 0. This implies that n divides m. Therefore m = n.

COROLLARY 3.4. Let $x \in A$. Suppose that λ is antisymmetric. The following assertions are equivalent:

- (1) x generates a nontrivial orthogonal summand;
- (2) x generates an orthogonal summand of order 2;
- (3) $\lambda(x, x)$ has order 2 in \mathbb{Q}/\mathbb{Z} .

PROOF. (2) \implies (1) is clear and (2) \iff (3) follows from Lemma 3.3. Suppose (1) holds. By Lemma 3.3, x generates a nontrivial subgroup B of the same order as the order of $\lambda(x, x)$ in \mathbb{Q}/\mathbb{Z} . Since $2\lambda(x, x) = 0$, B has order 2. **3.1. Symmetric linking pairings.** The paragraph is devoted to symmetric linking pairings.

PROPOSITION 3.5 (Symmetric linkings). Let (A, λ) be a nondegenerate symmetric linking pairing on a finite p-group. There exists an orthogonal splitting $(A, \lambda) = \bigoplus_k (A_k, \lambda_k)$ where each (A_k, λ_k) is a nondegenerate bilinear pairing such that A_k is

- (i) either a cyclic p-group,
- (ii) or a direct sum of two copies of a cyclic group of order 2^n . In this case, λ_k is represented by a matrix of the form $\begin{bmatrix} 0 & 2^{-n} \\ 2^{-n} & 0 \end{bmatrix}$ or $\begin{bmatrix} 2^{1-n} & 2^{-n} \\ 2^{-n} & 2^{1-n} \end{bmatrix}$.

REMARK 1.3. If p is odd, only the case (i) of Prop. 3.5 may occur.

PROOF. The proof goes by induction on |A|. If |A| = p then A is cyclic and the assertion holds. Let now p^n denote the period of A. We distinguish two cases:

<u>p</u> odd: we claim that there exists $x \in A$ such that $\lambda(x, x)$ has order exactly p^n in \mathbb{Q}/\mathbb{Z} . Otherwise, the order of $\lambda(x, x)$ divides p^{n-1} for all x; then the order of 2 $\lambda(x, y) = \lambda(x + y, x + y) - \lambda(x, x) - \lambda(y, y)$ also divides p^{n-1} (for all x, y); thus $p^{n-1}A \subset A^{\perp}$, contradicting nondegeneracy. So pick up $x \in A$ so that the order of $\lambda(x, x)$ is p^n . The cyclic subgroup B generated by x has order p^n . By Lemma 3.3, B is an orthogonal summand of A: $A = B \oplus B^{\perp}$. We apply the induction hypothesis to $B^{\perp} \subset A$.

<u>*p* even</u>: if an element x exists such that $\lambda(x, x)$ has order 2^n , the argument above applies. Consider the case when no such elements exists in A. Then nondegeneracy of λ ensures that there exist $x, y \in A$, both of order 2^n , such that $\lambda(x, y)$ has order exactly 2^n . So there exist even integers r and s such that $\lambda(x, x) = \frac{r}{2^n} \pmod{1}$ and $\lambda(y, y) = \frac{s}{2^n} \pmod{1}$. Let B denote the subgroup generated by x and y. Let $a \ x + b \ y \in B \cap B^{\perp}$. We have

$$0 = \lambda(ax + by, x) = a\lambda(x, x) + b\lambda(x, y) + \frac{ar}{2^n} = \frac{ar}{2^n} + \frac{b}{2^n} \mod 1.$$

It follows that $ar + b = 0 \mod 2^n$. Similarly the equality $\lambda(ax + by, y) = 0$ leads to $a + bs = 0 \mod 2^n$. We deduce that $a = b = 0 \mod 2^n$. Therefore, Bis the direct sum of the cyclic groups generated by x and y and $B \cap B^{\perp} = 0$. We conclude by again applying induction to B^{\perp} .

The statement about the matrix representatives of λ_k is a consequence of Lemma 3.6 below.

Denote by $\operatorname{Sym}_2(\mathbb{Z}/2^n\mathbb{Z})$ the algebra of two by two matrices with coefficients in $\mathbb{Z}/2^n\mathbb{Z}$ and by $\operatorname{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$ the group of two by two matrices with coefficients in $\mathbb{Z}/2^n\mathbb{Z}$ that are invertible over $\mathbb{Z}/2^n\mathbb{Z}$. For $1 \leq k \leq n$, define an equivalence relation $\sim_{\mathbb{Z}/2^k\mathbb{Z}}$ in $\operatorname{Sym}_2(\mathbb{Z}/2^n\mathbb{Z})$ by $A \sim_{\mathbb{Z}/2^k\mathbb{Z}} B$ if there exists $C \in \operatorname{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ such that ${}^tCAC = M \mod 2^k$.

LEMMA 3.6. Let $n \ge 1$. We have

$$\begin{bmatrix} 2r & u \\ u & 2s \end{bmatrix}_{\mathbb{Z}/2^n} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for any $r, s, u \in \mathbb{Z}$ with u odd.

PROOF. Note that for all $n \ge 1$,

(3.1)
$$\begin{bmatrix} 2r & u \\ u & 2s \end{bmatrix} \sim \begin{bmatrix} 2r & -u \\ -u & 2s \end{bmatrix}$$

and

(3.2)
$$\begin{bmatrix} 2r & u \\ u & 2s \end{bmatrix} \underset{\mathbb{Z}/2^n\mathbb{Z}}{\sim} \begin{bmatrix} 2s & u \\ u & 2r \end{bmatrix}$$

We proceed inductively on n. For n = 1, the result is trivial. For n = 2:

• If $2r \neq 2s \mod 4$, then by (3.2), we may assume that $2r = 0 \mod 4$ and $2s = 2 \mod 4$. Then

$$\begin{bmatrix} 0 & u \\ u & 2 \end{bmatrix} \underset{\mathbb{Z}/4\mathbb{Z}}{\sim} \begin{bmatrix} 0 & u \\ u & 2+2u \end{bmatrix} = \begin{bmatrix} 0 & u \\ u & 0 \end{bmatrix} \underset{\mathbb{Z}/4\mathbb{Z}}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(the last relation is either an equality or follows from (3.1.) \cdot If $2r = 2s \mod 4$, then applying (3.1) if necessary, we have

Γ	2r	u		2r	1	_ [$\overline{2}$	1	0	1	
	u	2r	\sim $\mathbb{Z}/4\mathbb{Z}$	1	2s		1	2	1	0	.

Now the result follows by repeated applications of the lemma 3.7.

LEMMA 3.7 (A version of Hensel's lemma). Let $n \ge 2$ and $A, B \in \operatorname{Sym}_2(\mathbb{Z}/2^n\mathbb{Z}) \cap \operatorname{GL}_2(\mathbb{Z}/2^n\mathbb{Z})$. Suppose that $A \underset{\mathbb{Z}/2^k\mathbb{Z}}{\sim} B$ for some $2 \le k \le n-1$. Then $A \underset{\mathbb{Z}/2^{k+1}\mathbb{Z}}{\sim} B$.

PROOF. There is $M_k \in GL_2(\mathbb{Z}/2^k)$ such that ${}^tM_kAM_k = B \mod 2^k$. We expect a solution M_{k+1} to the equation

(3.3)
$${}^{t}M_{k+1}AM_{k+1} = B \mod 2^{k+1}.$$

We look for a solution of the form $M_{k+1} = M_k + X_{k+1}$ where X_{k+1} is a matrix with coefficients in $\mathbb{Z}/2^{k+1}$ such that $X_{k+1} = 0 \mod 2^k$. Plugging this expression in (3.3) and expanding, we find that a necessary condition is that

(3.4)
$${}^{t}M_{k}AX_{k+1} + {}^{t}X_{k+1}AM_{k} = B - {}^{t}M_{k}AM_{k} \mod 2^{k+1}$$

This equation is of the form $UX + {}^{t}(UX) = H$, with $U = {}^{t}M_{k}A$ and $H = B - {}^{t}M_{k}AM_{k}$. A formal solution is $X = \frac{1}{2}U^{-1}H$. Note that U is

invertible over $\mathbb{Z}/2^k\mathbb{Z}$, hence over $\mathbb{Z}/2^{k+1}\mathbb{Z}$. Further, $H = B - {}^tM_kAM_k$ is 0 mod 2^k by hypothesis. Thus, $X_{k+1} = \frac{1}{2}U^{-1}H$ is a solution of (3.4) and $M_{k+1} = M_k + X_{k+1}$ is a solution of (3.3). Since M_{k+1} is invertible mod 2^k , it is also invertible mod 2^{k+1} , which concludes the proof.

The proof of Lemma 3.7 contains more than the statement of Proposition 3.5. Denote by $\sim_{\mathbb{Z}_2}$ the same equivalence relation but defined over the 2-adic integers \mathbb{Z}_2 .

COROLLARY 3.8. Any symmetric matrix M with coefficients in \mathbb{Z}_2 is equivalent (for $\sim \mathbb{Z}_2$) to a block-diagonal matrix with each block of one of the three following types:

$$\begin{bmatrix} a \end{bmatrix} (a \in \mathbb{Z}_2), \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

REMARK 1.4. Although Proposition 3.5 is fundamental and will be used systematically, in practice one needs to know how to deal with linking pairings that are not canonically split.

EXAMPLE 1.3. Let p, q, r be three pairwise coprime integers. Consider the cyclic linking pairing $\left(\frac{r}{pq}\right)$ defined on \mathbb{Z}/pq sending (1, 1) to $r/pq \mod 1$. By Proposition 3.5, this linking pairing must be isomorphic to an orthogonal sum of two cyclic pairings on \mathbb{Z}/p and \mathbb{Z}/q respectively. Let $(\alpha, \beta) \in \mathbb{Z}^2$ be a Bezout pair for (p, q) so that $\beta p + \alpha q = 1$. There is an isomorphism

 $\mathbb{Z}/p \times \mathbb{Z}/q \to \mathbb{Z}/pq, \ (u,v) \mapsto u\alpha q + v\beta p \mod pq$

whose inverse is

$$\mathbb{Z}/pq \to \mathbb{Z}/p \times \mathbb{Z}/q, \ x \mapsto (x \mod p, \ y \mod q).$$

Using the fact that $\alpha q + \beta p = 1$, we see that

$$(u\alpha q + v\beta p)^2 = u^2\alpha q + v^2\beta p \mod pq.$$

It follows that

(3.5)
$$\left(\frac{r}{pq}\right) \simeq \left(\frac{r\alpha}{p}\right) \oplus \left(\frac{r\beta}{q}\right).$$

Applying (3.5) to $\left(\frac{1}{21}\right)$ for instance, we obtain

$$\left(\frac{1}{21}\right) \simeq \left(\frac{1}{7}\right) \oplus \left(\frac{-2}{3}\right) = \left(\frac{1}{7}\right) \oplus \left(\frac{1}{3}\right).$$

More generally, let p_1, \ldots, p_n be pairwise coprime integers. For each $1 \leq i \leq n$, set $M_i = \prod_{j \neq i} p_j$ and let μ_i be an integer such that $\mu_i M_i = 1 \mod p_i$. The map

$$\prod_{i=1}^{n} \mathbb{Z}/p_i \to \mathbb{Z}/\prod_{i=1}^{n} p_i, \ (u_1, \dots, u_n) \mapsto \sum_{i=1}^{n} u_i M_i \mu_i$$

is an isomorphism whose inverse is

$$\mathbb{Z}/\prod_{i=1}^{n} p_i \to \prod_{i=1}^{n} \mathbb{Z}/p_i, \ x \mapsto (x \mod p_1, \dots, x \mod p_n).$$

Using the fact that $\mu_i M_i = 1 \mod p_i$, we see that

$$\left(\sum_{i=1}^n u_i M_i \mu_i\right)^2 = \sum_{i=1}^n u_i^2 M_i \mu_i \mod \prod_{i=1}^n p_i.$$

It follows that

(3.6)
$$\left(\frac{1}{\prod_{i=1}^{n} p_i}\right) \simeq \bigoplus_{i=1}^{n} \left(\frac{\mu_i}{p_i}\right).$$

For instance, applying (3.6) to $\left(\frac{1}{861}\right)$ (861 = 3 × 7 × 41) yields

$$\left(\frac{1}{861}\right) \simeq \left(\frac{1}{3}\right) \oplus \left(\frac{1}{7}\right) \oplus \left(\frac{1}{41}\right).$$

DEFINITION 1.5. Let $\lambda : A_2 \times A_2 \to \mathbb{Q}/\mathbb{Z}$ be an ε -linking pairing on a 2-group A_2 of period 2^k . The summand evaluation map is the map $\epsilon : A_2 \to \mathbb{Z}/2\mathbb{Z}$ defined by $\epsilon(x) = 1$ if x generates an orthogonal summand of order 2^k in A and $\epsilon_k(x) = 0$ otherwise.

LEMMA 3.9. The summand evaluation map is a homomorphism.

PROOF. Let $l: A_2 \times A_2 \to \mathbb{Z}/2^k\mathbb{Z}$ be the bilinear pairing defined by

$$\lambda(x,x) = \frac{l(x,x)}{2^k} \mod 1.$$

Then by Lemma 3.3, $\epsilon(x) = l(x, x) \mod 2$. Hence ϵ is a homomorphism.

COROLLARY 3.10. If λ is antisymmetric, the summand evaluation map is nontrivial only on groups of period 2.

PROOF. Apply Cor. 3.4.

The summand evaluation map extends to a map $A \times A \to \mathbb{Z}/2\mathbb{Z}$ for any ε -linking pairing.

LEMMA 3.11. The summand evaluation map is an invariant of isomorphism classes of ε -linking pairings.

The precise meaning of the Lemma is the following. If $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\lambda' : G' \times G' \to \mathbb{Q}/\mathbb{Z}$ are two ε -linking pairings related by an isomorphism $\varphi : G \to G'$ such that $\lambda' \circ \varphi^{\otimes 2} = \lambda$, then the respective summand evaluation maps are related by $\epsilon_{\lambda'} \circ \varphi = \epsilon_{\lambda}$.

Proof.

3.2. Antisymmetric linking pairings. This paragraph is devoted to the decomposition of antisymmetric linking pairings. According to Lemma 1.1 and the remark thereafter, the only difference between symplectic and antisymmetric linking pairings occurs on 2-groups. We begin with three examples of antisymmetric linking pairings: the first one is symplectic, the last two are antisymmetric nonsymplectic.

EXAMPLE 1.4 (symplectic linking pairing). Let p be a prime number and k a positive number. Let A and B be two copies of a cyclic group of order p^k . Choose a generator $x \in A$ and a generator $y \in B$. There is a uniquely defined symplectic linking pairing $H = H_{p,k}$ defined on $A \oplus B$ by

$$H(x,y) = \frac{1}{p^k} \mod 1, \quad H(x,x) = H(y,y) = 0.$$

EXAMPLE 1.5. The assignment

$$(1 \mod 2, 1 \mod 2) \mapsto \frac{1}{2} \mod 1$$

determines an antisymmetric linking pairing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. It is both symmetric and antisymmetric, but not symplectic. We denote it by $C_2 = (\frac{1}{2})$.

EXAMPLE 1.6 (a noncyclic antisymmetric nonsymplectic linking pairing). Let k be a positive number. Let A and B be two copies of a cyclic group of order 2^k . Choose a generator $x \in A$ and a generator $y \in B$. There is a uniquely defined antisymmetric linking pairing E_k defined on $A \oplus B$ by

$$E_k(x,x) = 0, \ E_k(x,y) = \frac{1}{2^k} \mod 1, \ E_k(y,y) = \frac{1}{2} \mod 1.$$

With respect to the system of generators (x, y) for $A \oplus B$, E_k is represented by the matrix $\begin{bmatrix} 0 & -1/2^k \\ 1/2^k & 1/2 \end{bmatrix}$.

It turns out that any symplectic linking pairing occurs as a finite orthogonal sum of pairings of the type of Example 1.4.

PROPOSITION 3.12 (Symplectic linking pairings). Let A be a nondegenerate finite linking p-group. There exists an orthogonal splitting $A = \bigoplus_k A_k$ where each A_k is a an orthogonal sum of a finite number of copies of $H_{p,k}$.

PROOF. We proceed by induction on the exponent p^n of A. Let $x \in A$ have maximal order p^n . Let G be the subgroup generated by x. Since λ is nondegenerate, there exists $y \in A$ such that $\lambda(x, y) = \frac{1}{p^n} \mod 1$. In particular, y has also order p^n . Since λ is symplectic, the subgroup H generated by y does not intersect nontrivially G. Thus G and H form a direct sum B in A. We claim that $\lambda|_{B\times B}$ is nondegenerate. Indeed, let $z = a' x + b' y \in B$ such that $\lambda(a x + b y, z) = 0$ for all $a, b \in \mathbb{Z}$. We find that

$$0 = \lambda(a \ x + b \ y, z) = ab'\lambda(x, y) + ba'\lambda(y, x) = \frac{ab' - a'b}{p^n} \mod 1$$

Hence $ab' - a'b = 0 \mod p^n$ for all $a, b \in \mathbb{Z}$. This implies that $a' = b' = 0 \mod p^n$, thus z = 0. We conclude by Lemma 1.3 that $A = B \oplus B^{\perp}$. The proof is now completed by applying the induction to B^{\perp} .

COROLLARY 3.13. Let A be a finite Abelian p-group. There exists a nondegenerate symplectic linking pairing λ on A if and only if the p-rank of A is even. If this is the case then λ is unique up to isomorphism.

PROOF. By the previous proposition, λ is isomorphic to an orthogonal sum of copies of $H_{p,k}$ whose rank is 2.

The following corollary will be used in the theory of the Weil representation.

COROLLARY 3.14. Given a symplectic linking pairing $\lambda : A \times A \to \mathbb{Q}/\mathbb{Z}$, there exists a bilinear pairing $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that

(3.7)
$$\lambda(x,y) = \beta(x,y) - \beta(y,x), \text{ for any } x, y \in A.$$

PROOF. Decompose $(A, \lambda) = \bigoplus_k (A_k, \lambda_k)$ where each component is a symplectic linking pairing of the form described in Prop. 3.12. It suffices to construct a form β satisfying (3.7) on $A = A_k$. Let x, y be two generators of A_k . Define

$$\begin{array}{rcl} \beta(x,y) &=& \lambda(x,y) \\ \beta(x,x) &=& \beta(y,y) &= 0 \\ \beta(y,x) &=& 0 \end{array}$$

and extend β to a bilinear map on A_k .

DEFINITION 1.6. A pairing satisfying the condition (3.7) of Cor. 3.14 will be called a *Seifert pairing* with respect to the symplectic form λ .

For a motivation for this terminology, see

COROLLARY 3.15. Given a symplectic form λ , the set of Seifert pairings with respect to λ is acted on freely and transitively by the set of symmetric linking pairings.

PROPOSITION 3.16 (Antisymmetric linking pairings). Let A be a nondegenerate (-1)-linking 2-group. Then A splits orthogonally as an orthogonal sum of copies of C_2 , $H_{2,k}$ and E_k .

PROOF. We follow Wall [97, §4]. Since (A, λ) is nonsingular and $x \mapsto \lambda(x, x)$ is a homomorphism, there is $\alpha \in A$ such that $\lambda(x, x) = \lambda(\alpha, x)$ for all $x \in A$. If $\alpha = 0$, Prop. 3.12 applies. So we may from now on assume that $\alpha \neq 0$. Since α has order 2, $\lambda(\alpha, \alpha)$ has order at most 2. If the order of $\lambda(\alpha, \alpha)$ is exactly 2, then by Cor. 3.4, α generates an orthogonal summand of order 2 in A which is C_2 . Hence $A = C_2 \oplus C_2^{\perp}$, and furthermore, C_2^{\perp} is symplectic and Prop. 3.12 again applies. It remains to consider the case when $\lambda(\alpha, \alpha) = 0$. Let r be the greatest integer such that $\alpha = 2^{r-1}x$ for some $x \in A$ (the *height* of α). If r = 1 then $\lambda(x, x) = 0$. For r > 1, since $\lambda(x, x) = \lambda(\alpha, x) = 2^{r-1} \lambda(x, x)$, we have $(2^{r-1} - 1) \lambda(x, x) = 0$ hence $\lambda(x, x) = 0$. Since α has order 2, x has order 2^r in A. By nondegeneracy of (A, λ) , there is $y \in A$ such that $\lambda(x, y) = \frac{1}{2^r} \mod 1$. This implies that y has order 2^r and the subgroups respectively generated by x and y do not intersect nontrivially, hence form a direct sum G in A. We claim that $\lambda|_{G\times G}$ is nondegenerate. Indeed, suppose that for any $a', b' \in \mathbb{Z}$,

$$\begin{aligned} 0 &= \lambda(ax + by, a'x + b'y) = aa'\lambda(x, x) + ab'\lambda(x, y) + ba'\lambda(y, x) + bb'\lambda(y, y) \\ &= 0 + (ab' - ba')\lambda(x, y) + bb'\lambda(2^{r-1}x, y) \\ &= \frac{ab' - ba'}{2^r} + \frac{bb'}{2} \mod 1. \end{aligned}$$

It follows that $a = b = 0 \mod 2^r$ and the claim follows. By Lemma 1.3, G is an orthogonal summand of A and $A = G \oplus G^{\perp}$. Remark that $(G, \lambda|_{G \times G})$

identifies to the antisymmetric (-1)-linking pairing E_r (see Example 1.6). Since $x \in G$, $\hat{\lambda}(x)$ vanishes on G^{\perp} so $\lambda|_{G^{\perp} \times G^{\perp}}$ is symplectic and Prop. 3.12 applies on G^{\perp} . This completes the proof.

REMARK 1.5. The proof is a bit more precise than the statement of the proposition: it provides a unique normal form for each antisymmetric linking pairing. With the notation of the proof above, if $\lambda(\alpha, \alpha) \neq 0$ then A splits as the orthogonal sum of C_2 and a symplectic linking pairing; if $\lambda(\alpha, \alpha) = 0$, then A splits as the orthogonal sum of E_r (where r is the height of α) and a symplectic linking pairing.

4. Classification results for linking pairings

The set $\mathfrak{M}^{\varepsilon}$ of isomorphism classes of ε -linking pairings has a monoid structure for the orthogonal sum. First we review the description of these monoids and their invariants. Then we generalize them to monoids of pointed linking pairings. All pairings are assumed to be nondegenerate.

The p-primary decomposition (Lemma 3.2) induces a decomposition

$$\mathfrak{M}^{\varepsilon} = \bigoplus_{p} \mathfrak{M}_{p}^{\varepsilon},$$

over all primes, where $\mathfrak{M}_p^{\varepsilon}$ denotes the monoid of isomorphism classes of ε -linking pairings over *p*-groups. It is therefore sufficient to describe each monoid $\mathfrak{M}_p^{\varepsilon}$ separately.

We begin with the description of \mathfrak{M}^- .

4.1. The monoid \mathfrak{M}^- .

THEOREM 4.1. Let p be an odd prime number. The monoid \mathfrak{M}_p^- is freely generated by $H_{p,k}, k \ge 1$.

PROOF. An antisymmetric linking pairing on a *p*-group with *p* odd must be symplectic (cf. $\S1$). Then the decomposition of symplectic linking pairings (Prop. 3.12) implies the result.

The case when p = 2 is due to C.T.C. Wall[97, §4].

THEOREM 4.2. The monoid \mathfrak{M}_2^- is generated by generators C_2 , E_k and $H_{2,k}$, $k \ge 1$ and relations

(4.1)
$$C_2 + C_2 = E_1$$
, $C_2 + E_k = C_2 + H_{2,k}$, $E_k + E_l = E_k + H_{2,l}$ for $k \leq l$.

PROOF. By Prop. 3.16 and Remark 1.5, any element in \mathfrak{M}_2^- has a unique normal form. Any element in \mathfrak{M}_2^- that consists of the sum at least two generators has a summand which identifies to the left side of one of the relations above. Writing the normal form for the left side of each of the relations yields the three relations as stated. The proof follows.

4.2. The monoid \mathfrak{M}^+ . Let $a, n \in \mathbb{Z}^{\times}$. We denote by $\left(\frac{a}{n}\right)$ the cyclic linking pairing defined on $\mathbb{Z}/n\mathbb{Z}$ that sends (1 mod 1, 1 mod 1) to $\frac{a}{n}$ mod 1. We begin with linking pairings on *p*-groups with *p* odd.

The following result is due to C.T.C. Wall [97, §5].

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THEOREM 4.3. Let p be an odd prime number. Let $n_p \in \mathbb{Z}$ denote an integer that is not a square mod p. The monoid \mathfrak{M}_p^+ splits as the orthogonal sum

$$\mathfrak{M}_p^+ = \bigoplus_{k \ge 1} \mathfrak{M}_{p,k}^+$$

where $\mathfrak{M}_{p,k}^+$ is the monoid of isomorphism classes of linking pairings on direct sums of copies of $\mathbb{Z}/p^k\mathbb{Z}$. Each $\mathfrak{M}_{p,k}^+$ is generated by two generators $\left(\frac{1}{p^k}\right)$ and $\left(\frac{n_p}{p^k}\right)$. The only relation is

(4.2)
$$\left(\frac{1}{p^k}\right) + \left(\frac{1}{p^k}\right) = \left(\frac{n_p}{p^k}\right) + \left(\frac{n_p}{p^k}\right), \ k \ge 1.$$

PROOF. Lemma 3.5 implies the orthogonal splitting. A little more attention reveals that the splitting of Lemma 3.5 is unique (for p odd) in the sense that each component in $\mathfrak{M}_{p,k}^+$ is uniquely determined. Furthermore $\mathfrak{M}_{p,k}^+$ is generated by cyclic linking pairings. Let $C_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$. Given a cyclic linking pairing $\left(\frac{a}{p^k}\right)$, an automorphism $x \mapsto fx$ of C_{p^k} will change aby af^2 and hence, the quadratic residue or quadratic nonresidue character of a is preserved. It follows that $\mathfrak{M}_{p,k}^+$ is generated by $\left(\frac{1}{p^k}\right), \left(\frac{n_p}{p^k}\right)$. Consider $(A, \lambda) \in \mathfrak{M}_{p,k}^+$. Choose a system (x_1, \ldots, x_r) of generators for A. Let $\lambda(x_i, x_j) = \frac{a_{ij}}{p^k} \mod 1$. The determinant

$$\sigma(\lambda) = \det(a_{ij})_{1 \leqslant i,j \leqslant r} = \begin{cases} +1 & \text{if it is a square mod } p; \\ -1 & \text{otherwise} \end{cases} \in C_{p^k}^{\times} / C_{p^k}^2$$

up to multiplication by a square mod p is an invariant of λ . Furthermore, it is multiplicative on orthogonal sums.

We shall use repeatedly a corollary of Hensel's lemma, namely the fact that $x \in \mathbb{Z}$ is a square mod p if and only if x is a square mod p^k for all $k \ge 1$ (see for instance [88, Chap. 2, § 2.2, Corollary 2]).

LEMMA 4.4. Let n_p be a nonsquare in $\mathbb{Z}/p^k\mathbb{Z}$. There exist $x_1, x_2 \in \mathbb{Z}/p^k\mathbb{Z}$ such that $x_1^2 + x_2^2 = n_p$.

PROOF. Suppose k = 1. Note that $-n_p - x_1^2$ takes $\frac{p+1}{2}$ distinct values as x_1 runs over $\mathbb{Z}/p\mathbb{Z}$. Since there are $\frac{p-1}{2}$ distinct squares mod p (see for instance [88, Chap. 1, § 3.1, Theorem 4], we deduce that the equation $x_1^2 + x_2^2 = n_p$ has a solution (x_1, x_2) in $\mathbb{Z}/p\mathbb{Z}$. The general statement follows from Hensel's lemma.

The matrix
$$\Phi = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$$
 defines an automorphism $\mathbb{Z}/p^k \mathbb{Z} \oplus \mathbb{Z}/p^k \mathbb{Z}$
 $\mathbb{Z}/p^k \mathbb{Z} \oplus \mathbb{Z}/p^k \mathbb{Z}$. Set $Q = \left(\frac{1}{p^k}\right) \oplus \left(\frac{1}{p^k}\right)$. Then
 $Q(\Phi(x,y)) = \frac{(x_1 \ x + x_2 \ y)^2}{p^n} + \frac{(-x_2 \ x + x_1 \ y)^2}{p^n}$
 $= \frac{(x_1^2 + x_2^2)x^2}{p^n} + \frac{(x_1^2 + x_2^2)y^2}{p^n} = \frac{n_p x^2}{p^n} + \frac{n_p y^2}{p^n}.$

This is the desired identity (4.2). It follows that if the rank of A (as a free $\mathbb{Z}/p^k\mathbb{Z}$ -module) is r, then there are exactly two isomorphism classes of linking pairings on A, namely

$$r\left(\frac{1}{p^k}\right)$$
 and $(r-1)\left(\frac{1}{p^k}\right) + \left(\frac{n_p}{p^k}\right)$.

In particular, the rank of A and the determinant $\sigma(\lambda)$ are complete invariants for $(A, \lambda) \in \mathfrak{M}_{p,k}^+$.

Consider $(A, \lambda) \in \mathfrak{M}_p^+$ with orthogonal decomposition $(A, \lambda) = \bigoplus_{k \ge 1} (A_k, \lambda_k)$ with $(A_k, \lambda_k) \in \mathfrak{M}_{p,k}^+$. For $k \ge 1$, let $\rho_k(A)$ be the rank of A_k as a free $\mathbb{Z}/p^k\mathbb{Z}$ -module. Set $\rho(\lambda)$ to be the collection of ranks $(\rho_k(A))_{k\ge 1}$. Set $\sigma_k(\lambda) = \sigma(\lambda_k) \in \{\pm 1\}$ (as defined in the proof of Th. 4.3) and $\sigma(A, \lambda) = (\sigma_k(A, \lambda))_{k\ge 1}$.

COROLLARY 4.5. The isomorphism class $(A, \lambda) \in \mathfrak{M}_p^+$ is determined by $\rho(\lambda), \sigma(\lambda)$.

DEFINITION 1.7. Let $\mathfrak{M}_p[n]$ be the set of linking pairings of order p^n for $n \ge 1$. For n = 0, $\mathfrak{M}_p[n] = 0$.

We give a combinatorial description of $\mathfrak{M}_p[n]$. First, we need a few classical definitions.

DEFINITION 1.8. A partition of an integer n is a nonincreasing sequence $c = (c_1, c_2, \ldots, c_l)$ of integers such that $c_1 + c_2 + \ldots + c_l = n$. Each c_i is called a part of the partition c. The multiplicity of an integer k is the number of occurrences of the integer k in the sequence, i.e., the number of parts c_j such that $c_j = k$. The total number l of parts is called the *length* of the partition. The set of partitions of n is denoted $\Lambda[n]$.

EXAMPLE 1.7. c = (5, 5, 3, 3, 2, 1) is a partition of 19. The multiplicity of the part 5 is 2. The length of the partition is 6.

EXAMPLE 1.8. Any permutation $\tau \in \mathfrak{S}_n$ determines, via its cycle decomposition, a partition of n: order the cycles in weakly decreasing lengths and collect the lengths of the respective cycles. Conversely, a partition of ndetermines a conjugacy class of \mathfrak{S}_n .

According to the classification of abelian groups, any finite p-group of order p^n is a finite sum of cyclic p-groups, whose product of orders equals p^n . Thus the isomorphism classes of finite p-groups of order p^n are in bijective correspondence with partitions of n. From Theorem 4.3 we deduce what

needs to be added in order to classify isomorphism classes of linking pairings on p-groups of order p^n .

DEFINITION 1.9. An overpartition of an integer n is a nonincreasing sequence $c = (c_1, c_2, \ldots, c_l)$ of integers such that $c_1 + c_2 + \ldots + c_l = n$ and such that each last occurrence of an integer may be overlined. The set of overpartitions of n is denoted $\tilde{\Lambda}[n]$.

EXAMPLE 1.9. $c = (5, 5, 3, \overline{3}, 2, \overline{1})$ is an overpartition of 19. Note also that any partition of n is an overpartition of n, thus $\Lambda[n] \subseteq \widetilde{\Lambda}[n]$.

PROPOSITION 4.6. There is a bijective correspondence between $\mathfrak{M}_p[n]$ and $\widetilde{\Lambda}[n]$, extending the bijective correspondence between the set of isomorphism classes of finite p-groups of order p^n and $\Lambda[n]$.

PROOF. Let $(A, \lambda) \in \mathfrak{M}_p[n]$. We associate to (A, λ) an overpartition as follows. The sequence $(\rho_k(\lambda))_{k \ge 1}$ is finite and verifies

$$\sum_k \rho_k(\lambda) = n$$

Drop all zeros in this sequence. This yields a sequence $(\rho_{k_j}(\lambda))_{j\geq 1}$ of nonzero natural integers. Reorder this sequence so that $k_1 \geq k_2 \geq k_3 \geq \cdots$. (This step is needed only to conform to the definition of a partition as a nonincreasing sequence of integers.) Consider the following partition: each k_j is a part with multiplicity $\rho_{k_j}(\lambda) \geq 1$. Overline the last occurrence of the part kif and only if $\sigma_k(\lambda) = -1$. This yields an overpartition of n. Conversely, to an overpartition c, we associate the isomorphism class of a linking pairing λ as follows. By Corollary 4.5, it suffices to specify the invariants $\rho_k(\lambda)$ and $\sigma_k(\lambda)$. Set

 $\begin{array}{lll} \rho_k(\lambda) &=& \text{multiplicity of } k \text{ in the partition } c \\ \text{and} & \sigma_k(\lambda) &=& \begin{cases} -1 & \text{if the last occurrence of } k \text{ is overlined} \\ +1 & \text{otherwise.} \end{cases}$

It is easy to verify that these two maps, at the level of isomorphism classes of linking pairings, are inverse to each other.

We consider now the case p = 2. For $k \ge 1$, we define two linking pairings $F_k(k \ge 1), G_k(k \ge 2)$ on $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z}$ by the two matrices of Prop. 3.5, (ii), respectively. These linking pairings are pairwise nonisomorphic since $|F_k| = |G_k| = 2^{2k}$ for all $k \ge 1$ and $\sigma_k(F_k) = +1$ and $\sigma_k(G_k) = -1$ for all $k \ge 2$.

REMARK 1.6. Let $k \ge 3$. Let n, n' be two odd integers. The linking pairings $\left(\frac{n}{2^k}\right)$ $(k \ge 3)$ and $\left(\frac{n'}{2^k}\right)$ are isomorphic if and only if $n = n' \mod 8$ (see [96, Chap. 5, §4]). In particular, there are 4 pairwise nonisomorphic linking pairings on a cyclic group of order 2^k .

A. Kawauchi and S. Kojima [51, Th. 0.1] give the presentation below of \mathfrak{M}_2^+ .

THEOREM 4.7. The monoid \mathfrak{M}_2^+ is generated by the following generators

(i) cyclic generators:
$$\left(\frac{1}{2}\right)$$
, $\left(\frac{\pm 1}{4}\right)$ and $\left(\frac{\pm 1}{2^k}\right)$ $(k \ge 3)$ and $\left(\frac{\pm 3}{2^k}\right)$ $(k \ge 3)$.
(ii) noncyclic generators: F_k $(k \ge 1)$, G_k $(k \ge 2)$

and the following relations:

(4.3)
$$\left(\frac{n_1}{2^k}\right) + \left(\frac{n_2}{2^k}\right) = \left(\frac{n_1+4}{2^k}\right) + \left(\frac{n_2+4}{2^k}\right) \quad (k \ge 3)$$

(4.4)
$$\left(\frac{n}{2^k}\right) + 2\left(\frac{-n}{2^k}\right) = \left(\frac{-n}{2^k}\right) + F_k \quad (k \ge 1)$$

(4.5)
$$3\left(\frac{n}{2^k}\right) = \left(\frac{-n+4}{2^k}\right) + G_k \quad (k \ge 2)$$

(4.7)
$$\left(\frac{n_1}{2^k}\right) + \left(\frac{n_2}{2^{k+1}}\right) = \left(\frac{n_1 + 2n_1}{2^k}\right) + \left(\frac{n_2 + 2n_1}{2^{k+1}}\right) \quad (k \ge 1)$$

(4.8)
$$\left(\frac{n}{2^k}\right) + F_{k+1} = \left(\frac{n+4}{2^k}\right) + G_{k+1} \quad (k \ge 1)$$

(4.9)
$$F_k + \left(\frac{n}{2^{k+1}}\right) = G_k + \left(\frac{n+4}{2^k}\right) \quad (k \ge 2)$$

(4.10)
$$\left(\frac{n_1}{2^k}\right) + \left(\frac{n_2}{2^{k+2}}\right) = \left(\frac{n_1+4}{2^k}\right) + \left(\frac{n_2+4}{2^{k+2}}\right) \quad (k \ge 1)$$

PROOF. That (i) and (ii) form a system of generators follows from Prop. 5.30. The eight relations can be verified by explicitly finding an isomorphism of linking pairings (Prop. 1.3 is useful for identifying orthogonal summands). The fact that the eight relations form a complete system of relations (i.e. generate any relation between linking pairings) is proved in [51].

Further study of the monoid \mathfrak{M}_2^+ is pursued in [67] (construction of a normal form) and [20] (alternative presentation of \mathfrak{M}_2^+ using Gauss invariants). We mention one open problem. In accordance with the previous paragraph, let $\mathfrak{M}_2[n]$ denote the set of isomorphism classes of linking pairings on 2-groups of order 2^n . Can one explicit a combinatorial construction of $\mathfrak{M}_2[n]$ in the same spirit as that of $\mathfrak{M}_p[n]$ for $p \neq 2$? We do not know a complete answer to that question. Here are a few remarks in that direction. From the presentation of \mathfrak{M}_2 , one observes that there is a natural embedding $\mathfrak{M}_p \to \mathfrak{M}_2$. The sequence of numbers $\mu_n = |\mathfrak{M}_2[n]|$ of isomorphism classes of linking pairings on 2-groups of order 2^n does not seem to identify to a known integer sequence. The program based on [20] that we wrote [21] computes recursively μ_n for any n. The first fifteen values are

$$\mu_1 = 1, \mu_2 = 4, \mu_3 = 6, \mu_4 = 14, \mu_5 = 20, \mu_6 = 43, \mu_7 = 59, \mu_8 = 108,$$

 $\mu_9 = 158, \mu_{10} = 265, \mu_{11} = 373, \mu_{12} = 600, \mu_{13} = 838, \mu_{14} = 1301, \mu_{15} = 1797.$ A combinatorial construction involves a generalization of the partition numbers.

5. Quadratic functions on Abelian groups

5.1. General quadratic functions.

DEFINITION 1.10. Let A, U be abelian groups. A quadratic function on A is a map $q: A \to U$ such that

$$b_q: (x, y) \mapsto b_q(x, y) = q(x+y) - q(x) - q(y)$$

is Z-bilinear on A. The (symmetric) linking pairing b_q is called the linking pairing *associated* to q; the map q is said to be a quadratic function over b_q .

REMARK 1.7. An additive map is a quadratic map with trivial associated linking pairing. The associated linking pairing b_q may be regarded as the 2-cocycle (akin to the "additivity defect") of the quadratic map q.

Note that a quadratic function q verifies q(0) = 0 (take x = y = 0 in the definition) and $b_q(x, x) = q(x) + q(-x)$ for all x (since 0 = q(x - x)).

EXAMPLE 1.10 (Cyclic group). Let $m \ge 1$ and $A = \mathbb{Z}/m\mathbb{Z}$. Let $\beta \in \mathbb{Z}$. The map $q: A \to \mathbb{Q}/\mathbb{Z}, \ y \mapsto \frac{\beta y^2}{2m} \mod 1$ defines a quadratic function if and only if β or m is even. [Proof: $0 = q(0) = q(m) = \frac{\beta m^2}{2m} = \frac{\beta m}{2} \mod 1$.]

DEFINITION 1.11. It will be convenient to denote -q the map defined by (-q)(x) = -q(x) and q^- the map defined by $q^-(x) = q(-x)$.

LEMMA 5.1. If q is a quadratic function then -q, q^- and $q+q^-$ are quadratic functions.

DEFINITION 1.12. The group of quadratic functions from A to U (for addition) is denoted Quad(A, U). If the value group U is understood, we write Quad(A).

REMARK 1.8. For applications in this monograph, either A will be a finitely generated free Abelian group (a lattice) and $U = \mathbb{Z}$, or A will be a finitely generated Abelian group and $U = \mathbb{Q}/\mathbb{Z}$.

DEFINITION 1.13. A quadratic function $q: A \to U$ is said to be *nondegen*erate if the associated bilinear pairing $b_q: A \times A \to U$ is nondegenerate.

EXAMPLE 1.11. Consider the example 1.10 above. The quadratic function $q : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \ y \mapsto \frac{\beta y^2}{2m} \mod 1$ is nondegenerate if and only if $\begin{cases} \beta \text{ and } m \text{ are coprime} & \text{if } m \text{ is odd;} \\ \beta \text{ and } 2m \text{ are coprime} & \text{if } m \text{ is even.} \end{cases}$

There are several motivations, from algebra as well as from topology, for studying possibly degenerate quadratic functions. For one possible motivation, see below Remark 1.16. We record a class of degenerate quadratic functions that canonically induce nondegenerate quadratic functions. See, however, Remark 1.10 below.

LEMMA 5.2. Let $q: A \to U$ be a quadratic function that $q(A^{\perp}) = 0$. Then q induces a nondegenerate quadratic function $\bar{q}: A/A^{\perp} \to U$ by $\bar{q}([x]) = q(x)$, $x \in A$.

REMARK 1.9. Quadratic functions satisfying the hypothesis of Lemma 5.2 are studied in §5.4. Lemma 5.2 is generalized in Chap. ??, §.

DEFINITION 1.14. Two quadratic functions $q : A \to U$ and $q' : A' \to U$ are *isomorphic* if there exists a group isomorphism $\phi : A \to A'$ such that $q'(\phi(x)) = q(x)$ for all $x \in A$.

EXAMPLE 1.12. The quadratic functions q and q^- are isomorphic.

EXAMPLE 1.13. Consider the example 1.10 above. If $\beta = \alpha^2 \mod m$, then quadratic functions defined on $\mathbb{Z}/m\mathbb{Z}$ respectively by $q(y) = \frac{\beta y^2}{2m} \mod 1$ and $q'(x) = \frac{\alpha x}{2m} \mod 1$ are isomorphic.

LEMMA 5.3. If two quadratic functions are isomorphic, then their associated linking pairings are isomorphic.

EXERCISE 1.1. The converse does not hold in general. For instance, $q(x) = \frac{x^2}{8} \mod 1$ and $q'(x) = \frac{5x^2}{8} \mod 1$ define two nonisomorphic quadratic forms on $\mathbb{Z}/4\mathbb{Z}$ with the same associated linking pairing (since $b_q(x, y) = b_{q'}(x, y) = \frac{xy}{4} \mod 1$).

DEFINITION 1.15. Given two quadratic functions $q : A \to \mathbb{Q}/\mathbb{Z}$ and $q' : A' \to \mathbb{Q}/\mathbb{Z}$, their orthogonal sum $(A, q) \oplus (A', q')$ is defined as the quadratic function $q \oplus q' : A \oplus A' \to \mathbb{Q}/\mathbb{Z}$ by

$$(q \oplus q')(x+y) = q(x) + q(y), \quad x \in A, \ y \in A'.$$

Let B be a subgroup of A. A quadratic function $q': B \to \mathbb{Q}/\mathbb{Z}$ is an orthogonal summand of $q: A \to \mathbb{Q}/\mathbb{Z}$ if

$$(A,q) = (B,q') \oplus (B^{\perp},q|_{B^{\perp}}).$$

The following result is an immediate consequence of the definition.

LEMMA 5.4. If a quadratic function (A, q) splits orthogonally $(A, q) = (B, q') \oplus (C, q'')$, then the associated linking pairing splits accordingly:

$$(A, b_q) = (B, b_{q'}) \oplus (C, b_{q''}).$$

REMARK 1.10. Lemma 5.2 does *not* state that the original quadratic function splits as the orthogonal sum of the induced quadratic function $(A/A^{\perp}, \bar{q})$ and the trivial quadratic function $(A^{\perp}, 0)$. In fact, the extension $0 \rightarrow A^{\perp} \rightarrow A \rightarrow A/A^{\perp} \rightarrow 0$ may not split. (The quadratic function defined by $q(x) = x^2/8 \mod 1$, $x \in A = \mathbb{Z}/8\mathbb{Z}$, provides such an example.)

DEFINITION 1.16. A quadratic function q on A is homogeneous if $q(n x) = n^2 q(x)$ for all $x \in A$. A homogeneous quadratic function shall also be called a quadratic form. The set of quadratic forms form a subgroup, denoted $\operatorname{Quad}^0(A, U)$, in the group of quadratic functions $\operatorname{Quad}(A, U)$. As before, we shall write simply $\operatorname{Quad}^0(A)$ if the value group U is understood.

LEMMA 5.5. A quadratic function q is homogeneous if and only if $q = q^{-}$.

PROOF. Necessity is clear. Conversely the identity $q(nx) = n^2q(x)$ is proved by induction on $n \ge 1$ by noticing that q((n+1)x) = q(nx+x) = $q(nx) + q(x) + n \ b_q(x,x) = n^2q(x) + q(x) + 2n \ q(x) = (n+1)^2 \ q(x)$. EXAMPLE 1.14. For any quadratic function q, the quadratic function $q + q^$ is homogeneous.

5.2. Quadratic refinements.

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DEFINITION 1.17. Given a symmetric bilinear pairing $\lambda : A \times A \to U$, a quadratic refinement (or enhancement) of λ is a quadratic function $q : A \to U$ such that $b_q = \lambda$. The set of quadratic refinements of λ will be denoted Quad (λ) .

A fairly general condition for a quadratic refinement to exist is given below (Proposition 5.6).

Let $\lambda : A \times A \to U$ be a linking pairing. Why would one consider quadratic refinements of λ ? Here is a possible algebraic motivation. (A motivation from topology will arise later.) Form the set $A \times U$ and define a new addition law by

$$(a,t) \cdot (a',t') = (a+a',t+t'+\lambda(a,a')), a,a' \in A, t,t' \in U.$$

It is readily verified that this defines an abelian group structure on $A \times U$. Furthermore, as both the injection $U \to A \times U$, $t \mapsto (0, t)$ and the natural projection $A \times U \to A$ are group homomorphisms, this group fits into the short exact sequence

$$(5.1) 0 \to U \to A \times U \to A \to 0.$$

Assume either that U is injective (for instance $U = \mathbb{Q}/\mathbb{Z}$) or that A is projective (for instance A is a lattice). Then the short exact sequence above is split (as a sequence of \mathbb{Z} -modules). Hence there exists a section $s : A \to A \times \mathbb{Q}/\mathbb{Z}$, $x \mapsto (x, q_s(x))$ that is a group homomorphism. It follows that q_s is a quadratic refinement of λ . In particular, quadratic refinements exist. Conversely, given a quadratic refinement q of λ , the map $s_q : x \mapsto (x, q(x))$ defines a section $A \to A \times U$. Hence we have proved:

PROPOSITION 5.6. If the short exact sequence (5.1) splits, then the set $Quad(\lambda)$ of quadratic refinements of a linking pairing λ are in bijective correspondence with sections of (5.1).

If q is a homogeneous quadratic refinement of a linking pairing $\lambda : A \times A \rightarrow U$, then $2q(x) + \lambda(x, x) = q(2x) = 4q(x)$, so $2q(x) = \lambda(x, x)$ for all $x \in A$. We deduce the following observation:

LEMMA 5.7. ?? If U has no 2-torsion, then there is at most one homogeneous quadratic refinement of a nonsingular linking pairing. If multiplication by 2 in invertible in U, a homogeneous quadratic refinement of a linking pairing exists and is unique.

EXAMPLE 1.15. Let $\lambda(x, y) = x \ y, \ x, y \in \mathbb{Z}$. This is a symmetric bilinear pairing on $A = \mathbb{Z}$ with values in $U = \mathbb{Z}$ that has no homogeneous quadratic refinement. Note that in accordance with Proposition 5.6, b does have a (nonhomogeneous) quadratic refinement, for instance

$$q(x) = \frac{x^2 + x}{2}, \ x \in \mathbb{Z},$$

defines a quadratic refinement $\mathbb{Z} \to \mathbb{Z}$ of λ .

The set $\text{Quad}(\lambda)$ of quadratic refinements over λ is freely and transitively acted on by Hom(A, U) by addition. Thus, if λ is nondegenerate, then $\text{Quad}(\lambda)$ is freely and transitively acted on by A via the formula

(5.2)
$$q \cdot x = q + \lambda(x, -) = q + \lambda(x) \in \text{Quad}(\lambda), \quad q \in \text{Quad}(\lambda), \quad x \in A.$$

The set $\text{Quad}^{0}(\lambda)$ of homogeneous quadratic refinements over λ is freely and transitively acted on by Hom(A,). If λ is nondegenerate, then $\text{Quad}^{0}(\lambda)$ is freely and transitively acted on by $A_{2} = \{x \in A \mid 2x = 0\}$ via the formula

(5.3)
$$q \cdot x = q + \lambda(x, -) = q + \lambda(x) \in \text{Quad}^{0}(\lambda), \quad q \in \text{Quad}(\lambda), \quad x \in A_{2}.$$

5.3. The groups \hat{A}^{quad} and A^{quad} .

DEFINITION 1.18. Let A be an Abelian group. Let QA be the abelian group freely generated by generators $t_x, t_{x,y}$ $(x, y \in G)$. The group \hat{A}^{quad} is the quotient of QA by the subgroup I generated by relations

$$t_{x,y} = t_{y,x}, \ t_{x+y} = t_x t_{x,y} t_y, \ t_{x,y+z} = t_{x,y} t_{x,z}, \ t_{x,0} = 1, \quad x, y, z \in G.$$

Since $t_{0,0} = t_{0,0}t_{0,0}$ (third relation), $t_{0,0} = 1$. Since $t_0 = t_0t_{0,0}t_0$ (second relation), $t_0 = 1$. Since $1 = t_{x,0} = t_{x,y}t_{x,-y}$, we have $t_{x,-y} = t_{x,y}^{-1}$. Hence $1 = t_0 = t_{x-x} = t_x t_{x,-x} t_{-x}$ so $t_{x,x} = t_x t_{-x}$ for all $x \in G$.

We keep the same notation for an element in QA and its image in \hat{A}^{quad} (if this does not seem to cause confusion). Although \hat{A}^{quad} is abelian, it will be useful to keep the multiplicative notation.

Recall the definition of the tensor product $A \otimes B$ of two Abelian groups A and B. The tensor product of A and B is an Abelian group $A \otimes B$ and a bilinear map $h : A \times B \to A \otimes B$ which solves the following "universal mapping problem":



For every Abelian group C and every bilinear map $b: A \times B \to C$, there exists a unique homomorphism $\tilde{b}: A \otimes B \to C$ making the diagram above commutes. The solution is unique up to an unique isomorphism.

The group \hat{A}^{quad} possesses a similar property.

PROPOSITION 5.8. The group \hat{A}^{quad} , together with the quadratic function $t: A \to \hat{A}^{\text{quad}}$, $x \mapsto t_x$, solves the following universal mapping problem:



For every quadratic function $q : A \to U$, there exists a unique homomorphism $\tilde{q} : \hat{A}^{\text{quad}} \to U$ such that the diagram above commutes.

PROOF. The map $g \mapsto t_g$ defines a quadratic function (written multiplicatively) since $t_{x+y} = t_x t_{x,y} t_y$ for all $x, y \in A$. Since QA is free, there is a unique homomorphism $QA \to U$ sending t_x to q(x) and $t_{x,y}$ to $b_q(x,y)$ for all $x, y \in A$. Since q is a quadratic function and b_q is the associated symmetric linear pairing, this homomorphism vanishes on the subgroup I, hence induces a map $\tilde{q} : \hat{A}^{\text{quad}} \to U$. Thus $q = \tilde{q} \circ t$. This proves existence of $(\hat{A}^{\text{quad}}, \tilde{q})$. Uniqueness follows as usual from universality.

LEMMA 5.9. The map

 $q \mapsto \tilde{q}$

is an isomorphism $\operatorname{Quad}(A, U) \to \operatorname{Hom}(\hat{A}^{\operatorname{quad}}, U)$.

PROOF. The map $q \mapsto \tilde{q}$ is clearly additive. The inverse map is $f \mapsto f \circ t$.

COROLLARY 5.10. The quadratic map $t: A \to \hat{A}^{\text{quad}}, x \mapsto t_x$ is injective.

PROOF. The identity map $A \to A$ is an additive map, hence a quadratic map $q: A \to A$, such that if $x \neq y$ then $\tilde{q}(t_x) = q(x) = x \neq y = q(y) = \tilde{q}(t_y)$. It follows that $t_x \neq t_y$ in \hat{A}^{quad} .

PROPOSITION 5.11. The assignment $A \mapsto \hat{A}^{\text{quad}}$ defines an additive faithful endofunctor of the category of abelian groups with homomorphisms.

PROOF. Let $\varphi : A \to B$ be a homomorphism between abelian groups. We define a map $\varphi^{\text{quad}} : \hat{A}^{\text{quad}} \to \hat{B}^{\text{quad}}$ as follows:

$$\varphi^{\text{quad}}(t_{x,y}) = t_{\varphi(x),\varphi(y)}, \ \varphi^{\text{quad}}(t_x) = t_{\varphi(x)} \quad x, y \in A.$$

It follows from the definition that the assignment is a homomorphism and preserves additivity, composition and the identity. For faithfulness, let $\varphi, \psi \in$ Hom(A, B) such that $\varphi^{\text{quad}} = \psi^{\text{quad}}$. This implies $t_{\varphi(x)} = t_{\psi(x)}$ for all $x \in A$. It follows from Cor. 5.10 that $\varphi = \psi$.

In particular, given a pair of abelian groups A, B and a homomorphism $\varphi: A \to B$, there is a commutative diagram

$$\begin{array}{c|c} \operatorname{Quad}(B,U) \xrightarrow{\simeq} \operatorname{Hom}(\hat{B}^{\operatorname{quad}},U) \\ \varphi & & & & \\ \varphi & & & & \\ \operatorname{Quad}(A,U) \xrightarrow{\simeq} \operatorname{Hom}(\hat{A}^{\operatorname{quad}},U) \end{array}$$

PROPOSITION 5.12. The group \hat{A}^{quad} , together with the quadratic map $A \rightarrow \hat{A}^{\text{quad}}, x \mapsto t_x$, also solves the following universal mapping problem:



For every quadratic map $q : A \to U$ with associated symmetric bilinear pairing $b_q : A \times A \to U$, there exists a unique homomorphism $\tilde{q} : \hat{A}^{\text{quad}} \to U$ such that the diagram above commutes, i.e. $\tilde{q} \circ t_{-,-} = b_q$.

PROOF. The map defined by $(x, y) \mapsto t_{x,y}$ is symmetric bilinear since in $\widehat{A}^{\text{quad}}$, the relations $t_{x,y} = t_{y,x}$ and $t_{x+y,z} = t_{x,z} t_{y,z}$ hold for all $x, y, z \in A$. The rest of the proof is similar to that of Prop. 5.8.

REMARK 1.11. The group \hat{A}^{quad} does not solve the following universal problem: for every symmetric bilinear pairing $b : A \times A \to U$, there exists a unique homomorphism $\tilde{q} : \hat{A}^{\text{quad}} \to U$ such that $\tilde{q} \circ t_{-,-} = b$. Indeed, if a homomorphism $\tilde{q} : \hat{A}^{\text{quad}} \to U$ exists, then it must specify $\tilde{q}(t_x)$ and not only $\tilde{q}(t_{x,y})$. In other words, the data of q(x) (for each $x \in A$), namely of the quadratic function, and not only $b_q(x, y)$ (for all $x, y \in A$), namely the associated symmetric bilinear pairing, is required. Not every symmetric bilinear pairing b determines uniquely a quadratic function q such that $b = b_q$. We shall return to this important feature a little later.

The next result is the "quadratic relation".

LEMMA 5.13. The following relation holds in \hat{A}^{quad} . For any finite sequence $x_1, \ldots, x_n \in A$,

(5.4)
$$t_{x_1+\dots+x_n} = \prod_{1 \leq i \leq n} t_{x_i} \prod_{1 \leq i < j \leq n} t_{x_i,x_j}.$$

PROOF. Induction on n from the relation $t_{x+y} = t_x t_y t_{x,y}$.

LEMMA 5.14. The following relation holds in \hat{A}^{quad} . For any integer n and $x \in A$,

(5.5)
$$t_{nx} = t_x^n t_{x,x}^{\frac{n(n-1)}{2}}.$$

PROOF. The identity for $n \in \mathbb{N}$ is the particular case $z_1 = \cdots = z_n$ in Lemma 5.13. Then $1 = t_{nx-nx} = t_{nx}t_{-nx}t_{-nx}nx = t_{nx}t_{-nx}t_{x,x}^{-n^2}$ so

$$t_{-nx} = t_{nx}^{-1} t_{x,x}^{n^2} = t_x^{-n} t_{x,x}^{-\frac{n(n-1)}{2}} t_{x,x}^{n^2} = t_x^{-n} t_{x,x}^{-\frac{n(n-1)}{2}}.$$

The identities above will be used several times in the sequel.

COROLLARY 5.15. The following properties holds in \widehat{A} :

(1) If $x \in A$ is a torsion element of order m, then $t_x \in \widehat{A}^{\text{quad}}$ is a torsion element of order dividing m (resp. dividing 2m) if m is odd (resp. if m is even).

(2) If $x, y \in A$ are torsion elements of order m and n respectively then $t_{x,y} \in \widehat{A}^{\text{quad}}$ is a torsion element of order dividing gcd(x, y).

(3) If A is torsion, then so is \hat{A}^{quad} .

PROOF. (1) Since $1 = t_{0,x} = t_{mx,x} = t_{x,x}^m$, $t_{x,x}$ has order dividing m. By Lemma 5.14, $t_x^k = t_{kx} t_{x,x}^{-\frac{k(k+1)}{2}}$. In particular, for k = 2m,

$$t_x^{2m} = t_{0,x} t_{x,x}^{-m(2m+1)} = 1$$

Hence the order of t_x divides 2m. If m is odd, then for k = m,

$$t_x^m = t_0 t_{x,x}^{-\frac{m(m+1)}{2}} = t_{x,x}^{-\frac{m(m+1)}{2}} = 1$$

since $\frac{m(m+1)}{2} = 0 \mod m$ if and only if m is odd. Hence the order of t_x divides m if m is odd.

(2) Since $t_{x,y}^m = t_{mx,y} = 1 = t_{x,ny} = t_{x,y}^n$, the order of $t_{x,y}$ divides both m and n, hence divides their gcd.

(3) follows from (1) and (2).

COROLLARY 5.16. The following properties holds in \widehat{A} :

(1) If $x \in A$ is a torsion element of order m and generates a direct summand of A, then $t_x \in \hat{A}^{\text{quad}}$ is a torsion element of order m (resp. 2m) if m is odd (resp. if m is even).

(2) If $x, y \in A$ are torsion elements of order m and n respectively generating direct summands of A such that $\mathbb{Z}x \cap \mathbb{Z}y = 0$ or $\mathbb{Z}x = \mathbb{Z}y$ then $t_{x,y} \in \widehat{A}^{\text{quad}}$ is a torsion element of order gcd(x, y).

PROOF. (1) Let $q : \mathbb{Z}x \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function such that $q(kx) = \frac{\beta k^2}{2m} \mod 1$ where $\beta = \begin{cases} 1 & \text{if } m = 0 \mod 2; \\ 2 & \text{if } m = 1 \mod 2. \end{cases}$ (Cf. example 1.10.) It is readily verified that the order of q(x) is m (resp. 2m) if m is odd (resp. if m is even). Moreover, q is nondegenerate, hence nonsingular on $\mathbb{Z}x$. Consider any extension q' on A. By the universal property of $\widetilde{A}^{\text{quad}}$ (Prop. ??), $\widetilde{q'}(t_x) = q'(x)$. Thus the order of t_x is a multiple of m (resp. 2m) if m is odd (resp. if m is even). By Corollary 5.15, the conclusion follows.

(2) Consider the case $\mathbb{Z}x \cap \mathbb{Z}y = 0$ first. Let $q : \mathbb{Z}x \oplus \mathbb{Z}y \to \mathbb{Q}/\mathbb{Z}$ be the quadratic function defined by $q(kx, ly) = \frac{kl}{\gcd(m,n)} \mod 1$. We have $b_q(x, y) = \frac{1}{\gcd(m,n)} \mod 1$. Extend q to a quadratic function q' to A. By the universal property of \hat{A}^{quad} (Prop. 5.8), $\tilde{q}'(t_{x,y}) = b_{q'}(x, y) = b_q(x, y)$. Thus the order of $t_{x,y}$ is a multiple of $\gcd(m, n)$. By Corollary 5.15, the conclusion follows.

In the remaining case, $\mathbb{Z}x = \mathbb{Z}y$ so there exists $k \in \mathbb{Z}$ such that y = kx with gcd(m, n) = 1. Consider the same quadratic functions q, q' introduced in the proof of (1). Then $b_q(x, kx) = \frac{\beta k}{m} \mod 1$ has order m. By the universal property of \widehat{A}^{quad} (Prop. 5.12), $\widetilde{q'}(t_{x,y}) = b_{q'}(x, y) = b_q(x, y) = b_q(x, kx)$. Hence the order of $t_{x,y}$ is a multiple of m. Conclude by applying again Corollary 5.15.

EXERCISE 1.2. Complete the proof of (1)-(2) above by justifying the existence of the claimed extensions of the quadratic functions on A. [Hint: use Lemma 3.3 and orthogonality.]

EXERCISE 1.3. Suppose that $(x_i)_{i \in I}$ is a minimal set of generators for A. Deduce a set of generators for \hat{A}^{quad} . Deduce that if $\operatorname{rank}(A) = k$ then $\operatorname{rank}(\hat{A}^{\text{quad}}) \geq \frac{k(k+1)}{2}$. Prove that the functor of Prop. 5.11 is not a full functor.

EXERCISE 1.4. Prove that $\hat{\mathbb{Z}}^{\text{quad}} \simeq \mathbb{Z} \times \mathbb{Z}$.

EXERCISE 1.5. Prove that

(5.6)
$$\widehat{\mathbb{Z}/n\mathbb{Z}}^{\text{quad}} \simeq \begin{cases} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \text{if } n \text{ is odd;} \\ \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \text{if } n \text{ is even.} \end{cases}$$

The symmetric tensor product $S^2(A)$ is defined as the quotient of $A \otimes A$ by the subgroup generated by $x \otimes y - y \otimes x$, $x, y \in A$. The symmetric tensor product $S^2(A)$, together with the symmetric bilinear map $A \times A \rightarrow$ $S^2(A), (x, y) \mapsto [x \otimes y]$, satisfies the following universal property: for every symmetric bilinear pairing $b : A \times A \to U$, there exists a unique homomorphism $\tilde{b} : S^2(A) \to U$ such that the following diagram is commutative:



In view of Prop. 5.12, the similarity with the symmetric tensor product is clear. The next result clarifies the relationship between the symmetric tensor product $S^2(A)$ and the group \hat{A}^{quad} .

PROPOSITION 5.17. There is a nonsplit short exact sequence

 $0 \to S^2(A) \to \hat{A}^{\text{quad}} \to A \to 0.$

PROOF. The assignment $t_{-,-}: (x, y) \to t_{x,y}$ from $A \times A \to \hat{A}^{\text{quad}}$ is a \mathbb{Z} -bilinear map, hence factors through $A \times A$. Since $t_{-,-}$ is symmetric, it factors further through $S^2(A)$. The corresponding map $j: S^2(A) \to \hat{A}^{\text{quad}}$ is injective.

It remains to identify the cokernel $\hat{A}^{\text{quad}}/j(S^2A)$. For this, note the identities

(5.7)
$$t_x t_y = t_{x+y} t_{x,y}^{-1} = t_{x+y} t_{-x,y}, \quad t_x t_y^{-1} = t_{x-y} t_{x,x-y}$$

It follows that $t_x t_y = t_{x+y}$ modulo $\operatorname{Im}(j)$ and $t_x t_y^{-1} = t_{x-y}$ modulo $\operatorname{Im}(j)$. Hence the map $p : \hat{A}^{\operatorname{quad}} \to A$ that assigns $p(t_{x,y}) = 0$ and $p(t_x) = x$ for all $x, y \in A$ is a well-defined epimorphism. Clearly Ker p contains the image $j(A \otimes A)$. Conversely, let $\omega \in \operatorname{Ker} p$. By the relations above, ω is a mixed product of elements of the (first) form $t_{x,y}^{\pm 1} = t_{\pm x,y}$ and elements of the (second) form $t_{z_j}^{\varepsilon_j}$, with $\varepsilon_j = \pm 1, j = 1, \ldots, r$. So $p(\omega) = p(t_{z_1}^{\varepsilon_1} \cdots t_{z_r}^{\varepsilon_r}) =$ $\varepsilon_1 z_1 + \cdots + \varepsilon_r z_r = 0$. We need to show that the product $t_{z_1}^{\varepsilon_1} \cdots t_{z_r}^{\varepsilon_r}$ can be rewritten as a product of elements of the first form only. This will be a consequence of the following lemma.

LEMMA 5.18. If $\varepsilon_1 z_1 + \cdots + \varepsilon_r z_r = 0$ then $t_{z_1}^{\varepsilon_1} \cdots t_{z_r}^{\varepsilon_r} = 1 \mod j(A \otimes A)$.

PROOF. For r = 1, $t_0 = 1$. For r = 2, this follows from the relations (5.7). For the general case, up to reordering, we may assume that for $1 \le i \le m \le r$, $\varepsilon_j = +1$ and for $m + 1 \le j \le r$, $\varepsilon_j = -1$. Set $u = z_1 + \cdots + z_m$ and $v = z_{m+1} + \cdots + z_r$. Applying twice the quadratic relation (5.4), we have

$$\prod_{\substack{1 \leq i \leq m}} t_{z_i} = t_u \prod_{\substack{1 \leq i < j \leq m}} t_{z_i, z_j}^{-1}$$
$$\prod_{\substack{m+1 \leq i \leq r}} t_{z_i}^{-1} = t_v^{-1} \prod_{\substack{m+1 \leq i < j \leq r}} t_{z_i, z_j}.$$

By the relation (5.7) above,

$$t_u t_v^{-1} = t_{u-v} t_{u-v,v} = t_0 t_{0,v} = 1 \cdot 1 = 1.$$

We deduce that

$$t_{z_1}^{\varepsilon_1}\cdots t_{z_r}^{\varepsilon_r} = \prod_{1\leqslant i\leqslant m} t_{z_i} \prod_{m+1\leqslant i\leqslant r} t_{z_i}^{-1} = \prod_{1\leqslant i< j\leqslant m} t_{z_i,z_j}^{-1} \prod_{m+1\leqslant i< j\leqslant r} t_{z_i,z_j}.$$

In particular, $t_{z_1}^{\varepsilon_1} \cdots t_{z_r}^{\varepsilon_r} \in j(A \otimes A)$ as desired.

Finally, it is easy to see that the epimorphism $p: \hat{A}^{\text{quad}} \to A$ has no section that is a homomorphism.

For the rest of this paragraph, A is a finite Abelian group and the quadratic functions take values in $U = \mathbb{Q}/\mathbb{Z}$.

LEMMA 5.19. Let $q: A \to U$ be a quadratic function. For any $x \in A$ and $n \in \mathbb{Z}$,

(5.8)
$$q(n \ x) = n \ q(x) + \frac{n(n-1)}{2} \ b_q(x,x).$$

PROOF. Apply \tilde{q} to the relation (5.5) of Lemma 5.14.

LEMMA 5.20. If x has odd (resp. even) order n in A, then the order of q(x) divides n (resp. divides 2n) in \mathbb{Q}/\mathbb{Z} .

PROOF. Apply \tilde{q} to the first assertion of Cor. 5.15.

DEFINITION 1.19. The group A^{quad} is defined as the quotient of \hat{A}^{quad} by the relation $t_{x,x} = t_x^2$, for all $x \in A$.

The group A^{quad} satisfies also a universal property for homogeneous quadratic functions.

PROPOSITION 5.21. The group A^{quad} , together with the homogeneous quadratic function $t: A \to A^{\text{quad}}$, $x \mapsto t_x$, solves the following universal mapping problem:



For every homogeneous quadratic function $q: A \to U$, there exists a unique homomorphism $\tilde{q}: A^{\text{quad}} \to U$ such that the diagram above commutes.

PROOF. The proof is similar to that of Prop. 5.8.

COROLLARY 5.22. The isomorphism $q \mapsto \tilde{q}$, $\text{Quad}(A, U) \to \text{Hom}(\hat{A}^{\text{quad}}, U)$ restricts to an isomorphism $\text{Quad}^0(A, U) \to \text{Hom}(A^{\text{quad}}, U)$ making the following diagram commutative:

$$\begin{array}{c|c} \operatorname{Quad}^{0}(A,U) \longrightarrow \operatorname{Hom}(A^{\operatorname{quad}},U) \\ & & & & \\ \operatorname{incl} & & & & \\ \operatorname{Quad}(A,U) \longrightarrow \operatorname{Hom}(\hat{A}^{\operatorname{quad}},U), \end{array}$$

where $p: \hat{A}^{\text{quad}} \to A^{\text{quad}}$ denotes the canonical projection.

EXERCISE 1.6. An isomorphism $\varphi : A \to A'$ between two quadratic functions $q : A \to U$ and $q' : A \to U$ such that $q = q' \circ \varphi$ induces an isomorphism $\phi : \hat{A}^{\text{quad}} \to \hat{A}'^{\text{quad}}$ such that $\tilde{q} = \tilde{q}' \circ \phi$. A similar statement is true for homogeneous quadratic functions and the group A^{quad} . Hint: complete the diagram



EXERCISE 1.7. Prove that

(5.9)
$$(\mathbb{Z}/n\mathbb{Z})^{\text{quad}} \simeq \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } n \text{ is odd;} \\ \mathbb{Z}/2n\mathbb{Z} & \text{if } n \text{ is even.} \end{cases}$$

The next proposition sums up the various relationships between the groups \hat{A}^{quad} , A^{quad} and symmetrized tensor products.

PROPOSITION 5.23. The following diagram has exact rows and columns and is commutative:



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PROOF. The nontrivial groups of the top row are all seen to be subgroups of the corresponding groups below in the middle row, so all columns are exact. The nontrivial maps of the top row are the restrictions of the corresponding nontrivial maps below in the middle row respectively. From this, the exactness of the top row, as well as that of the lower row, follows. The commutativity of the top two squares also follows, hence the lower two squares are commutative as well. The definition and exactness of the middle row is given by Prop. 5.17.

REMARK 1.12. None of the short exact sequences in Prop. 5.23 is split.

5.4. Tame and nontame quadratic functions. The following definition is specific to the setting of finite Abelian groups.

DEFINITION 1.20. A quadratic function $q: A \to \mathbb{Q}/\mathbb{Z}$ such that $q(A^{\perp}) = 0$ will be called *tame*.

The terminology is due to R. Taylor [92]; one finds also the term "non defective" (find references) in the litterature. Obviously a nondegenerate quadratic function is tame; but there are others. The following two lemmas determine tame quadratic functions.

LEMMA 5.24. A nontame quadratic function $q : A \to \mathbb{Q}/\mathbb{Z}$ determines a nontrivial homomorphism $q|_{A^{\perp}}$ of order dividing the exponent of A^{\perp} . If moreover q is homogeneous, then $q|_{A^{\perp}}$ has order two. In particular, if A^{\perp} has odd order, then any quadratic form on A is tame.

PROOF. Let $x, y \in A^{\perp}$. Then $q(x + y) = q(x) + q(y) + b_q(x, y) = q(x) + q(y) + 0 = q(x) + q(y)$. Furthermore, $q(nx) = nq(x) + \frac{n(n-1)}{2}b_q(x, x) = nq(x)$ for $n \in \mathbb{Z}$. Hence $q|_{A^{\perp}}$ is a nontrivial homomorphism of order dividing the exponent of A^{\perp} . Since q is moreover homogeneous, q(x) = q(-x) = -q(x) so 2q(x) = 0.

PROPOSITION 5.25. A quadratic function $q : A \to \mathbb{Q}/\mathbb{Z}$ is nontame if and only if it has a cyclic orthogonal summand B such that $q|_B$ is a nontrivial homomorphism.

PROOF. If $q|_B$ is a nontrivial homomorphism then $\hat{b}_q(B) = 0$, so $B \subseteq A^{\perp}$, so q is nontame. Conversely, suppose that q is nontame. Then $q|_{A^{\perp}}$ is a nontrivial homomorphism. Without loss of generality, we may assume that A is a p-group. Let $x \in A^{\perp}$ such that q(x) has maximal order p^n in \mathbb{Q}/\mathbb{Z} . Then $q(a \ x) = a \ q(x)$ for all $a \in \mathbb{Z}$. To complete the proof, we have to show that x generates an orthogonal summand in A. First we claim that x generates a direct summand in A. Let x_1, \ldots, x_r be a minimal complete system of generators for A so that $x = a_1 \ x_1 + a_2 \ x_2 + \cdots + a_r x_r$, where $a_1, \ldots, a_r \in \mathbb{Z}$. Note that $x \notin pA$ for otherwise, q(x) would not have maximal order. Hence at least one a_j , say a_1 , is coprime with p. It follows that $x = y_1 + a_2 \ x_2 + \cdots + a_r \ x_r$. The subgroup H generated by x_2, \ldots, x_r in A verifies

$$(5.10) A = \mathbb{Z} \ x \oplus H.$$
Next, since $x \in A^{\perp}$, x is orthogonal to H, so the decomposition (??) is orthogonal.

EXAMPLE 1.16. Let p be a prime number and $1 \leq k \leq n$ be two positive integers. The homomorphism $NT_{p,n}^k$ on $\mathbb{Z}/p^n\mathbb{Z}$ defined by

(5.11)
$$\operatorname{NT}_{p,n}^{k}(1 \mod p^{n}) = \frac{1}{p^{k}} \mod 1,$$

defines a nontame quadratic function. Any nontame quadratic function on a cyclic *p*-group is isomorphic to some $NT_{p,n}^k$

COROLLARY 5.26. A quadratic function $q: A \to \mathbb{Q}/\mathbb{Z}$ is nontame if and only if it has a cyclic orthogonal summand $NT_{p,n}^k$ for some prime p and integers $1 \leq k \leq n.$

REMARK 1.13. Any nontrivial group A carries a nontame quadratic function on it.

Because of homogeneousness, the description of nontame quadratic forms is simpler.

EXAMPLE 1.17. Let $n \ge 1$. There is exactly one quadratic form NT_n on $\mathbb{Z}/2^n\mathbb{Z}$ that is nontame. It is given by

(5.12)
$$\operatorname{NT}_n(1 \mod 2^n) = \frac{1}{2} \mod 1.$$

Note that $NT_n = NT_{2,n}^1$.

COROLLARY 5.27. A homogeneous quadratic function is nontame if and only it has a orthogonal summand NT_n , $n \ge 1$.

PROOF. In view of Prop. 5.25 and Cor. 5.26, it suffices to observe that the only nontrivial order 2 homomorphisms on cyclic 2-groups are precisely the ones described in Example 1.17.

PROPOSITION 5.28. Let $q: A \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function. There is a unique orthogonal decomposition of quadratic functions

$$(5.13) (A,q) = (A_t,q_t) \oplus (A_{nt},q_{nt})$$

such that

- (1) $q_t : A_t \to \mathbb{Q}/\mathbb{Z}$ is tame; (2) $A_{nt} \subseteq A^{\perp};$
- (3) $q_{\rm nt} = 0$ if and only if q is tame;
- (4) $(A_{\rm nt}, q_{\rm nt})$ is an orthogonal sum of nontame cyclic summands.

Furthermore, there are nonnegative integers $a_{p,n}^k \ge 0$, where $1 \le k \le n$ and p runs over the primes, such that

(5.14)
$$(A_{\mathrm{nt}}, q_{\mathrm{nt}}) = \bigoplus_{p} \bigoplus_{n} \bigoplus_{1 \le k \le n} a_{p,n}^{k} \operatorname{NT}_{p,n}^{k}$$

If q is homogeneous, then there exist $1 \leq n_1 < \cdots < n_r$ such that

(5.15)
$$(A_{\mathrm{nt}}, q_{\mathrm{nt}}) = a_1 \operatorname{NT}_{n_1} \oplus a_2 \operatorname{NT}_{n_2} \cdots \oplus a_r \operatorname{NT}_{n_r}$$

PROOF. We make an inductive use of the proof of the Prop. 5.25, splitting each time an orthogonal summand of maximal order. The process is finite and stops when the remaining summand is nontame. We obtain the desired decomposition (5.13) with the prescribed collection (5.14) of nontame cyclic summands if q is homogeneous (by Prop. ??). By construction, q splits on the annihilator as

$$q|_{A^{\perp}} = q|_{A^{\perp} \cap A_{\mathrm{t}}} \oplus q|_{A^{\perp} \cap A_{\mathrm{nt}}} = 0 \oplus q|_{A_{\mathrm{nt}}}$$

since $q(A^{\perp} \cap A_t) = 0$ and $A_{nt} \subseteq A^{\perp}$. An isomorphism between quadratic functions $q: A \to \mathbb{Q}/\mathbb{Z}$ and $q': A' \to \mathbb{Q}/\mathbb{Z}$ sends isometrically A^{\perp} onto A'^{\perp} , so the orthogonal summands are preserved. In particular it preserves each nontame cyclic orthogonal summand of the decomposition. The statement about uniqueness follows.

DEFINITION 1.21. In the decomposition (5.13) of Prop. 5.28, the tame quadratic function (A_t, q_t) is called the *tamed* quadratic function or the *tame part* of (A, q). The quadratic function (A_{nt}, q_{nt}) is called the *absolute nontame* quadratic function or the *absolute nontame part* of (A, q). We shall also say that a quadratic function is *absolutely nontame* or *linear* if it is an orthogonal sum of nontame cyclic summands.

REMARK 1.14. The absolute nontame part of a quadratic function is linear. However, a linear term may occur in a (nonorthogonal) decomposition of a nontame quadratic form. For instance, the quadratic form on $\mathbb{Z}/4\mathbb{Z}$ defined by

$$q(x) = \frac{x^2}{4} + \frac{x}{2} = -\frac{x^2}{4} \mod 1$$

is tame. An absolute nontame quadratic function is an orthogonal sum of linear summands.

COROLLARY 5.29. Two quadratic functions are isomorphic if and only if their tame parts are isomorphic and their absolute nontame parts are isomorphic. In particular, the integers $a_{p,n}^k$ are invariants of the isomorphism class of the quadratic function.

5.5. Quadratic functions and associated linking pairings. The set Quad(A) of all quadratic functions (including degenerate quadratic functions) defined on A is an additive group for the operation defined by

$$(q + q')(x) = q(x) + q'(x), \ x \in A.$$

(This operation is not to be confused with the orthogonal sum.) The map $q \mapsto b_q$ defines a projection onto the additive group Link(A) of all linking pairings defined on A. Note that Quad(A) contains as a subgroup the group Quad⁰(A) of all homogeneous quadratic functions. These groups fit into the following diagram with exact rows

We shall use repeatedly the following basic result.

PROPOSITION 5.30. The following assertions are equivalent:

- (1) A has odd order.
- (2) Multiplication by 2 in A is an automorphism.
- (3) The second row of (5.16) is split.
- (4) The first row of (5.16) is split.

PROOF. (1) \implies (2): Since the finite homomorphism $A \to A, x \mapsto 2x$ has trivial kernel, it must be an automorphism.

$$(2) \Longrightarrow (3)$$
: The map $s : \text{Link}(A) \to \text{Quad}^0(A), \lambda \mapsto s(\lambda)$ defined by

$$s(\lambda)(x) = \frac{1}{2}\lambda(x,x), \ x \in A$$

is a section.

 $(3) \Longrightarrow (4)$: Any section $\text{Link}(A) \to \text{Quad}^0(A)$ composed with the inclusion $\text{Quad}^0(A) \subset \text{Quad}(A)$ is a section.

 $(4) \Longrightarrow (1)$: Assume that A has even order. We show that there exists no section for the first row of (5.16). Let $x \in A$ of order 2^k with k maximal (i.e., the 2-valuation of the order of x is maximal among those of all elements of A). Then x generates a direct summand $\langle x \rangle$ of A, say $A = \langle x \rangle \oplus B$. Let $\lambda : A \times A \to \mathbb{Q}/\mathbb{Z}$ be the (degenerate) linking pairing defined by

$$\lambda(mx, nx) = \frac{mn}{2^k} \mod 1, \text{ for } m, n \in \mathbb{Z} \text{ and } \lambda(x, B) = \lambda(B, B) = 0.$$

Since λ is the orthogonal sum of a cyclic linking pairing and a trivial linking pairing, the decomposition $A = \langle x \rangle \oplus B$ is an orthogonal decomposition. Suppose by contradiction that there does exist a section $s : \text{Link}(A) \rightarrow \text{Quad}(A)$ splitting the first row (5.16). Then there exists $h \in \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ such that

$$s(\lambda) = q + h$$

where

$$q(m \ x+b) = \frac{m^2}{2^{k+1}} \mod 1, \text{ for any } m \in \mathbb{Z}, \ b \in B.$$

But

$$0 = s(0) = s(2^k \lambda) = 2^k s(\lambda) = 2^k q + 2^k h$$

Hence $2^k h = -2^k q \neq 0$ and $2^{k+1} h = -2^{k+1} q = 0$. It follows that h has order 2^{k+1} exactly. This contradicts that the 2-valuation of the order of x is maximal.

In the remainder of this section, all quadratic functions are assumed to be nondegenerate.

The set \mathfrak{MQ} of isomorphism classes of quadratic functions has a monoid structure for the orthogonal sum. The set \mathfrak{MQ}^0 of isomorphism classes of homogeneous quadratic functions is a submonoid of \mathfrak{MQ} .

The decomposition results for quadratic functions on a finite abelian group parallel those for linking pairings. In particular, there are decompositions

$$\mathfrak{MQ} = \bigoplus_p \mathfrak{MQ}_p, \quad \mathfrak{MQ}^0 = \bigoplus_p \mathfrak{MQ}_p^0$$

where \mathfrak{MQ}_p and \mathfrak{MQ}_p^0 are the submonoids of quadratic functions and homogeneous quadratic functions on finite *p*-groups.

LEMMA 5.31. The map $q \mapsto b_q$ induces a surjective monoid homomorphism $\mathfrak{MQ}_p^0 \to \mathfrak{M}_p^+$ for any prime p. For $p \neq 2$, it is an isomorphism.

PROOF. The first statement follows from the remarks at the end of §??. The second statement follows from Prop. 5.30.

5.6. Tensor product. The following construction of quadratic forms will appear frequently in topological applications.

LEMMA 5.32. Let $q : A \to \mathbb{Q}/\mathbb{Z}$ be a quadratic form and let $b : L \times L \to \mathbb{Z}$ be a symmetric bilinear pairing on a free abelian group L. There is a unique quadratic form $q \otimes b$ on $A \otimes L$ determined by

$$(q \otimes b)(x \otimes y) = q(x)b(y, y), x \in A, y \in L.$$

PROOF. Let $\{y_j\}_j$ be a \mathbb{Z} -basis for L. Then

$$(q \otimes b)(x_j \otimes y_j) = \sum_j q(x_j)b(y_j, y_j) + \sum_{j < k} b_q(x_j, x_k)b(y_j, y_k).$$

REMARK 1.15. The fact that q is a quadratic form (i.e. is homogeneous) is crucial to the fact that $q \otimes b$ is well-defined. This fact was overlooked in my article [15, Lemma 1].

DEFINITION 1.22. The quadratic form $q \otimes b$ determined by Lemma 5.32 is the *tensor product* of q and b.

REMARK 1.16. The nondegeneracy is not preserved by tensor product: the tensor product of a nondegenerate homogeneous quadratic function and a nondegenerate bilinear lattice may be degenerate. For instance, consider the tensor product of the nondegenerate quadratic form defined by $q: x \mapsto x^2/4 \mod 1$, $x \in \mathbb{Z}/2$ and the nondegenerate bilinear lattice defined by $b: (x, y) \mapsto 2xy, x \in \mathbb{Z}$. It is the quadratic form $q \otimes b$ defined on $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}$ by

$$(q \otimes b)(x \mod 2) = \frac{x^2}{2} \mod 1.$$

Clearly $q \otimes b$ is degenerate, in fact nontame.

CHAPTER 2

The discriminant construction

Theee discriminant construction is a classical construction (that dates back at least to Puppe and Burger) that measures the defect of unimodularity of a lattice. The defect consists of a linking group. The construction has nice properties: it is natural, surjective (any linking group can be realized in this fashion) and preserves orthogonal sums. The behavior on tensor products is more involved and is the main interest of this chapter, in view of the reciprocity formula.

1. Lattices

A *lattice* is a finitely generated free abelian group.

DEFINITION 2.1. An ε -symmetric bilinear lattice is an ε -symmetric bilinear form $f: V \times V \to \mathbb{Z}$ on a lattice V.

REMARK 2.1. We will use the short term ε -lattice, or even lattice, if the ε -symmetric bilinear pairing is implicit.

A lattice V generates over \mathbb{Q} a vector space $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$. An ε -lattice (V, f) extends to an ε -symmetric bilinear form $f_{\mathbb{Q}} : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$. The form $f_{\mathbb{Q}}$ is nonsingular if and only if f is nondegenerate. An ε -lattice (V, f) is said unimodular if f is nonsingular. The dual lattice is defined as

$$V^{\sharp} = \{ x \in V_{\mathbb{Q}} \mid f_{\mathbb{Q}}(x, V) \subseteq \mathbb{Z} \}.$$

A subgroup of a lattice V is finitely generated and free abelian and is called a *sublattice* of V. More generally, given a sublattice $S \subseteq V$, the dual lattice is defined as

$$S^{\sharp} = \{ x \in V_{\mathbb{Q}} \mid f_{\mathbb{Q}}(x, S) \subseteq \mathbb{Z} \}.$$

The map $\widehat{f}_{\mathbb{Q}} : x \mapsto f_{\mathbb{Q}}(x, -)$ restricts to a map between S^{\sharp} and $S^* = \text{Hom}_{\mathbb{Z}}(S, \mathbb{Z})$. This map is an isomorphism if f is nondegenerate. It follows that $S^{\sharp\sharp} = S$ for any sublattice S of V if and only if f is nondegenerate. One observes that if $S, T \subseteq V$ are sublattices, then

(1.1)
$$S \subseteq T \Rightarrow T^{\sharp} \subseteq S^{\sharp}, \quad (S+T)^{\sharp} = S^{\sharp} \cap T^{\sharp}.$$

A sublattice $S \subseteq V$ is *primitive* if the quotient group V/S is a lattice. Let (V, f) be a bilinear lattice.

EXAMPLE 2.1. The annihilator Ker $\hat{f} \subseteq V$ is a primitive sublattice of V. This is equivalent to

LEMMA 1.1. The quotient $\overline{V} = V/\text{Ker } \widehat{f}$ is a lattice.

PROOF. Clearly \overline{V} is finitely generated. Let $[x] = x + \text{Ker } \hat{f} \in \overline{V}$ such that n[x] = 0. Then $n \in Ker \hat{f}$. Thus

$$0 = f(n \ x, V) = n \ f(x, V).$$

Since $f(x, V) \subseteq \mathbb{Z}$ and \mathbb{Z} has no torsion, it follows that f(x, V) = 0. Hence $x \in \text{Ker } \hat{f}$ and [x] = 0.

For a given sublattice $S \subseteq V$, there is smallest primitive sublattice $\tilde{S} \subseteq V$ containing S. This lattice is called the *primitive hull* of S. The primitive hull of S has the same rank as S. The following observation is useful.

REMARK 2.2. An isomorphism $\varphi: S \to S'$ between sublattices of V and V' does not necessarily extend to an isomorphism $\tilde{\varphi}: \tilde{S} \to \tilde{S}'$ between their respective primitive hulls. For instance, take $V = V' = \mathbb{Z} \oplus \mathbb{Z}$, $S = \mathbb{Z} \oplus 0$ and $S' = 2\mathbb{Z} \oplus 0$. Clearly the map $x \mapsto 2x$ defines an isomorphism between S and S'. However, this map does not extend to an isomorphism between $\tilde{S} = S = \mathbb{Z}$ and $\tilde{S}' = \mathbb{Z}$.

LEMMA 1.2. An isomorphism $\varphi: S \to S'$ between primitive sublattices of V extends to an automorphism of V.

PROOF. Since V/S is free, the short exact sequence

 $0 \to S \to V \to V/S \to 0$

splits. Choose a section $s: V/S \to V$ so that the map

$$\psi: S \oplus V/S \to V, \ (x, y) \mapsto (x, s(y))$$

is an isomorphism. Similarly there is a section $s': V/S' \to V$ such that $\psi': (x, y) \mapsto (x, s'(y))$ is an isomorphism from $S' \oplus V/S'$ onto V.

Since S and S' are isomorphic primitive sublattices, there is an isomorphism $g:V/S\simeq V/S'$ of lattices.

Then $\psi' \circ (f \oplus g) \circ \psi^{-1}$ is an automorphism of V extending φ .

COROLLARY 1.3. An isomorphism $\varphi : S \to S'$ between sublattices of V extends to an automorphism of V if and only if it extends to an isomorphism $\tilde{S} \to \tilde{S}'$ between their primitive hull.

PROOF. In one direction, use Lemma 1.2. For the converse, let $\tilde{\varphi}: V \to V$ be the automorphism extending φ . We have to show that $\tilde{\varphi}(\tilde{S}) = \tilde{S}'$. An element y lies in \tilde{S} if and only if there is some $n \in \mathbb{Z}$ such that $ny \in S$. Let $y \in \tilde{S}$ such that $x = ny \in S$. We have $\varphi(x) = \tilde{\varphi}(ny) = n\varphi(y) \in S'$. Thus $\varphi(y) \in \tilde{S}'$. It follows that $\tilde{\varphi}(\tilde{S}) \subseteq \tilde{S}'$. The reverse inclusion is proved similarly using φ^{-1} .

Let G be a finitely generated abelian group. The quotient group FG = G/Tors G is a lattice. Let $S \subset FG$ be a lattice. A partial section $s: S \to G$ (that is, a map $s: S \to FG$ such that $p \circ s|_S = \text{id}_S$) does not necessarily extend to a full section $FG \to G$.

LEMMA 1.4. If S is primitive, then any partial section $s: S \to G$ extends to a section $s: S \to G$.

PROOF. Since V/S is free, the short exact sequence $0 \to S \to V \to V/S \to 0$ gives rise to an exact sequence

$$0 \longrightarrow \operatorname{Hom}(V/S, G) \longrightarrow \operatorname{Hom}(V, G) \longrightarrow \operatorname{Hom}(S, G) \longrightarrow 0.$$

2. Discriminant ε -linking pairings

An ε -lattice (V, f) is unimodular if f is nonsingular. It follows from the previous paragraph that (V, f) is unimodular if and only if $V^{\sharp} = V$. We are interested in studying the failure of f to be unimodular. A natural invariant is provided by the following

DEFINITION 2.2. To an ε -lattice (V, f), one associates an ε -linking pairing, called the *discriminant pairing*, $\lambda_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$ by the formula:

(2.1)
$$G_f = V^{\sharp}/V, \quad \lambda_f([x], [y]) = f_{\mathbb{Q}}(x, y) \mod 1.$$

The discriminant pairing (G_f, λ_f) is symmetric (resp. antisymmetric, resp. symplectic) if and only if (V, f) is symmetric (resp. antisymmetric, resp. symplectic). The discriminant construction arises from a particular class of free resolutions of length 1.

LEMMA 2.1. (G_f, λ_f) is nonsingular if and only if (V, f) is nondegenerate.

A basic result asserts that almost any nondegenerate ε -linking pairing can be produced by this construction [97, Theorem (6)]:

THEOREM 2.2. The assignment $(V, f) \mapsto (G_f, \lambda_f)$ is surjective onto the monoid of nondegenerate symmetric (resp. symplectic) linking pairings on finite abelian groups.

As an example, any unimodular lattice $(V^{\sharp} = V)$ yields the trivial linking pairing. Clearly the discriminant construction preserves (orthogonal) sum. It follows from these two observations that the discriminant pairing is unaffected by adding orthogonal summands of unimodular lattices. A converse is known since the work of Puppe. To state it in our setting, it is convenient to introduce some definitions about maps between lattices.

A bilinear lattice map between two bilinear lattices (V, f) and (W, g) is a map $\alpha : V \to W$ such that $g(\alpha(x), \alpha(y)) = f(x, y)$ for all $x, y \in V$. This is also denoted $\alpha^*g = f$ in the sequel. If α is injective, then we say that α is an embedding of bilinear lattices. If α is bijective, then α is an isomorphism of bilinear lattices. Two bilinear lattices (V, f) and (W, g)are stably equivalent if there exist unimodular bilinear lattices (U, h) and (U', h') such that $(V, f) \oplus (U, h)$ and $(W, g) \oplus (U', h')$ are isomorphic bilinear lattices. Any bilinear lattice map α extends in a unique fashion to a map $\alpha_{\mathbb{Q}} : V_{\mathbb{Q}} \to W_{\mathbb{Q}}$ and thus restricts to a map $V^{\sharp} \to W^{\sharp}$ and therefore induces

a map $[\alpha] : G_f = V^{\sharp}/V \to W^{\sharp}/W = G_g$. It follows that a stable equivalence induces an isomorphism on the induced discriminant linking pairings. The converse is also true:

THEOREM 2.3. Two nondegenerate linking pairings are isomorphic if and only if they lift to stably isomorphic bilinear lattices.

For a proof, see e.g., [23].

Our goal consists in recovering the product of two linking pairings from the discriminant of their lattices.

Let (V, f) and (W, g) be nondegenerate bilinear lattices. Set $Z = V \otimes W$ and define a (symmetric nondegenerate) bilinear pairing $f \otimes g : Z \times Z \to \mathbb{Z}$ by

$$(f \otimes g)(x \otimes y, x' \otimes y') = f(x, x')g(y, y')$$
 for $x, x' \in V, y, y' \in W$.

LEMMA 2.4. There is a natural isomorphism $V^{\sharp} \otimes W^{\sharp} \to Z^{\sharp}$.

There are also natural inclusion maps $V^{\sharp} \otimes W \to (V \otimes W)^{\sharp}$ and $V \otimes W^{\sharp} \to (V \otimes W)^{\sharp}$ (where the dual lattice of the target space refers to the bilinear pairing $f \otimes g$) which we shall use freely without further notice. In particular, we verify directly the fact that

(2.2)
$$(f_{\mathbb{Q}} \otimes g_{\mathbb{Q}})(V^{\sharp} \otimes W, V \otimes W^{\sharp}) \subseteq \mathbb{Z}.$$

More precisely:

LEMMA 2.5.

(2.3)
$$(V^{\sharp} \otimes W)^{\sharp} = V \otimes W^{\sharp} \text{ and } (V \otimes W^{\sharp})^{\sharp} = V^{\sharp} \otimes W.$$

The inclusion $V \otimes W^{\sharp} \subseteq (V^{\sharp} \otimes W)^{\sharp}$ is just the equality (2.2). The lemma asserts that this is an equality.

PROOF. It suffices to prove the first equality since one deduces the second one by using the fact that $Z^{\sharp\sharp} = Z$ (since $f \otimes g$ is nondegenerate). The desired equality will result from the following commutative diagram:



The top horizontal arrow is the natural inclusion. The vertical arrows are the tensor product of adjoint maps and the adjoint map of the tensor product of pairings respectively (and they can be identified once $V_{\mathbb{Q}} \otimes W_{\mathbb{Q}}$ is identified to $(V \otimes W)_{\mathbb{Q}}$). We claim that the vertical arrows are bijective maps. Since the map adjoint to $f_{\mathbb{Q}} \otimes g_{\mathbb{Q}}$ is bijective, it is sufficient to check that

$$(\widehat{f}_{\mathbb{Q}} \otimes \widehat{g}_{\mathbb{Q}})(V \otimes W^{\sharp}) = \operatorname{Hom}_{\mathbb{Z}}(V^{\sharp}, \mathbb{Z}) \otimes \operatorname{Hom}_{\mathbb{Z}}(W, \mathbb{Z})$$

and

$$\widehat{f_{\mathbb{Q}} \otimes g_{\mathbb{Q}}}((V^{\sharp} \otimes W)^{\sharp}) = \operatorname{Hom}_{\mathbb{Z}}(V^{\sharp} \otimes W, \mathbb{Z}).$$

Both identities follow from the nondegeneracy of $f_{\mathbb{Q}}$ and $g_{\mathbb{Q}}$.

We now consider the linking pairing

$$\lambda_{f\otimes g}: G_{f\otimes g} \times G_{f\otimes g} \to \mathbb{Q}/\mathbb{Z}.$$

The natural inclusion map $V^{\sharp} \otimes W \to (V \otimes W)^{\sharp}$ induces a homomorphism $j_f: G_f \otimes W \to G_{f \otimes g}$ by

$$j_f(x \pmod{V} \otimes y) = x \otimes y \pmod{Z}$$

where $x \in V^{\sharp}, y \in W$. Similarly, define a homomorphism $j_g : V \otimes G_g \to G_{f \otimes g}$ by

$$j_g(x \otimes y \pmod{W}) = x \otimes y \pmod{Z}$$

where $x \in V, y \in W^{\sharp}$.

LEMMA 2.6. We have

$$\lambda_{f\otimes g}\circ j_f^{\otimes 2}=\lambda_f\otimes g$$

and

$$\lambda_{f\otimes g}\circ j_g^{\otimes 2}=f\otimes \lambda_g$$

The following observation is a consequence of $f,\,g$ being nondegenerate.

LEMMA 2.7. The maps j_f and j_g are injective.

Set $A = j_f(G_f \otimes W) \subseteq G_{f \otimes g}$ and $B = j_g(V \otimes G_g) \subseteq G_{f \otimes g}$. LEMMA 2.8. The subgroups A and B are mutually orthogonal in $G_{f \otimes g}$: $A^{\perp} = B$.

PROOF. Consequence of definitions and (2.3).

Let $H = A \cap A^{\perp}$. We record the following consequence:

COROLLARY 2.9.
$$\lambda_{f\otimes g}|_{H\times H} = 0.$$

Assume that f and g are both nondegenerate. We now describe H in more details.

LEMMA 2.10. There are exact sequences

$$0 \longrightarrow V \otimes G_g \xrightarrow{j_g} G_{f \otimes g} \longrightarrow G_f \otimes W^{\sharp} \longrightarrow 0$$

and

$$0 \longrightarrow G_f \otimes W \xrightarrow{j_f} G_{f \otimes g} \longrightarrow V^{\sharp} \otimes G_g \longrightarrow 0.$$

PROOF. Let us identify Coker j_f :

$$\operatorname{Coker} j_f = \frac{G_{f \otimes g}}{j_f(V \otimes G_g)} = \frac{\frac{Z^{\sharp}}{Z}}{\frac{V^{\sharp} \otimes W}{Z}} \simeq \frac{Z^{\sharp}}{V^{\sharp} \otimes W} \simeq \frac{V^{\sharp} \otimes W^{\sharp}}{V^{\sharp} \otimes W} \simeq V^{\sharp} \otimes \frac{W^{\sharp}}{W} = V^{\sharp} \otimes G_g.$$

The identification of Coker j_g is similar.

LEMMA 2.11. There is a natural isomorphism $H^* \simeq G_f \otimes G_g$ and a short exact sequence

$$0 \longrightarrow H^{\perp} \stackrel{j}{\longrightarrow} G_{f \otimes g} \longrightarrow G_{f} \otimes G_{g} \longrightarrow 0.$$

PROOF. On the one hand, there is a short exact sequence

$$0 \to H^{\perp} \to G_{f \otimes g} \to H^* \to 0$$

On the other hand, $H^{\perp} = (A \cap A^{\perp})^{\perp} = A + A^{\perp}$. Thus

$$H^* = G_{f \otimes g} / (A + A^{\perp}).$$

There remains to see that the latter group is isomorphic to $G_f\otimes G_g.$ There is a natural epimorphism

$$G_{f\otimes g} \to G_f \otimes G_g.$$

Indeed, this map can be defined in two ways

$$p_2 \circ p_1 = q_2 \circ q_1$$

as the following commutative diagram with exact rows and columns indicates:



It follows that the kernel of the epimorphism is

$$\begin{split} &\operatorname{Ker}(p_2 \circ p_1) = p_1^{-1}(j_g(V \otimes G_g)) = j_f(G_f \otimes W) + j_g(V \otimes G_g) = q_1^{-1}(j_f(G_f \otimes W)) = \operatorname{Ker}(q_2 \circ q_1). \end{split}$$
 Thus the quotient map

(2.4)
$$\psi: G_{f \otimes g} / (A + A^{\perp}) \xrightarrow{\simeq} G_f \otimes G_g$$

is an isomorphism.

Remark. The isomorphism between $G_f \otimes G_g$ and H^* is the composition

$$G_f \otimes G_g \xrightarrow{\psi^{-1}} G_{f \otimes g} / (A + A^{\perp}) \xrightarrow{l} H^*,$$

where the isomorphism on the right is

$$l: x \mod (V \otimes W) \mapsto \lambda_{f \otimes q}(x, -)|_H.$$

Here is an alternative argument to show that $G_f \otimes G_g$ and $G_{f \otimes g}/(A + A^{\perp})$ are isomorphic. Define a natural map

$$G_f \otimes G_g \to G_{f \otimes g} / (A + A^{\perp})$$

by

$$(x \bmod V) \otimes (y \bmod W) \mapsto [(x \otimes y) \bmod (V \otimes W)]$$

where [-] denotes the element in $G_{f\otimes g}$ considered modulo $A + A^{\perp}$. It follows also from $A = \text{Im } j_f$ and $A^{\perp} = \text{Im } j_g$ that this map is injective. (Suppose that $u \in G_f \otimes G_g$ is sent to $0 \in G_{f\otimes g}/(A + A^{\perp})$. Then the image of u is represented by a sum of elements in $A + A^{\perp}$. Since $A = \text{Im } j_f$ and $A^{\perp} = \text{Im } j_g$, all these elements are of the form $(x \otimes y) \mod (V \otimes W)$ where either $x \in V$ or $y \in W$. Therefore $u = 0 \in G_f \otimes G_g$.) Surjectivity also follows from the definitions. It is easily seen to be ψ^{-1} .

LEMMA 2.12. The map $j_f: G_f \otimes W \to G_{f \otimes g}$ restricts to an isomorphism

$$j_f|_{\operatorname{Ker}(\widehat{\lambda_f}\otimes\widehat{g})}:\operatorname{Ker}(\widehat{\lambda_f}\otimes\widehat{g})\xrightarrow{\simeq} H$$

Similarly, the map $j_g: W \otimes G_g \to G_{f \otimes g}$ restricts to an isomorphism

$$j_g|_{\operatorname{Ker}(\widehat{f}\otimes\widehat{\lambda_q})}:\operatorname{Ker}(\widehat{f}\otimes\widehat{\lambda_g})\xrightarrow{\simeq} H.$$

PROOF. We prove the first isomorphism – the second one is similar. Since j_f is injective, it suffices to prove that $j_f(\operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) = H$.

First $j_f(\operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) \subseteq j_f(G_f \otimes W) = A$. Next, let $u \in \operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g})$ and $j_f(v) \in A$. We have

$$\lambda_{f\otimes g}(j_f(u), j_f(v)) = \lambda_{f\otimes g} \circ j_f(u, v) = (\lambda_f \otimes g)(u, v) = 0.$$

Hence $j_f(\operatorname{Ker}(\widehat{\lambda_f}\otimes \widehat{g}))$ and A are orthogonal, that is,

$$j_f(\operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) \subseteq A^{\perp}.$$

Therefore,

$$j_f(\operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g})) \subseteq A \cap A^{\perp} = H.$$

Conversely, let $j_f(x) = j_g(y) \in A \cap A^{\perp} = H$. Let $z \in G_f \otimes W$. Then

$$(\lambda_f \otimes g)(x,z) = \lambda_{f \otimes g}(j_f(x), j_f(z)) = \lambda_{(f \otimes g)}(j_g(y), j_f(z)) = 0.$$

(The first equality results from Lemma 2.6 and the third one from Lemma 2.8.) This proves that $x \in \operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g})$. Hence $H \subseteq j_f(\operatorname{Ker}(\widehat{\lambda_f} \otimes \widehat{g}))$. This achieves the proof.

Remark. The following diagram with exact rows and columns is commutative:



According to the "snake lemma", there is a Bockstein map

$$\beta: \operatorname{Ker}(\widehat{f} \otimes \operatorname{id}_{G_g}) \to \operatorname{Coker}(\widehat{f} \otimes \operatorname{id}_W)$$

connecting the exact sequence made of the maps of the first and the last row. Since $\operatorname{Ker}(\widehat{\lambda}_f \otimes \operatorname{id}_{W^*}) = 0$, the Bockstein map β is injective. Hence there is an exact sequence $0 \to \operatorname{Ker}(\widehat{f} \otimes \operatorname{id}_{G_g}) \to \operatorname{Coker}(\widehat{f} \otimes \operatorname{id}_W) \to \operatorname{Coker}(\widehat{\lambda}_f \otimes \operatorname{id}_{W^*})$. Since $G \otimes W = \operatorname{Coker}(\widehat{f} \otimes \operatorname{id}_W)$, β induces an isomorphism

$$\bar{\beta} : \operatorname{Ker}(\widehat{f} \otimes \widehat{\lambda}_g) = \operatorname{Ker}(\widehat{f} \otimes \operatorname{id}_{G_g}) \to \operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g}) = \operatorname{Ker}(\widehat{\lambda}_f \otimes \widehat{g}).$$

It follows from definitions that

$$\bar{\beta} = j_f^{-1}|_H \circ j_g|_{\operatorname{Ker}(\widehat{f} \otimes \operatorname{id}_{G_g})}$$

3. Tensor product of linkings

The previous section described (Lemma 2.11) an isomorphism $G_f \otimes G_g \to H^*$ that splits as the composition of two isomorphisms. The bilinear pairing $G/H^{\perp} \times H \to \mathbb{Q}/\mathbb{Z}$ defined by

$$l([x], y) = \lambda_{f \otimes q}(x, y), \quad x \in G, y \in H,$$

is nonsingular, so its left adjoint is an isomorphism $G/H^{\perp} \simeq H^*$. The second isomorphism is the map $\psi: G_{f\otimes g}/H^{\perp} \to G_f \otimes G_g$ induced by the quotient map $G_{f\otimes g} \to G_f \otimes G_g$. The composition $\hat{l} \circ \psi^{-1}: G_f \otimes G_g \to H^*$ is an isomorphism. In this section we relate $l: G/H^{\perp} \times H \to \mathbb{Q}/\mathbb{Z}$ to the tensor product of discriminant linkings $\lambda_f \otimes \lambda_g$ (defined in §2).

We first define a map

$$V^{\sharp} \times W \to \frac{V^{\sharp} \otimes W_{\mathbb{Q}}}{V^{\sharp} \otimes W + V \otimes W^{\sharp}}$$

by the assignment

$$m: (\xi, w) \mapsto \left[\xi \otimes \frac{w}{n}\right],$$

where n is the smallest nonnegative integer such that $n \xi \in V$.

LEMMA 3.1. This map induces a homomorphism $m: G_f \otimes W \to K$.

PROOF. We show that m is \mathbb{Z} -bilinear. Note that any element $\xi \in V^{\sharp}$ can be written as $\xi = \frac{\xi'}{n}$ where $\xi' \in V$ and $n \in \mathbb{Z}$. We further require ξ' to be indivisible: $\xi' \notin k V$ for all k > 1. This condition is equivalent to n being the smallest nonnegative integer such that $n \notin \in V$. We have

$$m\left(\frac{\xi'}{n},w\right) = \left[\frac{\xi'\otimes w}{n}\right].$$

It follows that

$$m\left(\frac{\xi'}{n}, w + w'\right) = m\left(\frac{\xi'}{n}, w\right) + m\left(\frac{\xi'}{n}, w'\right).$$

We now verify linearity on the left: let $\zeta \in V^{\sharp}$ that we write as $\zeta = \frac{\zeta'}{p}$ where $\zeta \in V$ is indivisible. Write

$$n = k \cdot n', \ p = k \cdot p', \ \text{with} \ k = \gcd(n, p).$$

Then

$$\begin{split} m\left(\frac{\xi'}{n} + \frac{\zeta'}{p}, w\right) &= m\left(\frac{p' \ \xi'}{n'p'k} + \frac{n' \ \zeta'}{n'p'k}, w\right) = \left[\frac{(p'\xi' + n'\zeta') \otimes w}{n'p'k}\right] \\ &= \left[\frac{p' \ \xi' \otimes w}{n'p'k} + \frac{n' \ \zeta' \otimes w}{n'p'k}\right] \\ &= \left[\frac{\xi' \otimes w}{n}\right] + \left[\frac{\zeta' \otimes w}{p}\right] \\ &= m\left(\frac{\xi'}{n}, w\right) + m\left(\frac{\zeta'}{p}, w\right) \end{split}$$

(We used in the second equality the fact that n' and p' are coprime, so that $p' \xi' + n' \zeta'$ is again indivisible in V.) Therefore m induces a group homomorphism (still denoted m) $V^{\ddagger} \otimes W \to K$. It follows from the definition that $V \otimes W \subseteq \text{Ker}(m)$. Hence m induces a homomorphism $G_f \otimes W \to K$.

LEMMA 3.2. $\operatorname{Ker}(\widehat{\lambda}_f \otimes \widehat{g}) = \operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g})$ is generated by all elements $x \otimes w \in G_f \otimes W$ such that $\widehat{g}(w) \in nW^*$ and $n \ x = 0$ for some $n \in \mathbb{Z}$.

PROOF. The subgroup identifies to $\operatorname{Tor}_1^{\mathbb{Z}}(G_f, G_g)$. In particular, it depends only on G_f and G_g . The result is clear if G_f is a finite cyclic group. In the general case, G_f is a sum of finite cyclic groups and we use the fact that $\operatorname{Tor}_1^{\mathbb{Z}}(A \oplus B, G_g) \simeq \operatorname{Tor}_1^{\mathbb{Z}}(A, G_g) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(B, G_g)$.

LEMMA 3.3. The map $m : G_f \otimes W \to K$ restricts to a map $m|_{\operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g})}$ whose image lies in $G_{f \otimes g}/H^{\perp} = \frac{V^{\sharp} \otimes W^{\sharp}}{V^{\sharp} \otimes W + V \otimes W^{\sharp}}$.

PROOF. Let $[\xi] \otimes w$ be a generator of $\operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g})$ as in Lemma 3.2: there is $n \in \mathbb{Z}$ such that $\widehat{g}(w) \in n$ W^* and $n \notin V$. Thus $\widehat{g}_{\mathbb{Q}}(\frac{w}{n}) = \frac{1}{n} \widehat{g}_{\mathbb{Q}}(w) \in W^*$, that is $\frac{w}{n} \in W^{\sharp}$. Hence $\xi \otimes \frac{w}{n} \in V^{\sharp} \otimes W^{\sharp}$ and

$$m([\xi] \otimes w) = \left[\xi \otimes \frac{w}{n}\right] \in \frac{V^{\sharp} \otimes W^{\sharp}}{V^{\sharp} \otimes W + V \otimes W^{\sharp}} = G_{f \otimes g}/H^{\perp}.$$

Set $\mu' = \mu'_f = m|_{\operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g})}.$

LEMMA 3.4. The map μ' is an isomorphism

$$\operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g}) \to G_{f \otimes g} / H^{\perp} = \frac{V^{\sharp} \otimes W^{\sharp}}{V^{\sharp} \otimes W + V \otimes W^{\sharp}}.$$

PROOF. The two groups are finite and isomorphic (Lemma 2.11 and Lemma 2.12). Hence it suffices to prove that μ' is onto. Choose orthogonal bases $e = (e_1, \ldots, e_n)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)$ for $(V_{\mathbb{Q}}, f_{\mathbb{Q}})$ and $(W_{\mathbb{Q}}, g_{\mathbb{Q}})$ respectively, so that there exist $a_i, b_j \in \mathbb{Z} - \{0\}$ $(1 \leq i \leq n, 1 \leq j \leq p)$, such that

$$V^{\sharp} = \bigoplus_{i} \frac{1}{a_{i}} \mathbb{Z} \ e_{i}, \quad W^{\sharp} = \bigoplus_{j} \frac{1}{b_{j}} \mathbb{Z} \ \varepsilon_{j}.$$

We have

$$V^{\sharp} \otimes W + V \otimes W^{\sharp} = \bigoplus_{i,j} \left(\frac{1}{a_i} \mathbb{Z} + \frac{1}{b_j} \mathbb{Z} \right) (e_i \otimes \varepsilon_j) = \bigoplus_{i,j} \frac{1}{\operatorname{lcm}(a_i, b_j)} \mathbb{Z}(e_i \otimes \varepsilon_j).$$

Therefore

$$\frac{V^{\sharp} \otimes W^{\sharp}}{V^{\sharp} \otimes W + V \otimes W^{\sharp}} = \frac{\bigoplus_{i,j} \frac{1}{a_i b_j} \mathbb{Z}(e_i \otimes \varepsilon_j)}{\bigoplus_{i,j} \frac{1}{\operatorname{lcm}(a_i, b_j)} \mathbb{Z}(e_i \otimes \varepsilon_j)} = \bigoplus_{i,j} \frac{1}{\operatorname{gcd}(a_i, b_j)} \mathbb{Z}/\mathbb{Z}[e_i \otimes \varepsilon_j].$$

Now we verify the identity:

$$\frac{1}{\gcd(a_i,b_j)}[e_i\otimes\varepsilon_j] = \left[\frac{e_i}{\gcd(a_i,b_j)}\otimes\frac{\varepsilon_j}{\gcd(a_i,b_j)}\right] = \mu'\left(\left[\frac{e_i}{\gcd(a_i,b_j)}\right]\otimes\varepsilon_j\right).$$

We define an isomorphism $\mu_f: H \to G_{f \otimes g}/H^{\perp}$ as the composition

$$\nu_f = \mu'_f \circ j_f^{-1}|_H : H \xrightarrow{j_f^{-1}} \operatorname{Ker}(\operatorname{id}_{G_f} \otimes \widehat{g}) \xrightarrow{\mu'_f} G/H^{\perp}.$$

There is a similar isomorphism $\nu_g: H \to G_{f \otimes g} / H^\perp$ defined as the composition

$$\nu_g = \mu'_g \circ j_g^{-1}|_H : H \xrightarrow{j_g^{-1}} \operatorname{Ker}(\widehat{f} \otimes \operatorname{id}_{G_g}) \xrightarrow{\mu'_g} G/H^{\perp}.$$

It follows from definitions that

 $\nu_f = \nu_g.$

Recall the isomorphism $\psi: G/H^{\perp} \to G_f \otimes G_g$ we defined in the previous paragraph. We are now ready to define an isomorphism $\mu: H \to G_f \otimes G_g$ as the composition

$$\mu = \psi \circ \nu.$$

THEOREM 3.5. For all $x \in G_{f \otimes q}/H^{\perp}, y \in H$,

$$l(x,y) = (\lambda_f \otimes \lambda_g)(\psi(x), \psi \circ \nu(y)) = (\lambda_f \otimes \lambda_g)(\psi(x), \mu(y)).$$

PROOF. Let $x = [\xi]$, $x' = [\xi'] \in G_f = V^{\sharp}/V$ and $y = [\zeta]$, $y' = [\zeta'] \in G_g = W^{\sharp}/W$. We have to show the equality $l(\psi^{-1}(x \otimes y), \mu^{-1}(x' \otimes y')) = (\lambda_f \otimes \lambda_g)(x \otimes y, x' \otimes y')$. We have $\psi^{-1}(x \otimes y) = [[\xi \otimes \zeta]] = [\xi \otimes \zeta] \mod H^{\perp}$. With no loss of generality, we may assume that x' generates a cyclic (direct) summand of G_f of order n. Hence we may assume that $x' \otimes y' = x' \otimes y''$ with $y'' = [\zeta'']$ of order dividing n. Thus $\mu^{-1}(x' \otimes y') = \mu(x' \otimes y'') = j_f(x' \otimes n\zeta'') = [\xi' \otimes n\zeta'']$.

We compute

$$l(\psi^{-1}(x \otimes y), \mu^{-1}(x' \otimes y')) = l([[\xi \otimes \zeta]], [\xi' \otimes n\zeta''])$$

= $\lambda_{f \otimes g}([\xi \otimes \zeta], [\xi' \otimes n\zeta''])$
= $(f_{\mathbb{Q}} \otimes g_{\mathbb{Q}})(\xi \otimes \zeta, \xi' \otimes n\zeta'') \mod 1$
= $f_{\mathbb{Q}}(\xi, \xi') \cdot g_{\mathbb{Q}}(\zeta, n\zeta'') \mod 1$
= $\lambda_f(x, x') \cdot \underbrace{g_{\mathbb{Q}}(\zeta, n\zeta'')}_{\in \mathbb{Z}}$.

On the other hand, using again the fact that

$$\left(\frac{a}{n} \mod 1\right) \otimes \left(\frac{b}{n} \mod 1\right) = \frac{a}{n} \cdot b \mod 1 = \frac{ab}{n} \mod 1$$

in $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \otimes \frac{1}{n}\mathbb{Z}/\mathbb{Z} = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$, we see that

$$\begin{aligned} (\lambda_f \otimes \lambda_g)(x \otimes y, x' \otimes y') &= \underbrace{\lambda_f(x, x')}_{\in \frac{1}{n} \mathbb{Z}/\mathbb{Z}} \otimes \underbrace{\lambda_g(y, y')}_{\in \frac{1}{n} \mathbb{Z}/\mathbb{Z}} \\ &= \lambda_f(x, x') \otimes (g_{\mathbb{Q}}(\zeta, \zeta'') \mod 1) \\ &= \lambda_f(x, x') \cdot n \ g_{\mathbb{Q}}(\zeta, \zeta'') \\ &= \lambda_f(x, x') \cdot g_{\mathbb{Q}}(\zeta, n\zeta''). \end{aligned}$$

This finishes the proof.

COROLLARY 3.6. The isomorphism class of $l: G/H^{\perp} \times H \to \mathbb{Q}/\mathbb{Z}$ does not depend on the particular presentations (V, f) and (W, g) and depends only on the linking pairings (G_f, λ_f) and (G_g, λ_g) respectively.

4. Wu classes and quadratic functions

We keep notations from the previous paragraph. We extend the discriminant construction to lattices endowed with a special element called a Wu class.

DEFINITION 2.3. A Wu class $v \in V^{\sharp}$ is any element $v \in V^{\sharp}$ such that

$$f(x,x) - f_{\mathbb{O}}(x,v) \in 2\mathbb{Z}$$
, for all $x \in V$.

A Wu class is *integral* if it lies in V. A bilinear lattice (V, f) is said to be even if $0 \in Wu(f)$.

LEMMA 4.1. Any bilinear lattice (V, f) has an integral Wu class.

PROOF. The map $x \mapsto f(x, x) \mod 2$ is a homomorphism. Assume that the discriminant of f is odd. Then f induces a nonsingular symmetric bilinear pairing on W = V/2V by $\overline{f}(x + 2V, y + 2V) = f(x, y) \mod 2$. Hence there exists $[v] \in V/2V$ such that $\overline{f}([x], [v]) = \overline{f}([x], [x])$. Lift [v]to some representative $v \in V$. Then v is a Wu class for (V, f). Suppose the discriminant of f is even. Then \overline{f} has discriminant 0 and its annihilator $A = \{[x] \in W \mid \overline{f}([x], W) = 0\}$ is nonzero. Then \overline{f} induces a nondegenerate, and hence nonsingular, linking pairing on $\overline{f'} : W/A \times W/A \to \mathbb{F}_2$. The previous argument yields a $[v] \in W/A$ such that $\overline{f'}([x], [v]) = \overline{f'}([x], [x])$

for all $[x] \in W/A$. Lifting [v] to a representative $v \in V$ yields an integral Wu class.

The set Wu(f) of Wu classes is freely and transitively acted on by V^{\sharp} . The action is given by the formula

$$z \cdot s = z + 2s, \quad z \in \operatorname{Wu}(f), \ s \in V^{\sharp}.$$

This action restricts to an action of V on the set $Wu^V(f)$ of integral Wu classes.

LEMMA 4.2. The map

$$\operatorname{Wu}(f) \times \operatorname{Wu}(g) \to \operatorname{Wu}(f \oplus g), \ (v, w) \mapsto v + w$$

is an affine isomorphism over the isomorphism

 $V^{\sharp} \oplus W^{\sharp} \to (V \oplus W)^{\sharp}.$

DEFINITION 2.4. To a bilinear lattice (V, f, v) equipped with a Wu class, one associates a quadratic function $\varphi_{f,v}: G_f \to \mathbb{Q}/\mathbb{Z}$ over the linking pairing λ_f by

(4.1)
$$\varphi_{f,v}(x+V) = \frac{1}{2} \left(f_{\mathbb{Q}}(x,x) - f_{\mathbb{Q}}(x,v) \right) \mod 1, \ x \in V^{\sharp}.$$

This quadratic function is the *discriminant quadratic function*.

The following two properties are immediate from the definition.

LEMMA 4.3. The discriminant quadratic function $\varphi_{f,v}$ is homogeneous if and only if v is an integral Wu class.

This property has a particular importance for the algebraic description of spin and spin^c-structures on closed 3-manifolds.

LEMMA 4.4. If (V, f) is nondegenerate, then $\varphi_{f,v}$ is nondegenerate.

REMARK 2.3. A direct computation yields

(4.2)
$$\varphi_{f,v+2s} = \varphi_{f,v} - \lambda_f([s]), \quad v \in \operatorname{Wu}(f), \ s \in V^{\sharp}.$$

LEMMA 4.5. The discriminant construction preserves orthogonal sums:

$$\varphi_{f \oplus g, v+w} = \varphi_{f, v} \oplus \varphi_{g, w},$$

for lattices f, g and Wu classes $v \in Wu(f), w \in Wu(g)$.

We now state two basic results in the theory of discriminant quadratic functions.

THEOREM 4.6. The assignment $(V, f, v) \mapsto (G_f, \varphi_{f,v})$ is surjective onto the monoid of quadratic functions on finite abelian groups. When restricted to even lattices, the assignment is surjective onto the monoid of homogeneous quadratic functions on finite abelian groups.

The equivalence relation on bilinear lattices can be extended to bilinear lattices equipped with Wu classes as follows. Say that (V, f, v) and (W, g, w) are strongly stably equivalent if there exist unimodular lattices (U, h, u) and (U', h', u') equipped with Wu classes $u \in Wu(h)$ and $u' \in Wu(h')$ respectively and an isomorphism $\psi : U \oplus V \to U' \oplus W$ such that $(h' \oplus g)(\psi(x), \psi(y)) = (h \oplus f)(x, y)$ for all $x, y \in U \oplus V$ and $\psi_{\mathbb{Q}}(u \oplus v) \in u' \oplus w + 2(U' \oplus W)$. The relation is an equivalence relation. It is verified that two strongly stably equivalent triples (V, f, v) and (W, g, w) give rise to isomorphic discriminant quadratic functions. A fundamental result consists in the converse.

THEOREM 4.7. [19, Prop. 3.1] Two nondegenerate quadratic functions on finite abelian groups are isomorphic if and only if they can be lifted to strongly stably equivalent bilinear lattices equipped with Wu classes.

Consider the lattice \mathbb{Z} equipped with the unimodular form ± 1 , sending (1, 1) to ± 1 , and the integral Wu class $1 \in \mathbb{Z}$. It is shown in [19, Cor. 3.5] that the strong stabilization in Th. 4.6 can be realized using only these unimodular lattices.

5. Tensor products and half-integral Wu classes

Let (V, f) and (W, g) be two nondegenerate bilinear lattices.

LEMMA 5.1. There is an injective map

$$\operatorname{Wu}(f) \times \operatorname{Wu}(g) \to \operatorname{Wu}(f \otimes g), \ (v, w) \mapsto v \otimes w.$$

PROOF. Let $v \in Wu(f), w \in Wu(g)$. For any $x \in V, y \in W$,

$$(f \otimes g)(x \otimes y, x \otimes y) - (f \otimes g)(v \otimes w, x \otimes y)$$

= $f(x, x)g(y, y) - f(v, x)g(w, y)$
= $\underbrace{\left(f(x, x) - f(v, x)\right)}_{\equiv 0 \mod 2} \underbrace{g(y, y)}_{\in \mathbb{Z}} + \underbrace{f_{\mathbb{Q}}(v, x)}_{\in \mathbb{Z}} \underbrace{\left(g(y, y) - g_{\mathbb{Q}}(w, y)\right)}_{\equiv 0 \mod 2}$
= 0 mod 2.

LEMMA 5.2. The group $V^{\sharp} \otimes W + V \otimes W^{\sharp}$ acts freely on Wu $(f \otimes g)$.

PROOF. The group $V^{\sharp} \otimes W + V \otimes W^{\sharp}$ is a subgroup of the group $V^{\sharp} \otimes W^{\sharp}$ acting freely on Wu $(f \otimes g)$.

This action is not transitive in general since the inclusion $V^{\sharp} \otimes W + V \otimes W^{\sharp} \subseteq V^{\sharp} \otimes W^{\sharp}$ is proper in general. Indeed, there is equality if and only if

$$V^{\sharp} \otimes W \ \cap \ V \otimes W^{\sharp} = V^{\sharp} \otimes W \ \cap \ (V^{\sharp} \otimes W)^{\sharp} = V \otimes W.$$

It will be convenient for our purpose to consider the action of a slightly bigger subgroup (cf. Cor. 1.15). First we describe a special subset of Wu classes. Consider the set S of Wu classes of the form $v \otimes w$ where $v \in Wu^V(f)$ or $w \in Wu^W(g)$ (i.e, at least one of the Wu classes v or w has to be integral). Consider first the difference $\Delta = v \otimes w - v' \otimes w'$ of two elements in S. Then

$$v \otimes w - v' \otimes w' = v \otimes w - v \otimes w' + v \otimes w' - v' \otimes w'$$

= $v \otimes (w - w') + (v - v') \otimes w$
= $0 \mod 2(V \otimes W^{\sharp} + V^{\sharp} \otimes W).$

This suggests the following definition. Let

$$Z' = \frac{1}{2} (V^{\sharp} \otimes W + V \otimes W^{\sharp}) \cap (V^{\sharp} \otimes W^{\sharp}).$$

Observe that $2Z' = V^{\sharp} \otimes W + V \otimes W^{\sharp}$.

DEFINITION 2.5. The set $\operatorname{Wu}^{1/2}(f \otimes g)$ of half-integral Wu classes consists of all $z \in \operatorname{Wu}(f \otimes g)$ such that there exist $s \in S$ and $t \in Z'$ such that z = s + 2t.

This is a subset of $Wu(f \otimes g)$. Consider a similar definition.

DEFINITION 2.6. The set $\operatorname{Wu}_0^{1/2}(f \otimes g)$ of special half-integral Wu classes consists of all $z \in \operatorname{Wu}(f \otimes g)$ such that there exist $s \in S$ and $t \in 2Z'$ such that z = s + 2t.

This is a subset of $Wu^{1/2}(f \otimes g)$. What we have proved is the following

LEMMA 5.3. The group Z' (resp. 2Z') acts freely and transitively on $\operatorname{Wu}^{1/2}(f \otimes g)$, resp. $\operatorname{Wu}^{1/2}_0(f \otimes g)$, by

$$x \cdot t = x + 2t, \quad x \in \operatorname{Wu}^{1/2}(f \otimes g), \quad t \in Z'(\operatorname{resp.} t \in 2Z')$$

and

$$\operatorname{Wu}_0^{1/2}(f \otimes g) \subseteq \operatorname{Wu}^{1/2}(f \otimes g) \subseteq \operatorname{Wu}(f \otimes g).$$

As observed above the inclusions are strict in general.

Remark. It follows from Lemma 5.3 that for any half-integral (resp. special half-integral) Wu class z, there exist a pair $(v, w) \in Wu^V(f) \times Wu^W(g)$ of *integral* Wu classes such that

 $z = v \otimes w + 2t$, for some unique $t \in Z'$ (resp. $t \in 2Z'$).

The main motivation for introducing the set of half-integral Wu class lies in Theorem 6.6 and Corollary 1.15.

6. The discriminant and the characteristic homomorphism

We keep notation from the previous paragraph. The next lemma is mostly a reminder of the definitions.

LEMMA 6.1. The image of $V^{\sharp} \otimes W$ under the canonical projection $Z^{\sharp} \rightarrow G_{f \otimes g} = Z^{\sharp}/Z$ is A.

As a consequence, we have

 $V^{\sharp} \otimes W \ \cap \ (V^{\sharp} \otimes W)^{\sharp} = V \otimes W \ \iff \ A \cap A^{\perp} = 0.$

Let $z \in Z \otimes \mathbb{Q}$ be a Wu class for $(Z, f \otimes g)$. The discriminant (eq. (4.1)) of $(Z, f \otimes g, z)$ produces a nondegenerate quadratic function $\varphi_{f \otimes g, z} : G_{f \otimes g} \to \mathbb{Q}/\mathbb{Z}$ where $G_{f \otimes g} = Z^{\sharp}/Z$.

Recall that the subgroup H in $G_{f\otimes g}$ consists in the intersection of $A = j_f(G_f \otimes W)$ and $A^{\perp} = j_g(V \otimes G_g)$. Note that H is also the image of $V^{\sharp} \otimes W \cap V \otimes W^{\sharp}$ under the canonical projection $Z^{\sharp} \to G_{f\otimes g}$.

LEMMA 6.2. $\varphi_{f\otimes g,z}|_H$ is a homomorphism $H \to \mathbb{Q}/\mathbb{Z}$.

PROOF. By Lemma 2.9, the associated linking pairing $\lambda_{f\otimes g}$ vanishes on $H \times H$.

According to [19, Th. 2.10], there is an affine isomorphism

$$\frac{\operatorname{Wu}(f \otimes g)}{2Z} \to \operatorname{Quad}(\lambda_{f \otimes g}), \ [z] \mapsto \varphi_{f \otimes g, z}$$

over the group isomorphism

$$G_{f\otimes g} \to G^*_{f\otimes g}, \ [s] \mapsto -\lambda_{f\otimes g}([s], -).$$

Here $G_{f\otimes g} = Z^{\sharp}/Z$ acts freely and transitively on $\frac{\operatorname{Wu}(f\otimes g)}{2Z}$ by the formula

$$[z] \cdot [s] = [z + 2s], \quad z \in Wu(f \otimes g), \quad s \in Z^{\sharp}$$

and $G_{f\otimes g}$ acts freely and transitively on $\text{Quad}(\lambda_{f\otimes g})$ by the usual formula (5.2). The isomorphism is affine in the sense that

(6.1)
$$\varphi_{f\otimes g,z} \cdot [s] = \varphi_{f\otimes g,[z]\cdot[-s]} = \varphi_{f\otimes g,z-2s}$$

for any $s \in Z^{\sharp}$.

LEMMA 6.3. The group $G_{f\otimes g}/(A + A^{\perp})$ acts freely and transitively on the quotient set $\frac{\operatorname{Wu}(f\otimes g)}{2(V^{\sharp}\otimes W + V\otimes W^{\sharp})}$.

We now investigate the dependency of the homomorphism of Lemma 6.2 on the Wu class. Recall that $G_{f\otimes g}/(A + A^{\perp}) = G_{f\otimes g}/H^{\perp}$ acts freely and transitively on $H^* = \operatorname{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z})$ by the formula

$$[x] \cdot \alpha = \alpha + \lambda_{f \otimes g}(x, -), \quad x \in G_{f \otimes g}, \ \alpha \in H^*.$$

LEMMA 6.4. The affine map

$$\operatorname{Wu}(f \otimes g) \to \operatorname{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z}), \ z \mapsto \varphi_{f \otimes q, z}|_{H}$$

induces an affine isomorphism

$$\frac{\operatorname{Wu}(f \otimes g)}{2(V^{\sharp} \otimes W + V \otimes W^{\sharp})} \to H^{*}$$

over the isomorphism

$$G_{f\otimes g}/H^{\perp} \to H^*, \ [x] \mapsto -\lambda_{f\otimes g}(x,-)|_H.$$

PROOF. To prove that the map is well-defined, it suffices to verify that $\varphi_{f\otimes g,z+2[k]}|_H = \varphi_{f\otimes g,z}|_H$ for $k \in (V^{\sharp} \otimes W) + (V \otimes W^{\sharp})$. This amounts to verifying that for $x \in (V^{\sharp} \otimes W) \cap (V \otimes W^{\sharp})$,

$$(f_{\mathbb{Q}} \otimes g_{\mathbb{Q}})(k, x) \in \mathbb{Z}$$

This follows by Lemma 2.5.

As noted before, the group $G_{f\otimes g}/H^{\perp}$ acts freely and transitively on both sets $\frac{\operatorname{Wu}(f\otimes g)}{2(V^{\sharp}\otimes W + V\otimes W^{\sharp})}$ and H^* . Let us verify that the induced map is affine: for any $x \in Z^{\sharp}$, $z \in \operatorname{Wu}(f \otimes g)$,

$$\varphi_{f\otimes g,z\cdot x}|_{H} = \varphi_{f\otimes g,z+2x}|_{H} = \varphi_{f\otimes g,z}|_{H} - \lambda_{f\otimes g}([x], -)|_{H} = \varphi_{f\otimes g,z}|_{H} \cdot [x].$$

We conclude by using the fact that $G_{f\otimes g}/H^{\perp}$ acts freely and transitively on both sets.

To sum up the results so far, we proved that the following diagram is commutative:



Here the downward left arrow is the natural epimorphism induced by the inclusion $Z \subseteq V^{\sharp} \otimes W + V \otimes W^{\sharp}$ and res_{H} denotes the restriction to the subgroup H.

COROLLARY 6.5. The quotient set

$$\frac{\operatorname{Wu}(f \otimes g)}{2(V^{\sharp} \otimes W + V \otimes W^{\sharp})}$$

has the structure of an Abelian group isomorphic to $G_f \otimes G_g$. In particular, the zero element is the unique class [z] such that $\varphi_{f \otimes g, z}|_H = 0$.

Remark. The quotient set

$$\frac{\operatorname{Wu}_{0}^{1/2}(f \otimes g)}{2(V^{\sharp} \otimes W + V \otimes W^{\sharp})}$$

is a singleton (that may or may not coincide with the class [z] such that $\varphi_{f\otimes g,z}|_{H} = 0$). In other words, under the affine map of Lemma 6.4, all special half-integral Wu classes correspond to the same element in H^* . Furthermore, this element has order at most 2 since $2Wu_0^{1/2}(f\otimes g) \subseteq 2(V^{\sharp}\otimes W + V\otimes W^{\sharp})$.

THEOREM 6.6. Let $z \in Wu_0^{1/2}(f \otimes g)$. The map $\varphi_{f \otimes g,z}|_H$ has order at most 2 in H^* and depends only on λ_f and λ_g .

DEFINITION 2.7. We call this map the characteristic homomorphism associated to (G_f, λ_f) and (G_g, λ_g) .

The subgroup generated by $\varphi_{f\otimes g,z}|_H$ in H^* therefore only depends on λ_f and λ_g .

DEFINITION 2.8. We call this subgroup the *characteristic* subgroup associated to λ_f and λ_g .

The characteristic subgroup is either trivial or has order 2.

COROLLARY 6.7. If $G_f \otimes G_g$ has odd order, then the characteristic subgroup is trivial.

PROOF. By Lemma 2.11, H^* has odd order; so has any subgroup of H^* , in particular the characteristic subgroup. By Th. 6.6, it must have order dividing 2, hence it is trivial.

A proof of Th. 6.6 relying on structural properties of the discriminant (§??) is given in §8. Alternatively, an explicit expression for the characteristic map $\varphi_{f\otimes g,z}|_H$ is derived in §10. (However, the proof relies partially on Th. 6.6.)

7. More on the discriminant

This paragraph is devoted to properties of the discriminant. They are used to prove Th. 6.6 in the next paragraph.

We begin with the following observations. There is the natural right action of the group $\operatorname{Aut}(V)$ of automorphisms of V on the set of nondegenerate symmetric bilinear pairings on V:

$$f \cdot \alpha = \alpha^* f = f \circ (\alpha^{\otimes 2}), \ \alpha \in \operatorname{Aut}(V).$$

Similarly, for a finite abelian group G, the group Aut(G) of automorphisms of G acts on the set Quad(G) of quadratic functions on G by the formula

$$q \cdot \beta = \beta^* q = q \circ \beta, \ q \in \text{Quad}(G), \ \beta \in \text{Aut}(G).$$

There is also an action of Aut(G) on the set of all linking pairings on G by a similar formula.

Let O(f) denote the automorphism group of f, that is, the isotropy subgroup of $\operatorname{Aut}(V)$ consisting of automorphisms fixing f. Let $O(\lambda_f)$ denote the automorphism group of λ_f , that is the isotropy subgroup of $\operatorname{Aut}(G_f)$ consisting of automorphisms fixing λ_f . Then $O(\Lambda_f)$ acts on the set of quadratic functions over λ_f , $\operatorname{Quad}(\lambda_f) \subseteq \operatorname{Quad}(G_f)$, by the same formula as above. Recall that any automorphism α of V (resp. fixing f) induces an automorphism $[\alpha]$ of G_f (resp. fixing λ_f). Hence the assignment

$$\alpha \mapsto [\alpha]$$

yields natural maps

$$\operatorname{Aut}(V) \to \operatorname{Aut}(G_f), \quad \operatorname{O}(f) \to \operatorname{O}(\lambda_f)$$

making the following diagram commutative



where the vertical arrows are canonical inclusions.

LEMMA 7.1. For any $\alpha \in \operatorname{Aut}(V)$,

$$v \in \operatorname{Wu}(f) \iff \alpha_{\mathbb{O}}^{-1}v \in \operatorname{Wu}(\alpha^* f).$$

PROOF. Direct computation.

Consider now the set $L_{Wu}(V)$ of all pairs (f, v) where $f : V \times V \to \mathbb{Z}$ is a lattice pairing as before and $v \in Wu(f)$. As a consequence of Lemma 7.1, the group Aut(V) acts on $L_{Wu}(V)$ by the formula:

(7.1)
$$(f,v) \cdot \alpha = (\alpha^* f, \alpha_{\mathbb{Q}}^{-1} v).$$

In particular, O(f) acts on Wu(f). Let $O(f_{\mathbb{Q}})$ denote the automorphism group of $f_{\mathbb{Q}}$. There is a restriction map on $O(f_{\mathbb{Q}})$ defined by $\alpha \mapsto \alpha|_{Wu(f)}$. Denote by O(Wu(f)) the image. The action of O(f) on Wu(f) yields a map

$$O(f) \rightarrow O(Wu(f)).$$

For $(f, v) \in L_{Wu}(V)$, let O(f, v) denote the isotropy subgroup of Aut(V) consisting of automorphisms fixing (f, v) under the action (7.1). Observe that there are natural embeddings

$$O(f, v) \to O(f), \ O(\varphi_{f,v}) \to O(\lambda_f)$$

fitting in the commutative diagram

$$O(f, v) \longrightarrow O(\varphi_{f, v})$$

$$\downarrow \qquad \qquad \downarrow$$

$$O(f) \longrightarrow O(\lambda_f)$$

Finally denote by $\operatorname{Aut}(L_{\operatorname{Wu}}(V))$ the symmetric group over the set $L_{\operatorname{Wu}}(V)$, by $\operatorname{Aut}(\operatorname{Quad}(\lambda_f))$ the symmetric group over the set $\operatorname{Quad}(\lambda_f)$ and by $\operatorname{Aut}(\operatorname{Quad}(G_f))$ the symmetric group over the set $\operatorname{Quad}(G_f)$.

There are natural maps between the various automorphism groups described above. The canonical inclusions

$$\operatorname{Wu}(f) \to L_{\operatorname{Wu}}(V), v \mapsto (f, v), \operatorname{Quad}(\lambda_f) \subseteq \operatorname{Quad}(G_f)$$

induce maps

$$O(Wu(f)) \to Aut(L_{Wu}(V)), Aut(Quad(\lambda_f)) \to Aut(Quad(G_f))$$

respectively.

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THEOREM 7.2. Let G be a finite abelian group. There exists a lattice V such that the formula

$$\varphi_{f,v} \cdot \alpha = \varphi_{(f,v) \cdot \alpha}, \quad (f,v) \in L_{Wu}(V), \ \alpha \in Aut(V)$$

defines an action of $\operatorname{Aut}(V)$ on $\operatorname{Quad}(G)$.

PROOF. According to [71], there is a lattice V such that the map

 $L_{\mathrm{Wu}}(V) \to \mathrm{Quad}(G), \ (f,v) \mapsto \varphi_{f,v}$

is surjective. The point that requires to be proved is that the formula for the action is independent of the particular choice (f, v) over $q = \varphi_{f,v}$.

Let $f, g: V \times V \to \mathbb{Z}$ be two bilinear lattices equipped with a Wu classe $v \in Wu(f)$ and $w \in Wu(g)$ respectively. Assume that both discriminant quadratic functions $\varphi_{f,v}$ and $\varphi_{g,w}$ lie in Quad(G). Let $\psi \in Aut(G)$. We say that (f, v) and (g, w) are strongly stably equivalent over ψ , denoted

$$(f,v) \sim (g,w),$$

if there is a strong stable equivalence between (f, v) and (g, w) that is realized by a lattice automorphism inducing the automorphism $\psi : G \to G$. (See §?? and Th. 4.7.)

PROPOSITION 7.3. With the notation above: $\psi^* \varphi_{g,w} = \varphi_{f,v}$ if and only if $(f,v) \underset{\psi}{\sim} (g,w)$.

A proof is easily derived from [19, Prop. 3.1].

LEMMA 7.4. For three symmetric bilinear pairings on a lattice V equipped with Wu classes: $(f, v) \underset{\psi}{\sim} (f', v'), (f', v') \underset{\psi'}{\sim} (f'', v'') \implies (f, v) \underset{\psi' \circ \psi}{\sim} (f'', v'').$

PROOF. Direct computation or consequence of Prop. 7.3.

Let K be a subgroup of Aut(G). We say that two quadratic functions $q, q' : G \to \mathbb{Q}/\mathbb{Z}$ are K-isomorphic (written $q \underset{K}{\sim} q'$) if there exists $\psi \in K$ such that $\psi^*q' = q$. Similarly, for two bilinear lattices $f, g : V \times V \to \mathbb{Z}$ equipped with Wu classes v, v' respectively, we say that (f, v) and (g, v') are K-isomorphic (written $(f, v) \underset{K}{\sim} (g, v')$) if there exists $\psi \in K$ such that $(f, v) \underset{\psi}{\sim} (g, v')$. An immediate consequence of Th. 4.7 is the following observation.

LEMMA 7.5. Let K be a subgroup of Aut(G). We have

$$\varphi_{f,v} \underset{K}{\sim} \varphi_{g,v'} \iff (f,v) \underset{K}{\sim} (g,v').$$

Recall that $\operatorname{Aut}(V)$ acts on $\operatorname{Quad}(V^{\sharp}/V)$ via the natural map $\operatorname{Aut}(V) \to \operatorname{Aut}(V^{\sharp}/V)$. The next observation is a sufficient condition for the equivalence relation \sim_{V} to be compatible with the action of $\operatorname{Aut}(V)$.

LEMMA 7.6. Let $q, q' : G \to \mathbb{Q}/\mathbb{Z}$ be two quadratic functions on G and let $K \lhd \operatorname{Aut}(G)$ be a normal subgroup in $\operatorname{Aut}(G)$. Then $q \underset{K}{\sim} q' \iff q \cdot \alpha \underset{K}{\sim} q' \cdot \alpha$ for any $\alpha \in \operatorname{Aut}(V)$.

PROOF. Suppose $q \underset{K}{\sim} q'$: there exists $\psi \in K$ such that $q' \circ \psi = q$. Since K is normal in Aut(G), $\psi' = \alpha^{-1} \circ \psi \circ \alpha \in K$ and $(q' \cdot \alpha) \circ \psi' = q \circ \alpha$.

End of proof of Theorem 7.2. Let $K \lhd \operatorname{Aut}(G)$. Applying Lemmas 7.5 and 7.6, we have

$$(f,v) \underset{K}{\sim} (g,w) \implies (f,v) \cdot \alpha \underset{K}{\sim} (g,w) \cdot \alpha \implies \varphi_{(f,v) \cdot \alpha} \underset{K}{\sim} \varphi_{(g,w) \cdot \alpha}$$

for any $\alpha \in \operatorname{Aut}(V)$. The result follows by taking $K = {\operatorname{id}_G}$.

Remark. The proof above shows that Theorem 7.2 generalizes as follows.

THEOREM 7.7. Let G be a finite abelian group and let $K \triangleleft \operatorname{Aut}(G)$. There exists a lattice V such that the formula

$$\varphi_{f,v} \cdot \alpha = \varphi_{(f,v) \cdot \alpha}, \quad (f,v) \in L_{\mathrm{Wu}}(V), \ \alpha \in \mathrm{Aut}(V)$$

induces an action of $\operatorname{Aut}(V)$ on the equivalence classes in $\operatorname{Quad}(G)$ for the relation $\underset{\nu}{\sim}$.

Theorem 7.2 is the case when K is trivial and equivalence classes are singletons. The other extreme case is when $K = \operatorname{Aut}(G)$ and the equivalence classes consist of isomorphic quadratic functions on G. There are other non-trivial intermediate cases since the automorphism group of a finite abelian group is nonsimple in general [78]. As an example, the automorphism group of $\mathbb{Z}/3 \times \mathbb{Z}/3$ is $\operatorname{GL}_2(\mathbb{Z}/3)$: both the subgroup $\operatorname{SL}_2(\mathbb{Z}/3)$ of matrices of determinant 1 and the subgroup of diagonal 2×2 matrices with coefficients in $\{\pm 1\}$ are normal.

Since the map

$$\operatorname{Wu}(f) \to \operatorname{Quad}(\lambda_f), \ z \mapsto \varphi_{f,z}$$

is surjective, the action of $\operatorname{Aut}(L_{\operatorname{Wu}}(V))$ on $\operatorname{Quad}(G_f)$ restricts to an action of $\operatorname{O}(\operatorname{Wu}(f))$ on $\operatorname{Quad}(\lambda_f)$ defined by

$$\varphi_{f,z} \cdot \alpha = \varphi_{f,\alpha^{-1}z}, \ z \in \operatorname{Wu}(f), \ \alpha \in \operatorname{O}(\operatorname{Wu}(f)).$$

Compatibility of the various actions is expressed by the commutative diagram



To verify that the cube is indeed commutative, the main point consists in verifying that for $\alpha \in \mathcal{O}(f)$,

$$\varphi_{f,z} \circ [\alpha] = \varphi_{\alpha * f, \alpha^{-1}z} = \varphi_{(f,z) \cdot \alpha} = \varphi_{f,z} \cdot \alpha,$$

which follows from our discussion above. In particular, there is an isomorphism

$$\varphi_{f,z} \cdot \alpha \simeq \varphi_{f,z}.$$

Next, we consider the tensor product $f \otimes g$. The group $\operatorname{Aut}(V)$ acts on $L_{\operatorname{Wu}}(V \otimes W)$ via the natural map

$$\operatorname{Aut}(V) \to \operatorname{Aut}(V \otimes W), \ \alpha \mapsto (\alpha \otimes 1_W).$$

Explicitly, the action is given by

$$(k,z) \cdot \alpha = ((\alpha \otimes 1_W)^* k, (\alpha \otimes 1_W)^{-1} z), \quad (k,z) \in L_{Wu}(V \otimes W), \ \alpha \in Aut(V).$$

It follows that $\operatorname{Aut}(V)$ acts on $\operatorname{Quad}(G_f)$.

Similarly, there is a natural inclusion map

$$O(f) \to O(f \otimes g), \ \alpha \mapsto \alpha \otimes 1_W.$$

It follows that O(f) acts on $O(Wu(f \otimes g))$ and on $O(\lambda_{f \otimes g})$. Finally, the composition

$$G_f \xrightarrow{-\otimes 1_W} G_f \otimes W \xrightarrow{j_f} G_{f \otimes g}$$

enables to define an action of $\operatorname{Aut}(G_f)$ on $\operatorname{Quad}(G_{f\otimes g})$. This action restricts to an action of $O(\lambda_f)$ on $\operatorname{Quad}(\lambda_{f\otimes g})$.

The cube above is still commutative if we replace f by $f \otimes g$ in all occurrences of f in the right face.

We note that $\operatorname{Aut}(V)$ also acts on the set of subgroups of $G_{f\otimes g}$ via the map $\operatorname{Aut}(V) \to \operatorname{Aut}(V \otimes W)$. Explicitly,

$$K \cdot \alpha = [\alpha \otimes 1_W]^{-1}(K), \ \alpha \in \operatorname{Aut}(V), \ K \subseteq G_{f \otimes g},$$

where $[\alpha \otimes 1_W]$ denotes the automorphism on $G_{f \otimes g}$ induced by the map $\alpha \otimes 1_W \in \operatorname{Aut}(V \otimes W)$. (The action is a right action so as to be consistent with the previous actions.)

LEMMA 7.8. The subgroup $H = j_f(G_f \otimes W) \cap j_g(V \otimes G_g)$ introduced in §?? is invariant under the action of Aut(V).

PROOF. Let $\alpha \in Aut(V)$. Let

$$[x \otimes w] = j_f([x] \otimes w) \in j_f(G_f \otimes W),$$

with $x \in V^{\sharp}$, $w \in W$. We have

 $[\alpha \otimes 1_W][x \otimes w] = [\alpha_{\mathbb{Q}} x \otimes w] = j_f([\alpha_{\mathbb{Q}} x] \otimes w) \in j_f(G_f \otimes W).$

Hence $j_f(G_f \otimes W)$ is invariant under α . A similar argument shows that $j_g(V \otimes G_g)$ is invariant under α . The lemma follows.

8. Proof of Theorem 6.6

We already know that $\varphi_{f\otimes g,z}|_H$ has order at most 2 for $z \in Wu_0^{1/2}(f \otimes g)$ and that it is independent of the particular choice of $z \in Wu_0^{1/2}(f \otimes g)$. For a fixed bilinear lattice g, we shall prove that $\varphi_{f\otimes g,z}|_H$ only depends on λ_f . (The argument is completely symmetric in g.)

First step: action of Aut(V) and O(f) on the homomorphism $\varphi_{f\otimes q,z}|_H$.

Recall the (right) action of $\operatorname{Aut}(V)$ on $\operatorname{Quad}(G_f)$ and on subgroups of G_f . Let $\alpha \in \operatorname{Aut}(V)$ act on $\varphi_{f \otimes g, z}$ and H. We have

$$(\varphi_{f\otimes g,z}\cdot\alpha)|_{H\cdot\alpha}=(\varphi_{f\otimes g,z}\cdot\alpha)|_{H},$$

according to Lemma 7.8. Thus if $\alpha \in O(f)$, then

$$\varphi_{f\otimes g,z}\cdot\alpha|_{H\cdot\alpha}=\varphi_{f\otimes g,z\cdot\alpha}|_{H\cdot\alpha}$$

Second step: if $z \in Wu_0^{1/2}(f \otimes g)$ then $\varphi_{f \otimes g, z}|_H$ is invariant under O(f).

The subset $\operatorname{Wu}_0^{1/2}(f \otimes g)$ is invariant under the action of O(f) on $\operatorname{Wu}(f \otimes g)$. The claim follows.

Third step: stabilization of f.

Let (U, u) be an unimodular lattice. We show that replacing f by $f \oplus u$ does not affect the homomorphism $\varphi_{f \otimes q, z}|_{H}$.

First, $\lambda_{f\oplus u} = \lambda_f \oplus \lambda_u = \lambda_f \oplus 0 = \lambda_f$ and $\lambda_{(f\oplus u)\otimes g} = \lambda_{f\otimes g} \oplus \lambda_{u\otimes g}$. Next, the monomorphism $j_{f\oplus u} : G_f \otimes W \to G_{f\otimes g} \oplus G_{u\otimes g}$ factors through the monomorphism $j_f : G_f \otimes W \to G_{f\otimes g}$ and the canonical inclusion $G_{f\otimes g} \to G_{f\otimes g} \oplus G_{u\otimes g}$ sending $x \in G_{f\otimes g}$ to (x, 0). In particular, the images of $j_{f\oplus u}$ and j_f coincide and are contained in $G_{f\otimes g} \oplus 0 \subseteq G_{f\otimes g} \oplus G_{u\otimes g}$. Denote by H' the new subgroup when f is replaced by $f \oplus u$. It follows that $H' \subseteq G_{f\otimes g} \oplus 0 \subseteq G_{f\otimes g} \oplus G_{u\otimes g}$ and is equal to H once $G_{f\otimes g} \oplus 0$ is identified to $G_{f\otimes g}$. Let z' be an arbitrary Wu class of $(f \oplus u) \otimes g = (f \otimes g) \oplus (u \otimes g)$ such that its restriction on $V \otimes W$ is z. Then

$$\varphi_{(f \oplus u) \otimes q, z'}|_{H'} = \varphi_{(f \otimes q) \oplus (u \otimes q), z'}|_{H'} = \varphi_{f \otimes g, z}|_{H} \oplus 0$$

Therefore, we have proved that $\varphi_{f\otimes g,z}|_H$ is invariant under O(f) and stabilization of f by unimodular lattices. It follows from [?] (see also [71] [23]) that $\varphi_{f\otimes g,z}|_H$ only depends on λ_f as claimed.

9. Remarks and useful formulas

Another proof of Theorem 6.6 results from the following observations. First,

for all $v \in Wu^V(f)$, $\varphi_{f \otimes g, v \otimes w} \circ j_f = \varphi_{f, v} \otimes g$

and similarly

for all
$$w \in Wu^W(g)$$
, $\varphi_{f \otimes g, v \otimes w} \circ j_g = f \otimes \varphi_{g, w}$.

Second, let $z \in Wu^{1/2}(f \otimes g)$ written as $z = v \otimes w + 2s$ with $v \in Wu^V(f)$ or $w \in Wu^W(g)$, and $s \in V^{\sharp} \otimes W + V \otimes W^{\sharp}$. Then

$$\varphi_{f\otimes g,z} \circ j_g|_{j_g^{-1}H} = f \otimes \varphi_{g,w}|_{j_g^{-1}H} = \varphi_{f,v} \otimes g|_{j_f^{-1}H} = \varphi_{f\otimes g,z} \circ j_f|_{j_f^{-1}H}.$$

A slightly more explicit expression is given by the formula:

(9.1)
$$\varphi_{f\otimes g,v\otimes w-2t}\circ j_f = \varphi_{f,v}\otimes g + (\lambda_f\otimes \widehat{g}_{\mathbb{Q}})([t])|_{G_f\otimes W},$$

where $v \in \operatorname{Wu}^V(f)$, $t \in V^{\sharp} \otimes W^{\sharp}$, $[t] \in G_f \otimes W^{\sharp}$. Here the map $(\widehat{\lambda}_f \otimes \widehat{g}_{\mathbb{Q}})([t])|_{G_f \otimes W}$ denotes the homomorphism induced by the map adjoint to the bilinear pairing

$$\lambda_f \otimes g_{\mathbb{Q}}|_{W^{\sharp} \times W} : (G_f \otimes W^{\sharp}) \times (G_f \otimes W) \to \mathbb{Q}/\mathbb{Z}$$

at $[t] \in G_f \otimes W^{\sharp}(1)$.

Similarly,

(9.2)
$$\varphi_{f\otimes g, v\otimes w-2t} \circ j_g = f \otimes \varphi_{g,w} + (\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_g)([t])|_{V\otimes G_g}$$

¹One should note at this point that the bilinear pairing $(\lambda_f \otimes g_{\mathbb{Q}})|_{G_f \otimes W^{\sharp} \times G_f \otimes W}$ is well defined, as $g_{\mathbb{Q}}(W^{\sharp}, W) \subseteq \mathbb{Z}$ acts by multiplication on $\lambda_f(G_f, G_f) \subset \mathbb{Q}/\mathbb{Z}$. A similar observation applies to the bilinear pairing $(f_{\mathbb{Q}} \otimes \lambda_g)|_{V^{\sharp} \otimes G_g \times V \otimes G_g}$ considered below.

where $w \in \operatorname{Wu}^W(g)$, $t \in V^{\sharp} \otimes W^{\sharp}$, $[t] \in V^{\sharp} \otimes G_g$. The map $(\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_g)([t])|_{V \otimes G_g}$ denotes the homomorphism induced by the map adjoint to the bilinear pairing

$$f_{\mathbb{Q}}|_{V^{\sharp} \times V} \otimes \lambda_g : V^{\sharp} \otimes G_g \times V \otimes G_g \to \mathbb{Q}/\mathbb{Z}$$

at $[t] \in V^{\sharp} \otimes G_g$.

In the cases when t lies in the smaller subgroup $V^{\ddagger} \otimes W$, the formula simplifies

(9.3)
$$\varphi_{f \otimes g, v \otimes w - 2t} \circ j_f = \varphi_{f, v} \otimes g + (\lambda_f \otimes g)([t], -),$$

where $v \in Wu^{V}(f), t \in V^{\sharp} \otimes W, [t] \in G_{g} \otimes W$. Similarly,

(9.4)
$$\varphi_{f\otimes g, v\otimes w-2t} \circ j_g = f \otimes \varphi_{g,w} + (f \otimes \lambda_g)([t], -),$$

where $w \in Wu^W(g), t \in V \otimes W^{\sharp}, [t] \in V \otimes G_f.$

10. The characteristic homomorphism: explicit form

Let (G, λ) and (G', λ') two (nondegenerate) linking pairings on finite abelian groups G and G' respectively. We define a map

$$\chi: G \times G' \to \mathbb{Z}/2$$

as follows: we set $\chi(x, y) = 1$ if x and y both generate an *orthogonal sum*mand of the same even order in G and G' respectively; we set $\chi(x, y) = 0$ otherwise. Note that the map χ depends on the linking pairings λ and λ' .

EXAMPLE 2.2. If G or G' has odd order, then $\chi = 0$.

EXAMPLE 2.3. If G and G' are both cyclic of order a power of 2, then $\chi(x, y) = 1$ if and only if both x and y are generators.

PROPOSITION 10.1. The map $\chi: G \times G' \to \mathbb{Z}/2\mathbb{Z}$ is bilinear.

Therefore χ induces a homomorphism $G \otimes G' \to \mathbb{Z}/2\mathbb{Z}$, still denoted χ .

PROOF. Although the proof is a consequence of the general theory of linking pairings in torsion Dedekind modules (cf. [22, Chap. 2]), we give an elementary proof based on the easy observation (cf. Lemma 3.3).

LEMMA 10.2. Let $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a linking pairing and let $x \in G$. The subgroup generated by x in G is an orthogonal summand if and only if x and $\lambda(x, x)$ have the same order in G and \mathbb{Q}/\mathbb{Z} respectively.

First, it is obvious that χ only depends on unordered pairs $(x, y) \in G \times G'$. Secondly, it is not hard to see that it is sufficient to consider 2-groups. Let $x, y \in G$ and $z \in G'$. Suppose first $\chi(x, z) = \chi(y, z) = 1$. We have to prove that $\chi(x + y, z) = 0$. By hypothesis, both x and y generate an orthogonal summand of even order 2^k in G and similarly z in G'. By Lemma 10.2, the order of $\lambda(x, x)$ and the order of $\lambda(y, y)$ in \mathbb{Q}/\mathbb{Z} coincide with the order of x (resp. of y) in G. Thus there are odd integers $a, b \in \mathbb{Z}$ such that $\lambda(x, x) = \frac{a}{2^k} \mod 1$ and $\lambda(y, y) = \frac{b}{2^k} \mod 1$. Hence for some $c \in \mathbb{Z}$,

$$\begin{split} \lambda(x+y,x+y) &= \lambda(x,x) + 2\lambda(x,y) + \lambda(y,y) \\ &= \frac{a}{2^k} + \frac{2c}{2^k} + \frac{b}{2^k} \\ &= \frac{a+2c+b}{2^k} \mod 1. \end{split}$$

Since a + 2c + b is even, $\lambda(x + y, x + y)$ is of order strictly less than 2^k . By Lemma 10.2 again, we conclude that x + y does not generate an orthogonal summand of order 2^k . Hence $\chi(x + y, z) = 0$.

Suppose next that $\chi(x,z) = \chi(y,z) = 0$. We have to show that $\chi(x+y,z) = 0$. Suppose the contrary. Then x + y generates an orthogonal summand of order 2^k in G. So $\lambda(x + y, x + y) = \frac{a}{2^k} \mod 1$ for some odd integer a. By our hypotheses, there are even integers $b, c, d \in 2\mathbb{Z}$ such that

$$\begin{split} \lambda(x+y,x+y) &= \frac{a}{2^k} &= \lambda(x,x) + 2 \cdot \lambda(x,y) + \lambda(y,y) \\ &= \frac{b}{2^k} + \frac{c}{2^k} + \frac{d}{2^k} \\ &= \frac{b+c+d}{2^k}. \end{split}$$

Since b + c + d is even, $\lambda(x + y, x + y)$ is of order strictly less than 2^k . This is a contradiction. Hence $\chi(x + y, z) = 0$.

Suppose finally that $\chi(x, z) = 1$ and $\chi(y, z) = 0$. Assume that the order of y divides the order of x. By Lemma 10.2, there exists an odd integer $a \in \mathbb{Z}$ and integers $b, c \in 2\mathbb{Z}$ such that $\lambda(x, x) = \frac{a}{2^k} \mod 1$, $\lambda(y, y) = \frac{2b}{2^k} \mod 1$ and $\lambda(x, y) = \frac{c}{2^k} \mod 1$. Hence

$$\begin{aligned} \lambda(x+y,x+y) &= \lambda(x,x) + 2 \cdot \lambda(x,y) + \lambda(y,y) \\ &= \frac{a}{2^k} + 2\frac{c}{2^k} + \frac{2b}{2^k} \\ &= \frac{a+2b+2c}{2^k} \mod 1. \end{aligned}$$

Since a + 2b + 2c is odd, we conclude that the order of $\lambda(x + y, x + y)$ in \mathbb{Q}/\mathbb{Z} equals the order of x + y in G, hence by Lemma 10.2, x + y generates an orthogonal summand of order 2^k in G, as z does in G'. This implies $\chi(x + y, z) = 1$.

EXAMPLE 2.4. Let (G, λ) be a linking pairing. Let n be a positive integer and $(\mathbb{Z}/n, \lambda_n)$ be a cyclic linking pairing. Denote by G[n] the subgroup of elements of order dividing n. Let

$$h: G[n] \to G \otimes \mathbb{Z}/n\mathbb{Z}$$

be the isomorphism defined by $h(x) = x \otimes (1 \mod n)$. The characteristic homomorphism $\chi : G \otimes \mathbb{Z}/n\mathbb{Z} \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}/2$ associated to λ and λ_n is given by

(10.1)
$$\chi(x) = \frac{n}{2}\lambda(h^{-1}(x), h^{-1}(x)), \quad x \in G \otimes \mathbb{Z}/n\mathbb{Z}.$$

This result provides a way to compute χ for any pair (λ, λ') of linking pairings: decompose λ' into an orthogonal sum of indecomposable linking pairings; disregard the noncyclic ones and the cyclic ones of odd order; using the remark above, the homomorphism χ is the orthogonal sum of the restriction of χ to the remaining components which is computed by the example above.

DEFINITION 2.9. A characteristic element associated to a pair (λ, λ') of linking pairings is an element $\theta \in G \otimes G'$ verifying

 $\chi = \widehat{\lambda \otimes \lambda'}(\theta).$

PROPOSITION 10.3. The characteristic element is unique and has order at most 2 in $G \otimes G'$.

PROOF. Since λ and λ' are nondegenerate, so is $\lambda \otimes \lambda'$. Being a linking pairing, $\lambda \otimes \lambda'$ is nonsingular. Thus there is a unique $\theta \in G \otimes G'$ of order at most 2 such that $\chi = \widehat{\lambda \otimes \lambda'}(\theta) \in \operatorname{Hom}(G \otimes G', \mathbb{Z}/2\mathbb{Z}) \subset (G \otimes G')^*$.

We now relate our previous construction (§??) to the characteristic homomorphism. Recall the isomorphism $\mu: H \to G_f \otimes G_g$ defined in §??.

THEOREM 10.4. Let $z \in Wu_0^{1/2}(f \otimes g)$. Then $\varphi_{f \otimes g, z}|_H = \chi \circ \mu$ where χ is the characteristic homomorphism associated to λ_f and λ_g .

PROOF. It is sufficient to verify the statement with $z = v \otimes w$ where v and w are integral Wu classes of f and g respectively. Then we verify that they coincide on the generators of the orthogonal summands of an orthogonal splitting of (G_f, λ_f) and (G_g, λ_g) .

We can now write down a general formula for the homomorphism $\varphi_{f\otimes g,z}|_H$ for an arbitrary Wu class $z = z_0 + 2t$, $z_0 \in Wu_0^{1/2}(f \otimes g)$, $t \in Z^{\sharp}$:

(10.2) $\varphi_{f\otimes g,z}|_{H} = \chi \circ \mu(-) - (\lambda_f \otimes \lambda_g)(\psi([t]), \mu(-)),$

where $\psi: G/H^{\perp} \to G_f \otimes G_g$ is the natural isomorphism defined in §2. We deduce

THEOREM 10.5. With the notation above, the homomorphism $\varphi_{f\otimes g, z_0+2t}|_H$ is zero if and only if

$$\chi = (\lambda_f \otimes \lambda_g)(\psi([t]), -).$$

In other words, $\varphi_{f \otimes g, z_0+2t}|_H$ is zero if and only if $\psi([t])$ is the characteristic element in $G_f \otimes G_q$ of the pair (λ_f, λ_q) .

CHAPTER 3

The classification of pointed quadratic functions

This chapter deals with the classification of pointed quadratic functions and linking groups. A pointed quadratic function is essentially a quadratic function with distinguished elements. The ideas in this chapter do not lead to a description of the monoid of isomorphism classes as in the case of linking pairings, but to a complete system of invariants.

1. The quadratic ring and Gauss sums

Let A be a finite Abelian group. It will be convenient to use the group \hat{A}^{quad} introduced in Chap. 1, §5. Recall that \hat{A}^{quad} is generated by symbols $t_x, t_{x,y}$ and "quadratic relations" (see Def. 1.18).

DEFINITION 3.1. The Gauss element associated to A is the element

(1.1)
$$\mathfrak{g}_A = \sum_{x \in A} t_x \in \mathbb{Z}[\widehat{A}^{\text{quad}}]$$

REMARK 3.1. The Gauss element plays a distinguished rôle in Abelian Topological Quantum Field Theory.

It follows from the definition of $\mathbb{Z}[\hat{A}^{\text{quad}}]$ that the Gauss element \mathfrak{g}_A is nonzero. In view of applications (...), it is convenient to introduce a ring closely related to $\mathbb{Z}[\hat{A}^{\text{quad}}]$ in which \mathfrak{g}_A is invertible.

DEFINITION 3.2. Consider the quotient Q of $\mathbb{Z}[\hat{A}^{\text{quad}}]$ by the two-sided ideal generated by the elements $\sum_{x \in A} t_{x,y}$ for all $y \neq 0$. The quadratic ring QR(A) is the localization of Q with respect to the multiplicative set $\{1, |A|, |A|^2, \ldots\}$.

In other words, QR(A) is the ring obtained from a certain quotient ring of $\mathbb{Z}[\hat{A}^{\text{quad}}]$ by inverting all powers of |A|. The extra relations imposed in $\mathbb{Z}[\hat{A}^{\text{quad}}]$ are natural (with respect to a morphism) and sufficient to make the Gauss element invertible.

Recall that the group algebra $\mathbb{Z}[\widehat{A}^{\text{quad}}]$ has an involution defined by $\overline{t_x} = t_x^{-1}$, $\overline{t_{x,y}} = t_{x,y}^{-1}$ $(x, y \in A)$ and linear extension. The ideal and the multiplicative set in the definition 3.2 are invariant under this involution. Hence the involution induces an involution in QR(A), denoted the same way.

PROPOSITION 1.1. The Gauss element \mathfrak{g}_A is invertible in QR(A) and

(1.2)
$$\mathfrak{g}_A^{-1} = \frac{1}{|A|} \,\overline{\mathfrak{g}}_A.$$

PROOF. We observe that

$$\mathfrak{g}_A \overline{\mathfrak{g}_A} = \sum_{w \in A} \sum_{x \in A} t_w t_x^{-1} = \sum_{x,w} t_{w-x} t_{x,w-x} = \sum_{x,y} t_y t_{x,y}$$
$$= \sum_{y \in A} \left(\sum_{x \in A} t_{x,y} \right) t_y = \sum_{y \neq 0} \left(\sum_{x \in A} t_{x,y} \right) t_y + |A| 1$$
$$= 0 + |A| = |A|.$$

Let $\hat{A}^{\text{quad}} \to U(1)$ be a character of \hat{A}^{quad} . Since $U(1) \subset \mathbb{C}^{\times}$, the universal property of the group ring $\mathbb{Z}[\hat{A}^{\text{quad}}]$ ensures that the character determines a unique ring homomorphism $\mathbb{Z}[\hat{A}^{\text{quad}}] \to \mathbb{C}$ (which is a \mathbb{Z} -algebra map). We are interested in characters of a special kind, namely those coming from a quadratic function $q: A \to U(1)$. Recall that any such quadratic function gives rise (by the universal property of \hat{A}^{quad}) to a character $\tilde{q}: \hat{A}^{\text{quad}} \to$ U(1). Since A is finite, it will be convenient to consider, as usual, a quadratic function $q: A \to \mathbb{Q}/\mathbb{Z}$ and post-compose it with a character $\chi: (\mathbb{Q}/\mathbb{Z}, +) \to$ $(U(1), \times)$. Any character of A with values in U(1) is a composition $\chi \circ q$, for some χ and q as above. We sum up these observations as follows.

LEMMA 1.2. A pair (χ, q) as above determines a unique \mathbb{Z} -algebra map $\operatorname{ev}_{\chi,q}$: $\mathbb{Z}[\widehat{A}^{\operatorname{quad}}] \to \mathbb{C}$.

The notation for the map is meant to suggest an "evaluation" using the pair (χ, q) , but $ev_{\chi,q}$ can be regarded as well as a substitution map. Whenever there is no need to emphasize the dependency on the character χ , we shall write ev_q or even simply ev.

REMARK 3.2. The map $\operatorname{ev}_{\chi,q}$ sends t_x to $\chi \circ q(x)$ and $t_{x,y}$ to $\chi \circ b_q(x,y)$ for all $x, y \in A$. In particular, the map is induced by the substitution (algebra) map $\mathbb{Z}[QA] \to \mathbb{C}$ where QA is the abelian group freely generated by $t_x, t_{x,y}, x, y \in A$.

DEFINITION 3.3. A pair (χ, q) that consists of a character $\chi : \mathbb{Q}/\mathbb{Z} \to \mathrm{U}(1)$ and a quadratic function $q : A \to \mathbb{Q}/\mathbb{Z}$ will be called *nondegenerate* if $\chi \circ q$ is nondegenerate.

REMARK 3.3. A pair (χ, q) is nondegenerate if and only if χ is injective and q is nondegenerate.

PROPOSITION 1.3. If the pair (χ, q) is nondegenerate, then it induces a unique \mathbb{Z} -algebra map $ev_{\chi,q} : QR(A) \to \mathbb{C}$.

PROOF. By Lemma. 1.2, there is an induced \mathbb{Z} -algebra map $\operatorname{ev}_{\chi,q}$: $\mathbb{Z}[\hat{A}^{\operatorname{quad}}] \to \mathbb{C}$. This map induces the desired map if and only if $\operatorname{ev}_{\chi,q}$ is zero on the ideal generated by $\sum_{x \in A} t_{x,y}$, for all $y \neq 0$. We have

$$\operatorname{ev}_{\chi,q}\left(\sum_{x\in A} t_{x,y}\right) = \sum_{x\in A} \chi(b_q(x,y)).$$

Denote by S the latter sum. We wish to prove that S = 0 if $y \neq 0$. This will be a consequence of the following observation.

LEMMA 1.4. The following relation holds in $\mathbb{Z}[\widehat{A}^{\text{quad}}]$: for all $y, z \in A$,

(1.3)
$$\left(\sum_{x\in A} t_{x,y}\right) \cdot (1 - t_{z,y}) = 0.$$

PROOF OF LEMMA. Write

$$\sum_{x \in A} t_{x,y} = \sum_{x \in A} t_{x+z,y} = \sum_{x \in A} t_{x,y} t_{z,y} = \left(\sum_{x \in A} t_{x,y}\right) t_{z,y}.$$

Applying $ev_{\chi,q}$ to the relation (1.3), we find that $S \cdot (1 - \chi(b_q(z, y))) = 0$. Suppose $y \neq 0$. Since b_q is nondegenerate, there exists $z \in A$ such that $b_q(z, y) \neq 0$. Since χ is injective, it follows that $1 - \chi(b_q(z, y)) \neq 0$ hence S = 0.

The following exercises are intended to highlight some differences between the group ring $\mathbb{Z}[\hat{A}^{\text{quad}}]$ and the quadratic ring QR(A). They also illustrate the need of introducing the quadratic ring QR(A).

EXERCISE 3.1. Prove that for any $y \in A$, $\sum_{x \in A} t_{x,y} \neq 0$ in $\mathbb{Z}[\hat{A}^{\text{quad}}]$. [Hint: apply $\text{ev}_{\chi,q}$ for a suitable pair (χ, q) .]

EXERCISE 3.2. Prove the following result. Let $q : A \to \mathbb{Q}/\mathbb{Z}$ a nondegenerate quadratic function. There exists an element $a = a(q) \in \mathbb{Z}[\hat{A}^{\text{quad}}]$ with the following properties:

- (1) $\mathfrak{g}_A \overline{\mathfrak{g}_A} a = |A| a.$
- (2) $\operatorname{ev}_{\chi,q}(a) \neq 0$ for any injective character $\chi : \mathbb{Q}/\mathbb{Z} \to \mathrm{U}(1)$.
- (3) *a* is a zero divisor in $\mathbb{Z}[\hat{A}^{\text{quad}}]$.

Since a(q) is a zero divisor in $\mathbb{Z}[\hat{A}^{\text{quad}}]$, the result (1) does not imply that \mathfrak{g}_A is invertible in the localization of $\mathbb{Z}[\hat{A}^{\text{quad}}]$ with respect to the multiplicative set $\{1, |A|, |A|^2, \ldots\}$.

DEFINITION 3.4. Let $q : A \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a finite abelian group and $\chi : (\mathbb{Q}/\mathbb{Z}, +) \to (\mathrm{U}(1), \times)$ a character. The image in \mathbb{C} of the Gauss element by $\mathrm{ev}_{\chi,q}$, denoted

$$\Gamma_{\chi}(A,q) = \operatorname{ev}_{\chi,q}(\mathfrak{g}_A) = \sum_{x \in A} (\chi \circ q)(x)$$

is called the *unnormalized Gauss sum* associated to (A, q, χ) . It is convenient to define also

$$\gamma_{\chi}(A,q) = |A|^{-\frac{1}{2}} |A^{\perp}|^{-\frac{1}{2}} \Gamma_{\chi}(A,q)$$

as the *normalized Gauss sum* associated to q. Whenever the character is understood from the context, we shall suppress it from the notation.

EXAMPLE 3.1. The map $\chi : \mathbb{Q}/\mathbb{Z} \to U(1)$ defined by $\chi(t) = \exp(2\pi i t)$ is a character. The corresponding Gauss sum is called the "classical" Gauss sum:

$$\Gamma(A,q) = \sum_{x \in A} \exp(2\pi i q(x)).$$

The original "Gauss sum" was introduced by C.F. Gauss in 1801: it is the sum of the example above for $A = \mathbb{Z}/p\mathbb{Z}$ where p is an odd prime and q is the quadratic form defined by $q(x) = x^2/p \mod 1$. Gauss proves in [34] that $\Gamma(A,q)^2 = (-1)^{\frac{p-1}{2}} p$. The applications of Gauss sums in number theory are ubiquitous. See for instance [5] for a survey. For a recent application, see [3].

1.1. Basic properties of Gauss sums. We begin with elementary and fundamental properties. The first one is the invariance of the Gauss sum of q under an isomorphism of q. The second one is the behaviour of the Gauss sum under orthogonal sum and sign reversal of quadratic functions. The third one is the behaviour of the Gauss sum under translation of the quadratic function by the action of the group of homomorphisms.

We consider the Gauss element first.

LEMMA 1.5. A homomorphism $\varphi : A \to B$ induces a homomorphism $\varphi^{\text{quad}} : \mathbb{Z}[\hat{A}^{\text{quad}}] \to \mathbb{Z}[\hat{B}^{\text{quad}}]$ such that $\varphi^{\text{quad}}(\mathfrak{g}_A) = |\text{Ker}(\varphi)| \cdot \mathfrak{g}_{\varphi(A)}.$

COROLLARY 1.6. An isomorphism of quadratic $\varphi : (A,q) \to (A',q')$ functions such that $q' \circ \varphi = q$ induces an isomorphism $\hat{\varphi} : \mathbb{Z}[\widehat{A}^{\text{quad}}] \to \mathbb{Z}[\widehat{A'}^{\text{quad}}]$ such that $\hat{\varphi}(g_A) = g_{A'}$ and $\operatorname{ev}_{q'} \circ \hat{\varphi} = \operatorname{ev}_q$.

Hence the Gauss element is to be seen as a kind of characteristic element as the level of the group ring $\mathbb{Z}[\hat{A}^{\text{quad}}]$.

COROLLARY 1.7. If $q: A \to \mathbb{Q}/\mathbb{Z}$ and $q': A' \to \mathbb{Q}/\mathbb{Z}$ are isomorphic quadratic functions, then $\gamma(A, q) = \gamma(A', q')$.

The next properties, immediate from the definition, are the behavior with respect to the orthogonal sum of quadratic functions and the natural involution (opposite) of quadratic functions:

Lemma 1.8.

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(1.4)
$$\gamma((A,q) \oplus (A',q')) = \gamma(A,q) \cdot \gamma(A',q'),$$

(1.5)
$$\gamma(A, -q) = \gamma(A, q).$$

These properties shall be used in the classical context of the Witt group of quadratic functions (See Chap...) and of the reciprocity.

Another useful property is the behaviour of the Gauss sum when a homomorphism is added to the quadratic function:

Lemma 1.9.

(1.6)
$$\gamma(A, q + \hat{b}_q(\alpha)) = \gamma(A, q) \ \chi(-q(\alpha)).$$

PROOF. One "completes the square"

$$q(x) + b_q(\alpha, x) = q(x + \alpha) - q(\alpha),$$

apply χ and sum over $x \in A$.

By summing over all elements in A the identity (1.6), we obtain the following corollary:

Corollary 1.10.

(1.7)
$$\sum_{\alpha \in A} \Gamma(A, q + \hat{b}_q(\alpha)) = |\Gamma(A, q)|^2.$$

The next classical property is the computation of the absolute value of the Gauss sum.

PROPOSITION 1.11. $|\gamma(A,q)| = 1$ if and only if $q|_{A^{\perp}} = 0$; $\gamma(A,q) = 0$ if and only if $q|_{A^{\perp}} \neq 0$.

In particular, the nullity of the Gauss sum detects whether the quadratic function is tame or not.

FIRST PROOF. Set $A' = A/A^{\perp}$ and denote by p the canonical projection $A \to A'$. By Lemma 1.5, $p^{\text{quad}}(\mathfrak{g}_A) = |A^{\perp}| \cdot \mathfrak{g}_{A'}$. Suppose first that the quadratic function is tame. Then it induces a nondegenerate quadratic function $q' : A/A^{\perp} \to \mathbb{Q}/\mathbb{Z}$ such that $q' \circ p = q$ (Lemma 5.2). Hence by Prop. 1.3, we can apply $\operatorname{ev}_{q'}$ to the identity $\mathfrak{g}_{A'}\overline{\mathfrak{g}_{A'}} = |A|$ and find that $|\Gamma(A',q')|^2 = \Gamma(A',q') \cdot \overline{\Gamma(A',q')} = |A'|$. Let $p^{\operatorname{quad}} : \operatorname{QR}(A) \to \operatorname{QR}(A')$ denote the ring map induced from the projection p. Then

$$\Gamma(A,q) = \operatorname{ev}_q(\mathfrak{g}_A) = \operatorname{ev}_{q'} \circ p^{\operatorname{quad}}(\mathfrak{g}_A) = \operatorname{ev}_{q'}\left(|A^{\perp}|\mathfrak{g}_{A'}\right) = |A^{\perp}|\Gamma(A',q').$$

Hence

$$|\Gamma(A,q)| = |A^{\perp}| |\Gamma(A',q')| = |A^{\perp}| |A'|^{\frac{1}{2}} = |A^{\perp}|^{\frac{1}{2}} |A|^{\frac{1}{2}},$$

which is the desired result. It remains to see that if q is nontame, then $\Gamma(A,q) = 0$. Let $A/A^{\perp} = \{[x_1], \ldots, [x_n]\}$. We compute in $\mathbb{Z}[\hat{A}^{\text{quad}}]$:

$$\mathfrak{g}_A = \sum_{i=1}^n \sum_{x \in A^\perp} t_{x_i+x} = \sum_{i=1}^n \sum_{x \in A^\perp} t_{x_i} t_x t_{x_i,x}.$$

Applying $ev_{\chi,q}$, we find that

$$\Gamma_{\chi}(A,q) = \sum_{i=1}^{n} \sum_{x \in A^{\perp}} \chi(q(x_i)) \, \chi(q(x)) \, \chi(b_q(x_i,x)) = \sum_{i=1}^{n} \sum_{x \in A^{\perp}} \chi(q(x_i)) \, \chi(q(x))$$
$$= \left(\sum_{i=1}^{n} \chi(q(x_i))\right) \left(\sum_{x \in A^{\perp}} \chi(q(x))\right)$$

Now observe that $q|_{A^{\perp}}$ is a nontrivial homomorphism, hence the right sum on the right hand side is zero.

SECOND PROOF. The proof uses twice the following classical argument:

LEMMA 1.12. Let $w : A \to U(1)$ be a homomorphism on a finite group A. Then

$$\sum_{x \in A} w(x) = \begin{cases} |A| & if \ w = 0; \\ 0 & otherwise. \end{cases}$$

PROOF OF LEMMA. Let $y \in A$. Since $x \mapsto yx$ is bijective,

$$\sum_{x\in A} w(x) = \sum_{x\in A} w(yx) = \sum_{x\in A} w(y) \ w(x).$$

Therefore,

$$(1-w(y)) \sum_{x \in A} w(x) = 0.$$

If there is some y such that $w(y) \neq 0$, the identity above implies $\sum_{x \in A} w(x) = 0$. Otherwise, the sum equals $1 + \cdots + 1 = |A|$.

By Cor. 1.10,

$$|\Gamma(A,q)|^2 = \sum_{\alpha \in A} \sum_{x \in A} \chi(q(x) + b_q(x,\alpha)).$$

Exchanging the sums, we find that

$$|\Gamma(A,q)|^2 = \sum_{x \in A} \chi(q(x)) \sum_{\alpha \in A} \chi(b_q(x,\alpha)).$$

By Lemma 1.12,

$$\sum_{\alpha \in A} \chi(b_q(x, \alpha)) = \begin{cases} |A| & \text{if } \hat{b}_q(x) = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$|\Gamma(A,q)|^2 = |A| \cdot \sum_{x \in A^{\perp}} e^{2\pi i q(x)}.$$

Note that $q|_{A^{\perp}}: A^{\perp} \to \mathbb{Q}/\mathbb{Z}$ is a homomorphism. By Lemma 1.12 again,

$$\sum_{x \in A^{\perp}} e^{2\pi i q(x)} = \begin{cases} |A^{\perp}| & \text{if } q|_{A^{\perp}} = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $|\Gamma(G,q)|^2 = |A| \cdot |A^{\perp}|$ or 0 according to whether $q|_{A^{\perp}} = 0$ or not. The result follows.

1.2. Gauss sums and the discriminant.

LEMMA 1.13. Let $q: A \to \mathbb{Q}/\mathbb{Z}$ be a homogeneous quadratic form and let α be an element of order 2 in A. Let $f: V \times V \to \mathbb{Z}$ be a symmetric bilinear lattice. For any integral Wu class $w \in \mathrm{Wu}^V(f)$,

(1.8)
$$\gamma(A \otimes V, (q \cdot \alpha) \otimes f) = \gamma(A \otimes V, q \otimes f) \ \chi(-(q \otimes f)(\alpha \otimes v)).$$

PROOF. We observe that

$$(q \cdot \alpha)(x) f(y,y) = (q(x) + b_q(\alpha)(x) f(y,y))$$

= $q(x) f(y,y) + b_q(\alpha,x) f(y,y)$
= $q(x) f(y,y) + b_q(\alpha,x) f(w,y)$
= $q(x) f(y,y) + (b_q \otimes f)(\alpha \otimes w, x \otimes y).$
Here in the third equality we used the fact that $f(w, y) = f(y, y) \mod 2$ for any Wu class $w \in Wu^V(f)$. Hence $(q \cdot \alpha) \otimes f = q \otimes f + b_q \otimes f(\alpha \otimes w)$. The result follows by applying Lemma 1.9 to $q \otimes f$ with element $\alpha \otimes w$.

Let $f: V \times V \to \mathbb{Z}$ and $g: W \times W \to \mathbb{Z}$ be two nondegenerate bilinear lattices. Recall that the subgroups $A = j_f(G_f \otimes W)$ and $B = j_g(V \otimes G_g)$ are mutually orthogonal in $G_{f \otimes g}$ with respect to the discriminant linking pairing $\lambda_{f \otimes g}$ (See §2, Lemma 2.8). As before, we set $H = A \cap B = A \cap A^{\perp}$. Recall also the natural isomorphism $\psi: G_{f \otimes g}/H \to G_f \otimes G_g$.

THEOREM 1.14. Let $z \in Wu(f \otimes g)$ be a Wu class. The following assertions are equivalent:

- (1) $\gamma(G_f \otimes W, \varphi_{f \otimes g, z} \circ j_f) \neq 0.$
- (2) $\gamma(V \otimes G_g, \varphi_{f \otimes g, z} \circ j_g) \neq 0.$
- (3) $\varphi_{f\otimes g,z}|_H = 0.$
- (4) The Wu class $z \in Wu(f \otimes g)$ is sent to 0 under the natural projection $Wu(f \otimes g) \to Wu(f \otimes g)/2(V^{\sharp} \otimes W + V \otimes W^{\sharp}).$
- (5) $\psi([t])$ is the characteristic element associated to (λ_f, λ_g) for a decomposition $z = z_0 + 2t$ with $z_0 \in \operatorname{Wu}_0^{1/2}(f \otimes g), t \in V^{\sharp} \otimes W^{\sharp}, [t] \in G_{f \otimes g}/H^{\perp}.$

PROOF. Since j_f is injective (Lemma 2.7), $(G_f \otimes W, \varphi_{f \otimes g, z} \circ j_f) \simeq (A, \varphi_{f \otimes g, z}|_A)$. The annihilator of $\varphi_{f \otimes g, z}|_A$ is $A \cap A^{\perp} = H$. Therefore, applying Cor. ?? to the quadratic function $\varphi_{f \otimes g, z}|_A$ gives the equivalence (1) \iff (3). A similar argument yields (2) \iff (3). The equivalence (3) \iff (4) follows from Lemma 6.4 (§4). The equivalence (3) \iff (5) is Theorem 10.5.

COROLLARY 1.15. If a Wu class $z \in Wu(f \otimes g)$ satisfies $\varphi_{f \otimes g, z}|_H = 0$ then $z \in Wu^{1/2}(f \otimes g)$.

COROLLARY 1.16. Let $(v, w) \in Wu^V(f) \times Wu^W(g)$. The following assertions are equivalent:

- (1) $\gamma(G_f \otimes W, \varphi_{f,v} \otimes g) \neq 0.$
- (2) $\gamma(V \otimes G_g, f \otimes \varphi_{g,w}) \neq 0.$
- (3) $\varphi_{f,v} \otimes g \mid_{\operatorname{Ker}(\widehat{\lambda}_f \otimes \widehat{g})} = 0.$
- (4) $f \otimes \varphi_{g,w} \mid_{\operatorname{Ker}(\widehat{f} \otimes \widehat{\lambda}_g)} = 0.$
- (5) The characteristic homomorphism χ associated to λ_f and λ_g is zero.
- (6) The characteristic element associated to λ_f and λ_g) is zero.

Gauss sums play a fundamental rôle in the classification of pointed linking pairings.

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2. Classification of pointed linking pairings

The use of Gauss sums in the classification of isomorphism classes of linking pairings goes back to Minkowski; it is proved in [51, Theorem 4.1] that Gauss sums form a complete system of invariants.

Using a different approach, we generalize this result to the classification of isomorphism classes of pointed linking pairings.

DEFINITION 3.5. A pointed linking pairing is a pair formed by a linking pairing (G, λ) and an element $c = (c_1, \ldots, c_n) \in G^n$ for some natural integer n. Two pointed linking pairings (G, λ, c) and (G', λ', c') are *isomorphic* if there is an isomorphism of linking pairings that sends c onto c'.

In the sequel of this paragraph, we fix a nondegenerate pointed linking pairing (G, λ, c) , with $c = (c_1, \ldots, c_n) \in G^n$, $n \ge 1$. Consider now a triple (V, h, s) where $h: V \to \mathbb{Z}$ is a homogeneous nondegenerate quadratic function on a lattice V and $s = (s_1, \ldots, s_n) \in (V^*)^n$. We form a new quadratic function on $V \otimes G$ defined by

(2.1)
$$h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)$$

where $h \otimes \lambda$ is the usual tensor product of a homogeneous quadratic function and a linking pairing and $s \otimes c = \sum_j s_j \otimes c_j \in V^* \otimes G$. Explicitly

$$(h \otimes \lambda) \left(\sum_{j} x_j \otimes y_j \right) = \sum_{j} h(x_j) \lambda(x_j, x_j) + \sum_{j < k} b_h(x_j, x_k) \lambda(x_j, x_k), \quad x_j \in V, \ y_j \in G$$

where b_h is the bilinear symmetric pairing associated to h. Here $(\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c)$ is the homomorphism $V \otimes G \to \mathbb{Q}/\mathbb{Z}$ defined by

$$(\mathrm{id}_{V^*}\otimes\widehat{\lambda})(s\otimes c)(x\otimes y) = \sum_i s_i(x)\lambda(c_i,y), \ x\in V, \ y\in G.$$

Let $\Gamma_{h,s}(\lambda, c) = \Gamma\left(V \otimes G, h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \hat{\lambda})(s \otimes c)\right)$ be the (unnormalized) Gauss sum associated to the quadratic function defined above by (2.1). It is convenient to consider as well the normalized Gauss sum $\gamma_{h,s}(\lambda, c) =$ $\gamma\left(V \otimes G, h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \hat{\lambda})(s \otimes c)\right).$

LEMMA 2.1. If $s \otimes c = (\hat{b}_h \otimes id_G)(y)$ for some $y \in V \otimes G$, then

(2.2)
$$\gamma_{h,s}(\lambda,c) = \gamma(V \otimes G, h \otimes \lambda) \cdot e^{-2\pi i (h \otimes \lambda)(y)}.$$

PROOF. According to the hypothesis,

$$\gamma_{h,s}(\lambda,c) = \gamma \left(V \otimes G, h \otimes \lambda + (\mathrm{id}_{V^*} \otimes \widehat{\lambda})(s \otimes c) \right)$$
$$= \gamma \left(V \otimes G, h \otimes \lambda + (\widehat{b}_h \otimes \widehat{\lambda})(y) \right)$$
$$= \gamma (V \otimes G, h \otimes \lambda) \cdot e^{-2\pi i (h \otimes \lambda)(y)} \qquad \text{by Lemma 1.9.}$$

Also we need to define invariants extracted from the group G. Recall that every linking pairing (G, λ) splits in an orthogonal decomposition into linking pairings

$$(G,\lambda) = \bigoplus_p (G_p,\lambda_p)$$

on *p*-groups, where *p* describes a finite subset of primes. Furthermore, every linking pairing (G_p, λ_p) splits in an orthogonal decomposition into linking pairings

$$(G_p, \lambda_p) = \bigoplus_{k \ge 1} (G_p^k, \lambda_p^k)$$

where each G_p^k is a free \mathbb{Z}/p^k -module and hence has a well defined rank. Set

$$\rho_p^k(\lambda) = \operatorname{rank} \, G_p^k \in \mathbb{N}.$$

Clearly the ranks $\rho_p^k(\lambda)$ depend only on the underlying group G, are additive under direct sums and only finitely many of them are non zero.

THEOREM 2.2. Two pointed linking pairings (G, λ, c) and (G', λ', c') with distinguished n-tuples $c \in G^n$ and $c' \in (G')^n$ are isomorphic if and only if the following conditions are satisfied:

- (1) $\rho_p^k(\lambda) = \rho_p^k(\lambda')$ for all prime p and all $k \ge 1$;
- (2) $\gamma_{h,s}(\lambda, c) = \gamma_{h,s}(\lambda', c')$ for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^n$.

We make a few observations on Th. 2.2. Condition (1) is purely grouptheoretic and does not involve the pairings nor the distinguished elements.

In condition (2), only a finite number of Gauss sums is required. However, it can be shown that one needs, in the most general case, to consider at least one rank 2 lattice (V, f).

If we replace the Gauss sums $\gamma_{h,s}(\lambda, c)$ by unnormalized Gauss sums $\Gamma_{h,s}(\lambda, c)$ in condition (2), then condition (1) becomes redundant. Taking the absolute value of appropriately chosen unnormalized Gauss sums yield the invariants of condition (1).

The classification of linking pairings without distinguished point (see [51, Th. 4.1]) is recovered from Th. 2.2 by taking n = 1 and setting c and c' to be the zero element of G and G' respectively. The Gauss sum $\gamma_{h,s}(\lambda, 0)$ then is just $\gamma(V \otimes G, h \otimes \lambda)$.

PROOF. We give an abridged proof here, referring to [22] for details. The proof is based on two lemmas.

The first lemma is a "reduction to linear algebra" based on the classification of linking pairings. Let $N \ge 1$. Recall that n denotes the number of distinguished elements. Denote by $\mathcal{R}_{N,n}$ (resp. \mathcal{R}_N) the vector space of matrices with N rows and n columns (resp. the vector space of square symmetric matrices of size N) with entries in \mathbb{Q}/\mathbb{Z} . For $r = (r_{jk})_{1 \le j,k \le N} \in \mathcal{R}_N$ and $r' \in \mathcal{R}_{N,n}$, set

 $S_{r,r'}(\lambda, c) = \{(x_1, \dots, x_N) \in G^N \mid \lambda(x_j, x_k) = r_{jk} \text{ and } \lambda(x_j, c_k) = r'_{jk} \}.$

This set is clearly finite and we denote its cardinality by $|S_{r,r'}(\lambda, c)|$.

LEMMA 2.3. Two pointed linking pairings (G, λ, c) and (G', λ', c') with n distinguished elements are isomorphic if and only if $\rho_p^k(\lambda) = \rho_p^k(\lambda')$ for all prime p and all $k \ge 1$ and $|S_{r,r'}(\lambda, c)| = |S_{r,r'}(\lambda', c')|$ for all matrices $r \in \mathcal{R}_N$ and $\mathcal{R}_{N,n}$ for N large enough.

The second lemma is classical.

LEMMA 2.4. A family of distinct characters is free over \mathbb{C} .

We interpret the unnormalized Gauss sums $\Gamma_{h,s}(\lambda, c)$ as characters and related them to the invariants $|S_{r,r'}(\lambda, c)|$. Fix a basis of V and identify hwith a square symmetric matrix of size N, each $s_j \in V^*$, $1 \leq j \leq N$, with a vector $(s_{jk})_{1 \leq k \leq N} \in \mathbb{Z}^N$. Then

(2.3)
$$\Gamma_{h,s}(\lambda,c) = \sum_{\substack{r \in \mathcal{R}_N \\ r' \in \mathcal{R}_{N,n}}} |S_{r,r'}(\lambda,c)| \exp\left(2\pi i \operatorname{Trace}\left(hr + sr'\right)\right).$$

The sum is finite since only finitely many terms are non zero. The maps

$$\kappa_{r,r'}: (h,s) \mapsto \exp\left(2\pi i \operatorname{Trace}\left(fr + sr'\right)\right)$$

are distinct characters, hence the family $(\kappa_{r,r'})_{r,r'}$ is free over \mathbb{C} . On the other hand, only finitely many sets $S_{r,r'}(\lambda, c)$ are non empty. Therefore there is a finite number M of homogeneous quadratic functions equipped with multiforms $(h^1, s^1), \ldots, (h^M, s^M)$ such that the matrix $\Phi = (\kappa_{r_j, r'_j}(h^k, s^k))_{1 \leq j,k \leq M}$ is invertible over \mathbb{C} . Set

$$\Gamma = (\Gamma_{h^j, s^j}(\lambda, c))_{1 \le j \le M}, \quad S = (|S_{r_j, r'_j}(\lambda, c)|)_{1 \le j \le M}.$$

We deduce from (2.3) the identity

$$\Gamma = \Phi \cdot S.$$

Since Φ is invertible, Γ determines S and conversely. The result then follows from Lemma 2.3.

The Gauss sum $\gamma_{h,s}(\lambda, c)$ and the quadratic function (2.1) can be interpreted using the discriminant construction as follows. First,

(2.4)
$$h \otimes \lambda + (\operatorname{id}_{V^*} \otimes \widehat{\lambda})(s \otimes a) = (b_h \otimes q) + (\widehat{b}_{h\mathbb{Q}} \otimes \widehat{\lambda})(\xi \otimes c),$$

where on the right hand side:

- q is a homogeneous quadratic refinement of λ ;
- $b_h: V \times V \to \mathbb{Z}$ denotes the symmetric bilinear pairing associated to the quadratic form $h: V \to \mathbb{Z}$;
- $\xi \in (V^{\sharp})^n$ is defined by $\widehat{f}_{\mathbb{Q}}(\xi_j) = s_j, \ 1 \leq j \leq n;$
- $\xi \otimes c = \sum_{j} \xi_j \otimes c_j;$

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• The map $(\hat{b}_{h\mathbb{Q}} \otimes \hat{\lambda})(\xi \otimes c) : V \times G \to \mathbb{Q}/\mathbb{Z}$ is induced by the map adjoint to $b_h \otimes \lambda$ at $\xi \otimes c$. It is defined by

$$(\hat{b}_{h\mathbb{Q}}\otimes\hat{\lambda})(\xi\otimes c)(x\otimes y)=(b_h)_{\mathbb{Q}}(\xi,x)\ \lambda(c,y)\ x\in V,\ y\in G.$$

It follows from (9.2) that the quadratic function on the right hand side identifies to $\varphi_{b_h \otimes g, v \otimes w - 2t} \circ j_g$ where (W, g, w) is a bilinear lattice equipped with an integral Wu class $w \in Wu(g)$ such that $(G, q) = (W^{\sharp}/W, \varphi_{g,w}),$ $v \in Wu(b_h)$ is a Wu class for b_h and $t \in V^{\sharp} \otimes W^{\sharp}$ is a lift of $\xi \otimes c \in V^{\sharp} \otimes G$. We conclude that

(2.5)
$$\gamma_{h,s}(\lambda,c) = \gamma(V \otimes G, \varphi_{b_h \otimes g, v \otimes w - 2t} \circ j_g).$$

COROLLARY 2.5. Two linking pairings (G, λ) and (G', λ') are isomorphic if and only if the following conditions are satisfied:

- (1) $\rho_p^k(\lambda) = \rho_p^k(\lambda')$ for all prime p and all $k \ge 1$;
- (2) $\gamma(V \otimes G, h \otimes \lambda) = \gamma(V \otimes G', h \otimes \lambda')$ for all lattices V equipped with a homogeneous quadratic function $h: V \to \mathbb{Z}$.

3. The classification of pointed quadratic functions

The results of the previous paragraph are generalized to pointed quadratic functions. A pointed quadratic function on a finite abelian group G consists of a quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$ equipped with $c = (c_1, \ldots, c_n) \in G^n$ for some integer $n \ge 0$. Two pointed quadratic functions (G, q, c) and (G', q', c') are *isomorphic* if there is an isomorphism of linking pairings that sends c_j onto c'_j , $1 \le j \le n$.

Before stating the theorem of this paragraph, we recall two simple definitions. Given a quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$, the difference $d_q(x) = q(x) - q(-x), x \in G$, defines a homomorphism $G \to \mathbb{Q}/\mathbb{Z}$, the homogeneity defect. This map is zero if and only if q is homogeneous. Recall that a quadratic function canonically induces an associated linking pairing b_q . Hence there is a well defined surjective ("forgetful") homomorphism

$$(G, q, c) \mapsto (G, b_q, c)$$

from the monoid of pointed (nondegenerate) quadratic functions (with distinguished n-tuples) to the monoid of pointed (nondegenerate) linking pairings (with distinguished n-tuples). We shall use a related but distinct homomorphism

$$(G, q, c) \mapsto (G, b_q, c \oplus \tilde{b}_q^{-1} d_q)$$

from the monoid of pointed (nondegenerate) quadratic functions with distinguished *n*-tuples to the monoid of pointed (nondegenerate) linking pairings with distinguished (n + 1)-tuples. Here the adjoint map $\hat{b}_q : G \to G^*$ is bijective hence $\hat{b}_q^{-1}d_q$ is a well defined element in G and $c \oplus \hat{b}_q^{-1}d_q$ denotes the (n + 1)-tuple obtained by adjoining the form $d_q \in G^*$ to the *n*-tuple $c = (c_1, \ldots, c_n)$ on the right. This latter map is not onto. (It is onto if we restrict the image to pointed linking pairings with (n + 1)-tuples of distinguished points whose last distinguished point lies in 2G.) THEOREM 3.1. Two pointed quadratic functions (G, q, c) and (G', q', c') with distinguished n-tuples are isomorphic if and only if the following conditions are satisfied:

- (1) ρ^k_p(b_q) = ρ^k_p(b_{q'}) for all prime p and all k ≥ 1;
 (2) γ_{h,s}(b_q, c ⊕ b̂_q⁻¹d_q) = γ_{h,s}(b_{q'}, c' ⊕ b̂_q⁻¹d_{q'}) for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^{n+1}$;
- (3) $\gamma(G,q) = \gamma(G',q')$ and $\gamma(G,q+\hat{b}_q(c_i)) = \gamma(G,q'+\hat{b}_{\sigma'}(c'_i)), i =$ $1, \ldots, n.$

Remarks similar to those to Th. 2.2 also apply to Th. 3.1.

PROOF. Clearly if the pointed quadratic functions are isomorphic then the conditions are verified. Let us prove the converse. Suppose the conditions are satisfied. Then the conditions of Th. 2.2 are satisfied. Therefore, $(G, b_q, c \oplus \hat{b_q}^{-1} d_q)$ and $(G', b_{q'}, c' \oplus \hat{b_{q'}}^{-1} d_{q'})$ are isomorphic pointed linking pairings. Explicitly, let $\phi : G \to G'$ an isomorphism such that $\phi^* b_{q'} = b_q$ and $\phi(c) = c'$ and $\phi^* d_{q'} = d_q$. Replacing the triple (G', q', c') by the isomorphic triple $(G, \phi^* q', \phi^* c') = (G, \phi^* q', c)$, it is enough to show that (G, q, c)and (G, ϕ^*q', c) are isomorphic. Note that $d_{\phi^*q'} = \phi^*d_{q'} = d_q$. So we may assume that (G, q, c) and (G, q', c) are two pointed quadratic functions over the same associated bilinear linking pairing, with the same homogeneity defect and the same distinguished elements satisfying the conditions (1), (2)and (3). Let us construct an isomorphism between (G, q, c) and (G, q', c). Since q and q' are quadratic functions over the same nondegenerate linking pairing, they differ by some $\alpha \in G$: $q' = q + \hat{b_q}(\alpha)$. The equality $d_q = d_{q'}$ implies that $2\alpha = 0$. Since $\gamma(G, q) = \gamma(G, q') = \gamma(G, q) e^{2\pi i q(\alpha)}$, we deduce that $q(\alpha) = 0$. Define a map $[\mathfrak{n}]: G \to \mathbb{Z}/2\mathbb{Z}$ by $b_q(\alpha, x) = \frac{[\mathfrak{n}](x)}{2} \mod 1$ for all $x \in G$. Clearly [n] is a homomorphism. Since q(a) = 0,

$$0 = q(2\alpha) = q(\alpha) + q(\alpha) + b_q(\alpha, \alpha) = b_q(\alpha, \alpha),$$

hence $[\mathfrak{n}](\alpha) = 0$. Consider the map

$$\psi: G \to G, \ x \mapsto x + \mathfrak{n}(x)\alpha$$

where $\mathfrak{n}(x) \in \mathbb{Z}$ is an arbitrary lift of $[\mathfrak{n}](x) \in \mathbb{Z}/2\mathbb{Z}$. Since

$$\psi^{2}(x) = \psi(x + \mathfrak{n}(x)\alpha) = x + \mathfrak{n}(x)\alpha + \mathfrak{n}(x + \mathfrak{n}(x)\alpha)\alpha$$

$$= x + \mathfrak{n}(x)\alpha + \mathfrak{n}(x)\alpha + \mathfrak{n}(x)\mathfrak{n}(\alpha)\alpha$$

$$= x + \mathfrak{n}(x) 2\alpha + 0$$

$$= x + 0$$

$$= x,$$

 ψ is an involutive automorphism of G. Furthermore,

$$q(\psi(x)) = q(x+\mathfrak{n}(x)\alpha) = q(x)+\mathfrak{n}(x) \ b_q(x,\alpha)+q(\mathfrak{n}(x)\alpha) = q(x)+b_q(x,\alpha)+0 = q'(x)$$
for any $x \in G$. Therefore $\psi^*q' = q$. Since

$$\gamma(G,q) \ e^{2\pi i q(c_i)} = \gamma(G,q + \hat{b}_q(c_i)) = \gamma(G,q' + \hat{b}_{q'}(c_i)) = \gamma(G,q') \ e^{2\pi i q'(c_i)} = \gamma(G,q) \ e^{2\pi q'(c_i)},$$

we deduce that $q(c_i) = q'(c_i) = q(c_i) + b_q(\alpha, c_i)$, hence $b_q(\alpha, c_i) = 0$. Hence $\psi(c_i) = c_i + \mathfrak{n}(c_i)\alpha = c_i$

for all i = 1, ..., n. Therefore ψ is an isomorphism between (G, q, c) and (G, q', c) as desired.

REMARK 3.4. The system of invariants of Th. 3.1 is minimal in the sense that if one equality among the equalities of conditions (1)-(3) is not satisfied then there is a pair of nonisomorphic pointed quadratic functions satisfying all the other equalities.

EXAMPLE 3.2. As an illustration of the previous remark, we point out that there exist nonisomorphic pointed quadratic functions (G, q, c) and (G, q', c')such that (G,q) and (G',q') are isomorphic and the associated pointed linking pairings (G, b_q, c) and $(G, b_{q'}, c')$ are isomorphic. Such an example is provided by

$$(\mathbb{Z}/16\mathbb{Z}, q(k \mod 16) = \frac{x^2 + 4x}{32} \mod 1, c = 1 \mod 16)$$

and $(\mathbb{Z}/16\mathbb{Z}, q(k \mod 16) = \frac{-7x^2 + 20x}{32} \mod 1, c' = 3 \mod 16).$

The map $x \mapsto 3x$ provides the isomorphism between the associated pointed linking pairings, the map $x \mapsto 5x$ provides the isomorphism between the quadratic functions, but there is no isomorphism between the pointed quadratic functions. In terms of invariants, one checks that $\gamma(G, q + \hat{b}_q(c)) \neq$ $\gamma(G, q' + \hat{b_{q'}}(c'))$. All other equalities in the statement of Th. 3.1 are satisfied.

COROLLARY 3.2. Suppose that $c \in (2G)^n$. Two pointed quadratic functions (G, q, c) and (G', q', c') with distinguished n-tuples are isomorphic if and only if the following conditions are satisfied:

- (1) ρ^k_p(b_q) = ρ^k_p(b_{q'}) for all prime p and all k ≥ 1;
 (2) γ_{h,s}(b_q, c ⊕ b̂_q⁻¹d_q) = γ_{h,s}(b_{q'}, c' ⊕ b̂_q⁻¹d_{q'}) for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^{n+1}$;
- (3) $\gamma(G,q) = \gamma(G',q').$

PROOF. Conditions (1) and (2) imply that $c' \in (2G)^n$. In the proof of Th. 3.1, the last equality of condition (3) is used only to ensure that $b_q(\alpha, c_i) = 0$. But the condition $c_i \in 2G$ already implies that equality for all $i=1,\ldots,n.$

REMARK 3.5. Cor. 3.2 applies in particular if the underlying group has odd order.

Another special case worth considering is the case of pointed homogeneous quadratic functions: it turns out that this case is analogous to the case of pointed linking pairings. Fix a nondegenerate pointed homogeneous quadratic function (G, q, c) with a distinguished *n*-tuple $c \in G$. Consider a triple (V, f, s) where (V, f) is a bilinear lattice and s an element in $(V^*)^n$. We form the quadratic function on $V \otimes G$

$$f \otimes q + (\mathrm{id}_{V^*} \otimes \widehat{b}_q)(s \otimes c)$$

and denote by

(3.1)
$$\gamma_{f,s}(q,c) = \gamma \left(V \otimes G, f \otimes q + (\mathrm{id}_{V^*} \otimes \widehat{b}_q)(s \otimes c) \right)$$

the corresponding Gauss sum.

COROLLARY 3.3. Two pointed homogeneous quadratic functions (G, q, c) and (G', q', c') with distinguished n-tuples are isomorphic if and only if the following conditions are satisfied:

- (1) $\rho_p^k(b_q) = \rho_p^k(b_{q'})$ for all prime p and all $k \ge 1$;
- (2) $\gamma_{f,s}(q,c) = \gamma_{f,s}(q',c')$ for all triples (V,f,s) of bilinear lattices (V,f) equipped with a multiform $s \in (V^*)^n$.

PROOF. Since q is homogeneous, $d_q = 0$, hence $\gamma_{h,s \oplus s_{n+1}}(b_q, c \oplus 0) = \gamma_{h,s}(b_q, c)$ for all triples (V, h, s) of lattices V equipped with a homogeneous quadratic function h and a multiform $s \in (V^*)^n$. Observe that

$$h \otimes b_q = b_h \otimes q$$

for any homogeneous quadratic function $h: V \to \mathbb{Z}$ and homogeneous quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$. Hence $\gamma_{h,s}(b_q, c) = \gamma_{b_h,s}(q, c)$. The second observation is that $\gamma(G, q) = \gamma_{f,0}(q, c)$ and $\gamma(G, q + \hat{b}_q(c_i)) = \gamma_{f,1}(q, c_i)$ with the pointed bilinear lattice $V = \mathbb{Z}$, f(1, 1) = 1, s = 0 and $s = 1_{\mathbb{Z}}$ respectively. The result follows from Th. 3.1.

REMARK 3.6. It is possible to give a proof of Corollary 3.3 along the lines of Th. 2.2. We leave it to the reader to prove directly that the isomorphism class of (G, q, c) is classified by condition (1) and the Gauss sums $\gamma_{f,s}(q, c)$ for all triples (V, f, s). Since $\gamma(G, q) = \gamma_{f,0}(q, c)$ for $V = \mathbb{Z}, f(1, 1) = 1$, this easily implies the result. As mentioned above, this line of proof is valid for homogeneous quadratic functions only.

As the particular case of quadratic functions with no distinguished element (or with trivial element), we recover the classification of quadratic functions [19, Th. 4.1].

COROLLARY 3.4. Two quadratic functions (G, q) and (G', q') are isomorphic if and only if $\gamma(G, q) = \gamma(G', q')$ and there is an isomorphism ϕ of their associated linking pairings such that $d_{q'} \circ \phi = d_q$.

REMARK 3.7. As illustrated in Example 3.2, the classification of pointed quadratic functions cannot be recovered by Corollary 3.4 and Theorem 2.2 alone.

4. Linking groups and pointed linking groups

We generalize the notion of linking pairings to allow non torsion elements in the underlying group. Let $n \ge 1$. First we extend slightly the definition linking pairings. A linking group is a pair (G, λ) where G is a finitely generated abelian group and λ : Tors $G \times \text{Tors } G \to \mathbb{Q}/\mathbb{Z}$ is a linking pairing. According to our terminology, a linking pairing is a torsion linking group (i.e., the underlying group G in the definition is a torsion group). If G has no torsion element, then the linking group (G, λ) reduces to the underlying group G. An *isomorphism* between linking groups (G, λ) and (G', λ') is an isomorphism $\phi : G \to G'$ of groups such that $\phi|_{\text{Tors } G}$: Tors $G \to \text{Tors } G'$ verifies $(\phi|_{\text{Tors } G})^*(\lambda') = \lambda$. In other words, an isomorphism of linking groups is a group isomorphism whose restriction to torsion induces an isomorphism of linking pairings.

EXAMPLE 3.3. Any bilinear lattice (V, f) induces a linking group $(G^f = \text{Coker } \hat{f}, \lambda^f)$ by formula (??). (See §??.) This linking group is called the *discriminant linking group*.

Two equivalent lattices (V, f) and (V', f') induce isomorphic discriminant linking pairings if and only the induced nondegenerate bilinear lattices (\bar{V}, \bar{f}) and (\bar{V}', \bar{f}') are stably equivalent; however the induced discriminant linking groups may be non isomorphic. A simple example is provided by $V = \mathbb{Z}$, f(x, y) = 2xy and $V' = \mathbb{Z} \oplus \mathbb{Z}$ and $f'(x \oplus x', y \oplus y') = 2xy$. We see on this example that $\operatorname{Coker} \hat{f} = \mathbb{Z}/2\mathbb{Z}$ while $\operatorname{Coker} \hat{f}' = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence the linking groups are not isomorphic while the linking pairings are.

A pointed linking group is a triple (G, λ, c) where (G, λ) is a linking group and $c \in G^n$ is a distinguished *n*-tuple. (In contrast with the definition of pointed linking pairings, G is now allowed to have non torsion - including distinguished - elements.) An *isomorphism* of pointed linking groups is an isomorphism of the underlying linking groups sending the distinguished *n*tuple to the distinguished *n*-tuple.

EXAMPLE 3.4. The first integral homology of a 3-manifold endowed with distinguished elements provides a fundamental example of pointed linking group.

We now derive a lemma in order to deal with isomorphism of pointed linking groups.

The classification of pointed linking groups can be essentially reduced to the classification of pointed linking pairings by means of the following lemma. For a map $f : A \to B$, we denote by $f^{\otimes n} : A^n \to B^n$ the *n*-ary cartesian product map induced by f.

LEMMA 4.1. Let (G, λ, c) and (G', λ', c') be two pointed linking groups with distinguished n-tuples $c \in G^n$ and $c \in G'^n$ respectively. The following assertions are equivalent:

(1) There is an isomorphism of pointed linking groups

$$(G, \lambda, c) \simeq (G', \lambda', c')$$

- (2) There are
 - (i) a group isomorphism $\nu : G/\text{Tors } G \to G'/\text{Tors } G'$ such that $\nu^{\otimes n}([c]) = [c'];$

(ii) two retractions $r: G \to \text{Tors } G$ and $r': G' \to \text{Tors } G'$ of the natural inclusions into G and G' respectively and an isomorphism ψ of pointed linking pairings

(Tors $G, \lambda, r^{\otimes n}(c)$) \simeq (Tors $G', \lambda', r'^{\otimes n}(c')$).

PROOF. In order to lighten notation, since it is clear when *n*-ary cartesian product is meant, we suppress the superscript $^{\otimes n}$. (1) \Rightarrow (2): clearly a pointed linking group isomorphism φ induces a linking group isomorphism $\varphi|_{\text{Tors }G}$ between (Tors G, λ) and (Tors G', λ'). Choose any retraction $r: G \to \text{Tors }G$ of $i: \text{Tors } G \to G$. Then $r' = \varphi|_{\text{Tors }G} \circ r \circ \varphi^{-1}$ is a retraction of $i': \text{Tors } G' \to G'$. Then $\varphi|_{\text{Tors }G}(r(c)) = r' \circ \varphi(c) = r'(c')$. (2) \Rightarrow (1): let $p: G \to G/\text{Tors } G, x \mapsto p(x) = [x]$ denote the canonical projection. Define similarly the canonical projection p' onto G'/Tors G'. The map

 $(r,p): G \to \text{Tors } G \oplus G/\text{Tors } G, \ x \mapsto (r(x), p(x) = [x])$

is a group isomorphism. There is a similar isomorphism $(r', p') : G' \to$ Tors $G' \oplus G'/$ Tors G'. Define an isomorphism $\varphi : G \to G'$ by the following composition

$$G \xrightarrow{(r,p)} \text{Tors } G \oplus G/\text{Tors } G \xrightarrow{\psi \oplus \nu} \text{Tors } G' \oplus G'/\text{Tors } G' \xrightarrow{(r',p')^{-1}} G'.$$

Thus $\varphi = (r', p')^{-1} \circ (\psi \oplus \nu) \circ (r, p)$. By construction $\varphi|_{\text{Tors } G} = \psi$ and thus it is an isomorphism of pointed linking pairings between $(\lambda, r(c))$ and $(\lambda, r'(c'))$. The isomorphism $\varphi : G \to G'$ induces an isomorphism $[\varphi] :$ $G/\text{Tors } G \to G'/\text{Tors } G'$ defined by $[\varphi]([x]) = [\varphi(x)]$ for all $x \in G$. By construction $[\varphi] = \nu$. The formula $s([x]) = x - r(x), x \in G$ defines unambiguously a section s of the canonical projection $p : G \to G/\text{Tors } G$. Define similarly a section s' of the canonical projection $p' : G' \to G'/\text{Tors } G'$ by $s'([x']) = x' - r'(x'), x' \in G'$. Then

 $\varphi \circ s([x]) = \varphi(x - r(x)) = \varphi(x) - \varphi(r(x)) = \varphi(x) - r' \circ \varphi(x) = s' \circ [\varphi]([x]),$

thus $\varphi \circ s = s' \circ [\varphi]$. It follows that

$$\begin{aligned} \varphi(c) &= \varphi(r(c) + c - r(c)) = \varphi(r(c)) + \varphi(s([c])) = r'(c') + s'([\varphi]([c])) \\ &= r'(c') + s'(\nu([c])) \\ &= r'(c') + s'([c']) \\ &= r'(c') + c' - r'(c') \\ &= c'. \end{aligned}$$

COROLLARY 4.2. Two linking groups (G, λ) and (G', λ') are isomorphic if and only if the groups G/Tors G and G'/Tors G' are isomorphic and the linking groups (Tors G, λ) and (Tors G', λ') are isomorphic.

COROLLARY 4.3. Two linking groups (G, λ) and (G', λ') are isomorphic if and only if the following conditions are satisfied:

- (1) rank $G = \operatorname{rank} G'$;
- (2) $\rho_p^k(\lambda) = \rho_p^k(\lambda')$ for all prime p and all $k \ge 1$;

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(3) $\gamma(V \otimes tG, h \otimes \lambda) = \gamma(V \otimes tG', h \otimes \lambda')$ for all lattices V equipped with a homogeneous quadratic function $h: V \to \mathbb{Z}$.

5. The isomorphism problem for pointed lattices

We now give a necessary and sufficient condition to solve the isomorphism problem for pointed lattices (pointed linking groups with trivial linking pairing and abelian free group). This is the Proposition 5.1 below. This condition will be used in the sequel to manufacture invariants of linking groups.

Let V be a lattice. The linear group GL(V) acts naturally on V in the usual way. Extend diagonally this action to any n-ary cartesian power of V.

PROPOSITION 5.1. Let V be a lattice. Two n-tuples $x = (x_1, \ldots, x_n) \in V^n$ and $y = (y_1, \ldots, y_n) \in V^n$ lie in the same orbit of $\operatorname{GL}_n(V)$ if and only if (5.1)

for any
$$N \in \mathbb{Z}$$
, for any $(a_1, \dots, a_n) \in \mathbb{Z}^n$, $\sum_j a_j \ x_j \in N \cdot V \Leftrightarrow \sum_{j \in J} a_j \ y_j \in N \cdot V$

To prove Proposition 5.1, we need a number of lemmas.

LEMMA 5.2. Let x_1, \ldots, x_n be \mathbb{Z} -independent elements in a lattice V. The sublattice S generated by x_1, \ldots, x_n is primitive if and only if (5.2)

for all
$$(a_1, \ldots, a_n) \in \mathbb{Z}^n$$
 with $gcd(a_1, \ldots, a_n) = 1$, $\sum_i a_i x_i \in V \setminus \bigcup_{k>1} k \cdot V$.

PROOF. Suppose that there is $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $gcd(a_1, \ldots, a_n) = 1$ such that $\sum_i a_i x_i \in k \cdot V$ for some k > 1. Let $y = \frac{1}{k} \sum_i a_i x_i \in V$. By hypothesis, $k \cdot y = \sum_i a_i x_i \in S$. We claim that $y \notin S$. Otherwise since the x_i 's are independent, $k | a_i$ for all i, which contradicts the fact that $gcd(a_1, \ldots, a_n) = 1$. Hence S is not primitive.

Conversely, suppose that S is not primitive. There exists $y \in V \setminus S$ such that $k \cdot y \in S$ for some k > 1. Consider the smallest integer k > 1 realizing this condition. Then there exist b_1, \ldots, b_n such that $k \cdot y = \sum_i b_i x_i$. Let $l = \gcd(b_1, \ldots, b_n)$. Since $y \notin S$ and by minimality of k, the integers k and l are coprime. Let $a_i = b_i/l$, $i = 1, \ldots, n$. By construction, $\gcd(b_1, \ldots, b_n) = 1$. We have

$$ky = l \sum_{i \in S} a_i x_i \,.$$

Since k and l are coprime, $\sum_{i} a_i x_i \in k \cdot V$. This is the desired result.

LEMMA 5.3. Let x_1, \ldots, x_n and y_1, \ldots, y_n be two families of independent elements in V satisfying the condition (5.1). Let S and S' be the sublattices generated by x_1, \ldots, x_n and y_1, \ldots, y_n respectively. Let $c_{ij} \in \mathbb{Q}$ be rational numbers, $1 \leq i, j \leq n$. The primitive hull of S is generated by $x'_i = \sum_j c_{ij}x_j$ if and only the primitive hull of S' is generated by $y'_i = \sum_j c_{ij}y_j$, $1 \leq i \leq n$.

PROOF. Consequence of the previous lemma.

LEMMA 5.4. With the same hypothesis and notation as in the previous lemma, there exists an automorphism $\varphi : V \to V$ such that $\varphi(x_i) = y_i$ for all *i*.

PROOF. Define $\varphi(x'_i) = y'_i$, $1 \leq i \leq n$. This defines an isomorphism between the respective primitive hulls \widetilde{S} and $\widetilde{S'}$. By Lemma 1.2, this isomorphism extends to an automorphism $\widetilde{\varphi} : V \to V$. Let x'_i , $1 \leq i \leq n$, be the generators of \widetilde{S} . There are rational numbers c_{ij} $(1 \leq i, j \leq n)$ such that $x'_i = \sum_j c_{ij}x_j$. The matrix $C = (c_{ij})_{1 \leq i, j \leq n}$ is invertible over \mathbb{Q} . Since some multiple of each x'_j lies in S, we deduce that the inverse matrix $C^{-1} = (d_{ij})_{1 \leq i, j \leq n}$ has integral coefficients. By the hypothesis, $y'_i = \sum_j c_{ij}x'_j$, $1 \leq i, j \leq n$, form a \mathbb{Z} -basis of generators for $\widetilde{S'}$. Therefore,

$$\varphi(x_i) = \varphi\left(\sum_j d_{ij} x'_j\right)$$
$$= \sum_j d_{ij} \varphi(x'_j)$$
$$= \sum_j d_{ij} y'_j$$
$$= y_i.$$

This is the desired result.

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PROOF OF PROPOSITION 5.1. Necessity is clear. Let J be a maximal subset of $\{1, \ldots, n\}$ such that the elements $x_j, j \in J$, are independent over \mathbb{Z} . Then the relation for N = 0 shows that the elements $y_j, j \in J$, are also independent over \mathbb{Z} . Applying the previous lemma yields an automorphism $\varphi \in \operatorname{GL}(V)$ such that $\varphi(x_i) = y_i$ for all $i \in J$. Let $k \notin J$. There is a relation

$$\sum_{j\in J} a_j \ x_j + a_k \ x_k = 0,$$

for some $a_k \neq 0$. Then

(5.3)
$$\sum_{j \in J} a_j \ y_j + a_k \ y_k = 0.$$

Since $y_j = \varphi(x_j)$ for $j \in J$, we deduce that

$$0 = \varphi\left(\sum_{j \in J} a_j \ x_j + a_k \ x_k\right) = \sum_{j \in J} a_j \ y_j + a_k \ \varphi(x_k).$$

Comparing this equality to (5.3), we deduce that

$$a_k y_k = a_k \varphi(x_k).$$

Since $a_k \neq 0$ and since V is torsion free, $y_k = \varphi(x_k)$.

6. The stable classification

We begin by recalling the classical results on the stable classification of lattices. Recall from §?? that by definition, two bilinear lattices are stably equivalent if they become isomorphic after adding to them some unimodular orthogonal summands. Furthermore, the map

$$(V, f) \mapsto (G_f, \lambda_f)$$

induces a bijective correspondence between stable equivalence classes of nondegenerate bilinear lattices and isomorphism classes of nondegenerate linking pairings (Th. 2.3).

The aim of this paragraph is to extend this result to the more general setting of pointed linking groups.

A pointed bilinear lattice V is a bilinear lattice equipped with a finite ordered collection c of elements $c_1, \ldots, c_n \in V^*$. Pointed bilinear lattices form a monoid for the orthogonal sum \oplus . Let (V, f, c) and (V, f, c) be two pointed bilinear lattices. A weak isomorphism between them is an isomorphism $\psi : (V, f) \to (V', f')$ of bilinear lattices such that $\psi^*(c') = c \mod \hat{f}(V)$. For instance, if the bilinear lattices are isomorphic in the usual sense and if the distinguished elements c and c' lie in $\hat{f}(V)$ and $\hat{f}(V')$ respectively, then the pointed bilinear lattices are weakly isomorphic.

EXAMPLE 3.5. Let (U, h) and (U', h') be unimodular bilinear lattices. If $\varphi : (U, h) \to (U', h')$ is an isomorphism of bilinear lattices, then $\varphi : (U, h, u) \to (U', h', u')$ is a weak isomorphism of pointed lattices for any $u \in U^n$ and any $u' \in U'^n$.

We say that (V, f, c) and (V', f', c') are stably equivalent if there exist pointed unimodular lattices (U, h, u) and (U', h', u') such that $(V, f, c) \oplus (U, h, u)$ and $(V', f', c') \oplus (U', h', u')$ are weakly isomorphic as pointed bilinear lattices.

Clearly stably equivalent pointed lattices induce isomorphic pointed linking groups. The main observation of this section lies in the converse and generalizes Th. 2.3.

THEOREM 6.1. Two pointed bilinear lattices (V, f, c) and (V', f', c') are stably equivalent if and only if there is an isomorphism

$$\phi : (\operatorname{Coker} \widehat{f}, \lambda^f, [c]) \to (\operatorname{Coker} \widehat{f'}, \lambda^{f'}, [c'])$$

of the induced discriminant linking groups. In fact, any isomorphism ϕ : $(H, \lambda, x) \rightarrow (H', \lambda', x')$ of linking groups lifts to a stable equivalence of pointed bilinear lattices.

PROOF. Let $\mathfrak{s} : (V, f, c) \oplus (U, h, u) \to (V', f', c') \oplus (U', h', u')$ be an explicit weak isomorphism realizing the two stable equivalence between the pointed bilinear lattices (V, f, c) and (V', f', c'). Since unimodular pointed lattices are sent via the discriminant construction to trivial linking pairing

(with trivial distinguished elements), this isomorphism induces an isomorphism

$$\operatorname{Coker}\,(\widehat{f'\oplus h'}) = \operatorname{Coker}\,\widehat{f'} \to \operatorname{Coker}\,\widehat{f} = \operatorname{Coker}\,\widehat{f\oplus h}$$

whose restriction to $G^{f'}$ = Tors Coker \hat{f} is an isomorphism of the linking pairings $\lambda^{f'}$ and λ^{f} .

Conversely, it suffices to prove the last statement. Let $\phi : (H, \lambda, x) \to (H', \lambda', x')$ be an isomorphism of linking groups. Set G = Tors H and G' = Tors H'. Using [23, Proof of Th. 4.1], lift $\phi|_G : G \to G'$ to a stable equivalence \mathfrak{s} between nondegenerate bilinear lattices (V, f) and (V', f'). Consider the following commutative diagram of extension of abelian groups

where *i* is the canonical inclusion and $[\phi] : F \to F'$ is the isomorphism induced by ϕ . The two horizontal short exact sequences are split. Choose a section $s : F \to H$ of *p*. Then $s' = \phi \circ s \circ [\phi]^{-1} : F' \to H'$ is a section of *p'*. Define

$$\tilde{V} = V \oplus s(F)^*, \quad \tilde{f} = f \oplus 0$$

and similarly $\tilde{V}' = V' \oplus s'(F')^*$, $\tilde{f}' = f' \oplus 0$. We see that $\tilde{\phi} = \mathfrak{s} \oplus \phi|_{s(F)}$ is an isomorphism between the lattices (\tilde{V}, \tilde{f}) and $(\tilde{V}', \tilde{f}' \text{ inducing the iso$ $morphism <math>\phi : (H, \lambda) \to (H', \lambda')$. Furthermore, let $\tilde{x} \in \tilde{V}^* = V^* \oplus s(F)$ and $\tilde{x}' \in \tilde{V'}^* = V'^* \oplus s'(F')$ be lifts of $x \in H = G \oplus s(F)$ and $x \in H' = G' \oplus s'(F')$ respectively. Clearly $\tilde{\phi}^*$ sends \tilde{x}' to $\tilde{x} \mod \tilde{f}(\tilde{V})$. Therefore, $\tilde{\phi}$ is a suitable stable equivalence between pointed bilinear lattices lifting the isomorphism between group linkings (H, λ, x) and (H', λ', x') .

In practice, the following corollary is useful. It shows that in the process of stabilization, one can restrict to a particular pointed unimodular lattice. Denote by ± 1 the bilinear lattice on \mathbb{Z} sending (1,1) to ± 1 .

COROLLARY 6.2. Two n-pointed bilinear lattices (V, f, c) and V', f', c') are stably equivalent if and only if they are related by a finite sequence

$$(V, f, c) \xrightarrow{o_1} \cdots \xrightarrow{o_r} \cdots \xleftarrow{o_r'} \cdots \xleftarrow{o_1'} (V', f', c')$$

of the following two operations:

- (i) *lattice isomorphisms*;
- (ii) orthogonal sum with $(\mathbb{Z}, \pm 1, 0)$.

Let us state the particular case of torsion groups.

COROLLARY 6.3. Two pointed nondegenerate bilinear lattices (V, f, c) and (V', f', c') are stably equivalent if and only if the discriminant linking pairings (G_f, λ_f) and $(G_{f'}, \lambda_{f'})$ are isomorphic. Furthermore, any isomorphism between two nondegenerate pointed linking pairings (G, λ, x) and (G', λ', x')

can be lifted to a stable equivalence of pointed bilinear lattices (V, f, \tilde{x}) and $(V', f', \tilde{x'})$.

REMARK 3.8. The notion of stable equivalence generalizes the notion of stable equivalence for bilinear lattices (without distinguished element) and is weaker than the notion of strongly stable equivalence defined for bilinear lattices equipped with Wu classes. It is indeed clear from the definition that if two triples (V, f, c) and (V', f', c') of bilinear lattices equipped with distinguished elements $v \in V^*$ and $v' \in V'^*$ that happen to be image of Wu classes (by $\hat{f}_{\mathbb{Q}}$ and $\hat{f}'_{\mathbb{Q}}$ respectively) are strongly stably equivalent, then they are stably equivalent. See Th. 2.2 and Th. 4.7 respectively.

CHAPTER 4

Lagrangians and Witt groups

The notion of Lagrangian was introduced in the previous chapter, §1. This chapter is devoted to the study of Lagrangians and the associated Witt groups. The material covered here is fairly classical and is used as a building block both towards the reciprocity formula in Chap. and the topological quantum field theory in Chap.

1. The Lagrangian category

Throughout this section, we deal with nondegenerate ε -symmetric bilinear pairings. As is customary in order to ease notation, given an abelian group A endowed with a bilinear pairing, we denote by -A the same group endowed with the opposite bilinear pairing.

We are first interested in defining a suitable category of Lagrangians in finitely generated abelian groups endowed with ε -symmetric bilinear pairings. Before we do so, consider the following three examples of Lagrangians.

EXAMPLE 4.1. Given any ε -symmetric bilinear pairing A, the diagonal Lagrangian is defined as $\text{Diag}(A) = \{(a, a) \mid a \in A\} \subseteq -A \oplus A$. The antidiagonal Lagrangian is defined as $\overline{\text{Diag}}(A) = \{(-a, a) \mid a \in A\} \subseteq -A \oplus A$.

EXAMPLE 4.2 (Lagrangian associated to a bilinear pairing preserving isomorphism). More generally, let $f : A \to B$ be a morphism. The graph of f, defined as $\operatorname{Graph}(f) = \{(a, f(a)) \mid a \in A\} \subset -A \oplus B$ is isotropic (resp. Lagrangian) if and only if f preserves the ε -symmetric bilinear pairings on A and B (resp. if and only if f is bijective and preserves the ε -symmetric bilinear pairings on A and B). The case $f = \operatorname{id}_A$ (resp. $f = -\operatorname{id}_A$) gives the diagonal Lagrangian (resp. the antidiagonal Lagrangian).

We now define the composition of Lagrangians.

LEMMA 1.1. Let A, B, C be three nonsingular ε -symmetric bilinear pairings on finitely generated abelian groups such that the map $V \mapsto V^{\perp}$ is involutive on subgroups. Let Λ be a Lagrangian in $-A \oplus B$ and let Λ' be a Lagrangian in $-B \oplus C$. The subset

 $\Lambda' \circ \Lambda = \{(a, c) \in -A \oplus C \mid \text{there is } b \in B \text{ such that } (a, b) \in \Lambda \text{ and } (b, c) \in \Lambda'\}$

is a Lagrangian in $-A \oplus C$.

The following definition is mainly needed for the proof of Lemma 1.1.

DEFINITION 4.1. Let A be a nondegenerate ε -symmetric bilinear pairing and let $I \subset A$ be isotropic. Let $B \subset A$ a subgroup. We define the *contraction* of B along I by

(1.2)
$$B^{I} = (B+I) \cap I^{\perp} \subseteq I^{\perp}.$$

EXERCISE 4.1. The following properties hold:

- (1) $B^I = (B \cap I^{\perp}) + I.$
- (2) Contraction and orthogonal "commute": $(B^I)^{\perp} = (B^{\perp})^I$.
- (3) If I is Lagrangian then $B^I = I$.
- (4) If B is isotropic (resp. Lagrangian) then so is B^{I} .

PROOF. Denote by a dot a ε -symmetric bilinear pairing. Let (a, c) and (a', c') be two elements in $\Lambda' \circ \Lambda$. There exist $b, b' \in B$ such that $(a, b) \in \Lambda$ and $(b, c) \in \Lambda$, such that $(a', b') \in \Lambda'$ and $(b', c') \in \Lambda'$. Then

$$(a, c) \cdot (a', c') = -a \cdot a' + c \cdot c' = -a \cdot a' + b \cdot b' - b \cdot b' + c \cdot c'$$

= $(a, b) \cdot (a', b') + (b, c) \cdot (b', c')$
= $0 + 0$
= $0.$

Hence $\Lambda' \circ \Lambda \subseteq (\Lambda' \circ \Lambda)^{\perp}$. Let us prove the converse. We consider the orthogonal sum $-A \oplus B \oplus -B \oplus C$. In this group lies the isotropic subgroup $H = 0 \oplus \text{Diag}(B) \oplus 0$ with orthogonal $H^{\perp} = A \oplus \text{Diag}(B) \oplus C$. The canonical projection $A \oplus B \oplus B \oplus C \to (A \oplus B \oplus B \oplus C)/H$ restricts to a projection $p : H^{\perp} \to H^{\perp}/H = A \oplus C$ which preserves orthogonality. Consider the subgroup $G = (\Lambda \oplus \Lambda')^H$. We have $p(G) = \Lambda' \circ \Lambda$. Since p preserves orthogonality, $p(G^{\perp}) = (\Lambda' \circ \Lambda)^{\perp}$. Since $\Lambda \oplus \Lambda'$ is Lagrangian, so is G.

EXERCISE 4.2. The composition of Lagrangians is associative. The diagonal Lagrangian plays the rôle of the identity.

DEFINITION 4.2. The category of Lagrangians $\operatorname{Lag}^{\varepsilon}(U)$ over U is defined as follows. An object is a nondegenerate ε -symmetric bilinear pairing on a finitely generated abelian group A with values in U. A morphism between two objects A and B is a triple (Λ, A, B) where Λ is a Lagrangian in $-A \oplus B$. The composition of two morphisms (Λ, A, B) and (Λ', B, C) is the morphism $(\Lambda' \circ \Lambda, A, C)$ defined by Eq. (1.1).

For more details, see for instance [46], [93, IV, §3]. It is convenient at times to identify A to $A \oplus 0$ and B to $0 \oplus B$ respectively as subgroups of $A \oplus B$. We now fix a category of Lagrangians.

Definition 4.3. A Lagrangian Λ in $-A \oplus B$ is decomposable if

$$\Lambda = (\Lambda \cap A) \oplus (\Lambda \cap B).$$

The definition obviously depends on the fixed decomposition $-A \oplus B$.

REMARK 4.1. The orthogonal sum of two Lagrangians $\Lambda \subset A$ and $\Lambda' \subset B$ is always decomposable in $-A \oplus B$. A general Lagrangian may not be

decomposable: for instance, the Lagrangian $\operatorname{Graph}(f)$ associated to a ε -symmetric bilinear pairing preserving isomorphism is not decomposable. In particular, the diagonal Lagrangian $\operatorname{Diag}(A) \subset -A \oplus A$ is not decomposable.

DEFINITION 4.4. Two Lagrangians L and L' are transverse in A if

$$L + L' = A.$$

REMARK 4.2. If L and L' are two transverse Lagrangians in A then $L \cap L' = 0$.

PROOF. Recall that the underlying ε -symmetric bilinear pairing is nondegenerate. We have

$$0 = A^{\perp} = (L + L')^{\perp} = L^{\perp} \cap L'^{\perp} = L \cap L'.$$

LEMMA 1.2. Let A, B be two objects in $Lag^{\varepsilon}(U)$. Let L, L' be a Lagrangian in $-A \oplus B$. If L is transverse to L' and if L' is decomposable, then L is decomposable.

PROOF. The inclusion $(L \cap A) + (L \cap B) \subseteq \Lambda$ always holds. To see the other inclusion, let $z \in L$. Since $L + L' = -A \oplus B$, there exists $w \in L'$ and $(a,b) \in A \times B$ such that z + w = a + b. Since L' is decomposable, there exists a decomposition w = w' + w'' with $w' \in A, w'' \in B$. Thus z = (a - w') + (b - w'') is a decomposition for z.

The two main cases we have in mind are $U = \mathbb{Z}$ (the category of Lagrangians in ε -lattices) and $U = \mathbb{Q}/\mathbb{Z}$ (the category of Lagrangians in ε -linking pairings).

DEFINITION 4.5. A Lagrangian category is *involutive* if for any object A and any subgroup $V \subseteq A$, $V^{\perp \perp} = V$.

If a Lagrangian category $\operatorname{Lag}^{\varepsilon}(U)$ is involutive then for any ε -symmetric bilinear pairing S and for any subgroups V, W of S,

$$(V \cap W)^{\perp} = V^{\perp} + W^{\perp}.$$

(This identity is the result of taking the orthogonal of the identity (1.1).)

LEMMA 1.3. The Lagrangian categories $\operatorname{Lag}^{\varepsilon}(\mathbb{Z})$ and $\operatorname{Lag}^{\varepsilon}(\mathbb{Q}/\mathbb{Z})$ are involutive.

In the rest of this paragraph, we consider only involutive Lagrangians categories. First we refine our criterion to find a Lagrangian (Lemma 1.2).

DEFINITION 4.6. A Lagrangian L is *split* if it is a direct summand.

EXAMPLE 4.3. The diagonal (resp. antidiagonal) Lagrangian is split. Proof: one direct summand of Diag(A) (resp. $\overline{\text{Diag}}(A)$) is $A \oplus 0 = \{(a, 0) \mid a \in A\}$.

REMARK 4.3. If $\varepsilon = +1$, a Lagrangian may not be split. As an example, the cyclic symmetric linking pairing $(\frac{1}{9})$ on $\mathbb{Z}/9\mathbb{Z}$ is Lagrangian; the only Lagrangian Λ is the cyclic subgroup generated by 3 mod 1, which is not a direct summand of $\mathbb{Z}/9\mathbb{Z}$. In particular, Λ does *not* have a Lagrangian direct complement.

REMARK 4.4. Let Λ be a split Lagrangian, it is not true that *any* direct complement of Λ is itself Lagrangian. Consider the symmetric linking pairing (1) defined on $A = \mathbb{Z}$. The diagonal Diag(A) is a Lagrangian in $-A \oplus A$ which has the cyclic subgroup B generated by $(0, 1) \in A$ as a complementary subgroup: Diag $(A) \oplus B = -A \oplus A$. Then B^{\perp} is generated by (1, 0), so that $B \cap B^{\perp} = 0$ and B is not even isotropic.

The following observation is a refinement of Lemma 1.2 in our context.

LEMMA 1.4. Let L be a Lagrangian in A and let W be isotropic in A. The following statements are equivalent:

- (1) W is a Lagrangian and L and W are transverse;
- (2) $A = L \oplus W$.

PROOF. If L and L' are transverse Lagrangians, then A splits as the direct sum of L and L' (Remark 4.2). Conversely, suppose that there is a direct summand W such that $\Lambda \oplus W = A$. Then

$$A = 0^{\perp} = (\Lambda \cap W)^{\perp} = \Lambda + W^{\perp}.$$

Since

$$0 = A^{\perp} = (\Lambda + W)^{\perp} = \Lambda \cap W^{\perp}.$$

A splits as the direct sum of Λ and W^{\perp} . Hence Lemma 1.2 applies.

LEMMA 1.5. Two Lagrangians L and L' are transverse if and only if $L \cap L' = 0$.

PROOF. We have $L+L' = L^{\perp} + L'^{\perp} = (L \cap L')^{\perp}$. We apply the involution property to conclude.

We have seen that a Lagrangian may not be split. There is an elementary and important case when this does hold automatically.

LEMMA 1.6. Suppose that A is a torsion free abelian group. Then every Lagrangian in A is split.

PROOF. Since A is a \mathbb{Z} -module, the hypothesis implies that A is free over \mathbb{Z} .

LEMMA 1.7. Assume that $\varepsilon = -1$. Given a set E of Lagrangians, there exists a Lagrangian L' tranverse to each Lagrangian in E.

PROOF. Consider the set of all isotropic subgroups intersecting trivially with each Lagrangian L in E. This set is not empty since it contains the trivial subgroup. Choose a maximal element L' with respect to the inclusion. Let L be a Lagrangian in E. Then

(1.3)
$$A = (L \cap L'')^{\perp} = L^{\perp} + L'^{\perp} = L + L'^{\perp}.$$

We want to show that L' is Lagrangian. Suppose first that $L'^{\perp} \subseteq L + L'$. Then it follows from (1.3) that $A = L \oplus L' = L \oplus L'^{\perp}$. Apply Lemma 1.2 to obtain the desired result. Now we claim that the inclusion $L'^{\perp} \subseteq L + L'$ necessarily holds. If not then there exists $x \in L'^{\perp}$ such that $x \notin L + L'$. Let $\langle x \rangle$ be the submodule generated by x. Since $\langle x \rangle \subseteq \langle x \rangle^{\perp}$ (here we use $\varepsilon = -1$), we see that

$$L' + \langle x \rangle \subseteq L'^{\perp} \cap \langle x \rangle^{\perp} = (L' + \langle x \rangle)^{\perp}.$$

Hence $L' + \langle x \rangle$ is isotropic, still intersects trivially L and contains L'. This contradicts maximality of L'.

REMARK 4.5. For symmetric bilinear pairings, Lemma 1.7 does not hold even for a single split Lagrangian. Let A be a \mathbb{Z} -module such that multiplication by 2 is not invertible endowed with any symmetric linking pairing. The diagonal Lagrangian Diag(A) in $-A \oplus A$ is split but none of the direct complements is a Lagrangian: there does not exist a Lagrangian transverse to Diag(A).

Given a pair of transverse Lagrangians L, L' in A, the ε -symmetric bilinear pairing on A induces a bilinear pairing $L \times A/L' \to U$.

PROPOSITION 1.8. Let A be a nonsingular ε -symmetric bilinear pairing. The group O(A) of automorphisms of A acts transitively on pairs of transverse Lagrangians.

The following observation is a preparation for a suitable refinement of $\text{Lag}^{\varepsilon}(U)$.

LEMMA 1.9. For an object $A \in Lag^{\varepsilon}(U)$, we denote by L_A a Lagrangian in A. Let $(A, L_A), (B, L_B), (C, L_C)$ be three pairs where A, B, C are objects in $Lag^{\varepsilon}(U)$. Let Λ be a Lagrangian in $-A \oplus B$ and let Λ' be a Lagrangian in $-B \oplus C$. If Λ is transverse to $L_A \oplus L_B$ and if Λ' is transverse to $L_B \oplus L_C$, then $\Lambda' \circ \Lambda$ is transverse to $L_A \oplus L_C$.

In short, the composition of transverse Lagrangians is transverse.

PROOF. We have to prove that $\Lambda_{N \circ M} \cap L_C = \Lambda_{N \circ M} \cap L_A = 0$. We prove that $\Lambda_{N \circ M} \circ L_C$, the other case is similar. Let $x \in \Lambda_{N \circ M} \cap L_C$. Write x = (a, c) as an element in $-A \oplus C$. Since $(a, c) \in L_C \subseteq C$, a = 0 and $c \in L_C$. Since $x = (0, c) \in \Lambda_{N \circ M}$, there exists $b \in B$ such that $(b, c) \in \Lambda_N$. Since Λ_N is transverse to the decomposable Lagrangian $L_B \oplus L_C$, by Lemma 1.2, Λ_N itself is decomposable. Hence $c \in \Lambda_N \cap C$. So finally $c \in \Lambda_N \cap L_C = 0$.

DEFINITION 4.7. The category of transverse Lagrangians $\operatorname{Lag}_{\operatorname{trans}}^{\varepsilon}(U)$ over U is defined as follows. An object in $\operatorname{Lag}_{\operatorname{trans}}^{\varepsilon}(U)$ is a pair (A, L_A) where A is an object in $\operatorname{Lag}^{\varepsilon}(U)$ and L_A is a Lagrangian in A. A morphism between two objects A and B is a Lagrangian Λ in $-A \oplus B$ such that Λ is transverse to $L_A \oplus L_B$.

There is a faithful forgetful functor $\operatorname{Lag}_{\operatorname{trans}}^{\varepsilon}(U) \to \operatorname{Lag}^{\varepsilon}(U)$ that "forgets" the extra Lagrangians and the transversality property.

2. Lagrangian ε -symmetric bilinear pairings and quadratic functions

Let $q: A \to U$ be a quadratic function on a finitely generated abelian group A with values in an abelian group U.

DEFINITION 4.8. A subgroup $\Lambda \subset A$ is *isotropic* with respect to q if $q(\Lambda) = 0$.

LEMMA 2.1. If Λ is isotropic with respect to q then Λ is isotropic with respect to b_q .

PROOF. Let $x, y \in \Lambda$. Then $b_q(x, y) = q(x+y) - q(x) - q(y) = 0$.. Hence $\Lambda \subset \Lambda^{\perp}$.

REMARK 4.6. The converse of Lemma 2.1 does not hold. The action (5.3) prevents this. For instance, on $A = \mathbb{Z}/4\mathbb{Z}$, consider the symmetric linking pairing $(\frac{1}{4})$. Then Diag(A) is a Lagrangian in $-A \oplus A$, hence is isotropic. The quadratic form q defined by

$$q(x,y) = \frac{x^2}{8} - \frac{y^2}{8} + \frac{x}{2} \mod 1, \ (x,y) \in -A \oplus A,$$

is a quadratic enhancement of $-A \oplus A$ but $q(1 \mod 4, 1 \mod 4) = \frac{1}{2} \mod 1 \neq 0$.

COROLLARY 2.2. If Λ is isotropic with respect to q then there is an induced quadratic function $\bar{q} : A/\Lambda \to U$ such that the following diagram is commutative:



In view of the above, the following definition is natural.

DEFINITION 4.9. A subgroup $\Lambda \subset A$ is Lagrangian with respect to q if $q(\Lambda) = 0$ and $\Lambda = \Lambda^{\perp}$. A quadratic function q is said to be Lagrangian (resp. split Lagrangian) if there exists a Lagrangian (resp. a split Lagrangian) with respect to q. The definition is similar for ε -symmetric linking pairings.

REMARK 4.7. A trivial quadratic function $q: A \to U$ (i.e., such that q(A) = 0) is Lagrangian.

The terms *metabolizer* and *metabolic* (instead of Lagrangian subgroup and Lagrangian quadratic function) are also used, see for instance [83]. On lattices, a quadratic function is split Lagrangian if and only if it is Lagrangian. The property of being (split) Lagrangian is preserved by isomorphisms of quadratic functions or ε -symmetric bilinear pairings.

We now fix an involutive Lagrangian category $\text{Lag}^{\varepsilon}(U)$. We denote by Quadr(U) (resp. $\text{Quadr}^0(U)$) the monoid of all tame quadratic functions (resp. of all tame quadratic forms) defined on a finitely generated abelian group with values in U. Similarly we denote by $\text{Symm}^{\varepsilon}(U)$ the monoid of all ε -symmetric linking pairings defined on a finitely generated abelian group with values in U. The operation is the orthogonal sum. These monoids can be turned into small additive categories in the obvious way.

We are interested in determining whether an ε -symmetric bilinear pairing (or a quadratic function) is Lagrangian, in other words, in the existence of morphisms in the category $\operatorname{Lag}^{\varepsilon}(U)$. We begin with elementary observations.

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LEMMA 2.3. Let $A \in \text{Quadr}(U)$.

- (1) A is (split) Lagrangian if and only if -A is (split) Lagrangian.
- (2) The orthogonal sum of two Lagrangians is a Lagrangian in the orthogonal sum.
- (3) If A is torsion and is a Lagrangian then each p-primary component is Lagrangian (for the quadratic function restricted to the p-primary component).

The analogous statement for quadratic forms and for ε -symmetric bilinear pairings holds.

PROOF. (1) Λ is a Lagrangian for A if and only it is for -A. (2) Let Λ and Λ' be Lagrangians for A and B respectively. We write Λ for $\Lambda \oplus 0$ and Λ' for $0 \oplus \Lambda'$ in $A \oplus B$. Then $\Lambda^{\perp} = \Lambda \oplus B$ and $\Lambda'^{\perp} = A \oplus \Lambda'$. Then

$$(\Lambda \oplus \Lambda')^{\perp} = \Lambda^{\perp} \cap \Lambda'^{\perp} = (\Lambda \oplus B) \cap (A \oplus \Lambda') = \Lambda \oplus \Lambda'.$$

(3) Let H be an isotropic subgroup in A. Let p be a prime and let A_p denote the p-primary component of A. Then

$$H \cap A_p \subseteq H^{\perp} \subseteq H^{\perp} + A_p^{\perp} = (H \cap A_p)^{\perp}$$

so $H \cap A_p$ is isotropic. Assume now that H is Lagrangian in A. We have

$$(H \cap A_p)^{\perp} = H^{\perp} + A_p^{\perp} = H + \bigoplus_{q \neq p} A_q$$
$$= \bigoplus_q (H \cap A_q) + \bigoplus_{q \neq p} A_q$$
$$= (H \cap A_p) \oplus \bigoplus_{q \neq p} (H \cap A_q) + \bigoplus_{q \neq p} A_q$$
$$= (H \cap A_p) \oplus \bigoplus_{q \neq p} A_q$$

It follows that when restricted to the *p*-component,

$$(H \cap A_p)^{\perp} \cap A_p = H \cap A_p.$$

Therefore the subgroup $H \cap A_p$ is a Lagrangian for the restriction A_p .

LEMMA 2.4. Let $A \in \text{Quadr}(U)$. Then the orthogonal sum $-A \oplus A$ is split Lagrangian.

The analogous statement for quadratic forms and for ε -symmetric bilinear pairings holds.

PROOF. Diag(A) is a split Lagrangian.

REMARK 4.8. If $A' \simeq A$ then $-A \oplus A'$ is split Lagrangian.

Let $(A, q) \in \text{Quad}(U)$. Denote by I(A) the set of all q-isotropic subgroups in A. There is a similarly defined set for a bilinear pairing. Any isotropic subgroup $H \subseteq I(A)$ induces a quadratic function \bar{q} on the subquotient H^{\perp}/H by the formula $\bar{q}([x]) = q(x)$ for all $x \in H^{\perp}$. DEFINITION 4.10. The quadratic function \bar{q} defined on H^{\perp}/H is called the *contraction* of q relatively to H.

The definition for ε -symmetric bilinear pairings is similar: $\overline{b}([x], [y]) = b(x, y), x, y \in H^{\perp}$. The quotient group H^{\perp}/H will always be implicitly endowed with the induced contraction form.

REMARK 4.9. If q is nondegenerate (resp. nonsingular) then all its contractions are nondegenerate (resp. nonsingular).

It turns out that contraction is a natural tool to "split off" Lagrangian summands.

PROPOSITION 2.5. Let $A \in \text{Quadr}(U)$. Let $H \in I(A)$. The orthogonal sum

 $-A \oplus H^{\perp}/H$

is split Lagrangian. There is an analogous statement for ε -symmetric bilinear pairings and quadratic forms.

Note that Lemma 2.4 is just the special case of Proposition 2.5 when A is nondegenerate and H = 0.

PROOF. In $A \oplus H^{\perp}/H$, the subgroup $\Delta = \{(x, [x]) \mid x \in H^{\perp}\}$ is a split Lagrangian.

If the subgroup H is not isotropic then the contraction with respect to H is not defined. However, the subgroup $H^{\perp} \cap H$ is *b*-isotropic with respect to an ε -symmetric linking *b* on *A* since $H^{\perp} \cap H \subseteq H + H^{\perp} = (H^{\perp} \cap H)^{\perp}$. As before, *b* induces a quotient ε -symmetric bilibnear pairing on $(H^{\perp} \cap H)^{\perp}/H^{\perp} \cap H$.

PROPOSITION 2.6. Let $(A, b) \in \text{Sym}^{\varepsilon}(U)$. Let H be any subgroup of A. The orthogonal sum

$$(A,b) \oplus \left(\frac{H^{\perp} + H}{H^{\perp} \cap H}, -\bar{b}\right)$$

is split Lagrangian.

PROOF. Apply the previous proposition to the form on the isotropic subgroup $H^{\perp} \cap H$.

REMARK 4.10. The analogous statement for quadratic functions is not automatic because a quadratic function q may not vanish on $H^{\perp} \cap H$. See Proposition 3.6 below. This fact will play a crucial rôle in the reciprocity formula and the topological quantum field theory associated to a quadratic form.

3. Witt groups

We define two basic relations in $\operatorname{Quadr}(U)$, $\operatorname{Quadr}^0(U)$ and $\operatorname{Sym}^{\varepsilon}(U)$.

3. WITT GROUPS

DEFINITION 4.11. Two quadratic functions $A, A' \in \text{Quadr}(U)$ are stably related (resp. split stably related) if $A \oplus \Lambda$ is isomorphic to $A' \oplus \Lambda'$ for suitable Lagrangian (resp. split Lagrangian) Λ and Λ' in Quadr(U). We write $A \stackrel{\text{stable}}{\sim} A'$. There is an analogous definition for $\text{Quadr}^0(U)$ and $\text{Sym}^{\varepsilon}(U)$.

The next definition is motivated by Lemma 2.4

DEFINITION 4.12. Two quadratic functions $A, A' \in \text{Quadr}(U)$ are Lagrangerelated (resp. split Lagrange-related) if $-A \oplus A'$ is Lagrangian (resp. split Lagrangian). We note $A \stackrel{\text{Lag}}{\sim} A'$.

In terms of the Lagrangian category introduced in $\S1$, A and A' are Lagrangerelated if and only if there is a morphism between A and A'.

LEMMA 3.1. Both the stable relation and the Lagrange relation are equivalence relations.

PROOF. Reflexivity and symmetry are obvious. Transitivity of the stable relation follows from the fact that the orthogonal sum of two Lagrangians is Lagrangian (Lemma 2.3, (2)). Transitivity of the Lagrange relation is the composition of morphisms in the Lagrangian category (Lemma 1.1).

Our goal is to show that the two equivalence relations are the same.

PROPOSITION 3.2. $A \stackrel{\text{stable}}{\sim} A'$ if and only if $A \stackrel{\text{Lag}}{\sim} A'$.

A key point in the proof of Proposition 3.2 is the following observation (which is a kind of a converse to Lemma 2.4). Let us temporarily call stably Lagrangian a pairing or a quadratic function stably related to a Lagrangian pairing.

LEMMA 3.3. If $A \in \text{Quadr}(U)$ is stably Lagrangian and nonsingular then it is Lagrangian. The analogous statement for $\text{Quadr}^0(U)$ and $\text{Sym}^{\varepsilon}(U)$ holds.

LEMMA. The proof is based on the observation that if N, P are two isotropic subgroups of A, then the induced quadratic functions on N^{\perp}/N and P^{\perp}/P respectively are Lagrange-related. Indeed, by Proposition 2.5, $A \stackrel{\text{Lag}}{\sim} N^{\perp}/N$ and $A \stackrel{\text{Lag}}{\sim} P^{\perp}/P$. Since $\stackrel{\text{Lag}}{\sim}$ is an equivalence relation, the claim follows.

Now suppose $A \stackrel{\text{stable}}{\sim} \Lambda$ with a Lagrangian $C \in \text{Quadr}(U)$. Then there exist Lagrangian B and Lagrangian D in Quadr(U) such that $A \oplus C \simeq D$. Choose Lagrangians $\Lambda \subset C$ and $\Lambda' \subset D$ respectively. The observation above applied to Λ considered as a subgroup of C shows that $A = \Lambda^{\perp}/\Lambda$ (orthogonality with respect to $A \oplus C$) is Lagrange-related to the trivial pairing $0 = \Lambda'^{\perp}/\Lambda'$, hence is Lagrangian.

PROPOSITION. Assume that $A \stackrel{\text{Lag}}{\sim} B$. By definition, there is a Lagrangian quadratic function N such that $-A \oplus B \simeq N$. Then

$$A \oplus N \simeq A \oplus -A \oplus B \simeq B \oplus -A \oplus A.$$

Since $-A \oplus A$ is Lagrangian (Lemma 2.4), this means that $A \stackrel{\text{stable}}{\sim} B$. Conversely, suppose that $A \stackrel{\text{stable}}{\sim} B$. Then there exist Lagrangian subgroups N, N' such that $A \oplus N \simeq B \oplus N'$. Thus by Lemma 2.4,

 $-A \oplus B \oplus -N \oplus N'$

is Lagrangian. Since $-N \oplus N'$ is already Lagrangian (Lemma 2.3, (2)), $-A \oplus B$ is stably Lagrangian. Therefore $-A \oplus B$ is Lagrangian by Lemma 3.3, hence $A \stackrel{\text{Lag}}{\sim} B$.

The respective sets of equivalence classes of $\operatorname{Quadr}(U)$, $\operatorname{Quadr}^0(U)$ and $\operatorname{Sym}^{\varepsilon}(U)$ are denoted $\mathfrak{WQ}(U), \mathfrak{WQ}^0(U)$ and $\mathfrak{W}^{\varepsilon}(U)$ respectively.

LEMMA 3.4. Each set defined above is an abelian group for the operation induced by the orthogonal sum \oplus .

PROOF. To see that there is a well defined operation induced by the orthogonal sum, we verify that \sim is compatible with \oplus . This is clear using the stable relation. Lemma 2.4 implies that every element [A] has an inverse, namely -[A] = [-A].

DEFINITION 4.13. The groups $\mathfrak{WQ}(U), \mathfrak{WQ}^0(U)$ and $\mathfrak{W}^{\varepsilon}(U)$ are the *Witt* groups of quadratic groups, homogeneous quadratic groups and ε -symmetric linking groups respectively with values in U. For simplicity, we shall denote by $\mathfrak{W}(U)$ the Witt group of symmetric linking groups.

The following remark is useful in the context of torsion.

LEMMA 3.5. There is a canonical decomposition

$$\mathfrak{WQ}(U) = \bigoplus_{p} \mathfrak{WQ}_{p}(U)$$

where $\mathfrak{WQ}_p(U)$ denotes the Witt group associated to the monoid of quadratic functions on p-primary subgroups with values in U. The analogous statement for other Witt groups holds.

PROOF. This follows from the orthogonal decomposition of an abelian group into its *p*-primary subgroups (Lemma 3.2).

PROPOSITION 3.6 (The basic alternative). Let $(A,q) \in \text{Quadr}(U)$ and let $N \subseteq A$ be a subgroup.

- Either $q(N \cap N^{\perp}) \neq 0;$
- or q induces quadratic functions $N/(N \cap N^{\perp})$ and $N^{\perp}/(N \cap N^{\perp})$ respectively such that in the Witt group $\mathfrak{WQ}(U)$,

(3.1)
$$[A] = \left[\frac{N}{N \cap N^{\perp}}\right] + \left[\frac{N^{\perp}}{N \cap N^{\perp}}\right].$$

PROOF. Let $H = N \cap N^{\perp}$ and assume that q(H) = 0. Then by Proposition 3.6, q induces a quadratic function H^{\perp}/H and $[A] = [H^{\perp}/H]$. By the involutive property,

$$H^{\perp} = N^{\perp} + N^{\perp \perp} = N^{\perp} + N = N + N^{\perp}.$$

Hence in $\operatorname{Quadr}(U)$,

$$H^{\perp}/H \simeq N/H \oplus N^{\perp}/H.$$

Therefore in the Witt group, $[A] = [H^{\perp}/H] = [N/H] \oplus [N^{\perp}/H]$, which is the desired result.

REMARK 4.11. One can define refined Witt groups by considering only split Lagrangians [51, §5]. The equivalence relation is considerably more restrictive; therefore the Witt groups obtained in this fashion are (much) larger and sit between the monoid of isomorphism classes and the Witt groups we have defined above.

We close this section by considering a degenerate quadratic function $q: A \to U$. We have already seen (Lemma 5.2) that if $q(A^{\perp}) = 0$ then q induces a nondegenerate quadratic function $\bar{q}: A/A^{\perp} \to U$. If the short exact sequence $0 \to A^{\perp} \to A \to A/A^{\perp} \to 0$ is split, then there is an isomorphism of quadratic groups

$$(A,q) \simeq (A/A^{\perp}, \bar{q}) \oplus (A^{\perp}, q|_{A^{\perp}}),$$

where $q|_{A^{\perp}}$ is the trivial quadratic form identically zero on A^{\perp} , hence is Lagrangian. Therefore, at the level of the Witt group $\mathfrak{WQ}(U)$, $[A,q] = [A/A^{\perp}, \bar{q}]$. But the hypothesis on the short exact sequence may not be satisfied in general (See Remark 1.10). Nevertheless, the equality in $\mathfrak{WQ}(U)$ holds in general:

LEMMA 3.7. Let $q : A \to U$ be a tame quadratic function. Then $[A,q] = [A/A^{\perp}, \bar{q}]$ in $\mathfrak{WQ}(U)$.

PROOF. Apply Prop. 2.5 to (A, q) with q-isotropic subgroup A^{\perp} .

It follows from Lemma 3.7 that we can restrict to nondegenerate quadratic functions in our study of the Witt group $\mathfrak{WQ}(U)$. However, it is useful to consider also nontame quadratic functions (in particular for topological applications). We shall discuss in §4.3 the appropriate "Witt setting" for general quadratic functions on finite Abelian groups.

4. The Witt group of finite quadratic abelian groups

4.1. Quadratic functions. Let p be a prime.

LEMMA 4.1 (Reduction Lemma). Let α, β be integers coprime with p. The following relation holds in $\mathfrak{WQ}_p(\mathbb{Q}/\mathbb{Z})$:

$$\begin{bmatrix} \frac{\alpha \ x^2}{p^n} + \frac{\beta \ x}{p^k} \end{bmatrix} = \begin{bmatrix} \frac{\alpha \ x^2}{p^{n-2}} + \frac{\beta \ x}{p^{k-1}} \end{bmatrix} \qquad \text{for } 0 \le \max(k,1) < n$$
$$\begin{bmatrix} \frac{\alpha \ x^2}{2^{n+1}} + \frac{\beta \ x}{p^k} \end{bmatrix} = \begin{bmatrix} \frac{\alpha \ x^2}{2^{n-1}} + \frac{\beta \ x}{2^{k-1}} \end{bmatrix} \qquad \text{for } 0 \le \max(k,2) < n$$

PROOF. We treat the case when p is odd. The case p = 2 is similar. Let q be the quadratic function defined by $q(x) = \frac{\alpha x^2}{p^n} + \frac{\beta x}{p^k}$ on $A = \mathbb{Z}/p^n\mathbb{Z}$. Under the hypotheses the subgroup $B = p^{n-1}A$ is q-isotropic. Hence q is Lagrangian equivalent to the contraction \bar{q} on B^{\perp}/B . It is readily verified that $(B^{\perp}/B, \bar{q})$ is isomorphic to the quadratic function q' defined on $\mathbb{Z}/p^{n-2}\mathbb{Z}$ by $q'(x) = \frac{\alpha}{p^{n-2}} + \frac{\beta}{p^{k-1}}$.

Looking at the order of the "linear part", we deduce the following facts.

COROLLARY 4.2. If $0 \le k \le 1$, then $\left[\frac{\alpha \ x^2}{p^n} + \frac{\beta \ x}{p^k}\right] = \left[\frac{\alpha \ x^2}{p^{n-2}}\right]$. In particular, if n is even then $\left[\frac{\alpha \ x^2}{p^n} + \frac{\beta \ x}{p^k}\right] = 0$.

PROOF. The first assertion is a direct consequence of the lemma. Now every nondegenerate linking pairing on a cyclic group A of order p^{2n} has Lagrangian $p^n A$. The second assertion follows.

4.2. Quadratic forms. We begin with the computation of the classical Witt group of quadratic forms on finite abelian groups. By Lemma 3.5, $\mathfrak{WQ}^0(\mathbb{Q}/\mathbb{Z})$ splits as the direct sum of $\mathfrak{WQ}_p^0(\mathbb{Q}/\mathbb{Z})$, p prime.

We record a fundamental observation due to F.Connolly [12, proof of Th. 1.13].

PROPOSITION 4.3. Let $q \in \mathfrak{MQ}^0$. Let p be a prime and denote by $q_p \in \mathfrak{MQ}_p^0$ the corresponding orthogonal summand. The following identities holds in \mathfrak{MQ}_p^0 :

 $q_p = -q_p \qquad \qquad if \ p = -1 \ \mathrm{mod} \ 4$ $q_p + q_p = (-q_p) + (-q_p) \qquad \qquad if \ p = -1 \ \mathrm{mod} \ 4$ $q_2 + q_2 + q_2 = (-q_2) + (-q_2) + (-q_2) + (-q_2).$

REMARK 4.12. The proposition applies in particular to linking pairings since the map $\mathfrak{MQ}^0 \to \mathfrak{M}^+$ is surjective.

PROOF. We follow [92, Lemma 1.15]. First work in the ring \mathbb{Z}_p of *p*-adic integers.

LEMMA 4.4. There exist $x_1, x_2, x_3, x_3 \in \mathbb{Z}_p$ such that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1$. Furthermore, if $p = -1 \mod 4$ then one can take $x_3 = x_4 = 0$; if $p = 1 \mod 4$ then one can take $x_2 = x_3 = x_4 = 0$;

PROOF. The equation $x^2 = a$ has a solution in \mathbb{Z}_p if and only if it has a solution in $\mathbb{Z}/p\mathbb{Z}$ (for p odd) or in $\mathbb{Z}/8\mathbb{Z}$ for p = 2 (see [**96**, Chap.5, §4] and use Hensel's lemma). If $p = 1 \mod 4$, then -1 is a square mod p so $x_1^2 = -1$ has a solution in \mathbb{Z}_p . If $p = -1 \mod 4$, then -1 is a nonsquare mod p. Therefore Lemma 4.4 applies. For p = 2, the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1$ has the solution $x_1 = x_2 = x_3 = 1, x_4 = 2$ in $\mathbb{Z}/8\mathbb{Z}$.

For (x_1, x_2, x_3, x_4) satisfying the condition of the lemma above, form the matrix

$$C = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4 & x_3 & -x_2 & -x_1 \end{bmatrix}$$

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with coefficients in \mathbb{Z}_p . A direct computation shows that

$$C \cdot C^T = (x_1^2 + x_2^2 + x_3^2 + x_4^2) I_4 = -I_4.$$

Now as a finite abelian group, A_p is a \mathbb{Z}_p -module: the action of \mathbb{Z}_p on A_p is the diagonal action of \mathbb{Z}_p on the direct sum of the cyclic *p*-groups (each of which regarded as embedded in \mathbb{Z}_p). So *C* defines an automorphism of $A_p \oplus A_p \oplus A_p \oplus A_p$. Let q' denote the quadratic form $q \oplus q \oplus q \oplus q$. The associated linking pairing is $b' = b \oplus b \oplus b \oplus b$.

A direct computation shows that

$$q'(Cx) = \left(\sum_{j=1}^{4} x_j^2\right) q'(x) = -q'(x)$$

for any x lying in a summand of $A_p \oplus A_p \oplus A_p \oplus A_p$. (Here we use the fact that q is homogeneous.) Using "anti-orthonormality" of the columns of C, we compute similarly that

$$b_{q'}(Cx, Cy) = -b_{q'}(x, y), \quad x, y \in A_p \oplus A_p \oplus A_p \oplus_p$$

Using $q'(x + y) = q'(x) + q'(y) + b_{q'}(x, y)$, we deduce that C induces an isomorphism between q' and -q'.

PROPOSITION 4.5. Let p be a prime number. There are canonical isomorphisms

$$\mathfrak{WQ}_p^0(\mathbb{Q}/\mathbb{Z}) \simeq \mathfrak{W}_p(\mathbb{Q}/\mathbb{Z}), \text{ for } p \neq 2,$$
$$\mathfrak{W}_p(\mathbb{Q}/\mathbb{Z}) \simeq \mathfrak{W}_p(\mathbb{F}_p) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = 1 \mod 4\\ \mathbb{Z}/4\mathbb{Z} & \text{if } p = 3 \mod 4\\ \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2 \mod 4 \end{cases}$$

where $\mathfrak{W}(\mathbb{F}_p)$ denotes the Witt group of symmetric linking pairings on the field \mathbb{F}_p .

PROOF. The first isomorphism is induced by the map $(A, q) \mapsto (A, b_q)$ which is an isomorphism if A has no even order element. The natural canonical injection

$$\mathbb{F}_p \to \mathbb{Q}/\mathbb{Z}, \ 1 \mapsto \frac{1}{p} \mod 1$$

induces a monoid homomorphism $\mathfrak{M}^+(\mathbb{F}_p) \to \mathfrak{M}_p^+$. This map induces a map at the level of Witt groups. Provide an inverse to this map as follows ([83, Chap. 5, §1, Theorem 1.5]). Let $(A, \lambda) \in \mathfrak{M}_p^+$. If pA = 0 then A is a \mathbb{F}_p -vector space. In particular, λ determines an element in $\mathfrak{W}(\mathbb{F}_p)$. Suppose that $p^n A = 0$ and $p^{n-1}A \neq 0$ for some $n \geq 2$. The subgroup $A_1 = p^{n-1}A$ is isotropic since $\lambda(p^{n-1}x, p^{n-1}y) = p^{2n-2}\lambda(x, y) = \lambda(p^n p^{n-2}x, y) = 0$. Set $B_1 = A_1^{\perp}/A_1$ and let $\overline{\lambda}$ be the induced linking pairing on B_1 . Applying Proposition 2.5, we have $[A, \lambda] = [B_1, \overline{\lambda}]$. Observe that $p^{n-1}B_1 = 0$. We can therefore reiterate this process a finite number of times and obtain a linking pairing $(B_k, \overline{\lambda}_k)$ such that $[A, \lambda] = [B_k, \overline{\lambda}_k]$ and $p B_k = 0$. Hence the procedure assigns a well-defined element in $\mathfrak{W}(R/\mathfrak{p})$. It remains to see that it induces a map $\mathfrak{W}_{\mathfrak{p}} \to \mathfrak{W}(R/\mathfrak{p})$. First, the procedure preserves orthogonal sums. Furthermore, we claim that Lagrangian linking pairings are sent to Lagrangian linking pairings. It suffices to see that if λ is Lagrangian, then so is $\overline{\lambda}$. Let $L \subseteq A$ be a Lagrangian for λ . Let $L_1 = (L \cap A_1^{\perp})/(L \cap A_1) \subseteq A_1^{\perp}/A_1 = B_1$. It follows that

$$(L \cap A_1^{\perp})^{\perp} = L^{\perp} + A_1^{\perp \perp} = L + A_1.$$

Hence

$$(L \cap A_1^{\perp})^{\perp} \cap A_1^{\perp} = (L + A_1) \cap A_1^{\perp} = L \cap A_1^{\perp},$$

where the last equality follows from the inclusion $A_1 \subseteq A_1^{\perp}$. Therefore, $L_1^{\perp} = L_1$ is a Lagrangian for $\bar{\lambda}$. This proves our claim. It is not hard to see that this map is the desired inverse.

In view of Theorem 4.3, $\mathfrak{W}(\mathbb{F}_p)$ has two generators $b = \lfloor 1/p \rfloor$ and $b' = \lfloor n_p/p \rfloor$ where n_p is a quadratic nonresidue mod p. If $p = 1 \mod 4$, then $\lfloor 1/p \rfloor = \lfloor -1/p \rfloor = -\lfloor 1/p \rfloor$ hence b = -b so b and b' have order 2. As there is no other nontrivial extra relation (the relation (4.2) gives 2b = 2b' = 0), we deduce that the map $b \mapsto (1,0), b' \mapsto (0,1)$ induces an isomorphism $\mathfrak{W}(\mathbb{F}_p) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $p = -1 \mod 4$, then -1 is a quadratic nonresidue mod p so $b' = \lfloor -1/p \rfloor = -\lfloor 1/p \rfloor = -b$. Prop. 4.3 yields 2b = -2b hence 4b = 0. The isomorphism $\mathfrak{W}(\mathbb{F}_p) = \mathbb{Z}/4\mathbb{Z}$ follows. For p = 2, $\mathfrak{W}(\mathbb{F}_2)$ is generated by $b = \lfloor 1/2 \rfloor$. Note that $\lfloor -1/2 \rfloor = \lfloor 1/2 \rfloor$ so b = -b and therefore 2b = 0. The isomorphism $\mathfrak{W}(\mathbb{F}_2) = \mathbb{Z}/2\mathbb{Z}$ follows.

REMARK 4.13. For a computation of the isometry classes of forms on \mathbb{F}_p (see [83, Chap. 2, §3]). One can derive from it (or from Theorems 4.3 and 4.7) an alternative computation of the Witt groups above.

REMARK 4.14. The Lagrangian used in the proof (to prove that the map $\mathfrak{W}_p(\mathbb{Q}/\mathbb{Z}) \to \mathfrak{W}_p(\mathbb{F}_p)$ is well defined) is nonsplit in general.

REMARK 4.15. The refined Witt groups (see Remark 4.11) do not harbor new torsion properties but are considerably larger. For instance, $[1/p^{2k}] = 0$ in $\mathfrak{W}_p(\mathbb{Q}/\mathbb{Z})$ since $p^k \mathbb{Z}/p^{2k} \mathbb{Z}$ is Lagrangian (not split), but $[1/p^{2k}]$ is not zero in the refined Witt group. See [51, Prop. 5.2] for a detailed computation.

The Gauss sum was introduced in Chap. 1 as an invariant of the isomorphism class of a quadratic function. A more precise result holds: it is actually an invariant of $\mathfrak{WQ}(\mathbb{Q}/\mathbb{Z})$.

PROPOSITION 4.6. Let (A, q) be a Lagrangian quadratic function on a finite abelian group. Then $\gamma(A, q) = 1$. In particular, γ induces a homomorphism $\mathfrak{WQ}(\mathbb{Q}/\mathbb{Z}) \to \mathbb{C}^{\times}$.

PROOF. Let x_1, \ldots, x_r be a complete set of coset representatives for A/L. Then the Gauss sum associated to q can be written as

$$\gamma(A,q) = |A|^{-\frac{1}{2}} \sum_{j=1}' \sum_{y \in L} \chi(q(x_j + y)).$$

After we replace $q(x_j + y) = q(x_j) + b_q(x_j, y) + q(y) = q(x_j) + b_q(x_j, y)$, the sum becomes

$$\gamma(A,q) = \sum_{j=1}^{\prime} \chi(q(x_j)) \sum_{y \in L} \chi(b_q(x_j,y)).$$

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Now for each fixed j, the map $\hat{b}_q(x_j) : y \mapsto \chi(b_q(x_j, y))$ is a homomorphism from L to U(1). If it is a trivial map, constant equal to 1, then the sum

(4.1)
$$\sum_{y \in L} \chi(b_q(x_j, y))$$

equals $1 + \cdots + 1 = |L|$. This happens if and only if $\hat{b}_q(x_j) = 0$ if and only if $x_j \in L^{\perp} = L$. If $x_j \notin L$, then $\hat{b}_q(x_j)$ is nonzero and by a classical argument (Lemma 1.12), the sum (4.1) is zero. Since exactly one representative x_j lies in L, we conclude that that

$$\gamma(A,q) = |A|^{-\frac{1}{2}} |L| = |A|^{-\frac{1}{2}} |A|^{\frac{1}{2}} = 1.$$

Since γ is multiplicative on orthogonal sums (1.8), the second statement immediately follows.

COROLLARY 4.7. Let $[A,q] \in \mathfrak{WQ}_p^0(\mathbb{Q}/\mathbb{Z})$. If p = 2 then $\gamma(A,q)^8 = 1$; if $p = -1 \mod 4$ then $\gamma(A,q)^4 = 1$; if $p = 1 \mod 4$ then $\gamma(A,q)^2 = 1$.

PROOF. Apply γ to the quadratic forms of Prop. 4.3, use multiplicativity on orthogonal sums and the fact that $\gamma(-q) = \overline{\gamma(q)}$ (Lemma 1.8). Since q is nondegenerate, $\gamma(q) \neq 0$. The result follows.

PROPOSITION 4.8. $\mathfrak{WQ}_2^0(\mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The generators can be represented by the cyclic quadratic forms $(\frac{1}{4})$ and $(\frac{1}{8})$ on $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ respectively.

4.3. The Witt monoid of finite quadratic functions. For nontame quadratic functions on finite Abelian groups, the stable relation is still an equivalence relation; the Lagrange relation may not be reflexive.

LEMMA 4.9. If $q : A \to \mathbb{Q}$ is a Lagrangian degenerate quadratic function then any Lagrangian contains the annihilator A^{\perp} and q is tame.

PROOF. Let $\Lambda \subset A$ be a Lagrangian. We have

$$\Lambda = \Lambda^{\perp} = \Lambda^{\perp \perp} = \Lambda + A^{\perp}.$$

Hence $A^{\perp} \subset \Lambda$. Since $q(\Lambda) = 0$, we deduce that $q|_{A^{\perp}} = 0$.

LEMMA 4.10. A quadratic function that has a nontame orthogonal summand is nontame. The orthogonal sum of two nontame (resp. tame) quadratic functions is nontame (resp. tame).

PROOF. Immediate from the definitions.

COROLLARY 4.11. If (A,q) is nontame then $(-A,q) \oplus (A,q)$ is not Lagrangian.

PROOF. $(-A, q) \oplus (A, q)$ is nontame by Lemma 4.10, hence cannot be Lagrangian by Lemma 4.9.

As a consequence, the Lagrange relation is not reflexive on nontame quadratic functions. Therefore, one considers the stable relation instead.

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LEMMA 4.12. The stable relation in the set of all quadratic functions is an equivalence relation. The set $\widetilde{\mathfrak{WQ}}(\mathbb{Q}/\mathbb{Z})$ of stable equivalence classes is an Abelian monoid. Nontame quadratic functions represent nontrivial elements and have no inverse.

PROOF. The proof of the first statement and second statement follows the same lines as that of Lemmas 3.1 and 3.4. Any quadratic function that has a nontame orthogonal summand is nontame, by Lemma 4.10, hence cannot be Lagrangian. The last statement follows.

DEFINITION 4.14. The monoid $\mathfrak{WQ}(\mathbb{Q}/\mathbb{Z})$ is called the *Witt monoid* of quadratic functions.

LEMMA 4.13. Two absolute nontame quadratic functions represent the same element in $\widetilde{\mathfrak{WQ}}(\mathbb{Q}/\mathbb{Z})$ if and only if they are isomorphic.

PROOF. One direction is obvious. Conversely, suppose that two absolute nontame quadratic functions verify A = A' in $\widetilde{\mathfrak{WQ}}(\mathbb{Q}/\mathbb{Z})$. Then there is an isomorphism $A \oplus \Lambda \simeq A' \oplus \Lambda'$ of quadratic functions for some Lagrangians Λ, Λ' . By Lemma 4.10, $A \oplus \Lambda$ and $A' \oplus \Lambda'$ remain nontame. By Lemma 4.9, $\Lambda_{\rm nt} = 0 = \Lambda'_{\rm nt}$. Therefore,

$$A = A_{\rm nt} = A_{\rm nt} \oplus 0 = (A \oplus \Lambda)_{\rm nt} \simeq (A' \oplus \Lambda')_{\rm nt} = A'_{\rm nt} \oplus 0 = A'_{\rm nt} = A'.$$

LEMMA 4.14. There is a canonical decomposition

$$\widetilde{\mathfrak{WQ}}(\mathbb{Q}/\mathbb{Z}) = \bigoplus_p \widetilde{\mathfrak{WQ}}_p(\mathbb{Q}/\mathbb{Z})$$

where $\widetilde{\mathfrak{WQ}}_p(\mathbb{Q}/\mathbb{Z})$ is the Witt monoid of all quadratic functions on finite *p*-groups. A similar decomposition holds for the Witt monoid $\widetilde{\mathfrak{WQ}}^0(\mathbb{Q}/\mathbb{Z})$ of all quadratic forms.

The following result relates the Witt monoid to the Witt group of quadratic forms:

PROPOSITION 4.15. Let p be a prime number. If $p \neq 2$, then there is a commutative diagram

If p = 2, there is a commutative diagram

with the following properties:

(1) all horizontal isomorphisms are induced by the orthogonal decomposition into tame and absolute nontame parts;

(2) the p-component $[A_p, q|_{A_p}]$ lies in the subgroup $\mathfrak{WQ}_p(\mathbb{Q}/\mathbb{Z})$ (resp. in the subgroup $\mathfrak{WQ}_p^0(\mathbb{Q}/\mathbb{Z})$) if and only if it is represented by a tame quadratic function (resp. a tame quadratic form);

(3) all vertical maps are induced by the natural inclusions;

(4) the monoid monomorphism $i \times j$ is induced on the first component by the natural inclusion $i: \mathfrak{WQ}_2^0(\mathbb{Q}/\mathbb{Z}) \to \mathfrak{WQ}_2(\mathbb{Q}/\mathbb{Z})$ and is defined on the second component by $j(a_1, a_2, a_3, \ldots) = (a_1, (a_2, 0), (a_3, 0, 0), \ldots)$.

PROOF. Let $q : A \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function. By Prop. 5.28, $(A,q) = (A_t,q_t) \oplus (A_{nt},q_{nt})$, where q_t is tame and q_{nt} is either identically zero (if and only if q is tame) or is an absolute nontame quadratic form. Hence in the Witt monoid, there is a decomposition

$$[A,q] = [A_{\rm t},q_{\rm t}] + [A_{\rm nt},q_{\rm nt}],$$

where the second component lies in the Witt monoid of all absolute nontame quadratic functions. By Lemma 4.13, the latter identifies with the monoid of isomorphism classes of absolute nontame quadratic functions. (In particular, $[A_{\rm nt}, q_{\rm nt}] = 0$ if and only if q is tame.) Therefore the top (resp. bottom) horizontal isomorphisms in the diagrams follow from the last statement of Prop. 5.28 for quadratic functions (resp. quadratic forms); the second component of each horizontal isomorphism consists of the collection of invariants associated to the absolute nontame part. (In particular, note that for p odd, all quadratic forms on a p-group are tame, hence the equality $\widetilde{\mathfrak{WQ}}_p^0 = \mathfrak{WQ}_p^0(\mathbb{Q}/\mathbb{Z}).$)

CHAPTER 5

Reciprocity

We derive a formula in the Witt group of torsion quadratic functions; presented as an alternative, it generalizes all previously known formulas of reciprocity. We keep notation from the previous paragraphs. The formula can be regarded as a far-reaching generalization of the classical Van der Blij formula. Hence we begin the Van der Blij formula first.

1. The van der Blij - Milgram formula

In 1959, F. van der Blij stated a formula relating the bilinear lattice to its discriminant function [8]. In short, this is a computation of the Gauss sum associated to a discriminant quadratic function. Since any quadratic function is a discriminant quadratic function (Th 4.6), this computation applies to any nondegenerate finite quadratic function. The computation is explicit in terms of the bilinear lattice lying over the quadratic function and yields a fundamental invariant of the quadratic function. The applications of this beautiful formula, many times rediscovered, lie in algebra and topology.

THEOREM 1.1. Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate quadratic function on a finite abelian group. Let (V, f, v) be any nondegenerate bilinear lattice equipped with a Wu class $v \in V^{\sharp}$ such that $(G_f, \varphi_{f,v}) = (G, q)$. Then

(1.1)
$$\gamma(G_f, \varphi_{f,v}) = \exp\left(\frac{2\pi i}{8}(\operatorname{sign}(f) - f_{\mathbb{Q}}(v, v))\right).$$

Recall that $\operatorname{sign}(f)$ denotes the signature of the lattice $(V, f) \otimes \mathbb{R}$. Van der Blij's original formula is the case v = 0 (for *even* lattices) and det $f = 1 \mod 2$.

As hinted above, the applications in algebra and topology are just too numerous to list. We shall content ourselves with a few obvious observations and consequences. Define

(1.2)
$$\beta(q) = \operatorname{sign}(f) - f_{\mathbb{Q}}(v, v) \mod 8.$$

COROLLARY 1.2. The rational residue $\beta(q) \in \mathbb{Q}/8\mathbb{Z}$ is an invariant of the isomorphism class of (G, q). Furthermore,

- 1. $\beta(q)$ is in fact an invariant of the Wu class of (G, q).
- 2. If q is homogeneous then $\beta(q) \in \mathbb{Z}/8\mathbb{Z}$.
- 3. If (V, f) is unimodular then $\beta(q) = 0$, i.e.,

 $\operatorname{sign}(f) = f(v, v) \mod 8$, for any Wu class $v \in \operatorname{Wu}(f)$.

5. RECIPROCITY

2. Proof of the van der Blij – Milgram formula

There are several proofs of the van der Blij formula. It is pointed out by C. T. C. Wall [98] that the original proof given by van der Blij has an analytical gap¹. The proof below is inspired from that of J. Milnor and M. Husemoller [66, Appendix 4] (see also J. Lannes and F. Latour, [56]).

In order to lighten the notation, we drop the notation for the symmetric bilinear pairing in this section. Let V be a nondegenerate symmetric bilinear lattice V equipped with a Wu class v. Any sublattice U inherits a structure of a symmetric bilinear lattice by restriction. Furthermore, since $U \subseteq V \subseteq V^{\sharp} \subseteq U^{\sharp}$, one has the inclusion Wu $(V) \subseteq$ Wu(U). Hence the sublattice U is also a symmetric bilinear lattice equipped with Wu class v. In particular, (V, v) and (U, v) induce discriminant quadratic functions, which we denote V^{\sharp}/V and U^{\sharp}/U respectively.

LEMMA 2.1. If U is a sublattice of index k > 0 in V, then

 $[U^{\sharp}/U] = [V^{\sharp}/V]$

in the Witt group $\mathfrak{WQ}(\mathbb{Q}/\mathbb{Z})$. In particular, the Witt class of the discriminant quadratic function is computable from any finite index sublattice.

PROOF. (Compare with [66, Appendix 4].) Consider the subgroup H = V/U in U^{\sharp}/U . Then $H^{\perp} = V^{\sharp}/U$. Thus $H \subseteq H^{\perp}$. Furthermore, since v is a Wu class for v, the discriminant quadratic form on U^{\sharp}/U vanishes on H. Therefore Proposition 2.5 applies: $[U^{\sharp}/U] = [H^{\perp}/H] = [V^{\sharp}/V]$.

Consider a lattice U in $V \otimes \mathbb{Q}$ such that $U \otimes \mathbb{Q} = V \otimes \mathbb{Q}$. Then both U and V contain the sublattice $U \cap V$ that has finite index in each of them. Hence by Lemma 2.1, $[U^{\sharp}/U] = [V^{\sharp}/V]$.

The symmetric bilinear form over the Q-vector space $V \otimes \mathbb{Q}$ is isomorphic to an orthogonal sum of 1-dimensional spaces (through the Gram-Schmidt procedure). Therefore it contains a lattice U which is an orthogonal sum of 1-dimensional spaces. By choosing a sublattice if necessary (for instance twice the previous one), we can suppose that U is even. We choose the Wu class 0 for U. By Lemma 4.2, each orthogonal 1-dimensional summand is also even. Since γ is an invariant of Witt groups, it suffices to verify the identity for a 1-dimensional even lattice $V = \mathbb{Z}$. Such a bilinear lattice is of the form $(2m) : (x, y) \mapsto 2mxy$ for some nonzero integer m. We endow it with the Wu class $0 \in \mathbb{Z}$. The associated discriminant quadratic function is $\varphi_0(x) = \frac{x^2}{4m} \mod 1$, $x \in \mathbb{Z}/m\mathbb{Z}$, with associated linking pairing $\lambda(x, y) = \frac{xy}{2m}$, $x, y \in \mathbb{Z}/m\mathbb{Z}$. Assume that m > 0. We are left to compute

(2.1)
$$\gamma(V^{\sharp}/V,\varphi_0) = \frac{1}{\sqrt{|m|}} \sum_{x \in \mathbb{Z}/m\mathbb{Z}} \exp\left(2\pi i \frac{x^2}{2m}\right)$$
$$= \exp\left(\frac{\pi i}{4}\right) = \exp\left(2\pi i \frac{\operatorname{sign}(m)}{8}\right).$$

¹This analytical gap can be possibly removed by means of Nikulin's theory.
Hence this agrees with the stated formula. If m < 0 then the Gauss sum is the conjugate of the previous one (2.1) so the stated formula is again verified. It remains to verify that the formula is verified if we change the Wu class. The set of Wu classes is $Wu(V) = 2V^{\sharp} = 2 \times \frac{1}{2m}\mathbb{Z} = \frac{1}{m}\mathbb{Z}$. Let $v = \frac{k}{m} \in Wu(V), k \in \mathbb{Z}$. The associated discriminant quadratic function is

$$\varphi_v(x) = \varphi_0(x) - \lambda_{\varphi_0}(x, k \mod 2m) = \frac{x^2}{4m} - \frac{kx}{2m} \mod 1, \ x \in \mathbb{Z}/m\mathbb{Z}.$$

Then

$$\gamma(V^{\sharp}/V,\varphi_v) = \gamma(V^{\sharp}/V,\varphi_0) \cdot \exp\left(2\pi i\varphi_0(k \mod 2m)\right) \quad \text{by Lemma 1.9}$$
$$= \exp\left(2\pi i\frac{\operatorname{sign}(m)}{8}\right) \cdot \exp\left(2\pi i\frac{k^2}{4m}\right)$$
$$= \exp\left(2\pi i\frac{\operatorname{sign}(m) - 2m \ k^2}{8}\right).$$

This is the desired identity and this concludes the proof.

3. The reciprocity identity

The reciprocity is a general identity involving tensor products and the discriminant construction. Let $f: V \times V \to \mathbb{Z}$ and $g: W \times W \to \mathbb{Z}$ be two nondegenerate bilinear lattices. Endow $f \otimes g$ with a Wu class $z \in (V \otimes W)^{\sharp}$. Recall that there are natural maps $j_f: G_f \otimes W \to G_{f \otimes g}$ and $V \otimes G_g \to G_{f \otimes g}$ respectively. The subgroups $A = j_f(G_f \otimes W)$ and $B = j_g(V \otimes G_g)$ are mutually orthogonal in $G_{f \otimes g}$ with respect to the discriminant linking pairing $\lambda_{f \otimes g}$.

THEOREM 3.1 (Reciprocity). The following identity holds in $\widetilde{\mathfrak{WQ}}(\mathbb{Q}/\mathbb{Z})$: (3.1) $[G_f \otimes W, \varphi_{f \otimes g, z} \circ j_f] = [G_{f \otimes g}, \varphi_{f \otimes g, z}] + [V \otimes G_g, -\varphi_{f \otimes g, z} \circ j_g].$

This is the most general version of reciprocity, expressed in the Witt monoid $\widetilde{\mathfrak{WQ}}(\mathbb{Q}/\mathbb{Z})$ of all quadratic functions on finite abelian groups.

PROOF. Apply Prop. 3.6 to the quadratic group $(G_{f\otimes g}, \varphi_{f\otimes g,z})$ with respect to the subgroup $A = j_f(G\otimes W)$: either $\varphi_{f\otimes g,z}|_{A\cap A^{\perp}}$ is not identically zero or

$$\begin{split} & [G_{f\otimes g}, \varphi_{f\otimes g,z}] = [A/A \cap A^{\perp}, \varphi_{f\otimes g,z}|_{A/A \cap A^{\perp}}] + [A^{\perp}/A \cap A^{\perp}, \varphi_{f\otimes g,z}|_{A^{\perp}/A \cap A^{\perp}}], \\ & \text{where, abusing notation, we denote by the same letter the quadratic function on } A \text{ (resp. } A^{\perp}) \text{ and the induced quadratic function on } A/A \cap A^{\perp} \text{ (resp. } A^{\perp}/A \cap A^{\perp}). \\ & \text{Consider the quadratic function } \varphi_{f\otimes g,z}|_A \text{ on } A. \text{ Let } H = A \cap A^{\perp}. \\ & \text{Then } H^{\perp} = (A + A^{\perp}) \cap A = A \text{ (orthogonality with respect to the associated linking pairing } \lambda_{f\otimes g}|_{A\times A}). \\ & \text{It follows from Prop. } 2.5 \text{ applied to the } \varphi_{f\otimes g,z}|_A \text{ -isotropic subgroup } H \text{ that} \end{split}$$

$$[A, \varphi_{f \otimes g, z}|_A] = [A/A \cap A^{\perp}, \varphi_{f \otimes g, z}|_{A/A \cap A^{\perp}}].$$

Similarly,

$$[A^{\perp}, \varphi_{f \otimes g, z}|_{A^{\perp}}] = [A^{\perp}/A \cap A^{\perp}, \varphi_{f \otimes g, z}|_{A^{\perp}/A \cap A^{\perp}}].$$

But

$$(A, \varphi_{f \otimes g, z}|_A) \simeq (G_f \otimes W, \varphi_{f \otimes g, z} \circ j_f)$$

and similarly

$$(G_{f\otimes g}, \varphi_{f\otimes g, z}|_{A^{\perp}}) \simeq (V \otimes G_g, \varphi_{f\otimes g, z} \circ j_g).$$

The result follows.

A corollary of Th. 3.1 is the following reciprocity formula [94, 1.3]. Below bar denotes complex conjugation.

Corollary 3.2.

$$(3.2) \quad \gamma(G_f \otimes W, \varphi_{f \otimes g, z} \circ j_f) = \gamma(G_{f \otimes g}, \varphi_{f \otimes g, z}) \cdot \overline{\gamma(V \otimes G_g, \varphi_{f \otimes g, z} \circ j_g)}.$$

Proof of Corollary 3.2. We apply Theorem 3.1, considering first the case when $\varphi_{f\otimes g,z}|_H$ is not identically zero on H. Since the annihilator of $\varphi_{f\otimes g,z}|_A$ is $A \cap A^{\perp} = H$, Corollary ?? implies that

 $0 = \gamma(A, \varphi_{f \otimes g, z}|_A) = \gamma(G_f \otimes W, \varphi_{f \otimes g, z} \circ j_f).$

A similar argument shows that $\gamma(V \otimes G_g, \varphi_{f \otimes g, z} \circ j_g) = 0$. Therefore the identity (3.2) is verified.

Suppose next that $\varphi_{f \otimes g, z}|_H$ is identically zero. Then by Corollary ??, the three Gauss sums appearing in (3.2) are nonzero. Applying the homomorphism $\gamma : \mathfrak{WQ}(\mathbb{Q}/\mathbb{Z}) \to U(1)$ to the relation (??) yields the desired relation.

COROLLARY 3.3. Let $v \in Wu(f)$ and $w \in Wu(g)$. For any $v_0 \in Wu^V(f)$ and $w_0 \in Wu^W(g)$,

$$(3.3)$$

$$\gamma\left(G_{f}\otimes W,\varphi_{f,v_{0}}\otimes g + (\widehat{\lambda}_{f}\otimes g)(\left[\frac{v_{0}-v}{2}\right]\otimes w)\right) =$$

$$\gamma(G_{f\otimes g,v\otimes w}) \overline{\gamma\left(V\otimes G_{g}, f\otimes \varphi_{g,w_{0}} + (\widehat{f}\otimes\widehat{\lambda}_{g})(v\otimes\left[\frac{w_{0}-w}{2}\right])\right)}.$$

PROOF. Apply Cor. 3.2 to the case $z = v \otimes w$. Noting that

 $v \otimes w = v_0 \otimes w + (v - v_0) \otimes w = v \otimes w_0 + v \otimes (w - w_0)$

with $v - v_0 \in 2V^{\sharp}$ and $w - w_0 \in 2W^{\sharp}$, we have, according to (9.1),

 $\varphi_{f\otimes g,v\otimes w}\circ j_f = \varphi_{f,v_0}\otimes g - (\lambda_f\otimes g)(\left[\frac{v-v_0}{2}\right]\otimes w, -)|_{G\otimes W}$ and, according to (9.2),

 $\varphi_{f\otimes g,v\otimes w} \circ j_g = f \otimes \varphi_{g,w_0} - (f \otimes \lambda_f)(v \otimes \left[\frac{w-w_0}{2}\right], -)|_{V \otimes G_g}.$ This yields the desired result.

COROLLARY 3.4. [13, Th. 3] For any $v_0 \in \operatorname{Wu}^V(f)$ and $w_0 \in \operatorname{Wu}^W(g)$, (3.4) $\gamma(G_f \otimes W, \varphi_{f,v_0} \otimes g) = \gamma(G_{f \otimes g,v \otimes w}) \overline{\gamma(V \otimes G_g, f \otimes \varphi_{g,w_0})}$.

PROOF. Cor. 3.3 with integral Wu classes $v = v_0$ and $w = w_0$.

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CHAPTER 6

The Weil representation of a finite abelian group

In 1964, in a remarkable paper [100], André Weil constructed a unitary representation associated to a symplectic locally compact abelian group. In a few decades the Weil representation has appeared to be a central object in mathematics, lying at the crossroads between the theory of theta functions, number theory, harmonic analysis and quantum mechanics.

In the third paragraph of his celebrated paper, André Weil makes in passing the following remark: "qu'il me soit permis, en passant, de signaler l'intérêt qu'il y aurait peut-être à examiner de plus près, du point de vue de la présente théorie, le cas des groupes finis." ("Let me mention in passing the interest that might lie in studying more closely, from the viewpoint of the present theory, the case of finite groups.")

The exact intent of Weil is not immediately clear, aside from a note to the 1946 paper [54] by H. Kloosterman¹. One possible interpretation is that the existence of the Weil representation for finite abelian groups is closely related to the existence of Abelian Topological Quantum Field Theories.

1. The Heisenberg group

1.1. The Heisenberg group associated to a Seifert form. Let A be an Abelian group, let C be a subgroup of \mathbb{C} and let $\omega : A \times A \to C$ be a nonsingular symplectic pairing. We are interested in the following two cases: (1) A is finitely generated free Abelian, C is the group of integers; (2) A is finite Abelian, C is the group \mathbb{Q}/\mathbb{Z} . We choose a form $\beta : A \times A \to C$ such that

(1.1)
$$\beta(x,y) - \beta(y,x) = \omega(x,y), \ \forall x, y \in A.$$

Such a pairing was called in Chapter 1, §3.2, a *Seifert pairing*. Recall that there is no uniqueness of the Seifert pairing β for a given symplectic form ω : adding an arbitrary symmetric bilinear pairing to a Seifert pairing produces another Seifert pairing associated to the same symplectic form.

The motivation for this terminology comes from the following example.

EXAMPLE 6.1. Let $\Sigma \subset S^3$ be an oriented smooth surface. Choose a collar $\operatorname{Int}(\Sigma) \times [-1,1] \subset S^3 - \partial \Sigma$. For a 1-cycle x representing an element x in $H_1(\Sigma) = H_1(\Sigma \times 0)$, denote by x^+ (resp. x^-) the 1-cycle representative

¹The Weil representation appears in this paper presumably for the first time. It was independently rediscovered by I.E. Segal [87] in 1960, followed by D. Shale [85].

corresponding to $x \times 1$ (resp. $x \times -1$) in the collar. The *Seifert form* is a bilinear pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ defined by

$$\beta(x,y) = \operatorname{lk}(x,y^+) \in \mathbb{Z}, \ x,y \in H_1(\Sigma),$$

where lk denotes the usual symmetric linking pairing of cycles in S^3 . Note that $lk(x, y^+) = lk(x^-, y)$. Hence

 $\beta(x,y) - \beta(y,x) = \operatorname{lk}(x,y^+) - \operatorname{lk}(y,x^+) = \operatorname{lk}(x,y^+) - \operatorname{lk}(x,y^-) = -x \bullet y$

where • denotes the intersection pairing on Σ (with the usual orientation convention)². Hence $-\beta$ satisfies the relation (1.1) where ω is the intersection pairing.



DEFINITION 6.1. The Heisenberg group $\mathscr{H}_{\beta}(A)$ associated to (A, β) is the extension of A defined as the set $A \times C$ endowed with the multiplication rule

$$(x,t) \cdot (y,t') = (x+y,t+t'+\beta(x,y)).$$

If β is understood, we suppress the subscript and write simply $\mathscr{H}(A)$.

Associativity follows from associativity in A and bilinearity of β ; the pair (0,0) of neutral elements in A and C is the neutral element of $\mathscr{H}(A)$; the inverse of (x,t) is $(-x, -t + \beta(x, x))$ for any $x \in A$ and $t \in C$.

It follows from the definition that $\mathscr{H}_{\beta}(A)$ lies in the exact sequence

$$(1.2) 0 \longrightarrow C \longrightarrow \mathscr{H}(A) \longrightarrow A \longrightarrow 0.$$

EXAMPLE 6.2. The multiplicative group of upper triangular 3×3 integral matrices with 1's on the diagonal is isomorphic to the Heisenberg group

 $\mathscr{H}(\mathbb{Z}^2)$ associated to the Seifert pairing defined by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

EXERCISE 6.1. Prove in detail the previous statement. What happens if the ground ring \mathbb{Z} is replaced by a more general ring R?

The following proposition shows that there is exactly one isomorphism class of Abelian Heisenberg group, the direct product group.

PROPOSITION 1.1. The following assertions are equivalent:

²The alternative formula $(x, y) \mapsto \operatorname{lk}(x^+, y)$ defines a Seifert form for $+\omega$. The latter is given for instance in [2, §, 9.5, Definition 12]. Here we follow [82, p. 201-202], [59, p. 53, def. 6.4].

(1) The short exact sequence

$$(1.3) \qquad 0 \longrightarrow C \longrightarrow \mathscr{H}(A) \longrightarrow A \longrightarrow 0$$

is split;

(2) $\omega = 0$ (β is symmetric);

(3) There is an group isomorphism $H_{\beta}(A) \simeq A \times C$.

PROOF. We prove $(1) \implies (2) \implies (3) \implies (1)$. Suppose that the extension $\mathscr{H}(A)$ is split. There exists then a (group-theoretic) section $s: A \to \mathscr{H}(A), s(a) = (s_1(a), s_2(a))$, with $s_1: A \to A$ and $s_2: A \to \mathbb{Q}/\mathbb{Z}$. Since s is a section, we must have $s_1(a) = a$. Since s is a group homomorphism,

$$s_2(a+a') = s_2(a) + s_2(a') + \beta(a,a'), \quad \forall a,a' \in A.$$

This implies that β is symmetric, or equivalently, $\omega = 0$. If β is symmetric, choose a quadratic function $q: A \to C$ over β . The map

 $s: A \times C \to \mathscr{H}_{\beta}(A), \ (a, c) \mapsto (a, c + q(a))$

is a group isomorphism between $A \times C$ (with direct product group structure) and $H_{\beta}(A)$. The remaining implication (3) \Longrightarrow (1) is obvious.

DEFINITION 6.2. Let Z denote the center of $\mathscr{H}(A)$.

The following observations are useful.

LEMMA 1.2. Let
$$X = (x, t), Y = (y, t') \in \mathscr{H}(A)$$
. Then

(1.4) $[X,Y] = (0,\omega(x,y)).$

The center of $\mathscr{H}(A)$ is $Z = 0 \times C$ and does not depend on the Seifert pairing. Furthermore, $[\mathscr{H}(A), \mathscr{H}(A)] \subseteq Z$ with equality if ω is nonsingular.

PROOF. The equality is a direct computation:

$$\begin{split} [X,Y] &= XYX^{-1}Y^{-1} \\ &= (x,t)(y,t')(-x,-t+\beta(x,x))(-y+-t'+\beta(y,y)) \\ &= (x+y,t+t'+\beta(x,y))(-x-y,-t-t'+\beta(x,x)+\beta(y,y)+\beta(-x,-y)) \\ &= (0,0+\beta(x,y)+\beta(x,x)+\beta(y,y)+\beta(x,y)-\beta(x+y,x+y)) \\ &= (0,\beta(x,y)-\beta(y,x)) \\ &= (0,\omega(x,y)). \end{split}$$

Thus $(x, t) \in Z$ if and only if $\omega(x, y) = 0$ for all $y \in A$. Since ω is nondegenerate, x = 0. Hence $Z = 0 \times C$. The other assertions follow.

REMARK 6.1. It follows from Lemma 1.2 that for a nonsingular symplectic form ω , $\mathscr{H}(A)$ is nilpotent of nilpotence class 2 and the exact sequence (1.2) identifies with the canonical exact sequence

$$0 \to [\mathscr{H}(A), \mathscr{H}(A)] \to \mathscr{H}(A) \to \mathscr{H}(A) / [\mathscr{H}(A), \mathscr{H}(A)] \to 0.$$

COROLLARY 1.3. Let Λ be a Lagrangian in A. The set $L = \Lambda \times C$ is a maximal abelian normal subgroup of $\mathcal{H}(A)$. Conversely, any maximal abelian subgroup G of $\mathcal{H}(A)$ is normal and of this form, i.e., there exists a Lagrangian Λ in A such that $G = \Lambda \times C$. PROOF. Clearly, L is a subgroup of $\mathscr{H}(A)$ with composition law (1.5)

 $(x,t)(y,t') = (x+y,t+t'+\beta(x,y)) = (y+x,t'+t+\beta(y,x)) = (y,t')(x,t),$ for any $x, y \in \Lambda$, $t, t' \in C$, since $0 = \omega(x,y) = \beta(x,y) - \beta(y,x)$. Hence L is abelian. According to Lemma 1.2,

(1.6)
$$(x,t)(y,t')(x,t)^{-1} = (0,\omega(x,y))(y,t') = (y,t'+\omega(x,y))$$

In particular, L is normal in $\mathscr{H}(A)$.

Now a subgroup G in $\mathscr{H}(A)$ is abelian if and only if for any $(x, t), (y, t') \in G$, the relation (1.5) holds if and only if $\beta(x, y) = \beta(y, x)$ if and only if $\omega(x, y) = 0$. The epimorphism $\mathscr{H}(A) \to A$, $(x, t) \mapsto x$ restricts to a group morphism $G \to A$. Thus the image of the latter is an isotropic subgroup of A. The last statement of the corollary follows.

COROLLARY 1.4. Let $L = \Lambda_1 \times C$ be a maximal abelian normal subgroup of $\mathscr{H}(A)$. For each Lagrangian Λ_0 transverse to Λ_1 , there is an isomorphism $\varphi : \Lambda_0 \to \mathscr{H}(A)/L$.

PROOF. Define $\varphi(a_0) = [(a_0, 0)]$ for $a_0 \in \Lambda_0$. This is a group morphism because

(1.7)

$$\varphi(a_0)\varphi(b_0) = [(a_0, 0)(b_0, 0)] = [(a_0 + b_0, \beta(a_0, b_0)] = [\underbrace{(0, \beta(a_0, b_0))}_{\in Z \subset L} (a_0 + b_0, 0)]$$
$$= [(a_0 + b_0, 0)] = \varphi(a_0 + b_0).$$

Since Λ_0 and Λ_1 are transverse, $A = \Lambda_0 \oplus \Lambda_1$ and there is a well defined projection p_0 onto Λ_0 with respect to Λ_1 . Define a map $\mathscr{H}(A) \to A_0$ by sending (x,t) to $p_0(x) = x_0$ for any $(x,t) \in \mathscr{H}(A)$. This map is a group morphism, sends L to 0 and thus induces a group morphism $\mathscr{H}(A)/L \to A_0$ inverse to φ .

A group isomorphism $\varphi: A \to A'$ induces an isomorphism

 $\varphi \times \mathrm{id}_C : (x,t) \mapsto (\varphi(x),t)$

from $\mathscr{H}_{\beta}(A)$ onto $\mathscr{H}_{(\varphi^{-1})*\beta}(A')$. However, it is not even necessary for two Heisenberg groups to be isomorphic that the Seifert pairings be isomorphic, as the following observations show.

REMARK 6.2. The map $(x,t) \mapsto (x,-t)$ is an isomorphism between the Heisenberg groups $H_{\beta}(A)$ and $H_{-\beta}(A)$.

EXERCISE 6.2. Provide examples of bilinear pairings $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that β is not isomorphic to $-\beta$. Find a necessary and sufficient condition in the case of ε -linking pairings. (Use the classification results, Chap. 1, §4.)

LEMMA 1.5 (Presentation of the Heisenberg group: free Abelian case). Assume that A is free Abelian, C is Z and $\omega : A \times A \to \mathbb{Z}$ is nonsingular. Let Λ_0 and Λ_1 two transverse Lagrangians in A. Let $\langle x_i, i \in I \rangle$ (resp. $\langle y_i, i \in I \rangle$) be a system of generators of Λ_0 (resp. Λ_1).

(1) The group $\mathscr{H}(A)$ is generated by $\{(x_i, 0) \mid i \in I\} \cup \{(y_i, 0), i \in I\}$.

(2) The group
$$\mathscr{H}(A)$$
 may be presented as

(1.8)
$$\langle u_i, v_j, i, j \in I \mid [u_i, [u_j, v_k]] = [v_i, [u_j, v_k]] = 1 \rangle.$$

The proof is an immediate generalization of the argument given by P. Kahn in [48] (whom he attributes to K. Brown) for the special case of the Heisenberg group $\mathscr{H}(\mathbb{Z}^2)$ considered in Example 6.2 above.

PROOF. For (1), let H be the subgroup generated by $\{X_i = \{(x_i, 0) \ i \in I\} \cup \{Y_j = (y_j, 0), \ j \in J\}$. According to Lemma 1.2, $[(x_i, 0), (y_j, 0)] = (0, \omega(x_i, y_j) \ (i, j \in I)$ which generate Z. Hence $Z \subseteq H$. Next,

$$(x_i, 0) \cdot (y_j, 0) \cdot \underbrace{(0, -\beta(x_i, y_j))}_{\in Z} = (x_i + y_j, \beta(x_i, y_j) \cdot (0, -\beta(x_i, y_j)) = (x_i + y_j, 0).$$

By transversality, $(w, 0) \in H$ for all $w \in A$. Using the relation (1.4) once more, we conclude that $(w, t) \in H$ for all $(w, t) \in A \times C$, i.e. $H = \mathscr{H}(A)$.

Let us prove (2). Let F be the free group generated by $u_i, v_j, i, j \in I$, R the relator normal subgroup presented in the statement of the theorem and let G be the quotient group F/R. According to Lemma 1.2, any pair x, y of elements in $\mathscr{H}(A)$ satisfies $[x, y] \in Z$ hence satisfies the relations of the presentation (1.8). Therefore there is a unique homomorphism $f : G \to \mathscr{H}(A)$ that sends $[u_i]$ to $(x_i, 0)$ and $[v_j]$ to $(y_j, 0), i \in I, j \in I$. It follows from (1) that the homomorphism is onto. Furthermore, the map f induces a commutative diagram

where all vertical arrows are epimorphisms. Clearly the abelianization G_{ab} of G is the free Abelian group on the generators $u_i, v_j, i, j \in I$, therefore $G_{ab} = G/[G, G]$ is isomorphic to $\Lambda_0 \oplus \Lambda_1 = A$. Thus the induced epimorphism $\overline{f} : G/[G, G] \to A$ is actually an isomorphism. Observe that [G, G]is generated by $[(x, 0), (y, 0)] = (0, \omega(x, y)), x, y \in \Lambda_0 \cup \Lambda_1$. It follows that [G, G] is generated by (0, 1), thus $[G, G] = \mathbb{Z}$. Hence the induced epimorphism $f|_{[G,G]} : [G, G] \to \mathbb{Z}$ is an isomorphism. Apply the 5-lemma to conclude that f is an isomorphism.

LEMMA 1.6 (Presentation of the Heisenberg group: finite case). Assume that A is finite Abelian, $\omega : A \times A \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate. Let C be the set of values of ω , i.e. $C = \omega(A, A)$. Let Λ_0 and Λ_1 two transverse Lagrangians in A. Let $\langle x_i, i \in I \rangle$ (resp. $\langle y_i, i \in I \rangle$) be a system of generators of Λ_0 (resp. Λ_1).

- (1) The symplectic form $\omega : A \times A \to C$ is nonsingular.
- (2) The finite Heisenberg group $\mathscr{H}(A)$ is generated by $\{(x_i, 0) \ i \in I\} \cup \{(y_i, 0), i \in I\}.$

6. THE WEIL REPRESENTATION OF A FINITE ABELIAN GROUP

(3) For an element $x \in A$, denote by ord(x) the order of x in A. The group $\mathscr{H}(A)$ may be presented as

(1.9)
$$\langle u_i, v_i, i \in I \mid u_i^{o(x_i)} = v_j^{o(y_j)} = [u_i, [u_j, v_k]] = [v_i, [u_j, v_k]] = 1 \rangle.$$

PROOF. Assertion (1) follows from Lemma 2.2. The remainder of the proof is similar to that of Lemma 1.5. $\hfill\blacksquare$

REMARK 6.3. Provided that the rank of A is even, there is a unique nonsingular symplectic pairing $\omega : A \times A \to C$ up to isomorphism. Lemmas 1.5 and 1.6 show that the Heisenberg group $H_{\beta}(A)$, up to isomorphism, does not depend on the choice of the Seifert form β . However, there is no canonical isomorphism between two Heisenberg groups $\mathscr{H}_{\beta}(A)$ and $\mathscr{H}_{\beta'}(A)$.

We need to make more explicit the general form of an automorphism of $\mathscr{H}(A)$. Let $\Phi : \mathscr{H}_{\beta}(A) \to \mathscr{H}_{\beta}(A)$ be an automorphism of $\mathscr{H}(A)$. Since as a set $\mathscr{H}(A) = A \times C$, write

$$\Phi(x,t) = (\varphi(x,t), \psi(x,t)), \ x \in A, t \in C$$

with $\varphi : A \times C \to A$ and $\psi : A \times C \to C$. First, since $A = \mathscr{H}(A)/[\mathscr{H}(A), \mathscr{H}(A)]$, Φ induces an automorphism $\overline{\Phi} : A \to A$ such that the diagram

$$\begin{aligned} \mathscr{H}(A) & \xrightarrow{p} A \\ & \downarrow \Phi & \qquad \qquad \downarrow \overline{\Phi} \\ \mathscr{H}(A) & \xrightarrow{p} A \end{aligned}$$

is commutative. Since $p:A\times C\to A$ is the canonical projection on the first factor, we find that

$$\varphi(x,t) = p \circ \Phi(x,t) = \Phi \circ p(x,t) = \Phi(x).$$

In particular, $\varphi(x, t)$ only depends on $x \in A$. Next,

$$\begin{aligned} (\Phi(x),\psi(x,t)) &= \Phi(x,t) = \Phi((0,t)\,(x,0)) = \Phi(0,t)\,\Phi(x,0) \\ &= (0,\psi(0,t))\,(\overline{\Phi}(x),\psi(x,0)) \\ &= (\overline{\Phi}(x),\psi(0,t) + \psi(x,0)). \end{aligned}$$

Hence there is a decomposition $\psi(x,t) = \psi(0,t) + \psi(x,0)$. On the one hand,

$$\begin{aligned} \Phi(x,t)\Phi(y,t') &= (\overline{\Phi}(x),\psi(0,t) + \psi(x,0)) (\overline{\Phi}(y),\psi(0,t') + \psi(y,0)) \\ &= (\overline{\Phi}(x) + \overline{\Phi}(x),\psi(0,t) + \psi(0,t') + \psi(x,0) + \psi(y,0) + \beta(\overline{\Phi}(x),\overline{\Phi}(y)). \end{aligned}$$

On the other hand,

$$\Phi((x,t) (y,t')) = \Phi(x+y,t+t'+\beta(x,y))$$

= $(\overline{\Phi}(x+y), \psi(x+y,t+t'+\beta(x,y)))$
= $(\overline{\Phi}(x+y), \psi(0,t+t'+\beta(x,y))+\psi(x+y,0)).$

Hence

$$(1.10) \psi(0,t) + \psi(0,t') + \psi(x,0) + \psi(y,0) + \beta(\overline{\Phi}(x),\overline{\Phi}(y)) = \psi(0,t+t'+\beta(x,y)) + \psi(x+y,0).$$

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Observe that

$$\begin{aligned} (0,\psi(0,t+u)) &= \Phi(0,t+u) = \Phi((0,t)(0,u)) = (0,\psi(0,t))(0,\psi(0,u)) \\ &= (0,\psi(0,t)+\psi(0,u)). \end{aligned}$$

Hence $t \mapsto \psi(0, t)$ is additive. We can therefore simplify the equality (1.10) and obtain the relation

$$\psi(x,0)+\psi(y,0)+\beta(\overline{\Phi}(x),\overline{\Phi}(y))=\psi(0,\beta(x,y))+\psi(x+y,0).$$

Equivalently,

(1.11)
$$\psi(x+y,0) - \psi(x,0) - \psi(y,0) = \beta(\overline{\Phi}(x),\overline{\Phi}(y)) - \psi(0,\beta(x,y)).$$

Note that (1.11) implies that the map $(x, y) \mapsto \beta(\overline{\Phi}(x), \overline{\Phi}(y)) - \psi(0, \beta(x, y))$ is symmetric bilinear. (The right hand side implies that it is bilinear, the left hand side implies that it is symmetric.) Thus the map $x \mapsto \psi(x, 0)$ is a quadratic function. Let us determine the restriction of Φ to the center $Z = 0 \times C$ of $\mathscr{H}(A)$. We have $\Phi|_Z(0, t) = (0, \psi(0, t))$. Now

$$\begin{aligned} (0,\psi(0,\omega(x,y))) &= \Phi(0,\omega(x,y)) = \Phi([(x,t),(y,t')]) \\ &= [\Phi(x,t),\Phi(y,t')] \\ &= [(\overline{\Phi}(x),\psi(x,t)),(\overline{\Phi}(y),\psi(y,t'))] \\ &= (0,\omega(\overline{\Phi}(x),\overline{\Phi}(y))), \end{aligned}$$

hence the relation

$$\psi(0,\omega(x,y))=\omega(\overline{\Phi}(x),\overline{\Phi}(y)).$$

In particular, $\psi(0,t) = t$ if and only if $\overline{\Phi}^* \omega = \omega$. Now the rôle of the symplectic group is apparent.

DEFINITION 6.3. The symplectic group over (A, ω) is

$$\operatorname{Sp}(A) = \{ s \in \operatorname{Aut}(A) \mid s^* \omega = \omega \}.$$

Let us sum up the result just proved.

LEMMA 1.7 (Automorphisms that are the identity on the center). An automorphism $\Phi : \mathscr{H}(A) \to \mathscr{H}(A)$ satisfies $\Phi|_Z = \mathrm{id}_Z$ if and only if $\overline{\Phi} \in \mathrm{Sp}(A)$.

We arrive at the following definition, inspired by that of Gurevich and Hadani in their study of the Weil representation in characteristic two [40].

DEFINITION 6.4. The affine symplectic group over (A, ω) is

(1.12)
$$\operatorname{ASp}(A) = \{ s \in \operatorname{Aut}(\mathscr{H}(A)) \mid s|_Z = \operatorname{Id}_Z \}.$$

We sum up and complete the results obtained so far.

PROPOSITION 1.8. The group ASp(A) is an extension of Sp(A) that fits into the short exact sequence

$$(1.13) \qquad 0 \longrightarrow \operatorname{Hom}(A, C) \longrightarrow \operatorname{ASp}(A) \longrightarrow \operatorname{Sp}(A) \longrightarrow 1.$$

Elements in ASp(A) can be presented as ordered pairs $(s,q), s \in Aut(A), q : A \rightarrow C$ acting on $\mathcal{H}(A)$ by

(1.14)
$$(s,q) \cdot (x,t) = (s(x), t+q(x)), \quad (x,t) \in \mathscr{H}(A).$$

The group law in ASp(A) is given by

(1.15) $(s',q') \cdot (s,q) = (s' \circ s, q' \circ s + q).$

Furthermore, the automorphism s and the quadratic function q are related by the formula

(1.16) $q(x+y) - q(x) - q(y) = \beta(s(x), s(y)) - \beta(x, y), \ \forall x, y \in A.$

In particular, each $s \in \text{Sp}(A)$ determines a linking pairing $\lambda_s : A \times A \to C$ defined by

(1.17)
$$\lambda_s(x,y) = \beta(s(x),s(y)) - \beta(x,y), \quad x,y \in A.$$

PROOF. It follows from the previous discussion that an automorphism Φ of $\mathscr{H}(A)$ decomposes as $\Phi(x,t) = (\overline{\Phi}(x), \psi(0,t) + \psi(x,0))$ where $t \mapsto \psi(0,t)$ is an additive group homomorphism and $x \mapsto \psi(x,0)$ is a quadratic function over symmetric bilinear pairing $(x,y) \mapsto \beta(s(x), s(y)) - \psi(0, \beta(x,y))$. Suppose that $\Phi \in ASp(A)$. Then $\psi(0,t) = t$ and the last two assertions of the Proposition follow for $s = \overline{\Phi}$ and $\psi(x,t) = t + q(x)$. it follows from Lemma 1.7 that the induced automorphism $\overline{\Phi}$ on A is symplectic. The corresponding map

$$\operatorname{ASp}(A) \to \operatorname{Sp}(A), \ \Phi \mapsto \overline{\Phi}$$

is a group homomorphism. Let us prove that it is onto. Let $s \in \text{Sp}(A)$. Choose any quadratic function q over the symmetric bilinear pairing $(x, y) \mapsto \beta(s(x), s(y)) - \beta(x, y)$. Define $\Phi(x, t) = (s(x), t + q(x))$. We claim that this defines an automorphism in ASp(A). First, $\Phi(0, t) = (0, t)$, i.e. Φ restricts to the identity on the center Z of $\mathscr{H}(A)$. Secondly, compute

$$\Phi((x,t) (y,t')) = \Phi(x + y, t + t' + \beta(x,y))$$

= $(s(x + y), t + t' + \beta(x,y) + q(x + y)))$
= $(s(x) + s(y), t + t' + q(x) + q(y) + \beta(s(x), s(y)))$
= $(s(x), t + q(x)) (s(y), t' + q(y))$
= $\Phi(x,t) \Phi(y,t').$

Thus Φ is a homomorphism. Thirdly, it is readily verified that the map $(x,t) \mapsto (s^{-1}(x), t - q(s^{-1}(x)))$ is Φ^{-1} . This proves that $\Phi \mapsto \overline{\Phi}$ is an epimorphism. Next, an automorphism Φ lies in the kernel of the map $\Phi \mapsto \overline{\Phi}$ if and only if $\overline{\Phi} = \operatorname{id}_A$ if and only if $\Phi(x,t) = (x,t+q(x))$ where q is quadratic over $(x,y) = \beta(\overline{\Phi}(x), \overline{\Phi}(y)) - \beta(x,y) = \beta(x,y) - \beta(x,y) = 0$ if and only if $q \in \operatorname{Hom}(A, C)$. This proves that the sequence (1.13) is exact. Lastly, the group law is obtained by writing, for $\Phi(x,t) = (s(x), t + q(x))$ and $\Phi'(x,t) = (s'(x), t + q'(x))$, the composition

$$\Phi' \circ \Phi(x,t) = \Phi(s(x), t + q(x)) = (s'(s(x)), t + q(x) + q'(s(x))).$$

REMARK 6.4. Our presentation of the Heisenberg groups in this paragraph uses the additive notation for the value group $C \subseteq \mathbb{C}$. Given a symplectic abelian group (A, ω) with Seifert form β , we may regard the symplectic pairing and the Seifert pairing as *bimultiplicative* pairings $A \times A \to \mathbb{C}^{\times}$ into the *multiplicative* group \mathbb{C}^{\times} . Then we could equivalently define the Heisenberg group associated to A by the same group law as before but on the underlying set $A \times \mathbb{C}^{\times}$. This leads to a Heisenberg group that contains an isomorphic image of the Heisenberg group $\mathscr{H}(A)$. Similarly, for finite abelian groups, one may choose the value group to be the multiplicative group $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. This leads to the Heisenberg group $\mathscr{H}^{U(1)}(A)$ that also contains an isomorphic image of $\mathscr{H}(A)$.

1.2. The short exact sequence for ASp(A). We take up the question whether the exact sequence (1.13) is split. According to Proposition 1.8, a set-theoretic section of the epimorphism $ASp(A) \rightarrow Sp(A)$ is represented by a map $Sp(A) \rightarrow ASp(A)$, $s \mapsto (s, q_s)$ where q_s is quadratic function over the symmetric bilinear map $s^*\beta - \beta$. The section is group-theoretic (i.e. is a homomorphism) if $q_{s's} = q_{s'} \circ s + q_s$.

PROPOSITION 1.9. Let A be a finite abelian group. The following assertions are equivalent:

- The affine symplectic group ASp(A) is a split extension of Sp(A) and Hom(A, Q/Z);
- (2) A has odd order.
- (3) Multiplication by 2 defines an automorphism of A;

PROOF. The proof is a relative version of the proof of Prop. 5.30.

Let us prove $(1) \Longrightarrow (2)$. Assume that A has even order. We shall show that there is no group-theoretic section $\operatorname{ASp}'(A) \to \operatorname{ASp}(A)$. Consider an element $x_0 \in A$ of order 2^k where k is maximal. By the classification of symplectic linking pairings (Prop. 3.12), there exists another element $x_1 \in A$ of order 2^k such that the subgroups B_0 and B_1 generated by x_0 and x_1 respectively are isotropic and do not intersect nontrivially, the subgroup $B_0 \oplus B_1$ is an orthogonal summand of (A, ω) and

$$\omega(x_0, x_1) = \frac{1}{2^k} \pmod{1}.$$

Define an element $s \in \text{Sp}(A)$ by setting

$$s(x_0) = x_0, \ s(x_1) = x_0 + x_1$$

and extending by the identity on the orthogonal complement of $B_0 \oplus B_1$ in A. Without loss of generality (cf. Corollary 3.14 and Corollary 3.15), we may assume that $\beta(x_0, x_1) = \omega(x_0, x_1)$ and $\beta(x_1, x_0) = \beta(x_0, x_0) = \beta(x_1, x_1) = 0$. Then $\lambda_s = s^*\beta - \beta$ is trivial except on $B_0 \oplus B_1$ where

$$\lambda_s(x_1, x_1) = \frac{1}{2^k} \mod 1, \ \lambda_s(x_0, x_0) = \lambda_s(x_0, x_1) = \lambda_s(x_1, x_0) = 0.$$

So λ_s is actually trivial everywhere except on the cyclic component B_1 where it identifies to $\left(\frac{1}{2^k}\right)$. By an immediate induction from (??), for any r,

$$(s,\lambda_s)^r = (s,\lambda_s)\cdots(s,\lambda_s) = (s^r,s^{r*}\beta - \beta).$$

Since $s^r(x_0) = x_0, s^r(x_1) = r x_0 + x_1$, a direct computation shows that (1.18)

$$s^{r*}\beta(x_1, x_1) - \beta(x_1, x_1) = \beta(r x_0 + x_1, r x_0 + x_1) - 0 = r \beta(x_0, x_1) = \frac{r}{2^k} \mod 1.$$

Suppose that there is a group-theoretic section σ : Sp(A) \rightarrow ASp(A) : $s \mapsto \sigma(s, \lambda_s) = (\sigma_1(s), \sigma_2(\lambda_s))$. Then $\sigma_1(s) = s$. Let $q_s = \sigma_2(\lambda_s)$. Since $q_s : A \to \mathbb{Q}/\mathbb{Z}$ is a quadratic function over λ_s , we must have

$$q_s(x_1) = \frac{1}{2^{k+1}} + \text{term of order dividing } 2^k \pmod{1},$$

by Lemma 5.20. Then for any r, the group-theoretic section sends $(s, \lambda_s)^r$ to

$$(s,q_s)^r = (s^r,(s^{r-1})^*q_s + \dots + s^*q_s + q_s).$$

Let $Q_r = (s^{r-1})^* q_s + \cdots + s^* q_s + q_s$. Note that Q_r is a quadratic function over $(s^r)^* \beta - \beta$. We compute that

$$(s^{l})^{*}q_{s}(x_{1}) = q_{s}(s^{l}(x_{1})) = q_{s}(l x_{0} + x_{1}) = q_{s}(x_{1}) + q_{s}(l x_{0}) + \lambda_{s}(l x_{0}, x_{1})$$
$$= q_{s}(x_{1}) + q_{s}(l x_{0}) + 0$$
$$= \frac{1}{2^{k+1}} + \text{term of order dividing } 2^{k} \pmod{1}$$

where we used Lemma 5.20 in the last equality. It follows that

 $Q_r(x_1) = \frac{r}{2^{k+1}} + \text{term of order dividing } 2^k \pmod{1}.$

In particular, Q_{2^k} is nontrivial. Therefore the group-theoretic section σ would send $(s, \lambda_s)^{2^k} = (\mathrm{id}_A, 0)$ (the trivial element in $\mathrm{ASp}'(A)$) to the non-trivial element $(s, q_s)^{2^k} = (\mathrm{id}_A, Q_{2^k})$ in $\mathrm{ASp}(A)$. This is the desired contradiction.

The implication $(2) \Longrightarrow (3)$ is standard (Prop. 5.30).

Let us prove (3) \implies (1). We should prove that the short exact sequence (1.13) is split. Define a map $\sigma : \operatorname{Sp}(A) \to \operatorname{ASp}(A)$ by $\sigma(s) = (s, q_s)$ with

$$q_s(x) = \frac{1}{2}\lambda_s(x, x), \ x \in A.$$

The map $q_s:A\to \mathbb{Q}/\mathbb{Z}$ is clearly a (homogeneous) quadratic function. We have

$$q_{s \circ s'}(x) = \frac{1}{2} \left(\beta(s \circ s'(x), s \circ s'(x)) - \beta(x, x) \right) \\ = \frac{1}{2} \left(\beta(s \circ s'(x), s \circ s'(x)) - \beta(s'(x), s'(x)) \right) \\ + \frac{1}{2} \left(\beta(s'(x), s'(x)) - \beta(x, x) \right) \\ = q_s \circ s'(x) + q_{s'}(x).$$

It follows that

$$\sigma(s \circ s') = (s \circ s', q_{s \circ s'}) = (s \circ s', q_s \circ s' + q_{s'}) = \sigma(s) \cdot \sigma(s').$$

Hence σ is a group-theoretic section of the short exact sequence (1.13).

REMARK 6.5. It is true for any Abelian group A that if multiplication by 2 is an automorphism of A, then the exact sequence is split. (The finiteness of A is not used for the implication $(3) \implies (1)$ in Proposition 1.9.)

PROPOSITION 1.10. Let A be a finitely generated free Abelian group. The affine symplectic group ASp(A) is a nonsplit extension of Sp(A) and $Hom(A, \mathbb{Z})$.

PROOF. A group-theoretic section $\sigma : \operatorname{Sp}(A) \to \operatorname{ASp}(A), s \mapsto (s, q_s)$ exists if and only if for each $s \in \operatorname{Sp}(A)$, there is a quadratic function $q_s : A \to \mathbb{Z}$ over $\lambda_s = s^*\beta - \beta$ that satisfies the relation $q_{ss'} = q_s \circ s' + q_{s'}$ for any $s, s' \in \operatorname{Sp}(A)$. Since q_s is quadratic over λ_s , we can write

$$q_s(x) = \frac{1}{2}(\lambda_s(x, x) + \phi_s(x)), \ x \in A,$$

for some homomorphism $\phi_s: A \to \mathbb{Z}$ satisfying the following two conditions:

(1.19)
$$\phi_{ss'} = \phi_s \circ s' + \phi_{s'}, \quad \phi_s(x) = \lambda_s(x, x) \mod 2$$

Therefore, a group-theoretic section exists if and only if for each $s \in \text{Sp}(A)$, there exists a homomorphism $\phi_s \in \text{Hom}(A, \mathbb{Z})$ satisfying (1.19).

Now the map $\operatorname{Sp}(A) \to \operatorname{Hom}(A, \mathbb{Z}/2\mathbb{Z}), s \mapsto \lambda_s \mod 2$ is well-defined. Since A is free, each homomorphism $x \mapsto \lambda_s(x, x) \mod 2$ individually lifts to a homomorphism $A \to \mathbb{Z}$. We shall show that there is no system of lifts $(\phi_s)_{s \in \operatorname{Sp}(A)}$ that satisfies (1.19).

It suffices to prove this for $A = \mathbb{Z}^2$, $\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\operatorname{Sp}(A) = \operatorname{SL}_2(\mathbb{Z})$. The two elements

$$s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad t = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

satisfy the relations

(1.20)
$$s^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = (st)^3$$

For each element $f \in \text{Sp}(A)$, let us write $\phi_f = [a_f \ b_f]$, for $a_f, b_f \in \mathbb{Z}$, acting on elements in \mathbb{Z}^2 (viewed as column vectors), so that $\phi_f(x, y) = a_f x + b_f y$, $x, y \in \mathbb{Z}$. According to (1.20), $\phi_{s^2} = \phi_{-Id} = \phi_{(st)^3}$, so that

(1.21)
$$a_{s^2} = a_{-\mathrm{Id}} = a_{(st)^3}, \ b_{s^2} = b_{-\mathrm{Id}} = b_{(st)^3}.$$

Then

$$\begin{bmatrix} a_{st} \ b_{st} \end{bmatrix} = \phi_{st} = \phi_s \circ t + \phi_t = \begin{bmatrix} a_f \ b_f \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a_t \ b_t \end{bmatrix}$$
$$= \begin{bmatrix} a_s + a_t + b_s \ b_s + b_t \end{bmatrix}.$$

Hence

(1.22)
$$a_{st} = a_s + a_t + b_s, \ b_{st} = b_s + b_t.$$

Similarly the relation

$$\phi_{(st)^2} = \phi_{st} \circ st + \phi_{st} = (\phi_s \circ t + \phi_t) \circ st + \phi_s \circ t + \phi_t$$
$$= \phi_s \circ (tst + t) + \phi_t \circ (st + \mathrm{Id})$$

yields

$$\begin{bmatrix} a_{(st)^2} & b_{(st)^2} \end{bmatrix} = \begin{bmatrix} a_s & b_s \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} a_t & b_t \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -a_s - 2b_s - b_t & a_s + a_t + b_s + b_t \end{bmatrix}.$$

122 6. THE WEIL REPRESENTATION OF A FINITE ABELIAN GROUP

2. The Heisenberg group and the discriminant

The goal of this paragraph is to study the Heisenberg group associated to the discriminant group. Let (V, ω) be a nondegenerate symplectic lattice. The discriminant construction (see Chap.) associates to (V, ω) a discriminant finite symplectic group $(G_{\omega}, \lambda_{\omega})$ that fits into the exact sequence

$$(2.1) 0 \longrightarrow V \longrightarrow V^{\sharp} \longrightarrow G_{\omega} \longrightarrow 0.$$

The intermediate group V^{\sharp} has a natural symplectic structure induced by ω , namely $\omega_{\mathbb{Q}}|_{V^{\sharp} \times V^{\sharp}} : V^{\sharp} \times V^{\sharp} \to \mathbb{Q}$.

Let $\beta: V \times V \to \mathbb{Z}$ be a Seifert form associated to (V, ω) . This determines a Heisenberg group $\mathscr{H}(V)$ for V. Then V^{\sharp} has also a Seifert form, namely $\beta_{\mathbb{Q}}|_{V^{\sharp} \times V^{\sharp}} : V^{\sharp} \times V^{\sharp} \to \mathbb{Q}$. We define the Heisenberg group $\mathscr{H}(V^{\sharp})$ associated to $(V^{\sharp}, \beta_{\mathbb{Q}}|_{V^{\sharp} \times V^{\sharp}})$ in the same way as before: it is the set $V^{\sharp} \times \mathbb{Q}$ endowed with the usual multiplication rule. There is a natural choice of Seifert forms:

LEMMA 2.1. There is a Seifert form $\beta : V \times V \to \mathbb{Z}$ associated to ω such that $\beta_{\mathbb{Q}}(V, V^{\sharp}) \subseteq \mathbb{Z}$ and $\beta_{\mathbb{Q}}(V^{\sharp}, V) \subseteq \mathbb{Z}$.

PROOF. The Seifert form constructed

We fix a Seifert form as in Lemma 2.1 in this paragraph. The last group $G_{\omega} = V^{\sharp}/V$ in the sequence (2.1) also inherits a Seifert form, namely the Seifert form defined by

$$\beta_{\omega}([x], [y]) = \beta_{\mathbb{Q}}(x, y) \mod 1, \quad x, y \in V^{\sharp}.$$

(Lemma 2.1 ensures that this is well defined.)

LEMMA 2.2. There is a short exact sequence

$$1 \longrightarrow \mathscr{H}(V) \longrightarrow \mathscr{H}(V^{\sharp}) \longrightarrow \mathscr{H}(G_{\omega}) \longrightarrow 1.$$

PROOF. The inclusion map $\mathscr{H}(V) \to \mathscr{H}(V^{\sharp})$ is the natural one provided by the set-theoretic inclusion. The conjugate formula (1.6) shows that $\mathscr{H}(V) \triangleleft \mathscr{H}(V^{\sharp})$. The natural projection map $\mathscr{H}(V^{\sharp})/\mathscr{H}(V) \to H(G_{\omega}), (w, t) \mapsto ([w], [t])$ is a group isomorphism.

We define a new symplectic group.

DEFINITION 6.5.

$$\operatorname{ASp}(V^{\sharp}, V) = \{ \Phi \in \operatorname{Aut}(\mathscr{H}(V^{\sharp}) \mid \Phi|_{\mathscr{H}(V)} \in \operatorname{ASp}(V) \}.$$

By definition, an automorphism $\Phi \in \operatorname{ASp}(V^{\sharp}, V)$ gives rise to an element $\overline{\Phi}|_{V} \in \operatorname{Sp}(V^{\sharp})$ and also to an element in $\operatorname{ASp}(V)$.

PROPOSITION 2.3. There is a commutative diagram with exact rows and columns



Second row: we define the first nontrivial arrow $\operatorname{ASp}(V) \to \operatorname{ASp}(V^{\sharp})$. Let $(s,q) \in \operatorname{ASp}(V)$. The automorphism $s \in \operatorname{Sp}(V)$ induces an automorphism $s_{\mathbb{Q}} \in \operatorname{Sp}(V_{\mathbb{Q}})$. This automorphism induces by restriction a monomorphism $s_{\mathbb{Q}} : V^{\sharp} \to V_{\mathbb{Q}}$. We claim that $s_{\mathbb{Q}}(V^{\sharp}) = V$ for any $s \in \operatorname{Sp}(V)$. Let $x \in V^{\sharp}$. We have

3. The Schrödinger representation

In this section, (A, ω) is a finite symplectic group endowed with a Seifert form $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$.

LEMMA 3.1. For any Lagrangian A_1 in A, there is a Lagrangian A_0 in Aand a Seifert form $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that

$$\begin{cases} A_0 \oplus A_1 = A, \\ \beta|_{A_0 \times A_1} = \omega|_{A_0 \times A_1}, \ \beta|_{A_1 \times A_0} = \beta|_{A_1 \times A_1} = \beta|_{A_0 \times A_0} = 0. \end{cases}$$

PROOF. See the proof of the corollary 3.14.

The purpose of this setting is to regard A_0 and A_1 as dual subgroups.

COROLLARY 3.2. The maps

 $a_0 \mapsto \beta(a_0, -)|_{A_1}, \quad a_1 \mapsto \beta(-, a_1)|_{A_0}$

are group isomorphisms $A_0 \simeq A_1^*$ and $A_1 \simeq A_0^*$.

In the sequel we fix a Lagrangian A_1 and another Lagrangian A_0 satisfying the conditions of the Lemma above. The precise choice for A_0 is actually irrelevant for our later purpose (see below §4, Corollary 6.7), but allows for a concrete description of the Schrödinger representation.

Let $\chi: Z \to U(1)$ be a character on the center of $\mathscr{H}(A)$. For each $a \in A$, define a character

$$\chi_a : A \to U(1), \ x \mapsto \chi(-\beta(x,a)).$$

Let L^2A_0 be the Hilbert space consisting of \mathbb{C} -valued functions over A_0 endowed with the positive definite hermitian product

(3.1)
$$\langle f,g\rangle \mapsto \sum_{x\in A_0} \overline{f(x)} g(x).$$

An orthonormal basis for L^2A_0 consists of the set of functions $\delta_x, x \in A_0$ defined by $\delta_x(y) = 1$ if x = y and $\delta_x(y) = 0$ otherwise. Recall that L^2A_0 is actually a commutative and associative algebra for the convolution product

$$f \star g (x) = \sum_{y \in A_0} f(x - y) g(y), \ x \in A_0.$$

We have $\delta_x \star \delta_y = \delta_{x+y}, x, y \in A_0$. The map

$$x \mapsto \delta_x$$

extends linearly to an algebra isomorphism $(\mathbb{C}[A_0], \cdot) \to (L^2 A_0, \star).$ Consider on L^2A_0 the following two operators:

- Translation: T_{a0}δ_x = δ_{x+a0} defined for a₀ ∈ A₀.
 Modulation: M_{a1}δ_x = χ_{a1} δ_x defined for a₁ ∈ A₁.

LEMMA 3.3. For any $f \in L^2(A_0)$, $a_0, z \in A_0, a_1 \in A_1$,

$$(T_{a_0}f)(z) = f(z-a_0), \quad (M_{a_1}f)(z) = \chi_{a_1}(z) \ f(z) = \chi(\beta(z,a_1))^{-1} \ f(z).$$

PROOF. The second statement is clear. The first statement follows from $\{\delta_x\}_{x\in A_0}$ being a basis for $L^2(A_0)$ and the identity $\delta_{x+a_0}(z) = \delta_x(z-a_0)$.

DEFINITION 6.6. For each $a = (a_0, a_1) \in A$, define the Weyl operator W_a : $L^2 A_0 \rightarrow L^2 A_0$ by

$$W_a = M_{a_1} T_{a_0}.$$

It is clear from the definition that $W_0 = \text{Id.}$ Furthermore, if $a = (a_0, a_1)$ and $b = (b_0, b_1)$ then

(3.2)
$$W_b W_a = \chi_{a_1} (b_0)^{-1} W_{b+a}$$

In particular, $W_{-a}W_a = \chi_{a_1}(a_0) \operatorname{Id} = W_a W_{-a}$, thus

(3.3)
$$W_a^{-1} = \chi_{a_1}(a_0)^{-1} W_{-a}$$

It follows from (3.2) that

(3.4)
$$W_b W_a = \chi_{a_1}(b_0)^{-1} \chi_{b_1}(a_0) \ W_a W_b.$$

LEMMA 3.4. For any $a = (a_0, a_1) \in A$,

(3.5)
$$W_a^* = \chi_{a_1}(a_0)^{-1} \ W_{-a} = W_a^{-1}$$

In particular, W_a is unitary.

PROOF. Using the inner product (3.1), one computes

$$\langle W_a f, g \rangle = \chi_{a_1}(a_0)^{-1} \langle f, W_{-a}g \rangle.$$

Hence $W_a^* = \chi_{a_1}(a_0)^{-1} W_{-a}$. The second equality is (3.3).

Let $U(L^2A_0)$ denote the space of unitary operators on L^2A_0 . The map

$$A \to U(L^2 A_0), \ a \mapsto W_a,$$

is a projective unitary representation in the sense that there is a cocycle

$$c(a,b) = \chi_{b_1}(a_0) = \chi(\beta(a,b))^{-1} \in \mathrm{U}(1)$$

such that $W_{a+b} = c(a, b)W_aW_b$. The group – known as the Mackey obstruction group associated to (A, c) – that consists of all pairs $(a, z) \in A \times U(1)$ endowed with the law

$$(a, z) \cdot (a', z') = (a + a', zz'c(a, b)^{-1}), \quad a \in A, z \in \mathcal{U}(1)$$

is precisely the Heisenberg group $\mathscr{H}_{\chi\circ\beta}(A) = \mathscr{H}^{\mathrm{U}(1)}(A)$ (cf. Remark 6.4). The map

$$\pi : \mathscr{H}(A) \to \mathrm{U}(L^2 A_0), \ (a,t) \mapsto \chi(t) \ W_a,$$

is a faithful unitary linear representation of $\mathscr{H}(A)$. This is the Schrödinger representation of the Heisenberg group.

LEMMA 3.5. The following properties hold:

- (1) The Weyl operators $W_a, a \in A$, form a basis of $\operatorname{End}_{\mathbb{C}}(L^2A_0)$.
- (2) The Schrödinger representation $\pi : \mathscr{H}(A) \to U(L^2A_0)$ is irreducible.
- (3) $\pi|_Z(0,t) = \chi(t)$ Id for all $t \in \mathbb{Q}/\mathbb{Z}$.

PROOF. We follow [74, Lemma 3.2], see also [44, p. 823].

(1) Define a map $\alpha : A \to \operatorname{End}_{\mathbb{C}}(L^2A_0)$ by

$$\alpha(x)\phi = W_x \ \phi \ W_x^{-1}.$$

Relation (3.2) ensures that this is an additive map, hence a representation. If $a = (a_0, a_1)$ and $b = (b_0, b_1)$ then the relation (3.4) implies that

$$\alpha(b)(W_a) = \chi_{a_1}(b_0)^{-1}\chi_{b_1}(a_0) W_a = \chi(\omega(a,b))^{-1} W_a.$$

So W_a is an eigenvector of α with eigencharacter. Since $a \mapsto \chi(\omega(a, -))^{-1}$ is an isomorphism of A onto $\operatorname{Hom}(A, \operatorname{U}(1))$, the eigenvectors W_a have distinct eigencharacters. It follows that $\{W_a, a \in A\}$ is a set of linearly independent elements of $\operatorname{End}_{\mathbb{C}}(L^2A_0)$. Since the cardinality of this set is

$$|A| = |A_0| \cdot |A_1| = |A_0| \cdot |A_0| = |A_0|^2 = \dim_{\mathbb{C}} \operatorname{End}_{\mathbb{C}} (L^2 A_0),$$

the set is a basis of $\operatorname{End}_{\mathbb{C}}(L^2A_0)$.

(2) A subspace of L^2A_0 invariant under all the Weyl operators W_a is invariant under End_C(L^2A_0), according to (1). Thus it is either 0 or L^2A_0 .

(3) Follows from definitions.

REMARK 6.6. An endomorphism commuting with all Weyl operators is a multiple of the identity. Proof: by Lemma 3.5 (statement (1)) and linearity, the endomorphism has to commute with all endomorphism, hence the result.

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The representation $\pi : \mathscr{H}(A) \to U(L^2A_0)$ induces a representation $\tilde{\pi} : \mathbb{C}[\mathscr{H}(A)] \to \operatorname{End}_{\mathbb{C}}(L^2A_0)$ by linear extension

$$\tilde{\pi}\left(\sum_{j}\lambda_j(a_j,t_j)\right) = \sum_{j}\lambda_j\pi(a_j,t_j).$$

By property (1) of Lemma 3.5, $\tilde{\pi}$ is surjective. The kernel is easily identified: it consists of the two-sided ideal J generated by all elements

 $(0,t)\cdot h-\chi(t)h,\quad (0,t)\in Z,\ h\in \mathscr{H}(A).$

DEFINITION 6.7. The reduced group algebra $V[\mathscr{H}(A)]$ of the Heisenberg group on A is defined as the quotient of $\mathbb{C}[\mathscr{H}(A)]$ by J.

As a consequence of our previous observation, the reduced group algebra of the Heisenberg group identifies to the algebra of endormorphisms of L^2A_0 .

COROLLARY 3.6. The representation $\tilde{\pi}$ induces an algebra isomorphism between the reduced group algebra $V[\mathscr{H}(A)]$ and $\operatorname{End}_{\mathbb{C}}(L^2A_0)$.

We now turn to the main result of this paragraph.

THEOREM 3.7 (Stone-Von Neumann-Mackey). For any irreducible unitary representation $\rho : \mathscr{H}(A) \to U(\mathcal{H})$ where \mathcal{H} is a Hilbert space such that $\rho|_Z(0,t) = \chi(t) \operatorname{Id}_{\mathcal{H}}$, there is an isometry $\Psi : L^2(A_0) \to \mathcal{H}$ such that

$$\Psi(\pi(h)f) = \rho(h)\Psi(f), \quad \text{for all } f \in L^2(A_0), \ h \in \mathscr{H}(A).$$

In short: up to unitary equivalence, there is a unique unitary irreducible representation $\pi : \mathscr{H}(A) \to \mathrm{U}(L^2A_0)$ such that $\pi|_Z(0,t) = \chi(t) \operatorname{Id}_{L^2A_0}$. The Schrödinger representation is essentially unique.

PROOF. We follow [74, Theorem 3.1] supplying details from [60, p.26-27]. Since π is faithful, we regard $\mathscr{H}(A)$ as embedded into $U(L^2A_0)$. By the previous lemma, $\operatorname{End}_{\mathbb{C}}(L^2A_0)$ is freely generated over \mathbb{C} by the Weyl operators $W_a, a \in A$. Therefore $\rho : \mathscr{H}(A) \to U(\mathcal{H})$ extends linearly to a representation $\tilde{\rho} : \operatorname{End}_{\mathbb{C}}(L^2A_0) \to \operatorname{End}_{\mathbb{C}}(\mathcal{H})$ by

(3.6)
$$\tilde{\rho}\left(\sum_{a}\lambda_{a}W_{a}\right) = \sum_{a}\lambda_{a}\rho(W_{a}).$$

This turns \mathcal{H} into a $\operatorname{End}_{\mathbb{C}}(L^2A_0)$ -module. Since ρ is irreducible, \mathcal{H} is a simple $\operatorname{End}_{\mathbb{C}}(L^2A_0)$ -module. On the one hand, since ρ is unitary,

 $\rho(W_a)^*\rho(W_a) = \mathrm{Id}_{\mathcal{H}} \text{ for all } a \in A.$

On the other hand, since W_a is unitary,

 $\rho(W_a^*)\rho(W_a) = \rho(W_a^*W_a) = \rho(\mathrm{Id}_{L^2A_0}) = \mathrm{Id}_{\mathcal{H}}.$

Hence $\rho(W_a^*) = \rho(W_a)^{-1} = \rho(W_a)^*$. It follows from (3.6) that $\tilde{\rho}(\phi^*) = \tilde{\rho}(\phi)^*$ for all $\phi \in \operatorname{End}_{\mathbb{C}}(L^2A_0)$.

For $a \in A_0$, let p_a be the orthogonal projector onto $\mathbb{C}\delta_a \subseteq L^2 A_0$ defined by $p_a(f) = \langle f, \delta_a \rangle \delta_a$. Then $p_a^2 = p_a$ and $p_a^* = p_a$. Furthermore, $\tilde{\rho}(p_a)^2 =$

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 $\tilde{\rho}(p_a^2) = \tilde{\rho}(p_a)$ and $\tilde{\rho}(p_a)^* = \tilde{\rho}(p_a^*) = \tilde{\rho}(p_a)$ thus $\tilde{\rho}(p_a)$ is an orthogonal projector. Since $\tilde{\rho}$ is injective, $\tilde{\rho}(p_a)$ has rank 1. In particular,

$$\tilde{\rho}(p_a)\mathcal{H} = \mathbb{C}v_0 \neq 0$$

for some vector $v_0 \in \mathcal{H}$, $||v_0||_{\mathcal{H}} = 1$. Define a map $\Psi : L^2 A_0 \to \mathcal{H}$ by setting

$$\Psi(\phi\delta_0) = \tilde{\rho}(\phi)v_0, \quad \phi \in \operatorname{End}_{\mathbb{C}}(L^2A_0)$$

and extending by \mathbb{C} -linearity. This map is well defined since

- (i) $\operatorname{End}_{\mathbb{C}}(V)\delta_0 = L^2 A_0;$
- (ii) if $\phi(\delta_0) = \phi'(\delta_0)$ then $\phi p_0 = \phi' p_0$, so $\tilde{\rho}(\phi) \tilde{\rho}(p_0) = \tilde{\rho}(\phi') \tilde{\rho}(p_0)$, hence $(\tilde{\rho}(\phi) - \tilde{\rho}(\phi')) \circ \tilde{\rho}(p_0) = 0$. Since $\tilde{\rho}(p_0) \neq 0$, we have $\tilde{\rho}(\phi)|_{\mathcal{H}_0} = \tilde{\rho}(\phi')|_{\mathcal{H}_0}$ so that $\tilde{\rho}(\phi)h = \tilde{\rho}(\phi')h$.

We claim that Ψ is surjective. Let $\phi_a \in \operatorname{End}_{\mathbb{C}}(L^2A_0)$ such that $\phi_a(\delta_0) = \delta_a$. Then $\phi_a p_0 \phi_a^* = p_a$ and therefore $\operatorname{Id}_{L^2A_0} = \sum_{a \in A_0} \phi_a p_a \phi_a^*$ Applying $\tilde{\rho} : \operatorname{End}_{\mathbb{C}}(L^2A_0) \to \operatorname{End}_{\mathbb{C}}(\mathcal{H})$ yields

$$\mathrm{Id}_{\mathcal{H}} = \sum_{a \in A_0} \tilde{\rho}(\phi_a) \tilde{\rho}(p_0) \tilde{\rho}(\phi_a)^*.$$

Hence any element $h \in \mathcal{H}$ decomposes as $h = \sum_{a \in A_0} \lambda_a \tilde{\rho}(\phi_a) v_0$. Surjectivity of Ψ follows.

Observe that $\Psi(\phi \delta_a) = \Psi(\phi \phi_a \delta_0) = \tilde{\rho}(\phi \phi_a) v_0 = \tilde{\rho}(\phi) \tilde{\rho}(\phi_a) v_0 = \tilde{\rho}(\phi) \Psi(\phi_a \delta_0) = \tilde{\rho}(\phi) \Psi(\delta_a)$. It follows that

(3.7)
$$\Psi(\phi f) = \tilde{\rho}(\phi)\Psi(f)$$
 for all $\phi \in \operatorname{End}_{\mathbb{C}}(L^2A_0)$ and $f \in L^2A_0$.

Using again that ρ is unitary, we deduce that $\Psi : L^2 A_0 \to \mathcal{H}$ is an isometry. Restricting the property (3.7) to $\mathscr{H}(A)$ yields the desired result.

DEFINITION 6.8. A model of the Schrödinger representation is an irreducible representation $\pi : \mathscr{H}(A) \to U(\mathcal{H})$ into a Hilbert space \mathcal{H} such that $\pi|_{Z}(0,t) = \chi(t)$ Id_{\mathcal{H}} for all $(0,t) \in Z$.

By the Stone-von Neumann Theorem, all models of the Schrödinger representation are equivalent. That there are several models is a key feature of the Schrödinger representation. Before explaining how this fact leads to the Weil representation, we present other models.

4. The Schrödinger representation as an induced representation

An alternative incarnation of *Schrödinger representation* is the representation of $\mathscr{H}(A)$ induced from a maximal abelian normal subgroup in $\mathscr{H}(A)$.

LEMMA 4.1. For any Lagrangian A_1 in A, the subset $L_1 = A_1 \times \mathbb{Q}/\mathbb{Z} \subset \mathscr{H}(A)$ is a maximal abelian normal subgroup.

In particular, L_1 contains the center $Z = 0 \times \mathbb{Q}/\mathbb{Z}$. Extend the character $\chi : Z \to U(1)$ to a character $\tilde{\chi}$ on L_1 by

$$\tilde{\chi} : L_1 \to \mathrm{U}(1), \ (x,t) \mapsto \chi(t).$$

We now assume that these choices are fixed throughout the construction to follow.

Consider the space of all functions $\mathscr{H}(A) \to \mathbb{C}$. Let \mathcal{H}_{A_1} be the subspace of functions $f : \mathscr{H}(A) \to \mathbb{C}$ such that

$$f(h \cdot l) = \tilde{\chi}(l)^{-1} f(h), \quad h \in \mathscr{H}(A), \ l \in L_1.$$

Note that if $f \in \mathcal{H}_{A_1}$, then |f| is invariant under right (and thus also left) translations by elements of L_1 , hence induces a map $A_0 = \mathscr{H}(A)/L_1 \to \mathbb{C}$.

By definition, the induced representation is given by the action of $\mathscr{H}(A)$ on \mathcal{H}_{A_1} by translations

$$\pi'(h)[f](x) = f(h^{-1}x), \quad f \in \mathcal{H}_{A_1}, \ h \in \mathscr{H}(A).$$

To see that this representation is equivalent to the previous representation π , observe first that any map $f: A_0 \to \mathbb{C}$ extends in a unique way to a map $\tilde{f} \in \mathcal{H}_{A_1}$ defined by

$$\tilde{f}((a_0,0)\cdot l) = \tilde{\chi}(l)^{-1} f(a_0), \quad a_0 \in A_0, \ l \in L_1.$$

PROPOSITION 4.2. The map $f \mapsto \tilde{f}$ is an isometry

$$L^2(A_0) \to \mathcal{H}_{A_1}, \ f \mapsto \widetilde{f},$$

which is an equivalence between the representations π and π' .

PROOF. The first assertion follows from the definition. For the second assertion, we have to verify the identify $\pi(h)[f] = \pi'(h)[\tilde{f}]$ for $h \in \mathscr{H}(A)$, $f \in L^2(A_0)$. To check that the two maps are equal, one has to verify that they coincide on an arbitrary element $x \in \mathscr{H}(A)$. Let x be such an element: it decomposes as $x = (a_0, 0)l$ where $a_0 \in A_0$ and $l \in L_1$ are uniquely determined by x. On the one hand, we have

$$\pi(h)[f](y) = \chi(t) M_{h_1} T_{h_0} f(y).$$

Hence

$$\widetilde{\pi(h)[f]}(x) = \widetilde{\pi(h)[f]}((a_0, 0)l) = \widetilde{\chi}(l)^{-1}\pi(h)[f](a_0)$$
$$= \widetilde{\chi}(l)^{-1}\chi(t) M_{h_1} T_{h_0}f(a_0).$$

On the other hand, we have

$$\pi'(h)[\tilde{f}](x) = \tilde{f}(h^{-1}x) = \tilde{f}((-h_0 - h_1, -t + \beta(h_0, h_1)(a_0, 0)l))$$

= $\tilde{f}(\underbrace{(a_0 - h_0, 0)}_{\in A_0} \underbrace{(0, -t + \beta(a_0, h_1))(-h_1, 0)l}_{\in L_1})$
= $\tilde{\chi}(l)^{-1} \chi(t) \chi(-\beta(a_0, h_1)) f(a_0 - h_0)$
= $\tilde{\chi}(l)^{-1} \chi(t) M_{h_1} T_{h_0} f(a_0).$

This is the desired equality and concludes the proof.

COROLLARY 4.3. The Schrödinger representation depends on the choice of one Lagrangian A_1 in A rather than on a pair of transverse Lagrangians of A.

REMARK 6.7. An alternative proof follows from the definition of the Weyl operator, where A_0 can be replaced by A/A_1 throughout the construction.

As our construction is based on the finite group A, there is yet another presentation of the induced model as a *quotient* of the group algebra of the Heisenberg group. Let $I = I_{A_1,\chi}$ be the two-sided ideal in $\mathbb{C}[\mathscr{H}(A)]$ generated by the set $\{h \cdot l - \tilde{\chi}(l) h, h \in \mathscr{H}(A), l \in L_1\}$. Set

$$\mathcal{H}(A_1) = \mathbb{C}[\mathscr{H}(A)]/I.$$

The Heisenberg group $\mathscr{H}(A)$ acts by translations on the left on the group algebra $\mathbb{C}[\mathscr{H}(A)]$:

$$h \cdot \delta_k = \delta_{hk}, \quad h, k \in \mathscr{H}(A).$$

Since I is stable under $\mathscr{H}(A)$, this action descends to a representation $\mathscr{H}(A) \times \mathscr{H}(A_1) \to \mathscr{H}(A_1)$:

$$\pi''(h)[\delta_k] = [h \cdot \delta_k] = [\delta_{hk}], \quad h, k \in \mathscr{H}(A).$$

The following proposition is similar to Prop. 4.2.

PROPOSITION 4.4. The map $\delta_{a_0} \mapsto [\delta_{a_0}]$ defines an equivalence

$$L^2 A_0 \to \mathcal{H}(A_1)$$

between the representations π and π'' .

PROOF. Let us show that $[\pi(h)\delta_{a_0}] = \pi''(h)[\delta_{a_0}]$ for any $h \in \mathscr{H}(A)$ and $a_0 \in A_0$. Let $h = (h_0 + h_1, t)$ with $h_0 \in A_0, h_1 \in A_1$. First, $\pi(h)\delta_{a_0} = \chi(t) \chi_{h_1}(-) \delta_{a_0+h_0} = \chi(t) \chi_{h_1}(a_0 + h_0) \delta_{a_0+h_0}$. Hence

$$[\pi(h)\delta_{a_0}] = \chi(t) \chi_{h_1}(a_0 + h_0) [\delta_{a_0 + h_0}].$$

Next, $ha_0 = (h_0 + h_1, t)(a_0, 0) = (a_0 + h_0, 0)(0, h_1, t - \beta(a_0 + h_0, h_1)).$ Hence
 $\pi''(h)[\delta_{a_0}] = [\delta_{ha_0}] = \chi(t - \beta(a_0 + h_0, h_1)) [\delta_{a_0 + h_0}] = [\pi(h)\delta_{a_0}].$

REMARK 6.8. There is a natural injective algebra map $j : \mathbb{C}[\mathscr{H}(A)] \to L^2\mathscr{H}(A)$ whose image consists of complex-valued functions on $\mathscr{H}(A)$ with finite support. Consider the left-invariant Haar measure μ on $\mathscr{H}(A)$ normalized so that $\mu(\mathscr{H}(A)) = 1$. The canonical injection inj : $\mathcal{H}_{A_1} \to L^2\mathscr{H}(A)$ is a section of the projection map $p : L^2\mathscr{H}(A) \to \mathcal{H}_{A_1}$ defined by

(4.1)
$$(pf)(x) = \int_{L_1} f(xl) \,\chi(l) \,d\mu(l)$$

Furthermore, the following diagram is commutative

$$\mathbb{C}[\mathscr{H}(A)] \xrightarrow{j} L^{2}\mathscr{H}(A)$$

$$\underset{\mathcal{H}(A_{1})}{\xrightarrow{\simeq}} \mathcal{H}_{A_{1}}$$

The canonical projection proj : $\mathbb{C}[\mathscr{H}(A)] \to \mathcal{H}(A) = \mathbb{C}[\mathscr{H}(A_1)]/I$ has no section corresponding to the inclusion map inj. (Formally, a section $s : \mathcal{H}(A_1) \to \mathbb{C}[\mathscr{H}(A)]$ can be defined as

$$s([\delta_x]) = \sum_{l \in L_1} \chi(l)^{-1} \, \delta_{xl}$$

but then the target space must be enlarged accordingly to include functions with infinite support and then, it must coincide with $L^2\mathcal{H}(A)$.)

5. The Weil representation

5.1. The intertwining operator. According to the previous paragraph, the Schrödinger representation can be regarded as an induced functional representation defined as a map $\pi_{\Lambda} : \mathscr{H}(A) \to U(\mathcal{H}_{\Lambda}))$ (noted π' in the previous paragraph). This map depends on the choice of a Lagrangian Λ in A. Another choice of Lagrangian Λ' leads to another Schrödinger representation $\pi_{\Lambda'} : \mathscr{H}(A) \to U(\mathcal{H}_{\Lambda'})$. By the Stone-Von Neumann-Mackey theorem, the two representations are unitary equivalent: there exists an isometry $\rho_{\Lambda',\Lambda} \in \operatorname{Hom}_{\mathscr{H}(A)}(\mathcal{H}_{\Lambda}, \mathcal{H}_{\Lambda'})$ such that the diagram

$$\begin{aligned} \mathscr{H}(A) \times \mathcal{H}_{\Lambda} & \xrightarrow{\pi_{\Lambda}} \mathcal{H}_{\Lambda} \\ & \operatorname{id} \times \rho_{\Lambda',\Lambda} \\ & \downarrow \\ \mathscr{H}(A) \times \mathcal{H}_{\Lambda'} & \xrightarrow{\pi_{\Lambda'}} \mathcal{H}_{\Lambda'} \end{aligned}$$

is commutative. In formula,

$$\rho_{\Lambda',\Lambda}\pi_{\Lambda}(h)[f] = \pi_{\Lambda'}(h)\rho_{\Lambda',\Lambda}f, \quad h \in \mathscr{H}(A), \ f \in \mathcal{H}_{\Lambda}.$$

In short:

$$\rho_{\Lambda',\Lambda}\pi_{\Lambda}=\pi_{\Lambda'}\rho_{\Lambda',\Lambda}.$$

This relation alone determines the map $\rho_{\Lambda',\Lambda}$ up to a multiplicative constant (See Remark 6.6). Such a map is called an *intertwining operator*.

LEMMA 5.1. Suppose that Λ and Λ' are transverse in A. Up to a multiplicative constant,

(5.1)
$$\rho_{\Lambda',\Lambda}[f](h) = \sum_{l \in \Lambda'} f(h \cdot (l,0))$$

PROOF. We need to verify that $\rho[f] \in \mathcal{H}_{\Lambda'}$. Let $L' = \Lambda' \times \mathbb{Q}/\mathbb{Z}$ the maximal normal subgroup associated to Λ' . Let $x = (u, t) = (u, 0)(0, t) \in L'$. Then $f(hx(l, 0)) = f(h \cdot (u, 0) \cdot (0, t) \cdot (l, 0)) = f(h \cdot (u, 0)(l, 0) \cdot (0, t)) = \chi(t)^{-1} \cdot f(h \cdot (u, 0)(l, 0)) = \tilde{\chi}(x)^{-1}f(h \cdot (u, 0)(l, 0))$. Then by summing over $l \in \Lambda'$, we see that $\rho[f](hx) = \tilde{\chi}(x)^{-1} \cdot \rho[f](h)$. Thus $\rho[f] \in \mathcal{H}_{\Lambda'}$. Next, since the action is by translations, ρ is a $\mathcal{H}(A)$ -map.

Consider now three pairwise transverse Lagrangians Λ , Λ' and Λ'' in A. They give rise to three intertwining operators $\rho_{\Lambda',\Lambda}$, $\rho_{\Lambda'',\Lambda'}$ and $\rho_{\Lambda'',\Lambda}$ respectively. Both $\rho_{\Lambda'',\Lambda}$ and $\rho_{\Lambda'',\Lambda'} \circ \rho_{\Lambda',\Lambda}$ are intertwiners of $\mathcal{H}_{\Lambda''}$ and \mathcal{H}_{Λ} . Since \mathcal{H}_{Λ} is irreducible (Lemma 3.5), it follows from Schur's lemma that there exists $C(\Lambda'', \Lambda', \Lambda) \in U(1)$ such that

(5.2)
$$\rho_{\Lambda'',\Lambda'} \circ \rho_{\Lambda',\Lambda} = C(\Lambda'',\Lambda',\Lambda) \ \rho_{\Lambda'',\Lambda}.$$

As noticed by A. Weil, it turns out that the cocycle $C(\Lambda'', \Lambda', \Lambda)$ can be expressed as a Gauss sum.

Suppose Λ'', Λ' and Λ are pairwise transverse Lagrangians. Since $\Lambda \oplus \Lambda'' = A$, there is a well-defined projection $p_{\Lambda'',\Lambda} : A \to \Lambda''$ on Λ'' with respect to Λ .

LEMMA 5.2. Suppose Λ'' and Λ are transverse. Then

$$C(\Lambda'',\Lambda',\Lambda) = \sum_{l'\in\Lambda'} \chi(\beta(l',p_{\Lambda'',\Lambda}(l')).$$

Note that the map $q_{\Lambda'',\Lambda',\Lambda} : l' \mapsto \beta(l', p_{\Lambda'',\Lambda}(l'))$ is a homogeneous quadratic form on Λ' . Thus Lemma 5.2 can be rephrased as

$$C(\Lambda'', \Lambda', \Lambda) = \gamma_{\chi}([q_{\Lambda'', \Lambda', \Lambda}])$$

where $[q_{\Lambda'',\Lambda',\Lambda}]$ denotes the Witt class of $q_{\Lambda'',\Lambda',\Lambda}$ in WQ(\mathbb{Q}/\mathbb{Z}).

PROOF. Let $\tilde{\delta}_0 \in \mathcal{H}_{\Lambda}$ the extension of the map δ_0 (defined in Prop. 4.2). Note that the support of $\tilde{\delta}_0$ is $Z \cdot \Lambda = \Lambda \times \mathbb{Q}/\mathbb{Z}$. We apply the identity (5.2) to $\tilde{\delta}_0$ evaluated at x = 0. Since Λ and Λ' are transverse, $Z \cdot \Lambda$ and $Z \cdot \Lambda'$ intersect exactly on the center Z. Hence

$$\rho_{\Lambda',\Lambda}[\tilde{\delta}_0](0) = \sum_{l' \in \Lambda'} \tilde{\delta}_0(l',0) = \tilde{\delta}_0(0) = 1.$$

Thus

$$C(\Lambda'', \Lambda', \Lambda) = \rho_{\Lambda'', \Lambda'}(\rho_{\Lambda', \Lambda}[\tilde{\delta}_0])(0)$$

=
$$\sum_{l'' \in \Lambda''} \sum_{l' \in \Lambda'} \tilde{\delta}_0((l'', 0)(l', 0))$$

=
$$\sum_{l'' \in \Lambda''} \sum_{l' \in \Lambda'} \tilde{\delta}_0(l'' + l', \beta(l'', l')).$$

Now $\tilde{\delta}_0(a,t) \neq 0$ if and only if $a \in \Lambda$. Thus the only nonzero terms in the sum above occur when $l'' + l' \in \Lambda$. This is equivalent to $l'' = -p_{\Lambda'',\Lambda}(l')$. Let us compute this term:

$$\tilde{\delta}_0(\underbrace{l'-p_{\Lambda'',\Lambda}(l')}_{\in\Lambda},\beta(-p_{\Lambda'',\Lambda}(l'),l')) = \chi(\beta(-p_{\Lambda'',\Lambda}(l'),l'))^{-1} = \chi(\beta(p_{\Lambda'',\Lambda}(l'),l'))$$

Summing over all $l' \in \Lambda'$ yields the desired result.

We turn now to the case when Lagrangians are not necessarily pairwise transverse. The formula above (5.1) for $\rho_{\Lambda',\Lambda}$ is no longer applicable. We proceed as follows. By Lemma 1.7, there is a Lagrangian Λ'' which is transverse both to Λ and to Λ' . Thus both $\rho_{\Lambda'',\Lambda}$ and $\rho_{\Lambda',\Lambda''}$ are well defined. We could then define

$$\rho_{\Lambda',\Lambda} = \rho_{\Lambda',\Lambda''} \circ \rho_{\Lambda'',\Lambda}$$

provided that we could show that the operator $\rho_{\Lambda',\Lambda}$ does not depend on the particular choice of Λ'' among all Lagrangians transverse to both Λ and Λ' . Let Λ_1 and Λ_2 be two such Lagrangians. We wish to show that $\rho_{\Lambda',\Lambda_1} \circ \rho_{\Lambda_1,\Lambda} = \rho_{\Lambda',\Lambda_2} \circ \rho_{\Lambda_2,\Lambda}$. By Lemma 1.7, there exists a Lagrangian Λ_3 such that Λ_3 is transverse to $\Lambda_1, \Lambda_2, \Lambda_1$ and Λ_2 . We observe that

$$\rho_{\Lambda',\Lambda_1} \circ \rho_{\Lambda_1,\Lambda} = \rho_{\Lambda',\Lambda_1} \circ C(\Lambda_1,\Lambda_3,\Lambda)^{-1} \rho_{\Lambda_1,\Lambda_3} \circ \rho_{\Lambda_3,\Lambda}$$
$$= C(\Lambda_1,\Lambda_3,\Lambda)^{-1} \rho_{\Lambda',\Lambda_1} \circ \rho_{\Lambda_1,\Lambda_3} \circ \rho_{\Lambda_3,\Lambda}$$
$$= C(\Lambda_1,\Lambda_3,\Lambda)^{-1} C(\Lambda',\Lambda_1,\Lambda_3) \rho_{\Lambda',\Lambda_3} \circ \rho_{\Lambda_3,\Lambda}.$$

Up to a new higher cocycle, the products $\rho_{\Lambda',\Lambda_1} \circ \rho_{\Lambda_1,\Lambda}$ and $\rho_{\Lambda',\Lambda_3} \circ \rho_{\Lambda_3,\Lambda}$ coincide.

5.2. Definition of the Weil representation. Let $\pi : \mathscr{H}(A) \to U(\mathcal{H})$ be a model of the Schrödinger representation. By definition, the affine symplectic group ASp(A) acts by automorphisms on the Heisenberg group $\mathscr{H}(A)$:

$$\operatorname{ASp}(A) \times \mathscr{H}(A) \to \mathscr{H}(A), \ (s,h) \mapsto s(h).$$

For each $s \in ASp(A)$, the map $\pi \circ s$ defines a new representation of the Heisenberg group. This representation is still irreducible unitary and verifies

$$\pi \circ s|_Z(0,t) = \pi|_Z(s(0,t)) = \pi|_Z(0,t) = \chi(t) \operatorname{Id}_{\mathcal{H}}$$

for any $t \in \mathbb{Q}/\mathbb{Z}$. Hence by Theorem 3.7, the representations π and $\pi \circ s$ are unitary equivalent: there exists a unitary operator (defined up to a unitary scalar) $\rho_s \in \mathcal{H} \to \mathcal{H}$ such that

(5.3)
$$\rho_s(\pi(h)f) = \pi(s(h))(\rho_s f), \quad \forall h \in \mathscr{H}(A), \ \forall f \in \mathcal{H}.$$

Equivalently,

(5.4)
$$\pi \circ s = \rho_s \ \pi \ \rho_s^{-1}.$$

The Weil representation is the map

$$\operatorname{ASp}(A) \to \operatorname{U}(\mathcal{H}), s \mapsto \rho_s.$$

This definition depends on the choice, for each $s \in ASp(A)$, of a unitary operator $\rho_s \in U(\mathcal{H})$ verifying (5.4).

LEMMA 5.3. The Weil representation is a projective representation in the sense that for any $s, s' \in ASp(A)$, there exists $c(s, s') \in U(1) = S^1$ such that (5.5) $\rho_{ss'} = c(s, s') \rho_s \rho_{s'}$.

 $p_{ss} = c(s,s) p_{s}p_{s}^{s}.$

The map $(s,s^\prime)\mapsto c(s,s^\prime)$ is a 2-cocycle satisfying the identity

(5.6)
$$c(s_0, s_1s_2)c(s_1, s_2) = c(s_0s_1, s_2)c(s_0, s_1), \quad \forall s_0, s_1, s_2 \in ASp(A).$$

PROOF. For $s, s' \in ASp(A)$,

$$\rho_s \rho_{s'} \pi(h) = \rho_s \pi^{s'}(h) \rho_{s'} = \rho_s \pi(s'(h)) \rho_{s'} = \pi^s(s'(h)) \rho_s \rho_{s'} = \pi(ss'(h)) \rho_s \rho_{s'}$$
$$= \pi^{ss'}(h) \rho_s \rho_{s'}.$$

Set
$$C(s,s') = \rho_{ss'}^{-1} \rho_s \rho_{s'} \in U(L^2 A_0)$$
. Then
 $C(s,s')\pi(h) = \rho_{ss'}^{-1} \rho_s \rho_{s'}\pi(h) = \rho_{ss'}^{-1} \pi^{ss'}(h) \rho_s \rho_{s'} = \pi(h) \rho_{ss'}^{-1} \rho_s \rho_{s'} = \pi(h) C(s,s')$

Since π is irreducible, Schur's lemma implies that C(s, s') = c(s, s') Id_{\mathcal{H}} for some multiple $c(s, s') \in \mathbb{C}^{\times}$. Since C(s, s') is unitary, $c(s, s') \in U(1)$.

The cocycle identity is derived from associativity by writing down the equality $\rho_{s_0 \cdot (s_1 s_2)} = \rho_{(s_0 s_1) \cdot s_2}$ and applying (5.5).

There is the natural question of *linearization*: can one choose the operators $(\rho_s)_{s \in ASp(A)}$ in such a way that the corresponding Weil representation is linear? Specifically, is there a map $b: A \to U(1)$ such that $s \mapsto b(s) \rho_s$ is linear? Such a map exists if and only if b(ss') c(s, s') = b(s) b(s') for all $s, s' \in ASp(A)$. The cocycle c is a *coboundary* in this case.

In general there is a construction of a central extension of ASp(A) that has defined on it a linear representation induced by the projective Weil representation and the cocycle c. The set $ASp(A)_c$ of all pairs $(s, t) \in ASp(A) \times U(1)$ becomes a group (the Mackey obstruction group associated to c) when endowed with the operation

$$(s,t) \cdot (s',t') = (ss',tt'c(s,s')^{-1}), \quad s \in Asp(A), \ t \in U(1).$$

Clearly the group $ASp(A)_c$ is a central extension of ASp(A) and fits into the short exact sequence

(5.7)
$$1 \to U(1) \to \operatorname{ASp}(A)_c \to \operatorname{ASp}(A) \to 1$$

The map, induced by the projective Weil representation, defined by

$$\operatorname{ASp}(A)_c \to U(L^2 A_0), \ (s,t) \mapsto t \ \rho_s$$

is a linear representation.

One can ask for a smallest group U, a map $u : U \to U(1)$ and a group G_c and a map $g : G_c \to ASp(A)_c$ such that there is a commutative diagram with exact sequences

$$1 \longrightarrow U(1) \longrightarrow \operatorname{ASp}(A)_c \longrightarrow \operatorname{ASp}(A) \longrightarrow 1$$
$$\begin{array}{c} u \\ u \\ 1 \longrightarrow U \longrightarrow G_c \longrightarrow \operatorname{ASp}(A) \longrightarrow 1. \end{array}$$

By taking U = 1, we see the following

PROPOSITION 5.4. The Weil representation is linearizable if and only if the short exact sequence (5.7) splits.

THEOREM 5.5. If A had odd order, then the Weil representation is linearizable. If A had even order, then the projective Weil representation lifts to a linear representation of the double cover of ASp(A).

5.3. A computation of the Weil cocycle. Our goal is to describe a canonical choice for the Weil representation $s \mapsto \rho_s$ and to describe the corresponding cocycle explicitly. We shall use the induced representation model $\pi_{A_1} : \mathscr{H}(A) \to U(\mathcal{H}_{A_1})$ for each Lagrangian A_1 .

The group ASp(A) acts on functions on $\mathcal{H}(A)$ by

$$(s, f) \mapsto s \cdot f$$
, with $(s \cdot f)(h) = f(s^{-1}(h)), h \in \mathcal{H}(A)$.

Now if $f \in \mathcal{H}_{A_1}$, then $s \cdot f \in \mathcal{H}_{s(A_1)}$. More precisely, s acts as a unitary operator $r_s : \mathcal{H}_{A_1} \to \mathcal{H}_{s(A_1)}$. Furthermore, we observe a property of the family of unitary operators $(r_s)_{s \in ASp(A)}$ and intertwiners:

LEMMA 5.6. The diagram

$$\begin{array}{c} \mathcal{H}_{A_1} \xrightarrow{\rho_{A_2,A_1}} & \mathcal{H}_{A_2} \\ \downarrow^{r_s} & \downarrow^{r_s} \\ \mathcal{H}_{s(A_1)} \xrightarrow{\rho_{s(A_2),s(A_1)}} & \mathcal{H}_{s(A_2)}. \end{array}$$

is commutative.

It is then natural to consider, similarly as in the previous paragraph, a "new" representation $\pi_{A_1}^s : \mathscr{H}(A) \to \mathrm{U}(\mathcal{H}_{s(A_1)})$ defined by

$$\pi_{A_1}^s(h) = \pi_{s(A_1)}(s(h)), \ h \in \mathscr{H}(A).$$

This is again a model of the Schrödinger representation. We observe furthermore that

$$r_s(\pi_{A_1}(h)[f]) = \pi^s_{A_1}(h)[r_s(f)], \quad h \in \mathscr{H}(A), \ f \in \mathcal{H}_{A_1}$$

which shows that r_s is a unitary equivalence between the representations π_{A_1} and $\pi_{A_1}^s$:

$$r_s \pi_{A_1} r_s^{-1} = \pi_{A_1}^s.$$

We define for $s \in ASp(A)$,

$$\rho_s = \rho_{A_1,s(A_1)} \circ r_s : \mathcal{H}_{A_1} \to \mathcal{H}_{A_1}.$$

LEMMA 5.7. The assignment $s \mapsto \rho_s$ defines a linear map

$$\operatorname{ASp}(A) \to \operatorname{U}(\mathcal{H}_{A_1})$$

that satisfies the identity

$$\rho_s \pi_{A_1}(h) \rho_s^{-1} = \pi_{A_1}(s(h)), \quad h \in \mathscr{H}(A), \ s \in \mathrm{ASp}(A).$$

REMARK 6.9. The map ρ depends on a fixed Lagrangian A_1 in A.

PROPOSITION 5.8. The cocycle $(s_1, s_2) \mapsto c(s_1, s_2)$ of the Weil representation defined by

$$\rho_{s_1s_2} = c(s_1, s_2) \ \rho_{s_1}\rho_{s_2}$$

is given by

(5.8)
$$c(s_1, s_2) = C(A_1, s_1A_1, s_1s_2A_1)^{-1}.$$

PROOF. We compute

$$\begin{split} \rho_{s_1} \rho_{s_2} &= \rho_{A_1, s_1(A_1)} \circ r_{s_1} \circ \rho_{A_1, s_2(A_1)} \circ r_{s_2} \\ &= \rho_{A_1, s_1(A_1)} \circ \rho_{s_1(A_1), s_1 s_2(A_1)} \circ r_{s_1} \circ r_{s_2} \\ &= C(A_1, s_1 A_1, s_1 s_2 A_1) \ \rho_{A_1, s_1 s_2(A_1)} \circ r_{s_1 s_2} \\ &= C(A_1, s_1 A_1, s_1 s_2 A_1) \ \rho_{s_1 s_2}. \end{split}$$

We applied Lemma 5.6 in the second equality and used the fact that $r_{s_1s_2} = r_{s_1}r_{s_2}$ in the third equality. The desired result follows.

6. The Maslov index for finite groups

We define here the Maslov index for an ordered triple of Lagrangians in a finite symplectic group (A, ω) . We adapt the original definition by Kashiwara as described by Lion and Vergne in [**60**, I, §1.5] to our setting. For other generalizations see also the work of Kamgarpour and Thomas [**49**].

Let A be a symplectic abelian group. Let A_0, A_1, A_2 be three Lagrangians in A. Consider the abelian group $A_0 \oplus A_1 \oplus A_2$. The *Maslov index* is defined as the Witt class of the homogeneous quadratic form Q defined on $A_0 \oplus A_1 \oplus A_2$ by

$$Q(a_0 + a_1 + a_2) = \omega(a_0, a_1) + \omega(a_1, a_2) + \omega(a_2, a_0), \quad (a_0, a_1, a_2) \in A_0 \times A_1 \times A_2.$$

In the case when A is a lattice, the Witt group is the Witt group of integral quadratic forms is isomorphic to \mathbb{Z} , the isomorphism being given by the signature. In the case when A is a finite abelian group, the Witt group is the Witt group \mathfrak{WQ} of finite quadratic forms is isomorphic to $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We denote the Maslov index of (L_0, L_1, L_2) by $\mu(L_0, L_1, L_2) \in \mathfrak{WQ}$.

The following two properties are consequences of the definition. The first property states that the Maslov index is invariant under circular permutation:

(6.1)
$$\mu(L_0, L_1, L_2) = -\mu(L_1, L_0, L_2) = -\mu(L_0, L_2, L_1).$$

From the classification of finite symplectic pairings, we see that the symplectic group Sp(A) acts transitively on pairs of transverse Lagrangians. The second property states the Maslov index is invariant under the action of the symplectic group:

(6.2)
$$\forall s \in \text{Sp}(A), \ \mu(s \ L_0, s \ L_1, s \ L_2) = \mu(L_0, L_1, L_2).$$

A more subtle property of the Maslov index is the chain relation.

PROPOSITION 6.1. Let A_0, A_1, A_2, L be four Lagrangians. The Maslov index verifies the relation

(6.3)
$$\mu(A_0, A_1, A_2) = \mu(A_0, A_1, L) + \mu(A_1, A_2, L) + \mu(A_2, A_0, L).$$

The chain relation is a relation in the Witt group. In the classical setting (when A is a lattice or a vector space), the Maslov index is an integer and the chain relation has a geometric interpretation. See [**60**, I, §1.5.8] for a proof. (Autres références...)

7. The Weil representation of a finite quadratic form

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a homogeneous quadratic form on a finite abelian group G with associated linking pairing $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$. Let (V, ω) be a symplectic lattice equipped with a Seifert form $\beta : V \times V \to \mathbb{Z}$. Then the form

$$b_q \otimes \omega : G \otimes V \times G \otimes V \to \mathbb{Q}/\mathbb{Z}$$

is nondegenerate and alternate, hence is defines a symplectic form on $G \otimes V$.

LEMMA 7.1. The form $b_q \otimes \beta : G \otimes V \times G \otimes V \to \mathbb{Q}/\mathbb{Z}$ is a Seifert form for $b_q \otimes \omega$.

PROOF. Let $x, x' \in G$ and $y, y' \in V$. We compute

$$(b_q \otimes \beta)(x \otimes y, x' \otimes y') - (b_q \otimes \beta)(x' \otimes y', x \otimes y) = b_q(x, x')\beta(y, y') - b_q(x', x)\beta(y', y)$$
$$= b_q(x, x')\beta(y, y') - b_q(x, x')\beta(y', y)$$
$$= b_q(x, x')(\beta(y, y') - \beta(y', y))$$
$$= b_q(x, x')\omega(y, y')$$
$$= (b_q \otimes \omega)(x \otimes y, x' \otimes y')$$

where we used that b_q is symmetric in the second equality.

Denote by O(q) the group of automorphisms of G fixing q. Recall that for $s \in Sp(V), \lambda_s \in Link(V)$ denotes the linking pairing defined by $\lambda_s = s^*\beta - \beta$.

PROPOSITION 7.2. There is a well defined monomorphism

$$O(q) \otimes Sp(V) \to ASp(G \otimes V), \ \alpha \otimes s \mapsto (\alpha \otimes s, q \otimes \lambda_s).$$

In particular, there is a well defined monomorphism

$$\operatorname{Sp}(V) \to \operatorname{ASp}(G \otimes V), \ s \mapsto (\operatorname{id}_G \otimes s, q \otimes \lambda_s).$$

PROOF. The point is to verify that $(\alpha \otimes s, q \otimes \lambda_s) \in ASp(G \otimes V)$. Clearly $\alpha \otimes s \in Sp(G \otimes V)$. We compute

$$\begin{aligned} b_{q\otimes\lambda_s}(x\otimes y, x'\otimes y') &= (b_q\otimes\lambda_s)(x\otimes y, x'\otimes y') \\ &= b_q(x, x') \cdot \left(\beta(sy, sy') - \beta(y, y')\right) \\ &= b_q(x, x') \ \beta(sy, sy') - b_q(x, x') \ \beta(y, y') \\ &= b_q(\alpha x, \alpha x') \ \beta(sy, sy') - b_q(x, x') \ \beta(y, y') \\ &= (b_q\otimes\beta) \left((\alpha\otimes s)(x\otimes y), (\alpha\otimes s)(x'\otimes y')\right) - (b_q\otimes\beta)(x\otimes y, x'\otimes y') \end{aligned}$$

Suppose that $V = L_0 \oplus L_1$ is a Lagrangian decomposition of V. Then $G \otimes V = (G \otimes L_0) \oplus (G \otimes L_1)$ is a Lagrangian decomposition of $G \otimes V$. Composing the map of Prop. 7.2 with the Weil representation defined in the previous section gives a projective representation

$$O(q) \otimes Sp(V) \to U(L^2(G \otimes L_0)), (\alpha, s) \mapsto \rho_{\alpha \otimes s, q \otimes \lambda_s}.$$

This is the Weil representation of the quadratic form q.

The groups O(q) and Sp(V), viewed as subgroups of $ASp(G \otimes V)$, are mutual centralizers. They form a prominent instance of an *reductive dual pair*.

8. Particular cases and examples

Several particular cases of the Weil representation of the quadratic form q are of interest. The representation $O(q) \otimes Sp(V) \rightarrow U(L^2(G \otimes L_0))$ considered in the previous paragraph restricts to a representation

$$\operatorname{Sp}(V) \to \operatorname{U}(L^2(G \otimes L_0)), s \mapsto \rho_{\operatorname{id}_G \otimes s, q \otimes \lambda_s}.$$

For simplicity, we denote this representation by $s \mapsto \rho_s$. We describe this representation in terms of generators for Sp(V).

Let $g \ge 1$. Let $V = \mathbb{Z}^{2g}$ endowed with the canonical symplectic form. Then $\operatorname{Sp}(V)$ identifies with the symplectic group

$$\operatorname{Sp}_{2g}(\mathbb{Z}) = \{ M \in \operatorname{GL}_{2g}(\mathbb{Z}), \ M^T \Omega M = \Omega \}, \ \Omega = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} \}$$

also called Siegel's modular group. (Our convention follows the left action notation: automorphisms act on the left on groups.) The Seifert form is

$$\beta = \left[\begin{array}{cc} 0 & 0 \\ 1_g & 0 \end{array} \right].$$

The lattice V has a canonical Lagrangian decomposition

$$V = L_0 \oplus L_1,$$

where $L_0 = \{x \in \mathbb{Z}^{2g} \mid \forall g \leq j \leq 2g, x_j = 0\}$ and $L_1 = \{y \in \mathbb{Z}^{2g} \mid \forall 1 \leq j \leq g, y_j = 0\}$. For $x = (x_0, x_1), y = (y_0, y_1) \in L_0 \times L_1$,

$$\beta(x,y) = \langle x_1, y_0 \rangle$$

where $\langle -, - \rangle$ denotes here the canonical symmetric positive definite product on \mathbb{Z}^{g} . Examples of integral symplectic matrices are (8.1)

$$\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \begin{bmatrix} 1_g & 0 \\ B & 1_g \end{bmatrix} \text{ with } B = B^T \text{ integral}, \begin{bmatrix} A^T & 0 \\ 0 & A^{-1} \end{bmatrix} \text{ with } A \in \mathrm{GL}_g(\mathbb{Z}).$$

REMARK 6.10. The set of all matrices of the three types above generates $\operatorname{Sp}_{2g}(\mathbb{Z})$, see [89] and [7]. Furthermore, the set of each type generates a subgroup of $\operatorname{Sp}_{2g}(\mathbb{Z})$ which has a group theoretic section into $\operatorname{ASp}_{2g}(\mathbb{Z})$.

We explicit below the map

$$\operatorname{Sp}_{2g}(\mathbb{Z}) \to \operatorname{U}(L^2(G \otimes L_0)) = \operatorname{U}(L^2(G^g)), \ s \mapsto \rho_s.$$

PROPOSITION 8.1. The Weil representation $\rho : \operatorname{Sp}_{2g}(\mathbb{Z}) \to U(L^2(G^g))$ is determined by the following formulas:

(8.2)
$$\rho\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} f(x) = |G \otimes L_0|^{-1/2} \sum_{y \in G \otimes L_0} \chi((b_q \otimes \beta)(y, x)) f(y).$$

(8.3)
$$\rho\left(\begin{array}{cc}1_g & 0\\B & 1_g\end{array}\right)f(x) = \chi(-(q\otimes B)(x))f(x)$$

(8.4)
$$\rho \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix} f(x) = f((1_G \otimes A^T)^{-1}x).$$

REMARK 6.11. The proposition has a generalization (with essentially the same proof below) for an arbitrary commutative locally compact group G endowed with some Haar measure (a Borel measure invariant under translations which is unique up to a scalar multiple). In this case, the group $G \otimes L_0$ inherits a Haar measure $d\mu$ and the first relation reads

(8.5)
$$\rho\left(\begin{array}{cc} 0 & -1_g \\ 1_g & 0 \end{array}\right)f(x) = \int_{G\otimes L_0} \chi((b_q\otimes\beta)(y,x)) f(y) \ d\mu(y).$$

PROOF. The three operators defined in the proposition are unitary (for the hermitian product defined by (3.1)). The rest of the proof consists in verifying the identity (5.4). We have to verify that

$$\rho_s(\pi^s(h)f) = \pi(h)(\rho_s f), \quad \forall h \in H(G \otimes V), \ \forall f \in L^2(G \otimes L_0)$$

with s being one of the three symplectomorphisms above. Set $G_0 = G \otimes L_0$ and $G_1 = G \otimes L_1$. Write (s_0, s_1) the image of $a = (a_0, a_1) \in G_0 \oplus G_1$ by $\mathrm{id}_G \otimes s \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Let $h = (a_0, a_1, t) \in H(G \otimes V)$. Recall that s acts on h as

$$s \cdot h = (s_0, s_1, t + (q \otimes \lambda_s)(t)).$$

We have

$$\pi^{s}(h)f(x) = \pi(s_{0}, s_{1}, t + q \otimes \lambda_{s}(a_{0}, a_{1}))f(x)$$

= $\chi(t + q \otimes \lambda_{s}(a_{0}, a_{1})) W_{s_{0}, s_{1}}f(x)$
= $\chi(t + q \otimes \lambda_{s}(a_{0}, a_{1})) \chi_{s_{1}}(x) f(x + s_{0})$
= $\chi(t + q \otimes \lambda_{s}(a_{0}, a_{1})) \chi(b_{q} \otimes \beta(s_{1}, x)) f(x + s_{0}).$

Let g(x) be

$$\pi(a_0, a_1, t) f(x) = \chi(t) \ \chi((b_q \otimes \beta)(a_1, x)) \ f(x + a_0)$$

Consider now each case separately. It will be convenient to denote in this paragraph by $\langle -, - \rangle$ the symmetric bilinear product on \mathbb{Z}^g associated to the $g \times g$ identity matrix.

In the first case: $s_0 = a_1, s_1 = -a_0$. Then

$$\lambda_s(x,y) = -(\langle x_0, y_1 \rangle + \langle x_1, y_0 \rangle).$$

This symmetric bilinear pairing admits a quadratic enhancement defined on V by

$$x \mapsto -\langle x_0, x_1 \rangle = -\beta(x, x).$$

It follows that $q \otimes \lambda_s(a, b) = -(b_q \otimes \beta)(a, b)$ for all $a, b \in G \otimes V$. We compute

$$\rho_s g(x) = |G \otimes L_0|^{-1/2} \sum_{y \in G \otimes L_0} \chi(b_q \otimes \beta(y, x)) g(y)$$

= $|G \otimes L_0|^{-1/2} \chi(t) \sum_{y \in G \otimes L_0} \chi(b_q \otimes \beta(y, x + a_1)) f(y + a_0)$
= $|G \otimes L_0|^{-1/2} \chi(t) \sum_{y' \in G \otimes L_0} \chi(b_q \otimes \beta(y' - a_0, x + a_1)) f(y').$

On the other hand, setting $h = \rho_s f$, we have

$$\pi^{s}(a,t)h(x) = \pi(a_{1},-a_{0},t-q\otimes\lambda_{s}(a_{0},a_{1}))h(x)$$

$$= \chi(t-b_{q}\otimes\beta(a_{0},a_{1}))\chi(b_{q}\otimes\beta(-a_{0},x))h(x+a_{1})$$

$$= |G\otimes L_{0}|^{-1/2}\chi(t)\chi(b_{q}\otimes\beta(-a_{0},x+a_{1}))\cdot$$

$$\cdot\sum_{y\in G\otimes L_{0}}\chi(b_{q}\otimes\beta(y,x+a_{1}))f(y)$$

$$= \rho_{s}g(x).$$

In the second case: $s_0 = a_0$, $s_1 = Ba_0 + a_1$. Then

$$\lambda_s(x,y) = x_0^T B y_0, \ x, y \in V.$$

It follows that $(q \otimes \lambda_s)(a_0, a_1) = (q \otimes B)(a_0)$. We compute

$$\rho_s g (x) = \chi(-(q \otimes B)(x)) g(x)$$

= $\chi(t) \chi(b_q \otimes \beta(a_1, x)) \chi(-(q \otimes B)(x)) f(x + a_0).$

Let $h = \rho_s(f)$. We have

$$\pi^{s}(a,t)h(x) = \pi(a_{0}, Ba_{0} + a_{1}, t + (q \otimes \lambda_{s})(a_{0}, a_{1}))h(x)$$

= $\chi(t + (q \otimes B)(a_{0})) \chi(b_{q} \otimes \beta(Ba_{0} + a_{1}, x)) h(x + a_{0})$
= $\chi(t + (q \otimes B)(a_{0})) \chi(b_{q} \otimes \beta(Ba_{0} + a_{1}, x)) \chi(-q \otimes B(x + a_{0})) f(x + a_{0})$
= $\chi(t) \chi(b_{q} \otimes \beta(a_{1}, x)) \chi(-(q \otimes B)(x)) f(x + a_{0})$
= $\rho_{s}g(x).$

Here we used in the penultimate equality the fact that $(b_q \otimes \beta)(Ba_0, x) = (b_q \otimes \beta)(0 \oplus Ba_0, x \oplus 0) = (b_q \otimes B)(a_0, x).$

In the third case: $s_0 = (1_G \otimes A^T)a_0$, $s_1 = (1_G \otimes A^{-1})a_1$. Then $\lambda_s = 0$. It follows that $q \otimes \lambda_s = 0$. We compute

$$\rho_s g(x) = g((1_G \otimes A^{-T})x)$$

= $\chi(t) \ \chi(b_q \otimes \beta(a_1, (1_G \otimes A^{-T})x)) \ f((1_G \otimes A^{-T})x + a_0)$

Let $h = \rho_s f$. We have

$$\pi^{s}(a,t)h(x) = \pi((1_{G} \otimes A^{T})a_{0}, (1_{G} \otimes A^{-1})a_{1},t)h(x)$$

= $\chi(t) \chi(b_{q} \otimes \beta((1_{G} \otimes A^{-1})a_{1},x))h(x + (1_{G} \otimes A^{T})a_{0}).$

Since
$$h(u) = f((1_G \otimes A^{-T})u)$$
, we deduce that
 $\pi^s(a,t)h(x) = \chi(t) \chi(b_q \otimes \beta((1_G \otimes A^{-1})a_1,x)) f((1_G \otimes A^{-T})(x + (1_G \otimes A^T)a_0))$
 $= \chi(t) \chi(b_q \otimes \beta((1_G \otimes A^{-1})a_1,x)) f((1_G \otimes A^{-T})x + a_0)$
 $= \chi(t) \chi(b_q \otimes \beta(a_1, (1_G \otimes A^{-T})x)) f((1_G \otimes A^{-T})x + a_0)$
 $= \rho_s g(x).$

We begin with the Weil representation associated to the group $SL_2(\mathbb{R})$ (corresponding to the case when the genus of the surface is 1) and then we describe the general case.

Let $\mathrm{SL}_2(\mathbb{R})$, resp. $\mathrm{SL}_2(\mathbb{Z})$, be the multiplicative group of 2 by 2 matrices with real coefficients (resp. with integer coefficients) and determinant equal to one. Let $H = \{\tau = u + iv \in \mathbb{C} | v > 0\}$ be the upper half plane. The formula

$$(M,\tau) \mapsto M \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \tau \in H$$

defines a transitive (resp. discontinuous) action of $SL_2(\mathbb{R})$ (resp. $SL_2(\mathbb{Z})$) on H. It is well known (see for instance [88, Chap. VII; Théorème 2]) that

$$S = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \quad \text{and} \quad T = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$$

generate $SL_2(\mathbb{Z})$ with relations

$$S^2 = (ST)^3, \ (ST)^6 = 1.$$

The group $SL_2(\mathbb{R})$ admits a double cover, called the *metaplectic group* $Mp_2(\mathbb{R})$. This group is realized as the set of pairs

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}), \ \tau \mapsto f_M(\tau)$$

where $f_M(\tau)$ is a holomorphic solution of the equation $c\tau + d = f_M(\tau)^2$. In other words, $\tau \mapsto f_M(\tau)$ is a function defined in H as a holomorphic square root of the holomorphic function $\tau \mapsto c\tau + d$. Elements in Mp₂(\mathbb{R}) obey the associative multiplication law

(8.6)
$$(M, f_M(\tau)) \cdot (N, f_M(\tau)) = (MN, f_M(N \cdot \tau) f_N(\tau))$$

which turns $Mp_2(\mathbb{Z})$ into a group with unit

$$1_{Mp_2(\mathbb{Z})} = \left(\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], 1 \right).$$

Let $Mp_2(\mathbb{Z})$ be the inverse image of $SL_2(\mathbb{Z})$ under the covering map $Mp_2(\mathbb{R}) \to SL_2(\mathbb{R})$. The following lemma is a consequence of well known facts about $SL_2(\mathbb{Z})$.

LEMMA 8.2. The group $Mp_2(\mathbb{Z})$ is generated by the two elements

$$\hat{S} = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sqrt{\tau} \right) \quad and \quad \hat{T} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1 \right)$$

with relations

$$\hat{Z} = \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \sqrt{-1} \right) = \hat{S}^2 = (\hat{S}\hat{T})^3, \ \hat{Z}^2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \right), \ \hat{Z}^4 = 1_{\mathrm{Mp}_2(\mathbb{Z})}$$

The order 4 element \hat{Z} generates the center of $Mp_2(\mathbb{Z})$.

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a finite abelian group Gsuch that $\gamma(G,q) \neq 0$. There is a *unitary* representation $\rho_q : \operatorname{Mp}_2(\mathbb{Z}) \to \operatorname{Aut}(\mathbb{C}[G])$, called the *Weil representation*, associated to (G,q). Let $(\mathfrak{e}_g)_{g\in G}$ be the standard basis of the group ring $\mathbb{C}[G]$ so that $\mathfrak{e}_g \cdot \mathfrak{e}_h = \mathfrak{e}_{g+h}$ (where dot denotes the (convolution) product of $\mathbb{C}[G]$). Then $\rho = \rho_q$ is defined by the action on the generators $\hat{S}, \hat{T} \in \mathrm{Mp}_2(\mathbb{Z})$ by

(8.7)
$$\rho(\hat{S})\mathfrak{e}_g = |G|^{-\frac{1}{2}}\gamma(G,q) \cdot \sum_{h\in G} \exp(-2\pi i \ b_q(g,h)) \ \mathfrak{e}_h$$

(8.8)
$$\rho(\hat{T})\mathfrak{e}_g = \exp(2\pi i q(q))\mathfrak{e}_g$$

One computes that

$$\rho(\hat{Z})\mathfrak{e}_g = \gamma(G,q)^2 \mathfrak{e}_{-g}.$$

For a proof using relations of $SL_2(\mathbb{Z})$ that these formulas indeed define a full projective action of $SL_2(\mathbb{Z})$, see [72].

Let $\hat{A} = (A, f_A)$ be a preimage of $A \in \mathrm{SL}_2(\mathbb{Z})$. Then according to (8.6), the other preimage of A is $\hat{A}\hat{Z}^2 = (A, -f_A)$. Since $\rho(\hat{Z}^2) = \rho(\hat{Z})^2 = \gamma(G,q)^4$ id_{C[G]}, we see that ρ induces a linear representation of $\mathrm{SL}_2(\mathbb{Z})$ if $\gamma(G,q)^4 = 1$ and only a projective representation of $\mathrm{SL}_2(\mathbb{Z})$ otherwise. For a homogeneous quadratic function q, the Gauss sum $\gamma(G,q)$ is an eighth root of unity. It follows that the corresponding cocycle $c(A,B) = c_q(A,B)$ lies in $\{-1, +1\}$ for q is homogeneous. It can be computed as follows. Choose first a canonical preimage \hat{A} in $\mathrm{Mp}_2(\mathbb{Z})$ of each $A \in \mathrm{SL}_2(\mathbb{Z})$ by using a fixed branch cut for the argument of f_A . Then set $\rho(A) = \rho(\hat{A})$. We have by definition

(8.9)
$$\rho(AB) = c(A, B) \ \rho(A) \ \rho(B), \ A, B \in \mathrm{SL}_2(\mathbb{Z}).$$

It follows from this definition and (8.6) that

$$c(A,B) = \rho(AB, f_{AB}(\tau)) \ \rho(AB, f_A(B\tau)f_B(\tau))^{-1} = f_{AB}(\tau) \ f_A(B\tau)^{-1} \ f_B(\tau)^{-1}.$$

The second equality follows from the fact that the second term differs only by the choice of the square roots and a different choice introduces only a sign factor. For the same reason, we see that c(A, B) is independent of the actual value of τ . The explicit computation of ρ for an arbitrary element $\hat{A} \in \text{Mp}_2(\mathbb{Z})$ is carried out in [84] and [91].

For $g \ge 1$, let

$$\operatorname{Sp}_{2\mathfrak{g}}(\mathbb{R}) = \{ M \in \operatorname{GL}_{2g}(\mathbb{R}) \mid M^T J M = J \}, J = \begin{bmatrix} 0 & 1_g \\ -1_g & 0 \end{bmatrix},$$

the symplectic group over \mathbb{R} . This group contains a most important discrete subgroup $\operatorname{Sp}_{2g}(\mathbb{Z})$, consisting of symplectic matrices with integer coefficients, called Siegel's modular group. Exemples of integral symplectic matrices are (8.10)

$$\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix} \text{ with } B = B^T \text{ integral}, \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix} \text{ with } A \in \mathrm{GL}_g(\mathbb{Z})$$

The set of matrices above generates $\operatorname{Sp}_{2g}(\mathbb{Z})$, see [89]. Let H_g denote the set of $g \times g$ symmetric matrices with complex coefficients the imaginary part of which is definite positive (Siegel's half space). The formula

$$(M,Z) \mapsto M\langle Z \rangle = (AZ+B)(CZ+D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad Z \in H_g$$

defines a transitive (resp. discontinuous) action of $\operatorname{Sp}_{2g}(\mathbb{R})$ (resp. $\operatorname{Sp}_{2g}(\mathbb{Z})$) on H_g . Since $\pi_1(\operatorname{Sp}_{2g}(\mathbb{R})) = \mathbb{Z}$, the symplectic group admits a double cover $\operatorname{Mp}_{2g}(\mathbb{R})$, called the metaplectic group. This group can be realized as pairs

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \operatorname{Sp}_{2\mathfrak{g}}(\mathbb{R}), \quad Z \mapsto \sqrt{\det(CZ + D)}.$$

Here the map $Z \mapsto \sqrt{\det(CZ + D)}$ is a holomorphic square root of the holomorphic map $Z \mapsto \det(CZ + D)$ (See for instance [**30**, Chap. I, Remarks 2.3 and 3.1]). The group multiplication is given by the same formula as above, except that the action is replaced by the action of $\operatorname{Sp}_{2g}(\mathbb{R})$ on H_g . The metaplectic group over \mathbb{Z} is defined as the inverse image of $\operatorname{Sp}_{2g}(\mathbb{Z})$ under the covering map $\operatorname{Mp}_{2g}(\mathbb{R}) \to \operatorname{Sp}_{2g}(\mathbb{Z})$. The elements

$$\hat{S} = \left(\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}, \sqrt{\det(Z)} \right), \ \hat{T} = \left(\begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}, 1 \right) \text{ and } \hat{U} = \left(\begin{bmatrix} A & 0 \\ 0 & (A^T) - 1 \end{bmatrix}, \sqrt{\det(A^{-1})} \right)$$

lying above (8.10) generate $Mp_{2g}(\mathbb{Z})$.

Remarkably, the Weil representation extends to the metaplectic group $\operatorname{Mp}_{2g}(\mathbb{Z})$ for any $\mathfrak{g} \geq 1$. This is part of the content of the Theorem below. This extension is based on tensor product as follows. Let $(\mathfrak{e}_x)_{x\in G\otimes\mathbb{Z}^g}$ be a basis for $\mathbb{C}[G\otimes\mathbb{Z}^g]$. The symbol 1_g used to denote the $g \times g$ identity matrix shall also be used to denote the canonical positive definite bilinear symmetric pairing

$$(a,b) \mapsto \sum_{1 \leq j \leq g} a_j b_j, \quad a,b \in \mathbb{Z}^g.$$

Define a map $\rho = \rho_q : \operatorname{Mp}_{2q}(\mathbb{Z}) \to \operatorname{Aut}(\mathbb{C}[G \otimes \mathbb{Z}^g])$ by

$$(8.11) \quad \rho(\hat{S})\mathfrak{e}_x = |G|^{-\frac{g}{2}}\gamma(G,q)\sum_{y\in G\otimes\mathbb{Z}^g}\exp(-2\pi i(b_q\otimes 1_G)(x,y))\mathfrak{e}_y.$$

$$(8.12) \quad \rho(\hat{T})\mathfrak{e}_x = \exp(2\pi i(x\otimes R)(x))\mathfrak{e}_y.$$

 $(8.12) \quad \rho(T)\mathfrak{e}_x = \exp(2\pi i (q \otimes B)(x)) \mathfrak{e}_x$

(8.13)
$$\rho(U)\mathfrak{e}_x = \mathfrak{e}_{(1_G \otimes A^T)^{-1}x}$$

These formulas specialize to the case g = 1 which is the case considered above. It is not immediately clear that these formulas fit to yield a representation of the metaplectic group. This, however, will be seen below as a consequence of the previous section.

Part 2

Abelian topological quantum field theory
CHAPTER 7

Linking pairings and Lagrangians

1. The intersection pairing

Let $n \ge 1$. Consider a connected compact oriented 2n-manifold X, possibly with boundary. Consider the homomorphism defined by the composition

(1.1)
$$H_n(X, \partial X) \xrightarrow{\simeq} H^n(X) \longrightarrow \operatorname{Hom}(H_n(X), \mathbb{Z}),$$

where the first map on the left is the isomorphism given by Poincaré-Lefschetz duality. This map determines a bilinear pairing $a_X : H_n(X, \partial X) \times H_n(X) \to \mathbb{Z}$ whose adjoint map is the homomorphism (1.1).

Consider the projection homomorphism map $j_*: H_n(X) \to H_n(X, \partial X)$.

DEFINITION 7.1. The intersection pairing associated to X is the bilinear pairing $\iota_X : H_n(X) \times H_n(X) \to \mathbb{Z}$ defined by

$$\iota_X(x,y) = a_X(j_*(x),y), \quad x,y \in H_n(X).$$

It follows from Poincaré-Leftschetz duality that the intersection pairing on X is $(-1)^n$ -symmetric. In particular, if X is a 4-manifold (n = 4), ι_X is symmetric. If X is a surface (2-manifold), ι_X is antisymmetric. A geometrical interpretation of the intersection linking pairing is as follows. Let $x, y \in H_n(X)$. Represent x, y by disjoint and transverse smooth *n*-cycles u, v respectively. Then

(1.2)
$$\iota_X(x,y) = \sum_{p \in u \cap v} \varepsilon(p) \in \mathbb{Z}$$

where $\varepsilon(p) = +1$ or -1 according to whether the orientation of the sum $T_p u \oplus T_p v$ of the tangent spaces at p matches the orientation of the tangent space $T_p X$ at p or not. The geometric definition makes apparent that ι_X is $(-1)^n$ -symmetric.

REMARK 7.1. If ∂X is nonempty then the intersection pairing ι_X may be degenerate. See ... for an example. If ∂X is empty then the intersection pairing is nondegenerate.

2. The linking pairing of a (2n-1)-manifold

Let $n \ge 2$. Let M be any connected compact oriented (2n - 1)-manifold with boundary ∂M . Poincaré-Lefschetz duality and the (torsion) universal coefficient theorem lead to the following sequence of isomorphisms

Tors
$$H_{n-1}(M) \simeq \text{Tors } H^n(M, \partial M) \simeq \text{Tors } \text{Ext}(H_{n-1}(M, \partial M), \mathbb{Z})$$

 $\simeq \text{Hom}(\text{Tors } H_{n-1}(M, \partial M), \mathbb{Q}/\mathbb{Z}).$

There is therefore a nonsingular bilinear pairing

 a_M : Tors $H_{n-1}(M) \times$ Tors $H_{n-1}(M, \partial M) \to \mathbb{Q}/\mathbb{Z}$.

A geometrical definition of this pairing is as follows. First define linking numbers for cycles. Let x be an integral (n-1)-cycle in M and let ybe a relative (n-1)-cycle in $(M, \partial M)$ representing homology classes in Tors $H_{n-1}(M)$ and Tors $H_{n-1}(M, \partial M)$ respectively. We may assume that xand y are in general position. There exists $k \in \mathbb{Z}$ and a n-chain C in M such that $k x = \partial C$. We may further assume that C and y intersect transversally, i.e., in a finite number of points away from the boundary.

DEFINITION 7.2. The rational linking number of x and y to be

$$\operatorname{lk}_M(x,y) = \frac{C \cdot y}{k} \in \mathbb{Q}$$

where we denote by a small dot the algebraic intersection number.

EXAMPLE 7.1. Let $M = S^1 \times D^2$ be the solid torus. Let \star denote an arbitrary point on S^1 . Let $x = \star \times \partial D^2$ be a meridian and $y = S^1 \times 0$ be a longitude. Clearly x bounds a disc $C = \star \times D^2$ and y is a boundary modulo $\partial M = S^1 \times \partial D^2$ (the torus surface). Since C and y intersect in exactly one point, it follows that for a suitable choice of orientations, $lk_M(x, y) = +1$.

EXAMPLE 7.2. Consider the link L in the 3-sphere S^3 formed by two transversal great circles J and K (the *Hopf link*). It can be obtained as the closure of the braid σ_1^2 in Artin's notation. The linking number of the components is ± 1 according to the choice of orientations.

More generally, if x and y represent homologically trivial elements in $H_{n-1}(M)$, then $lk_M(x, y)$ is an integer.

EXERCISE 7.1. The following exercise leads to a useful property of the linking number $lk_M(x, y)$ in the case when M is a rational homology *n*-sphere.

1. There is a Poincaré-Alexander-Lefschetz duality isomorphism $H_{n-1}(x; \mathbb{Q}) \simeq H_{n-1}(M-x; \mathbb{Q})$. [See e.g. Bredon [9, VI, 8, Cor. 8.4] and use the long exact sequence associated to (M, M - x).]

2. The linking number $lk_M(x, y)$ only depends on the rational (n - 1)-homology class of y in the complement of x in M.

PROPOSITION 2.1. The following identity holds:

 $a_M([x], [y]) = \operatorname{lk}_M(x, y) \mod 1.$

DEFINITION 7.3. The linking pairing

 λ_M : Tors $H_{n-1}(M) \times$ Tors $H_{n-1}(M) \to \mathbb{Q}/\mathbb{Z}$

associated to M is defined by

$$\lambda_M = a_M \circ (\mathrm{id} \times j_*),$$

where j_* : Tors $H_{n-1}(M) \to \text{Tors } H_{n-1}(M, \partial M)$ denotes the natural projection homomorphism.

PROPOSITION 2.2. If j_* : Tors $H_{n-1}(M) \to \text{Tors } H_{n-1}(M, \partial M)$ is an isomorphism then the linking pairing λ_M is nonsingular.

COROLLARY 2.3. If M is closed then the linking pairing λ_M is nonsingular. PROPOSITION 2.4. The linking pairing λ_M of M^{2n-1} is $(-1)^n$ -symmetric.

PROOF. This fact already holds at the level of the linking number. Let C, C' be two chains such that $\partial C = k x$ and $\partial C' = l y$. Then

 $0 = \partial (C \cdot C') = \partial C \cdot C' + (-1)^{n-1}C \cdot \partial C' = k x \cdot C' + (-1)^{n-1}C \cdot l y.$

Because of dimensions, $x \cdot C' = C' \cdot x$. Dividing the equality above by kl yields $lk_M(y,x) + (-1)^{n-1} lk_M(x,y) = 0$. Since $lk_M(x,y) = \frac{x \cdot C'}{l} = \frac{C \cdot y}{k}$, the result follows.

It follows that for a closed oriented 3-manifold M, $(H_1(M), \lambda_M)$ is a linking group.

3. The relation between linking and intersection pairings

The intersection pairing on a connected oriented 2n-manifold X^{2n} and the linking pairing of its boundary $M^{2n-1} = \partial X$ are related via the discriminant construction (Chapter ...). We describe here this relation. In general, the relation involves a certain subquotient of Tors $H_{n-1}(M)$. We begin with a few observations.

Consider the long exact sequence for the pair (X, M)(3.1)

$$\rightarrow H_n(X) \xrightarrow{j} H_n(X, M) \xrightarrow{\partial} H_{n-1}(M) \xrightarrow{i} H_{n-1}(X) \xrightarrow{j} H_{n-1}(X, M) \rightarrow$$

The linking pairing λ_M can be computed on the image of $\partial : H_n(X, M) \to H_{n-1}(M)$. Let $x, y \in H_n(X, M)$ such that $\partial x, \partial y \in \text{Tors } H_{n-1}(M)$. Let $r, s \ge 1$ such that $r \partial x = s \partial y = 0$. Exactness of (3.1) at $H_n(X, M)$ provides $\tilde{x}, \tilde{y} \in H_n(X)$ such that

$$j(\tilde{x}) = r \ x, \ \ j(\tilde{y}) = s \ y.$$

The following lemma shows that the intersection pairing of X determines λ_M on $\operatorname{Im}(\partial) \cap \operatorname{Tors} H_{n-1}(M)$.

Lemma 3.1.

(3.2)
$$\lambda_M(\partial x, \partial y) = -\frac{1}{r}(\tilde{x} \cdot y) = -\frac{1}{s}(x \cdot \tilde{y}) = -\frac{1}{rs}\iota_X(\tilde{x}, \tilde{y}),$$

where dot denotes intersection product between $H_n(X)$ and $H_n(X, M)$.

The equalities in (3.2) are understood to hold in \mathbb{Q}/\mathbb{Z} .

PROOF. Let U, V be relative cycles in (X, M) representing x and y respectively. By assumption, $r \partial x = 0$, then $r \partial V$ bounds an n-chain C in X, thus r U - C is an integral n-cycle in X representing \tilde{x} . Assuming transversality (as always), the algebraic intersection

$$(r \ U - C) \cdot V$$

is a well-defined integer. Then (cf. Remark ??)

$$\frac{1}{rs}\iota_X(\tilde{x},\tilde{y}) = \frac{1}{r}(\tilde{x}\cdot y) = \frac{1}{r} (r \ U - C) \cdot V.$$

Since

$$\lambda_M(\partial x, \partial y) = \frac{C \cdot V}{r} \mod 1,$$

the result follows.

It is sometimes useful to provide a reformulation in terms of rational extensions. Denote by a subscript \mathbb{Q} rational extension. Since $\partial x, \partial y$ are torsion elements, there exists $x', y' \in H_n(X; \mathbb{Q}) = H_n(X) \otimes \mathbb{Q}$ such that $j_{\mathbb{Q}}(x') = x$ and $j_{\mathbb{Q}}(y') = y$ in $H_n(X, M; \mathbb{Q}) = H_n(X, M) \otimes \mathbb{Q}$. Then

(3.3)
$$\lambda_M(\partial x, \partial y) = -(\iota_X)_{\mathbb{Q}}(x', y') \mod 1.$$

COROLLARY 3.2. The intersection pairing of X determines the linking pairing of M on $\operatorname{Im}(\partial) \cap \operatorname{Tors} H_{n-1}(\partial X)$.

The group

$$\bar{F}_n(X) = \frac{H_n(X)}{\operatorname{Ker}(j_X) + \operatorname{Tors} H_n(X)},$$

endowed with the nondegenerate symmetric bilinear pairing denoted $\bar{\iota}_X$ induced by ι_X , is a lattice. Therefore, we can apply the discriminant construction to it and obtain a discriminant linking pairing $(G_{\bar{\iota}_X}, L_{\bar{\iota}_X})$ which we denote (G_M, L_M) . Since $\bar{\iota}_X$ is non-degenerate, G_M is finite.

The sequence (3.1), when restricted to torsion subgroups of the middle groups, induces a complex

(3.4)

$$\rightarrow$$
 Tors $H_n(X, M) \xrightarrow{\partial_t}$ Tors $H_{n-1}(M) \xrightarrow{i_t}$ Tors $H_{n-1}(X) \longrightarrow$

Poincaré duality implies that the diagram

$$H_n(X, M) \xrightarrow{(-1)^n \partial} H_{n-1}(M)$$

$$\simeq \Big| PD \qquad PD \Big| \simeq$$

$$H^n(X) \xrightarrow{i^*} H^n(M)$$

is commutative (See for example [9, Chap. VI, §9.2]). The sign $(-1)^n$ is induced by the usual convention of orientation ("outward normal first") for (X, M). It follows from definitions and naturality of universal coefficients that

(3.5)
$$\lambda_M(\partial_t(x), y) = (-1)^n \ \check{\iota}_X(x, i_t(y)).$$

In particular, since both λ_M and $\check{\iota}_X$ are non-singular,

(3.6)
$$\operatorname{Im}(\partial_t)^{\perp} = \operatorname{Ker}(i_t).$$

Hence $\operatorname{Im}(\partial_t) \subseteq \operatorname{Ker}(i_t) = \operatorname{Im}(\partial_t)^{\perp}$, i.e., $\operatorname{Im}(\partial_t)$ is λ_M -isotropic.

PROPOSITION 3.3. The linking pairing on $\operatorname{Ker}(i_t)/\operatorname{Im}(\partial_t)$ induced by λ_M is canonically isomorphic to $(G_M, -L_M)$.

PROOF. Denote by H(X, M) the (acyclic) complex (3.1) and by Tors H(X, M) the subcomplex (3.4). Let F(X, M) be the quotient complex, completing the short exact sequence of complexes

$$0 \to \text{Tors } H(X, M) \to H(X, M) \to F(X, M) \to 0.$$

The induced homology long exact sequence reduces, because of acyclicity of H(X, M), to a natural homology isomorphism

$$H_*(F(X,M)) \simeq H_{*-1}(\text{Tors } H(X,M)),$$

induced by the connecting homomorphism. In particular, denoting by j_X : $F_n(X) \to F_n(X, M)$ the natural map induced by the adjoint of the intersection pairing. we have:

$$\frac{\operatorname{Ker}\,\partial}{\operatorname{Im}\,j_X}\simeq\frac{\operatorname{Ker}\,i_t}{\operatorname{Im}\,\partial_t}.$$

Then

$$G_M = \text{Coker } j_X \simeq \frac{\text{Ker } i_t}{\text{Im } \partial_t}.$$

That λ_M induces a linking pairing on $\operatorname{Ker}(i_t)/\operatorname{Im}(\partial_t)$ – which we continue to denote λ_M – is a consequence of the λ -isotropy of $\operatorname{Im}(\partial_t)$. Let $x, y \in$ $H_n(X, M)$ such that $\partial x, \partial y \in \operatorname{Tors} H_{n-1}(X)$. Pick $x', y' \in H_n(X; \mathbb{Q})$ such that $j_{\mathbb{Q}}(x') = x$ and $j_{\mathbb{Q}}(y') = y$. Letting $[x], [y] \in \operatorname{Coker} j_X$ be the images respectively of x, y, we have

$$L_M([x], [y]) = (\iota_X)_{\mathbb{Q}}(x, y) \mod 1.$$

Applying (3.3), we deduce that $-L_M$ and λ_M are canonically isomorphic.

COROLLARY 3.4. In the Witt groups \mathfrak{W} and \mathfrak{W}^s ,

[Tors
$$H_{n-1}(M), \lambda_M$$
] = -[G_M, L_M].

REMARK 7.2. There are various particular cases, however, when the equality holds at the level of linking pairings themselves and not only in the Witt group. See...

PROOF. Since $\operatorname{Ker}(i_t) = \operatorname{Im}(\partial_t)^{\perp}$, this is a consequence of Propositions 3.3 and 2.5.

It is convenient to consider also non-canonical splittings of the linking pairing. Let $F_n(X, M)$ be a free abelian subgroup of $H_n(X, M)$ such that

$$H_n(X, M) = F_n(X, M) \oplus \text{Tors } H_n(X, M).$$

Set $F = \partial F_n(X, M)$. The arguments above show that the linking pairing splits as

(Tors
$$H_{n-1}(M), \lambda_M$$
) = $(F, \lambda_M|_{F \times F}) \oplus (F^{\perp}, \lambda)$.

By Corollary ??, $\operatorname{Im}(\partial_t) \subseteq F^{\perp}$.

The following result is a slight improvement of Corollary 3.4, due to Gilmer [].

PROPOSITION 3.5. The linking pairing (F^{\perp}, λ) is weakly metabolic.

PROOF. First, assume

- the natural map Tors $H_n(X) \to \text{Tors } H_n(X, M)$ is injective.
- the natural map Tors $H_{n-1}(X) \to \text{Tors } H_{n-1}(X, M)$ is surjective.
- $H_{n-1}(M)$ is torsion.

There is an exact sequence

 $0 \succ \mathrm{Tors}\ H_n(X) \succ \mathrm{Tors}\ H_n(X,M) \succ F^{\perp} \succ \mathrm{Tors}\ H_{n-1}(X) \succ \mathrm{Tors}\ H_{n-1}(X,M) \succ 0$

We shall show that $D = \text{Im}(\partial_t)$ is a metabolizer for (F^{\perp}, λ) . In the exact sequence above, Poincaré duality and the universal coefficient theorem provide isomorphisms Tors $H_n(X) \simeq \text{Tors } H_{n-1}(X, M)$ and Tors $H_n(X, M) \simeq \text{Tors } H_{n-1}(X)$. Thus, by exactness,

$$|F^{\perp}| = \frac{|\text{Tors } H_{n-1}(X)|^2}{|\text{Tors } H_n(X)|^2} \text{ and } |D| = \frac{|\text{Tors } H_{n-1}(X)|}{|\text{Tors } H_n(X)|}$$

Hence $|F^{\perp}| = |D|^2$, which implies that $D = D^{\perp}$ (orthogonality in F^{\perp}).

In the general case, one can modify by surgery (X, M) without changing the intersection pairing of X and the linking pairing of M, in such a way that the hypotheses above are satisfied.

We now consider the particular case when $H_*(M; \mathbb{Q}) = H_*(S^{2n-1}; \mathbb{Q})$. By definition, M is a rational homology sphere. Then all homology groups $H_k(M), 1 \leq k \leq 2n-2$, are torsion groups, hence, by compactness, finite. In particular, $H_{n-1}(M)$ is a torsion finite group. Then Ker $i_t = \text{Ker } i = \text{Im } \partial$. Thus the linking pairing L_M is actually defined on $G_M = \text{Im } \partial/\text{Im } \partial_t$.

THEOREM 3.6. Let M be a rational homology 2n - 1-sphere and let X be a connected compact oriented 2n-manifold such that $\partial X = M$. Then $\lambda_M = \lambda_1 \oplus \lambda_2$ where λ_1 has presentation with rank dim $H_n(X; \mathbb{Q})$ and signature $\sigma(X)$ and λ_2 is metabolic.

Set $\overline{H}_{n-1}(M) = H_{n-1}(M)/\mathrm{Im}(\partial_t)$. There is an induced complex

$$F_n(X) \longrightarrow F_n(X, M) \xrightarrow{\partial} \bar{H}_{n-1}(M) \longrightarrow H_{n-1}(X)$$

For ease of notation, continue to denote λ_M the induced linking pairing on $\overline{H}_{n-1}(M)$. Note that $G_M = \operatorname{Im}(\overline{\partial})$ and that the induced linking pairing on G_M is identified with L_M . Assume, furthermore, that ι_X is non-degenerate, i.e., $\operatorname{Rad}(\iota_X) = 0$. Then Lemma 3.1 shows that

$$l = \lambda_M |_{B \times B}$$

is non-singular. It follows from Lemma ?? that B is an orthogonal summand for $(\bar{H}_{n-1}(M), \lambda_M)$. Thus there is a canonical orthogonal splitting $(\bar{H}_{n-1}(M), \lambda_M) = (B, l) \oplus (B^{\perp}, l').$

CHAPTER 8

Three-manifolds and Lagrangians

This chapter is devoted to the study of Lagrangians that arise in the homology of 3-manifolds.

1. Lagrangians

Let M be a compact 3-manifold with boundary ∂M .

1.1. Linking number and linking pairing. We consider a slight generalization of the setting introduced in the previous chapter. The idea appears in a paper by J. Morgan and D. Sullivan¹ [70, §6] and has been rediscovered since by several authors. Suppose that M is equipped with a subgroup $\Lambda \subset H_1(\partial M)$, isotropic with respect to the intersection pairing on $H_1(\partial M)$. Let $Z(\Lambda)$ be the set of 1-cycles in M such that their homology classes induce torsion elements in $H_1(M)/i_*(\Lambda)$. Let $x, y \in Z(\Lambda)$ be two 1-cycles in general position in M. Unravelling the definition, we see that there exist $n \in \mathbb{Z}$, a 1-cycle z whose homology class lies in Λ and a 2-chain C in M such that $n \ x = i_*z + \partial C$. Assuming that C is in general position with respect to y and denoting by a dot algebraic intersection in M, we define the linking number

$$lk_{\Lambda}(x,y) = \frac{C \cdot y}{n} \in \mathbb{Q}.$$

LEMMA 1.1. lk_{Λ} is well-defined, symmetric and bilinear.

REMARK 8.1. The linking number takes values in \mathbb{Z} if and only if one of the cycles is a boundary modulo a 1-cycle whose homology class lies in Λ . In particular, if $[x] \in i_*(\Lambda)$, then $lk_{\Lambda}(x, y) \in \mathbb{Z}$.

PROOF. If $n \ x = i_* z' + \partial C'$ is another decomposition, then $\partial(C - C') = i_*(z' - z)$ represents an element in $i_*(\Lambda)$. In particular, C - C' is a relative 2-cycle, i.e. represents an element in $H_2(M, \partial M)$. Let $p \in \mathbb{Z}$ such that $p \ [y] = i_*(w)$ in $H_1(M)$ for some $w \in \Lambda$. Thus the algebraic intersection $(C - C') \cdot p \ y$ is computed using the homological intersection product $\bullet : H_2(M, \partial M) \times H_1(M) \to \mathbb{Z}$. Let $a = [C - C'] \in H_2(M, \partial M)$. Since this product takes value in \mathbb{Z} , no torsion occurs. Hence

$$(C - C') \cdot y = \frac{1}{p} (a \cdot p \ y) = \frac{1}{p} (a \bullet_M i_*(w)) = \frac{1}{p} (\partial a \bullet_{\partial M} w) = 0.$$

¹It may have been introduced even earlier, but so far this is the earliest reference I could find.

The third equality is a well-known property of intersection products with respect to the long exact sequence associated to $(M, \partial M)$. In the last equality, the product is the intersection product on $H^1(\partial M)$. Since both ∂a and w lie in the same isotropic Λ , their product vanishes. This proves that lk_{Λ} is well defined.

Since lk_{Λ} is defined for 1-cycles, it is a bilinear pairing. To see that the pairing is symmetric, let C' be a 2-chain in M and z' a 1-cycle whose homology class lies in $i_*(\Lambda)$ such that $p \ y = i_*z' + \partial C'$. Assume transversality, the intersection of two 2-chains C and C' is a 1-cycle in M. Hence

$$0 = \partial(C \cdot C') = \partial C \cdot C' - C \cdot \partial C' = (nx - i_*z) \cdot C' - C \cdot (ny - i_*z')$$
$$= n \ x \cdot C' - C \cdot p \ y - i_*z \cdot C' + C \cdot i_*z'$$
$$= n \ x \cdot C' - C \cdot p \ y.$$

Dividing by n p gives $0 = lk_{\Lambda}(y, x) - lk_{\Lambda}(x, y)$.

REMARK 8.2. If Λ, Λ' are two isotropic subgroups of $H_1(\partial M)$ such that $i_*\Lambda \subseteq i_*\Lambda'$, then $Z(\Lambda) \subseteq Z(\Lambda')$. The latter inclusion induces an epimorphism $G_{\Lambda}M \to G_{\Lambda'}M$ that in turn induces a homomorphism $T_{\Lambda}M \to T_{\Lambda'}M$. It follows that for any 1-cycles $x, y \in Z(\Lambda)$ in general position in M, there exists an integer k such that

$$\lambda_{\Lambda'}(x,y) = k \,\lambda_{\Lambda}(x,y).$$

EXERCISE 8.1. Give an example such that the induced homomorphism $T_{\Lambda}M \to T_{\Lambda'}M$ is not onto.

Given the special role that Lagrangians play in 3-cobordisms (see §...), we are interested in the special case when Λ is a Lagrangian.

LEMMA 1.2. Let M be any connected compact oriented 3-manifold with boundary ∂M . Let Λ be a Lagrangian in $H_1(\partial M)$. Let

$$i_*: H_1(\partial M) \to H_1(M)$$

be the homomorphism induced by inclusion. The linking pairing on M induces a linking pairing

$$\lambda_{\Lambda} : \operatorname{Tors} \left(H_1(M)/i_*(\Lambda) \right) \times \operatorname{Tors} \left(H_1(M)/i_*(\Lambda) \right) \to \mathbb{Q}/\mathbb{Z}$$

defined by

$$\lambda_{\Lambda}([a], [b]) = \operatorname{lk}_{\Lambda}(a, b) \mod 1.$$

REMARK 8.3. By a Poincaré-Lefschetz duality argument, Ker i_* is a Lagrangian in $H_1(\partial M)$. In the case when $\partial M = \emptyset$ or when $\Lambda = \text{Ker } i_*$, we recover the usual linking pairing. See [**37**, Chap. 4], [**9**, VI,10,Problem 8].

PROOF. See [70, §6]. Another proof follows from Theorem 1.6 below.

DEFINITION 8.1. Given a connected compact oriented 3-manifold M with boundary ∂M endowed with a Lagrangian Λ in $H_1(\partial M)$, we set

$$G_{\Lambda}M = H_1(M)/i_*(\Lambda), \ T_{\Lambda}M = \text{Tors } G_{\Lambda}M.$$

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According to Lemma 1.2, $T_{\Lambda}M$ carries a linking pairing λ_{Λ} .

DEFINITION 8.2. We say that a Lagrangian $\Lambda \subset H_1(\partial M)$ is essential in M if the inclusion map $i_* : H_1(\partial M) \to H_1(M)$ restricted to Λ induces an isomorphism onto its image $i_*(\Lambda)$.

We say that a Lagrangian $\Lambda \subset H_1(\partial M)$ has co-finite inclusion if the quotient $G_{\Lambda} = H_1(M)/i_*(\Lambda)$ is finite.

EXERCISE 8.2. If $H_1(M)$ is finite then ∂M is a union of 2-spheres and the only Lagrangian is 0. [Use the homology long sequence associated to $(M, \partial M)$ with rational coefficients.]

EXAMPLE 8.1. Let $M = T^2 \times [-1, 1]$, the cylinder over the 2-torus $T^2 = S^1 \times S^1 = \partial (B^2 \times S^1)$. The boundary of M consists of two copies (with opposite orientations) of the torus:

$$T_{-} = T \times 0, \ T_{+} = T \times 1$$

2-torus. Let m_{\pm} denote the meridian $\partial B^2 \times \star \times \{\pm 1\} \subset T_{\pm} \subset \partial M$. Let l_{\pm} denote the longitude $\star \times S^1 \times \{\pm 1\} \subset T_{\pm} \subset \partial M$. There are two inclusion homomorphisms

 $(i_{-})_*: H_1(T_{-}) \to H_1(M), \ (i_{+})_*: H_1(T_{+}) \to H_1(M),$

which combine into a third inclusion homomorphism

 $i_*: H_1(\partial M) = H_1(-T_-) \oplus H_1(T_+) \to H_1(M), \ (x, y) \mapsto (i_-)_*(x) + (i_+)_*(y).$ In particular, the Lagrangian Ker i_* coincides with the antidiagonal Lagrangian in $-H_1(T) \oplus H_1(T)$. It does not have co-finite inclusion. The Lagrangian Λ generated in 1-homology by $2m_- + l_-$ and l_+ has the co-finite inclusion property. In $H_1(M), \ (i_-)_*[l_-] = (i_+)_*[l_+]$ and $(i_-)_*[m_-] = (i_+)_*[m_+]$ and $H_1(M)$ is generated by $i_*[m_+]$ and $i_*[l_+]$. Hence

$$G_{\Lambda} = H_1(M)/i_*\Lambda = \langle [m_+], [l_+] \rangle / \langle [2m_+] + [l_+], [l_+] \rangle$$
$$= \langle [m] \rangle / \langle 2[m] \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

EXAMPLE 8.2. Consider the closed 3-manifold M obtained from $S^1 \times S^1 \times S^1$ by drilling out a small solid torus (contained in a small 3-ball). Then ∂M is a 2-torus and $H_1(M) = \mathbb{Z}^4$. No Lagrangian $\Lambda \subset H_1(\partial M) = \mathbb{Z}^2$ has the co-finite inclusion property. In particular, there are essential Lagrangians that do not have the co-finite inclusion property.

Below is an important special case when essential Lagrangian and Lagrangian with co-finite inclusion coincide.

LEMMA 1.3. Suppose that M is a cylinder over an oriented handlebody of genus $g \ge 1$. Let Λ be a Lagrangian in $H_1(\partial M)$. The following assertions are equivalent:

- (1) Λ has co-finite inclusion;
- (2) Λ is essential in M;

(3) The inclusion homomorphism $i_* : H_1(\partial M) \to H_1(M)$ preserves the rank of Λ ;

(4) $i_*\Lambda$ is a rank 2g sublattice in $H_1(M)$.

The following two remarks relate to the case when the Lagrangian is decomposable (see Def. 4.3, Chap. 4) with respect to a fixed decomposition $H_1(-\Sigma_1) \oplus H_1(\Sigma_2)$.

REMARK 8.4. A decomposable Lagrangian need not have co-finite inclusion. For instance, the Lagrangian Λ generated by $[m_-] \in H_1(-T_-)$ and $[m_+] \in H_1(T_+)$ is a (rank 2) decomposable sublattice of $H_1(T_-) \oplus H_1(T_+)$; however, $(i_-)_*[m_-] = (i_+)_*[m_+]$, hence $i_*(\Lambda)$ has only rank 1 in $H_1(T_-) \oplus H_1(T_+)$ so

$$G_{\Lambda}M = H_1(M)/i_*\Lambda \simeq \mathbb{Z}$$

where l is the standard longitude of T, so it does not have co-finite inclusion.

REMARK 8.5. An essential (resp. co-finite inclusion) Lagrangian need not be decomposable. Let $\Sigma = T$, the 2-torus, and $M = \Sigma \times [0, 1]$. It follows from the definition that $\Lambda = \{(x, 2x) \mid x \in H_1(\Sigma)\}$ is a non-decomposable Lagrangian in $H_1(-\Sigma) \oplus H_1(\Sigma)$. We have $(i_-)_*[m_-] = (i_+)_*[m_+]$ and $(i_-)_*[l_-] = (i_+)_*[l_+]$. So $i_*\Lambda$ is generated by 3[m] and 3[l], hence

$$H_1(M)/i_*(\Lambda) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

Therefore Λ has co-finite inclusion.

EXERCISE 8.3. Show that in the example above (Remark 8.5), there is no decomposable Lagrangian Λ' in $H_1(-\Sigma) \oplus H_1(\Sigma)$ such that $G_{\Lambda}M \simeq G_{\Lambda'}M$.

1.2. A geometric description of decomposable Lagrangians. Consider a connected oriented 3-manifold M with non-empty boundary ∂M . Such a manifold gives rise to a Lagrangian Ker $i_* \subset H_1(\partial M)$ where $i_* : H_1(\partial M) \to H_1(M)$ denotes the inclusion homomorphism. See for instance [9, VI, Th. 10.4].

DEFINITION 8.3. The Lagrangian Ker i_* is called the *topological Lagrangian* associated to $(M, \partial M)$.

The emphasis here in the definition is that the topological Lagrangian depends only on the topology of $(M, \partial M)$.

In general, there are many other Lagrangians in $H_1(\partial M)$; if we regard M as some Lagrangian decorated cobordism (see...), the topological Lagrangian generally differ from the Lagrangian Λ_M associated to the cobordism. One way to see that is to remark that Ker i_* does not have to be decomposable.

EXAMPLE 8.3. Let again $M = T^2 \times [0, 1]$, the cylinder over the 2-torus $T^2 = S^1 \times S^1$. The boundary of M consists of two copies

$$T_{-} = T \times 0, \ T_{+} = T \times 1$$

(with opposite orientations) of the 2-torus. The Lagrangian Ker i_* is generated by pairs $(-x, x) \in -H_1(T) \oplus H_1(T)$. It is not decomposable. Denote as before (Example 8.1) by

$$[m_+], [l_+] \in H_1(T_+)$$

the homological classes represented by the meridian and longitude respectively. They clearly generate $H_1(T_{\pm})$. Any pair of primitive elements of $H_1(T_{\pm})$ forms a symplectic basis of $H_1(T_{\pm})$ if and and only if it is uniquely

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represented by a matrix $A \in \operatorname{SL}_2(\mathbb{Z})$ with respect to the basis (l_{\pm}, m_{\pm}) . In particular, there is a one-to-one correspondence between Lagrangians in $H_1(T_{\pm})$ and primitive elements in $H_1(T_{\pm})$. Let now $\Lambda \in H_1(\partial M) = -H_1(T_-) \oplus H_1(T_+)$ be a Lagrangian. It is generated by two independent primitive elements. If one of these two elements can be taken to be in $H_1(T_{\pm})$ then (since it is primitive) it generates a Lagrangian $\Lambda_{\pm} \in H_1(T_{\pm})$. If both elements can be taken in $H_1(T_-)$ and $H_1(T_+)$ respectively, then Λ is a decomposable Lagrangian and $\Lambda = \Lambda_- \oplus \Lambda_+$. Conversely, any decomposable Lagrangian $\Lambda \subset -H_1(T_-) \oplus H_1(T_+)$ is generated by two primitive elements in $H_1(T_-)$ and $H_1(T_+)$ respectively. For instance,

$$\Lambda_{11} = \langle l_- \rangle \oplus \langle l_+ \rangle, \ \Lambda_{12} = \langle l_- \rangle \oplus \langle m_+ \rangle, \ \Lambda_{21} = \langle m_- \rangle \oplus \langle l_+ \rangle, \ \Lambda_{22} = \langle m_- \rangle \oplus \langle m_+ \rangle$$

are four distinct decomposable Lagrangians of $H_1(\partial M) = H_1(-T_-) \oplus H_1(T_+)$.

It is not hard to extend the observation of the previous example.

LEMMA 1.4. Let $(M, \Sigma_{-}, \Sigma_{+})$ be a 3-cobordism with ∂M consisting of exactly two connected components, Σ_{-} and Σ_{+} . A Lagrangian

$$\Lambda \subset H_1(\partial M)$$

is decomposable with respect to the decomposition

$$H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$$

if and only if Λ is generated by elements represented by simple closed oriented curves in ∂M .

EXAMPLE 8.4. In the example 8.3 above for $M = T \times [0, 1]$, the topological Lagrangian associated to M is generated by pairs $(-x, x) \in -H_1(T_-) \oplus H_1(T_+), x \in H_1(T)$. None of these pairs (except the trivial one) can be represented by *one* single simple closed oriented curve in ∂M .

EXAMPLE 8.5. In the example given in Remark 8.5, the Lagrangian Λ is generated by $([m_-], 2[m_+])$ and $([l_-], 2[l_+])$. None of these generators can be represented by a simple closed curves in ∂M .

PROOF. The representation of generators of Λ by simple closed oriented curves is a sufficient condition: each simple closed curve must lie in one single connected component of ∂M and in particular will induce a welldefined homological class in $H_1(\Sigma_-)$ or in $H_1(\Sigma_+)$. Since Λ is Lagrangian, there are exactly g_{\pm} simple closed curves in Σ_{\pm} where g_{\pm} denotes the genus of Σ_{\pm} . The result follows. Conversely, if Λ is decomposable then $\Lambda_{\pm} =$ $\Lambda \cap H_1(\Sigma_{\pm})$ is a Lagrangian in $H_1(\Sigma_{\pm})$. Such a Lagrangian is generated by a system of g independent primitive elements in $H_1(\Sigma_{\pm})$. An element in $H_1(\Sigma_{\pm})$ is primitive if and only if it is represented by a simple closed oriented nonseparating curve (see e.g., [64]). Hence the result.

1.3. Lagrangians and gluings. Consider a disjoint union H of oriented solid handlebodies H_1, \ldots, H_r and an orientation preserving homeomorphism $f: \cup_j \partial H_j \to \partial M$. Let

$$M = M \cup_f -H$$



FIGURE 1.1. A handlebody H, its meridians and the collars of its meridians.

be the closed oriented 3-manifold obtained by gluing the handlebodies to ∂M via f. Let Λ_f be the Lagrangian generated in $H_1(\partial M)$ by the images by f of the meridians of $\partial H_1, \ldots, \partial H_r$.

LEMMA 1.5. The Lagrangian Λ_f is decomposable and essential in M. Conversely any decomposable and essential Lagrangian $\Lambda \in H_1(\partial M)$ is obtained as Λ_f for some orientation preserving homeomorphism $f : \cup_j \partial H_j \to \partial M$.

Let j denote the map $M \to \widetilde{M}$ induced by the inclusion $M \subset M \cup H$ and the gluing. The following observation is useful.

THEOREM 1.6. Let x, y be two 1-cycles in general position in M such that some of their multiples lie in $i_*\Lambda$. Then

 $lk_{\Lambda}(x,y) = lk_{\widetilde{M}}(j_*x,j_*y).$

In particular, there is a linking pairing isomorphism

(Tors
$$H_1(M), \lambda_{\widetilde{M}}) \simeq (T_\Lambda M, \lambda_\Lambda).$$

In particular, λ_{Λ} is nondegenerate.

... . .

PROOF. Consider first the case when ∂M is a connected handlebody of genus $g \ge 1$. (The case when g = 0 is trivial.) Then H is another handlebody of genus g, so that ∂H is closed surface of genus g. Choose compressing discs D_1, \ldots, D_g in H for ∂H such that $m_1 = \partial D_1, \ldots, m_g = \partial D_g$ are meridians in ∂H . Choose disjoint collars N_1^0, \ldots, N_g^0 of D_1, \ldots, D_g respectively in H so that $\partial N_1^0, \ldots, \partial N_g^0$ are collars of m_1, \ldots, m_g respectively in ∂H . Set $N^0 = \cup_j N_j^0$. Note that $H - N^0$ is a topological closed 3-ball. See Figure 1.1.

Let N be the image of N^0 under the canonical projection map $\pi : M \coprod N_0 \to M \cup_f N^0$. Set $M' = M \cup N = M \cup_f N^0$ and $M \cap N = \pi(\partial N^0) \simeq \partial N^0$. The relevant part of the Mayer-Vietoris sequence reads

$$H_1(M \cap N) \xrightarrow{(i_*, j_*)} H_1(M) \oplus H_1(N) \longrightarrow H_1(M \cup N) \longrightarrow H_0(M \cap N)$$

Note that N is homeomorphic to N^0 and is contractible since each component N_j^0 is contractible, hence $H_1(N) = 0$. Thus the image of (i_*, j_*) reduces to $i_*(H_1(M \cap N))$ in $H_1(M)$. Now $i_*(H_1(M \cap N))$ is generated by the image of the meridian m viewed inside M. This is $i_*\Lambda$. It follows that $H_1(M') = H_1(M \cup N) \simeq H_1(M)/i_*H_1(M \cap N) = H_1(M)/i_*\Lambda$. Now \tilde{M} is obtained from M' by gluing a 3-ball onto M' since $H - N_0$ is 3-ball. Hence $H_1(\tilde{M}) = H_1(M') \simeq H_1(M)/i_*\Lambda = G_{\Lambda}$. Under this identification, the linking pairing on Tors $H_1(\tilde{M})$ coincides with the linking pairing on $T_{\Lambda}M$.

The general case (when ∂M consists of several components) is completely similar. Since the linking pairing (Tors $H_1(\tilde{M}), \lambda_{\tilde{M}}$) is the linking pairing of the closed 3-manifold \tilde{M} , it is nonsingular (Cor. 2.3) and the last statement of the lemma follows.

EXAMPLE 8.6. Let M be the solid torus $S^1 \times D^2$. Its boundary is $\partial M = S^1 \times \partial D^2 = S^1 \times S^1$. The first integral homology of ∂M is freely generated by a meridian $m = \star \times \partial D^2$ and a longitude $l = S^1 \times \star$. Let Λ be the Lagrangian generated by the longitude $l = S^1 \times \star \subset S^1 \times \partial D^2$. Then $G_{\Lambda}M = H_1(M)/i_*\Lambda = 0$, the linking numbers $lk_{\Lambda}(x, y)$ are integers and the linking pairing lk_{Λ} is trivial. Consider a homeomorphism $f : \partial (S^1 \times D^2) \to$ M sending the meridian $\star \times \partial D^2$ to the longitude l of M. Then $\widetilde{M} = S^3$. Hence linking numbers are usual linking numbers in S^3 (hence are integers) and the linking pairing on S^3 is trivial since the homology of S^3 is trivial.

EXAMPLE 8.7. Consider the same solid torus $M = S^1 \times D^2$. Let n be a nonzero integer and m and l the meridian and longitude as before. Let $\Lambda = \Lambda_n$ be the Lagrangian in $H_1(\partial M)$ generated by m + n l. Then $H_1(M)/i_*\Lambda \simeq \mathbb{Z}/n\mathbb{Z}$ and $lk_{\Lambda}(p \ [l], q \ [l]) = \pm \frac{pq}{n} \mod 1$. If $f : \partial(S^1 \times D^2) \to M$ is a homeomorphism sending the meridian $\star \times D^2$ to m + n l in M, then \widetilde{M} is the lens space L(n, 1) and we recover the cyclic linking pairing on L(n, 1) in this fashion.

Under the hypothesis of this paragraph, any quadratic enhancement q_{Λ} of λ_{Λ} is nondegenerate. We observe that any quadratic enhancement can be regarded as partially induced by a relative spin structure s on the 3-manifold M with boundary. We describe it as follows. With the same notation as above, endow the disjoint union of oriented solid handlebodies H_1, \ldots, H_r with relative spin structures s_1, \ldots, s_r respectively, in such a way that

$$s|_{\partial M} = \cup_i f^* s_i|_{\partial H_i}$$

Then gluing the handlebodies to M via f yields a closed 3-manifold M with spin structure \tilde{s} . Any quadratic enhancement of λ_{Λ} is obtained as the quadratic form induced by \tilde{s} for some suitable choice of s_1, \ldots, s_r .

REMARK 8.6. We regard the Lagrangian Λ as a kind of algebraic remnant of the gluing. Topologically it is easier to think in terms of gluings; algebraically (specifically in relation with the Weil representation), it is easier to think in terms of Lagrangians. 1.4. Seifert pairings. In this paragraph we show how a cylinder over a surface endowed with a Lagrangian gives rise to a pairing on the 1-homology of the surface. Let $\Sigma \times [0,1]$ be a cylinder over a closed surface Σ . Let $\Sigma_+ = \Sigma \times \{1\}$ and $\Sigma_- = \Sigma \times \{0\}$. We regard Σ as embedded in $\Sigma \times [0,1]$ via the inclusion $\Sigma \to \Sigma \times \{1/2\} \subset \Sigma \times [0,1]$. The cylinder structure provides Σ with a natural bicollar. For a 1-cycle $x \in \Sigma = \Sigma \times \{1/2\}$, we denote by x^+ the 1-cycle corresponding to $x \times \{1\}$ and by x^- the 1-cycle corresponding to $x \times \{1\}$ and by x^- the 1-cycle corresponding to $x \times \{0\}$. Denote as usual the inclusion homomorphisms by $i^{\pm}: H_1(\Sigma_{\pm}) \to H_1(\Sigma \times [0,1])$. Let Λ be a Lagrangian in $H_1(\partial(\Sigma \times [0,1])) = H_1(-\Sigma \times 0) \oplus H_1(\Sigma \times 1)$. We keep the same notation for the inclusion homomorphisms followed by the projection map $H_1(\Sigma \times [0,1]) \to G_{\Lambda} = H_1(\Sigma \times [0,1])/i_*\Lambda$.

We shall now assume that Λ is decomposable and essential.

LEMMA 1.7. The assignment

(1.1)
$$\beta([x], [y]) = \operatorname{lk}_{\Lambda}(i_*^- x^-, i_*^+ y^+)$$

where x and y are two representative cycles in general position in Σ , defines a bilinear pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Q}$.

PROOF. We have to verify that (??) is a well-defined pairing carried by $H_1(\Sigma)$. We claim that $lk_{\Lambda}(i_*^-x^-, i_*^+y^+)$ depends only on the homology class of x in $H_1(\Sigma)$. If we replace x by a homologous cycle $x' = x + \partial z$ in Σ then

$$i_*^- x'^- = i_*^- x^- + i_*^- (\partial z^-) = i_*^- x + \partial i_*^- z^-$$

so $i_*^- x'^-$ is a cycle homologous to $i_*^- x^-$ in the complement of $i_*^+ y^+$. By Lemma 1.5, Λ is generated by the image by some homeomorphism f of the meridians of a standard handlebody H of the same genus as that of Σ . Setting $\widetilde{M} = M \cup_f -H$, we have, according to Theorem 1.6,

$$\lambda_{\Lambda}(i_*^-x^-, i_*^+y^+) = \lambda_{\tilde{M}}(j_*i_*^-x^-, j_*u_*^+x^+).$$

Moreover, since $H_1(M)$ is finite, M is a rational homology 3-sphere. The claim on the linking number follows (cf. Exercise 7.1).

DEFINITION 8.4. The bilinear pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Q}$ defined above is the *bilinear pairing associated to the pair* (Σ, Λ) .

REMARK 8.7. If $G_{\Lambda}M = 0$, then the bilinear pairing is an integral bilinear pairing $H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$.

Question. If one assumes only that Λ is essential, does (1.1) still define a bilinear pairing as stated in Lemma ?? ?

EXAMPLE 8.8 (The Hopf Seifert pairing). Let $(M, \Sigma_{-}, \Sigma_{+})$ be the trivial cylinder over the standard 2-torus Σ . Recall that $\partial M = -\Sigma_{-} \cup \Sigma_{+}$. Let mand l be the standard meridian and the standard longitude of Σ respectively, forming a geometric symplectic basis for $\Sigma = \Sigma \times 1/2$. We denote by m^{-} , l^{-} (resp. m^{+}, l^{+}) the curves m, l respectively pushed onto Σ_{-} (resp. Σ_{+}).

The following relations hold:





FIGURE 1.2. The cylinder over the torus equipped with a geometric symplectic basis on the components Σ_{-} and Σ_{+} .

$$m^{+} \bullet_{\Sigma_{+}} l^{+} = +1 = m^{+} \bullet_{\partial M} l^{+}.$$
$$m^{-} \bullet_{\Sigma_{-}} l^{-} = +1 = -m^{-} \bullet_{-\Sigma_{-}} l^{-} = -m^{-} \bullet_{\partial M} l^{-}$$

Let Λ be the Lagrangian in $H_1(\partial M)$ generated by $[m^-]$ and $[l^+]$. Then

$$lk_{\Lambda}(i_*^+m^+, i_*^-l^-) = m^+ \bullet_{\Sigma_+} l^+ = +1.$$

Glue two copies of a standard solid torus $H = S^1 \times D^2$ of genus g to $\Sigma \times [0,1]$ as follows. We glue the first solid torus via a homeomorphism $-\partial H \to \Sigma \times \{1\}$ sending the meridian of ∂H to l^+ . We glue the second solid torus via a homeomorphism $\partial H \to \Sigma \times \{0\}$ sending the meridian of ∂H to m^- . The resulting closed 3-manifold is S^3 . We call this gluing the *Hopf gluing*. By Theorem 1.6, for any disjoint 1-cycles x, y in $\Sigma \times [0, 1]$,

$$\mathrm{lk}_{\Lambda}(x,y) = \mathrm{lk}_{S^3}(j_*x,j_*y).$$

In particular, $\beta([l], [m]) = lk_{\Lambda}(i_*l^-, i_*^+m^+)$ is the linking number of a positive Hopf link in S^3 . For this reason we shall call the Lagrangian Λ above the *Hopf Lagrangian*.

These considerations extend obviously to the case of an oriented closed connected surface Σ of arbitrary genus.

DEFINITION 8.5. Let $\Sigma \times [0, 1]$ be the cylinder over the standard surface of genus $g \ge 1$. Let H_g denote the standard oriented handlebody of genus g. Let $f_-: \Sigma_{g_-} \to \Sigma \times \{0\}$ be the identity and let $f_+: \Sigma_{g_+} \to \Sigma \times \{1\}$ be an orientation preserving homeomorphism sending the *j*-th meridian of $\Sigma_{g_-} = \partial H_g$ onto the *j*-th longitude of $\Sigma \times \{0\}$. The Hopf gluing is defined as the gluing that consists in gluing two handlebodies to the cylinder $\Sigma \times [0, 1]$ via the parametrization f_{-} and f_{+} on the bases:

$$S^3 = H_g \cup_{f_-} \Sigma \times [0,1] \cup_{f_+ \circ \operatorname{mir}} -H_g.$$

The Lagrangian $\Lambda \subseteq H_1(\partial(\Sigma \times [0, 1]))$ generated by the meridians of $\Sigma \times \{0\}$ and the longitudes of $\Sigma \times \{1\}$ is called the *Hopf Lagrangian*.

The Hopf Lagrangian Λ is decomposable and essential and $G_{\Lambda}(\Sigma \times [0, 1]) = 0$.

LEMMA 1.8. If Λ is the Hopf Lagrangian then the bilinear pairing β_{Λ} : $H_1(\Sigma) \times H_1(\Sigma) \rightarrow \mathbb{Z}$ is a Seifert pairing with respect to the intersection pairing on $-\Sigma$.

PROOF. Since $G_{\Lambda}(\Sigma \times [0, 1]) = 0$, the bilinear pairing β is integral. The proof that β induces the intersection pairing, i.e. that

$$-x \bullet_{\Sigma} y = \beta(x, y) - \beta(y, x),$$

is completely similar to that of Example 6.1, Chap. 6, §1.

CHAPTER 9

Abelian skein theory

In this section we give a topological ("skein") interpretation of the Heisenberg algebra and the Schrödinger representation of the Heisengerg algebra. These results are interesting by themselves because they provide state modules with extra structures. They give a completely skein-theoretic approach to the construction of Abelian TQFTs and they are building blocks used in the proof of Theorem 2.3. For this, we develop an appropriate calculus, called skein calculus, pioneered by J. H. Przytycki and V.G. Turaev.

1. Heisenberg skein modules: preliminary construction

Skein modules are certain modules (actually some of them are algebras) built from links with extra structures in three-manifolds. The algebraic properties of skein modules reflects the topology of the ambient three-manifold. The Heisenberg skein modules introduced in this section will be shown to be closely related to the Heisenberg groups.

Let M be an oriented compact 3-manifold. According to Moise's theorem [68], any topological 3-manifold has an essentially unique smooth structure. We shall use implicitly this fact without further comment. We shall need a few basic definitions on knot theory as well. For more details, we refer to the introductory chapters of the monographs of D. Rolfsen [82] and G. Burde and H. Zieschang [11]. A knot in M is a smooth embedding of a circle in M. More generally, a link in M is a smooth embedding of a finite collection of pairwise disjoint circles in M. A framed link in M is a smooth embedding of a finite collection of pairwise disjoint annuli in M. We shall frequently abuse notation and identify a link (resp. framed link) with its image in the ambient manifold M. In particular, the components of a framed link are thought of as a collection of pairwise disjoint annuli. A link is oriented if each of its componments is assigned an orientation. Two knots are *parallel* if they form the boundary of a framed knot in this previous sense. We shall use the blackboard convention for the drawing of framed knots and links: the annulus determining the framing on a component is understood to lie in the plane of the figure. It will be also convenient to think of a framed link as a link endowed with a unit vector field, the vector field pointing towards the parallel knot of the original knot. An isotopy of a (framed) link L in M is a smooth 1-parameter family φ_t of framed links such that φ_0 is the embedding defining L. Two (framed) links L and L' are *isotopic* in M if they can be included into one isotopy, i.e. if there exists a smooth 1-parameter family φ_t of framed links such that φ_0 is the embedding defining L and φ_1

is the embedding defining L'. We shall frequently abuse notation and not distinguish between a link and the isotopy class it represents.

DEFINITION 9.1. Let G be a finite Abelian group and let L be an oriented link in M. A basic colour for L is a map col : $\pi_0(L) \to G$ from the set of components of L to the group G. A *basically coloured* link in M is a link in M together with a basic colour.

In this chapter, we shall simply write "colour" instead of "basic colour".

We need to define two sets of "indeterminates" based on G.

DEFINITION 9.2. The set \tilde{G} is defined by the disjoint union

$$\tilde{G} = \{ t_g, t_g^{-1} \mid g \in G \} \cup \{ t_{g,h}, t_{g,h}^{-1} \mid \{g,h\} \subseteq G \}.$$

The set \check{G} is defined by the disjoint union

$$\check{G} = \{t_g, t_g^{-1} \mid g \in G\} \cup \{t_{g,h}, t_{g,h}^{-1} \mid g \neq h, \{g,h\} \subseteq G\} = \tilde{G} - \{t_{g,g}, t_{g,g}^{-1} \mid g \in G\}.$$

Let $\mathbb{Z}[\tilde{G}^{\text{quad}}]$ (resp. $\mathbb{Z}[\check{G}^{\text{quad}}]$) denote the multivariable Laurent polynomial algebra over the indeterminates in \tilde{G}^{quad} (resp. in \check{G}^{quad}). By definition, the following relations hold:

$$t_{g} \cdot t_{g}^{-1} = t_{g}^{-1} \cdot t_{g} = 1$$
$$t_{g,h} = t_{h,g}$$
$$t_{g,h} \cdot t_{g,h}^{-1} = t_{g,h}^{-1} \cdot t_{g,h} = 1.$$

DEFINITION 9.3. We define the free module $\mathscr{L}(M)$ over $\mathbb{Z}[\tilde{G}^{\text{quad}}]$ generated by the set of all isotopy classes of oriented framed coloured links in M, including the empty link denoted \varnothing . Similarly, for $k \ge k$, we define the free module $\mathscr{L}_k(M)$ over $\mathbb{Z}[\tilde{G}^{\text{quad}}]$ generated by the set of all isotopy classes of oriented framed coloured k-component links in M. In particular, $\mathscr{L}_1(M)$ is just the free module generated by the set of all isotopy classes of oriented framed coloured knots in M.

Note that $\mathscr{L}(M) = \{\varnothing\} \cup \bigcup_{k \ge 1} \mathscr{L}_k(M).$

The figure below represents two oriented framed coloured links $X_+(g, h)$ and $X_-(g, h)$ which are identical except in a small embedded ball in M where they look exactly as shown, where one arc is part of a component coloured by an element $g \in G$ and the other arc is part of a component coloured by $h \in G$.



Consider now the elements

(1.1)
$$X_{+}(g,h) - t_{g,h}X_{-}(g,h), \ X_{-}(g,h) - t_{g,h}^{-1}X_{+}(g,h)$$

where $X_+(g, h)$ and $X_-(g, h)$ are two oriented framed coloured links which are identical except in a small embedded ball D in M where they look exactly as shown in the figure, where one arc is part of a component coloured by an element $g \in G$ and the other arc is part of a component coloured by $h \in G$. The possibility g = h is accepted, whether the arcs belong to distinct components or not.

The next figure below represents an arbitrary oriented framed link $X_0(g,g)$ in M, where the interior of a small embedded ball D^3 is specified as shown.



We consider also the elements

(1.2)
$$X_{+}(g,g) - t_{q}X_{0}(g,g), \ X_{-}(g,g) - t_{a}^{-1}X_{0}(g,g)$$

where $X_{\pm}(g,g)$ and $X_0(g,g)$ are two oriented framed coloured links which are identical except in a small embedded ball in M where they look exactly as shown in the figure. (Note that the number of components of $X_{\pm}(g,g)$ is the number of components of $X_0(g,g)$ plus or minus one.)

Let X be an element in $\mathscr{L}(M)$ represented by an oriented framed coloured link L. Denote by $X \cup O$ the element in $\mathscr{L}(M)$ that consists of the topologically disjoint union of L and an extra annulus that bounds in M a disc disjoint from L (the trivially framed unknot) with an arbitrary color. Consider finally the element

$$(1.3) X - X \cup O.$$

We accept for X the possibility that X be the empty link.

Let $\mathscr{S}(M)$ be the submodule spanned by all elements of the three kinds enumerated above, respectively by (1.1), (1.2) and (1.3). The corresponding relations are called *skein relations*.

DEFINITION 9.4. The Heisenberg skein premodule $\mathscr{A}^{o}(M)$ is the quotient

 $\mathscr{L}(M)/\mathscr{S}(M)$

Elements of $\mathscr{A}^{\mathrm{o}}(M)$ are called *skeins*. The skein represented by an oriented framed link L will be denoted by $\langle L \rangle$. If we need to emphasize the color, we include it in the notation. For instance, if K is an oriented framed knot, then $\langle K(g) \rangle$ denotes the skein represented by K coloured with $g \in G$. We set $\mathscr{A}^{\mathrm{o}}_{\mathbb{C}}(M) = \mathscr{A}^{\mathrm{o}}(M) \otimes \mathbb{C}$. The *empty skein*, noted $\langle \varnothing \rangle$, is the skein induced by the empty link. EXAMPLE 9.1. If O denotes an annulus (coloured by an arbitrary color) bounding a disc in M, then $\langle O \rangle = \langle \emptyset \rangle$ (relation (1.3)).

LEMMA 1.1. The following relations hold in $\mathscr{A}^{o}(M)$:

(1) The commutativity relations: for all $g, h, k, l \in G$,

 $t_g t_h = t_h t_g, \ t_{g,h} = t_{h,g}, \ t_g t_{h,k} = t_{h,k} t_g, \ t_{g,h} t_{k,l} = t_{k,l} t_{g,h}.$

(2) The doubling relations: $t_q^2 = t_{g,g}$ for all $g \in G$.

PROOF. (1) The order in which the skein relations are processed is irrelevant. (2) The following relations hold in $\mathscr{A}^{o}(M)$:

$$X_{+}(g,g) = t_{g}X_{0}(g), \quad X_{+}(g,g) = t_{g,g}X_{-}(g,g) = t_{g,g}t_{g}^{-1}X_{0}(g,g).$$

Therefore

$$t_g X_0(g,g) = t_{g,g} t_g^{-1} X_0(g,g)$$

for any skein. The result follows.

COROLLARY 1.2. The skein premodule $\mathscr{A}^{\circ}(M)$ is a module over the Laurent polynomial algebra $\mathbb{Z}[\check{G}^{\text{quad}}]$.

PROOF. The skein module $\mathscr{A}^{\mathrm{o}}(M)$ inherits a $\mathbb{Z}[\tilde{G}^{\mathrm{quad}}]$ -module structure from $\mathscr{L}(M)$. The previous lemma shows that the map determined by

$$t_{g,g} \mapsto t_g^2, \ \mathbb{Z}[\tilde{G}^{\text{quad}}] \to \mathbb{Z}[\check{G}^{\text{quad}}]$$

induces on $\mathscr{A}^{o}(M)$ a structure of $\mathbb{Z}[\check{G}^{quad}]$ -module.

EXAMPLE 9.2. Consider the 3-manifold $M = S^3$ (or an integral homology 3-sphere). Let $L = L_1 \cup \cdots \cup L_n$ be an oriented framed link coloured with g_1, \ldots, g_n . Then

$$\left\langle L\right\rangle = \prod_{1\leqslant i < j\leqslant n} t_{g_i,g_j}^{\mathrm{lk}(L_i,L_j)} \cdot \prod_{1\leqslant i\leqslant n} t_{g_i}^{\mathrm{lk}(L_i,L_i')} \left\langle \varnothing \right\rangle$$

where L'_i denotes the component parallel to L_i determined by the framing. It follows that $\mathscr{A}^{\circ}(S^3) \simeq \mathbb{Z}[\check{G}^{\text{quad}}].$

EXAMPLE 9.3. Consider the case when G is the trivial group. We can set $t_0 = t$ and $t_{00} = u$. According to the lemma, $t^2 = u$. The relations of the first kind are $X_+ = tX_-$, $X_- = t^{-1}X_+$. The relations of the second kind are $X_+ = uX_0$, $X_- = u^{-1}X_0$. The Heisenberg premodule $\mathscr{A}^{\circ}(M)$ is a $\mathbb{Z}[t, t^{-1}]$ -module.

EXAMPLE 9.4. Consider the oriented framed and coloured two component link L in the solid torus (oriented handlebody of genus one) $S^1 \times D^2$ as pictured in Fig. 1.1. Denote as usual by $m = \star \times \partial D^2$ a meridian of $S^1 \times D^2$ and by $l = S^1 \times \star$ a longitude of $S^1 \times D^2$. Color the meridian by an element $g \in G$ and the longitude by an element $h \in G$.

Fig. 1.1 shows that in the Heisenberg skein premodule $\mathscr{A}^{\rm o}(S^1 \times D^2)$, the following relation holds:

$$\langle L \rangle = t_{q,h} \langle l(h) \rangle.$$

Note that both a relation of the first type (1.1) and a relation of the third type (1.3) were used.



FIGURE 1.1. A skein relation in $S^1 \times D^2$. After applying a skein relation, the loop labelled by g becomes trivial in $S^1 \times D^2$.

EXERCISE 9.1. In general, nontrivial identities are produced on a link diagram by a combination of Reidemeister moves and skein moves.

Consider the skein $\langle L \rangle$ represented by two parallel framed knots with opposite orientations in an arbitrary 3-manifold M and coloured with the same color as represented here. Prove that $\langle L \rangle = \langle \varnothing \rangle$.



LEMMA 1.3. \mathscr{A}° is a covariant functor from the category of oriented 3manifolds and isotopy classes of smooth orientation preserving inclusion maps to the category of modules and maps over $\mathbb{Z}[\check{G}^{quad}]$. In particular, if there is an orientation preserving diffeomorphism between M and N then $\mathscr{A}^{\circ}(M)$ and $\mathscr{A}^{\circ}(N)$ are isomorphic modules. In particular, the mapping class group of M acts by automorphisms on $\mathscr{A}^{\circ}(M)$.

PROOF. Consider a smooth embedding $f: M \to N$. If $\Phi: S^1 \times [0,1] \to M$ is an isotopy between two knots $\phi_0 = \Phi(-,0)$ qnd $\phi_1 = \Phi(-,1)$, then $f \circ \Phi: S^1 \times [0,1] \to N$ is an isotopy between $f \circ \phi_0$ and $f \circ \phi_1$. Thus f induces a module map $f_*: \mathscr{L}(M) \to \mathscr{L}(N)$. The inclusion map sends $\mathscr{S}(M)$ to $\mathscr{S}(N)$, i.e. preserves the skein relations as well, hence induces a map $\mathscr{A}^{\circ}(M) \to \mathscr{A}^{\circ}(N)$. Clearly the assignment is associative and sends the identity id_{M} to the identity $\mathrm{id}_{\mathscr{A}^{\circ}(M)}$. It follows that if $f: M \to N$ is an orientation preserving diffeomorphism, then the induced map $f_*: \mathscr{A}^{\circ}(M) \to \mathscr{A}^{\circ}(N)$ is a module isomorphism.

Suppose that f and g are two isotopic smooth embeddings $M \to N$ related by an isotopy $\Psi : M \times [0,1] \to N$ such that $f = \Psi(-,0)$ and $g = \Psi(-,1)$. Let $\Phi : S^1 \to M$ be an isotopy between two knots $\phi_0 = \Phi(-,0)$ and $\phi_1 = \Phi(-,1)$. Then

 $S^1 \times [0,1] \xrightarrow{\Phi} M \times [0,1] \xrightarrow{\Psi} N$

defines an isotopy between $f \circ \phi_0$ and $g \circ \phi_1$. Thus the isotopy class of f induces a well-defined map $f_* : \mathscr{L}(M) \to \mathscr{L}(N)$. It follows that there is a well-defined action of the mapping class group of M on the group of module automorphisms of $\mathscr{A}^{\mathrm{o}}(M)$.

The following result is the fundamental example that we shall be concerned with in this chapter. PROPOSITION 1.4. Let Σ be a closed oriented surface of genus g and Han oriented handlebody of genus g such that $\partial H = \Sigma$. There is a natural surjective module map $\mathscr{A}^{\circ}(\Sigma \times [0,1]) \to \mathscr{A}^{\circ}(H)$.

PROOF. Regarding the cylinder $\Sigma \times [0,1]$ as a collar of ∂H in H (that is, $\partial H = \Sigma \times \{0\}$) provides an inclusion $\Sigma \times [0,1] \subset H$. By Lemma 1.3, this inclusion induces a module map $\mathscr{A}^{\circ}(\Sigma \times [0,1]) \to \mathscr{A}^{\circ}(H)$. To see that this map is onto, observe that any framed link L in H can be isotoped in H so that L lies in $\Sigma \times [0,1] \subset H$.

Consider the 3-manifold $M = \Sigma \times [0,1]$ where Σ is a compact oriented surface. Provide M with the product orientation as usual. The product of two elements $L, L' \in \mathscr{L}(\Sigma \times [0,1])$ is defined by uniformly compressing L in $\Sigma \times [0,1/2]$, respectively L' in $\Sigma \times [1/2,1]$, and juxtaposing in $\Sigma \times [0,1] =$ $\Sigma \times ([0,1/2] \cup [1/2,1])$. The result $L \cdot L'$ is clearly an oriented framed coloured link in $\Sigma \times [0,1]$.

DEFINITION 9.5. The product

$$(1.4) (L,L') \mapsto L \cdot L'$$

induces a product on $\mathscr{A}(\Sigma \times [0,1])$, called the *skein product*.

The skein product turns $\mathscr{A}^{o}(\Sigma \times [0, 1])$ into an associative algebra with the empty skein $\langle \varnothing \rangle$ being the unit. By definition, $\langle L \rangle^{0} = \langle \varnothing \rangle = 1$.

Note. We write down product identities from left to right in accordance with our definition above of the skein product in $\Sigma \times [0, 1]$. (This convention will lead to the Schrödinger representation as a right action.) For instance, the figure opposite represents the product link a m b inside $\Sigma \times [0, 1]$. (For simplicity, the drawing of $\Sigma \times 0$ is omitted.)

Convention. We use the right hand rule convention for the orientation of handlebodies (that are embedded in \mathbb{R}^3), with the oriented normal (last) vector pointing towards the eye of the reader, as depicted opposite. All pictures are drawn following that convention.



An oriented genus two handlebody.

EXAMPLE 9.5. Consider the skein $\langle L \rangle$ in the cylinder $T \times [0,1]$ over the torus T^2 represented by the two component oriented framed coloured link L as pictured in Fig. 1.2. Denote as usual by m the meridian (here coloured by $g \in G$) and by l (here coloured by h) the longitude of $T = \partial S^1 \times D^2$. Then Fig. 1.2 shows that the following relations hold in the Heisenberg skein algebra $\mathscr{A}^{\circ}(T \times [0,1])$:

$$\langle L \rangle = \langle l(h) \rangle \cdot \langle m(g) \rangle = t_{gh} \langle m(g) \rangle \cdot \langle l(h) \rangle.$$

This example shows that $\mathscr{A}(T \times [0, 1])$ is not commutative.



FIGURE 1.2. A skein relation in $T \times [0, 1]$.

EXAMPLE 9.6. Consider the skein $\langle L \rangle$ in the cylinder $T \times [0, 1]$ over the torus T represented by the two component oriented framed coloured link L as pictured in the left side of Fig. 1.3. This is the same oriented framed link as in the previous example except that the two components are coloured by the same element $g \in G$. An application of the second skein relation is pictured in Fig. 1.3. Observe that the integral homology of the link in



FIGURE 1.3. Another skein relation in $T \times [0, 1]$.

 $T \times [0,1]$ is unchanged after the skein relation.

REMARK 9.1. If a link $L \subset \Sigma \times [0, 1]$ has a projection on Σ without crossings then its *n*-th power $\langle L \rangle^n$ is represented by *n* parallel copies of *L*. Furthermore, in this case, the $\langle L \rangle$ has an inverse $\langle -L \rangle$ where -L denotes *L* with the reversed orientation (this follows from Exercise 9.1 and our convention $\langle \varnothing \rangle = 1$).

However, if L has no projection on Σ without crossings then $\langle L \rangle^n$ is not represented in general by n parallel copies of L.

2. A skein multivariable polynomial

We now proceed to generalize Examples 9.4 and 9.5.

PROPOSITION 2.1. Let Σ be a compact oriented connected surface such that $\partial \Sigma$ consists of at most one component. Let Λ be the Hopf Lagrangian in

 $H_1(\partial(\Sigma \times [0,1]))$. Let L be a framed oriented coloured link in $\Sigma \times [0,1]$. There is an oriented framed coloured link $L_0 \subseteq \Sigma \times \{1/2\}$, unique up to isotopy, such that

(2.1)
$$\langle L \rangle = \prod_{\substack{\text{pairs}\\j,\ell \in \pi_0(L)\\j \neq \ell}} t_{\operatorname{col}(j),\operatorname{col}(\ell)}^{\operatorname{lk}_{\Lambda}(j,\ell)} \cdot \prod_{\ell \in \pi_0(L)} t_{\operatorname{col}(\ell)}^{\operatorname{lk}_{\Lambda}(\ell,\ell')} \cdot \langle L_0 \rangle \in \mathscr{A}^{\mathrm{o}}(\Sigma \times [0,1])$$

with the following properties:

- (1) The oriented links L and L_0 represent the same homology class: $[L] = [L_0]$ in $H_1(\Sigma)$.
- (2) The components of the oriented framed link L_0 consist of parallel copies of standard (trivially framed) meridians and longitudes in $\Sigma \times [0,1]$ (with possibly reversed orientation). Furthermore, all parallel components of L_0 have the same orientation.

PROOF. Consider a component ℓ of L. Regard $\Sigma \times [0,1]$ as the connected sum of g copies of $T \times [0,1]$ where T is the usual torus, possibly with one single disc removed. It will be convenient to consider a link diagram, using the projection of the link on $\Sigma \times 1/2$. Using isotopy (second Reidemeister move on the diagram) and the second skein relation, we see that ℓ is skein equivalent to a (possibly multi-component) link $\tilde{\ell}$ whose components lie individually in no more than one $T \times [0,1]$. See below for an elementary example.



FIGURE 2.1. Simplifying a link using skein moves.

Using the third skein relation we disregard the individual trivial components (like the one in the right side of Fig. 2.1). We are left with a new link L with non trivial components.

Using an isotopy if necessary, we may assume that the projection of the components onto $\Sigma \times \{1\}$ is generic. By the previous step, we may assume that the projection of each component lies either on a torus minus a disc or on a torus minus two disjoint discs. Furthermore, by using the skein relations,

we may assume that they have no crossing point in $\Sigma \times \{1\}$. Decompose each torus minus a disc into the connected sum of a pair of pants (denoted P) and a cylinder (denoted C). Similarly decompose each torus minus two discs into a connected sum of two pairs of pants.

LEMMA 2.2. The following skein relations hold:



The first relation is understood to hold in $P \times I$ and the second and third relations are understood to hold in $C \times I$.

PROOF OF LEMMA 2.2. The first relation from left to right is obtained as follows. First pull down from the curve a small band under the \uparrow , bring it back up on the other side, so as to create two crossing points with opposite sign (second Reidemeister move). Apply the second skein relation at each crossing. This creates a trivial component, which we discard using the third skein relation, and the two components depicted in the figure. The second relation is an application of the second skein relation. The third relation from left to right is obtained by creating two crossing points with opposite sign (second Reidemeister move) and applying the second skein relation at each crossing. Discarding the trivial component created, we obtain the component depicted on the right.

As a consequence, we can remove components that are parallel to the boundary component.

COROLLARY 2.3. The following relation holds in $(T - B^2) \times I$:



PROOF. Apply the first relation of Lemma 2.2 and Exercise 9.1.

It follows that one can transform the original link L into a new link L_0 whose components are all non trivial and pairwise disjoint, non parallel to the boundary component, such that each one of them is isotopic to a

trivially framed meridian or longitude in $\Sigma \times \{1/2\}$. By Exercise 9.1, all parallel components have the same orientation. Hence L_0 satisfies condition (2).

Let us compute the polynomial coefficient of the reduced link L_0 . Each skein relation of the first type (resp. of the second type) applied to a crossing point $c \in \Sigma$ between two distinct components j and l (resp. between two arcs belonging to the same component l) contributes to the polynomial by a factor $t_{col(j),col(l)}^{\varepsilon(c)}$ (resp. by a factor $t_{col(l)}^{\varepsilon(c)}$). (The sign $\varepsilon(c) \in \{-1,1\}$ is +1 or -1according to whether the orientation of $T_c\Sigma$ coincides or not with $T_c v \oplus T_c w$ where v is the unit tangent vector at c of the outgoing undercrossing arc and w is the unit tangent vector at c of the outgoing overcrossing arc.) The skein relation of the third type does not change the linking nor the framing numbers of the remaining components. The skein relations and isotopy do not change the homology class, hence condition (1) is satisfied.

Finally, the procedure produces a reduced link L_0 that satisfies (1) and (2). Let L_1 be another oriented framed coloured link determined by the same procedure that satisfies (1) and (2). By (2), L_1 consists of a union of parallel copies of meridians m_i and/or longitudes l_j , $1 \leq i, j \leq g$. Let a_i (resp. b_j) be the number of parallel copies of the *i*-th meridian (resp. *j*-th longitude), with minus sign if the orientation is reversed. Then by (1), $\sum_{1 \leq i,j \leq g} (a_i [m_i] + b_j [l_j]) = [L_1] = [L] = [L_0]$. It follows that L_0 has the same decomposition into parallel copies of oriented meridians and longitudes. Thus L_0 and L_1 are isotopic.

Observe that the set of colors of the components of L_0 (resp. L_1) is a subset of the set of colors of L.

Consider a finite sequence of isotopies and skein relations that leads from the original link to the reduced link. Consider two non trivial components j and l of the original link such that $lk_{\Lambda}(j,l) \neq 0$. Since finally the linking number between the two components is zero, we conclude that that the coefficient after the last step is $t_{col(j),col(l)}^{lk_{\Lambda}(j,l)}$. A similar observation shows that a single non trivial component j contributes exactly $t_{lk_{\Lambda}(\ell,\ell')}^{col(\ell)}$. Hence the expression of the polynomial P_L is as stated.

COROLLARY 2.4. The Laurent polynomial P_L associated to a link L is an invariant of the skein $\langle L \rangle$. In particular, it is an invariant of framed isotopy of L.

DEFINITION 9.6. The Laurent polynomial

(2.2)
$$P_L = \prod_{\substack{\text{pairs}\\ j, \ell \in \pi_0(L)\\ j \neq \ell}} t_{\operatorname{col}(j), \operatorname{col}(\ell)}^{\operatorname{lk}_\Lambda(j,\ell)} \cdot \prod_{\ell \in \pi_0(L)} t_{\operatorname{col}(\ell)}^{\operatorname{lk}_\Lambda(\ell,\ell')} \in \mathbb{Z}[\check{G}^{\operatorname{quad}}]$$

associated to the framed oriented coloured link L is called the *linking number* skein polynomial of L.



EXAMPLE 9.7. For the skein $\langle L \rangle$ of Example 9.5 (see Fig. 1.1), we have $P_L = t_{g,h}$.

COROLLARY 2.5. If L is a framed oriented coloured link in B^3 or S^3 then

$$\langle L \rangle = \prod_{\substack{\text{pairs}\\ j, \ell \in \pi_0(L)\\ j \neq \ell}} t_{\operatorname{col}(j), \operatorname{col}(\ell)}^{\operatorname{lk}_\Lambda(j, \ell)} \cdot \prod_{\ell \in \pi_0(L)} t_{\operatorname{col}(\ell)}^{\operatorname{lk}_\Lambda(\ell, \ell')}.$$

In particular the skein algebras $\mathscr{A}^{o}(B^{3})$ and $\mathscr{A}^{o}(S^{3})$ are both isomorphic to $\mathbb{Z}[\tilde{G}^{quad}].$

PROOF. Given a link L in S^3 , we can assume that L misses some small ball inside S^3 and therefore lies in a 3-ball B^3 . The 3-ball B^3 is diffeomorphic to $B^2 \times [0,1]$. Thus $\mathscr{A}^{\circ}(B^3) \simeq \mathscr{A}^{\circ}(B^2 \times [0,1])$ (Lemma 1.3). We apply Proposition 2.1 to a disc $\Sigma = B^2$, so $\langle L_0 \rangle = \langle \varnothing \rangle = 1$.

We record the behaviour of the linking number skein polynomial under the skein product.

LEMMA 2.6. Let J and L be two oriented framed coloured links in $\Sigma \times [0, 1]$. Then

$$P_{J\cdot L} = \prod_{\substack{\text{pairs}\{j,\ell\}\\(j,\ell)\in\pi_0(J)\times\pi_0(L)}} t_{\operatorname{col}(j),\operatorname{col}(l)}^{\operatorname{lk}_\Lambda(j,\ell)} P_J P_L.$$

PROOF. There are three Hopf Lagrangians: let Λ_0 (resp. Λ_1) be associated to the Hopf gluing applied to the first copy of $\Sigma \times [0, 1]$ which contains J (resp. applied to the second copy of $\Sigma \times [0, 1]$ which contains L). Finally let Λ be associated to the Hopf gluing applied to $\Sigma \times [0, 1] \simeq \Sigma \times [0, 1] \cup \Sigma \times [0, 1]$. Abusing notations and denoting by the same letter a component in possibly three distinct manifolds, we have $lk_{\Lambda_0}(j, \ell) = lk_{\Lambda}(j, \ell)$ for any pair j, ℓ of components of J. (if $j = \ell$, the framing number $lk_{\Lambda}(\ell, \ell')$ is meant.) Similarly, $lk_{\Lambda_1}(j, \ell) = lk_{\Lambda}(j, \ell)$ for any pair j, ℓ of components of L. Since $\pi_0(JL)$ is the disjoint union of $\pi_0(J)$ and $\pi_0(L)$, we compute by means of Lemma 2.1, the skein polynomial P_{JL} and the product of skein polynomials $P_J P_L$. Comparing the two yields the desired formula.

COROLLARY 2.7. For any oriented framed coloured link in $\Sigma \times [0, 1]$,

$$P_{-L} = P_L$$

where -L denotes the same coloured link with reversed orientation of all of its components.

PROOF. Colors are unchanged and orientation reversal of all components of the link does not affect linking and framing numbers. Hence the skein polynomial is unchanged.

COROLLARY 2.8. For any oriented framed coloured link in $\Sigma \times [0, 1]$,

$$P_{L^{\min}} = P_L^{-1}$$

where L^{\min} denoted the mirror image of the link L with the same colors.

PROOF. Follows from the fact that for the mirror link, all linking and framing numbers are the opposite of those of the original link.

3. Heisenberg skein modules and algebras

We fix a closed oriented surface Σ of genus g with its geometric symplectic basis $(m_1, l_1, \ldots, m_q, l_q)$ so that

$$m_i \bullet m_j = l_i \bullet l_j = 0, \quad m_i \bullet l_j = \delta_{ij}, \quad 1 \leq i, j \leq g$$

with trivial framing on each simple closed curve.

LEMMA 3.1. The Heisenberg skein algebra $\mathscr{A}^{o}(\Sigma \times [0,1])$ is a free $\mathbb{Z}[\check{G}^{quad}]$ module whose basis consists of all elements of the form

(3.1)
$$\prod_{j=1}^{g} \prod_{l=1}^{r_j} \langle \varepsilon(j) m_j(x_{j,l}) \rangle \cdot \prod_{j=1}^{g} \prod_{l=1}^{s_j} \langle \epsilon(j) l_j(y_{j,l}) \rangle$$

where the indices (numbers of parallel copies) $r_1, \ldots, r_g, s_1, \ldots, s_g$ lie in \mathbb{N} , the indices $\varepsilon(j), \epsilon(j)$ lie in $\{\pm 1\}$ (the minus sign meaning reversed orientation) and the colors $x_{j,l}$ and $y_{j,l}$ lie in G.

The Heisenberg skein premodule $\mathscr{A}^{o}(H)$ of a genus g oriented handlebody H is a free $\mathbb{Z}[\check{G}^{quad}]$ -module with basis

$$\prod_{j=1}^{g}\prod_{l=1}^{s_j} \langle \epsilon(j)l_j(y_{j,l})\rangle,$$

where each longitude with possible reversed orientation is coloured with an arbitrary element of G.

PROOF. The first statement follows from Prop. 2.1. For the second statement, view the solid handlebody H as containing the cylinder over the closed oriented surface Σ so that ∂H identifies with one of the bases, say $\Sigma \times 0$, of the cylinder over Σ . Let $\langle L \rangle$ be a skein in $\mathscr{A}^{\circ}(H)$. By isotopying L is necessary, we may assume that L lies in $\Sigma \times [0,1] \subset H$. By the previous argument, $\langle L \rangle$ is proportional to an element of the form (3.6). Now in $\mathscr{A}^{\circ}(H)$, each trivially framed meridian becomes a trivial knot so $\langle m \rangle = \langle \varnothing \rangle$. This gives the desired result.

REMARK 9.2. There is a slight abuse of notation in the second statement of Prop. 3.1. Indeed, $\mathscr{A}^{o}(H)$ has not been given yet any natural product structure. What the second statement really means is that any skein in Hcan be geometrically represented by a disjoint union of parallel longitudes (with the standard orientation) arbitrarily coloured.

Let L be any coloured oriented framed link (possibly empty) and let K be an oriented framed knot in M such that $L \cap K = \emptyset$. The orientation and the framing of K determine an oriented knot K'. Extend the framing of K to a framing for K'. Let $g, h \in G$ be arbitrary colors for K and K' respectively. Consider the element

(3.2)
$$\langle L \cup K(g) \cup K'(h) \rangle - \langle L \cup K(g+h) \rangle$$

in $\mathscr{A}^{\mathrm{o}}(M)$.

Consider similarly the element

$$(3.3) \qquad \langle L \cup K(0) \rangle - \langle L \rangle$$

in $\mathscr{A}^{\mathrm{o}}(M)$.

DEFINITION 9.7. The Heisenberg skein module $\mathscr{A}(M)$ is the quotient of $\mathscr{A}^{o}(M)$ by the submodule generated by all elements (3.2), (3.3).

PROPOSITION 3.2. The map that assigns to a skein $\langle L \rangle \in \widehat{\mathscr{A}}(M)$ its homology class

$$[L] = \sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell] \in H_1(M; G)$$

induces by \mathbb{Z} -linear extension a natural \mathbb{Z} -linear epimorphism

$$\mathscr{A}(M) \to \mathbb{Z}[H_1(M;G)].$$

PROOF. Assigning to an isotopy class of a framed oriented coloured link L in M its homology class

(3.4)
$$[L] = \sum_{\ell \in \pi_0(M)} \operatorname{col}(\ell) \otimes [\ell] \in H_1(M; G)$$

determines a well-defined Z-linear epimorphism

$$\mathscr{L}(M) \to \mathbb{Z}[H_1(M;G)].$$

It is not hard to verify that the skein relations do not change the 1-homology of L with coefficients in G. The first skein relation performed on L does not affect the number of components and both the colours and the integral 1homology of the components are unchanged. Hence the first skein relation leaves invariant $[L] \in H_1(M; G)$. The second skein relation performed on (a crossing of a regular projection of) L does affect the number of components: it splits one component into two components if there is initially one component self-crossing and it merges two components into one component if there are initially two distinct components (with the same colour) crossing. The algebraic sum of the integral 1-homology of the two components after the skein relation is performed (resp. before the skein relation is performed) is equal to the integral 1-homology of the single component before the skein relation is performed (resp. after the skein relation is performed). Since the colour remains constant, we conclude that the quantity $[L] = \sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell]$ is unchanged under the second skein relation. The invariance under the third skein relation is clear since we only add a trivial (hence homologically trivial) component.

Consider now the extra relations. For the element described by (3.2), its 1-homology class is

$$\sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell] + g \otimes [K] + h \otimes [K'] - \sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell] - (g \otimes h) \otimes [K] = 0$$

since [K] = [K']. Consider the element described by (3.3): its 1-homology class is

$$\sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell] - 0 \otimes [K] - \sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell] = 0.$$

Therefore the natural map $\mathscr{A}^0(M) \to \mathbb{Z}[H_1(M;G)]$ induces a \mathbb{Z} -linear epimorphism $\mathscr{A}(M) \to \mathbb{Z}[H_1(M;G)]$.

PROPOSITION 3.3. In $\mathscr{A}(M)$, the following relations hold for arbitrary elements $g, h, k \in G$:

$$t_{g,h} t_{g,k} = t_{g,h+k}, \ t_{g,h} t_{k,h} = t_{g+k,h}, \ t_g t_{g,h} t_h = t_{g+h}, \ t_{-g,h} = t_{g,h}^{-1}, \ t_{-g} = t_g.$$

PROOF. Consider the following skein identity:

$$t_{g,h} t_{g,k} \bigvee_{g}^{h+k} t_{g,k} \bigvee_{g}^{h+k} = t_{g,h} \downarrow_{g}^{h+k} \downarrow_{g}^{h+k} = \downarrow_{g}^{h+k} \downarrow_{g}^{h+k} = \downarrow_{g}^{h+k} \downarrow_{g}^{h+k} = \downarrow_{g}^{h+k} \downarrow_{g}^{h+k}$$

The identity $t_{g,h}t_{g,k} = t_{g,h+k}$ follows. The second identity follows since $t_{g,h} = t_{h,g}$. For the identity $t_{g+h} = t_g t_h t_{g,h}$, we have on the one hand the identity



On the other hand, we also have the skein identity

$$g + h \qquad g +$$

Comparing the two identities yields the relation $t_{g+h} = t_g t_h t_{g,h}$. Now

$$1 = \langle \varnothing \rangle = \bigcup_{g} = \bigcup_{g} = t_{g,0} \bigcup_{g} = t_{g,0} \bigcup_{g} = t_{g,0} \bigcup_{g} = t_{g,0} \langle \varnothing \rangle = t_{g,0} \langle \varnothing \rangle = t_{g,0} \langle \vartheta \rangle =$$

Thus $t_{g,0} = t_{0,g} = 1$. It follows that $1 = t_{g,0} = t_{g,-h}t_{g,h}$ so $t_{g,-h} = t_{g,h}^{-1}$. Similarly $t_{-g,h} = t_{g,h}^{-1}$. Since $t_0^2 = t_{0,0} = 1$, we see that $t_0 = \pm 1$. In fact,

$$1 = \langle \varnothing \rangle = \left\langle \bigcirc^{\circ} \bigcirc \right\rangle = t_0 \left\langle \bigcirc^{\circ} \bigcirc \right\rangle = t_0 \left\langle \varnothing \right\rangle = t_0.$$

Thus

$$1 = t_0 = t_{g-g} = t_g t_{-g} t_{g,-g} = t_g t_{-g} t_{g,g}^{-1} = t_g t_{-g} t_g^{-2} = t_g^{-1} t_{-g}$$

hence $t_{-g} = t_g$.

The following result is a direct consequence of Prop. 3.3 and the definition of G^{quad} (Chap. 1, §5).

COROLLARY 3.4. $\mathscr{A}(M)$ is a $\mathbb{Z}[G^{\text{quad}}]$ -module.

For instance, if $G = \mathbb{Z}$ then G^{quad} is generated by t_1 with no extra relation. Hence $G^{\text{quad}} = \mathbb{Z} = G$ and $\mathscr{A}(M)$ is a $\mathbb{Z}[t^{\pm 1}]$ -module.

From Lemma 3.1, we can describe further the structure of the Heisenberg algebra $\mathscr{A}(\Sigma \times [0,1])$.

PROPOSITION 3.5. The Heisenberg skein algebra $\mathscr{A}(\Sigma \times [0,1])$ is a free $\mathbb{Z}[G^{\text{quad}}]$ -module whose basis consists of all elements of the form

(3.6)
$$\prod_{j=1}^{g} \langle l_j(y_j) \rangle \cdot \prod_{j=1}^{g} \langle m_j(x_j) \rangle$$

where the colors x_i and y_i lie in G.

The Heisenberg skein module $\mathscr{A}(H_g)$ of the genus g oriented handlebody is a free $\mathbb{Z}[G^{\text{quad}}]$ -module whose basis consists of all possible disjoint unions of longitudes arbitrarily coloured by elements in G.

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic form with associated bilinear pairing $b: G \times G \to \mathbb{Q}/\mathbb{Z}$. Let $\chi : \mathbb{Q}/\mathbb{Z} \to \mathrm{U}(1)$ be a character.

DEFINITION 9.8. The group $U_{q,\chi} = \chi \circ q(G)$ is a finite subgroup of U(1): it is the *finite unitary group* associated to (q,χ) . We shall write U if the quadratic form and the character are understood from the context.

Recall that to the quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ is associated a group homomorphism $\tilde{q}: G^{\text{quad}} \to \mathbb{Q}/\mathbb{Z}$ (Chap. 1, §5).

DEFINITION 9.9. The evaluation map $ev_{q,\chi}: G^{quad} \to U$ is defined by

We shall write ev if the quadratic form and the character are understood from the context.

Here is an explicit formula for ev, given an element $P \in G^{\text{quad}}$ (represented as an element in \tilde{G}^{quad} , i.e. as a monic monomial):

$$ev(P) = P(t_g = \chi(q(g)), t_{g,h} = \chi(b(g,h)), \ g, h \in G).$$

LEMMA 3.6. The evaluation map ev is a group homomorphism and $U = ev(G^{quad})$.

PROOF. The two statements follow from the definition and universal property of \tilde{q} .

By the group ring construction, the map ev induces a ring morphism, still denoted ev: $\mathbb{Z}[G^{\text{quad}}] \to \mathbb{Z}[U]$. Regarding $\mathbb{Z}[U]$ as a $\mathbb{Z}[G^{\text{quad}}]$ -algebra, we define the $\mathbb{Z}[U]$ -module by the "ring change"

$$\mathscr{A}_{q,\chi}(M) = \mathscr{A}(M) \otimes_{\mathbb{Z}[G^{\text{quad}}]} \mathbb{Z}[U].$$

DEFINITION 9.10. The $\mathbb{Z}[U]$ -module $\widehat{\mathscr{A}}_{q,\chi}(M)$ is called the *reduced Heisenberg skein module*. To lighten notation, we shall simply drop the subscripts and simply write $\widehat{\mathscr{A}}(M)$ when the quadratic form q and the character χ are understood.

REMARK 9.3. There is a natural \mathbb{Z} -map $\mathscr{A}(M) \to \widehat{\mathscr{A}}(M)$ defined by $\langle L \rangle \mapsto \langle L \rangle \otimes 1$ where 1 is the unit element in U.

An immediate consequence of the definition is that the reduced Heisenberg skein module of a surface is also an algebra.

LEMMA 3.7. $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ is a $\mathbb{Z}[U]$ -algebra and the map

 $e:\mathscr{A}(\Sigma\times[0,1])\to\widehat{\mathscr{A}}(\Sigma\times[0,1]),\ \langle L\rangle\mapsto\langle L\rangle\otimes 1_U$

is a ring surjective map that satisfies

$$e(P \cdot \langle L \rangle) = ev(P) \cdot e(\langle L \rangle), \quad P \in G^{quad}.$$

REMARK 9.4. One may concretely regard the reduced Heisenberg skein module as the Heisenberg skein module obtained after "evaluation", i.e. after "evaluating" the indeterminate $t = \chi \circ \tilde{q}(t)$ for all $t \in G^{\text{quad}}$. This evaluation is equivalent to the evaluation

(3.8)
$$t_g = \chi(q(g)), \quad t_{g,h} = \chi(b(g,h)), \quad g,h \in G.$$

In other words, in the reduced skein module, (1) we replace two parallel oriented framed knots by one of them and add their original colors; (2) we evaluate $t_g = \chi(q(g))$ and $t_{g,h} = \chi(b_q(g,h))$; (3) Any framed oriented knot coloured with 0 can be erased. The first relation allows in particular to replace *n* parallel copies of an oriented framed link by one copy of them coloured with *n* times the original color and conversely for any $n \in \mathbb{N}$.

For the following definition, consider the natural ring map $\mathbb{Z}[U] \to \mathbb{C}$ that sends a formal \mathbb{Z} -linear combination of unitary elements to its actual complex value. Using this map, one may regard \mathbb{C} as a $\mathbb{Z}[U]$ -algebra.

DEFINITION 9.11. The reduced Heisenberg skein module with complex coefficients is $\widehat{\mathscr{A}}_{\mathbb{C}}(M) = \widehat{\mathscr{A}}(M) \otimes_{\mathbb{Z}[U]} \mathbb{C}.$

REMARK 9.5. The reduced Heisenberg skein module $\widehat{\mathscr{A}}_{\mathbb{C}}(M)$ is a \mathbb{C} -vector space. The reduced Heisenberg skein module $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1])$ is a \mathbb{C} -algebra.

PROPOSITION 3.8. Let L be an oriented framed coloured link in $\Sigma \times [0, 1]$. Let $\hat{\theta}_L = \sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes \ell$ the framed 1-cycle determined by L in $\Sigma \times [0, 1]$. Denote by $[L] = \theta_L$ its homology class in $H_1(\Sigma; G)$. Then

(3.9)
$$\langle L \rangle = \chi((q \otimes \operatorname{lk}_{\Lambda})(\widehat{\theta}_{L})) \cdot \langle L_{0} \rangle \text{ in } \mathscr{A}(\Sigma \times [0,1])$$

where L_0 is an oriented framed coloured link such that

(1) The oriented links L and L_0 represent the same homology class: $[L] = [L_0]$ in $H_1(\Sigma; G)$.

(2) The components of the oriented framed link L₀ consist of parallel copies of standard (trivially framed) meridians and longitudes in Σ × {1/2} (with possibly reversed orientation). In particular, for any pair of components l, l' of L₀, lk_Λ(l, l') = 0. Furthermore, all parallel components (meridian or longitude) of L₀ have the same orientation.

PROOF. We apply Prop. 2.1 to L in $\mathscr{A}(\Sigma \times [0,1])$. Evaluating the skein polynomial P_L (2.2) at $(t_g = \chi(q(g))$ and $t_{gh} = \chi(b(g,h))$, we observe that

$$P_L\left(t_g = \chi(q(g)), t_{gh} = \chi(b(g, h)), \ g, h \in G\right) = \chi((q \otimes \operatorname{lk}_\Lambda)(\widehat{\theta}_L)).$$

This gives the desired formula.

Let H_g be an oriented handlebody of genus g, so that $\partial H_g = \Sigma_g$. Recall the geometric symplectic basis $(m_1, l_1, \ldots, m_g, l_g)$ for the surface Σ . Then (l_1, \ldots, l_g) is a geometric basis for the first homology of H_g .

THEOREM 3.9. The map defined by

 $\langle L \rangle \mapsto [L]$

defines a $\mathbb{Z}[U]$ -linear isomorphism $\widehat{\mathscr{A}}(\Sigma \times [0,1]) \to \mathbb{Z}[U][H_1(\Sigma;G)]$. In particular, the Heisenberg skein algebra $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ is a free $\mathbb{Z}[U]$ -module with basis

$$\langle l_1 \rangle \cdots \langle l_g \rangle \cdots \langle m_1 \rangle \cdots \langle m_g \rangle,$$

where the geometric elements of the symplectic basis are coloured with arbitrary elements of G.

Let Λ_1 be the Lagrangian in $H_1(\Sigma_g)$ generated by the longitudes. The map defined by

$$\langle L \rangle \mapsto [L]$$

defines a $\mathbb{Z}[U]$ -linear isomorphism $\widehat{\mathscr{A}}(H_g) \to \mathbb{Z}[U][G \otimes \Lambda_1]$. In particular, the Heisenberg skein module $\widehat{\mathscr{A}}(H_g)$ of the genus g oriented handlebody is a free $\mathbb{Z}[U]$ -module whose basis elements consist of disjoint unions of longitudes coloured with arbitrary elements of G.

PROOF. Follows from Prop. 3.5.

In other words, an element of the basis of $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ is an arbitrary product of the skeins of the symplectic basis. Similarly, an element of the basis of $\widehat{\mathscr{A}}(H_g)$ is an arbitrary product of the skeins of the basis of the first homology of H_g . Whether a given skein appears in the product is determined by the color.

COROLLARY 3.10. Let g denote the genus of Σ . Then rank $\widehat{\mathscr{A}}(\Sigma \times [0,1]) = |G|^{2g}$ and rank $\widehat{\mathscr{A}}(H_q) = |G|^{g}$.

EXAMPLE 9.8. $\widehat{\mathscr{A}}(B^3) = \widehat{\mathscr{A}}(S^3) = \mathbb{Z}[U].$

COROLLARY 3.11. The map $\langle L \rangle \mapsto [L]$ induces \mathbb{C} -linear isomorphisms

$$\mathscr{A}_{\mathbb{C}}(\Sigma \times [0,1]) \simeq \mathbb{C}[H_1(\Sigma;G)], \quad \mathscr{A}_{\mathbb{C}}(H_q) \simeq \mathbb{C}[G \otimes \Lambda_1].$$

COROLLARY 3.12. $\widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1])$ is a \mathbb{C} -algebra of dimension $|G|^{2g}$ and $\widehat{\mathscr{A}_{\mathbb{C}}}(H_g)$ is a vector space over \mathbb{C} of dimension $|G|^g$.

DEFINITION 9.12. The product structure on $\widehat{\mathscr{A}}(H_g)$ is defined as the one induced by the product in $\mathbb{Z}[U][G \otimes \Lambda_1]$ on the generators given in Th. 3.9.

LEMMA 3.13. The product structure on $\widehat{\mathscr{A}}(H_g)$ turns $\widehat{\mathscr{A}}(H_g)$ into an algebra isomorphic to $\mathbb{Z}[U][G \otimes \Lambda_1]$. In particular, the reduced skein algebra $\widehat{\mathscr{A}}(H)$ on the genus g handlebody is commutative.

The product has a simple geometric meaning.

PROPOSITION 3.14. Let L and L' be two oriented framed links in H_g . If L and L' are topologically disjoint in H_g or if L' is parallel to L then

(3.10)
$$\langle L \cup L' \rangle = \langle L \rangle \cdot \langle L' \rangle.$$

In particular, this explains our previous (abuse of) notation (see Remark 9.2) for $\langle l_1 \rangle \cdots \langle l_g \rangle = \langle l_1 \cup \cdots \cup l_g \rangle$.

PROOF. According to Prop. 3.8, any link is skein equivalent (up to an element in U) to a disjoint union of oriented trivially framed coloured longitudes. Therefore it suffices to verify the identity for an oriented framed coloured link that is a disjoint union of oriented framed coloured longitudes $l_1 \ldots, l_g$. Let $[l_i] \in H_1(H_g; G)$ denote the 1-homology class of the *i*-th longitude. Then

$$[L] = [l_1(\operatorname{col}(l_1)) \cup \cdots \cup l_g(\operatorname{col}(l_g))] = \operatorname{col}(l_1) \otimes [l_1] + \cdots + \operatorname{col}(l_g) \otimes [l_g].$$

This justifies our previous notation: $\langle L \rangle = \langle l_1(\operatorname{col}(l_1)) \rangle \cdots \langle l_g(\operatorname{col}(l_g)) \rangle = \langle l_1(\operatorname{col}(l_1)) \cup \cdots \cup l_g(\operatorname{col}(l_g)) \rangle$. If l_i and l'_i denote the same trivially framed oriented longitude l with different colors $x, y \in G$, then $\langle l_i \rangle \cdot \langle l'_i \rangle$ can be geometrically represented by the union of l_i and a parallel copy coloured with x and y respectively. This framed oriented link is then skein equivalent to l_i coloured with x + y. Hence

$$\langle l_i(x) \rangle \cdot \langle l'_i(y) \rangle = \langle l_i(x+y) \rangle.$$

Since for a longitude l' parallel to l (determined by the framing of l), $\langle l'(y) \rangle = \langle l(y) \rangle$, the definition of the reduced Heisenberg module implies that

$$\langle l(x) \rangle \cdot \langle l'(y) \rangle = \langle l(x+y) \rangle = \langle l(x) \cup l'(y) \rangle.$$

The result follows.

REMARK 9.6. If the links are not topologically disjoint and not parallel, then the formula (3.10) does not hold in general. For instance, consider the Hopf link $L \cup L'$ inside the handlebody of genus 2 depicted below (left). Denote by $x, y \in G$ the respective colors of the components. Then $\langle L \rangle \cdot \langle L' \rangle = \langle l_1 \rangle \cdot \langle l_2 \rangle = \langle l_1 \cup l_2 \rangle$. But

$$(\bigcirc) = \exp(2\pi i b_q(x,y)) \cdot (\bigcirc) .$$
Hence $\langle L \cup L' \rangle = \exp(2\pi i b_q(x, y)) \cdot \langle l_1 \cup l_2 \rangle$. So if $b_q(x, y) \neq 0 \pmod{1}$, then $\langle L \cup L' \rangle \neq \langle L \rangle \cdot \langle L' \rangle$.

COROLLARY 3.15. There is a \mathbb{C} -algebra isomorphism $\widehat{A}_{\mathbb{C}}(H_q) \simeq \mathbb{C}[G \otimes \Lambda_1]$.

EXERCISE 9.2. Does the linear isomorphism $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1]) \to \mathbb{C}[H_1(\Sigma;G))]$ of Th. 3.9 extend to an algebra isomorphism ? Show that the algebra structure on $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1])$ defined geometrically above is not compatible with this linear isomorphism.

4. Relation to the Heisenberg group algebras

Our goal is to compare the Heisenberg skein algebra to the group algebra of a suitable Heisenberg group. We begin by observing that the multiplicative structure on skeins in $\Sigma \times [0, 1]$ is preserved at the level of 1-homology.

LEMMA 4.1. The assignment

 $L \mapsto [L]$

defines a map from the set of of isotopy classes of oriented framed coloured links in $\Sigma \times [0,1]$ to the group $H_1(\Sigma; G)$ that satisfies

(4.1)
$$[L \cdot L'] = [L] + [L'].$$

PROOF. Since $[L] = \sum_{\ell \in \pi_0(L)} \operatorname{col}(\ell) \otimes [\ell]$, it suffices to observe that $\pi_0(L \cdot L')$ is the disjoint union of $\pi_0(L)$ and $\pi_0(L')$.

LEMMA 4.2. The map $L \mapsto [L]$ induces an algebra homomorphism $\mathscr{A}(\Sigma \times [0,1]) \to \mathbb{Z}[H_1(\Sigma;G)].$

PROOF. Extend $L \mapsto [L]$ linearly to a Z-map from the algebra freely generated over Z by isotopy classes of of oriented framed coloured links in $\Sigma \times [0,1]$ (with the product defined earlier) to $\mathbb{Z}[H_1(\Sigma;G)]$. Then we let G^{quad} act trivially on $\mathscr{L}(\Sigma \times [0,1])$ and extend to $\mathscr{L}(\Sigma \times [0,1])$ by setting

$$[P \cdot L] = [L]$$

for any $P \in G^{\text{quad}}$ and any framed oriented coloured link L in $\Sigma \times [0, 1]$. This induces a \mathbb{Z} -map $\mathscr{A}(\Sigma \times [0, 1]) \to \mathbb{Z}[H_1(\Sigma; G)]$ such that $[P \langle L \rangle] = [L]$, for any $P \in G^{\text{quad}}$ and any skein $\langle L \rangle$. It is readily checked that this is an algebra homomorphism.

Our goal is to define Heisenberg groups associated to $H_1(\Sigma; G)$. First recall the Hopf Seifert form defined on $H_1(\Sigma)$: we associate to $H_1(\partial(\Sigma \times [0, 1]))$ the Hopf Lagrangian Λ that is generated by the meridians of $\Sigma \times 0$ and a maximal independent set of longitudes of $\Sigma \times 1$. The Hopf Seifert form is defined by $\beta_{\Lambda}([\ell], [\ell']) = \operatorname{lk}_{\Lambda}(i_*^-\ell_-, i_*^+\ell'_+)$ and induces the integral intersection pairing. (See Chap. 7, §1.4.) Secondly, we define a bilinear pairing as follows. Let $x, y \in H_1(\Sigma; G)$. Lift x, y to disjoint G-colored links $L, L' \subset \Sigma$. Set

(4.2)
$$c_{\Lambda}(x,y) = \prod_{\substack{\text{pairs } \{\ell,\ell'\}\\(\ell,\ell') \in \pi_0(L) \times \pi_0(L')}} t_{\operatorname{col}(\ell),\operatorname{col}(\ell')}^{-\beta_{\Lambda}([\ell],[\ell'])} \in G^{\operatorname{quad}}$$

where $\pi_0(L)$ denotes the set of components of L.

LEMMA 4.3. The formula (4.2) defines a Seifert form

 $c_{\Lambda}: H_1(\Sigma; G) \times H_1(\Sigma; G) \to G^{\text{quad}}$

associated to the symplectic form ω defined by

$$\omega(x,y) = \prod_{\substack{\text{pairs } \{\ell,\ell'\}\\(\ell,\ell')\in\pi_0(L)\times\pi_0(L')}} t_{\operatorname{col}(\ell),\operatorname{col}(\ell')}^{[\ell]\bullet[\ell']},$$

where the integral homology classes $[\ell]$ and $[\ell']$ verify

$$x = \sum_{\ell} \operatorname{col}(\ell) \otimes [\ell], \quad y = \sum_{\ell'} \operatorname{col}(\ell) \otimes [\ell'].$$

PROOF. independent of the various choices. (TBC)

DEFINITION 9.13. The unreduced Heisenberg group associated to $H_1(\Sigma; G)$ is the group $\mathscr{H}(H_1(\Sigma; G)) = H_1(\Sigma; G) \times G^{\text{quad}}$ endowed with the product

$$(x, P) \cdot (y, Q) = (x + y, c_{\Lambda}(x, y) P Q).$$

One verifies as in Chap. 6, §1, that $\mathscr{H}(H_1(\Sigma; G))$ is indeed a group.

REMARK 9.7. For G finite, the Heisenberg group $\mathscr{H}(H_1(\Sigma;G))$ is finite.

There is a well-defined map Φ from the set of isotopy classes of framed oriented coloured links in $\Sigma \times [0, 1]$ to $\mathscr{H}(H_1(\Sigma; G))$ induced by the formula

$$\Phi(L) = ([L], P_L), \quad L \in \mathscr{L}(\Sigma \times [0, 1]),$$

where P_L is the skein polynomial defined in Prop. 2.1. The group G^{quad} acts on itself in the natural way. That action extends naturally to an action on $\mathscr{H}(H_1(\Sigma; G))$ by regarding it as the natural action of the center $0 \times G^{\text{quad}}$ by multiplication:

(4.3)
$$Q \cdot (x, P) = (0, Q) \cdot (x, P) = (x, Q \cdot P), \quad P, Q \in G^{\text{quad}}, x \in H_1(\Sigma; G).$$

The map $Q \mapsto (0, Q)$ extends to a group ring map from $\mathbb{Z}[G^{\text{quad}}]$ into the center of $\mathbb{Z}[\mathscr{H}(H_1(\Sigma; G))]$. This turns $\mathbb{Z}[\mathscr{H}(H_1(\Sigma; G))]$ into a $\mathbb{Z}[G^{\text{quad}}]$ -algebra.

LEMMA 4.4. The map Φ induces a $\mathbb{Z}[G^{\text{quad}}]$ -algebra map:

$$\Phi: \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))].$$

PROOF. First, $\Phi : \mathscr{L}(\Sigma \times [0,1]) \to \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$ induces a map $\Phi : \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$ since $\Phi(L) = ([L], P_L)$ depends only on $\langle L \rangle$. Secondly, by the previous paragraph, $\Phi(P \langle L \rangle) = P \Phi(\langle L \rangle)$ for any $P \in \mathbb{Z}[G^{\text{quad}}]$. Thirdly, the induced map Φ is multiplicative since

$$\Phi(L \cdot L') = ([L \cdot L'], P_{L \cdot L'}) = ([L] + [L'], c_{\Lambda}(L, L') P_L P_{L'}) = \Phi(L) \Phi(L').$$

THEOREM 4.5. The map

 $\Phi: \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$

is a $\mathbb{Z}[G^{\text{quad}}]$ -algebra isomorphism.

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PROOF. We construct a map $\Psi : \mathbb{Z}[\mathscr{H}(H_1(\Sigma; G))] \to \mathscr{A}(\Sigma \times [0, 1])$ as follows. Let $(x, P) \in \mathscr{H}(H_1(\Sigma; G))$. The 1-homology class $x \in H_1(\Sigma; G) =$ $H_1(\Sigma) \otimes G$ can be lifted an oriented coloured link \tilde{L} in $\Sigma \times [0, 1]$ such that $x = [\tilde{L}]$ with the following prescriptions:

- \hat{L} consists of a disjoint union of unlinked and trivially framed standard oriented meridians (in $\Sigma \times 0$) and standard oriented longitudes (in $\Sigma \times 1$);
- all parallel components have the same orientation.

Consider now $P = \prod_{s \in S} t_s^{k_s} \in G^{\text{quad}}$. Lift P to a monomial in $\mathbb{Z}[S]$. We use the monomial P to modify $\tilde{L} \subset \Sigma \times [0,1]$ in such a way that the linking and framing numbers of the colored components match the corresponding partial degrees of P in the following fashion. First compare the colors of \tilde{L} and those of P. For each color $s \in G$ such that $k_s \neq 0$ and that does not appear in the list of colors of \tilde{L} , then we create an unknotted trivially framed component $\ell \subset \Sigma \times [0,1]$ with color s and topologically disjoint from the other components. Next, given two unlinked components ℓ and ℓ' of L with colors $\operatorname{col}(\ell)$ and $\operatorname{col}(\ell')$ respectively, we link them algebraically $k_{\mathrm{col}(\ell),\mathrm{col}(\ell')} = \deg_{\mathrm{col}(\ell),\mathrm{col}(\ell')} P$ times. Similarly, each one-component ℓ with color $\operatorname{col}(\ell)$ is now framed algebraically $k_{\operatorname{col}(\ell)} = \deg_{\operatorname{col}(\ell)} P$ times. Let L be the oriented framed coloured link thus obtained. We set $\Psi(x, P) = \langle L \rangle$. It is easy to see that the map Ψ is well-defined: indeed, the indeterminacy in the lift L lies solely in the possibility of parallel components and can be removed by requiring the lift L not to have any multiple parallel components. Using skein relations, it is clear that $P_L = P$. Furthermore, [L] = [L] = x. It follows that $\Psi = \Phi^{-1}$.

DEFINITION 9.14. The core of $\mathscr{A}(\Sigma \times [0,1])$ is

$$C\mathscr{A}(\Sigma \times [0,1]) = \{ P \langle L \rangle \mid P \in G^{\text{quad}}, L \in \mathcal{L}(\Sigma \times [0,1]) \}.$$

COROLLARY 4.6. The core $C\mathscr{A}(\Sigma \times [0,1])$ identifies to the Heisenberg group $\mathscr{H}(H_1(\Sigma;G))$. In particular, it has a group structure, its group algebra is $\mathscr{A}(\Sigma \times [0,1])$ and all elementary skeins in $\Sigma \times [0,1]$ are invertible.

PROOF. There is a commutative diagram

Hence Φ sends bijectively and multiplicatively $C\mathscr{A}(\Sigma \times [0,1]))$ to $\mathscr{H}(H_1(\Sigma;G))$, the Heisenberg group. The result follows.

REMARK 9.8. PLEASE CHECK THIS ! Using the isomorphism Φ , one can describe the inverse of an element $P \langle L \rangle \in C\mathscr{A}(\Sigma \times [0,1])$. The inverse of an element $(x, P) \in \mathscr{H}(H_1(\Sigma; G))$ is given by

$$(x, P)^{-1} = (-x, P^{-1}), \quad x \in H_1(\Sigma; G), \ P \in G^{\text{quad}}.$$

(Use the product law in $\mathscr{H}(H_1(\Sigma; G))$ and the fact that $c_{\Lambda}(-x, x) = c_{\Lambda}(x, x)^{-1} =$ 1. PLEASE CHECK THIS !) Using Φ^{-1} and Corollary 2.8, we deduce that

$$\left(P\left\langle L\right\rangle\right)^{-1} = P^{-1}\left\langle L^{\min}\right\rangle.$$

REMARK 9.9. The definition 9.14 suggests the following question. Is the core the full group of units (i.e., the group of all invertible elements) of $\mathscr{A}(\Sigma \times [0,1])$? Theorem 4.5 reduces this question to the question of determining the group of units of $\mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$. In the case when G has torsion, it can be answered negatively. (I do not know of a description of the group of units of $\mathscr{A}(\Sigma \times [0,1])$.) See Exercise 9.3. I do not know the answer for $G = \mathbb{Z}$. In this case, $\mathscr{H}(H_1(\Sigma))$ is torsion free and one expects that the group ring has no nontrivial units (i.e., the unit group coincides with $\pm \mathscr{H}(H_1(\Sigma;G))$). See [**50**].

EXERCISE 9.3. Let G be an abelian symplectic group endowed with a Seifert form. We assume that G has a nonzero torsion element. The purpose of the exercise is to show under this hypothesis, that the group of units of $\mathscr{A}(\Sigma \times [0,1])$ is strictly larger than the core $C\mathscr{A}(\Sigma \times [0,1])$.

1. Show that G^{quad} has a nonzero torsion element. [Hint: Use Corollary 5.15.]

2. Deduce that $\mathscr{H}(H_1(\Sigma; G))$ has a nonzero torsion element x.

3. Show that there exists an element $y \in \mathscr{H}(H_1(\Sigma; G))$ such that $y^{-1}xy \notin \langle x \rangle$.

4. Let *n* be the order of *x*. Show that the element $z = (x-1)y(1+x+x^2+\cdots+x^{n-1})$ is not zero in $\mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$.

5. Show that $z^2 = 0$.

6. Deduce that 1-z is an invertible element (of infinite order) in $\mathbb{Z}[\mathscr{H}(H_1(\Sigma; G))]$. Conclude.

Consider the particular case $G = \mathbb{Z}$. In this case, $G^{\text{quad}} = \mathbb{Z}$. The cocycle is a map $c_{\Lambda} : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}[t, t^{-1}]$ given by

$$c_{\Lambda}(x,y) = t^{\beta_{\Lambda}(x,y)}, \quad x,y \in H_1(\Sigma)$$

where $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ is the usual Seifert form inducing the intersection form on $H_1(\Sigma)$.

COROLLARY 4.7. [36, Th. 5.4] The map $\Phi : \mathscr{A}(\Sigma \times [0,1]) \to \mathbb{Z}[\mathscr{H}(H_1(\Sigma))]$ is a $\mathbb{Z}[t,t^{-1}]$ -algebra isomorphism. Furthermore,

$$C\mathscr{A}(\Sigma \times [0,1]) = \{ t^k \langle L \rangle \mid k \in \mathbb{Z}, L \in \mathscr{L}(\Sigma \times [0,1]) \}.$$

The first statement was first proved by R. Gelca and A. Uribe in [35, Theorem 5.6] (with ground field \mathbb{C} rather than the ring \mathbb{Z}). R. Gelca calls the group $C\mathscr{A}(\Sigma \times [0, 1])$ the *linking number skein group*.

Recall that we are given a quadratic form $q : G \to \mathbb{Q}/\mathbb{Z}$, with associated linking pairing $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ and a character $\chi : \mathbb{Q}/\mathbb{Z} \to U(1)$. Recall that U denotes the finite subgroup $\chi \circ q(G)$ in U(1). We define three closely related Heisenberg groups associated to $H_1(\Sigma; G)$.

DEFINITION 9.15. The Heisenberg group associated to the Seifert pairing $b \otimes \beta$ (with value group \mathbb{Q}/\mathbb{Z}) is denoted $\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma; G))$. The Heisenberg group associated with $\chi \circ (b \otimes \beta)$ (with value group U(1)) is denoted $\mathscr{H}^{U(1)}(H_1(\Sigma; G))$ and is called the *unitary Heisenberg group*. The Heisenberg group associated with $\chi \circ (b \otimes \beta)|^U$ (with value group U) is denoted $\mathscr{H}^U(H_1(\Sigma; G))$ and is called the *finite unitary Heisenberg group*.

The Heisenberg groups are related by homomorphisms

$$\mathscr{H}(H_1(\Sigma;G)) \longrightarrow \mathscr{H}^U(H_1(\Sigma;G)) \longrightarrow \mathscr{H}^{\mathrm{U}(1)}(H_1(\Sigma;G)).$$
$$\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))$$

The left horizontal arrow is $\operatorname{id} \times \chi \circ \tilde{q}|^U$, the right horizontal arrow is induced by the inclusion $U \subset U(1)$. The left diagonal arrow is $\operatorname{id} \times \tilde{q}$, the right diagonal arrow is $\operatorname{id} \times \chi$.

Recall that the composition $\chi \circ \tilde{q}$ is just the "evaluation" map ev.

LEMMA 4.8. The map $e = id \times ev$ is a group homomorphism $\mathscr{H}(H_1(\Sigma; G)) \rightarrow \mathscr{H}^U(H_1(\Sigma; G))$ that satisfies

$$e(P \cdot x) = ev(P) \cdot e(x), \quad P \in G^{quad}, \ x \in \mathscr{H}(H_1(\Sigma; G)).$$

The analogous statement for the full unitary group also holds.

As a ring, $\mathbb{Z}[\mathscr{H}^U(H_1(\Sigma; G))]$ is a \mathbb{Z} -algebra. Since $Z = 0 \times U$ is the center of $\mathscr{H}^U(H_1(\Sigma; G))$, U acts by multiplication on $\mathscr{H}^U(H_1(\Sigma; G))$. However, this does not quite turn $\mathbb{Z}[\mathscr{H}^U(H_1(\Sigma; G))]$ into a $\mathbb{Z}[U]$ -algebra. The reason is that, since

$$-1 \in U \cap \mathbb{Z} \subset \mathbb{Z}[U],$$

multiplication of elements in $\mathbb{Z}[\mathscr{H}^U(H_1(\Sigma; G))]$ by -1 is not just formal; it is to be identified to the action of -1 on them. Namely, one should have

 $-1(h, u) = (h, -u), h \in H_1(\Sigma; G), u \in U.$

This identity does not hold in $\mathbb{Z}[\mathscr{H}^U(H_1(\Sigma; G))]$, so we are led to consider the appropriate quotient. Let J_U denote the two-sided ideal in $\mathbb{Z}[U][\mathscr{H}^U(H_1(\Sigma; G))]$ generated by the elements

$$u(h, u') - (h, uu'), \quad h \in H_1(\Sigma; G), \ u, u' \in U.$$

DEFINITION 9.16. We define the quotient algebra

$$V(\mathscr{H}^U(H_1(\Sigma;G))) = \mathbb{Z}[U][\mathscr{H}^U(H_1(\Sigma;G))]/J_U.$$

The quotient algebra $V(\mathscr{H}^U(H_1(\Sigma; G)))$ is finite.

EXERCISE 9.4. Let J_{-} denote the two-sided ideal in $\mathbb{Z}[\mathscr{H}(H_{1}(\Sigma;G))]$ generated by -1(h, u) - (h, -u), for all $h \in H_{1}(\Sigma;G)$ and $u \in U$. The natural inclusion map $\mathbb{Z}[\mathscr{H}^{U}(H_{1}(\Sigma;G))] \to \mathbb{Z}[U][\mathscr{H}(H_{1}(\Sigma;G))]$ induces a $\mathbb{Z}[U]$ -algebra isomorphism

$$\mathbb{Z}[\mathscr{H}^{U}(H_{1}(\Sigma;G))]/J_{-}\simeq V[\mathscr{H}^{U}(H_{1}(\Sigma;G))].$$

COROLLARY 4.9. The map e induces a natural ring map

$$\mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))] \to V(\mathscr{H}^U(H_1(\Sigma;G)))$$

such that $e(P \cdot x) = ev(P) \cdot e(x), P \in \mathbb{Z}[G^{quad}], x \in \mathbb{Z}[\mathscr{H}(H_1(\Sigma; G))].$

PROOF. The desired map is the composition

$$\mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))] \to \mathbb{Z}[\mathscr{H}^U(H_1(\Sigma;G))] \to V(\mathscr{H}^U(H_1(\Sigma;G))).$$

Here the left arrow is the linear extension of e (which is a ring homomorphism) and the right arrow is the ring projection (cf. exercise 9.4). The map transfers the $\mathbb{Z}[G^{\text{quad}}]$ -algebra structure of $\mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$ to a $\mathbb{Z}[U]$ -algebra structure on $V(\mathscr{H}^U(H_1(\Sigma;G)))$.

THEOREM 4.10. There is a $\mathbb{Z}[U]$ -algebra isomorphism

$$\widehat{\Phi}: \widehat{\mathscr{A}}(\Sigma \times [0,1]) \to \mathbb{Z}[\mathscr{H}^U(H_1(\Sigma;G))]$$

such that the diagram

$$\begin{aligned} \mathscr{A}(\Sigma \times [0,1]) & \stackrel{\Phi}{\longrightarrow} \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))] \\ e \middle| & & & \downarrow^e \\ \widehat{\mathscr{A}}(\Sigma \times [0,1]) & \stackrel{\widehat{\Phi}}{\longrightarrow} V(\mathscr{H}^U(H_1(\Sigma;G))) \end{aligned}$$

is commutative, where the left vertical arrow is the natural ring projection map e induced by ev in Lemma 3.7.

PROOF. The map $\widehat{\Phi}$ is defined by

$$\widehat{\Phi}(\langle L \rangle_{\widehat{\mathscr{A}}(\Sigma \times [0,1])}) = ([L], \operatorname{ev}(P_L)).$$

It follows from definitions that the diagram is commutative. It remains to show that $\hat{\Phi}$ is invertible. We already know that Φ is invertible by Th. 4.5. We define an algebra map $\hat{\Psi}$ by setting

$$\widehat{\Psi}(x,t) = e \circ \Phi^{-1}(e^{-1}(x,t)), \ x \in H, \ t \in U.$$

The map is well-defined, i.e. does not depend on the lift $(x, P) \in \mathscr{H}(H_1(\Sigma; G))$ chosen so that $t = \operatorname{ev}(P)$. It follows from definitions that $\hat{\Phi}^{-1} = \hat{\Psi}$. Thus $\hat{\Phi}$ is invertible.

DEFINITION 9.17. We define $V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma;G))] = V[\mathscr{H}^U(H_1(\Sigma;G))] \otimes \mathbb{C}$.

The following two exercises provide alternative constructions of the reduced group algebra of Definition 9.17.

EXERCISE 9.5. Let $J_{\mathbb{C}}$ denote the two-sided ideal in $\mathbb{C}[\mathscr{H}^U(H_1(\Sigma;G))]$ generated by

 $u(h, u') - (h, uu'), \quad h \in H_1(\Sigma; G), \ u, u' \in U.$

There is a \mathbb{C} -algebra isomorphism

$$V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma;G))] \simeq \mathbb{C}[\mathscr{H}^U(H_1(\Sigma;G))]/J_{\mathbb{C}}.$$

EXERCISE 9.6. We relate the quotient of the group algebra defined above to the reduced group algebra defined in Chap. 6, (§3, Def. 6.7). Let J be the two-sided ideal in $\mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma; G))]$ generated by the elements

$$(0,t) \cdot x - \chi(t)x, \ (0,t) \in \mathbb{Z}, \ x \in \mathscr{H}(H_1(\Sigma;G)).$$

There is a \mathbb{C} -algebra isomorphism

$$V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma;G))] \simeq \mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))]/J.$$

PROOF. The map id $\times (\chi|^U)^{-1} : \mathscr{H}^U(H_1(\Sigma; G)) \to \mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma; G))$ is injective. So the \mathbb{C} -linear extension

$$\mathbb{C}[\mathscr{H}^{U}(H_{1}(\Sigma;G))] \to \mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_{1}(\Sigma;G))]$$

is an injective \mathbb{C} -algebra map which sends $J_{\mathbb{C}}$ into J. Therefore, it induces an injective \mathbb{C} -algebra map

$$f: V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma;G))] \to \mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))]/J.$$

Define a \mathbb{C} -linear map $\mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))] \to \mathbb{C}[\mathscr{H}^U(H_1(\Sigma;G))]$ by

$$(h,t) \mapsto \chi(t)(h,1), \quad h \in H_1(\Sigma;G), \ t \in \mathbb{Q}/\mathbb{Z}.$$

Since $(h,t) = (0,t)(h,0) = \chi(t)(h,0) \in \mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))]/J$, we verify that this map induces a map $g: \mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))]/J \to V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma;G))]$ such that $g = f^{-1}$.

COROLLARY 4.11. The following diagram of algebras is commutative:

where the top horizontal arrow is a $\mathbb{C}[G^{\text{quad}}]$ -algebra isomorphism and the bottom horizontal arrow is a \mathbb{C} -algebra isomorphism.

For each Lagrangian Λ_1 of $H_1(\Sigma)$, there is a Schrödinger representation $\pi_{\Lambda_1} : \mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma; G)) \to U(L^2(G \otimes H_1(\Sigma)/\Lambda_1))$ induced from the character χ on the maximal abelian subgroup $(G \otimes \Lambda_1) \times \mathbb{Q}/\mathbb{Z}$ in $H_1(\Sigma; G) \times \mathbb{Q}/\mathbb{Z}$. (See Chap. 6, §4.) For simplicity, we write $\Lambda_0 = H_1(\Sigma)/\Lambda_1$, identifying the quotient to a fixed Lagrangian.

We choose the Lagrangian Λ_1 to be the Lagrangian generated by meridians of Σ . The Lagrangian Λ_0 can be chosen to the Lagrangian generated by a maximal independent set of longitudes of Σ . If Σ is a surface with a maximal independent set of distinguished longitudes (in particular if Σ is a standard surface, see...), then one can take Λ_0 to be the Lagrangian generated by this set of longitudes of Σ . The following construction is independent of the particular choice for the Lagrangian Λ_0 such that $\Lambda_0 \oplus \Lambda_1 = H_1(\Sigma)$.

LEMMA 4.12. There is a unique \mathbb{C} -linear extension

 $\tilde{\pi}_{\Lambda_1}: V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma; G))] \to \operatorname{End}_{\mathbb{C}}(L^2(G \otimes \Lambda))$

of the Schrödinger representation making the diagram

commute.

PROOF. This is essentially Corollary 3.6. Extend π_{Λ} by \mathbb{C} -linearity to a linear map $\mathbb{C}[\mathscr{H}(H_1(\Sigma; G))] \to \operatorname{End}_{\mathbb{C}}(L^2(G \otimes \Lambda_0))$, that we continue to denote π_{Λ} . We have $\pi_{\Lambda}((0, t) \cdot h) = \pi_{\Lambda}(0, t) \pi_{\Lambda}(h) = \chi(t)\pi_{\Lambda}(h)$, so $\pi_{\Lambda}(J) = 0$. The result follows.

A crucial observation remains to be made at this point. According to Chap. §6, end of §4, a model for the Schrödinger representation is given by the algebra

$$\mathcal{H}(H_1(\Sigma;G)) = \mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))]/I$$

where I is the two-sided ideal generated by all elements

$$l \cdot h - \chi(l)h, h \in \mathscr{H}(H_1(\Sigma; G)), l \in G \otimes H_1(\Sigma)/\Lambda \times \mathbb{Q}/\mathbb{Z}.$$

Clearly $J \subseteq I$ so there is a natural projection map $V[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma; G))] \to \mathcal{H}(H_1(\Sigma; G))$. It is not hard to piece the definitions together to see that this map fits into the following commutative diagram

where the left vertical arrow is the epimorphism induced by inclusion

$$\Sigma \times [0,1] \subseteq H$$

of the cylinder over $\Sigma = \partial H$ (viewed as a collar of ∂H) inside the handlebody H; the middle vertical arrow is the projection map induced by the inclusion $J \subseteq I$; the right vertical arrow is the projection map defined, on the basis of Weyl operators for $\operatorname{End}_{\mathbb{C}}(L^2(G \otimes \Lambda))$, by

$$W_{(a_0,a_1)} \mapsto \chi(a_1)\delta_{a_0}, \quad (a_0,a_1) \in H_1(\Sigma;G).$$

These three maps are vector spaces epimorphisms. All other maps are algebra isomorphisms: the top maps are defined by Theorem 4.10 and Corollary 3.6; the bottom left map is the map induced by $\hat{\Phi}$; the bottom right map is defined by Prop. 4.4 (Chap. 6, §3).

EXERCISE 9.7. Show that the vertical arrows are not algebra maps.

By definition, $L^2(G \otimes \Lambda)$ is an $\operatorname{End}_{\mathbb{C}}(L^2(G \otimes \Lambda))$ -module and $\mathcal{H}(H_1(\Sigma; G))$ is a $V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma; G))]$ -module: the action of $V_{\mathbb{C}}[\mathscr{H}(H_1(\Sigma; G))]$ on $\mathcal{H}(H_1(\Sigma; G))$ is induced by the action of $\mathbb{C}[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma; G))]$ on itself.

It follows from the diagram above that there is also a $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ -module structure on $\widehat{\mathscr{A}}(H)$. In the next paragraph we give a simple geometric interpretation of this action, describe it using Abelian skein theory and relate it to the Schrödinger representation.

5. The Schrödinger representation from Abelian skein theory

Let M be a compact oriented 3-manifold with boundary Σ . In this paragraph, we explain how the skein module $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ acts on the skein module $\widehat{\mathscr{A}}(M)$. We then identity this action to "the" Schrödinger representation, as defined in Chap. 6, §3.

There is a natural gluing $M' = M \cup (\Sigma \times [0,1])$ defined by identifying $\partial M = \Sigma$ with $-\Sigma \times 0$. The result is a "thickened" manifold M' which is homeomorphic to M. Fix such a homeomorphism so that we may identify M' and M.



LEMMA 5.1. The Heisenberg skein algebra $\mathscr{A}(\Sigma \times [0,1])$ acts on the right on $\mathscr{A}(M)$ by the map

$$\mathscr{A}(M) \times \mathscr{A}(\Sigma \times [0,1]) \to \mathscr{A}(M \cup (\Sigma \times [0,1])) = \mathscr{A}(M'),$$

defined by

$$(\langle L \rangle, \langle L' \rangle) \mapsto \langle L \cup L' \rangle.$$

This action induces an action α_{Λ} of the reduced Heisenberg skein algebra $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ on $\widehat{\mathscr{A}}(M)$.

REMARK 9.10. The action is a right action because of our convention for the product of links. Recall that the product of links in $\Sigma \times [0, 1]$ is written from left to right, starting with the link whose projection on [0, 1] is the lowest. Since the gluing is performed on $\Sigma \times 0 = \partial M$, we write down from left to right the link in M and then the link in $\Sigma \times [0, 1]$. This ensures that α is a morphism (rather than an anti-morphism), i.e. a true action¹.

PROOF. Since M and M' are homeomorphic, Lemma 1.3 ensures that we can identify $\widehat{\mathscr{A}}(M')$ and $\widehat{\mathscr{A}}(M)$. The result is a consequence of the definitions.

¹We could have decided to perform the gluing on $\Sigma \times 1$ in order to get a left action. The reason we did not choose this convention is that with this convention, the gluing of a standard oriented handlebody $M = H_g$ (embedded in \mathbb{R}^3) with a cylinder $\partial M \times [0, 1]$ is "external": if we visualize it inside $S^3 = \mathbb{R}^3 \cup \{\infty\}$, then the thickened handlebody $M' = (\partial M \times [0, 1]) \cup M$ contains the point at infinity.

Let $H = H_g$ denote² an oriented handlebody of genus g such that $\partial H = \Sigma$. The following proposition gives a geometric (skein-theoretic) description of the Schrödinger representation.

PROPOSITION 5.2. The action of $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ on $\widehat{\mathscr{A}}(H_g)$ is induced by the Schrödinger representation π_{Λ_1} . More precisely, the following diagram is commutative:

$$\begin{aligned} \widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1]) & \xrightarrow{\alpha_{\Lambda}} \operatorname{End}_{\mathbb{C}}(\widehat{\mathscr{A}}(H)) \\ & \widehat{\Phi} \Big| \simeq & \simeq \Big| \\ V[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_{1}(\Sigma;G))] & \xrightarrow{\tilde{\pi}_{\Lambda}} \operatorname{End}_{\mathbb{C}}(L^{2}(G \otimes \Lambda)) \\ & & \uparrow & & \uparrow \\ & & & \uparrow & \\ \mathscr{H}(H_{1}(\Sigma;G)) & \xrightarrow{\pi_{\Lambda}} \operatorname{U}(L^{2}(G \otimes \Lambda)) \end{aligned}$$

where the first vertical descending arrow is the isomorphism of Theorem 4.10, the second vertical descending arrow is the isomorphism induced by 1-homology (See Th. 3.9).

PROOF. We have already seen that $\widehat{\mathscr{A}}(H_g)$ is isomorphic to the algebra $L^2(G \otimes \Lambda) = \mathbb{C}[G \otimes \Lambda]$ (Theorem 3.9). We have also seen that $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1])$ is isomorphic via $\widehat{\Phi}$ to the quotient $V[\mathscr{H}^{\mathbb{Q}/\mathbb{Z}}(H_1(\Sigma;G))]$ of the finite Heisenberg algebra $\mathscr{H}(H_1(\Sigma;G))$ (Theorem 4.10). By Lemma 4.12, It suffices to identify the representation on the image of the finite Heisenberg algebra $\mathscr{H}(H_1(\Sigma;G))$. Since the Schrödinger representation is induced from the subgroup $(\Lambda_1 \otimes G) \times \mathbb{Q}/\mathbb{Z}$, it suffices to identify the representation on this subgroup. A typical element in this subgroup is an elementary skein whose 1-homology lies in $\Lambda_1 \otimes G$.

Let therefore $\langle m_i(x) \rangle$ denote the skein in $\Sigma \times [0,1]$ represented by the *i*-th meridian of $\Sigma \times 1 \subset \Sigma \times [0,1]$ and coloured by $x \in G$. Let $\langle l_j(y) \rangle_H$ denote the skein in the handlebody H represented by the *j*-th longitude of H and coloured by $y \in G$. Using skein calculus, we verify that

$$\langle m_i(x) \rangle \cdot \langle l_j(y) \rangle_H = \begin{cases} \langle l_j(y) \rangle_H & \text{if } i \neq j; \\ \chi(b(x,y)) \langle l_j(y) \rangle_H & \text{if } i = j. \end{cases}$$

(In the case when Σ has genus 1, a proof follows from Fig. 1.1. The general case is similar.) Note that $\beta_{\Lambda}(i_*m_i, i_*l_j) = \delta_{ij}$, so that

$$\langle m_i(x) \rangle \cdot \langle l_j(y) \rangle_H = \chi(\delta_{ij} \ b(x,y)) \ \langle l_j(y) \rangle_H.$$

Therefore, if L is an arbitrary disjoint union of parallel meridians in $\Sigma \times 1$ and L' an arbitrary element in $\mathscr{S}(H)$,

(5.1)
$$\langle L \rangle_{\Sigma} \cdot \langle L' \rangle_{H} = \chi \Big((b \otimes \beta_{\Lambda})([L], [L']) \Big) \langle L' \rangle_{H}.$$

This is the desired result.

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²It can be taken to be the standard handlebody of genus q without loss of generality.

REMARK 9.11. Even though it is not needed, it is instructive to verify geometrically how the other elements induced by longitudes act. Let $\langle l_i(x) \rangle_{\Sigma}$ denote the skein in $\Sigma \times [0, 1]$ represented by the *i*-th longitude of $\Sigma \times 1 \subset$ $\Sigma \times [0, 1]$ and coloured by $x \in G$. Let $\langle l_j(y) \rangle_{H_g}$ denote the skein in H_g represented by the *j*-th standard longitude of H_g and coloured by $y \in G$. Clearly,

$$\langle l_i(x) \rangle \cdot \langle l_j(y) \rangle_H = \langle l_i(x) \rangle_H \cdot \langle l_j(y) \rangle_H.$$

Therefore, if L is an arbitrary disjoint union of parallel longitudes in $\Sigma \times 1$ and L' an arbitrary element in $\mathscr{S}(H)$,

(5.2)
$$\langle L \rangle_{\Sigma} \cdot \langle L' \rangle_{H} = \langle L \rangle_{H} \cdot \langle L' \rangle_{H}.$$

Recall that $\delta_{\langle L \rangle_H \cdot \langle L' \rangle_H} = \delta_{[L]} \star \delta_{[L']} = \delta_{[L]+[L']}$. Therefore, longitudes acts as translations as expected.

Tables of actions (three parts)

6. The Weil representation of a quadratic form from Abelian skein theory

6.1. The action of the isometry group O(q) on $\widehat{\mathscr{A}}(M)$. Let M be an oriented compact 3-manifold. The automorphism group $\operatorname{Aut}(G)$ of G acts on the polynomial algebra $\mathbb{Z}[\tilde{G}^{\text{quad}}]$ over the indeterminates in $U \cup V = \{t_g, t_g^{-1}, t_{g,h}, t_{a,h}^{-1}, g, h \in G\}$ by acting on the labels of the indeterminates:

$$\psi \cdot t_g^{\pm 1} = t_{\psi(g)}^{\pm 1}, \ \psi \cdot t_{g,h}^{\pm 1} = t_{\psi(g),\psi(h)}^{\pm 1}, \ \psi \in \operatorname{Aut}(G), \ g,h \in G.$$

More generally, the group $\operatorname{Aut}(G)$ acts on the $\mathbb{Z}[\tilde{G}^{\operatorname{quad}}]$ -module $\mathscr{L}(M)$ freely generated isotopy classes of framed oriented coloured links in M defined in §1. If we represent an elementary skein coloured by $g \in G$ by $\langle L(g) \rangle$, then the action is given by

$$\psi \cdot P\langle L(g) \rangle = (\psi \cdot P)\langle L(\psi(g)) \rangle, \quad P \in \mathbb{Z}[\tilde{G}^{\text{quad}}].$$

This action evidently preserves the submodule generated by the skein relations and therefore induces an action on the Heisenberg skein premodule $\mathscr{A}^{o}(\Sigma \times [0,1])$ by acting on the colors of the skeins. The action also preserves the submodules generated by the elements 3.2, 3.3 and 3.5 respectively. Therefore, the action induces an action

$$\operatorname{Aut}(G) \times \mathscr{A}(M) \to \mathscr{A}(M).$$

Consider the isometry group O(q), the subgroup of Aut(G) that consists of automorphisms of G preserving the quadratic form q. Each isometry $\psi \in O(q)$ leaves invariant the specializations $[t_g] = \chi(q(g))$ and $[t_{g,h}] = \chi(b(g,h))$, that is,

$$\psi \cdot [t_g] = [t_{\psi(g)}] = [t_g], \quad \psi \cdot [t_{g,h}] = [t_{\psi(g),\psi(h)}] = [t_{g,h}].$$

The previous action induces an action

$$O(q) \times \widehat{\mathscr{A}}(M) \to \widehat{\mathscr{A}}(M)$$

that affects only colors.

6.2. The action of the mapping class group. The previous paragraph §5 relies on one fixed identification

$$M \cup (\Sigma \times [0,1]) \xrightarrow{\sim} M$$

in order to identify (a model of) the Schrödinger representation (Lemma 5.1 and Prop. 5.2). There is, however, some extra freedom in this identification. The mapping class group $\mathfrak{M}(\Sigma)$ acts on the skein algebra $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ by

$$([h], \langle L \rangle) \mapsto h_* \langle L \rangle = \langle (h \times \mathrm{id}_{[0,1]})(L) \rangle,$$

where $h \in \text{Homeo}^+(\Sigma)$ and $L \in \mathscr{L}(\Sigma \times [0,1])$. Thus each mapping class [h] yields an automorphism of $\mathscr{A}(\Sigma \times [0,1])$ and $\widehat{\mathscr{A}}(\Sigma \times [0,1])$ respectively.

The following lemma relates this action to the symplectic group $\operatorname{Sp}(H_1(\Sigma))$. The group $\operatorname{Sp}(H_1(\Sigma))$ acts on $H_1(\Sigma; G)) = H_1(\Sigma) \otimes G$ as symplectomorphisms on $H_1(G)$ and as the identity on G; this action induces an action of $\operatorname{Sp}(H_1(\Sigma))$ on the finite Heisenberg algebra $\mathscr{H}(H_1(\Sigma; G))$ by acting on the 1-homology.

LEMMA 6.1. The action of $\mathfrak{M}(\Sigma)$ on $\mathscr{A}(\Sigma \times [0,1])$ factors through $\mathrm{Sp}(H_1(\Sigma))$.

PROOF. We verify that the diagram

$$\mathfrak{M}(\Sigma) \times \mathscr{A}(\Sigma \times [0,1]) \longrightarrow \mathscr{A}(\Sigma \times [0,1])$$

$$\star \times \Phi \bigvee \qquad \simeq \bigvee \Phi$$

$$\mathrm{Sp}(H_1(\Sigma)) \times \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))] \longrightarrow \mathbb{Z}[\mathscr{H}(H_1(\Sigma;G))]$$

is commutative: via the upper right corner, we find

$$\Phi\left(h_{\ast}\langle L\rangle\right) = \left(\left[h_{\ast}\langle L\rangle\right], P_{h(L)}\right).$$

Via the low left corner, we find

$$h_*\Psi(\langle L\rangle) = h_*([L], P_L) = (h_*[L], P_L).$$

The identity $[h_*\langle L \rangle] = h_*[L]$ follows from definitions. The identity $P_{h(L)} = P_L$ follows from the invariance of $L \mapsto P_L$ under positive homeomorphisms of $\Sigma \times [0, 1]$. The conclusion follows.

6.3. The Weil representation as a skein multiplication. The commutative diagram appearing in the proof of Lemma 6.1 can be extended as follows.

Here $ASp(H_1(\Sigma; G))$ is the group defined in Chap. 6. Recall that there is an epimorphism

$$\operatorname{ASp}(H_1(\Sigma; G)) \to \operatorname{Sp}(H_1(\Sigma; G))$$

onto the symplectic group $\operatorname{Sp}(H_1(\Sigma; G))$ for the symplectic form $b \otimes \omega$ (where ω is the intersection form on $H_1(\Sigma)$). The inclusion $\operatorname{O}(q) \times \operatorname{Sp}(H_1(\Sigma)) \to \operatorname{ASp}(H_1(\Sigma; G))$ is lifts the natural map $(\varphi, s) \mapsto \varphi \otimes s$ (see Chap. 6, §7, Prop. 7.2).

Accordingly, for each $s \in ASp(H_1(\Sigma))$, there is an action

$$\bullet_s:\widehat{\mathscr{A}_{\mathbb{C}}}(H)\times\widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma\times[0,1])\to\widehat{\mathscr{A}_{\mathbb{C}}}(H)$$

which defines a representation $\alpha_{\Lambda}^s : \widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1]) \to \operatorname{End}_{\mathbb{C}}(\widehat{\mathscr{A}}(H))$. By Prop. 5.2, α_{Λ}^s enjoys the same property as α_{Λ} does. Since it is also induced by the Schrödinger representation, it follows from Lemma 4.12 and the Stone-von Neumann Theorem that the two representations are equivalent: there exists $\rho_s \in \operatorname{End}_{\mathbb{C}}(\widehat{\mathscr{A}}(H))$ such that the following diagram

$$\begin{aligned} \widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1]) \times \widehat{\mathscr{A}_{\mathbb{C}}}(H) & \stackrel{\bullet}{\longrightarrow} \widehat{\mathscr{A}_{\mathbb{C}}}(H) \\ & & \downarrow^{\mathrm{id} \times \rho_s} & \qquad \qquad \downarrow^{\rho_s} \\ \widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1]) \times \widehat{\mathscr{A}_{\mathbb{C}}}(H) & \stackrel{\bullet_s}{\longrightarrow} \widehat{\mathscr{A}_{\mathbb{C}}}(H) \end{aligned}$$

is commutative. In other words,

(6.1)
$$\rho_s(\sigma \cdot \sigma') = \sigma \bullet_s \rho_s(\sigma') \quad \text{or} \quad \rho_s(\alpha_\Lambda(\sigma)\sigma') = \alpha^s_\Lambda(\sigma)\rho_s(\sigma')$$

for any $\sigma \in \widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1])$, $\sigma' \in \widehat{\mathscr{A}_{\mathbb{C}}}(H)$. Now by the commutative diagram (4.4), there are equivariant isomorphisms of algebras

$$\operatorname{End}_{\mathbb{C}}(\widehat{\mathscr{A}}_{\mathbb{C}}(H)) \simeq \operatorname{End}_{\mathbb{C}}(L^2(G \otimes \Lambda)) \simeq \widehat{\mathscr{A}}(\Sigma \times [0,1])$$

such that the natural action of $\operatorname{End}_{\mathbb{C}}(\widehat{\mathscr{A}}(H))$ on $\widehat{\mathscr{A}}_{\mathbb{C}}(H)$ is sent to the (skein) action of $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1])$ on $\widehat{\mathscr{A}}_{\mathbb{C}}(H)$. This implies the following result.

THEOREM 6.2. Let $s \in ASp(H_1(\Sigma; G))$. There is some skein

$$R_s \in \widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1])$$

such that

$$\rho_s(\sigma) = R_s \cdot \sigma, \quad \sigma \in \widehat{\mathscr{A}_{\mathbb{C}}}(H).$$

In particular, R_s is determined up to a multiplicative unit by the relation

(6.2)
$$R_s \cdot \sigma = s(\sigma) \cdot R_s, \quad \sigma \in \widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1]).$$

In other words, the Weil representation ρ_s is a skein R_s acting by skein product on $\widehat{\mathscr{A}}_{\mathbb{C}}(H)$; as a skein in $\Sigma \times [0,1]$, it also acts on $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1])$. We sum up the correspondence in the following table.

	$ ho_s$	R_s
algebra	$\operatorname{End}_{\mathbb{C}}(L^2(G\otimes \Lambda))$	$\widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1])$
Weil representation	acts on $L^2(G\otimes\Lambda)$	acts on $\widehat{\mathscr{A}_{\mathbb{C}}}(H)$
	as a unitary map	induced by skein product
Linear extension	acts on $\operatorname{End}_{\mathbb{C}}(L^2(G \otimes \Lambda))$	acts on $\widehat{\mathscr{A}_{\mathbb{C}}}(\Sigma \times [0,1])$
	by composition	by skein product

REMARK 9.12. There exists an unreduced version of Theorem 6.2 (without "hats") such that the following diagram



commutes.

6.4. The Weil representation of a quadratic form from Abelian skein theory. Since the action of $\mathfrak{M}(\Sigma)$ does not affect colors, nor does the action of $\operatorname{Sp}(H_1(\Sigma))$. It follows that the actions of O(q) and $\operatorname{Sp}(H_1(\Sigma))$ on $\widehat{\mathscr{A}}_{\mathbb{C}}(\Sigma \times [0,1])$ commute. In fact, their images are maximally commuting subgroups of $\operatorname{ASp}(H_1(\Sigma;G))$ as the map

 $(\varphi, s) \mapsto \varphi \otimes s$

induces a injective homomorphism

 $O(q) \otimes Sp(H_1(\Sigma)) \hookrightarrow ASp(H_1(\Sigma;G))$

lifting the natural injective homomorphism

$$O(q) \otimes Sp(H_1(\Sigma)) \hookrightarrow Sp(H_1(\Sigma;G)).$$

Here is an explicit formula for the action of $O(q) \times Sp(H_1(\Sigma))$ on $\mathscr{A}(\Sigma \times [0,1])$: for any lift $\tilde{s} \in \mathfrak{M}(\Sigma)$ of $s \in Sp(H_1(\Sigma))$,

$$(\varphi,s)\cdot \langle L(g)\rangle = \langle \tilde{s}_*L(\varphi(x))\rangle, \quad \varphi \in \mathcal{O}(q), \langle L\rangle \in \mathscr{A}(\Sigma \times [0,1]), g \in G.$$

Alternatively, the action can be expressed on the group algebra of the Heisenberg group:

$$(\varphi, s) \cdot (x, P) = ((\varphi \otimes s)_* x, \varphi \cdot P), \varphi \in \mathcal{O}(q), \ x \in H_1(\Sigma; G), \ P \in G^{\text{quad}}.$$

COROLLARY 6.3. For any $(\varphi, s) \in O(q) \times Sp(H_1(\Sigma))$, there exist skeins $R_{\varphi \otimes s}, R_{\varphi}, R_s \in \mathscr{A}(\Sigma \times [0, 1])$ such that up to a multiplicative constant,

(6.3)
$$R_{\varphi \otimes s} = R_{\varphi} \cdot R_s = R_s \cdot R_{\varphi}$$

and for any $\sigma \in \mathscr{A}(\Sigma \times [0,1])$,

(6.4)
$$\rho_{\varphi \otimes s}(\sigma) = R_{\varphi \otimes s} \cdot \sigma.$$

6. THE WEIL REPRESENTATION OF A QUADRATIC FORM FROM ABELIAN SKEIN THEORY

PROOF. The existence of the skein $R_{\varphi \otimes s} \in \mathscr{A}(\Sigma \times [0, 1])$ for any $\varphi \in O(q)$ and any $s \in ASp(H_1(\Sigma; G))$ is ensured by Theorem 6.2, as well as the second property (6.4). Let

$$R_{\varphi} = R_{\varphi \otimes \mathrm{id}}, \ R_s = R_{\mathrm{id} \otimes s}.$$

Then $R_{\varphi \otimes s} = R_{(\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes s)} = R_{\varphi} \cdot R_s.$ Similarly $R_{\varphi \otimes s} = R_s \cdot R_{\varphi}.$

6.5. Computations. We begin with the action of the mapping class group.

LEMMA 6.4. Let τ_m be the right Dehn twist about a meridian m on Σ . Then

$$R_{\tau_m} = |G|^{-\frac{1}{2}} \sum_{x \in G} t_x \langle m(x) \rangle.$$

The equality above is to be understood up to a multiplicative complex number of absolute value 1.

PROOF. The existence of R_{τ_m} is ensured by Corollary 6.3. It is sufficient to verify the identity (6.2) on the basis elements of $\mathscr{A}(\Sigma \times [0,1])$ which are products of skeins that consist of a meridian (resp. longitude). If σ has trivial geometrical intersection number with m, then $\tau_m(\sigma) = \sigma$. Since $\langle \sigma \rangle$ can be written using meridians and longitudes that have geometrical intersection number zero with m, the skeins $\langle m(x) \rangle$ and σ commute. Hence $R_{\tau_m} \cdot \sigma = \sigma \cdot R_{\tau_m}$. If the skein σ consists of the meridian m with some color z then $(\tau_m)_*(\sigma) = \sigma$, so

$$R_{\tau_m} \cdot \sigma = |G|^{-\frac{1}{2}} \sum_x t_x \langle m(x) \rangle \langle m(z) \rangle$$

= $|G|^{-\frac{1}{2}} \sum_x t_x \langle m(x+z) \rangle$
= $|G|^{-\frac{1}{2}} \sum_x t_x \langle m(z) \rangle \langle m(x) \rangle$
= $\langle m(z) \rangle |G|^{-\frac{1}{2}} \sum_x t_x \langle m(x) \rangle$
= $(\tau_m)_*(\sigma) \cdot R_{\tau_m}.$

Finally, let $\sigma = \langle l(y) \rangle$. First, as illustrated in the figure below (in the case Σ is a torus), $\langle \tau_m(l) \rangle = t_y \langle m(y) \rangle \langle l(y) \rangle$.



FIGURE 6.1. Computation of the skein of the right Dehn twist applied to a colored longitude.

The left hand side of (6.2) is

LHS =
$$R_{\tau_m} \cdot \langle l(y) \rangle = |G|^{-\frac{1}{2}} \sum_x t_x \langle m(x) \rangle \cdot \langle l(y) \rangle.$$

The right hand side of (6.2) is

$$\begin{split} \operatorname{RHS} &= \tau_{m*}(\langle l(y) \rangle) \cdot R_{\tau_m} = \langle \tau_m(l) \rangle \cdot R_{\tau_m} \\ &= t_y \langle m(y) \rangle \langle l(y) \rangle |G|^{-\frac{1}{2}} \sum_x t_x \langle m(x) \rangle \\ &= |G|^{-\frac{1}{2}} \sum_x t_y t_x \langle m(y) \rangle \langle l(y) \rangle \langle m(x) \rangle \\ &= |G|^{-\frac{1}{2}} \sum_x t_y t_x \langle m(y) \rangle t_{x,y} \langle m(x) \rangle \langle l(y) \rangle \\ &= |G|^{-\frac{1}{2}} \sum_x t_y t_x t_{x,y} \langle m(y+x) \rangle \langle l(y) \rangle \\ &= |G|^{-\frac{1}{2}} \left(\sum_x t_{x+y} \langle m(y+x) \rangle \right) \langle l(y) \rangle \\ &= |G|^{-\frac{1}{2}} \sum_z t_z \langle m(z) \rangle \langle l(y) \rangle \\ &= \mathrm{LHS}. \end{split}$$

EXERCISE 9.8. In order to compute R_{τ_m} , write down R_{τ_m} as an element in $\mathscr{A}(\Sigma \times [0,1])$, use the identity (6.2) on basis elements and verify that this indeed leads to the formula above (up to a multiplicative constant).

CHAPTER 10

The Abelian invariant of a closed 3-manifold

Throughout this chapter, M is a closed oriented connected 3-manifold.

We define a topological invariant $\tau(M)$ of M. A version of this invariant is derived from a modular (or quasi-modular) category in the sense of V. Turaev [93] or from a generalization of the invariant introduced by H. Murakami, T. Ohtsuki and M. Okada [69].

This invariant has a number of remarkable properties (...).

1. The definition from the skein theoretic framework

The construction of $\tau(M)$ is based on the technique of surgery, well known in topology. This technique enables to represent closed 3-manifolds by framed links in the familiar Euclidean 3-space (or the 3-sphere S^3). Specifically the 3-manifold M is presented as the result of surgery along a framed link L in $S^3 = \mathbb{R} - \{\infty\}$. The invariant $\tau(M)$ is derived from topological invariants of the link L satisfying some additional invariance property (invariance under the Kirby moves). In our situation, the topological invariants of L are the Abelian skein invariants $\langle L \rangle$ corresponding to different colorings of its components.

The skein invariants depend on the choice of a character $\chi : \mathbb{Q}/\mathbb{Z} \to \mathrm{U}(1)$.

In this chapter, $\chi : \mathbb{Q}/\mathbb{Z} \to U(1)$ denotes the fixed character defined by $\chi(t) = \exp(2\pi i t)$.

1.1. Surgery on links in the 3-sphere. Let B^2 be a closed unit 2disc so that its boundary $\partial B^2 = S^1$ is the unit circle. Consider a tubular neighborhood T of a knot k in S^3 . A meridian m is a simple closed curve in ∂T that bounds a disc in T. A longitude l is a simple closed curve on ∂T that transversally intersects once a fixed meridian and bounds a surface in $S^3 - T$. The 1-homology of ∂T is generated by the homology classes of mand l respectively. Hence the homology class of any simple closed curve γ on ∂T that is transverse to m and l, satisfies

$$[\gamma] = (\gamma \cdot m) [m] + (\gamma \cdot l) [l] = ([\gamma] \bullet [m]) [m] + ([\gamma] \bullet [l]) [l].$$

A framed knot in S^3 may be regarded as a smooth embedding of $\Sigma_1 = S^1 \times B^2$ in S^3 together with its parallel, the image of the standard longitude $S^1 \times \{0\}$. Consider such an embedded solid torus $j: S^1 \times B^2 \hookrightarrow S^3$ in the 3-sphere. Remove from S^3 the interior of $j(S^1 \times B^2)$ and glue back the solid

torus $B^2 \times S^1$ along $j|_{S^1 \times S^1}$. The result

$$\chi(S^{3}, L) = (S^{3} - \operatorname{Int}(S^{1} \times B^{2})) \cup_{i} (B^{2} \times S^{1})$$

is a closed 3-manifold, known as surgery along the torus in S^3 : it depends only on the isotopy class of the framed knot represented by j (see [37, Chaps. 4 and 5, §5.3], [82, Chap. 9]). In particular, it does not depend on the orientation of the surgery link L (but depends on the orientation of S^3). More generally, the procedure is carried out on framed links. A fundamental result due to W. Lickorish [58] and A. Wallace [99] asserts that any closed oriented 3-manifold M (up to orientation preserving homeomorphism) is the result of surgery on some framed link L: there exists a framed link $L \subset S^3$ such that $M = \chi(S^3, L)$. Via the theory of handles, this result also follows from the theorem of V. Rokhlin [81] asserting that every closed oriented 3-manifold bounds a compact oriented 4-manifold. (See [37, §5.4]). In order to state the other fundamental result that we use to define the invariant, we need to introduce two geometric operations (and their inverse) on links in S^3 . The first operation consists in creating a topologically disjoint unknot with framing ± 1 . The reverse operation consists in annihilating a topologically disjoint unknot with framing ± 1 . To describe the second operation, one needs to define first the band sum operation: consider two knots J and K in S^3 and a band embedding $b: [0,1] \times [0,1] \to S^3$ such that $b([0,1] \times [0,1]) \cap J = b([0,1] \times \{0\})$ and $b([0,1] \times [0,1]) \cap K = b([0,1] \times \{1\})$. Replacing the arc $b([0,1] \times \{0\})$ of J by $b(\{0,1\} \times [0,1])$ yields a new knot $J\sharp_b K$, denoted the band sum of J and K along b. The second operation consists in creating a parallel K' of some component K and replacing a link component J by a band sum of J and K'. This second operation is also called "sliding" a link component over another one (this is particularly relevant given the interpretation of this move in terms of handle sliding, see [37, Chap. 4]). This operation is schematically represented below by Fig. 1.1 below. (Note that this operation is well defined regardless of the orientation of the components.)

Kirby's theorem [53] states that two framed links in S^3 give rise to homeomorphic 3-manifolds if and only if they are related by isotopy in S^3 and a finite sequence of the two moves described above. As a consequence, an invariant of a framed link L is an invariant of the closed 3-manifold $\chi(S^3, L)$ if and only if it is invariant under both Kirby moves.

REMARK 10.1. It is useful in practice to orient the components of the surgery link L, even though the resulting 3-manifold $\chi(S^3, L)$ does not depend on the choice of the orientation.

1.2. Abelian invariant of a closed 3-manifold. Let $L = L_1 \cup \cdots \cup L_m$ be an oriented and framed link in S^3 presenting a closed oriented connected 3-manifold M. We may assume that L avoids a small ball inside S^3 ; then L lies inside a 3-ball $B^3 = S^3 - B^3$. For each coloring of the components of L with elements of G, there is an associated skein $\langle L \rangle \in \mathscr{A}(B^3)$. The invariant $\langle \cdot \rangle$ of links extends by linearity to links whose components are labelled by a element in $L^2(G)$, that is, a formal \mathbb{C} -linear combination of elements of G. The set $L^2(G)$ of labels can be regarded as the set of



FIGURE 1.1. The sliding move.

generalized colors. A label in G (in the sense of the previous chapter) will be henceforth called a *simple color*. For any *n*-component oriented framed link L, there is a linear *n*-form

$$L^2(G)^n \to \mathbb{C}, \ (c_1, \dots, c_n) \mapsto \langle L(c_1, \dots, c_n) \rangle,$$

where we denote by $L(c_1, \ldots, c_n)$ the link L whose (ordered) components are colored by c_1, \ldots, c_n respectively.

A special color plays a fundamental role with respect to the Kirby moves:

$$\Omega_G = \sum_{g \in G} \delta_g.$$

LEMMA 1.1. Any knot K colored with Ω_G satisfies the sliding property: for any knot J colored with color $c \in L^2(G)$,

(1.1)
$$\langle J(c) \cup K(\Omega) \rangle = \langle (J\sharp_b K)(c) \cup K(\Omega) \rangle$$

where \sharp_b denotes the handle sliding operation along a band b.

PROOF. One needs to compute the new framing number and the new linking number after the handle slide. Specifically one needs to compare the framing number of the knot $J \sharp_b K$ to that of J and to compare the linking number of $J \sharp_b K$ and K to that of J and K. We have

$$\operatorname{fr}(J\sharp_b K) = \operatorname{fr}(J) + \operatorname{fr}(K) + 2 \operatorname{lk}(J, K)$$

and

$$lk(J\sharp_b K, K) = lk(J, K) + fr(K).$$

For simple colors g, c, we have

$$\langle (J\sharp_b K)(c) \cup K(g) \rangle = t_c^{\operatorname{fr}(J\sharp K) - \operatorname{fr}(J)} t_{g,c}^{\operatorname{lk}(J\sharp K) - \operatorname{lk}(J,K)} \langle J(c) \cup K(g) \rangle$$

$$= t_c^{\operatorname{fr}(K) + 2\operatorname{lk}(J,K)} t_{g,c}^{\operatorname{fr}(K)} \langle J(c) \cup K(g) \rangle$$

$$= t_c^{\operatorname{fr}(K) + 2\operatorname{lk}(J,K)} t_{g,c}^{\operatorname{fr}(K)} t_g^{\operatorname{fr}(K)} t_{g,c}^{\operatorname{lk}(J,K)} \langle J(c) \rangle$$

$$= t_{g+c}^{\operatorname{fr}(K)} t_c^{2\operatorname{lk}(J,K)} t_{g,c}^{\operatorname{lk}(J,K)} \langle J(c) \rangle$$

$$= t_{g+c}^{\operatorname{fr}(K)} t_{g+c,c}^{\operatorname{lk}(J,K)} \langle J(c) \rangle.$$

Hence

$$\begin{split} \langle (J\sharp_b K)(c) \cup K(\Omega) \rangle &= \sum_{g \in G} \langle (J\sharp_b K)(c) \cup K(g) \rangle \\ &= \sum_{h \in G} t_h^{\mathrm{fr}(K)} t_{h,c}^{\mathrm{lk}(J,K)} \langle J(c) \rangle \\ &= \sum_{h \in G} \langle K(h) \rangle \langle J(c) \rangle \qquad \text{by Example 9.2} \\ &= \sum_{h \in G} \langle J(c) \cup K(h) \rangle \\ &= \langle J(c) \cup K(\Omega) \rangle. \end{split}$$

The general case when c is a generalized color immediately follows.

1.3. Definition and first properties. Let $L = L_1 \cup \cdots \cup L_m$ be an oriented and framed link in S^3 presenting M by surgery in S^3 . The *linking matrix* A_L is the matrix whose nondiagonal (i, j)-entry is the linking number of L_i and L_j in S^3 if $i \neq j$ and whose diagonal (j, j)-entry is the framing number of L_j in S^3 . The linking matrix is symmetric and integral. Let $\operatorname{sign}_+(L)$ (resp. $\operatorname{sign}_-(L)$, resp. $\operatorname{sign}_0(L)$) denote the number of positive (resp. negative, resp. null) eigenvalues of $A_L \otimes \mathbb{R}$. Set $\operatorname{sign}_+(L) - \operatorname{sign}_-(L)$ and $\operatorname{rank}(L) = \operatorname{rank}(A_L)$. Note that $\operatorname{rank}(L) = \operatorname{sign}_+(L) + \operatorname{sign}_-(L)$ and $m = \operatorname{rank}(L) + \operatorname{sign}_0(L)$. Finally let U_{\pm} denote the oriented unknot with framing ± 1 .

We begin with a few observations about U_+ .

Lemma 1.2.

$$\langle U_{\pm}(\Omega) \rangle = \sum_{x \in G} t_x^{\pm 1}.$$

PROOF. Since $\langle U_{\pm}(x) \rangle = t_x^{\pm 1} \langle \varnothing \rangle = t_x^{\pm 1}$, the result follows.

COROLLARY 1.3. If q is nondegenerate, then $\langle U_{\pm}(\Omega) \rangle$ is invertible in $\mathscr{A}_{\mathbb{C}}(S^3)$.

Proof.

Lemma 1.4.

$$\langle U_{\pm}(\Omega) \rangle = \sum_{g \in G} \chi(\pm q(g)) = \Gamma_{\chi}(G, \pm q).$$

If $q|_{\text{Ker }\hat{b}} = 0$ (in particular, if q is nondegenerate) then $\langle U_{\pm}(\Omega) \rangle \neq 0$.

In particular, if $\chi(t) = \exp(2i\pi t)$ and q is nondegenerate, then $\langle U_{\pm}(\Omega) \rangle = \Gamma(G, \pm q) = \Gamma(G, q)^{\pm 1}$.

DEFINITION 10.1. We denote $L(\Omega)$ the framed link L each component of which is colored by Ω , i.e., we write $L(\Omega)$ instead of $L(\Omega, \ldots, \Omega)$.

THEOREM 1.5. The skein

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PROPOSITION 1.6. The complex number

 $\tau_{\gamma}(M) = |G|^{-\operatorname{sign}_{0}(L)/2} \langle U_{+}(\Omega) \rangle^{-\operatorname{sign}_{+}(L)} \langle U_{-}(\Omega) \rangle^{-\operatorname{sign}_{-}(L)} \langle L(\Omega) \rangle$

is a topological invariant of M.

To simplify notation, we shall write $\tau(M)$ instead of $\tau_{\chi}(M)$ if the character χ is self-understood.

PROOF. Lemma 1.1 implies the invariance under the sliding operation. For the invariance under the other move that replace L by the topologically disjoint union $L \coprod U_{\pm}$ of L and an unknot U_{\pm} with framing ± 1 , we observe that

$$\langle (L [U_{\pm})(\Omega) \rangle = \langle L(\Omega) \rangle \langle U_{\pm}(\Omega) \rangle.$$

Since $\operatorname{sign} \pm (L \coprod U_{\pm}) = \operatorname{sign}_{\pm}(L) + 1$ and $\operatorname{sign}_{\pm}(L \coprod U_{\mp}) = \operatorname{sign}_{\pm}(L)$, the result follows.

REMARK 10.2. If we use the normalized special color $\Omega' = \frac{1}{|G|} \sum_{g \in G} \delta_g$ instead of Ω then we obtain $\tau'(M) = |G|^{\operatorname{sign}_0(L)/2} \tau(M)$.

REMARK 10.3. The invariant τ is extended to non connected closed 3manifolds by setting $\tau(M \coprod N) = \tau(M) \cdot \tau(N)$ for any closed 3-manifolds M and N.

The invariant has a remarkable property with respect to connected sum and orientation reversal.

PROPOSITION 1.7. For closed oriented 3-manifolds M and N,

(1.2)
$$\tau(M \sharp N) = \tau(M) \cdot \tau(N).$$

PROOF. The connected sum $M \sharp N$ can be presented as the surgery on S^3 on two topologically disjoint links J and K. It follows that $\langle J(x) \cup K(x) \rangle = \langle J(x) \rangle \cdot \langle K(x) \rangle$ for any color $x \in L^2(G)$, in particular for the special color Ω . Another consequence is that $A_{J \cup K} = A_J \oplus A_K$, so $\operatorname{sign}_{\pm}(J \cup K) = \operatorname{sign}_{+}(J) + \operatorname{sign}_{+}(K)$. The result follows.

PROPOSITION 1.8. For any closed oriented 3-manifold M,

(1.3)
$$\tau(-M) = \overline{\tau(M)}.$$

PROOF. If $M = \chi(S^3, L)$ then $-M = \chi(S^3, L^{\min})$ where L^{\min} denotes the mirror image of the link L. Then $A_{L^{\min}} = -A_L$ hence $\operatorname{sign}_0(L^{\min}) = \operatorname{sign}_0(L)$ and $\operatorname{sign}_+(L^{\min}) = -\operatorname{sign}_{\pm}(L)$,

$$\tau(-M) = |G|^{-\operatorname{sign}_0(L^{\operatorname{mir}})/2} \langle U_+(\Omega) \rangle^{-\operatorname{sign}_+(L^{\operatorname{mir}})} \langle U_-(\Omega) \rangle^{-\operatorname{sign}_-(L^{\operatorname{mir}})} \langle L^{\operatorname{mir}}(\Omega) \rangle$$
$$= |G|^{-\operatorname{sign}_0(L)/2} \langle U_+(\Omega) \rangle^{-\operatorname{sign}_-(L)} \langle U_-(\Omega) \rangle^{-\operatorname{sign}_+(L)} \overline{\langle L(\Omega) \rangle}$$
$$= \overline{\tau(M)}$$

where we used Lemma 1.4 in the last equality.

We explicit the invariant $\tau(M)$ in terms of the quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$.

PROPOSITION 1.9. If q is nondegenerate then

(1.4)
$$\tau(M,q) = \gamma(G,q)^{-\operatorname{sign}(L)} |G|^{-m/2} \sum_{x \in G \otimes \mathbb{Z}^m} \exp(2\pi i (q \otimes A_L)(x))$$

PROOF. By definition $\langle L(\Omega) \rangle = \sum_{x \in G^m} \langle L(x) \rangle$. By Prop. 3.8,

$$\langle L(x) \rangle = \exp\left(2\pi i \, q \otimes \operatorname{lk}\left(\sum_{j} x_{j} \otimes L_{j}\right)\right) = \exp(2\pi i \, q \otimes A_{L}(x)).$$

The normalization term is

$$\begin{split} \langle U_{+}(\Omega) \rangle^{-\operatorname{sign}_{+}(L)} \langle U_{-}(\Omega) \rangle^{-\operatorname{sign}_{-}(L)} &= \Gamma(G,q)^{-\operatorname{sign}_{+}(L)} \overline{\Gamma(G,q)^{-\operatorname{sign}_{-}(L)}} \\ &= |\Gamma(G,q)|^{-\operatorname{sign}_{+}(L) - \operatorname{sign}_{-}(L)} \gamma(G,q)^{-\operatorname{sign}(L)} \\ &= |\Gamma(G,q)|^{-\operatorname{rank}(L)} \gamma(G,q)^{-\operatorname{sign}(L)} \\ &= |G|^{-\operatorname{rank}(L)/2} \gamma(G,q)^{-\operatorname{sign}(L)} \end{split}$$

The result follows.

The explicit expression (1.4) for $\tau(M,q)$ shows that $\tau(M,q)$ depends on q only up to isomorphism:

COROLLARY 1.10. For any $\psi \in \operatorname{Aut}(G)$, $\tau(M, \psi^* q) = \tau(M, q)$.

For the dependency of $\tau(M,q)$ under the action of $\operatorname{Hom}(tH_1(M),\mathbb{Z}/2\mathbb{Z})$, resp. $\operatorname{Hom}(tH_1(M),\mathbb{Q}/\mathbb{Z})$, see §3 and §??.

The case when G is a cyclic group was studied first by H. Murakami, T. Ohtsuki and M. Okada [69].

PROPOSITION 1.11. Let N be a positive integer. Let q_N be the homogeneous quadratic form defined on $\mathbb{Z}/N\mathbb{Z}$ by

$$q_N(1) = \begin{cases} 1/N \pmod{1} & \text{if } N \text{ is odd;} \\ 1/(2N) \pmod{1} & \text{if } N \text{ is even;} \end{cases}$$

Then $Z_N(M) = \tau(M, q_N)$.

The formula of Theorem 1.9 allows for explicit computations of the invariant in terms of surgery presentations.

EXERCISE 10.1. Using surgery presentations and the corresponding linking matrices, show that:

$$\tau(S^3) = 1, \ \tau(S^2 \times S^1) = |G|^{1/2}.$$

1.4. The relation with the linking group of M. Further analysis of the invariant $\tau(M, q)$ involves the linking group $(H_1(M), \lambda_M)$. It is shown in [13, Th. 1] that $\tau(M)$ depends only on the linking pairing λ_M on the torsion subgroup $tH_1(M)$ and the first Betti number $b_1(M) \in \mathbb{N}$. A more explicit relation is derived from the reciprocity formula [13, Th. 4].

THEOREM 1.12. Let $q_M : tH_1(M) \to \mathbb{Q}/\mathbb{Z}$ be a quadratic enhancement of the linking pairing λ_M . Let (V, f, v) be a lift of the quadratic group (G, q) to some lattice bilinear group endowed with an integral Wu class $v \in Wu(f)$. Then

(1.5) $\tau(M) = |H^1(M;G)|^{\frac{1}{2}} \gamma(tH_1(M),q_M)^{-f(v,v)} \gamma(V \otimes tH_1(M),f \otimes q_M).$

PROOF. The proof consists in an application of the reciprocity formula. See [13, Th. 4] for further details.

It follows that if $\tau(M) \neq 0$ then

(1.6) $|\tau(M)| = |H^1(M;G)|^{1/2}.$

In particular, we can make the following observation. Define an oriented closed 3-manifold M to be an *integral homology sphere* if $H_*(M) = 0$.

COROLLARY 1.13. *M* is an integral homology sphere if and only if $\tau(M, q) = 1$ for all quadratic groups (G, q).

PROOF. Since $H_1(M) = 0$, the linking group $(H_1(M), \lambda_M)$ is trivial. The converse follows from (1.6). Alternatively apply Prop. ?? to S^3 endowed with the trivial linking pairing.

The vanishing of $\tau(M,q)$ can also be described explicitly [13, Th. 6]. For ease of notation, denote by $b: G \times G \to \mathbb{Q}/\mathbb{Z}$ the linking pairing associated to q.

COROLLARY 1.14. The following assertions are equivalent:

- (1) $\tau(M,q) = 0;$
- (2) The characteristic element of $(G \otimes tH_1(M), b \otimes \lambda_M)$ is nontrivial;
- (3) The linking pairings (G, b) and $(tH_1(M), \lambda_M)$ have an isomorphic orthogonal cyclic summand of even order.

PROOF. Since q is nondegenerate, $\tau(M, q)$ is zero if and only if the Gauss sum $\gamma(V \otimes tH_1(M), f \otimes q_M)$ is zero. Now the quadratic form q_M can be presented as the discriminant quadratic form $\varphi_{g,w}$ associated to a lattice W with symmetric bilinear pairing g and integral Wu class w. Apply Corollary 1.16: the characteristic element of $b = \lambda_f$ and $\lambda_g = \lambda_M$ is nontrivial. This proves $(1) \Longrightarrow (2)$. The equivalence $(2) \Longrightarrow (3)$ follows from the definition of the characteristic element (or homomorphism).

The formula (1.5) allows for a complete description of the invariant $\tau(M)$, as obtained in [14].

THEOREM 1.15. Let M and N be two closed oriented connected 3-manifolds. The following assertions are equivalent:

(1) There is a linking group isomorphism $(H_1(M), \lambda_M) \simeq (H_1(N), \lambda_N);$ (2) $\tau(M, q) = \tau(N, q)$ for all quadratic groups $(G, q : G \to \mathbb{Q}/\mathbb{Z}).$ PROOF. (1) \Longrightarrow (2): Let ψ be a linking group isomorphism $(H_1(M), \lambda_M) \simeq (H_1(N), \lambda_N)$ so that $\psi^* \lambda_N = \lambda_M$. Lift ψ to an isomorphism $(H_1(M), q_M) \simeq (H_1(N), q_N)$, for some homogeneous quadratic forms q_M and q_N over λ_M and λ_N respectively. This is done as follows. First lift λ_N to a homogeneous quadratic form q_N (this is possible since $\text{Quad}^0(\lambda_N)$ is nonempty, see §??). Then let $q_M = \psi^* q_N$ so that ψ becomes an isomorphism $(H_1(M), q_M) \simeq (H_1(N), q_N)$. It follows that $H^1(M)$ and $H^1(N)$ are isomorphic and

$$\gamma(V \otimes \mathsf{t}H_1(M), f \otimes q_M) = \gamma(V \otimes \mathsf{t}H_1(N), f \otimes q_N)$$

for any lattice (V, f). Hence formula (1.5) shows that $\tau(M, q) = \tau(N, q)$.

(2) \implies (1): The idea is to consider $\tau(M,q)$ as a Gauss invariant of the linking group $(H_1(M), \lambda_M)$. Specifically we wish to apply Corollary 4.3. For this, we need to verify that the three hypotheses are satisfied. Since $|H^1(M,G)| = |H^1(N;G)|$, the first hypothesis is satisfied.

Therefore $\tau(M, q)$ depends on the one hand, on the quadratic form (G, q) up to isomorphism, and on the other hand, $\tau(M, q)$ classifies the linking group of M up to isomorphism.

In view of Theorem 1.12, the torsion groups G (endowed with the quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$) and $H_1(M)$ (endowed with the linking pairing) play similar rôles. Therefore, one is led to present τ as an algebraic pairing – a viewpoint adopted in [17].

2. The Abelian invariant as an algebraic pairing

Let $\mathfrak{Q}^0(0)$ denote the set of isomorphism classes of nondegenerate homogeneous quadratic forms on finite abelian groups. We define a related set $\mathfrak{M}(0)$ as follows. An element in $\mathfrak{M}(0)$ is represented by a linking group (V, f)(see §??). Two linking groups represent the same element in $\mathfrak{M}(0)$ if they are isomorphic. Both sets $\mathfrak{Q}^0(0)$ and $\mathfrak{M}(0)$ are monoids for the orthogonal sum \oplus , with the trivial form being the neutral element. By [**51**, Theorem 6.1], any linking group (H, λ) is realized as the linking group of a closed oriented 3-manifold M, that is, $\lambda_M = \lambda$, $H_1(M) = H$. Define a pairing on $\mathfrak{M}(0) \times \mathfrak{Q}^0$ by

(2.1)
$$\langle H, \lambda; G, q \rangle = \tau(M, q)$$

for any closed oriented 3-manifold M realizing the linking group (H, λ) .

PROPOSITION 2.1. The pairing $\langle -; - \rangle : \mathfrak{M}(0) \times \mathfrak{Q}^{0}(0)$ is well defined and bimultiplicative.

PROOF. The discussion above on $\tau(M,q)$ ensures that $\langle H, \lambda; q \rangle$ is independent of the particular choice of such a 3-manifold M. In view of Proposition 1.9, the pairing is indeed multiplicative on orthogonal sums with respect to $\mathfrak{Q}^0(0)$. Theorem 1.12 also implies that the pairing is multiplicative on orthogonal sums with respect to $\mathfrak{M}(0)$: the quadratic enhancement of the orthogonal sum of two linking groups (H, λ) and (H', λ) can be taken as

the orthogonal sum of two quadratic enhancements q and q' of λ and λ' respectively.

One can alternatively present the pairing $\langle -, - \rangle$ in a purely algebraic fashion, without reference to the 3-manifold, as follows. Let (W,g) be any bilinear lattice that lifts the linking group $(H, -\lambda)$ via the discriminant construction. Then

(2.2)
$$\langle H, \lambda; G, q \rangle = \gamma(G, q)^{-\operatorname{sign}(g)} \gamma(G \otimes W, q \otimes g) |G \otimes H|^{\frac{1}{2}}.$$

Note that $G \otimes H$ is finite since G is finite.

Let (tH, q_H) be a quadratic enhancement of the linking pairing $(tH, -\lambda)$ and let (V, f, v) be a lift of the quadratic group (G, q) as before. By Theorem 1.12, the relation above can be rewritten as

(2.3)
$$\langle H, \lambda; G, q \rangle = \gamma(\mathfrak{t}H, q_H)^{f(v,v)} \gamma(V \otimes H, f \otimes q_H) |G \otimes H|^{\frac{1}{2}}$$

Bimultiplicativity of the pairing $\langle -, - \rangle$ means that

$$\langle (H,\lambda) \oplus (H',\lambda'); G,q \rangle = \langle H,\lambda; G,q \rangle \cdot \langle H',\lambda'; G,q \rangle,$$

and

$$\langle H, \lambda; (G,q) \oplus (G',q') \rangle = \langle H, \lambda; G,q \rangle \cdot \langle H,\lambda; G',q' \rangle$$

The following result, proved in [17, Th. 1], shows how the linking pairing λ_M is recaptured through $\tau(M)$. See also [14].

THEOREM 2.2. The pairing $\langle -, - \rangle : \mathfrak{M}(0) \times \mathfrak{Q}^0(0) \to \mathbb{C}$ is nondegenerate.

3. Extension to Spin structures

A spin structure σ on M is a trivialization considered up to homotopy of the tangent bundle over the 1-skeleton M^1 that extends over the 2-skeleton M^2 of M. Spin structures on a closed oriented 3-manifold M exist because of the vanishing of the Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$. The set Spin(M) of spin structures on M is in bijective correspondence with the group $H^1(M; \mathbb{Z}/2\mathbb{Z})$. More precisely, there is a canonical action

 $\operatorname{Spin}(M) \times H^1(M; \mathbb{Z}/2\mathbb{Z}) \to \operatorname{Spin}(M), \ (\sigma, \alpha) \mapsto \sigma \cdot \alpha$

that is free and transitive. A spin structure σ on M induces a canonical quadratic refinement

$$q_{\sigma}: \mathrm{t}H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

of the linking pairing λ_M . The map

$$\operatorname{Spin}(M) \to \operatorname{Quad}(\lambda_M), \ \sigma \mapsto q_{\sigma}$$

is affine over the natural epimorphism $H^1(M; \mathbb{Z}/2\mathbb{Z}) \to \text{Hom}(tH_1(M), \mathbb{Z}/2\mathbb{Z})$ in the sense that

$$q_{\sigma \cdot \alpha} = q_{\sigma} + \tilde{\alpha}, \ \alpha \in H^1(M; \mathbb{Z}/2\mathbb{Z}).$$

For all these facts, see for instance [53], [65], [37], [61]. With this fact in mind, it is interesting to observe that Theorem 1.12 has the following interpretation.

THEOREM 3.1. Let (V, f, v) be a lift of the quadratic group (G, q) to some lattice bilinear group endowed with an integral Wu class $v \in Wu(f)$. For any spin structure $\sigma \in Spin(M)$,

(3.1)
$$\tau(M) = |H^1(M;G)|^{\frac{1}{2}} \gamma(\mathsf{t}H_1(M), q_\sigma)^{-f(v,v)} \gamma(V \otimes \mathsf{t}H_1(M), f \otimes q_\sigma).$$

In other words, the computation of $\tau(M)$ uses the choice of a spin structure on M but it is independent of that particular choice.

The results of the previous paragraph suggest to refine the construction. A topological invariant τ^{spin} of (M, σ) may be defined as follows. Fix a linking pairing, i.e. an element $(H, \lambda) \in \mathfrak{M}(0)$.

PROPOSITION 3.2. The complex number

(3.2)
$$\tau^{\text{spin}}(M,\sigma) = \langle H, \lambda; tH_1(M), q_\sigma \rangle$$

is a topological invariant of the pair (M, σ) .

In other words, we reverse roles: the original quadratic form q of the definition is now that induced by the spin structure σ and the linking pairing that was previously induced by the 3-manifold M, is now algebraic.

REMARK 10.4. By means of (2.2), the expression for τ^{spin} can be made explicit.

(3.3)

$$\tau^{\operatorname{spin}}(M,\sigma) = \gamma(\operatorname{t} H_1(M), q_{\sigma})^{-\operatorname{sign}(g)} \gamma(\operatorname{t} H_1(M) \otimes W, q_{\sigma} \otimes g) |\operatorname{t} H_1(M) \otimes H|^{\frac{1}{2}},$$

where $(G_g, \lambda_g) = (H, \lambda).$

PROPOSITION 3.3. Let $\alpha \in tH_1(M)$ have order 2. Then

(3.4)
$$\tau^{\rm spin}(M,\sigma\cdot\alpha) = e^{2\pi i q_{\sigma}(\alpha)} \left(\sigma(g) - g(w,w)\right) \tau^{\rm spin}(M,\sigma)$$

for any integral Wu class $w \in Wu^W(g)$.

Here we identified $\alpha \in tH_1(M)$ with $\widehat{\lambda}_M(\alpha) \in \text{Hom}(tH_1(M), \mathbb{Z}/2\mathbb{Z})$. It is true that if $b_1(M) > 0$ then there are several lifts of α in $H^1(M; \mathbb{Z}/2\mathbb{Z})$; it is part of the statement of Proposition 3.3 that the left hand side of (3.4) is independent of the particular choice of lift.

REMARK 10.5. The quantity $\sigma(g) - g(w, w) \mod 8$ is the Brown invariant¹ of the discriminant homogeneous quadratic form $\varphi_{g,w}$, which is a quadratic refinement of λ .

EXERCISE 10.2. The result above implies that the right hand side of (3.4) must be independent of the choice of the quadratic refinement and depends only on the linking pairing λ . Let us try to verify this fact directly. Changing the quadratic refinement is equivalent to choosing a new integral Wu class w + 2t for g with $2t \in 2W^{\sharp} \cap W$. Compute

$$g(w + 2t, w + 2t) - g(w, w) = g(w, 2t) + g(2t, w) + g(2t, 2t)$$

= 4[g_Q(w, t) + g_Q(t, t)].

¹The original Brown invariant was defined for even symmetric bilinear pairings, i.e. when w = 0.

Now $q_{\sigma}(\alpha)$ has order at most 4 since α has order 2. Therefore, one must have

$$4[g(w,t) + g_{\mathbb{Q}}(t,t)] = 0 \mod 4.$$

This implies that $g_{\mathbb{Q}}(w,t) + g_{\mathbb{Q}}(t,t) \in \mathbb{Z}$. But $g_{\mathbb{Q}}(w,t) \in \mathbb{Z}$ so the computation implies that $g_{\mathbb{Q}}(t,t) \in \mathbb{Z}$, which cannot hold in general. Explain why this computation does not contradict Proposition 3.3.

For the following result, we denote the invariant by $\tau^{\text{spin}}(M, \sigma; H, \lambda)$ so as to express the dependence of the invariant on the choice of a linking group (H, λ) .

PROPOSITION 3.4. Let (M, σ_M) and (N, σ_N) two closed oriented connected spin 3-manifolds. The following assertions are equivalent:

(1) There is a quadratic group isomorphism $(H_1(M), q_M) \simeq (H_1(N), q_N);$ (2) $\tau^{\text{spin}}(M, \sigma; H, \lambda) = \tau^{\text{spin}}(N, \sigma_N; H, \lambda)$ for any $(H, \lambda) \in \mathfrak{M}(0).$

PROOF. See [17, Th. 2].

A nice consequence is that τ^{spin} classifies Y^{spin} equivalent spin manifolds in Massuyeau's spin refinement of the Goussarov-Habiro theory (see [61]).

4. Extension to Spin^c structures

The idea of extending the invariant τ to Spin^c structures is formally similar to the extension to Spin structures. A Spin^c structure on an oriented closed connected 3-manifold M is a complex structure (considered up to homotopy) on the 2-skeleton M^2 that extends to M. For references on Spin^c structures, see [**37**] and [**18**]. The set Spin^c(M) of Spin^c structures on M is acted on freely and transitively by $H^2(M; \mathbb{Z})$.

The basic observation consists in the interpretation of Spin^c structures as quadratic refinements of the linking pairing, as in [18, Th. 2.3]. First define a modified linking pairing

$$\lambda'_M: H_2(M; \mathbb{Q}/\mathbb{Z}) \times H_2(M; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

by the formula

$$\lambda'_M = \lambda_M \circ (\beta_M \times \beta_M),$$

where $\beta_M : H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M)$ denotes the Bockstein homomorphism. This modified linking pairing is nondegenerate if and only if M is a rational homology 3-sphere. There is a natural $H^2(M; \mathbb{Z})$ -embedding

(4.1)
$$\operatorname{Spin}^{c}(M) \to \operatorname{Q}(\lambda'_{M}), \ s \mapsto q^{s}$$

See [18, §2] for further details. To the Chern class $c(\sigma) \in H^2(M)$ corresponds the difference $d_{q^{\sigma}} : H_2(M; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ defined by $d_{q^{\sigma}}(x) = q^{\sigma}(x) - q^{\sigma}(-x) = \langle c(\sigma), x \rangle, \ x \in H_2(M; \mathbb{Q}/\mathbb{Z})$. A spin^c structure σ on M is torsion if its associated Chern class $c(\sigma) \in H^2(M)$ is torsion. The quadratic refinement q^{σ} is nondegenerate if and only if q^{σ} vanishes on $H_2(M) \otimes \mathbb{Q}/\mathbb{Z}$ if and only if σ is torsion. In this case, the quadratic function q^{σ} factors through a unique quadratic refinement of the usual linking pairing λ_M . In

particular, this is the case if M is a rational homology 3-sphere. As in [15], we shall consider only *torsion* spin^c structures.

A Spin structure induces naturally a Spin^c structure, hence there is a natural map Spin(M) \rightarrow Spin^c(M). This map is injective if M is a rational homology 3-sphere. The image of this map is the set of Spin^c structures σ with trivial Chern class $c_1(\sigma) = 0$. This map and the maps above fit into the commutative diagram

$$\begin{array}{c} \operatorname{Spin}(M) \longrightarrow \operatorname{Spin}^{c}(M) \\ \downarrow & \downarrow \\ \operatorname{Quad}(\lambda_{M}) \longrightarrow \operatorname{Quad}(\lambda'_{M}). \end{array}$$

The extension of the invariant τ to Spin^c structures is not obvious, however, because a priori one has to define a tensor product involving *non homogeneous* quadratic functions. If we try to mimic the definition of the spin case, then we run into the problem of defining the tensor product $q \otimes g$ where qis a possibly non homogeneous quadratic function². An alternative product (and the corresponding extension) is proposed in [17], but the extension in question is shown to fail to have the property of classifying degree 0 invariants of complex spin structures (See [19, §3] for the foundations of the theory of finite type invariants of complex spin structures). Another extension suggested at the end of the same paper corresponds to the Gaussian invariant used in the classification of general quadratic functions described here in §?? (Th. 3.1).

 $^{^{2}}$ Lemma 1 of [15] applies in fact only to homogeneous quadratic functions.

CHAPTER 11

The invariant τ for an oriented framed link in a closed 3-manifold

The Abelian invariant $\tau(M)$ of the chapter extends to an invariant of an oriented framed link in M. The idea consists in extending the Kirby calculus to closed 3-manifolds equipped with oriented framed links. The extension is natural and the skein calculus developped in §suits well that extension.

In particular, we prove a conjecture stated in [?]...

1. The Kirby calculus for oriented framed links in 3-manifolds

There is a "Kirby calculus" generalized to the setting of oriented framed links in closed oriented connected 3-manifolds. Consider a pair of oriented framed links

$$L = L_1 \cup \cdots \cup L_m, \quad J = J_1 \cup \cdots \cup J_n$$

sitting inside the 3-sphere S^3 . Performing surgery on the framed link Lyields a closed 3-manifold $M = \chi(S^3, L)$ that contains an oriented framed link $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ induced by the link $J \subset S^3$. Two oriented framed links (M, \mathcal{L}_M) and (N, \mathcal{L}_N) in closed oriented 3-manifolds are *equivalent* if there is an orientation preserving diffeomorphism $f : M \to N$ such that $f(\mathcal{L}_M) = \mathcal{L}_N$. (Here the framed links should be understood collections of small embedded annuli.) Note that if two links \mathcal{K} and \mathcal{L} are isotopic in N (as oriented framed ordered links), then (N, \mathcal{K}) and (N, \mathcal{L}) are equivalent. One observes that a slight generalization of the Lickorish–Wallace theorem holds: up to equivalence, any pair (M, \mathcal{L}) consisting of a closed oriented connected 3-manifold M and an n-component oriented framed link $\mathcal{L} \subset M$ is the result of surgery on a framed, partially oriented link $L_1 \cup \cdots \cup L_m \cup J_1 \cup \cdots J_n$ where

- the surgery is performed on the framed link L;
- the link $J = J_1 \cup \cdots \cup J_n$ is oriented.

It is useful to think of such a surgery as a "partial surgery": namely the surgery is performed on the first framed link L and nothing is done on the remaining components (that is the components of the oriented link J).

There is also an analog of Kirby's theorem. In order to state it, we briefly describe a number of reversible moves. The first move is the stabilization move (creation or annihilation of a trivial component with ± 1 -framing, unlinked from all other component) and involves only the first link L (the surgery link) indicated in Fig. 1.1. The second move comes in two forms:

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the first handle slide is one component of the first link L sliding over another component of L; the second handle slide is one component of the second link J sliding over one component of L. See Fig. 1.1 below.



FIGURE 1.1. The sliding moves for links in 3-manifolds.

THEOREM 1.1. Two pairs of disjoint framed links (L, J) and (L', J') in S^3 present equivalent pairs (M, \mathcal{L}) if and only if, up to reordering of the components of L and up to isotopy, they are related by a finite sequence of stabilization and handle slide moves.

REMARK 11.1. The classical Kirby theorem corresponds to the special case $J = J' = \emptyset$.

PROOF. This is straightforward from the usual Kirby's theorem. For each Kirby move $L_1 \to L_2$, we consider the corresponding diffeomorphism $\chi(S^3, L_1) \to \chi(S^3, L_2)$ of 3-manifolds and write down combinatorially the image of the link \mathcal{L} in $\chi(S^3, L_2)$.

2. Definition and invariance

Let $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ be an oriented and framed link in a closed oriented connected 3-manifold M. According to the previous section, the pair (M, \mathcal{L}) is presented by surgery by a pair

$$L = L_1 \cup \cdots \cup L_m, \ J = J_1 \cup \cdots \cup J_n$$

of disjoint framed oriented links in S^3 : the manifold M is obtained by surgery on L as before while the framed link \mathcal{L} in M^3 is the image of the framed link J after the surgery is performed on L. We order the set of components of J, so that after the surgery the set of components of \mathcal{L} inherit that order.

Now we fix as in the previous chapter a quadratic group $(G, q : G \to \mathbb{Q}/\mathbb{Z})$. Furthermore, we ascribe to each component of the oriented link \mathcal{L} a color $c \in L^2(G)$. **PROPOSITION 2.1.** The complex number

$$\tau(M,\mathcal{L}(c)) = |G|^{-\operatorname{sign}_0(L)/2} \langle U_+(\Omega) \rangle^{-\operatorname{sign}_+(L)} \langle U_-(\Omega) \rangle^{-\operatorname{sign}_-(L)} \langle L(\Omega) \cup J(c) \rangle$$

is a topological invariant of (M, \mathcal{L}) .

PROOF. The invariance under stabilization is the same as for Prop. 1.6. The invariance under the handle slide move follows from Lemma 1.1.

Proposition 2.1 generalizes Proposition 1.6. The invariant $\tau(M)$ considered in the previous chapter is recovered if the link \mathcal{L} is empty or the color c is trivial:

$$\tau(M) = \tau(M, \emptyset) = \tau(M, \mathcal{L}(0)).$$

Similarly as in the previous chapter, we give an explicit expression for $\tau(M, \mathcal{L})$ in terms of the quadratic group (G, q). It suffices to consider the case when the colors of the components of \mathcal{L} are simple. Simple colors are identified with elements of G. For each $1 \leq j \leq n$, ascribe a color $c_j \in G$ to \mathcal{L}_j . Choose an arbitrary orientation and order for the surgery link L. The number m is the number of components of L and the number n is the number of components of L and the number n is the number of components of L and the number n is the number of components of J. Denote by $A = A_{L\cup J}$ the linking matrix of the link $L \cup J$ in S^3 . This is an integral symmetric matrix of size m + n. Denote as before by $\operatorname{sign}(L) \in \mathbb{Z}$ the signature of $A_L \otimes \mathbb{R}$. The tensor product $q \otimes A$ can be regarded as a homogeneous quadratic form $G^{m+n} = G^m \times G^n \to \mathbb{Q}/\mathbb{Z}$.

PROPOSITION 2.2. Let q be nondegenerate and let $c = (c_1, \ldots, c_n)$ be an n-uple of simple colors. Then

(2.1)
$$\tau(M, \mathcal{L}(c)) = \gamma(G, q)^{-\operatorname{sign}(L)} |G|^{-m/2} \sum_{x \in G^m} \exp(2\pi i (q \otimes A_{L \cup J})(x, c)).$$

REMARK 11.2. For a subset $J \subseteq \{1, \ldots, n\}$, let $\mathcal{L}^J = \bigcup_{j \in J}$ denote the corresponding sublink of \mathcal{L} . If \mathcal{L} is ordered, then \mathcal{L}^J is also ordered. If $c = (c_i)_{1 \leq i \leq n} \in (L^2(G))^n$ is the color vector for the ordered link \mathcal{L} , then $c^J = (c_j)_{j \in J} \in (L^2(G))^{|J|}$ is the color vector for \mathcal{L}^J . Let $c \in G^n$ be a color vector such that $c_i = 0$ if $i \notin J$. Then

(2.2)
$$\tau(M, \mathcal{L}(c)) = \tau(M, \mathcal{L}^J(c^J)).$$

PROPOSITION 2.3. For pairs of links (M, \mathcal{L}) and (N, \mathcal{L}') ,

$$\tau((M, \mathcal{L}(c)) \sharp (N, \mathcal{L}'(c'))) = \tau(M, \mathcal{L}(c)) \cdot \tau(N, \mathcal{L}'(c')).$$

PROOF. One can present $(M, \mathcal{L}) \sharp (N, \mathcal{L}')$ as the disjoint union

$$L \cup L' \cup J \sharp J' \subset S^3$$

where L and L' are topologically disjoint framed links in S^3 such that $\chi(S^3, L) = M$ and $\chi(S^3, L') = N$.

3. The vanishing of the invariant

We give a necessary and sufficient condition for $\tau(M, \mathcal{L}(c))$ to vanish. This condition has a particular importance because of its decisive rôle in the construction of the topological quantum field theory in §.... This condition is expressed by means of the tensor product of linking pairings and the characteristic homomorphism. We assume throughout this section that the quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate and that the color of each component of \mathcal{L} is simple. The nondegenerate symmetric bilinear pairing associated to the quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ is denoted b_q .

THEOREM 3.1. $\tau(M, \mathcal{L}(c))$ is nonzero if and only if the class

$$\sum_j c_j \otimes [\mathcal{L}_j]$$

in $H_1(M;G)$ is the characteristic element of (G, b_q) and $(tH_1(M), \lambda_M)$, i.e., if and only if

$$\theta_{b_q,\lambda_M} = \sum_j c_j \otimes [\mathcal{L}_j].$$

Recall (cf. §??) that the characteristic element $\theta = \theta_{b_q,\lambda_M}$ lives in $G \otimes tH_1(M)$.

PROOF. First, the fact that $\sum_j c_j \otimes [\mathcal{L}_j]$ must lie in $G \otimes tH_1(M)$ is a necessary condition for $\tau(M, \mathcal{L}; q, c)$ to be nonzero is proved in [15, Th. 3 (1)].

Next, we observe from (??) that $\tau(M, \mathcal{L}; q, c) = 0$ if and only if the last Gauss sum on the right hand of (??) is zero. Developping the term $q \otimes A_{L\cup J}(x, c)$ in terms of the block decomposition of the matrix $A_{L\cup J} = \begin{bmatrix} A_L & A_{L,J} \\ A_{J,L} & A_J \end{bmatrix}$ enables to rewrite the Gauss sum as a product of a nonzero complex number and the Gauss sum

$$\gamma(G \otimes \mathbb{Z}^m, q \otimes A_L + (b_q \otimes A_{J,L})(-,c)) = \gamma(G \otimes W, q \otimes g + (\widehat{b}_q \otimes \widehat{g}_{\mathbb{Q}})([\omega]),$$

where g denotes the symmetric bilinear map on $W = \mathbb{Z}^m$ determined the $m \times m$ linking matrix A_L and $\omega \in G \otimes W^{\sharp}$ is a lift of $\sum_j c_j \otimes [\mathcal{L}_j] \in G \otimes W^{\sharp}/W = G \otimes \text{Tors } H_1(M)$. Now we apply the formula (9.1) to obtain the identity

$$q \otimes g + (b_q \otimes g_{\mathbb{Q}})(-,\omega) = \varphi_{f \otimes q, v \otimes w - 2t} \circ j_f,$$

where (V, f, v) is a bilinear lattice equipped with an integral Wu class v for f such that $(G_f, \varphi_{f,v}) = (G, q)(^1)$, where w is a Wu class for g and where $t \in V^{\sharp} \otimes W^{\sharp}$ is a lift of $\omega \in G \otimes W^{\sharp}$.

Finally we apply Th. 1.14, condition (4): $\gamma(G \otimes W, \varphi_{f \otimes g, v \otimes w - 2t} \circ j_f) \neq 0$ if and only if $\psi(\omega) = \theta \in G_f \otimes G_g$ is the characteristic element for $\lambda_f \otimes \lambda_g = b_q \otimes \lambda_M$. This is the desired result.

¹Such a triple exists by Th. 4.6.

COROLLARY 3.2. If $\tau(M, \mathcal{L}; q, c) \neq 0$ then $\sum_j c_j \otimes [\mathcal{L}_j]$ lies in $G \otimes \text{Tors } H_1(M)$ and has order at most 2. In particular, if $\tau(M, \mathcal{L}, q, c) \neq 0$ and at least one of the two groups G or $\text{Tors } H_1(M)$ has odd order, then $\sum_j c_j \otimes [\mathcal{L}_j] = 0$.

PROOF. The characteristic element θ lies by definition in $G \otimes \text{Tors } H_1(M)$ and satisfies $2 \theta = 0$ (see the end of §??). The first statement follows. The second statement is a consequence of the first one.

COROLLARY 3.3. The following assertions are equivalent:

- (1) $\tau(M,q) \neq 0;$
- (2) The characteristic element θ_{b_q,λ_M} is zero;
- (3) (G, b_q) and (Tors $H_1(M), \lambda_M$) have no common orthogonal cyclic summand of even order.

PROOF. Apply Th. 3.1 with $\mathcal{L} = \emptyset$. Then $\sum_j c_j \otimes \mathcal{L}_j = 0$. This gives (1) \Leftrightarrow (2). The equivalence (2) \Leftrightarrow (3) follows from the definition of the characteristic element.

4. Classification results: topology

In this paragraph, we investigate the sensitivity of the invariant $\tau(M, \theta)$ to the topology of (M, \mathcal{L}) .

There is a well defined notion of linking numbers of cycles whenever they represent torsion elements in homology. A *framing* of a smooth 1-cycle Z in M is a framing on each of its components. We denote by Z' the parallel copy of Z in M. The framing allows to define a number $q_M^{\text{fr}}(Z) \in \mathbb{Q}$ by the formula

$$q_M^{\rm fr}(Z) = \frac{1}{2} {\rm lk}_M(Z, Z') \in \mathbb{Q}.$$

If Z does not represent a torsion element in $H_1(M)$, then $q_M^{\text{fr}}(Z)$ is undefined.

THEOREM 4.1. Let (M, \mathcal{L}) and (N, \mathcal{J}) be two closed oriented connected 3manifolds equipped with oriented and framed n-component links \mathcal{L} and \mathcal{J} . The following two assertions are equivalent:

I. (i) There is an isomorphism

$$(H_1(M), \lambda_M, [\mathcal{L}_1], \dots, [\mathcal{L}_n]) \simeq (H_1(N), \lambda_N, [\mathcal{J}_1], \dots, [\mathcal{J}_n])$$

of pointed linking groups;

(ii) The rational linking and framing numbers are equal:

 $\operatorname{lk}_M(\mathcal{L}_i, \mathcal{L}_j) = \operatorname{lk}_N(\mathcal{J}_i, \mathcal{J}_j) \text{ and } q_M^{\operatorname{fr}}(\mathcal{L}_i) = q_N^{\operatorname{fr}}(\mathcal{J}_i) \text{ for all } 1 \leq i < j \leq n.$

II. $\tau(M, \mathcal{L}, q, c) = \tau(N, \mathcal{J}; q, c)$ for any quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$ equipped with $c \in G^n$.

The proof is dealt with in the next section. We first consider two simples examples of applications of Theorem 4.1.

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EXAMPLE 11.1. Consider the two pairs of links in S^3 representing two oriented knots (M, \mathcal{L}) and (N, \mathcal{J}) respectively (see Fig. 4.1). Since the surgery links are the same, $M = N = (S^1 \times S^2) \ \sharp \ L(5,1)$ (the connected sum of $S^1 \times S^2$ and a lens space). Set an arbitrary framing $\star \in \mathbb{Z}$ for the component $J_1 \subset S^3$ (in red in the figure; it should be the same for both components labelled J_1).



FIGURE 4.1. Two pairs of links representing two oriented knots in $(S^1 \times S^2) \not\equiv L(5,1)$.

We have

$$H_1(M) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}, \quad \lambda_M(x,y) = \frac{xy}{5} \mod 1.$$

Under the identification above, $[\mathcal{L}] = (5,1)$ and $[\mathcal{J}] = (5,3)$ respectively. The extended linking matrices are $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & \star \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 5 & 3 \\ 2 & 3 & \star \end{bmatrix}$ respec-

tively.

Let (G, q, c) be a pointed quadratic form. The characteristic element for (b_a, λ_M) is zero since Tors $H_1(M)$ has odd order. Hence $c \otimes [\mathcal{L}]$ is characteristic if and only if $c \otimes [\mathcal{L}]$ is zero. Consider the case when G is cyclic of order n. Let k be the 5-valuation of n. Let $n' = \begin{cases} n & \text{if } k = 0, 1; \\ n/5 & \text{otherwise.} \end{cases}$. Then $c \otimes [\mathcal{L}] = 0$ if and only if $c = 0 \mod n'$ (if and only if $c \otimes [\mathcal{J}] = 0$). Let n = 25 and $c = 5 \mod 25$. Using the definition (??) of τ and the observation above on the characteristic element, we find that

$$\tau(M,\mathcal{L};q,c) \neq \tau(M,\mathcal{J};q,c).$$

Thus there is no isomorphism $\phi: H_1(M) \to H_1(M)$ such that $\phi([\mathcal{L}]) = [\mathcal{J}]$ and $\phi^*(\lambda_M) = \lambda_M$.

EXAMPLE 11.2. Consider again the two oriented and framed knots \mathcal{L} and \mathcal{J} in $S^1 \times S^2$ presented in Fig. ??. Here $M = N = S^1 \times S^2$. Let (G, q, c) be a pointed quadratic form. It is not hard to see that $c \otimes [\mathcal{L}]$ is characteristic if and only if c = 0 in G. We conclude immediately that

$$\tau(S^1 \times S^2, \mathcal{L}; q, c) = \tau(S^1 \times S^2, \mathcal{J}; q, c).$$

It follows that there exists an isomorphism $\phi: H_1(M) \to H_1(M)$ such that $\phi([\mathcal{L}]) = [\mathcal{J}]$ and $\phi^*(\lambda_M) = \lambda_M$. (This isomorphism is actually induced by a diffeomorphism of the pair $(S^1 \times S^2, \mathcal{L})$, as described in the Example ??.)

COROLLARY 4.2. Let (M, \mathcal{L}) and (N, \mathcal{J}) be two closed oriented connected 3-manifolds with framed oriented n-component links \mathcal{L} and \mathcal{J} . Assume that none of the components of \mathcal{L} represents a torsion element in $H_1(M)$. Then the following assertions are equivalent:

I. There is an isomorphism

$$(H_1(M), \lambda_M, [\mathcal{L}_1], \dots, [\mathcal{L}_n]) \simeq (H_1(N), \lambda_N, [\mathcal{J}_1], \dots, [\mathcal{J}_n])$$

of pointed linking groups.

II. $\tau(M, \mathcal{L}, q, c) = \tau(N, \mathcal{J}; q, c)$ for any quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$ equipped with $c \in G^n$.

5. The proof of the classification theorem

Both implications² will be derived from the formula of [15, Th. 3] which we recall and slight adapt to our notation. To state this formula, there are a number of choices to make (although the final result does not depend on the particular choices made). Choose a spin structure s on M, inducing a homogeneous quadratic refinement q^s : Tors $H_1(M) \to \mathbb{Q}/\mathbb{Z}$ of the linking pairing λ_M on M. Choose a lattice pairing (V, f, v) equipped with an integral Wu class such that $(V^{\sharp}/V, \varphi_{f,v}) = (G, q)$.

Assume first that $\sum_j c_j \otimes \mathcal{L}_j \in G \otimes \text{Tors } H_1(M)$. Choose a framed 1-cycle $\eta = \sum_j \xi_j \otimes \mathcal{L}_j$ by lifting the coefficients $c_j \in G = V^{\sharp}/V$ to $\xi_j \in V^{\sharp}$. Then $[\eta] \in V^{\sharp} \otimes \text{Tors } H_1(M)$. Evaluating η against the Wu class $v \in V$ yields a framed integral 1-cycle $\eta_v = \sum_j f_{\mathbb{Q}}(v,\xi_j)\mathcal{L}_j$. Note that this cycle represents a torsion element. There is an invariant of framed 1-cycles δ_s defined by the following conditions (see [15, §2.3, Lemma 14]):

- (1) δ_s is a \mathbb{Z} -homomorphism and takes values in $\{0, 1/2\} = \frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z};$
- (2) δ_s depends on the spin structure s on M;
- (3) δ_s vanishes exactly on framed 1-cycles for which the spin structure s and the framing are compatible.

In the case when the framed cycle σ represents a torsion element, we have

$$\delta_s(\sigma) = q^{\rm fr}(\sigma) - q^s([\sigma]).$$

In particular, for $\sigma = \eta_v$, we have

$$\delta_s(\eta_v) = \sum_j f_{\mathbb{Q}}(v,\xi_j) \delta_s(\mathcal{L}_j).$$

Hence

(5.1)
$$\delta_s(\eta_v) = \sum_j f_{\mathbb{Q}}(v,\xi_j) \Big(q^{\mathrm{fr}}(\mathcal{L}_j) - q^s([\mathcal{L}_j]) \Big) \in 1/2\mathbb{Z}/\mathbb{Z}.$$

In particular, if $v = 0 \mod 2$, then $\delta_s(\eta_v) = 0$.

²This is not strictly necessary for the direct implication (I) \implies (II); however the idea of using characteristic elements is a key ingredient in the converse.

Using the fact that η also has a framing, we can slightly generalize the definition of $q^{\rm fr}$ above by defining

$$(f \otimes q^{\mathrm{fr}})(\eta) = \sum_{j} f_{\mathbb{Q}}(\xi_j, \xi_j) \ q^{\mathrm{fr}}(\mathcal{L}_j) + \sum_{j < k} f_{\mathbb{Q}}(\xi_j, \xi_k) \ \mathrm{lk}_M(\mathcal{L}_j, \mathcal{L}_k) \in \mathbb{Q}.$$

A fundamental formula [15, Th. 3] is the relation

(5.2)

$$\tau(M,\mathcal{L};q,c) = e^{2\pi i \left((f \otimes q^{\mathrm{fr}})(\eta) - \delta_s(\eta_v) \right)} \gamma(\mathrm{Tors} \ H_1(M),q_s)^{-f_{\mathbb{Q}}(v,v)}$$

$$\gamma\left(V \otimes \mathrm{Tors} \ H_1(M), f \otimes q_s + \left(\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_M \right)([\eta]) \right) |H^1(M;G)|^{1/2}.$$

Here $f \otimes q_s$ denotes the quadratic function $V \otimes \text{Tors } H_1(M) \to \mathbb{Q}/\mathbb{Z}$ over $f \otimes \lambda_M$ defined by

$$x \otimes y \mapsto f(x, x) q_s(y).$$

The map $(\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_M)([\eta])$ is the map on $V \otimes \text{Tors } H_1(M)$ induced by the map adjoint to the linking pairing $f \otimes \lambda_M$ at $[\eta] \in V^{\sharp} \times \text{Tors } H_1(M)$. Explicitly

$$V \otimes \text{Tors } H_1(M) \to \mathbb{Q}/\mathbb{Z}, \ z \mapsto (f_{\mathbb{Q}} \otimes \lambda_M)([\eta], z).$$

Note that this map is well defined and non-trivial in general since $[\eta] \in V^{\sharp} \otimes \text{Tors } H_1(M)$.

Let us prove the implication $(I) \Longrightarrow (II)$. Consider the case

$$\sum_{j} c_{j} \otimes [\mathcal{L}_{j}] \notin G \otimes \text{Tors } H_{1}(M)$$

first. By Cor. 3.2, $\tau(M, \mathcal{L}; q, c) = 0 = \tau(N, \mathcal{J}; q, c)$, which is the desired result. Consider next the case

$$\sum_{j} c_j \otimes [\mathcal{L}_j] \in G \otimes \text{Tors } H_1(M).$$

Since $\phi^* \lambda_N = \lambda_M$, choose a spin structure s' on N such that $\phi^* q_N^{s'} = q_M^s$. Thus $q_N^{s'} \simeq q_M^s$ and their associated Gauss sums are equal. Set $\eta_M = \sum_j \xi_j \otimes \mathcal{L}_j$ and $\eta_N = \sum_j \xi_j \otimes \mathcal{J}_j$. The isomorphism $\phi : H_1(M) \to H_1(N)$ induces an isomorphism $1_{V^{\sharp}} \otimes \phi$ sending $[\eta_M] \in V^{\sharp} \otimes H_1(M)$ to $[\eta_N] \in V^{\sharp} \otimes H_1(N)$. The isomorphism $1_{V^{\sharp}} \otimes \phi$ induces an isomorphism

$$f \otimes q_M^s + (\hat{f}_{\mathbb{Q}} \otimes \hat{\lambda}_M)([\eta_M]) \simeq f \otimes q_N^s + (\hat{f}_{\mathbb{Q}} \otimes \hat{\lambda}_N)([\eta_N]).$$

Hence their associated Gauss sums are equal. In view of $H_1(M) \simeq H_1(N)$, the equality $|H^1(M;G)| = |H^1(N;G)|$ is clear. Finally it follows from (iii) that

 $(f \otimes q_M^{\rm fr})(\eta_M) = (f \otimes q_N^{\rm fr})(\eta_N)$

and it follows from (i) and the definition of s' that

$$\delta(\eta_M) = \delta(\eta_N).$$

This is the desired result.

We now prove the converse.
Step 1: we recover the homology by taking the absolute value of $\tau(M;q) = \overline{\tau(M, \emptyset; q, 0)}$. By Corollary 3.3, if (G, q) and (Tors $H_1(M)$) have no common orthogonal cyclic summand of even order, then

$$|\tau(M;q)| = |H^1(M;G)|^{1/2} = |H_1(M) \otimes G|^{1/2}.$$

By allowing G to vary, we recover all p-components of $H_1(M)$ for all odd primes p. For 2-components, for each $k \ge 1$ and $G_k = \mathbb{Z}/2^k\mathbb{Z}$, we compute $\tau(M;q)$. By Corollary 3.3, we know that $H_1(M)$ has G_k as an orthogonal summand if and only if $\tau(M;q) = 0$. We only need to know the number of such summands. We observe that given any finite abelian group G, the group $G \oplus G$ carries a hyperbolic linking with no cyclic orthogonal summand. Lift this linking to any quadratic form q. Then $|\tau(M;q)|^2 =$ $|H^1(M;G \oplus G)| = |H^1(M;G)|^2$. By allowing G to vary over all 2-groups, we recover all 2-components of $H_1(M)$. Finally we recover in this fashion the isomorphism class of $H_1(M)$. (For more details on this step, see [14].)

Step 2: For c = 0, the formula (??) simplifies to $\overline{(5.3)}$

$$\tau(M,\mathcal{L};q,0) = \tau(M,q) = \frac{\gamma(V \otimes \text{Tors } H_1(M), f \otimes q^s)}{\gamma(\text{Tors } H_1(M), q^s)^{f(v,v)}} \cdot |H^1(M;G)|^{1/2}.$$

Using the discriminant construction (§??, Th. 4.6), we may choose any bilinear even lattice (V, f, 0) equipped with Wu class $v = 0 \in Wu^V(f)$ such that $(G_f, \varphi_{f,0}) = (G, q)$. Then the formula simplifies further to

(5.4)
$$\frac{\tau(M,q)}{|H^1(M;G)|^{1/2}} = \gamma(V \otimes \text{Tors } H_1(M), f \otimes q^s)$$

(5.5)
$$= \gamma(V \otimes H_1(M), \frac{1}{2}f \otimes \lambda_M).$$

For the second equality above, we regard $\frac{1}{2}f$ as the homogeneous quadratic function defined by $(\frac{1}{2}f)(x) = \frac{f(x,x)}{2} \in \mathbb{Z}$ for all $x \in V$ and we use the fact that 2 $q^s(y) = \lambda_M(y,y)$ for all $y \in H_1(M)$. We now apply the classification theorem (Th. 2.2) to the Gauss sums $\tau(M,q) = \gamma_{1/2} f_{,0}(\lambda_M,0)$ (this is the special case when the distinguished element is trivial): we recover the isomorphism class of the linking pairing λ_M . It follows from Step 1 and Corollary 4.2 that we recover the isomorphism class of the linking group $(H_1(M), \lambda_M)$.

<u>Step 3</u>: we show how to detect whether a \mathbb{Z} -linear combination $\sum_{j \in J} a_j[\mathcal{L}_j]$ is torsion in $H_1(M)$ and if it is, we show how to compute its order.

Let $J \subseteq \{1, \ldots, n\}$. Recall the corresponding ordered sublink $\mathcal{L}^J = \bigcup_{j \in J} \mathcal{L}_j$ of \mathcal{L} . Let $(a_j)_{j \in J} \in \mathbb{Z}^{|J|}$. We prove that τ detects whether the \mathbb{Z} -linear combination $\sum_{j \in J} a_j[\mathcal{L}_j]$ is torsion in $H_1(M)$. Note that it actually suffices to detect whether $\sum_{j \in J} a_j [\mathcal{L}_j] = 0$ in $H_1(M)$. We use the following fact from *p*-adic numbers. Let

$$\widehat{\mathbb{Z}} = \lim_{\longrightarrow} \mathbb{Z}/n\mathbb{Z} = \{(x_k)_{k \ge 1} \in \prod_{k \ge 1} \mathbb{Z}/k\mathbb{Z}, \text{ for all } n | m, x_m = x_n \mod n\}.$$

Note that the natural homomorphism

$$\mathbb{Z} \to \prod_{k \ge 1} \mathbb{Z}/k\mathbb{Z}, \ x \mapsto (x \mod k)_{k \ge 1}$$

factors through a map $\mathbb{Z} \to \widehat{\mathbb{Z}}$.

LEMMA 5.1. Let H be an abelian group. The map $H \to \widehat{\mathbb{Z}} \otimes H$ is injective.

Applying this lemma to $H = H_1(M)$ for the particular element $\sum_j a_j [\mathcal{L}_j]$ yields the

COROLLARY 5.2. (5.6) $\sum_{j\in J} a_j \left[\mathcal{L}_j\right] = 0 \text{ if and only if } \sum_{j\in J} (a_j \mod k) \otimes [\mathcal{L}_j] = 0 \in \mathbb{Z}/k\mathbb{Z}, \text{ for all } k \ge 1.$

Let $k \ge 1$. Set $C_k = \mathbb{Z}/k\mathbb{Z}$. By 1_k , we denote $(1 \mod k) \in C_k$. Let $1_k^* \in C_k^*$ be the unique form defined by $1_k^*(1_k) = \frac{1}{k} \mod 1$. Let $G_k = C_k \oplus C_k^*$. Define $q_k : G_k \to \mathbb{Q}/\mathbb{Z}$ by $q_k(x, \alpha) = \alpha(x)$. The quadratic form q_k is hyperbolic. In particular, (G_k, q_k) has no cyclic orthogonal summand. Hence the characteristic homomorphism vanishes: $\chi_{bq_k,\lambda_M} = 0$. Equivalently the characteristic element is zero. Denote by $c^J = (c_j)_{j \in J} \in G_k^{|J|}$ the color vector associated to the ordered sublink \mathcal{L}^J . By the vanishing condition (Th. 1.14), $\tau(M, \mathcal{L}^J; q_k, c^J) \neq 0$ if and only if $\sum_{j \in J} c_j \otimes [L_j] = 0$. This holds for any color vector $c^J = (c_j)_{j \in J}$ and for any $k \ge 1$. In particular, for each $k \ge 1$, we take $c_j = c_{j,k} = a_j \ (1_k, 1_k^*) \in G_k$. For each $k \ge 1$, set $c_k^J = (c_{j,k})_{j \in J} \in G_k^{|J|}$. We have $\sum_{j \in J} c_j \otimes [\mathcal{L}_j] = 0$ if and only if $\sum_{j \in J} (a_j \mod k) \otimes [\mathcal{L}_j] = 0$. By corollary 5.2, we conclude that

$$\sum_{j \in J} a_j[\mathcal{L}_j] = 0 \text{ if and only if for all } k \ge 1, \ \tau(M, \mathcal{L}^J; q_k, c_k^J) \neq 0.$$

In particular the invariant τ detects whether $\sum_{j \in J} a_j [L_j] = 0$, as claimed. The order of $\sum_{j \in J} a_j [\mathcal{L}_j]$ is the smallest $n \ge 1$ such that $\sum_{j \in J} na_j [\mathcal{L}_j] = 0$. It follows that it is the smallest $n \ge 1$ such that $\tau(M, \mathcal{L}^J; q_k, n c_k^J) \neq 0$.

As a particular case, τ detects whether a given component \mathcal{L}_j represents a torsion element in $H_1(M)$ of fixed finite order.

<u>Step 4</u>. Set $F_1(M) = H_1(M)/\text{Tors } H_1(M)$. We show how to detect whether any \mathbb{Z} -linear combination $\sum_{j \in J} a_j[\mathcal{L}_j]$ projects onto a primitive element in $F_1(M)$. (Recall that a primitive element in a lattice V is an element $x \in V$ that can be completed to a \mathbb{Z} -basis (x, x_2, \ldots, x_n) of V.)

We use the notation of the previous step.

LEMMA 5.3. Let $k \ge 1$. Let $c = c^J \in G_k^{|J|}$ denote the color vector defined by $c_j = c_{j,k} = a_k \ (1_k, 1_k^*) \in G_k$ for all $j \in J$.

(1) The element $\sum_{j \in J} a_j[L_j]$ projects onto a nonzero element in $k F_1(M)$ if and only if the following two conditions are verified:

5. THE PROOF OF THE CLASSIFICATION THEOREM

- (1.1) $\tau(M, \mathcal{L}; q_k, c^J) \neq 0$ for $c_j = a_k (1_k, 1_k^*) \in G_k, j \in J$.
- (1.2) For any positive integer k' not multiple of k, for any nonzero $c^{J} \in (G_{k'})^{|J|}, \tau(M, \mathcal{L}; q_{k'}, c^{J}) = 0.$
- (2) The element $\sum_{j \in J} a_j[\mathcal{L}_j]$ projects onto a primitive element in $F_1(M)$ if and only if $\tau(M, \mathcal{L}; q_k, c^J) = 0$ for all $k \ge 1$.

Step 5: we show how to recover $q^{\text{fr}}(\mathcal{L}_j) \in \mathbb{Q}$ for all $1 \leq j \leq n$ such that $[\mathcal{L}_j] \in \text{Tors } H_1(M)$. By Step 3, we know which components represents torsion elements. Henceforth we assume that there is at least one component \mathcal{L}_j that represents a torsion element in $H_1(M)$. Consider the set S of all pairs (G, c) where G is a finite Abelian group and $c = (c_1, \ldots, c_n) \in G^n$ is an n-tuple of colors such that $[\theta] = c \otimes [\mathcal{L}] = \sum_j c_j \otimes [\mathcal{L}_j] = 0$. Let $(G, c) \in S$. By Theorem 4.6, choose a triple (V, f, v) over (G, q). Since $[\theta] = 0 \in G \otimes H_1(M)$, we choose a lift η such that $[\eta] = 0$. Then the formula (5.2) reduces to (5.7)

$$\tau(M,\mathcal{L};q,c) = e^{2\pi i \left((f \otimes q^{\mathrm{fr}})(\eta) - \delta(\eta) \right)} \cdot \frac{\gamma(V \otimes \mathrm{Tors} \ H_1(M), f \otimes q^s)}{\gamma(\mathrm{Tors} \ H_1(M), q^s)^{f(v,v)}} \cdot |H^1(M;G)|^{\frac{1}{2}}.$$

Since we know the isomorphism class of λ_M , we can freely choose a spin refinement q^s of λ_M . Hence we can compute the Gauss sums of the right hand side of (5.7). We already know the order $|H^1(M;G)|$ by Step 1. Since $[\theta] = 0$, we know that the invariant does not vanish if and only if (G,q)and (Tors $H_1(M), \lambda_M$) have no common cyclic orthogonal summand of even order. Let (V, f, v) be a triple over (G, q) satisfying this condition and the condition that $(G, c) \in S$. Then we recover

$$\exp 2\pi i \left((f \otimes q^{\rm fr})(\eta) - \delta(\eta) \right).$$

In particular for v = 0, we have $\delta(\eta) = 0$, hence we recover the term

$$\exp 2\pi i (f \otimes q^{\rm tr})(\eta).$$

We now prove our claim. Let $1 \leq j \leq n$. Let d be the order of Tors $H_1(M)$. Set

$$\alpha = \begin{cases} 1 & \text{if } d \text{ is even;} \\ 2 & \text{if } d \text{ is odd.} \end{cases}$$

For each $N \ge 2$, consider the pair formed by the group $G = \mathbb{Z}/\alpha d^N \mathbb{Z}$ and the colors defined by

$$c_k = \begin{cases} \alpha d \mod \alpha d^N & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Clearly

$$[\theta] = c \otimes [\mathcal{L}] = c_j \otimes [\mathcal{L}_j] = 1 \otimes \alpha d \ [\mathcal{L}_j] \in \mathbb{Z}/\alpha d^N \mathbb{Z} \otimes H_1(M).$$

Thus $(G,c) \in S$ since $d [\mathcal{L}_j] = 0$ in Tors $H_1(M)$. Consider the bilinear lattice $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(x,y) = \alpha d^N x \cdot y$. Observe that f is always even (so that v = 0 is a Wu class) and $G_f = G$. Note that $q = \phi_{f,0}$ is a quadratic form over a cyclic group of order αd^N . Thus (G,q) and (Tors $H_1(M), \lambda_M$) have no common cyclic orthogonal summand. We lift $\theta = (\alpha d \mod \alpha d^N) \otimes \mathcal{L}_j$ to $\eta = \alpha d/\alpha d^N \otimes \mathcal{L}_j = 1/d^{N-1} \otimes \mathcal{L}_j$. Given our choice of (V, f, 0), the argument above applies: we recover the term

$$\exp\left(2\pi i (f \otimes q^{\mathrm{fr}})(\eta)\right) = \exp\left(2\pi i f_{\mathbb{Q}}(\xi_j, \xi_j) q^{\mathrm{fr}}(\mathcal{L}_j)\right)$$
$$= \exp\left(2\pi i \frac{\alpha}{d^{N-2}} q^{\mathrm{fr}}(\mathcal{L}_j)\right).$$

Hence we recover $\frac{\alpha}{d^{N-2}}q^{\text{fr}}(\mathcal{L}_j) \mod 1$ for all $N \ge 2$. The lemma below (well known in the context of *p*-adic numbers when *d* is prime) implies that we recover $\alpha q^{\text{fr}}(\mathcal{L}_j) \in \mathbb{Q}$ and hence $q^{\text{fr}}(\mathcal{L}_j) \in \mathbb{Q}$.

LEMMA 5.4. Let $d \ge 2$. Let Q_d be the inverse limit of $\mathbb{Q}/\epsilon d^N \mathbb{Z}$. The map

$$\mathbb{Q} \to Q_d, \ r \mapsto (r \mod d^N)_{N \ge 1}$$

is injective.

The following fact is also well known: a sequence $0 \leq r_N < d^N$ of rational numbers such that $r_N = r_{N+1} \mod d^N$ for all $N \ge 1$ corresponds to a rational number $r \in \mathbb{Q}$ provided that there exists $N_0 \ge 0$ such that $r_N = r_{N+1}$ for all $N > N_0$.

<u>Step 6</u>. Let $p: H_1(M) \to F_1(M)$ denote the canonical projection. We show that the invariant detects the isomorphism class of the pointed (plain) lattice $(F_1(M), p([\mathcal{L}_1]), \ldots, p([\mathcal{L}_n]))$. From Step 1, we know $\rho = \operatorname{rank}(F_1(M))$. From the previous two steps, we can find in a finite number of steps the (unique) maximal subset $I \subseteq \{1, \ldots, n\}$ such that $p([\mathcal{L}_i]) \neq 0$ in $F_1(M)$ for each $i \in I$.

For all $c \in G^{|I|}$, $\tau(M, \mathcal{L}; q, c^I) = \tau(M, \mathcal{L}^I; q, c)$. By Step 4, the invariant detects whether any \mathbb{Z} -combination $\sum_i a_i p([\mathcal{L}_i])$ lies in $kF_1(M)$. Apply the classification of pointed plain lattices (Prop. 5.1) to deduce the isomorphism class of $(F_1(M), p([\mathcal{L}_1]), \ldots, p([\mathcal{L}_n]))$. This proves our claim.

<u>Step 7</u>. Let $r: H_1(M) \to \text{Tors } H_1(M)$ be a retraction. We claim that the invariant τ determines the isomorphism class of the pointed linking pairing $(\lambda_M, r([\mathcal{L}])).$

Preliminary step: let (G, q, c) be a pointed quadratic form. We prove that for any choice of a 4-tuple (V, f, v, ξ) such that $(G_f, \varphi_{f,v}, [\xi]) = (G, q, c)$,

(5.8)

$$\gamma(V \otimes \text{Tors } H_1(M), f \otimes q_{s_M} + (\hat{f}_{\mathbb{Q}} \otimes \hat{\lambda}_M)([\eta_M])) =$$

= $\gamma(V \otimes \text{Tors } H_1(N), f \otimes q_{s_N} + (\hat{f}_{\mathbb{Q}} \otimes \hat{\lambda}_N)([\eta_N])).$

Set

$$\theta_M = \sum_k c_k \otimes \mathcal{L}_k, \ \theta_N = \sum_k c_k \otimes \mathcal{J}_k \in G \otimes H_1(M).$$

First, notice that according to Th. 3.1, if $[\theta_M]$ is non-characteristic then $[\theta_N]$ is also non-characteristic and both Gauss sums in (5.8) vanish (whatever our choice for (V, f, v, ξ)). Hence (5.8) is verified.

Next, suppose that $[\theta_M]$ is characteristic. Then $[\theta_N]$ also is characteristic. There is an isomorphism ψ : (Tors $H_1(M), \lambda_M$) \rightarrow (Tors $H_1(N), \lambda_N$) of linking pairings (by Step 2), hence $id_{G_f} \otimes \psi$ is also an isomorphism of linking pairings

 $(G_f \otimes \text{Tors } H_1(M), \lambda_f \otimes \lambda_M) \simeq (G_f \otimes \text{Tors } H_1(N), \lambda_f \otimes \lambda_N).$

Since the characteristic element is preserved under linking isomorphisms (by (??)), this is actually an isomorphism

 $(G_f \otimes \text{Tors } H_1(M), \lambda_f \otimes \lambda_M, [\theta_M]) \simeq (G_f \otimes \text{Tors } H_1(N), \lambda_f \otimes \lambda_N, [\theta_N])$

of pointed linking pairings. Observe that $\mathrm{id}_{G_f} \otimes \psi$ lifts to an isomorphism $\mathrm{id}_{V^{\sharp}} \otimes \psi : V^{\sharp} \otimes \mathrm{Tors} \ H_1(M) \simeq V^{\sharp} \otimes \mathrm{Tors} \ H_1(N)$. Thus for any lift ξ of $c \in G^n$,

$$(V^{\sharp} \otimes \text{Tors } H_1(M), f \otimes \lambda_M, [\eta_M]) \simeq (V^{\sharp} \otimes \text{Tors } H_1(N), f \otimes \lambda_N, [\eta_N]).$$

This isomorphism lifts to an isomorphism of pointed quadratic forms $(f \otimes q_{s_M}, [\eta_M]) \simeq (f \otimes q_{s_M}, [\eta_N])$. Therefore, (5.8) is again verified. This completes the preliminary step.

Since F_1M is free, there is a commutative diagram of split exact sequences

with retractions $1_{V^{\sharp}} \otimes r$ and $1_G \otimes r$ respectively. If follows that $\sum_k c_k \otimes [\mathcal{L}_k] \in G \otimes \text{Tors } H_1(M)$ if and only if $\sum_k c_k \otimes [\mathcal{L}_k] = \sum_k c_k \otimes r[\mathcal{L}_k]$. Thus

$$[\eta_M] = \sum_k \xi_k \otimes [\mathcal{L}_k] \in V^{\sharp} \otimes \text{Tors } H_1(M) \implies [\eta_M] = \sum_k \xi_k \otimes r[\mathcal{L}_k].$$

Let (V, f, 0) be an even lattice over (G, q) (cf. Th. 4.6). For any homogeneous quadratic form q, we have

$$f \otimes q = \frac{1}{2} f \otimes b_q,$$

where $\frac{1}{2}f$ denotes the homogeneous quadratic function defined on the lattice V by $(1/2 f)(x) = \frac{f(x,x)}{2} \in \mathbb{Z}$ for all $x \in V$. In particular,

$$f\otimes q_{s_M}=\frac{1}{2}f\otimes \lambda_M.$$

Now equality (5.8) reads

(5.9)
$$\gamma_{h,s}(\lambda_M, r([\mathcal{L}_1]), \dots, r([\mathcal{L}_n])) = \gamma_{h,s}(\lambda_N, r([\mathcal{J}_1]), \dots, r([\mathcal{J}_n])),$$

where $h = \frac{1}{2}f$ and $s = \hat{f}_{\mathbb{Q}}(\xi)|_{V^n} = (\hat{f}_{\mathbb{Q}}(\xi_1)|_V, \dots, \hat{f}_{\mathbb{Q}}(\xi_n)|_V) \in (V^*)^n$. We apply Theorem 2.2: Step 1 ensures that condition (1) is satisfied and (5.9) ensures that condition (2) is satisfied. The isomorphism

(Tors $H_1(M), \lambda_M, r([\mathcal{L}_1]), \ldots, r([\mathcal{L}_n])) \simeq$ (Tors $H_1(N), \lambda_N, r([\mathcal{J}_1]), \ldots, r([\mathcal{J}_n]))$) follows.

Step 8. Steps 7 and 8 imply that the hypotheses of Lemma 4.1 are satisfied. We conclude that there is an isomorphism of pointed linking groups

 $(H_1(M), \lambda_M, [\mathcal{L}_1], \dots, [\mathcal{L}_n]) \simeq (H_1(N), \lambda_N, [\mathcal{J}_1], \dots, [\mathcal{J}_n]).$

This concludes the proof.

6. The extension of the monoid pairing

Theorem 4.1 suggests extending the monoid pairing $\langle -, - \rangle : \mathfrak{M}^+ \times \mathfrak{Q}^0 \to \mathbb{C}$ defined in §??. This section is devoted to the construction of this extension. Our main result is that this extended pairing is nondegenerate.

Let $n \ge 0$. Let $\mathfrak{M}^+(n)$ denote the monoid of *n*-pointed linking groups. In other words, $\mathfrak{M}^+(n)$ consists of triples (H, λ, ℓ) where *H* is a finitely generated abelian group, λ : Tors $H \times \text{Tors } H \to \mathbb{Q}/\mathbb{Z}$ is a linking pairing and $\ell = (\ell_1, \ldots, \ell_n) \in H^n$ is an *n*-tuple of distinguished elements. The operation is the expected one, induced componentwise by orthogonal sum and addition. There is a natural embedding $\mathfrak{M}^+(n) \to \mathfrak{M}^+(p)$ for any $n \le p$ defined by adding p - n zeros on the right on the distinguished *n*-tuple to form a distinguished *p*-tuple. We define also a monoid $\mathfrak{Q}^0(n)$ that consists of pairs (q, c) where $q: G \to \mathbb{Q}/\mathbb{Z}$ is a nondegenerate homogeneous quadratic function on a finite abelian group *G* and *c* is a distinguished element in G^n . Clearly, $\mathfrak{Q}^0 = \mathfrak{Q}^0(0)$ embeds in $\mathfrak{Q}^0(n)$ in the usual way for any n > 0. An ordered link $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ in a closed connected oriented 3-manifold *M* induces an element

$$[\mathcal{L}] = ([\mathcal{L}_1], \dots, [\mathcal{L}_n]) \in H_1(M)^n$$

and therefore determines an element $(H_1(M), \lambda_M, [\mathcal{L}]) \in \mathfrak{M}^+(n)$.

LEMMA 6.1. Let $(H, \lambda, \ell) \in \mathfrak{M}^+(n)$. There exists a closed oriented 3-manifold M equipped with an oriented and ordered link $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n \subset M$ such that $(H_1(M), \lambda_M, [\mathcal{L}]) = (H, \lambda, \ell)$.

PROOF. By [51, Th. 6.1], any (nondegenerate) linking pairing can be realized as the linking pairing of a closed oriented 3-manifold. Actually, the 3-manifold can be chosen as a rational homology 3-sphere $M'(b_1(M') = 0)$. Let m be the rank of H. One needs to modify M' to another 3-manifold M such that $b_1(M) = m$ so that $H_1(M) = H$. The simplest way to do this is to make connected sums with $S^1 \times S^2$: $M = M' \sharp \sharp_{j=1}^m S^1 \times S^2$. Then $b_1(M \sharp \sharp_{j=1}^m S^1 \times S^2) = b_1(M) + m \ b_1(S^1 \times S^2) = 0 + m \ 1 = m$. It remains to choose an ordered oriented link $\mathcal{L} \subset M'$ such that its components represent prescribed homology classes ℓ_1, \ldots, ℓ_n . Since the dimension of each components is one, we can achieve this component by component. Since the codimension of each component in M is two, we can ensure that the components are pairwise disjoint.

Let $n \ge 0$. We define a pairing $\mathfrak{M}^+(n) \times \mathfrak{Q}^0(n) \to \mathbb{C}$ by

(6.1)
$$\langle H, \lambda, \ell; q, c \rangle = \tau(M, \mathcal{L}; q, c)$$

where M is any closed oriented 3-manifold equipped with a link $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n \subset M$ provided by the lemma above. For n = 0 (no distinguished element), this coincides with the previous definition (see §??). For $n \ge 1$

(when there is at least one distinguished element), the definition is unambiguous only if we fix the framing $lk_M(\mathcal{L}_j, \mathcal{L}'_j)$ of torsion components of \mathcal{L} in M. We require that for each $1 \leq j \leq n$,

$$\operatorname{lk}_M(\mathcal{L}_j, \mathcal{L}'_j) = \frac{+1}{N_j} \in \mathbb{Q}$$

where N_j is the order of the homology class $[\mathcal{L}_j]$ in $H_1(M)$. We refer to this framing as the *reference framing* of \mathcal{L} . For non torsion components, the framing is undefined. Theorem 4.1 ensures that the pairing $\langle -; - \rangle$ is well defined by (6.1).

THEOREM 6.2. The pairing

$$\mathfrak{M}^+(n) \times \mathfrak{Q}^0(n) \to \mathbb{C}, \quad ((H,\lambda,\ell),(q,c)) \mapsto \langle \lambda,\ell;q,c \rangle$$

is bilinear, left and right nondegenerate.

Bilinearity is meant with respect to the operations

 $(H,\lambda,\ell)\oplus(H',\lambda',\ell')=(H\oplus H',\lambda\oplus\lambda',\ell\oplus\ell')$

and

$$(q,c) \oplus (q',c') = (q \oplus q', c \oplus c').$$

Clearly the pairing above generalizes the pairing introduced in \S ??:

 $\big\langle H,\lambda,\varnothing;q,c\big\rangle=\big\langle H,\lambda;q,\varnothing\big\rangle=\big\langle H,\lambda;0\big\rangle=\big\langle H,\lambda,\ell;q,0\big\rangle.$

PROOF. Bilinearity follows from the definition. The 4.1 asserts that $\langle -, - \rangle$ is left nondegenerate. It remains to prove that it is also right nondegenerate.

Step 1: we recover G by taking the absolute value of $\tau(M, \mathcal{L}; q, c)$. This step is completely symmetric to the first step of the proof of Th. 4.1. We have

$$|\langle \lambda, m, \ell; q, \mathbf{c} \rangle| = |H^1(M; G)|^{1/2} = |G \otimes \text{Tors } H_1(M)|^{1/2}$$

if G and Tors $H_1(M)$ have no common cyclic orthogonal summand of even order (and is zero otherwise). According to the previous lemma, for any pointed linking pairing $(\lambda, \ell) \in \mathfrak{M}(n)$, there is a closed oriented rational homology 3-sphere M equipped with an oriented link \mathcal{L} such that $(\lambda, 0, \ell) =$ $(\lambda_M, 0, [\mathcal{L}])$. Endow \mathcal{L} with the reference framing. By appropriate choices of (Tors $H_1(M), \lambda_M$), we recover all p-components of G, hence the isomorphism class of G itself.

Step 2: we establish a formula for $\tau(M, \mathcal{L}; q, c)$ in a particular case (first proved in [19, Cor. 4]). Let $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n \subset M$ be an oriented link whose components are all homologically trivial: $[\mathcal{L}_j] = 0$ in $H_1(M)$ for $1 \leq j \leq n$. We have $[\eta] = \sum_k \xi_k \otimes [\mathcal{L}_k] = 0$ for any lift of $[\theta_M] = \sum_k c_k \otimes [\mathcal{L}_k] = 0$. Furthermore,

$$\delta_{s}(\eta_{v}) = \sum_{j} f_{\mathbb{Q}}(v,\xi_{j}) \left(q^{\mathrm{fr}}(\mathcal{L}_{j}) - q^{s}([\mathcal{L}_{j}])\right) = \sum_{j} f_{\mathbb{Q}}(v,\xi_{j}) \left(q^{\mathrm{fr}}(\mathcal{L}_{j}) - 0\right) = \sum_{j} f_{\mathbb{Q}}(v,\xi_{j}) q^{\mathrm{fr}}(\mathcal{L}_{j}).$$

Since each component is homologically trivial, all framing and linking numbers are integers. It follows that

$$(q \otimes \operatorname{lk}_M)(\theta_M) = (\varphi_{f,v} \otimes \operatorname{lk}_M) \left(\sum_j c_j \otimes \mathcal{L}_j\right) = (f \otimes q^{\operatorname{fr}})(\eta) - \delta_s(\eta_v) \mod 1.$$

It follows from the formula (5.2) that

(6.2)
$$\tau(M,\mathcal{L};q,c) = \exp\left(2\pi i(q\otimes \mathrm{lk}_M)(\theta)\right) \tau(M,q).$$

In particular, if M is a Z-homology 3-sphere, $\tau(M,q) = 1$. Hence

(6.3)
$$\tau(M,\mathcal{L};q,c) = \exp\bigg(2\pi i (q\otimes \mathrm{lk}_M)(\theta)\bigg).$$

for an empty (or zero-framed algebraically split) link, we have

$$\langle \lambda, m, \varnothing; q, c \rangle = \langle \lambda, m; q \rangle.$$

We have noted earlier that the pairing $\langle -, - \rangle : \mathfrak{M}^+ \times \mathfrak{Q}^0 \to \mathbb{C}$ is right nondegenerate. Hence the (isomorphism class of the) homogeneous quadratic form (G, q) is determined. In particular, the Gauss sum $\gamma(G, q)$ is recovered.

Step 3: consider the case when M is a \mathbb{Z} -homology 3-sphere with an oriented link $\mathcal{L} \subset M$. Then it can be shown ([15, corollary 5]) that

$$\langle \lambda_M, 0, [\mathcal{L}]; q, c \rangle = \tau(M, \mathcal{L}; q, c) = \exp\left(2\pi i (q \otimes A_{\mathcal{L}})(c)\right)$$

where $A_{\mathcal{L}}$ denotes the $n \times n$ symmetric integral linking matrix of \mathcal{L} in M. Hence by varying the zero-framed link \mathcal{L} in M, we can realize any symmetric integral $n \times n$ matrix with zeros on the diagonal. It follows that $q(c_j)$, $b_q(c_j, c_k) \in \mathbb{Q}/\mathbb{Z}, 1 \leq j, k \leq n$, are all determined.

Step 4: given any pointed linking pairing (λ, ℓ) , we realize it as the pointed linking pairing $(\lambda_M, [\mathcal{L}])$ associated to a closed oriented 3-manifold M equipped with a zero-framed oriented link \mathcal{L} . The pair (M, \mathcal{L}) itself can be realized as a pair (L, J) of disjoint links in S^3 where L is a framed m-component link (on which the surgery is performed) and J is an oriented framed ncomponent link (giving rise to \mathcal{L} once the surgery on L is performed). Such a pair determines a linking matrix $A_{L\cup J}$. This symmetric integral matrix decomposes as

$$A_{L\cup J} = \left[\begin{array}{cc} A_L & A_{L,J} \\ A_{J,L} & A_J \end{array} \right]$$

where A_L is the linking $m \times m$ matrix of L in S^3 , A_J is the linking $n \times n$ matrix of J in S^3 and $A_{J,L} = A_{L,J}^t$ is the $n \times m$ matrix of the linking numbers of the components of L with the components of J in S^3 . Hence, for $x \in G^m$ and $c \in G^n$,

$$(q \otimes A_{L \cup J})(x, c) = (q \otimes A_L)(x) + (b_q \otimes A_{J,L})(x, c) + (q \otimes A_J)(c),$$

where $q \otimes A_{L \cup J}$ is regarded as a quadratic function on $G \otimes \mathbb{Z}^{m+n} = G^{m+n} = G^m \times G^n$, $q \otimes A_L$ is regarded as a quadratic function on $G \otimes \mathbb{Z}^m = G^m$,

 $q \otimes A_J$ is regarded as a quadratic function on $G \otimes \mathbb{Z}^n = G^n$ and $b_q \otimes A_{J,L}$ is regarded as a bilinear pairing $(G \otimes \mathbb{Z}^m) \times (G \otimes \mathbb{Z}^n) = G^m \times G^n \to \mathbb{Q}/\mathbb{Z}$. It follows from the previous step that we recover the term

$$(q \otimes A_J)(c) = \sum_k q(c_k) \, \mathrm{lk}_{S^3}(J_k, J'_k) + \sum_{k < l} b_q(c_k, c_l) \, \mathrm{lk}_{S^3}(J_k, J_l).$$

Step 5: with the notation previously introduced, we have

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$$\frac{\tau(M,\mathcal{L};q,c)}{\gamma(G,q)^{-\operatorname{sign}(L)}|G|^{-m/2}} = \sum_{\mathbf{x}\in G^m} \exp(2\pi i (q \otimes A_{L\cup J})(\mathbf{x},\mathbf{c})).$$

By Steps 1 and 3, we know the factors $\gamma(G,q)^{-\operatorname{sign}(L)}$ and $|G|^{-m/2}$. Therefore, we recover the Gauss sum

$$\sum_{\mathbf{x}\in G^m} \exp(2\pi i (q \otimes A_{L\cup J})(\mathbf{x}, \mathbf{c})) = \sum_{\mathbf{x}\in G^m} \exp\left(2\pi i \Big((q \otimes A_L)(x) + (b_q \otimes A_{J,L})(x, c) + (q \otimes A_J)(c)\Big)\Big).$$

According to Step 4, we know the term $(q \otimes A_L)(c)$. Hence we recover the Gauss sum

$$\sum_{x \in G^m} \exp\left(2\pi i \Big((q \otimes A_L)(x) + (b_q \otimes A_{J,L})(x,c)\Big)\Big).$$

Since we know its absolue value $|G \otimes \text{Tors } H_1(M)|^{\frac{1}{2}}$ from Step 1, we recover the Gauss sum

 $\gamma(G \otimes \mathbb{Z}^m, q \otimes A_L + (b_q \otimes A_{J,L})(-,c)) = \gamma(G \otimes W, q \otimes g + (\hat{b}_q \otimes \hat{g}_{\mathbb{Q}})([\omega]),$

where g denotes the symmetric bilinear map on $W = \mathbb{Z}^m$ determined the $m \times m$ linking matrix A_L and $\omega \in G \otimes W^{\sharp}$ is a lift of $\theta = \sum_j c_j \otimes [L_j] \in G \otimes W^{\sharp}/W = G \otimes \text{Tors } H_1(M)$. It follows that we recover all Gauss sums $\gamma_{g,s}(q,c)$ for all bilinear pairings (W,g). Therefore, applying the classification result for homogeneous quadratic functions (Corollary 3.3) yields the desired result.

CHAPTER 12

Abelian topological quantum field theory

Let $M = (M, \Sigma_{-}, \Sigma_{+})$ be a connected compact oriented 3-cobordism. In other words, M is a connected compact oriented 3-manifold such that

$$\partial M = \Sigma_+ \coprod -\Sigma_-$$

The surfaces Σ_+ and Σ_- , called the *bases* of the cobordism, are closed and oriented. We also write

$$\Sigma_+ = \partial_+ M, \ \Sigma_- = \partial_- M.$$

If each base is connected, we say that the cobordism is *elementary*. Note that a given orientation restricts to an orientation on each of the connected component. The opposite orientation is denoted by a minus sign. Each connected component Σ of the base carries a nondegenerate symplectic pairing

$$H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}, \ ([a], [b]) \mapsto [a] \bullet [b]$$

in the first homology with integral coefficients, namely the *intersection pairing*. If Σ is not connected (if the cobordism is not elementary), the pairing may be degenerate. Recall that $H_1(-\Sigma) = -H_1(\Sigma)$ is the same space as $H_1(\Sigma)$ but carries the symplectic pairing *opposite* to that of $H_1(\Sigma)$.

We regard two cobordisms as equivalent if they are equivalent by an orientation preserving homeomorphism that is the identity on the boundary. As is well-known, 3-cobordisms form a category Cob where objects are closed oriented surfaces and morphisms are oriented 3-cobordisms and the composition is provided by the gluing along a common base. The composition will be denoted by \circ . For our purpose, this category has the right notion of morphisms but too many objects (too many noncanonical objects). In order to reduce the number of objects (to make them more canonical), we need to enrich somewhat this category.

Once we have defined the right category, we extend the invariant defined in the previous chapters to cobordisms. There are actually two equivalent constructions.

The first one is based on the following idea: glue to each base of a cobordism $(M, \Sigma_{-}, \Sigma_{+})$ a finite union of standard handlebodies. This yields a closed 3-manifold \tilde{M} with a pair of distinguished framed oriented links L^{+}, L^{-} in $-\Sigma_{-} \cup \Sigma_{+}$ (the links are the images of the meridians of the handlebodies respectively). Let g^{-} (resp. g^{+}) be the total genus of Σ_{-} (resp. Σ_{+}). Color the links with some elements $x^{-}, x^{+} \in G^{g_{-}} \times G^{g_{+}}$. According to the previous chapter, to a pair $(\tilde{M}, L^{-} \cup L^{+})$ is associated a topological invariant $\tau(\tilde{M}, L^{-} \cup L^{+}, (g^{-}, g^{+})) \in \mathbb{C}$. By varying the colors and using a

normalization factor, we obtain a linear operator $\tau(M)$ associated to the cobordism M.

The second construction uses only linking invariants associated to the cobordism M itself. It relies ultimately on the reciprocity.

Finally the invariant has special functorial properties: it is a topological quantum field theory (TQFT). It can be approximately regarded as a functor from an appropriate category of cobordisms to the category of finite-dimensional \mathbb{C} -vector spaces.

1. The parametrized cobordism category

Let g be a natural number. For each g, we fix an oriented handlebody H_q of genus g as follows. The *standard* handlebody H_0 of genus 0 is the unit closed 3-ball $D^3 \subset \mathbb{R}^3$. For $g \ge 1$, consider a union U_g of g circles C_1, \ldots, C_g of radius 1 in the plane z = 0 of \mathbb{R}^3 centered on the x-axis such that $C_j \cap C_{j+1}$ is the singleton $\{(2j, 0, 0)\}$ for $1 \leq j \leq g - 1$. Consider the discs in the plane y = 0 centered at (2j, 0, 0), $1 leq j \leq g$, of radii 1/2. These discs are orthogonal to the circles C_j . Consider the solid of revolution generated by these circles by revolving them around the axes x = 2j-1, y = 0 respectively. By definition, this is the standard handlebody H_g of genus g. It is a closed tubular neighborhood of U_g in \mathbb{R}^3 symmetric with respect to the planes z = 0 and y = 0. The circle centered at $(2j - 1, 0, 0), 1 \leq j \leq g$, of radius 1/2, lies in the boundary of H_g : it is the standard *j*-th longitude of H_g . The circle of radius 1 in the plane x = 2j - 1 centered at (2j - 1, 1, 0), $1 \leq j \leq g$, is the standard *j*-th meridian of H_g . It also lies in the boundary of H_q . All the standard handlebodies are orientable as submanifolds of \mathbb{R}^3 : we choose the orientation so that the outward normal vector is last. The map mir : $(x, y, z) \mapsto (x, y, -z)$ is an orientation reversing homeomorphism that restricts to H_q . It is the *standard* orientation reversing homeomorphism of H_q . A closed oriented surface is *standard* if Σ is empty or if there exists $g \in \mathbb{N}$ such that $\Sigma_g = \partial H_g$. The standard *j*-th longitude, resp. meridian, of Σ_q is the standard *j*-th longitude, resp. meridian, of H_q . The orientation of each meridian is chosen so that it moves counterclockwise in the plane x = 2j - 1. The orientation of each longitude is chosen so that it moves counterclockwise in the plane z = 0. The unit vector field tangent to the standard meridians of Σ_q is normal to the standard longitudes. Consider a standard longitude and push it slightly in the direction of this vector field. This yields a circle parallel to the standard longitude that lies in Σ_g , which we call the standard parallel longitude.

DEFINITION 12.1. A standard 3-cobordism is a 3-cobordism $(M, \Sigma_{-}, \Sigma_{+})$ whose bases are finite disjoint unions of standard surfaces.

EXAMPLE 12.1. The standard handlebody H_g can be regarded as the standard cobordism $(H_g, -\Sigma_g, \emptyset)$ or (H_g, \emptyset, Σ) .

The definition of a standard cobordism is rigid. It is completely dependent of our geometrical model of a closed surface of genus g. The idea is to

make all the gluings and all the computations using standard cobordisms and standard surfaces.

Two standard cobordisms $(M, \Sigma_{-}, \Sigma_{+})$ and $(N, \Sigma'_{-}, \Sigma'_{+})$ such that $\Sigma_{+} = \Sigma'_{-}$ can be composed by gluing along the common basis: $N \circ M = N \cup M$. Composition is an associative operation.

We need to define an equivalence relation on cobordisms. A first attempt could be to define a weak equivalence between two standard cobordism as an orientation preserving homeomorphism that sends the bottom (resp. top) base to the bottom (resp. top) base. If two standard cobordisms are weakly equivalent then their bottom and top bases respectively coincide. We certainly want this property to hold. As any homeomorphism restricts to a homeomorphism of the boundary, the only difference between this definition and a general homeomorphism is to distinguish between top and bottom bases. So we are left with the following definition.

DEFINITION 12.2. An *equivalence* between two standard cobordisms is an orientation preserving homeomorphism that induces the identity on the bases.

Denote equivalence by \sim .

LEMMA 1.1. Let M, M', N, N' be standard cobordisms. Suppose that $N \circ M$ and $N' \circ M'$ are well-defined. If $M \sim M'$ and $N \sim N'$ then $N \circ M \sim N' \circ M'$.

PROOF. Trivial.

LEMMA 1.2. Equivalence classes of standard cobordisms form a small category with finite coproducts (disjoint unions).

PROOF. Cobordisms are morphisms between two standard surfaces (possibly empty). Formally speaking, an object is a finite sequence of elements in $\{\star, 0, 1, 2, \ldots\}$. We set $\Sigma_{\star} = \emptyset$. For instance, a morphism between g_{-} (one term sequence) and g_{+} (one term sequence) is represented by an elementary standard cobordism $(M, \Sigma_{g_{-}}, \Sigma_{g_{+}})$. In general, a morphism between $(g_{1}^{-}, \ldots, g_{r}^{-})$ and $(g_{1}^{+}, \ldots, g_{s}^{+})$ is a standard cobordism with bottom base (resp. top base) a surface with connected closed components of genus $g_{1}^{-}, \ldots, g_{r}^{-}$ (resp. of genus $g_{1}^{+}, \ldots, g_{s}^{+}$) respectively. The notion of equivalence enables to have an identity cobordism for each object $g \in \{\star, 0, 1, 2, \ldots\}$: it is empty if the object is \star or it is the cylinder $\Sigma_{g} \times [0, 1]$ otherwise.

The category of equivalence classes of standard cobordisms is denoted Cob^0 .

DEFINITION 12.3. A parametrized 3-cobordism is an oriented 3-cobordism $(M, \Sigma_{-}, \Sigma_{+})$ equipped with two orientation preserving homeomorphisms $f_{-}: \Sigma_{g_{-}} \to \Sigma_{-}$ and $f_{+}: \Sigma_{g_{+}} \to \Sigma_{+}$ respectively.

EXAMPLE 12.2. The standard handlebody H_g of genus g can be regarded as a parametrized 3-cobordism $(H_g, \emptyset, \Sigma_g)$ parametrized by the identity $f_+ = id_{\Sigma_g}$. More generally, any standard cobordism provides an example of a trivially parametrized cobordism with identity parametrizations.

DEFINITION 12.4. Let $(M, \Sigma_{-}, \Sigma_{0})$ and $(N, \Sigma_{1}, \Sigma_{+})$ be two parametrized cobordisms such that there is an orientation preserving homeomorphism sending Σ_{0} to Σ_{1} . Let g be the genus of Σ_{0} . Denote by $f_{0} : \Sigma_{g} \to \Sigma_{0} \subset M$ and $f_{1} : \Sigma_{g} \to \Sigma_{1} \subset N$ the parametrizations of Σ_{0} and Σ_{1} respectively. The composition $N \circ M$ of N and M is defined as

$$N \circ M = N \coprod_{f_1} \Sigma_g \times [-1, 1] \coprod_{f_0} M.$$

Here the identifications are given by $(s, -1) = f_0(s)$ for all $s \in \Sigma_g$ and $(s, 1) = f_1(s)$ for all $s \in \Sigma_g$.

Loosely speaking, we glue a cylinder (over the basis Σ_g) to the disjoint union $N \prod M$ via the respective parametrizations. See Fig. 1.1.



FIGURE 1.1. Gluing two parametrized cobordisms.

DEFINITION 12.5. Let (M, Σ_+, Σ_-) and $(N, \Sigma'_+, \Sigma'_-)$ be two parametrized cobordisms. Let g_+ (resp. g_-) denote the genus of the surface Σ_+ (resp. g_-). Let f_-, f_+, f'_-, f'_+ be the parametrizations of $\Sigma_-, \Sigma_+, \Sigma'_-, \Sigma'_+$ respectively. An *equivalence* between M and N is an orientation preserving homeomorphism $F: M \to N$ inducing homeomorphisms on the bases such that $F|_{\Sigma_-} \circ f_- = f'_-$ and $F|_{\Sigma_+} \circ f_+ = f'_+$.

LEMMA 1.3. Let M, M', N, N' be parametrized cobordisms. Suppose that $N \circ M$ and $N' \circ M'$ are well-defined. If $M \sim M'$ and $N \sim N'$ then $N \circ M \sim N' \circ M'$.

PROOF. Let F_N (resp. F_M) be an equivalence between N and N' (resp. between M and M'). Let g be the genus of the bottom base of N which coincides with the genus of the top base of M. Define a map

$$\tilde{F}: N \coprod \Sigma \times [-1, 1] \coprod M \to N' \coprod \Sigma_g \times [-1, 1] \coprod M'$$

by

$$\tilde{F}(x) = \begin{cases} F_N(x) & \text{if } x \in N \\ x & \text{if } x \in \Sigma \times [-1, 1] \\ F_N(x) & \text{if } x \in M. \end{cases}$$

This map induces an orientation preservation homeomorphism $F: N \circ M \rightarrow N' \circ M'$ which commutes with the parametrizations, hence is an equivalence between $N \circ M$ and $N' \circ M'$.

LEMMA 1.4. Equivalence classes of parametrized cobordisms form a small category with finite coproducts (disjoint unions).

PROOF. An object is a finite sequence of elements in $\{\star, 0, 1, 2, \ldots\}$. We set $\Sigma_{\star} = \emptyset$. For a finite sequence $g = (g_1, \ldots, g_r)$, let Σ_g denote the disjoint union of the standard surfaces $\Sigma_{g_1}, \ldots, \Sigma_{g_r}$. A morphism between (g_1^-, \ldots, g_r^-) and (g_1^+, \ldots, g_s^+) is represented by a triple (M, f_-, f_+) where M is an oriented cobordism $(M, \Sigma_-, \Sigma_+), f_- : \Sigma_{g_-} \to \Sigma_-$ and $f_+ : \Sigma_{g_+} \to \Sigma_+$ are orientation preserving homeomorphisms. The identity morphism is represented by a cylinder with identity parametrizations on the bases. The other axioms are easily verified.

The category of equivalence classes of parametrized cobordisms is denoted Cob^{par}.

Any parametrized cobordism $(M, \Sigma_{-}, \Sigma_{+})$ yields a standard cobordism $(\widetilde{M}, \Sigma_{g_{-}}, \Sigma_{g_{+}})$ as follows. We set g_{\pm} to be the genus of Σ_{\pm} . (If Σ_{\pm} is empty, then we choose Σ_{q_+} to be empty.) We define

$$\tilde{M} = (\Sigma_{g_{-}} \times [0, 1]) \prod_{f_{-}} M \prod_{f_{+}} (\Sigma_{g_{+}} \times [0, 1]).$$

The identification are given by $(s,1) = f_{-}(s)$ for all $s \in \Sigma_{q_{-}}$ and (s,0) = $f_+(s)$ for all $s \in \Sigma_{g_+}$. Clearly \tilde{M} is a standard cobordism $(\tilde{M}, \Sigma_{g_-}, \Sigma_+)$ obtained by gluing the cylinders over Σ_{g_-} (resp. Σ_{g_+}) to Σ_- (resp. Σ_+) by means of the parametrization f_- (resp. f_+) along $\Sigma_{g_-} \times \{1\}$ (resp. along $\Sigma_{g_+} \times \{0\}).$

LEMMA 1.5. If M and N are two equivalent parametrized cobordisms then M and N are two equivalent standard cobordisms.

PROOF. Suppose there is an equivalence $F: (M, \Sigma_{-}, \Sigma_{0}) \to (N, \Sigma'_{-}, \Sigma'_{+})$ between two parametrized cobordisms. It follows that the genus of the bases coincide: $g_{-} = g'_{-}$ and $g_{+} = g'_{+}$. Define a disjoint union of maps

$$\tilde{F}: (\Sigma_{g_+} \times [0,1]) \coprod M \coprod (\Sigma_{g_-} \times [0,1]) \to (\Sigma_{g'_+} \times [0,1]) \coprod N \coprod (\Sigma_{g'_-} \times [0,1])$$

by

bу

$$\tilde{F}(x) = \begin{cases} x & \text{if } x \in \Sigma_{g_+} \times [0,1] \\ F(x) & \text{if } x \in M \\ x & \text{if } x \in \Sigma_{g_-} \times [0,1]. \end{cases}$$

Since F commute with parametrizations, the maps glue together to induce a map $(\tilde{M}, \Sigma_{g_-}, \Sigma_{g_+}) \to (\tilde{N}, \Sigma_{g'_-}, \Sigma'_{g_+})$ between standard cobordisms. The map is easily seen to be an orientation preserving homeomorphism. By construction, it preserves pointwise the bases. It is therefore an equivalence.

PROPOSITION 1.6. The assignment $M \mapsto \tilde{M}$ induces a covariant full functor $\operatorname{Cob}^{\operatorname{par}} \to \operatorname{Cob}^{0}$.

PROOF. Lemma 1.5 implies that the assignment is well defined at the level of equivalence classes. Let $F : \operatorname{Cob}^{\operatorname{par}} \to \operatorname{Cob}^{0}$ denote the corresponding assignment. The identity morphism of the object $g = (g_1, \ldots, g_r)$ in $\operatorname{Cob}^{\operatorname{par}}$ is represented by the cylinder $C = \Sigma_g \times [0, 1]$ with the identity as parametrization of the bases. It follows from the definition that $\tilde{C} = C$ in Cob^{0} . Thus F sends the identity morphism of g in $\operatorname{Cob}^{\operatorname{par}}$ to the identity morphism of g in Cob^{0} . The identity $F(N \circ M) = F(N) \circ F(M)$ follows from Fig. 1.2. By Example 12.2, any standard cobordism is realized as the



FIGURE 1.2. The composition $\tilde{N} \circ \tilde{M}$.

image of a trivially parametrized cobordism. Hence F is a full functor.

2. The Lagrangian cobordism category

Let A be a symplectic lattice. As is customary, we denote by -A the same underlying lattice A with the symplectic pairing opposite to that of A. In particular, if Σ is a closed surface, then $H_1(-\Sigma) = -H_1(\Sigma)$. Any orientation preserving homeomorphism $\Sigma \to \Sigma$ induces a symplectomorphism $H_1(\Sigma) \to$ $H_1(\Sigma)$.

The following definition should be seen as a motivation.

DEFINITION 12.6. A Lagrangian cobordism is an oriented compact 3-cobordism $M, \Sigma_{-}, \Sigma_{+})$ endowed with

- (1) Lagrangians $A^- \subseteq H_1(\Sigma_-)$ and $A^+ \subseteq H_1(\Sigma_+)$.
- (2) A Lagrangian $\Lambda_M \subset H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$ such that Λ_M is transverse to the Lagrangian $A^- \oplus A^+$ in $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$.

REMARK 12.1. The condition (2) is equivalent to $\Lambda \cap A^- = \Lambda \cap A^+ = 0$. This follows from Lemma 1.5.

REMARK 12.2. The Lagrangian Λ_M associated to the cobordism M is decomposable. This follows from Lemma 1.2. EXAMPLE 12.3 (Meridional and longitudinal Lagrangians). For any standard surface Σ_g , there are two distinguished and transverse Lagrangians, namely the Lagrangian generated in 1-homology by the meridians of Σ_g (the standard meridional Lagrangian) and the Lagrangian generated in 1homology by the longitudes of Σ_g (the standard longitudinal Lagrangian).

EXAMPLE 12.4 (The trivial Lagrangian cylinder). Let $M = \Sigma \times [0, 1]$ be the oriented cylinder over a standard surface $\Sigma = \Sigma_g$. We endow $\Sigma \times \{0\}$ (resp. $\Sigma \times \{1\}$ with the longitudinal Lagrangian A^+ (resp. A^-) in $H_1(\Sigma \times \{1\})$ (resp. $H_1(\Sigma \times \{0\})$) generated in 1-homology by the longitudes l_1, \ldots, l_g of $\Sigma \times \{1\}$ (resp. $H_1(\Sigma \times \{0\})$). We endow M with the Lagrangian Λ_M generated in $H_1(\partial M) = -H_1(\Sigma \times \{0\}) \oplus H_1(\Sigma \times \{1\})$ in 1-homology by the meridians of $\Sigma \times \{1\}$ and $\Sigma \times \{0\}$. Clearly Λ_M is transverse to $A^- \oplus A^+$. The cobordism acquires a structure of Lagrangian cobordism called the trivial Lagrangian cylinder.

EXAMPLE 12.5 (Lagrangian cobordism associated to a parametrization). Let $f: \Sigma_g \to \Sigma$ be an orientation preserving homeomorphism (a parametrization). We associate to f a Lagrangian cobordism C(f) as follows. As a cobordism, C(f) is obtained by gluing two cylinders $\Sigma_g \times [0, 1]$ and $\Sigma \times [0, 1]$ via f:

$$C(f) = \Sigma \times [0,1] \coprod_{f} \Sigma_{g} \times [0,1]$$

with the identification (s,1) = (f(s),0) for all $s \in \Sigma$. Hence C(f) is an oriented cobordism between Σ_g and Σ . To the bottom base we associate the standard longitudinal Lagrangian $A^- \subseteq H_1(\Sigma_g)$. To the top base we associated the Lagrangian $f(A^-) \subseteq H_1(\Sigma)$. To the cobordism C(f), we associate $\Lambda = \text{Graph}(f_*)$. Clearly Λ is transverse to $A^- \oplus f(A^-)$. The case when $f = \text{id}_{\Sigma_g}$ yields the diagonal Lagrangian for Λ and the trivial Lagrangian cylinder.

DEFINITION 12.7. The composition of two Lagrangian cobordisms $(M, \Sigma_{-}, \Sigma_{0})$ and $(N, \Sigma_{0}, \Sigma_{+})$ is defined as the underlying composition of the two cobordisms $N \circ M = N \cup M$ endowed with the Lagrangians of the bottom base of M and the top base of N and the Lagrangian $\Lambda_{N \circ M} = \Lambda_{N} \circ \Lambda_{M}$. (See Chap. 5, Lemma 1.1, for the composition of Lagrangians.)

LEMMA 2.1. The composition of two Lagrangian cobordism is a Lagrangian cobordism.

PROOF. This follows from Lemma 1.9.

DEFINITION 12.8. Two Lagrangian cobordisms are equivalent if there exists a cobordism equivalence (an orientation preserving homeomorphism that restricts to the identity on the boundary) between them sending Lagrangian onto Lagrangian.

Equivalence classes of Lagrangian cobordisms form a category Cob^{lag} with trivial Lagrangian cylinders being the identity morphisms. The assignment

$$(M, \Sigma_{-}, \Sigma_{+}) \mapsto (\Lambda, H_1(\Sigma_{-}), H_1(\Sigma_{+}))$$

is a functor $\operatorname{Cob}^{\operatorname{lag}} \to \operatorname{Lag}_{\operatorname{trans}}^{-1}(\mathbb{Z})$. As before the category $\operatorname{Cob}^{\operatorname{lag}}$ has too many objects. We now modify this category.

DEFINITION 12.9. A Lagrangian decorated cobordism is an oriented 3-cobordism $(M, \Sigma_{-}, \Sigma_{+})$ endowed with

- (1) A pair of isotopy classes of oriented framed links $L^- \subseteq \Sigma_-$ and $L^+ \subseteq \Sigma_+$ such that the subgroups A^- and A^+ generated in 1-homology by the components of L_- and L_+ respectively are Lagrangians in $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$ respectively.
- (2) A Lagrangian $\Lambda \subset H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$ such that Λ is transverse to the Lagrangian $A^- \oplus A^+$ in $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$.

EXAMPLE 12.6 (The trivial Lagrangian decorated cobordism). Let $M = \Sigma_g \times [0, 1]$ be the trivial Lagrangian cobordism. If we provide M with the two underlying sets of longitudes in $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{1\}$, then M is called the trivial Lagrangian decorated cobordism.

EXAMPLE 12.7 (Lagrangian decorated cobordism associated to a parametrized cobordism). Any parametrized cobordism $(M, \Sigma_{-}, \Sigma_{+})$ yields a Lagrangian decorated cobordism. The cobordism is topologically the same. Let L^{+} be the oriented framed link formed by the images of the longitudes of $\Sigma_{g_{+}}$ by the parametrization $f_{+}: \Sigma_{g_{+}} \to \Sigma_{+}$. Similarly let L^{-} be the oriented framed link formed by the longitudes of $\Sigma_{g_{-}}$ by the parametrization $f_{-}: \Sigma_{g_{-}} \to \Sigma_{-}$. Let Λ_{M} be the Lagrangian in $H_{1}(\partial M)$ generated in 1-homology by the images of the meridians of $\Sigma_{g_{+}}$ and $\Sigma_{g_{-}}$ under the parametrizations f_{+} and f_{-} respectively. Clearly Λ_{M} and $A^{-} \oplus A^{+}$ are transverse. This provides M with the structure of a Lagrangian decorated cobordism.

DEFINITION 12.10. Two decorated Lagrangian cobordisms M and N are equivalent if there is an orientation preserving homeomorphism $f: M \to N$ sending the top base (resp. the bottom base) to the top base (resp. the bottom base), sending the oriented framed link in the top base (resp. in the bottom base) to the oriented framed link in the top base (resp. in the bottom base) and sending the Lagrangian Λ_M to the Lagrangian Λ_N .

We now define the composition of two Lagrangian decorated cobordisms using standard surfaces.

Let $(M, \Sigma_{-}, \Sigma_{0})$ and $(N, \Sigma_{1}, \Sigma_{+})$ be two Lagrangian decorated cobordisms such that there is an orientation preserving homeomorphism $\Sigma_{0} \to \Sigma_{1}$. Let g be the genus of Σ_{0} . Let L^{0} (resp. L^{1}) be an oriented framed link inside Σ_{0} (resp. Σ_{1}). Let $1 \leq j \leq g$. Denote by l_{j} the j-th longitude of Σ_{g} . Choose orientation preserving homeomorphisms $f_{0} : \Sigma_{g} \to \Sigma_{0}$ and $f_{1} : \Sigma_{g} \to \Sigma_{1}$ such that $f_{0}(l_{i}) = L_{i}^{0}$ and $f_{1}(l_{i}) = L_{i}^{1}$, $i = 1, \ldots, g$. The composition $N \circ M$ of M and N is defined by

$$N \circ M = N \coprod_{f_1} \Sigma_g \times [-1, 1] \coprod_{f_0} M$$

with the identifications $(s, -1) = f_0(s)$ and $(s, 1) = f_1(s)$ for all $s \in \Sigma_g$.

REMARK 12.3. Each longitude m_j , $1 \leq j \leq g$, has a natural parallel in Σ : at each point $p \in m_j$, there is a well defined positive normal vector of length 1. The framing of the links L^0 and L^0 is inherited from the framing of the longitudes m_j , $1 \leq j \leq g$, in Σ_g .

REMARK 12.4. The choice of the homeomorphisms f_0 and f_1 is not unique in general. One should complete the definition of f_0 and f_1 by defining for instance on meridians and extending completely the definition of each homeomorphism to Σ_g . However, we shall soon see that a different choice will eventually lead to an equivalent Lagrangian decorated cobordism.

LEMMA 2.2. The composition of two Lagrangian decorated cobordisms has a natural structure of Lagrangian decorated cobordism.

PROOF. Two Lagrangian decorated cobordisms $(M, \Sigma_{-}, \Sigma_{0})$ and $(N, \Sigma_{1}, \Sigma_{+})$ are composable if and only if there is an orientation preserving homeomorphism $\Sigma_{0} \to \Sigma_{1}$ sending the link L^{0} to the link L^{1} (up to isotopy in Σ_{1}). The link associated to the bottom base (resp. top base) of $N \circ M$ is the link L^{-} (resp. L^{+}) associated to Σ_{-} (resp. to Σ_{+}). The links L^{-} and L^{+} generate Lagrangians $A^{-}(M)$ and $A^{+}(N)$ in $H_{1}(\Sigma_{-})$ and $H_{1}(\Sigma_{+})$ respectively.

The only point consists in *defining* the Lagrangian $\Lambda_{N \circ M}$ in $H_1(N \circ M)$ associated to the Lagrangian decorated cobordism $N \circ M$ so that it be transverse to $A^-(M) \oplus A^+(M)$. Let Λ_M and Λ_N be the Lagrangians associated to M and N respectively. Our gluing depends on the intermediate links in Σ_0 and Σ_1 , so we cannot define $\Lambda_{N \circ M} = \Lambda_N \circ \Lambda_M$ as in the case of Lagrangian cobordisms.

The cobordism $N \circ M$ can be written as the composition of cobordisms (in the sense of morphisms of Cob)

$$N \circ M = N \circ C(f_1) \circ C(f_0^{-1}) \circ M$$

where $C(f_1)$ and $C(f_0^{-1})$ are Lagrangian cobordisms associated to f_1 and f_2 respectively. (See Example 12.5.) Now regard M and M as Lagrangian cobordisms as well. We have expressed $N \circ M$ as the composition of four Lagrangian cobordisms, hence $N \circ M$ is a Lagrangian cobordism. Therefore (in the usual sense of composition of Lagrangians, by Lemma 2.1)

$$\Lambda_{N \circ M} = \Lambda_N \circ \operatorname{Graph}(f_{1*}) \circ \operatorname{Graph}(f_{0*}^{-1}) \circ \Lambda_M,$$

where f_{0*} and f_{1*} denote the symplectomorphisms induced in 1-homology by f_0 and f_1 respectively, is a Lagrangian transverse to $A^-(M)$ and $A^+(M)$.

We are therefore forced to define the composition $\Lambda_{N \circ M}$ by the formula above which proves the result.

The composition is associative and the trivial Lagrangian decorated cobordisms represent the identity morphisms.

PROPOSITION 2.3. The equivalence classes of Lagrangian decorated cobordisms form a category, denoted $\operatorname{Cob}_{fr}^{lag}$.

PROOF. First, one needs to verify that the composition of two Lagrangian decorated cobordisms, up to equivalence, does not depend on the particular homeomorphisms f_0 and f_1 chosen to send the set of components of the top base of the first cobordism to the components of the bottom base of the second cobordism. Associativity is a routine verification.

Example 12.7 shows that a Lagrangian decorated cobordism \tilde{M} is naturally associated to a parametrized cobordism M.

PROPOSITION 2.4. The assignment $M \mapsto \tilde{M}$ induces a full covariant functor $\operatorname{Cob}^{\operatorname{par}} \to \operatorname{Cob}^{\operatorname{lag}}_{\operatorname{fr}}.$

PROOF. Let $(M, \Sigma_{-}, \Sigma_{+})$ be a decorated Lagrangian cobordism with transverse Lagrangians $\Lambda_M \subseteq H_1(\partial M)$ and $A^- \oplus A^+ \subseteq H_1(\partial M)$ respectively. We show that this cobordism is represented by a parametrized cobordism. First the symplectic group acts transitively on pairs of transverse Lagrangians.

we need to find parametrizations $f_-: \Sigma_{g_-} \to \Sigma_-$ and $f_+: \Sigma_{g_+} \to \Sigma_+$ such that the images of meridians

REMARK 12.5. We could introduce a more general category $\widetilde{Cob}_{fr}^{lag}$ by removing the condition of transversality for the Lagrangians. For all practical purposes, the category $\operatorname{Cob}_{fr}^{\operatorname{lag}}$ will suffice.

VOIR SI PLUTOT SUR LA SECTION SUR LES PAIRINGS....

3. Axioms for a 3-dimensional TQFT

The notion of a topological quantum field theory (the emphasis is on "topological") is due to E. Witten [101]. M. Atiyah gives in [1] the first axiomatic definition of a TQFT. We give here the relevant definitions in dimension 3 with ground field $\mathbb C$ sufficient for our purposes.

We start with the definition of a modular functor, introduced by G. Segal [86]. The notion of modular functor relies on a suitable category of surfaces. Generally, the surfaces are required to be oriented and to have some extra structure. In the present setting, typically the extra structure will be a parametrization by a standard surface, an oriented framed link inside the surface or a Lagrangian in the 1-homology. For a general axiomatization of these extra structures, the reader is referred to [93].

DEFINITION 12.11. A 2-dimensional Modular Functor, in short a 2-MF, is a covariant functor from the category of closed oriented surfaces (2-manifolds), possibly with extra structure, and structure preserving homeomorphisms to the category of finite-dimensional vector spaces and isomorphisms such that

(1) To each (structure preserving) homeomorphism of closed oriented surfaces $f: \Sigma \to \Sigma'$ is assigned an isomorphism of vector spaces $f_{\sharp}: \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$. The vector space $\mathcal{T}(\Sigma)$ is called the space of states of Σ .

(2) τ is tensor multiplicative with respect to disjoint union: for two (structure preserving) homeomorphisms of closed oriented surfaces $f: \Sigma_0 \to \Sigma_1$ and $g: \sigma'_0 \to \Sigma'_1$, there is q commutative diagram

(3) τ is dual involutive with respect to orientation reversal: for any (structure preserving) homeomorphism of closed oriented surfaces $f: \Sigma \to \Sigma'$, there is a commutative diagram

$$\begin{array}{c} \mathcal{T}(-\Sigma') \xrightarrow{f_{\sharp}^{-1}} \mathcal{T}(-\Sigma) \\ \downarrow \simeq & \downarrow \simeq \\ \mathcal{T}(\Sigma')^* \xrightarrow{f_{\sharp}^*} \mathcal{T}(\Sigma)^* \end{array}$$

Another equivalent way of stating this is the existence, for any closed oriented surface Σ , of a nondegenerate bilinear pairing

$$d_{\Sigma}: \mathcal{T}(\Sigma) \times \mathcal{T}(-\Sigma) \to \mathbb{C}$$

satisfying the naturality conditions with respect to (structure preserving) homeomorphisms.

- (4) The state space of the empty set (viewed as a closed surface) is \mathbb{C} .
- (5) The isomorphisms of (2) (4) compatible with each other and with the canonical commutativity, associativity and unit isomorphisms arising from the disjoint union of two surfaces, of three surfaces and the union of a surface and the empty surface respectively.

We are now ready for the definition of a TQFT. We begin with Atiyah's version. We assume that we are given a category of 3-cobordisms. Given two vector spaces V and W, we shall often use the canonical linear isomorphism $V^* \otimes W \to \operatorname{Hom}(V, W)$ induced by the map $(\varphi, w) \mapsto \varphi(-)w$. The inverse isomorphism is given by $\operatorname{Hom}(V, W) \to V^* \otimes W, f \mapsto \sum_j e_j^* \otimes f(e_j)$ for any basis (e_j) of V.

DEFINITION 12.12. An anomaly free 3-dimensional Topological Quantum Field Theory, in short a 3-TQFT, consists in the following assignments:

- (1) A 2-dimensional modular functor τ that assigns to a (structure preserving) homeomorphism $f: \Sigma \to \Sigma'$ an isomorphism of finitedimensional vector spaces $f_{\sharp}: \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$.
- (2) To any compact oriented 3-manifold M is assigned a vector $\tau(M) \in \mathcal{T}(\partial M)$.

These assignments are subject to the following requirements:

(a) τ is functorial: for any positive homeomorphism $f: M \to N$ of compact oriented 3-manifolds,

$$(f|_{\partial M})_{\sharp}(\tau(M)) = \tau(N).$$

(b) τ is tensor multiplicative with respect to disjoint union: for two compact oriented 3-manifolds M and N, the isomorphism

$$\mathcal{T}(\partial M \coprod \partial N) \simeq \mathcal{T}(M) \otimes \mathcal{T}(N)$$

sends $\tau(M \coprod N)$ to $\tau(M) \otimes \tau(N)$.

(c) Behaviour with respect to gluing: consider a compact oriented 3manifold M such that $\partial M = N \coprod P \coprod Q$ and an orientation reversion homeomophism $f: N \to -P$. Denote by M' the compact oriented 3-manifold obtained from M by identifying N with -Pvia f. Then the composition

$$\mathcal{T}(N) \otimes \mathcal{T}(P) \otimes \mathcal{T}(Q) \xrightarrow{f_{\sharp} \otimes \mathrm{id}_{P} \otimes \mathrm{id}_{Q}} \mathcal{T}(P)^{*} \otimes \mathcal{T}(P) \otimes \mathcal{T}(Q) \xrightarrow{d_{P} \otimes \mathrm{id}_{Q}} \mathcal{T}(Q)$$

sends $\tau(M)$ to $\tau(M')$ in $\mathcal{T}(Q)$.

(d) Normalization on the cylinder: for a closed oriented surface Σ , $\tau(\Sigma \times [0, 1])$ is sent to $id_{\tau(\Sigma)}$ under the composition

$$\mathcal{T}(\partial(\Sigma \times I)) = \mathcal{T}(\Sigma \coprod -\Sigma) \simeq \mathcal{T}(\Sigma) \otimes \mathcal{T}(-\Sigma) \simeq \operatorname{Hom}(\mathcal{T}(\Sigma), \mathcal{T}(\Sigma)).$$

(e) Normalization on the 2-sphere and the 3-ball: $\mathcal{T}(S^2) = \mathbb{C}$ and $\mathcal{T}(B^3) = 1 \in \mathbb{C}$.

An immediate consequence of the axioms is that a 3-TQFT yields a numerical topological invariant of closed 3-manifolds, for if M is a closed 3-manifold, then $\tau(M) \in \mathcal{T}(\partial M) = \mathcal{T}(\emptyset) = \mathbb{C}$; and if $f: M \to N$ is a homeomorphism then functoriality implies that

$$\tau(N) = (f|_{\partial M})_{\sharp}(\tau(M)) = (f|_{\varnothing})_{\sharp}(\tau(M)) = \mathrm{id}_{\mathbb{C}}(\tau(M)) = \tau(M).$$

LEMMA 3.1. If (M, Σ, Σ') is a compact oriented 3-cobordism, that is $\partial M = -\Sigma \coprod \Sigma'$, then $\tau(M)$ identifies naturally to a linear operator $\mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$. Furthermore, if $F: (M, \Sigma_{-}, \Sigma_{+}) \to (N, \Sigma'_{-}, \Sigma'_{+})$ is a cobordism isomorphism then there is a commutative diagram

$$\begin{array}{c} \mathcal{T}(\Sigma_{-}) \xrightarrow{\tau(M)} \mathcal{T}(\Sigma_{+}) \\ (F|_{\Sigma_{-}})_{\sharp} \downarrow & \downarrow (F|_{\Sigma_{+}})_{\sharp} \\ \mathcal{T}(\Sigma'_{-}) \xrightarrow{\tau(N)} \mathcal{T}(\Sigma'_{+}). \end{array}$$

PROOF. The vector space $\mathcal{T}(\Sigma)^* \otimes \mathcal{T}(\Sigma')$ is canonically identified to Hom $(\mathcal{T}(\Sigma), \mathcal{T}(\Sigma'))$, so the first assertion is verified. By definition, F preserves the bases, i.e. sends Σ_- onto Σ'_- and Σ_+ onto Σ'_+ . Therefore $F|_{\partial M} = F|_{-\Sigma_-} \coprod F|_{\Sigma_+}$. By functoriality (axiom (a)), $(F|_{\partial M})_{\sharp}$ identifies to

$$(F|_{\Sigma_{-}}^{-1})^{*}_{\sharp} \otimes (F|_{\Sigma_{+}})_{\sharp} : \mathcal{T}(\Sigma_{-})^{*} \otimes \mathcal{T}(\Sigma_{+}) \to \mathcal{T}(\Sigma_{-}')^{*} \otimes \mathcal{T}(\Sigma_{+}')$$

via the commutative diagram



By functoriality, $(F|_{\partial M})_{\sharp}\tau(M) = \tau(N)$. The desired identity is the commutativity of the last square below.

LEMMA 3.2. Let $M = (M, \Sigma_{-}, \Sigma_{+})$ and $N = (N, \Sigma'_{-}, \Sigma'_{+})$ be two 3-cobordisms and $f : \Sigma_{+} \to \Sigma'_{-}$ be a positive homeomorphism. Let $N \circ_{f} M$ denote the cobordism obtained by identifying Σ_{+} and Σ'_{-} along f. Then Then

$$\tau(N \circ_f M) = \tau(N) \circ f_{\sharp} \circ \tau(M).$$

PROOF. The composition $N \circ_f M$ is obtained from $N \coprod M$ by identifying Σ_+ and Σ'_- along f. According to the gluing axiom, $\tau(N \circ_f M)$ is the image of $\tau(N \coprod M)$ under the composition

$$(3.1)$$

$$\mathcal{T}(-\Sigma_{-} \coprod \Sigma_{+} \coprod -\Sigma'_{-} \coprod \Sigma'_{+}) \simeq \mathcal{T}(\Sigma_{-})^{*} \otimes \mathcal{T}(\Sigma_{+}) \otimes \mathcal{T}(\Sigma'_{-})^{*} \otimes \mathcal{T}(\Sigma'_{+})$$

$$\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes f_{\sharp}^{*} \otimes \mathrm{id}}{\simeq} \mathcal{T}(\Sigma_{-})^{*} \otimes \mathcal{T}(\Sigma_{+}) \otimes \mathcal{T}(\Sigma_{+})^{*} \otimes \mathcal{T}(\Sigma'_{+})$$

$$\to \mathcal{T}(\Sigma_{-})^{*} \otimes \mathcal{T}(\Sigma'_{+}) \simeq \mathcal{T}(-\Sigma_{-} \coprod \Sigma'_{+}).$$

The desired identity follows from the commutativity of the square

We are now ready to state the desired version of the definition of a 3-TQFT (due to V. Turaev [93]).

DEFINITION 12.13. A 3-TQFT consists in the following assignments:

- (1) A 2-dimensional modular functor τ .
- (2) To each 3-cobordism $M = (M, \Sigma_{-}, \Sigma_{+})$ is assigned a linear operator $\tau(M) : \mathcal{T}(\Sigma_{-}) \to \mathcal{T}(\Sigma_{+}).$

These assignments are subject to the following requirements:

(i) Naturality: given two 3-cobordisms $M = (M, \Sigma_{-}, \Sigma_{+})$ and $N = (N, \Sigma'_{-}, \Sigma'_{+})$, there is a commutative square

(*ii*) Multiplicativity on disjoint union: given two 3-cobordisms $M = (M, \Sigma_{-}, \Sigma_{+})$ and $N = (N, \Sigma'_{-}, \Sigma'_{+})$, there is a commutative square

$$\begin{array}{c} \mathcal{T}(\Sigma_{-}\coprod\Sigma_{-}') \xrightarrow{\tau(M\coprod N)} \mathcal{T}(\Sigma_{+}\coprod\Sigma_{+}') \\ \simeq & \downarrow \\ \mathcal{T}(\Sigma_{-}) \otimes \mathcal{T}(\Sigma_{-}') \xrightarrow{\tau(M)\otimes\tau(N)} \mathcal{T}(\Sigma_{+}) \otimes \mathcal{T}(\Sigma_{+}') \end{array}$$

(*iii*) Fonctoriality (gluing): given two 3-cobordisms $M = (M, \Sigma_{-}, \Sigma_{+})$, $N = (N, \Sigma'_{-}, \Sigma'_{+})$ and a (structure preserving) homeomorphism $f : \Sigma_{+} \to \Sigma'_{-}$, let $N \circ_{f} M$ be the cobordism obtained by identifying Σ_{+} and Σ'_{-} along f. Then there exists $k \in \mathbb{C} - \{0\}$ such that

$$\tau(N \circ_f M) = k \ \tau(N) \circ f_{\sharp} \circ \tau(M).$$

The complex number k is called the *anomaly* of the TQFT.

(*iv*) Normalization (on the trivial cobordism): $\tau(\Sigma \times [0, 1], \Sigma, \Sigma) = \operatorname{id}_{\tau(\Sigma)}$.

The TQFT is said *anomaly free* if the complex number in (iii) can always be taken to be 1.

PROPOSITION 3.3. An anomaly free 3-TQFT in the sense of Definition 12.13 is an anomaly free 3-TQFT in the sense of Definition 12.12 and conversely.

PROOF. In the direction Def. 12.12 \longrightarrow Def. 12.13, consider a 3cobordism (M, Σ, Σ') . The image of $\tau(M) \in \mathcal{T}(-\Sigma \coprod \Sigma')$ under the isomorphism

$$\mathcal{T}(-\Sigma \coprod \Sigma') \simeq \mathcal{T}(\Sigma)^* \otimes \mathcal{T}(\Sigma') \simeq \operatorname{Hom}(\mathcal{T}(\Sigma), \mathcal{T}(\Sigma'))$$

the linear operator assigned in Def. 12.13 (axiom (2)). The naturality and gluing axioms follow from lemmas 3.1 and 3.2. Multiplicativity on disjoint union (axiom (ii)) follows from the corresponding axiom (b). Normalization on the cylinder corresponds under the definitions (axiom (iv) and (d) respectively). Conversely, for the direction Def. 12.13 \longrightarrow Def. 12.12, assuming an anomaly free 3-TQFT in the sense of Def. 12.13, let M be an oriented compact 3-manifold with boundary ∂M . We regard it as a cobordism $M = (M, \emptyset, \partial M), \tau(M) \in \text{Hom}(\mathbb{C}, \mathcal{T}(\partial M))$. Then $\tau(M)(1) \in \mathcal{T}(\partial M)$ is the vector assigned in Def. 12.12 (axiom (2)). Reading backwards the commutative diagrams of the proofs of lemmas 3.1 and 3.2, we see that the naturality and gluing axioms are satisfied.

REMARK 12.6. We have not tried to give a minimal set of axioms for a 3-TQFT. For instance, one need not a priori assume that the assignment of an isomorphism $f_{\sharp}: \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$ to a homeomorphism $f: \Sigma \to \Sigma'$ be a functor: this follows from the other axioms of Def. 12.12, see [4, §4.2, Th. 4.2.3]. Note that this relies strongly on the anomaly free gluing (axiom (c) in Def. 12.12).

As we have seen, a 3-TQFT gives rise in particular to \mathbb{C} -valued topological invariants of closed oriented 3-manifolds. We are interested in the converse: given a \mathbb{C} -valued topological invariant of 3-manifolds, can it be extended to a 3-TQFT ? It turns out that under given two simple conditions, the answer is almost positive. Let us say that a *weak* TQFT ? is the same object as a TQFT, except that it does not necessarily satisfy the multiplicativity property with respect to disjoint union.

PROPOSITION 3.4. Let τ_0 be \mathbb{C} -valued topological invariant of closed oriented 3-manifolds. Assume that for any closed oriented 3-manifolds M and N,

(3.2)
$$\tau_0(M \coprod N) = \tau(M) \cdot \tau(N)$$

and

(3.3)
$$\tau_0(-M) = \overline{\tau(M)}$$

where bar denotes complex conjugation. Then there exists a unique extension τ of τ_0 to a weak 3-TQFT.

The first condition (3.2) is generally easy to satisfy: in general, a topological invariant of a closed 3-manifold is defined first on a connected closed 3-manifold. Then the equality (3.2) extends the definition of the invariant τ to non necessarily connected closed 3-manifolds. This is exactly how we proceeded in Chapter 10, §1.3, Remark 10.3.

PROOF. We follow the construction in [6]. First extend τ_0 by linearity to the vector space over \mathbb{C} freely generated by all closed oriented 3manifolds (possibly with extra structures). For a closed oriented surface Σ , let $T(\Sigma)$ the vector space over \mathbb{C} whose basis consists of all oriented compact 3-manifolds whose boundary is parametrized by a (structure preserving) homeomorphism on a standard surface (or a finite disjoint union of standard surfaces). For $M \in T(\Sigma)$ with parametrization f and $N \in T(-\Sigma)$ with parametrization g, we set $M \cup_{\Sigma} N = M \cup_{g \circ f^{-1}} N$. Define a bilinear pairing

$$T(\Sigma) \otimes T(-\Sigma) \to \mathbb{C}, \ \langle M, N \rangle = \tau_0(M \cup_{\Sigma} N).$$

Let $\mathcal{T}(\Sigma)$ be the quotient of $T(\Sigma)$ by the left annihilator of $\langle -, - \rangle$. Then the induced bilinear pairing $d_{\Sigma} : \mathcal{T}(\Sigma) \otimes \mathcal{T}(-\Sigma) \to \mathbb{C}$ is nondegenerate. Now if M is a cobordism between Σ and Σ' , then $\tau(M) : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$ is defined as follows. Let [N] be a generator of $\mathcal{T}(\Sigma)$. Then

$$\tau(M)[N] = [N \cup_{\Sigma} M] \in \mathcal{T}(\Sigma').$$

It is easy to check that τ satisfies all the axioms of a weak anomaly free TQFT.

However, in general, one does need the multiplicativity axiom to hold as well as finite-dimensionality of the spaces of states $\mathcal{T}(\Sigma)$. To this end, an additional sufficient condition is required in [93, III §4], but seems hard to verify in practice on a candidate invariant τ_0 . In the case when the state spaces are finite-dimensional, then the bilinear pairing d_{Σ} is nonsingular (which is the desired property in order for the duality property $\mathcal{T}(-\Sigma) \simeq \mathcal{T}(\Sigma)^*$ to hold).

[This and the corollary may be replaced in the paragraph on representations of mapping class groups.

PROPOSITION 3.5. Given a structure preserving homeomorphism $f: \Sigma \to \Sigma'$, the isomorphism $f_{\sharp}: \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$ depends only on the isotopy class of f.

PROOF. Consider an isotopy

$$\Sigma \times [0,1] \to \Sigma', (x,t) \mapsto f_t(x)$$

between two structure preserving homeomorphisms $f_0 = f$ and $f_1 = g$. We wish to show that $f_{\sharp} = g_{\sharp}$. It suffices to show that if $F : \Sigma \to \Sigma$ is a structure preserving homeomorphism isotopic to the identity, then $F_{\sharp} = \operatorname{id}_{\mathcal{T}(\Sigma)}$. (Set $F = g^{-1}f$, apply the result to F so that by fonctoriality, $g_{\sharp}^{-1} \circ f_{\sharp} = (g^{-1} \circ f)_{\sharp} = F_{\sharp} = \operatorname{id}$ so $f_{\sharp} = g_{\sharp}$.)

Extend f_t to a homeomorphism $\Phi : \Sigma \times [0,1] \to \Sigma \times [0,1]$ by $(s,t) \mapsto (f_t(s),t)$. This is an isomorphism of cobordism. Naturality and normalization imply that $\mathrm{id} \circ \mathrm{id} = F_{\sharp} \circ \mathrm{id}$ hence the result.

COROLLARY 3.6. TFQT Mapping class group

Comments on equivalence. From invariants to TQFTs... etc.]

4. Construction I: filling in

This section is devoted to a first construction of the abelian TQFT associated to a quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group. This construction relies on "closing" cobordisms by standard handlebodies. The ground cobordism category is the category Cob^{par} of parametrized cobordisms.

4.1. Surgery presentation for parametrized 3-cobordisms. Just as any closed 3-manifold can be presented by surgery on S^3 along some framed link, we wish to state an analogous result for a compact oriented parametrized 3-cobordism. One should recover the previous statement if the boundary is empty. We shall follow and adapt slightly Turaev's description [93, IV, §2.3]. Another related approach is developed by Matveev and Polyak in [63].

Let $(M, \Sigma_{-}, \Sigma_{+})$ be an oriented compact parametrized 3-cobordism. The easiest solution is to fill in M by gluing handlebodies along the parametrizations of the bases and apply the previous statement (Wallace and Lickorish

theorem) to the resulting closed 3-manifold \widetilde{M} : there exists a framed link L such that \widetilde{M} is the surgery on S^3 along L. A closed tubular neighborhood of L is a finite union of embedded solid tori. The surgery link is a parallel of L lying in $\partial N(L)$ in S^3 (each component lying on the boundary of a solid torus), which we may regard as the boundary of $S^3 - \text{Int}(N(L))$. The complement $S^3 - \operatorname{Int}(N(L))$ is embedded inside \widetilde{M} . By isotoping the original link L in S^3 , we may assume that the surgery link avoids the handlebodies embedded in \widetilde{M} . Recall that the bases are parametrized, so we consider the images of the standard longitudes. Since the standard longitudes are oriented and trivially framed, the images inherit an orientation and a framing, hence form an oriented framed link inside M. Note that the link consists of the union of two distinguished sublinks corresponding to the bases of the cobordism. This link lies in the complement of N(L) so can also be regarded as a link $L^+ \cup L^-$ in S^3 . Note that the links L^+ and L^- come endowed with distinguished neighborhoods, namely the embedded handlebodies of the top and bottom bases respectively. First we isotop the handlebodies in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ so that the top handlebodies lie in $\mathbb{R}^2 \times [1/2, 1]$, the bottom handlebodies lie in $\mathbb{R}^2 \times [0, 1/2]$ and the link L lies in the complement of the handlebodies inside $\mathbb{R}^2 \times [0,1]$. Then by means of isotopy, we realize each genus g handlebody H as the union $H = N(R) \cup N(\cup_j L_j^{\pm})$ where

- (1) R is a rectangle (the direct product of two intervals) and N(R) is a closed regular 3-dimensional neighborhood of R;
- (2) $(L_j^{\pm})_{j \in J}$ is a set of g oriented band (for H) and $N(\cup_j L_j^{\pm})$ is a closed regular 3-dimensional neighborhood of $\cup_j L_j^{\pm}$ whose components are g solid tori.
- (3) $\partial R \cap \cup_j L_j^{\pm}$ consist of 2g aligned small intervals and $\partial N(R) \cap N(\cup_j L_j^{\pm})$ consist of 2g discs whose boundaries are (images of the standard) meridians lying in a plane parallel to $\mathbb{R}^2 \times 0$.

Such a triple (L^-, L, L^+) verifying these conditions is called a (surgery) link presentation of the cobordism M. The union $(L^- \cup L \cup L^+) - \operatorname{Int}(N(R))$ is a framed tangle $T^- \cup L \cup T^+$ with g^- bottom strands and g^+ top strands. The triple (T^-, L, T^+) is called a *tangle presentation* of the 3-cobordism M. Clearly the two presentations are equivalent: a tangle presentation gives rise to a link presentation with distinguished handlebodies and a link presentation. In particular, to a surgery presentation is associated an extended linking matrix.

Conversely, suppose given a triple (L_{-}, L, L_{+}) of disjoint oriented framed links in S^3 . Let g_{\pm} the number of components of L_{\pm} . There is a positive homeomorphism f_{\pm} sending the standard oriented trivially framed multilongitude of the standard handlebody $H_{g_{\pm}}$ of genus g_{\pm} to L_{\pm} . The homeomophism f_{\pm} extends to an embedding, still denoted f_{\pm} , of the standard handlebody $H_{g_{\pm}}$ of genus g_{\pm} into S^3 . We may assume that the images of f_{-} and f_{+} are disjoint.



FIGURE 4.1. A surgery presentation of an oriented compact 3-cobordism between two connected bases of genus 2. The figure on the right is a tangle version; the figure on the left is the link version.

Cutting out the interior of the handlebodies $f_{-}(\text{Int}H_{g_{-}})$ and $f_{+}(\text{Int}H_{g_{+}})$ from S^3 yields an oriented compact 3-cobordism E with bottom base Σ_{-} and top base Σ_{+} . The link L naturally lies in E and keeps its framing. Surgery on E along L results in a compact oriented 3-cobordism M with bases $\partial_{-}M = \Sigma_{-}$ and $\partial_{+}M = \Sigma_{+}$. Note that the bases are endowed with parametrizations induced by f_{+} .

EXERCISE 12.1. Generalize the description above to the case of a cobordism with non-connected bases.

PROPOSITION 4.1. Any compact oriented parametrized 3-cobordism $(M, \Sigma_{-}, \Sigma_{+})$ can be realized as the surgery on S^{3} along some triple (H_{-}, L, H_{+}) as described above: $M = s(S^{3}; L_{-}, L, L_{+})$.

DEFINITION 12.14. A triple (L_-, L, L_+) such as in Prop. 4.1 is called a surgery presentation for $M = s(S^3; L_-, L, L_+)$.

4.2. The composition of cobordisms in terms of surgery presentation. Consider a parametrized 3-cobordism P = NM obtained from 3-cobordisms M and N by gluing along a positive homeomorphism $\partial_+ M \rightarrow$ $\partial_- N$ commuting with parametrizations of the bases. Suppose that M is given a surgery presentation (T^-, L, T^+) and N is given a surgery presentation (T'^-, L', T'^+) . Define a new triple

$$(T'^{-}, L', T'^{+}) \circ (T^{-}, L, T^{+}) = (T^{-}, L \cup (T'^{-} \circ T^{+}) \cup L', T'^{+})$$

by composing the intermediate tangles (note that the composability of the cobordisms implies the composability of the tangles). See Fig. 4.2.

PROPOSITION 4.2. The composition $(T'^-, L', T'^+) \circ (T^-, L, T^+)$ of tangle presentations of M and N is a tangle presentation of the parametrized 3-cobordism NM.

4.3. Definition of the TQFT. Let $(M, \Sigma_{-}, \Sigma_{+})$ be a Lagrangian decorated cobordism. For simplicity we shall assume first that Σ_{-} and Σ_{+} are connected.

By definition, M is endowed with two parametrizations $f_-: \Sigma_{g_-} \to \Sigma_-$ and $f_+: \Sigma_{g_+} \to \Sigma_+$. Let $H_+ = (\Sigma_+ \times I) \cup_{f_+} (-H_{g_+})$ be the 3-manifold obtained by gluing a cylinder to the standard handlebody via the parametrization



FIGURE 4.2. The surgery presentation of a composition between two cobordisms.

map $f_+: \Sigma_{g_+} \to \Sigma_+ \times \{1\}$. Similarly let $H_- = (\Sigma_- \times I) \cup_{f_-} H_{g_-}$ be the 3-manifold obtained by gluing a cylinder to the standard handlebody via the parametrization map $f_-: \Sigma_{g_-} \to \Sigma_- \times \{0\}$. We define the *state space* associated to the bottom and top bases by

(4.1)
$$\mathcal{T}(\Sigma_{\pm}) = \mathscr{A}(H_{\pm}).$$

Thus as a vector space, $\mathcal{T}(\Sigma_{\pm})$ is generated by the 1-homologies of the standard colored longitudes and is isomorphic to $L^2(G \otimes \Lambda_{\pm})$ where Λ_{\pm} denotes the abelian group freely generated by the 1-homology of standard longitudes in $H_{g_{\pm}}$. In particular, $\mathcal{T}(\Sigma_{\pm})$ has dimension $|G|^{g_{\pm}}$. Note that the state space $\mathcal{T}(\Sigma_{\pm})$ depends on the parametrization.

LEMMA 4.3. A structure preserving homeomorphism $f: \Sigma \to \Sigma'$ between parametrized surfaces gives rise to an isomorphism $f_{\sharp}: \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$ induced at the level of the skein spaces.

The homeomorphism is assumed to preserve the extra structure of the parametrization: if $h : \Sigma_g \to \Sigma$ and $h' : \Sigma_g \to \Sigma'$ are parametrizations, then $f \circ h = h'$.

PROOF. For simplicity assume that Σ and Σ' both are connected and have genus g. Extend $f: \Sigma \to \Sigma'$ to a homeomorphism $(f \times \mathrm{id}_{[0,1]}) \coprod \mathrm{id}_{H_g} :$ $(\Sigma \times [0,1]) \coprod H_g \to (\Sigma' \times [0,1]) \coprod H_g$. By assumption on f, this homeomorphism extends to a homeomorphism

$$H = (\Sigma \times [0,1]) \cup_h H_g \to (\Sigma' \times [0,1]) \cup_{h'} H_g = H'.$$

This map induces an isomorphism f_{\sharp} at the level of the skein spaces (cf. Lemma 1.3).

Consider the closed oriented manifold

(4.2)
$$M = H_{g_{-}} \cup_{f_{-}} M \cup_{f_{+}} H_{g_{-}}$$

obtained by gluing two standard handle bodies to ${\cal M}$ via the parametrization maps.



FIGURE 4.3. Closing off by standard handlebodies.

Note that

$$\widetilde{M} \simeq H_{-} \cup_{\Sigma_{-}} M \cup_{\Sigma_{+}} H_{+}$$

Let \mathcal{L}^- (resp. \mathcal{L}^+) be the image by f_- (resp. by f_+) in Σ_- (resp. in Σ_+) of the standard longitudes of ∂H_{g_-} (resp. ∂H_{g_+}). We denote by the same symbol the induced links in \widetilde{M} after the gluing. Since a standard longitude has a preferred standard parallel, each link comes with a framing. We define \mathbb{C} -linear map $\tau(M) : \mathcal{T}(\Sigma_-) \to \mathcal{T}(\Sigma_+)$ by (4.3)

$$\tau(M) \left\langle \mathcal{L}^{-}(c_{-}) \right\rangle = |G|^{-g_{+}/2} \sum_{c_{+} \in G^{g_{+}}} \tau\left(\widetilde{M}, \mathcal{L}^{-}(c_{-}) \cup \mathcal{L}^{+}(c_{+})\right) \left\langle \mathcal{L}^{+}(c^{+}) \right\rangle$$

for any $c_{-} \in G^{g_{-}}$.

In the general case, when the bases are not necessarily assumed to be connected, each base is a disjoint union of connected closed oriented surfaces $\Sigma_1^-, \ldots, \Sigma_{r_-}^-$ and $\Sigma_1^+, \ldots, \Sigma_{r_+}^+$, with genera $g_1^-, \ldots, g_{r_-}^-$ and $g_1^+, \ldots, g_{r_+}^+$ respectively. Note that r_- and r_+ are the number of components of Σ_- and Σ_+ respectively. We say that

$$g_{\pm} = g_1^{\pm} + \dots + g_{r_{\pm}}^{\pm}$$

is the total genus of the base Σ_{\pm} . In accordance with the axioms, we set $\mathcal{T}(\Sigma_{\pm}) = \bigotimes_{k=1}^{r_{\pm}} \mathcal{T}(\Sigma_{k}^{\pm})$. We fill in the bases of M in the same way as above, using the parametrizations for each connected component of Σ_{-} and Σ_{+} , respectively. Call the resulting manifold \widetilde{M} . As before, we consider the images of the standard longitudes by the parametrizations. The only difference is that \mathcal{L}^{\pm} has $g_{\pm} = g_{1}^{\pm} + \cdots + g_{r_{\pm}}^{\pm}$ components. Then we define the cobordism operator $\tau(M)$ by the same formula as above (4.3) paying attention to the fact that now g denotes the total genus.

THEOREM 4.4. The assignment $\tau : \operatorname{Cob}^{\operatorname{par}} \to \operatorname{Vect}_{\mathbb{C}}, (M, \Sigma_{-}, \Sigma_{+}) \mapsto \tau(M)$ is a 3-TQFT.

The complete proof of this result is a detailed version of [16, Theorem 2] and consists in a verification of the axioms of a TQFT (Def. 12.13). We need an auxiliary construction formalizing surgery in the setting of cobordisms in order to present a proof of Theorem 4.4. We do this in the next section.

4.4. Proof of Theorem 4.4.

PROOF. Consider the rule that assigns to a parametrized surface Σ its state space $\mathcal{T}(\Sigma) = \widehat{\mathscr{A}}(H)$ (see (4.1)) and to an orientation and parametrization preserving homeomorphism $f: \Sigma \to \Sigma'$ the linear map $f_{\sharp}: \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$. By Lemma 4.3, f_{\sharp} is a linear isomorphism. This rule is easily seen to be functorial and to satisfy the conditions (1), (2) and (4) of the definition of a modular functor (Def. 4.3). The multiplicativity property on disjoint unions of closed surfaces follows from the definitions. For disjoint unions of cobordisms $(M, \Sigma_{-}, \Sigma'_{+})$ and $(N, \Sigma'_{-}, \Sigma'_{+}), \tau(M \mid N)$ is a linear map

$$\mathcal{T}(\Sigma_{-}) \otimes \mathcal{T}(\Sigma'_{-}) \simeq \mathcal{T}(\Sigma_{-} \coprod \Sigma'_{-}) \to \mathcal{T}(\Sigma_{+} \coprod \Sigma'_{+}) \simeq \mathcal{T}(\Sigma_{+}) \otimes \mathcal{T}(\Sigma'_{+}).$$

Consider the image by the parametrization in Σ_{-} and Σ'_{-} respectively of the respective standard longitudes. The image splits as disjoint union of framed colored links in Σ_{-} and Σ'_{-} respectively. Under the isomorphism $\mathcal{T}(\Sigma_{-} \coprod \Sigma'_{-}) \simeq \mathcal{T}(\Sigma_{-}) \otimes \mathcal{T}(\Sigma'_{-})$, the image becomes the corresponding tensor product of skeins. The naturality of the assignment $M \mapsto \tau(M)$ (axiom (i)) follows from definitions, as well as the multiplicativity on disjoint unions (axiom (ii)).

Let us prove the normalization axiom (axiom (iv)). First consider the trivial cobordism $(\Sigma \times [0, 1], \Sigma, \Sigma)$ for a connected genus g closed surface Σ . First, by definition $\mathcal{T}(\Sigma) = \hat{A}(H) = \mathbb{C}[G \otimes \Lambda]$ where H is an oriented handlebody such that $\partial H = \Sigma$. We apply formula (4.3). Recall that each parametrization is the identity.

Claim 1.
$$\widetilde{M} = \begin{cases} S^3 & \text{if } g = 0; \\ \sharp_{i=1}^g S^2 \times S^1 & \text{if } g \ge 1. \end{cases}$$

Proof of Claim 1. Consider the case g = 0. The cylinder is a cylinder over the 2-sphere S^2 . Gluing a 3-ball along $-\partial_-(S^2 \times [0,1]) = S^2$ yields another 3-ball. Gluing another 3-ball to its boundary yields the 3-sphere S^3 . Consider now the case g = 1. Consider the first parametrization of $\Sigma_- = -\partial H = -\partial_-(\Sigma \times [0,1])$, sending meridian onto meridian. The result of the identification is $H \cup (\Sigma \times [0,1]) \simeq H$. After identification along the parametrization of $\Sigma = \partial_+(\Sigma \times [0,1])$, the identification space is the gluing of two solid tori with identification of the meridians, hence $S^2 \times S^1$. The case $g \ge 2$ is handled similarly using connected sums.

The picture below represents a tangle presentation of a parametrized oriented compact cobordism, that consists of g oriented and 0-framed bottom arcs, g oriented and 0-framed top arcs and and oriented 0-framed link in S^3 that consists of g individually unknotted circles.



Claim 2. The picture is a surgery presentation in S^3 for the standard cylinder $\Sigma \times [0, 1]$ of genus g.

Proof of Claim 2. See [93, IV, §2.6] for a detailed proof.

Each $\mathcal{L}^{-}(c_{-})$ (resp. $\mathcal{L}^{-}(c_{-})$) is a standard colored multi-longitude of Σ . In order to compute (4.3), we need to compute $\tau(\widetilde{M}, \mathcal{L}^{-}(c_{-}) \cup \mathcal{L}^{+}(c_{+}))$. In order to do this, we apply the formula (2.2). Consider the surgery presentation above of $(\widetilde{M}, \mathcal{L}^{-}(c_{-}) \cup \mathcal{L}^{+}(c_{+}))$. The extended linking matrix A for the ordered pair of links $(L_0, L^- \cup L^+)$ is read off from the picture above:

$$A = \begin{bmatrix} 0 & -I_g & I_g \\ -I_g & 0 & 0 \\ I_g & 0 & 0 \end{bmatrix},$$

where I_g is the identity $g \times g$ matrix. It follows that $(q \otimes A)(x, c^-, c^+) = (b_q \otimes I_g)(c^+ - c^-, x), x \in G^g$. Note that the map $x \mapsto (b_q \otimes I_g)(c^+ - c^-, x)$ is a group homomorphism $G^g \to \mathbb{Q}/\mathbb{Z}$. Hence

$$\sum_{x \in G^g} \chi((q \otimes A)(x, c^-, c^+)) = \begin{cases} |G|^g & \text{if } c^+ - c^- = 0; \\ 0 & \text{if } c^+ - c^- \neq 0. \end{cases}$$

Applying the surgery formula (2.2), we find that

$$\tau(\tilde{M}, \mathcal{L}^{-} \cup \mathcal{L}^{+}, q) = \gamma(G, q)^{-\sigma(L_{0})} |G|^{-g/2} \sum_{x \in G^{g}} \chi((b_{q} \otimes A)(c^{+} - c^{-}, x))$$
$$= \delta_{c^{+}, c^{-}} |G|^{g/2}.$$

We deduce from the formula for the cobordism (4.3) that

$$\tau(M)\langle \mathcal{L}^{-}(c_{-})\rangle = \langle \mathcal{L}^{+}(c_{-})\rangle, \quad c_{-} \in G^{g}.$$

Since $\mathcal{T}(\Sigma_{-}) = \mathcal{T}(\Sigma_{+}) = \mathcal{T}(\Sigma)$, we deduce that $\tau(\Sigma \times [0, 1]) = \mathrm{id}_{\mathcal{T}(\Sigma)}$. This proves the normalization axiom for a connected closed surface. The general case of a cylinder of a possibly non-connected closed surface is similar.

We now verify the gluing axiom (iii). Consider two parametrized cobordisms $(M, \Sigma_{-}, \Sigma_{+})$ and $(N, \Sigma'_{-}, \Sigma'_{+})$ with a positive homeomorphism $\Sigma_{+} \rightarrow$ Σ'_{-} identifying the top basis of the first cobordism with the bottom basis of the second cobordism, commuting with the parametrizations of the bases. Denote by NM the composition of the two cobordisms, obtained by identifying Σ_{+} with Σ'_{-} . We shall use the tangle presentation for cobordisms. Assume that M is given a tangle presentation (T^{-}, L, T^{+}) and N is given a tangle presentation (T'^{-}, L', T'^{+}) . Then according to Prop. 4.2 $(T^{-}, L \cup T'^{-} \circ T^{+} \cup L', T'^{+})$ is a tangle presentation for NM. Let A be the extended linking matrix associated to (T'^{-}, L', T'^{+}) . Note that both A and B have a natural 3×3 block decomposition:

It follows from definitions that the extended linking matrix C for the composition $(T^-, L \cup T'^- \circ T^+ \cup L', T'^+)$ is

Let $\mathbf{x} = [c, x, x_{\circ}, x', c']$ where $c \in \mathbb{Z}^{|\pi_0(L^-)|}, x \in \mathbb{Z}^{|\pi_0(L)|}, x_{\circ} \in \mathbb{Z}^{|\pi_0(T' \circ T^+)|} = \mathbb{Z}^{|\pi_0(L')|} = \mathbb{Z}^{|\pi_0(L')|}, x' \in \mathbb{Z}^{|\pi_0(L')|}$ and $c' \in \mathbb{Z}^{|\pi_0(L'+)|}$. Then

$$\mathbf{x} C \mathbf{x}^{\mathrm{T}} = \begin{bmatrix} c & x & x_{\circ} \end{bmatrix} A \begin{bmatrix} c \\ x \\ x_{\circ} \end{bmatrix} + \begin{bmatrix} x_{\circ} & x' & c' \end{bmatrix} B \begin{bmatrix} x_{\circ} \\ x' \\ c' \end{bmatrix}.$$

It follows that

$$(q \otimes C)(\mathbf{x}) = (q \otimes A)(c, x, x_{\circ}) + (q \otimes B)(x_{\circ}, x', c').$$

Hence

$$(4.4)$$

$$\sum_{x,x_{\circ},x'} \chi((q \otimes C)(\mathbf{x})) = \sum_{x,x_{\circ},x'} \chi((q \otimes A)(c,x,x_{\circ})) \chi((q \otimes B)(x_{\circ},x',c'))$$

$$= \sum_{x_{\circ}} \sum_{x} \chi((q \otimes A)(c,x,x_{\circ})) \sum_{x'} \chi((q \otimes B)(x_{\circ},x',c'))$$

Note that the number of components of the new surgery link (for $N\overline{M}$) is $|\pi_0(L \cup T'^- \circ T^+ \cup L')| = |\pi_0(L)| + |\pi_0(T'^- \circ T^+)| + |\pi_0(L')| = |\pi_0(L)| + g_+(M) + |\pi_0(L')|$. The signature of the linking matrix for the new surgery link is sign $(L \cup T'^- \circ T^+ \cup L')$. Comparing the expressions for $\tau(N)$, $\tau(M)$ and $\tau(NM)$, we see that the expression for $\tau(NM)$ contributes the term $|G|^{-\frac{g_+(NM)}{2}} |G|^{-\frac{|\pi_0(L \cup T'^- \circ T^+ \cup L')|}{2}} = |G|^{-\frac{g_+(N)+g_+(M)+|\pi_0(L)|+|\pi_0(L')|}{2}}$, which is exactly the contribution of $\tau(N) \circ \tau(M)$ in powers of |G|. We deduce from

(4.4) and the definition of τ for links (2.1) that

 $\tau(NM) = \gamma(G,q)^{-(\operatorname{sign}(L \cup T'^{-} \circ T^{+} \cup L') - \operatorname{sign}(L') - \operatorname{sign}(L))} \tau(N) \circ \tau(M).$

Recall that for a nondegenerate quadratic form q, the Gauss sum $\gamma(G, q)$ is an 8-th root of unity, hence is not zero. This completes the proof of the gluing axiom (axiom (*iii*)).

It remains to complete the verification that τ is a 2-MF, namely the condition (3) of the definition, which asserts the existence of a nondegenerate bilinear pairing $d_{\Sigma} : \mathcal{T}(\Sigma) \times \mathcal{T}(-\Sigma) \to \mathbb{C}$ satisfying the naturality conditions with respect to isomorphisms. Let Σ be a closed surface. The cylinder $C_{\Sigma} =$ $\Sigma \times [0,1]$ can be regarded as the cobordism $(C, \Sigma \coprod -\Sigma, \emptyset)$. Accordingly, Cgives rise to an operator $d_{\Sigma} : \mathcal{T}(\Sigma \coprod -\Sigma) = \mathcal{T}(\Sigma) \otimes \mathcal{T}(-\Sigma) \to \mathbb{C}$. Naturality follows from definitions. We have already seen that the operator associated to C regarded as a cobordism $(C, -\Sigma, \Sigma)$ is $\mathrm{id}_{\mathcal{T}(C)}$ (normalization axiom). Then nondegeneracy of d_{Σ} follows from [93, III, §2.3]¹.

5. The restorative construction: counting cycles

The construction is the main result of the chapter. It is based on the reciprocity (Chapter 5). The previous construction provides a description of the cobordism in terms of a topological Ansatz based on the skein theory of topological handlebodies and external gluing. The construction in this section provides a description of the cobordism operator in terms of invariants of the 3-dimensional topology of the cobordism itself.

First we fix some auxiliary algebraic data based on the discriminant construction (Chapter 2). As before is provided a homogeneous nondegenerate quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group. We shall use the discriminant construction in the form of a presentation of (G, q) given by a triple (V, f, v) where $f: V \times V \to \mathbb{Z}$ is a (nondegenerate) bilinear lattice equipped with an integral Wu class $v \in Wu^V(f)$. We note that there are many choices for the presentation (V, f, v) and for the lift of coefficients according to a given presentation of (G, q) as a discriminant quadratic function. However, the outcome of our construction will be independent of all these choices.

Consider a 3-cobordism $(M, \Sigma_{-}, \Sigma_{+})$ in $\operatorname{Cob}_{\operatorname{Lag}}^{\operatorname{fr}}$. Recall the free abelian groups $A^{-} = A^{-}, A^{+}$ generated by the framed 1-cycles L_{i}^{-} , $(1 \leq i \leq g_{-})$ and L_{i}^{+} , $(1 \leq i \leq g_{+})$ respectively. We shall use the same notation, A^{-} and A^{+} , to denote the Lagrangian they generate in $H_{1}(\Sigma_{-})$ and $H_{1}(\Sigma_{+})$ respectively. Furthermore, $A^{-} \oplus A^{+}$ is a (decomposable) Lagrangian in $H_{1}(\partial M) = -H_{1}(\Sigma_{-}) \oplus H_{1}(\Sigma_{+})$. Define two state modules by

$$\mathcal{T}(\Sigma_{\pm}) = \mathbb{C}[G \otimes A^{\pm}].$$

These are simply vector spaces of formal combinations over \mathbb{C} of certain framed 1-cycles in ∂M (with coefficients in G) of dimensions $|G|^{g_{-}}$ and $|G|^{g_{+}}$ respectively. (The algebra structure on $\mathcal{T}(\Sigma_{+})$ will not be used in this

¹Since the proof requires only naturality of τ , the normalization axiom (axiom (*iv*)) and functoriality (gluing axiom (*iii*)).

paragraph.) Recall that as part of the definition, the cobordism M comes with a Lagrangian Λ in $H_1(\partial M) = -H_1(\Sigma_-) \oplus H_1(\Sigma_+)$, which is transverse to $A^- \oplus A^+$.

There are two nondegenerate linking pairings. One is the pairing $b_q: G \times G \to \mathbb{Q}/\mathbb{Z}$ associated to the quadratic function $q: G \to \mathbb{Q}/\mathbb{Z}$; the other is $\lambda_M: T_{\Lambda}M \times T_{\Lambda}M \to \mathbb{Q}/\mathbb{Z}$ where $T_{\Lambda}M = \text{Tors} (H_1(M)/i_*\Lambda)$ (see Lemma 1.2 and Lemma ??). According to Chap. 2, §10 (see Def. 2.9), there is a characteristic element

$$\theta = \theta_{b_a, \lambda_M} \in G \otimes T_\Lambda M \subset G \otimes G_\Lambda M$$

of order at most 2.

There are natural maps $i^{\pm}_*: G \otimes A^{\pm} \to G \otimes G_{\Lambda}$ defined as the composition

$$G \otimes A^{\pm} \xrightarrow{\subset} G \otimes Z_1(\Sigma_{\pm}) \longrightarrow G \otimes Z_1(M) \longrightarrow G \otimes \frac{H_1(M)}{i_*\Lambda} = G \otimes G_{\Lambda}$$

where $Z_1(M)$ denotes the group of 1-cycles in M and the middle map is induced by the inclusion homomorphism. For each $x \in G \otimes A^-$, let

$$\mathcal{H}(x) = \{ y \in G \otimes A^+ \mid i_*^+([y]) - i_*^-([x]) = \theta \in G \otimes G_\Lambda M \}.$$

An informal (and probably more inspiring) way of defining the set $\mathcal{H}(x)$ is to declare that a 1-cycle $y \in G \otimes A^+$ lies in $\mathcal{H}(x)$ if and only if when viewed inside M, the difference of the cycles x and y lies in the class defined by the characteristic element $\theta \in G \otimes T_{\Lambda}M$. Since G is finite, the set $\mathcal{H}(x)$ is finite. Note that $\mathcal{H}(x)$ can be empty.

EXAMPLE 12.8. In the case $T_{\Lambda}M = 0$, then $\lambda_{\Lambda} = 0$ and $\theta = 0$. Thus

$$\mathcal{H}(x) = \{ y \in G \otimes A^+ \mid i_*^+([y]) - i_*^-([x]) = 0 \in G \otimes G_\Lambda M \}.$$

EXAMPLE 12.9. In the case $G_{\Lambda}M = 0$, then for all $y \in G \otimes A^+$, the equality $i_*^+([y]) - i_*^-([x]) = 0$ in $G \otimes G_{\Lambda}M = 0$ is satisfied. Hence $\mathcal{H}(x) = G \otimes A^+$.

The 1-cycle $\kappa_{xy} = i^+_*(y) - i^-_*(x)$ (with coefficients in G) inherits a framing from the original framings of x and y. This cycle lifts to a framed oriented 1-cycle $\tilde{\kappa}_{xy}$ with coefficients in V^{\sharp} by lifting coefficients. Similarly the characteristic element $\theta \in G \otimes T_{\Lambda}M$ lifts to an element $\tilde{\theta} \in V^{\sharp} \otimes T_{\Lambda}M$.

Note that there exists a lift $\widetilde{\kappa}_{xy}$ such that $[\widetilde{\kappa}_{xy}] = \widetilde{\theta} \in V^{\sharp} \otimes T_{\Lambda}M$ if and only if $[\kappa_{xy}] = \theta \in G \otimes T_{\Lambda}M$ if and only if $y \in \mathcal{H}(x)$.

Let $\tilde{\kappa}_v$ denote the framed 1-cycle obtained by evaluating (coefficients of) $\tilde{\kappa}_{xy}$ against the integral Wu class $v \in Wu^V(f) \subset V$: $\tilde{\kappa}_v = (\hat{f}_{\mathbb{Q}} \otimes \mathrm{id})(v \otimes \mathrm{id})(\tilde{\kappa}_{xy})$. We shall need to use the invariant $\delta_s(\tilde{\kappa}_v) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ defined by (5.1). (As it was defined, this invariant depends on the choice of a spin structure, but the spin structure was used only to select a quadratic enhancement of the linking pairing.) We denote this quadratic form by q_{Λ} . We denote the resulting invariant by $\delta_{\Lambda}(\tilde{\kappa}_v) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

(5.1)

$$C_{\Lambda}(M) = \gamma(T_{\Lambda}M, q_{\Lambda})^{-f_{\mathbb{Q}}(v,v)}$$

$$\gamma\left(V \otimes T_{\Lambda}M, f \otimes q_{\Lambda} + \left(\widehat{f}_{\mathbb{Q}} \otimes \widehat{\lambda}_{\Lambda}\right)(\widetilde{\theta})\right) |G \otimes G_{\Lambda}|^{1/2} |G|^{-g_{+}/2}.$$

The number $C_{\Lambda}(M) \in \mathbb{C}$ is independent of the lift $\tilde{\theta}$ for θ . It is nonzero because of Theorem 3.1.

We define a linear operator $\tau(M) = \tau(M,q) : \mathcal{T}(\Sigma_{-}) \to \mathcal{T}_{+}(\Sigma_{+})$ by setting

(5.2)
$$\tau(M)x = C_{\Lambda}(M) \sum_{y \in \mathcal{H}(x)} \exp\left(2\pi i \left((f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \delta_{\Lambda}(\widetilde{\kappa}_{v})\right)\right) y$$

for any $x \in G \otimes A^-$ and extending by \mathbb{C} -linearity.

In an informal way (forgetting about the choice of the Lagrangian), we may say that $\tau(M)x$ computes a weighted sum of cycles in Σ_+ almost homologous to x when viewed inside M.

The major result of the paper [16] is the following theorem.

THEOREM 5.1. The assignment $(M, \Sigma_{-}, \Sigma_{+}) \mapsto \tau(M)$ defines a TQFT in dimension 3.

The assignment actually takes a cobordism (a morphism in the category $\operatorname{Cob}_{\operatorname{Dec}}^{\operatorname{fr}}$) to a unitary linear operator (cobordism invariant operator). In particular, $\tau(M)$ depends on a Lagrangian $\Lambda \in H_1(\partial M)$. It is part of the statement of Th. 5.1 that if M and N are equivalent, then $\tau(M) = \tau(N)$.

A consequence of the classification theorem (Th. 4.1).

Let us record a particular case of Theorem 5.1.

COROLLARY 5.2. Suppose that G or $T_{\Lambda}M$ has odd order. Then (5.3)

$$C_{\Lambda}(M) = \gamma(T_{\Lambda}M, q_{\Lambda})^{-f_{\mathbb{Q}}(v,v)} \gamma\left(V \otimes T_{\Lambda}M, f \otimes q_{\Lambda}\right) |G \otimes G_{\Lambda}|^{1/2} |G|^{-g_{+}/2}.$$

and for any $x \in G \otimes A^-$,

(5.4)
$$\tau(M)x = C_{\Lambda}(M) \sum_{y \in \mathcal{H}(x)} \exp(2\pi i (q \otimes \operatorname{lk}_{\Lambda}(\kappa_{xy}))) y.$$

PROOF. Either hypothesis implies that the characteristic element θ is zero in homology. Then each cycle κ_{xy} representing θ can be written as a linear combination, with coefficients in G, of boundaries, say $\kappa_{xy} = \sum_k c_k \otimes$ L_k . Lift coefficients $c_k \in G$ to coefficients $\xi_k \in V^{\sharp}$ and get a cycle $\tilde{\kappa}_{xy}$ with coefficients in V^{\sharp} . Recall that (G, q) is the discriminant quadratic form
derived from (V, f, v). It follows that

$$\begin{aligned} (q \otimes \mathrm{lk}_{\Lambda})(\kappa_{xy}) &= \sum_{k} q(c_{k}) \mathrm{lk}_{\Lambda}(L_{k}, L'_{k}) + \sum_{j < k} b_{q}(c_{j}, c_{k}) \mathrm{lk}_{M}(L_{j}, L_{k}) \\ &= (f \otimes q^{\mathrm{fr}})(\sum_{k} \xi_{k} \otimes L_{k}) - \sum_{k} f_{\mathbb{Q}}(\xi_{k}, v) \frac{1}{2} \mathrm{lk}_{\Lambda}(L_{k}, L'_{k}) \\ &= (f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \sum_{k} f_{\mathbb{Q}}(\xi_{k}, v) \left(\frac{1}{2} \mathrm{lk}_{\Lambda}(L_{k}, L'_{k}) - 0\right) \\ &= (f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \sum_{k} f_{\mathbb{Q}}(\xi_{k}, v) \left(\frac{1}{2} \mathrm{lk}_{\Lambda}(L_{k}, L'_{k}) - q_{\Lambda}([L_{k}])\right) \\ &= (f \otimes q^{\mathrm{fr}})(\widetilde{\kappa}_{xy}) - \delta_{\Lambda}(\widetilde{\kappa}_{v}). \end{aligned}$$

Since in homology we can take $\tilde{\theta} = 0 \in V^{\sharp} \otimes T_{\Lambda}M$, $C_{\Lambda}(M)$ is as stated in (5.3) and the result stated follows.

COROLLARY 5.3. If
$$T_{\Lambda}M = 0$$
 then
(5.5) $\tau(M)x = |G \otimes G_{\Lambda}|^{\frac{1}{2}}|G|^{-g_{+}/2} \sum_{\substack{y \in G \otimes A^{+}\\i_{*}^{+}([y])=i_{*}^{-}([x])}} \exp(2\pi i(q \otimes \operatorname{lk}_{\Lambda})(\kappa_{xy})) y.$

A complete proof of Th. 5.1 was given in [16] for v = 0. The same proof carries over in the general case. Here we wish to make a few comments. The crucial point in the proof of Theorem 5.1 lies in the exact behavior of τ under the gluing of 3-cobordisms. To the best of the author's knowledge, this is done in an indirect way: the 3-cobordisms are viewed as boundaries of 4-cobordisms and the composition of the 3-cobordisms is computed as the boundary of the composition of 4-cobordisms; then Wall's corrective formula for the signature of the composition is used to compute the anomaly². Wall's formula involves the Leray-Maslov index. However, the Leray-Maslov index is an invariant of a triple of Lagrangians (with respect to an antisymmetric bilinear pairing). We deduce the following result:

THEOREM 5.4. The assignment $(M, \Sigma_{-}, \Sigma_{+}) \mapsto \tau(M)$ defines a TQFT in dimension 4k - 1 for any $k \ge 1$.

PROOF. Each closed (4k-2)-manifold is naturally equipped with its intersection pairing $H_{2k-1}(\Sigma) \times H_{2k-1}(\Sigma) \to \mathbb{Z}$ which is antisymmetric. In addition, it is equipped with the isotopy class of framed (2k-1)-cycles that generate a Lagrangian. A cobordism M between Σ_- and Σ_+ is equipped with a Lagrangian $\Lambda \subset H_{2k-1}(\partial M) = H_{2k-1}(-\Sigma_-) \oplus H_{2k-1}(\Sigma_+)$. The definitions of linking number \mathbb{Ik}_{Λ} , linking pairing λ_{Λ} and quadratic enhancement q_{Λ} are the same. Then formula (5.2) makes sense for a (4k-1)-cobordism M with boundary $\partial M = -\Sigma_- \coprod \Sigma_+$. All the axioms for a TQFT are easily seen to hold except maybe the gluing axiom. Let $N \circ M$ be the composition of two cobordisms (M, Σ_-, Σ) and (N, Σ, Σ_+) . Let A_-, A, A_+ the respective Lagrangians in $H_{2k-1}(\Sigma_-), H_{2k-1}(\Sigma)$ and $H_{2k-1}(\Sigma_+)$. Consider the standard Lagrangians $\Lambda_M = \operatorname{Ker}(i_*: H_{2k-1}(\partial M) \to H_{2k-1}(M))$ and $\Lambda_N = \operatorname{Ker}(i_*: H_{2k-1}(\partial N) \to H_{2k-1}(N))$ respectively. The subspaces

$$\lambda_{-} = (\Lambda_{M})_{*}A_{-} = \{ y \in H_{2k-1}(\Sigma) \mid (x, y) \in \Lambda_{M} \text{ for some } x \in A_{-} \},\\ \lambda_{+} = (\Lambda_{N})^{*}A_{+} = \{ y \in H_{2k-1}(\Sigma) \mid (y, z) \in \Lambda_{N} \text{ for some } z \in A_{+} \}$$

²The exact computation is not needed, for instance, if one is interested only in the projective representation of the mapping class groups; in this case, Wall's formula is not necessary.

are Lagrangians in $H_{2k-1}(\Sigma)$. As in the case of 3-cobordisms, one finds that $\tau(N \cap M) = \gamma(C, q)^{-\mu(\lambda, A, \lambda_+)} \tau(N) \circ \tau(M)$

$$\gamma(\mathbf{N} \cup \mathbf{N}) = \gamma(\mathbf{G}, q) \quad \forall \quad \gamma(\mathbf{N}) \cup \gamma(\mathbf{M}),$$

where $\mu(\lambda_{-}, A, \lambda_{+}) \in \mathbb{Z}$ denotes the Leray-Maslov index (see for [97] and [93, IV,4]) of the three Lagrangians $\lambda_{-}, A, \lambda_{+}$ in $H_{2k-1}(\Sigma)$.

REMARK 12.7. A careful reader may notice that not only the cobordism invariant map $\tau(M) : \mathcal{T}(\Sigma_{-}) \to \mathcal{T}(\Sigma_{+})$ but the state modules $\mathcal{T}(\Sigma_{\pm})$ themselves depend on the extra structure on Σ_{\pm} , namely the oriented framed links L^{\pm} . The key dependency is that of the cobordism invariant operator. If we think of the link $L \subset H_1(\Sigma)$ as playing the rôle of a *base* of a fixed Lagrangian, we can identify the module of states $\mathcal{T}(\Sigma)$ to a fixed vector space (thought of as a *color module*). Suppose first that Σ has genus g and let A be the Lagrangian generated in homology by the components of L. Consider the canonical isomorphism

 $\operatorname{can}_L : \mathbb{C}[G^g] \to \mathbb{C}[G \otimes A] = \mathcal{T}(\Sigma), \quad (x_1, \ldots, x_g) \mapsto x_1 \otimes L_1 + \cdots + x_g \otimes L_g.$ Define the *color module* of Σ to be $T(\Sigma) = \mathbb{C}[G^g]$. Suppose next that Σ consists of several connected components $\Sigma_1, \ldots, \Sigma_r$. We define $T(\Sigma)$ to be the (non-ordered) tensor product of all the color modules of the components: $T(\Sigma) = \bigoplus_{j=1}^r T(\Sigma_j)$. The isomorphism can $T(\Sigma) \to \mathcal{T}(\Sigma)$ is defined to be the (non-ordered) tensor products of the isomorphisms corresponding to the components. Then we may define the cobordism invariant operator as a map

 $T(\Sigma_{-}) \to T(\Sigma_{+}), \ \tau'(M) = \operatorname{can}_{L^{+}}^{-1} \circ \tau(M) \circ \operatorname{can}_{L^{-}}.$

In particular, a cylinder (with extra structures at its bases) on a surface Σ of genus g gives rise to an operator $T(\Sigma) \to T(\Sigma)$. This is especially relevant in the next section when we derive representations of the mapping class group of surfaces from this TQFT.

CHAPTER 13

The return of the Weil representation

According to the general theory, any TQFT in dimension 3 yields a projective representation of the mapping class group of surfaces. We shall outline the procedure in our setting and proceed to the explicit computation of the representation. Then we state in a more precise form the identification with the Weil representation.

1. The mapping class group and parametrized cylinders

Let Σ be an oriented connected compact surface of genus g without boundary. Let $\mathcal{M}(\Sigma)$ denote the *mapping class group* of Σ , that is the group that consists of isotopy classes of orientation preserving homeomorphisms of Σ . We begin with a tautological representation of $\mathcal{M}(\Sigma)$.

DEFINITION 13.1. A parametrized cylinder C_{φ} over Σ is an oriented cylinder $\Sigma \times [0, 1]$ equipped with a homeomorphism $\varphi : \Sigma \times 0 \to \Sigma \times 1$.

REMARK 13.1. A parametrized cylinder C_{φ} over Σ is equivalently defined as the oriented cylinder $\Sigma \times [0, 1]$ equipped with a homeomorphism $\Sigma \rightarrow \Sigma \times 0$, parameterizing the bottom base. Thus a parametrized cylinder is a particular parametrized cobordism $(\Sigma \times [0, 1], \Sigma \times \{1\}, \Sigma \times \{0\})$ where the top base is parametrized by the identity and the bottom base is parametrized by a fixed homeomorphism.

DEFINITION 13.2. An equivalence between parametrized cylinders is an orientation preserving homeomorphism $\Phi: C_{\phi} \to C_{\psi}$ such that

(1.1)
$$\Phi|_{\Sigma \times \{1\}} = \mathrm{id}_{\Sigma \times \{1\}}, \Phi|_{\Sigma \times \{0\}} \circ \phi(x) = (\psi(x), 0) \text{ for all } x \in \Sigma.$$

REMARK 13.2. Two parametrized cylinders are equivalent if and only if they are equivalent as parametrized cobordisms.

Denote by $Cyl(\Sigma)$ the set of parametrized cylinders up to equivalence.

LEMMA 1.1. The map $\varphi \mapsto C_{\varphi}$ from the group of homeomorphisms to the set of parametrized cylinders induces a map

$$\mathfrak{M}(\Sigma) \to \operatorname{Cyl}(\Sigma), \ [\varphi] \mapsto [C_{\varphi}].$$

PROOF. Let $(x,t) \mapsto \varphi_t(x)$ be an isotopy between two homeomorphisms φ_0 and φ_1 of Σ . We need to show that there is an equivalence between C_{φ_0} and C_{φ_1} . The map

$$\Phi: \Sigma \times [0,1] \to \Sigma \times [0,1], \ (x,t) \mapsto (\varphi_t \varphi_0^{-1}, t)$$

is clearly a level-preserving homeomorphism and commutes with parametrizations. $\hfill\blacksquare$

LEMMA 1.2. The set $\operatorname{Cyl}(\Sigma)$ is equipped with a product defined as follows: for $[C_{\varphi}], [C_{\psi}] \in \operatorname{Cyl}(\Sigma)$, let

$$C_{\varphi} \circ C_{\psi} = \left(C_{\varphi} \bigsqcup C_{\psi} \right) / \sim,$$

where for any $x \in \Sigma$, $C_{\varphi} \ni (\varphi(x), 0) \sim (x, 1) \in C_{\psi}$. In other words, $C_{\varphi} \cdot C_{\psi}$ is the cylinder $\Sigma \times [0, 1]$ obtained by identifying the top base of a cylinder with the bottom base of a cylinder via φ and compressing the result in the t-coordinate. Define a product in $Cyl(\Sigma)$ by

$$[C_{\varphi}] \circ [C_{\psi}] = [C_{\varphi} \circ C_{\psi}].$$

In other words, the top base of C_{ψ} is identified via φ with the bottom base of C_{φ} . With this product, $Cyl(\Sigma)$ becomes a group so that the natural map

$$\mathfrak{M}(\Sigma) \to \operatorname{Cyl}(\Sigma)$$

is a group isomorphism.

PROOF. The product at the level of $\operatorname{Cyl}(\Sigma)$ is the composition of (equivalence classes of) parametrized cobordisms. Thus the map $[\varphi] \mapsto [C_{\varphi}]$ is multiplicative. There is obviously a map from the set of parametrized cylinders over Σ to the mapping class group $\mathfrak{M}(\Sigma)$ that sends C_{φ} to $[\varphi]$. We claim that this map induces a map $\operatorname{Cyl}(\Sigma) \to \mathfrak{M}(\Sigma)$. Let $\Phi : C_{\phi} \to C_{\psi}$ be an equivalence between two parametrized cylinders. Let $\varphi_t(x) = \Phi|_{\Sigma \times \{t\}}$ for all $x \in \Sigma$. Then $\varphi_1(x) = \operatorname{id}_{\Sigma}$ and $\varphi_0(x) \circ \varphi = \psi$. Therefore $f_t = \varphi_t \circ \varphi$ defines an isotopy between ψ (t = 0) and φ (t = 1). Hence $[\varphi] = [\psi]$. So we have just defined an inverse map $\operatorname{Cyl}(\Sigma) \to \mathfrak{M}(\Sigma), [C_{\varphi}] \mapsto [\varphi]$. Cyl (Σ) inherits its group structure from that of $\mathfrak{M}(\Sigma)$.

REMARK 13.3. This implies that any obvious generalization of $Cyl(\Sigma)$, for instance homology cylinders, will contain the mapping class groups.

DEFINITION 13.3. A geometric symplectic basis for $H_1(\Sigma)$ is a system of 2g oriented simple closed curves $(m_1, l_1, \ldots, m_g, l_g)$ on Σ such that

- (1) the complement $\Sigma (m_1 \cup \ldots \cup m_g)$ is connected;
- (2) the system $([m_1], [l_1], \ldots, [m_g], [l_g])$ of their 1-homology classes is a symplectic basis for the intersection pairing $\bullet : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$.

LEMMA 1.3. For any system (m_1, \ldots, m_r) of oriented simple closed curves whose complement is connected, there exists a geometric symplectic basis extending it.

PROOF. See [27, §1.3].

We call the curves m_1, \ldots, m_g (resp. l_1, \ldots, l_g) meridians (resp. longitudes).

REMARK 13.4. We could have as well defined meridians and longitudes as images by some parametrization of meridians and longitudes of a *standard* surface (Cf. $\S1$).

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The surface Σ shall always be equipped with a geometric symplectic basis. See Fig. 1.1.



FIGURE 1.1. The standard oriented surface Σ of genus g equipped with 2g oriented simple closed curves and g-1 unoriented simple closed curves. The geometric symplectic basis of $H_1(\Sigma)$ is $(m_1, l_1, \ldots, m_q, l_q)$.

DEFINITION 13.4. A Lagrangian cylinder C_{Λ} over an oriented compact surface Σ is an oriented cylinder $C = \Sigma \times [0,1]$ equipped with Lagrangians $A^- \subseteq H_1(\Sigma_-), A^+ \subseteq H_1(\Sigma_+)$ and $\Lambda \subseteq H_1(\partial C)$ such that $\Lambda \oplus A^- \oplus A^+ = H_1(\partial C)$.

A Lagrangian cylinder is a special case of a Lagrangian decorated cobordism (and hence, a Lagrangian cobordism). As a consequence, a parametrized cylinder induces a Lagrangian cylinder (cf. Prop. 2.4) as we recall now. Let C_{φ} be a parametrized cylinder. As a parametrized cobordism, C_{φ} comes equipped with two sets of oriented links L^- and L^+ respectively: L^- consists of the image of the longitudes l_1^-, \ldots, l_g^- in Σ_- by $\varphi : \Sigma_- \to \Sigma_-$ while L^+ consists of the longitudes l_1^+, \ldots, l_g^+ in Σ_+ . The links L^- and L^+ are framed by the framing given by a small positive collar of L^- and L^+ in Σ_- and Σ_+ . The links L^- and L^+ generate Lagrangians A_- and A_+ in $H_1(\Sigma_-) = H_1(\Sigma)$ and in $H_1(\Sigma_+)$ respectively. The Lagrangian $\Lambda = \Lambda_{\varphi}$ is generated by the image by φ of the meridians in Σ_- and by the meridians in Σ_+ . (LA PARTIE SUR SEIFERT A ETE DEPLACEE.)

2. The modular representation

According to the TQFT τ defined in the previous chapter, the cylinder M(h) gives rise to a cobordism map $\tau(h) : \mathcal{T}(\Sigma_{-}) \to \mathcal{T}(\Sigma_{+})$. Note that $\tau(h) \circ \tau(h^{-1})$ is a multiple of $\tau(\mathrm{id}_{\Sigma})$, hence a multiple of the identity, it follows that $\tau(h)$ is invertible.

The isomorphism $\tau(h)$ depends on the quadratic form q: when we need to emphasize this, we write $\tau_q(h)$. It follows from Lemma 1.2 that the cobordism M(h) depends only on the mapping class $[h] \in \mathcal{M}(\Sigma)$.

Any oriented closed surface can be endowed with a geometric symplectic basis. Since cylinders form a very particular class of cobordisms, we can be more specific about our choices here. We choose two *geometric* symplectic bases $(m_1^-, l_1^-, \ldots, m_g^-, l_g^-)$ and $(m_1^+, l_1^+, \ldots, m_g^+, l_g^+)$ for $H_1(\Sigma_-)$ and $H_1(\Sigma_+)$ respectively, as follows. Since any cobordism M(h) is a cylinder over Σ , the natural inclusion map

 $i^{\pm}: \pm \Sigma_{\pm} \to \Sigma \times \{\frac{1}{2}\} \subset \Sigma \times [0,1] = M(h)$

is a positive embedding which induces an isomorphism

$$i_*^{\pm}: H_1(\pm \Sigma_{\pm}) \xrightarrow{\simeq} H_1(M(h)).$$

We require the geometric symplectic bases to verify

$$i^-_*(l^-_j)=i^+_*(l^+_j),\ i^-_*(m^-_j)=i^+_*(m^+_j).$$

In particular, let

$$m_1 = i_*^+([m_1^+]), \dots, m_g = i_*^+(m_g^+), l_1 = i_*^+(l_1^+), \dots, l_g = i_*^+(l_g^+).$$

The set $[m_1], [l_1], ..., [m_g], [l_g]$ is a Z-basis for $H_1(M(h))$.

DEFINITION 13.5. The Lagrangian generated in $H_1(M(h))$ by l_1, \ldots, l_g is the longitudinal Lagrangian and is denoted Υ . The state module $T(\Sigma)$ associated to a standard oriented closed surface Σ is the group algebra $\mathbb{C}[G \otimes \Upsilon]$. More generally, the state module associated to an oriented closed surface endowed with a Lagrangian $A \subseteq H_1(\Sigma)$ is $T(\Sigma) = \mathbb{C}[G \otimes A]$.

For a standard surface, the state module $T(\Sigma)$ plays the rôle of a reference state module. If $f: \Sigma_g \to \Sigma$ is a parametrization of Σ sending Υ to Λ ...

By definition $\tau(h)$ depends on the choice of geometrical Lagrangians $A^$ and A^+ in Σ_- and Σ_+ . In the particular setting of parametrized cylinders, A^- (resp. A^+) is the lattice generated by the oriented framed link $h(l_1^-), \ldots, h(l_g^-)$ (resp. by l_1^+, \ldots, l_g^+). Note that A^- and A^+ both identify to Υ in $H_1(M(h))$ via the maps i_*^- and i_*^+ respectively. With this identification in mind, we can set $T(\Sigma_-) = T(\Sigma_+) = \mathbb{C}[G \otimes \Upsilon]$.

The following result is a consequence of Theorem 5.1.

LEMMA 2.1 (Modular representation). The map

$$\mathcal{M}(\Sigma) \to \operatorname{Aut}(T(\Sigma)), \ [h] \mapsto \tau_q(h)$$

defines a projective representation of $\mathcal{M}(\Sigma)$.

The fact that the representation is projective and not just linear is a consequence of the non trivial anomaly in the TQFT.

Let us describe more explicitly the modular representation τ_q above in terms of the formulas (5.1) and (5.2) for the TQFT given in the previous section.

By construction, $i_*(\Lambda)$ is the subgroup generated by $i_*^-h_*([m_1^-]), \ldots, i_*^-h_*([m_g^-])$ and $i_*^+([m_1^+]), \ldots, i_*^+([m_q^+])$. Writing in the basis of $H_1(M)$

$$i_*^- h_*([m_j^-]) = \sum_k (a_{jk} [l_k] + b_{jk} [m_k]), \ k = 1, \dots, g,$$

we find that $G_{\Lambda} = H_1(M(h))/i_*\Lambda$ is the abelian group generated by $[l_1], \ldots, [l_g]$ with relations $\sum_k a_{jk} [l_k]$. In particular, G_{Λ} has rank at most g.

Therefore, $T_{\Lambda}M(h) = \text{Tors Coker}(a_{jk})_{1 \leq j,k \leq g}$ depends only on the matrix $(a_{jk})_{1 \leq j,k \leq g}$. According to the previous section, this matrix has a simple interpretation:

$$a_{jk} = \operatorname{lk}_{\Lambda} \left(i_*^- h_*(m_j^-), i_*^+(l_k^+) \right) \in \mathbb{Z}.$$

On the other hand, linking numbers inside M(h) between images of longitudes are rational in general.

The pair $(G_{\Lambda}, \lambda_{\Lambda})$ is the discriminant linking group associated to the matrix $(a_{jk})_{1 \leq j,k \leq g}$. The characteristic element $\theta \in G \otimes G_{\Lambda}$ is the characteristic element associated to λ_{Λ} and λ_q . Any choice $x = (x_1, \ldots, x_g) \in G^g$ of colors determines an oriented framed 1-cycle $\sum_j x_j \otimes (i^- \circ h)(l_j^-)$ in M(h) with coefficients in G. Similarly any choice $y = (y_1, \ldots, y_g) \in G^g$ of colors determines an oriented framed 1-cycle $\sum_j y_j \otimes i^+(l_j^+)$ in M(f) with coefficients in G. Given $x = (x_1, \ldots, x_g) \in G^g$, we set

$$\mathcal{H}(x) = \left\{ y = (y_1, \dots, y_g) \in G^g \mid \underbrace{i_*^+ \left(\sum_j y_j \otimes [l_j^+] \right) - i_*^- \left(\sum_j x_j \otimes h_*([l^-]) \right)}_{[\kappa_{xy}]} = \theta \right\}.$$

Choose a lattice presentation (V, f, v) for (G, q) and a quadratic enhancement (G, q_{Λ}) of (G, λ_{Λ}) . Then we may lift θ to an element $\tilde{\theta} \in V^{\sharp} \otimes T_{\Lambda}M(h)$ as before. More generally, we may lift 1-cycles with coefficients in G to 1-cycles with coefficients in V^{\sharp} . In particular, for any choices of colors x, y, we obtain a 1-cycle $\tilde{\kappa}_{xy}$ with coefficients in V^{\sharp} .

We define a normalization coefficient C(h) using (5.1):

(2.1)
$$C(h) = C_{\Lambda}(M(h)).$$

Given $x \in G^g$ and $y \in \mathcal{H}(x)$, we define the *phase weight* associated to y by

$$\Omega(x,y) = \exp\left(2\pi i \left(\left(f \otimes q^{\text{fr}}\right)(\widetilde{\kappa}_{xy})\right) - \delta_v(\widetilde{\kappa}_v)\right)$$

Let $e_x, x \in G^g$ be the standard basis of $\mathbb{C}[G^g]$. Then

(2.2)
$$\tau(f)e_x = C(h) \cdot \sum_{y \in \mathcal{H}(x)} \Omega(x, y) \ e_y$$

As noted above, it is a consequence of Th. 5.1 that the map

(2.3)
$$\mathcal{M}(\Sigma) \to \operatorname{Aut}(\mathbb{C}[G^g]), \ [h] \mapsto \tau(h)$$

is a projective representation of $\mathcal{M}(\Sigma)$.

PROPOSITION 2.2. The representation $[h] \mapsto \tau(h)$ is unitary and factors through the symplectic linear representation

$$\mathcal{M}(\Sigma_g) \to \mathrm{Sp}(H_1(\Sigma)), \ [h] \mapsto h_*$$

induced by homology.

PROOF. We first show that $\tau(h)$ only depends on $h_* \in \text{Sp}(H_1(\Sigma))$.

(Since $\tau(M) = 1$ if M is an integral homology 3-sphere, the representation $[h] \mapsto \tau(h)$ factors through the symplectic representation.)

The following theorem is our main goal. It asserts that the Abelian TQFT representation based on a finite quadratic form q is essentially the Weil representation associated to q. Together with our description of the Abelian TQFT representation (2.2), it provides a new description of the Weil representation.

We first define the Weil representation in the appropriate setting. The group G is endowed with its quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$, which turns it into a quadratic group and in particular, into a linking group. The homology group $H_1(\Sigma)$ is endowed with its intersection form \bullet , which turns it into a symplectic group. In particular, the group $G \otimes H_1(\Sigma)$ becomes a symplectic group with the symplectic form $b_q \otimes \bullet$.

We endow the standard surface Σ with the Hopf Lagrangian Λ and the corresponding Hopf Seifert pairing $\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ (see §??), defined by

(2.4)
$$\beta([x], [y]) = \operatorname{lk}_{\Lambda}(i_*^- x^-, i_*^+ y^+)$$

where as usual $i_*^{\pm} : H_1(\Sigma_{\pm}) \to H_1(\Sigma \times [0,1])/i_*\Lambda$ denotes the inclusion homomorphisms. This Seifert pairing β induces a Seifert pairing

 $b_q \otimes \beta : H_1(\Sigma; G) \times H_1(\Sigma; G) \to \mathbb{Q}/\mathbb{Z}$

for the symplectic form $b_q \otimes \bullet$ which we still call the Hopf Seifert pairing.

One needs to select a Lagrangian in $G \otimes H_1(\Sigma) = H_1(\Sigma; G)$. We choose the Lagrangian L_0 in $H_1(\Sigma)$, which induces the Lagrangian $G \otimes L_0$ in $H_1(\Sigma; G)$.

Define a character $\chi : H_1(\Sigma; G) \to \mathbb{C}^{\times}$ by $\chi = \exp\left(2\pi i (b_q \otimes \beta)\right)$.

Consider the Weil representation

$$\rho: \operatorname{Sp}(H_1(\Sigma)) \to \operatorname{U}(L^2 G \otimes L_0) = \operatorname{U}(\mathbb{C}[G \otimes L_0])$$

associated to Lagrangian L_0 and character χ (see §...).

THEOREM 2.3 ("Weil = TQFT"). The TQFT representation $[h] \mapsto \tau(h)$ factors through the Weil representation $\rho : \operatorname{Sp}(H_1(\Sigma)) \to \operatorname{U}(\mathbb{C}[G \otimes L_0])$. In other words, the diagram

$$\mathcal{M}(\Sigma) \xrightarrow{\tau} \mathrm{U}(\mathbb{C}[G^g])$$

$$\downarrow^{\rho} \qquad \uparrow$$

$$\mathrm{Sp}(H_1(\Sigma)) \longrightarrow \mathrm{ASp}(H_1(\Sigma;G))$$

is commutative.

3. A direct proof of "Weil=TQFT" theorem

In order to identify the Weil representation, we use Proposition 8.1. It is therefore sufficient to identify the Weil representation on generators of $\operatorname{Sp}(H_1(\Sigma))$. We use the list (8.1) provided by Remark 6.10. In the sequel,

we endow $H_1(\Sigma)$ with a geometric symplectic basis (See Fig. 1.1). There are three types of generators of $\operatorname{Sp}(H_1(\Sigma))$, so there are three cases to consider. *First case*. Consider a diffeomorphism $h: \Sigma_- \to \Sigma_-$ such that $h(m_j^-) = l_j^-$ and $h(l_j^-) = -m_j^-$, $1 \leq j \leq g$. With respect to the symplectic basis $([m_1^-], \ldots, [m_g^-], [l_1^-], \ldots, [l_g^-])$, we have

$$\operatorname{Mat}_{[m_i^-],[l_i^-]}(h_*) = \left[\begin{array}{cc} 0 & -1_g \\ 1_g & 0 \end{array} \right].$$

Hence *h* represents a generator of $\operatorname{Sp}(H_1(\Sigma))$ of the first type. By definition, Λ is the Lagrangian generated by m_j^+ and $h(m_j^-) = -l_j^-$, $1 \leq j \leq g$. Thus $i_*\Lambda$ is generated by $i_*([m_j^+]) = [m_j]$ and $i_*h_*([m_j^-]) = -i_*([l_j^-]) = -[l_j]$, $1 \leq j \leq g$. It follows that $G_{\Lambda} = H_1(M(h))/i_*\Lambda = 0$. In particular $T_{\Lambda}M(h) = 0$, $\lambda_{\Lambda} = 0$ and $\theta = 0$. Since $G \otimes G_{\lambda} = 0$, for any $x \in G^g$, the set $\mathcal{H}(x)$ consists of all elements $y \in G^g$. Let

$$\kappa_{xy} = i_*^+ \left(\sum_j y_j \otimes l_j^+\right) - i_*^- \left(\sum_j x_j \otimes h(l_j^-)\right)$$
$$= i_*^+ \left(\sum_j y_j \otimes l_j^+\right) + i_*^- \left(\sum_j x_j \otimes m^-\right)$$

be the corresponding framed 1-boundary with coefficients in G. Since

$$lk_{\Lambda}(i_{*}^{+}l_{j}^{+},i_{*}^{+}l_{j}^{+'}) = lk_{\Lambda}(i_{*}^{-}m_{j}^{-},i_{*}^{-}m_{j}^{-'}) = 0,$$

we find

$$(q \otimes \operatorname{lk}_{\Lambda})(\kappa_{xy}) = (q \otimes \operatorname{lk}_{\Lambda})(\sum_{j} y_{j} \otimes i_{*}^{+} l_{j}^{+} + x_{j} \otimes i_{*}^{-} m_{j}^{-})$$
$$= \sum_{j < k} b_{q}(y_{j}, x_{k}) \operatorname{lk}_{\Lambda}(i_{*}^{+} l_{j}^{+}, i_{*}^{-} m_{k}^{-}).$$

We have

 $lk_{\Lambda}(i_{*}^{+}l_{j}^{+},i_{*}^{-}m_{k}^{-}) = lk_{\Lambda}(i_{*}^{-}m_{k}^{-},i_{*}^{+}l_{j}^{+}) = lk_{\Lambda}(i_{*}^{+}m_{k}^{+},i_{*}^{+}l_{j}^{+}) = m_{k}^{+} \bullet_{\Sigma_{+}} l_{j}^{+} = \delta_{jk}.$ Now apply Corollary 5.3. We obtain

$$\rho(h_*) \ e_x = |G|^{-g/2} \ \sum_{y \in G \otimes L_0} \chi((b_q \otimes \beta)(y, x)) \ e_y.$$

This is the formula (8.2) as desired.

Second case. We have to consider a diffeomorphism $h: \Sigma_{-} \to \Sigma_{-}$ such that with respect to the symplectic basis $([m_1^-], \ldots, [m_q^-], [l_1^-], \ldots, [l_q^-])$,

$$\operatorname{Mat}_{[m_i^-],[l_i^-]}(h_*) = \begin{bmatrix} 1_g & 0\\ B & 1_g \end{bmatrix}$$

where B is an symmetric integral square matrix of size g. Note that

$$\begin{bmatrix} 1_g & 0\\ B & 1_g \end{bmatrix} \cdot \begin{bmatrix} 1_g & 0\\ B' & 1_g \end{bmatrix} = \begin{bmatrix} 1_g & 0\\ B+B' & 1_g \end{bmatrix}$$

and

$$\chi(q \otimes (B + B')) = \chi(q \otimes B + q \otimes B') = \chi(q \otimes B) \cdot \chi(q \otimes B').$$

It follows that it suffices to verify the formula for an elementary symmetric integral matrix B. Let $1 \leq i < j \leq g$. Let E_{ij} denotes the elementary matrix defined by $(E_{ij})_{kl} = \delta_{ik} \cdot \delta_{jl}$. Consider the case when $B = E_{ij} + E_{ji}$. Then Λ is generated by $[m_k^+]$ and

$$h_*([m_k^-]) = m_k^- + \sum_p (\delta_{ip}\delta_{jk} + \delta_{jp}\delta_{ik})l_p = \begin{cases} m_i + l_j & \text{if } k = i; \\ m_j + l_i & \text{if } k = j; \\ m_j & \text{if } k \notin \{i, j\} \end{cases}$$

for $1 \leq k \leq g$. Thus $G_{\Lambda}M = H_1(M(h))/i_*\Lambda$ is the free abelian group of rank g-2 generated by $[l_1], \ldots, [\widehat{l_i}], [\widehat{l_j}], \ldots, [l_g]$. (Here \uparrow denotes deletion.) In particular, $T_{\Lambda}M$ is trivial, $\lambda_{\Lambda} = 0$ and $\theta = 0$. By definition, $y \in \mathcal{H}(x)$ if and only if $i^+_*([y]) = i^-_*([x])$ in $G \otimes G_{\Lambda}M$. Since $G_{\Lambda}M$ is free, $\operatorname{Ker}(G \otimes H_1(M) \to G \otimes G_{\Lambda}M) = G \otimes \Lambda$. It follows that the map $i^\pm_*|_{G \otimes A^\pm} : G \otimes A^\pm \to G \otimes G_{\Lambda}M$ are injective. (Here we use the fact that Λ and $A^- \oplus A^+$ are transverse Lagrangians in $H_1(\partial M(h))$.) It follows that the equation $i^+_*([y]) = i^-_*([x])$ has a unique solution in y. With the identification $A^+ = L_0 = A^-$, this solution identifies y = x.

Third case.

3.1. A second proof.

4. A few computations and examples

5. A modular category

For the definition of a modular category and the construction of quantum invariants of 3-manifolds from modular categories, we refer to [93, Chap. 2, Chap. 3]. Let $b: G \times G \to \mathbb{Q}/\mathbb{Z}$ be a biadditive pairing on a finite group G and $a: G \to \mathbb{Q}/\mathbb{Z}$ be a homomorphism such that 2a = 0. In [93, p.77] (the second "toy example"), V. Turaev defines a modular tensor category $\mathcal{C}(G, a, b)$ as follows.

- Objects are elements of G.
- The set of morphisms is \mathbb{C} between two identical elements of G and the singleton $\{0\}$ otherwise.
- Composition of morphisms is the usual product in \mathbb{C} .
- Tensor product of morphisms is induced by the usual products in G and \mathbb{C} respectively.
- The dual object to an object $g \in G$ is g^{-1} .
- The braiding $c_{g,h} : g \otimes h = gh \rightarrow hg = h \otimes g$ is defined as $\exp(2\pi i b(g,h))$ where b is a fixed bilinear pairing.
- The twist $t_q: g \to g$ is defined as $\exp(2\pi i(b(g,g) + a(g)))$ where a.
- The associator $(gh)k \rightarrow g(hk)$ is the identity (trivial).

This category is known to be a *semi-simple ribbon* category with the set of *simple* objects being G itself. Indeed, the (xy)-th entry of the S-matrix is

 $Tr(c_{x,y} \circ c_{y,x}) = \exp(2\pi i (b(x,y) + b(x,y) + a(x) + a(y))).$

By definition a semi-simple category (over \mathbb{C}) is modular if the S-matrix is invertible over \mathbb{C} .

PROPOSITION 5.1. Let (G, a, b) be a triple as above with $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ nondegenerate. The following three statements are equivalent:

(1) The form

$$b: G \times G \to \mathbb{Q}/\mathbb{Z}, \ (x, y) \mapsto b(x, y) + b(y, x)$$

is non-degenerate.

- (2) a = 0 and G has no cyclic (left or right) orthogonal summand of even order.
- (3) The semi-simple category $\mathcal{C}(G, a, b)$ is modular.

PROOF. Preliminary observation: using adjoint maps one sees that \tilde{b} is nondegenerate if and only if the linking

$$(x, y) \mapsto \dot{b}(x, y) + a(x) + a(y)$$

is nondegenerate, which implies a = 0. $(1) \Longrightarrow (2)$: suppose (1) satisfied. Suppose that G has a cyclic orthogonal summand of even order 2k generated by $x \in G$. Then by [15, Lemma 28], x and $\tilde{b}(x,x) = 2b(x,x)$ have the same order 2k in \mathbb{Q}/\mathbb{Z} . But this would imply that b(x,x) has order 4k, a contradiction since 2k b(x,x) = b(2k x,x) = b(0,x) = 0. $(2) \Longrightarrow (1)$: suppose that the linking \tilde{b} is degenerate, i.e., $\operatorname{Ann}(\tilde{b}) = \{x \in G \mid \tilde{b}(x,-) = 0\} \neq \{0\}$. It induces a nondegenerate linking \tilde{b}' on the quotient $G/\operatorname{Ann}(\tilde{b})$. Any section of the projection $G \to G/\operatorname{Ann}(\tilde{b})$ induces an isomorphism of linkings

$$(G, \tilde{b}) \simeq (G/\operatorname{Ann}(\tilde{b}), \tilde{b}') \oplus (\operatorname{Ann}(\tilde{b}), 0).$$

By the previous result, since \tilde{b}' is nondegenerate, $G/\operatorname{Ann}(\tilde{b}')$ has no cyclic orthogonal summand of even order. On the other hand, there is $x_0 \in \operatorname{Ann}(\tilde{b}) - \{0\}$ such that $\tilde{b}(x_0, -) = 0$. In particular,

$$b(x_0, x_0) = b(x_{0}, x_0) + b(x_0, x_0) = 0,$$

hence 2 $b(x_0, x_0) = 0$. This implies that the order of x_0 is even. It follows that $\operatorname{Ann}(\tilde{b})$ has even order. Note that the direct sum decomposition of $\operatorname{Ann}(\tilde{b})$ is orthogonal. Hence $\operatorname{Ann}(\tilde{b})$ contains an orthogonal cyclic summand of even order. This is the desired result. (3) \Longrightarrow (1): modularity is equivalent to the invertibility of the S-matrix. The latter is equivalent to the nondegeneracy of $(x, y) \mapsto \tilde{b}(x, y) + a(x) + a(y)$ which implies a = 0 and nondegeneracy of \tilde{b} by the preliminary observation. The converse follows from [10, Prop. 1.1].

Any triple (G, a, b) as above gives rise to a homogeneous quadratic form qon G defined, as the sum of the polarization of b and the homomorphism a, by: $q(x) = b(x, x) + a(x), x \in G$. In this case, the bilinear linking associated to q is the symmetrized form built from b. Conversely: LEMMA 5.2. Any homogeneous quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ is the sum of the polarization of some bilinear pairing b and a homomorphism $a: G \to \mathbb{Q}/\mathbb{Z}$ such that 2a = 0 if and only if G has no cyclic orthogonal summand of even order.

PROOF. If G has odd order then the image of q is an odd (cyclic) subgroup of \mathbb{Q}/\mathbb{Z} . Thus we can define $b(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$ for $x, y \in G$. Then q(x) = b(x, x) for all $x \in G$. If G has even order, consider any quadratic form q on G that is nondegenerate. Then by [15, Lemma 29], there exists $x \in G$ of order k such that q(x) has order 2k in \mathbb{Q}/\mathbb{Z} , while b(x, x) + a(x) has order dividing k for any bilinear pairing $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ and homomorphism $a : G \to \mathbb{Q}/\mathbb{Z}$. Therefore, q is not the sum of the polarization of b and the homomorphism a.

We consider now the case when C(G, a, b) is not modular. A weaker condition than the invertibility of the S-matrix is known in order to construct a topological invariant from a semi-simple ribbon category (see [10, Prop. 1.6], [13, Appendix A]):

(5.1)
$$\sum_{x \in G} \exp(2\pi i (b(x, x) + a(x))) \neq 0.$$

This is a Gauss sum. Using Lemma ??, we see that Condition (5.1) is satisfied if and only if

(5.2) b(x,x) + a(x) = 0 for all $x \in G$ such that $2x \in Ann(b) = Ker(b)$.

Without loss of generality, assume that b is nondegenerate. Then (5.2) becomes

(5.3) b(x,x) + a(x) = 0 for all $x \in G$ such that 2x = 0.

Since $x \mapsto b(x, x)$ is a homomorphism on $\{x \in G \mid 2x = 0\}$, the condition (5.2), viewed as an equation in a, has always a solution in $a \in G^*$.

The next step consists in extending a quantum invariant to a topological quantum field theory. The modularity is used in a crucial way in this extension. However, in the case of $\mathcal{C}(G, a, b)$, the invertibility of the S-matrix is used to ensure that the resulting topological quantum field theory is nondegenerate (i.e., the cobordism invariant on a cylinder may vanish). It is not hard to see that in the case when $\mathcal{C}(G, a, b)$ is semi-simple and satisfies (5.1), there is a topological quantum field theory associated to $\mathcal{G}(G, a, b)$.

We mention an alternative modular category that produces (possibly up to a normalization real factor) the invariant τ above. The category in question – as opposed to the one above – is not strict and includes a non trivial associator $a: (xy)z \to x(yz)$ defined as $a_{x,y,z} = \exp(2\pi i h(x, y, z)), x, y, z \in$ G, where h is a map : $G \times G \times G \to \mathbb{Q}/\mathbb{Z}$. The braiding is still defined as above $c_{x,y}: xy \to yx$ by $c_{x,y} = \exp(2\pi i b(x, y))$ but b is no longer necessarily biadditive. The maps b and h are required to satisfy the hexagon identity. It turns out that the pair (b, h) is an Abelian 3-cocycle in the sense of Eilenberg and McLane [24]. Computations of Abelian Eilenberg-McLane cohomology were performed by S. Eilenberg and S. McLane themselves [24] and in particular the identification of $H^3(A^1(G); \mathbb{Q}/\mathbb{Z})$ where $A^1(G)$ is a certain cell complex associated to G, and the set of homogeneous quadratic functions $G \to \mathbb{Q}/\mathbb{Z}$. Further computations were performed by A. Joyal and R. Street [47], F. Quinn [77] and others. The details of the construction of a modular category from this data are worked out by F. Quinn [77] and S.D. Stirling in [90]. The fundamental construction of Eilenberg and McLane has been generalized by C. Ospel to a nonabelian ("quasi-abelian") setting in [73], which in turn, has been further used by V. Turaev to construct enriched modular categories to incorporate a group action [95].

CHAPTER 14

Solutions and hints to Exercises

Solution to Exercise 1.1. If the quadratic functions are isomorphic, then 5 is a square mod 8, which is easily verified to be not true. ■

Solution to Exercise 6.1. Let H be the group defined by

$$H = \left\{ \left[\begin{array}{rrr} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{array} \right] \mid a, b, c \in \mathbb{Z} \right\}$$

and usual matrix multiplication. Define a map $\varphi: \mathbb{Z}^2 \times \mathbb{Z} \to H$ by

$$\varphi(a,b,c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

We compute that

$$\begin{aligned} \varphi(a,b,c) \cdot \varphi(a',b',c') &= \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a+a' & c+c'+ab' \\ 0 & 1 & b+b' \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus defining the Seifert form by $\beta((a, b), (a', b')) = ab'$, i.e. in matrix form, $\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, shows that φ induces a group isomorphism between $\mathscr{H}_{\beta}(\mathbb{Z}^2)$ and H. Only the ring structure of \mathbb{Z} is used, so the result remains valid if we replace the ground ring \mathbb{Z} by any commutative ring.

Solution to Exercise 3.1. The trivial map $ev_{\chi,0}$, sending A to 1 extends to the augmentation map $\varepsilon : \mathbb{Z}[\hat{A}^{\text{quad}}] \to \mathbb{Z}$ sending $\sum_{z \in \hat{A}^{\text{quad}}} c_z z$ to $\sum_{z \in \hat{A}^{\text{quad}}} c_z$. Since $\varepsilon (\sum_{x \in A} t_{x,y}) = |A| \neq 0$ (in \mathbb{Z}), $\sum_{x \in A} t_{x,y} \neq 0$.

Solution to Exercise 3.2. Since q is nondegenerate, for each $y \neq 0$, there exists $z \in A$ such that $b_q(z, y) \neq 0$. Form

$$a = a(q) = \prod_{y \neq 0} (1 - t_{z,y}).$$

Note that $ev(a) = \prod_{y \neq 0} (1 - \chi(b_q(z, y)) \neq 0$ in \mathbb{C} for any injective character. In particular, $a \neq 0$. Let us prove that a is a zero divisor. Since A is finite, $t_{z,y}$ is an element of finite order k. Note that $1 + t_{z,y} + \cdots + t_{z,y}^k$ is a nonzero element in $\mathbb{Z}[\hat{A}]$ (apply the augmentation map ε). Thus the identity

$$(1 - t_{z,y}) \cdot (1 + t_{z,y} + \dots + t_{z,y}^{k-1}) = 1 - t_{z,y}^k = 1 - 1 = 0$$

shows that $1 - t_{z,y}$ is a zero divisor.

Next, use the identity occurring in the proof of Proposition 1.1, namely

$$\mathfrak{g}_A \overline{\mathfrak{g}_A} = |A| + \sum_{y \neq 0} \left(\sum_{x \in A} t_{x,y} \right) t_y.$$

It follows from Lemma 1.4 (relation (1.3)) that

$$\mathfrak{g}_A \overline{\mathfrak{g}_A} a = |A| a + \sum_{y \neq 0} \underbrace{\left(\sum_{x \in A} t_{x,y}\right)}_{=0} a t_y = |A| a.$$

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