Diagrammatic Quantum Reasoning: Completeness and Incompleteness

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CNRS, Loria, Nancy, France

Workshop on Topology and Languages, Toulouse, June 2016

Diagrammatic Language for Reasoning in Quantum Computing

ZX-Calculus[1]

[1] B. Coecke, R. Duncan. Interacting quantum observables. ICALP'08.

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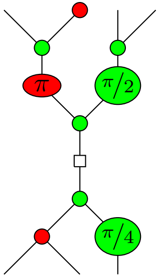

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Categorical Quantum Mechanics[2]

- Proving properties: protocols, algorithms, models of quantum computing.
  Proof assistant software: Quantomatic.

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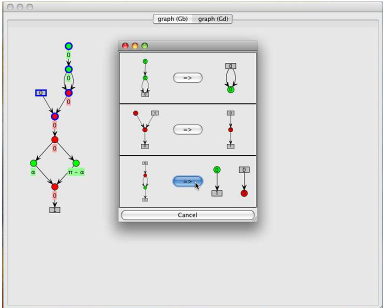

Diagrammatic Language for Reasoning in Quantum Computing

ZX-Calculus[1]

Categorical Quantum Mechanics[2]

- Proving properties: protocols, algorithms, models of quantum computing.
  Proof assistant software: Quantomatic.
- Foundations:
  entanglement, causality
  aximatisation of quantum mechanics.
- Pedagogical.

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Motivating Example:
Post-Selected Teleportation

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Post-Selected Teleportation

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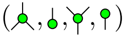
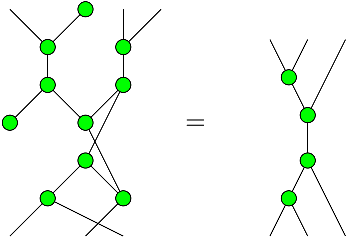
Frobenius Algebras

- (special commutative) Frobenius algebra 



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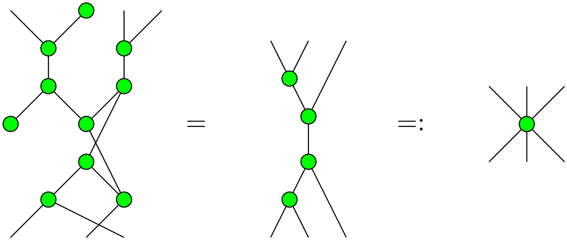

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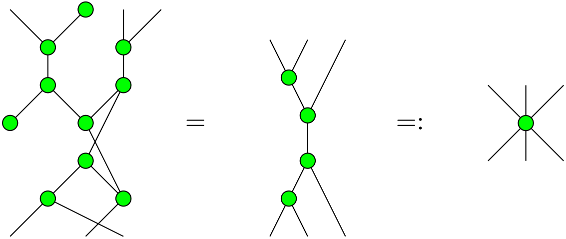
- (special commutative) Frobenius algebra (, , ,) , in bijection with orthonormal basis in **FdHilb** [Coecke,Pavlovic,Vicary'13[3]]



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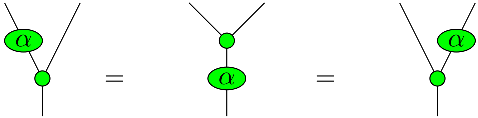
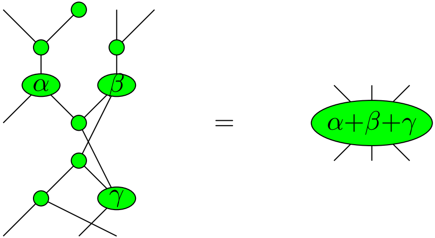
- Frobenius Algebra with Phases

Phase:





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Complementary basis

Frobenius algebra

Frobenius algebra

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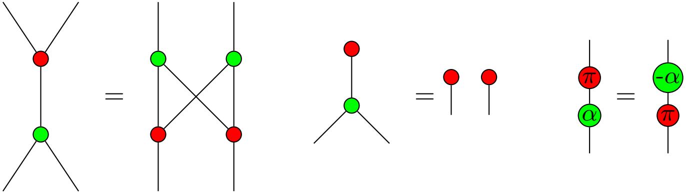

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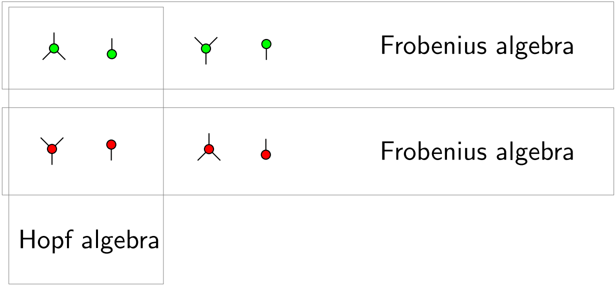

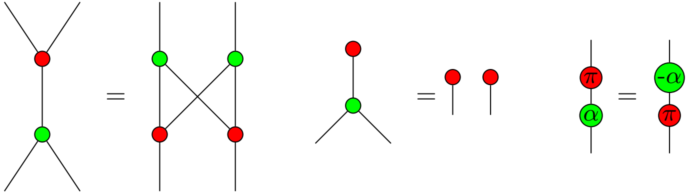

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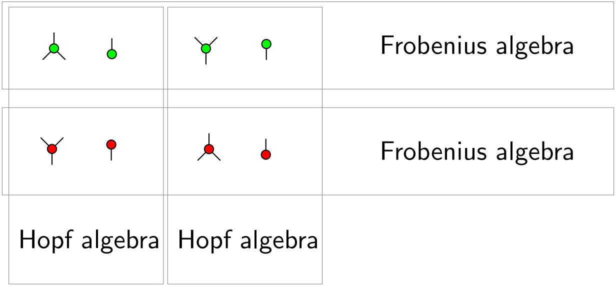

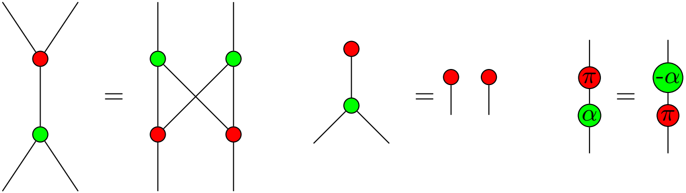

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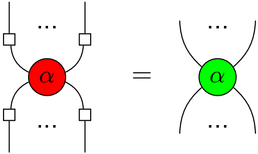
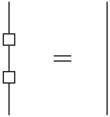
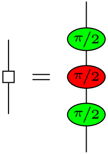
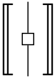




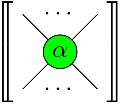


Universality, Soundness, and Completeness

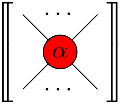
Universality



$$= \begin{cases} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} =: |+\rangle \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} =: |-\rangle \end{cases}$$

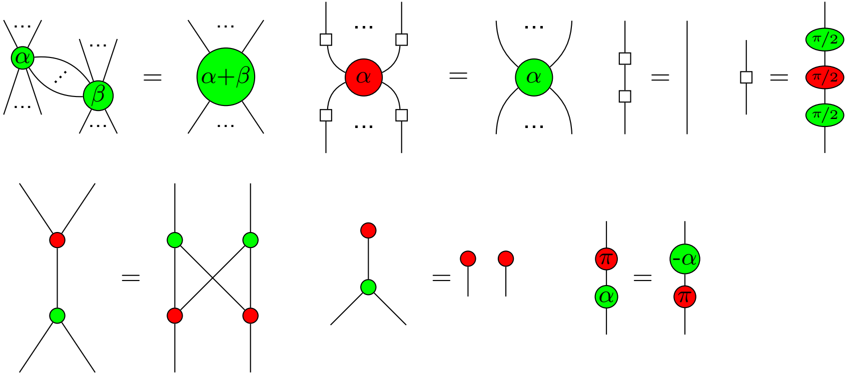


$$= \begin{cases} |0\dots0\rangle \mapsto |0\dots0\rangle \\ |1\dots1\rangle \mapsto e^{i\alpha}|1\dots1\rangle \end{cases}$$

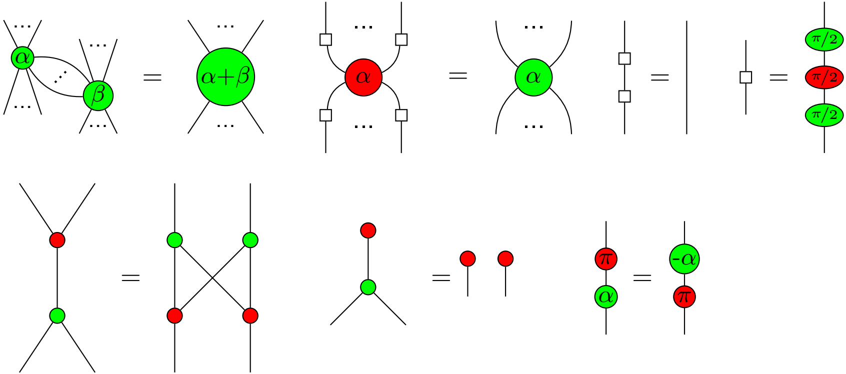


$$= \begin{cases} |+\dots+\rangle \mapsto |+\dots+\rangle \\ |-\dots-\rangle \mapsto e^{i\alpha}|-\dots-\rangle \end{cases}$$

- **Universality**: for any n-qubit linear map U, $\exists D$ s.t. $[\![D]\!] = U$.
- $\pi/4$-**fragment is approximately universal**: $\forall\epsilon > 0$ and any n-qubit linear map U, $\exists D$ with angles multiple of $\pi/4$ s.t. $|| [\![D]\!] - U|| < \epsilon$.
- $\pi/2$-**fragment is not (approximately) universal**.



- **Soundness**: $(ZX \vdash D_1 = D_2) \Rightarrow (\llbracket D_1 \rrbracket \simeq \llbracket D_2 \rrbracket)$
  where $\llbracket D_1 \rrbracket \simeq \llbracket D_2 \rrbracket$ if it exists a non zero $s \in \mathbb{C}$ s.t. $\llbracket D_1 \rrbracket = s \llbracket D_2 \rrbracket$



- **Soundness**: $(ZX \vdash D_1 = D_2) \Rightarrow ([\![D_1]\!] \simeq [\![D_2]\!])$
  where $[\![D_1]\!] \simeq [\![D_2]\!]$ if it exists a non zero $s \in \mathbb{C}$ s.t. $[\![D_1]\!] = s\,[\![D_2]\!]$
- **Completeness:** $([\![D_1]\!] \simeq [\![D_2]\!]) \overset{?}{\Longrightarrow} (ZX \vdash D_1 = D_2)$

"The most fundamental open problem related to the ZX-calculus is establishing its completeness properties for some of the calculus' variants" ▸ CQM wiki

Completeness of the $\pi/2$-fragment

Theorem [Backens'12[4]] Completeness of the $\pi/2$ fragment of the ZX-calculus.

$\forall D_1, D_2$ involving angles multiple of $\pi/2$ only,

$$[\![D_1]\!] \simeq [\![D_2]\!] \;\Leftrightarrow\; (ZX \vdash D_1 = D_2)$$

[4] M. Backens. The ZX-calculus is complete for stabilizer quantum mechanics. New J. Phys. 16 (2014) 093021

Incompleteness of ZX-calculus

Theorem [Schröder, Zamdzhiev'14[5]]. ZX-calculus is incomplete for Qubit Quantum Mechanics.

Proof.

[5] C. Schröder de Witt, V. Zamdzhiev. The ZX-calculus is incomplete for quantum mechanics. EPTCS 172, 2014

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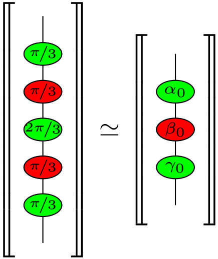

$$\alpha_0 = -\arccos\left(\frac{5}{2\sqrt{13}}\right), \beta_0 = -2\arcsin\left(\frac{\sqrt{3}}{4}\right), \gamma_0 = \arcsin\left(\frac{\sqrt{3}}{4}\right) - \alpha_0$$

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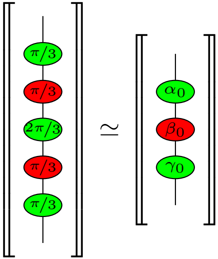

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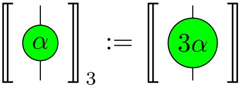

If $ZX \vdash D_1 = D_2$ then $[\![D_1]\!]_3 \simeq [\![D_2]\!]_3$.

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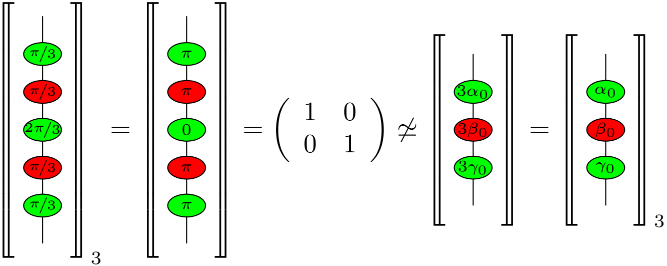

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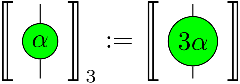

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(In)-completeness

- Completeness of the $\pi/2$-fragment [Backens'12]

- Incompleteness for Qubit QM [Schröder,Zamdzhiev'14]
  No obvious way to extend the ZX-calculus

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- Completeness of the 1-qubit $\pi/4$-fragment (path diagrams) [Backens'14[6]]

[6] M. Backens. The ZX-calculus is complete for the single-qubit Clifford+T group. EPTCS 172, 2014.

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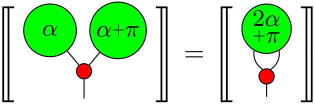
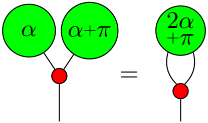
- Completeness of the 1-qubit $\pi/4$-fragment (path diagrams) [Backens'14[6]]
- Incompleteness of the $\pi/4$-fragment [Perdrix, Wang'16[7]]

[6] M. Backens. The ZX-calculus is complete for the single-qubit Clifford+T group. EPTCS 172, 2014.
[7] S. Perdrix, Q. Wang. Supplementarity is necessary for quantum diagram reasoning. MFCS'16

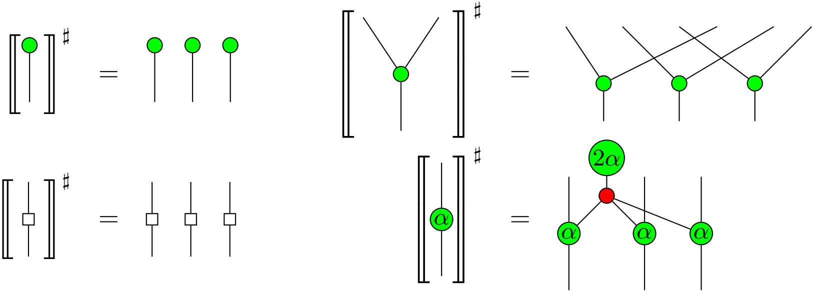
Supplementarity, a candidate for incompleteness



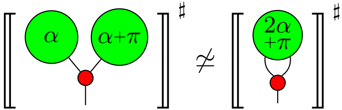
- Inspired by [Coecke,Edwards'10]: supplementarity.
- Can be proven in ZX when $\alpha = \pm\frac{\pi}{2}$.

Theorem: $\Big(ZX \vdash$  $\Big) \quad \Leftrightarrow \quad \alpha = 0 \bmod \frac{\pi}{2}$

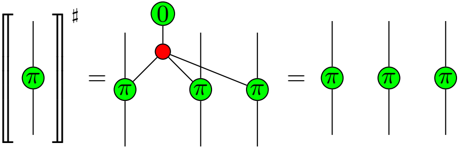
Alternative interpretation



Soundness: $(ZX \vdash D_1 = D_2) \Rightarrow [\![D_1]\!]^\sharp \simeq [\![D_2]\!]^\sharp$
Counterexample: $\forall \alpha \neq 0 \bmod \frac{\pi}{2}$,



Sound interpretation (1)



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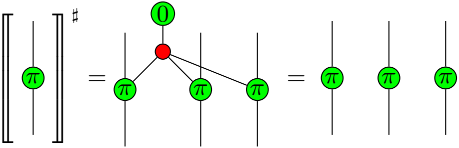

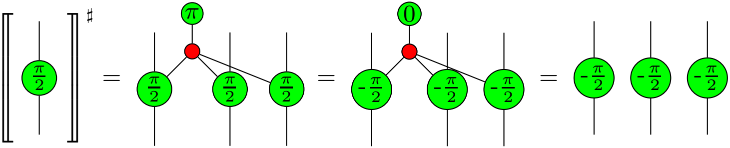

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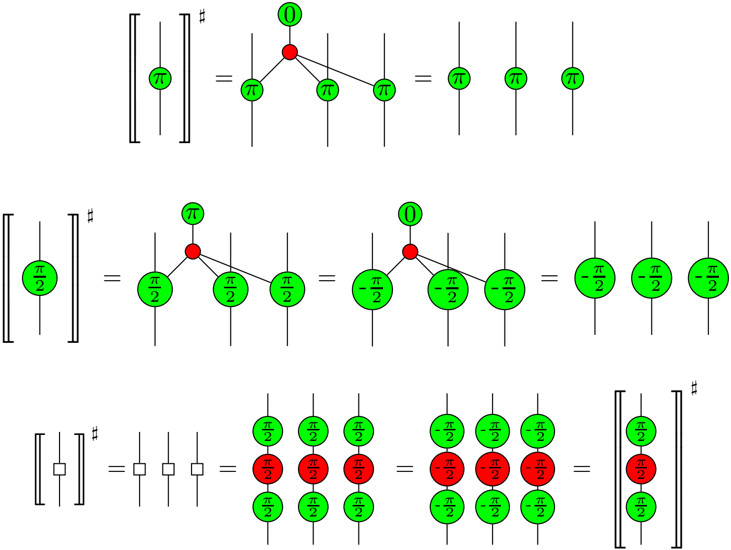
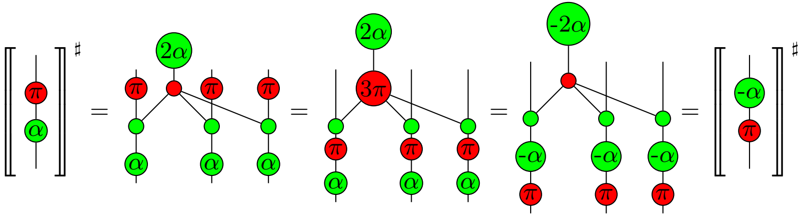
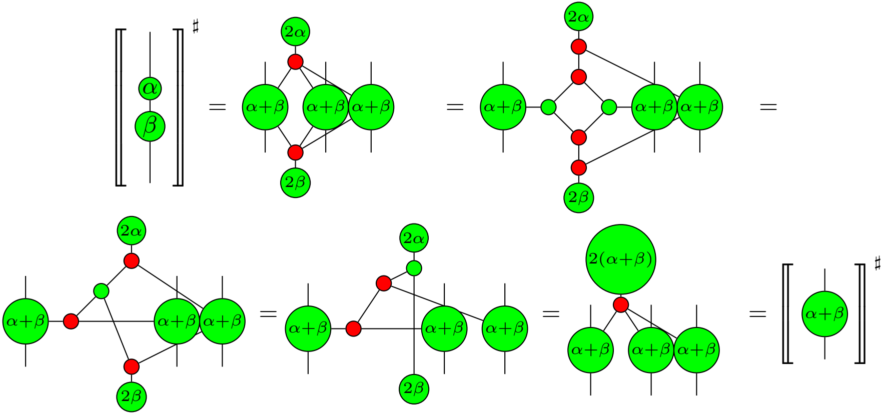


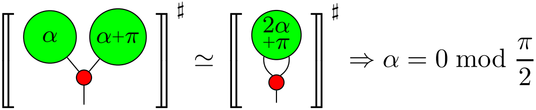
Sound interpretation (2)



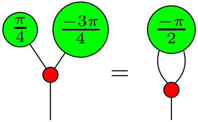
Sound interpretation (3)



Incompleteness

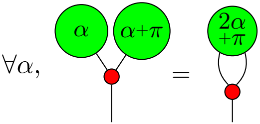


Corollary: $\frac{\pi}{4}$-fragment of ZX-calculus is not complete as the following equation cannot be derived:

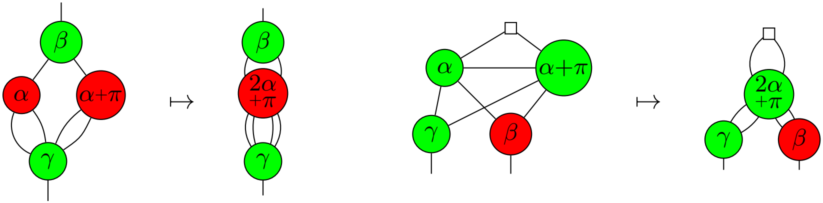


Graphical interpretation

Theorem. In ZX-calculus, antiphase twins can be merged if and only if



where two dots are antiphase twins if:
– they have the same colour;
– the difference between their angles is π;
– they have the same neighbourhood.

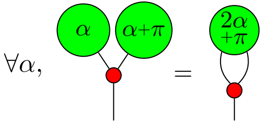


Conclusion

- $\frac{\pi}{4}$-fragment of ZX-calculus is completeness [Backens'12]
- Incompleteness in general [Schröder,Zamdzhiev'14]

  No obvious way to extend the ZX-calculus
- $\frac{\pi}{4}$-fragment is incompleteness [Perdrix,Wang'16]

Supplementarity as an axiom: ZX-calculus := ZX-calculus +'Supplementarity'



Open question.

Is $\frac{\pi}{4}$-fragment of ZX-calculus +'Supplementarity' complete?