QUADRATIC SIEGEL DISKS WITH SMOOTH BOUNDARIES.

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ABSTRACT. We show the existence of angles $\alpha \in \mathbb{R}/\mathbb{Z}$ such that the quadratic polynomial $P_\alpha(z) = e^{2i\pi\alpha}z + z^2$ has a Siegel disk with C^∞ boundary. This result was first announced by R. Pérez-Marco in 1993.

1. INTRODUCTION.

In this article, we are concerned with the dynamics of quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. It is known that there exists a unique formal power series ϕ_α of the form $\phi_\alpha(z) = z + b_2z^2 + b_3z^3 + \ldots$, such that

$$\phi_\alpha(e^{2i\pi\alpha}z) = P_\alpha(\phi_\alpha(z)).$$

We denote by $r_\alpha \geq 0$ the radius of convergence of the series ϕ_α. It is known (see [Y] for example) that $r_\alpha > 0$ for Lebesgue almost every $\alpha \in \mathbb{R}$, and more precisely, $r_\alpha > 0$ if and only if α satisfies the so-called Bruno condition (see definition 2 below).

From now on, we assume that $r_\alpha > 0$. In that case, the map $\phi_\alpha : \mathbb{D}(0, r_\alpha) \to \mathbb{C}$ is univalent, its image Δ_α is called the *Siegel disk* of P_α and r_α is called the *conformal radius* of the Siegel disk. The Siegel disk is the maximum domain on which P_α is conjugate to the rotation $R_\alpha : z \mapsto e^{2i\pi\alpha}z$. It is also the connected component of $\mathbb{C} \setminus J(P_\alpha)$ which contains 0, where $J(P_\alpha)$ is the Julia set of P_α, i.e., the closure of the set of repelling periodic points. Figure 1 shows the Julia sets of the quadratic polynomials P_α, for $\alpha = \sqrt{2}$ and $\alpha = \sqrt{10}$. Both polynomials have a Siegel disk colored grey.

In this article, we investigate the structure of the boundary of the Siegel disk. It is known since Fatou that this boundary is contained in the closure of the forward orbit of the critical point $\omega_\alpha = -e^{2i\pi\alpha}/2$ (for example, see [Mi] theorem 11.17). By plotting a large number of points in the forward orbit of ω_α, we should therefore get a good idea of what those boundaries look like.

In 1983, Herman [He1] proved that when α satisfies the so-called Herman condition (this is the optimal arithmetic condition to ensure that all analytic circle diffeomorphisms with rotation number α are analytically linearizable in a neighborhood of the circle), then the critical point actually belongs to the boundary of the Siegel disk. We will not define this condition in this article. Using a construction due to Ghys, Herman [He2] also proved the existence of quadratic polynomials P_α for which the boundary of the Siegel disk is a quasi-circle which does not contain the critical point. Afterwards, following an idea by Douady [D] and using work by Świątek [S] (see also [P2]), he proved that when α is diophantine of exponent 2, the boundary of the Siegel disk is a quasi-circle containing the critical point. In [McM], McMullen shows that the corresponding Julia sets have Hausdorff dimension less than 1 and that when α is a quadratic irrational, the boundary of the Siegel disk

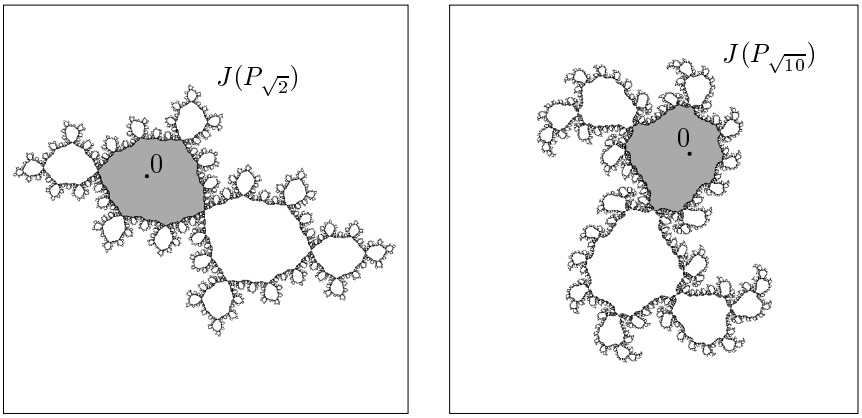

FIGURE 1. Left: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{2}}z + z^2$. Right: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{10}}z + z^2$. In both cases, there is a Siegel disk.

is self-similar about the critical point. More recently, Petersen and Zakeri [PZ] proved that for Lebesgue almost every $\alpha \in \mathbb{R}/\mathbb{Z}$, the boundary is a Jordan curve containing the critical point. Moreover, when α is not diophantine of exponent 2, this Jordan curve is not a quasi-circle (see [PZ]).

In [PM], Pérez-Marco proves that there exist maps univalent in $\mathbb{D}$ having Siegel disks compactly contained in $\mathbb{D}$ whose boundaries are C^∞ Jordan curves. This result is very surprising and very few people suspected that such a result could be true. As Pérez-Marco notices in [PM], the boundary cannot be an analytic Jordan curve since the linearizing map would extend across the boundary by Schwarz reflection. Pérez-Marco even produces examples where an uncountable number of intrinsic rotations extend univalently in a neighborhood of the closure of the Siegel disk. Pérez-Marco's results have several nice corollaries (see [PM]). For example, it follows that there exist analytic circle diffeomorphisms which are C^∞ linearizable but not analytically linearizable. This answers a question posed by Katok in 1970.

In 1993, in a seminar at Orsay, Pérez-Marco announced the existence of quadratic polynomials having Siegel disks with smooth boundaries. According to Pérez-Marco, his proof is rather technical. In this article, we will present a different approach to the existence of such quadratic polynomials.

Definition 1. *We say that the boundary of a Siegel disk Δ_α is accumulated by small cycles if any neighborhood of $\overline{\Delta_\alpha}$ contains a (whole) periodic cycle of P_α.*

Main Theorem. *Assume $\alpha \in \mathbb{R}$ is a Bruno number, $\ell \geq 0$ is an integer, $r \in]0, r_\alpha[$, $\eta > 0$ and $\varepsilon > 0$ are real numbers. Let $u_\alpha : \mathbb{R} \to \mathbb{C}$ be the $\mathbb{Z}$-periodic function $t \mapsto \phi_\alpha(re^{2i\pi t})$. Then there exists a Bruno number α' such that the following properties hold:*

- $|\alpha' - \alpha| < \eta$,
- $r_{\alpha'} = r$,
- *the linearizing map $\phi_{\alpha'} : \mathbb{D}(0,r) \to \Delta_{\alpha'}$ extends continuously to a function $\phi_{\alpha'} : \overline{\mathbb{D}(0,r)} \to \overline{\Delta_{\alpha'}}$ mapping the boundary of $\mathbb{D}(0,r)$ to the boundary of $\Delta_{\alpha'}$,*

- *the $\mathbb{Z}$-periodic function $u_{\alpha'} : \mathbb{R} \to \mathbb{C}$ defined by $u_{\alpha'}(t) = \phi_{\alpha'}(re^{2i\pi t})$ is C^∞, injective and its derivative does not vanish (thus the boundary of the Siegel disk is a smooth Jordan curve) and*
- *the functions u_α and $u_{\alpha'}$ are close in the following sense:*

$$\sup_{j=0..\ell} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u_{\alpha'}}{\partial t^j}(t) - \frac{\partial^j u_\alpha}{\partial t^j}(t) \right| < \varepsilon.$$

Finally, we may choose α' so that the boundary of the Siegel disk $\Delta_{\alpha'}$ is accumulated by small cycles.

Remark. When the polynomial P_α is not linearizable, i.e., $r_\alpha = 0$, it is known that 0 is accumulated by small cycles. It may be the case that the boundary of the Siegel disk of a quadratic polynomial is always accumulated by small cycles.

Corollary 1. *There exist quadratic polynomials with Siegel disks whose boundaries do not contain the critical point.*

Proof. Let α be a Bruno number, choose $r \in]0, r_\alpha[$ sufficiently small so that $\phi_\alpha(\partial\mathbb{D}(0,r)) \subset \mathbb{D}(0,1/10)$ and then choose $\varepsilon < 1/10$. Then, the boundary of the Siegel disk $\Delta_{\alpha'}$ given by theorem A is contained in $\mathbb{D}(0,1/5)$. Therefore, the critical point $\omega_{\alpha'} = -e^{2i\pi\alpha'}/2$ cannot belong to the boundary of the Siegel disk $\Delta_{\alpha'}$. ■

Remark. Our proof of existence of quadratic Siegel disks whose boundaries do not contain critical points is totally independent from Herman's proof.

Corollary 2. *The set $\mathcal{S} \subset \mathbb{R}$ of real numbers α for which P_α has a Siegel disk with smooth boundary is dense in $\mathbb{R}$ and has uncountable intersection with any open subset of $\mathbb{R}$.*

Proof. Given any Bruno number α and any $\eta > 0$, the conformal radius $r_{\alpha'}$ is an arbitrary real number in the interval $]0, r_\alpha[$ and so, the intersection of $\mathcal{S}$ with the interval $]\alpha - \eta, \alpha + \eta[$ is uncountable. The proof is completed since the set of Bruno numbers is dense in $\mathbb{R}$. ■

The idea of the proof of the main theorem is the following. We start with a Bruno number α such that P_α has a Siegel disk Δ_α with conformal radius r_α and we let $\phi_\alpha : \mathbb{D}(0, r_\alpha) \to \Delta_\alpha$ be the linearizing map. We denote by p_k/q_k the approximants to α given by the continued fraction of α. Finally we choose real numbers r and r_1 such that $0 < r < r_1 < r_\alpha$.

We then exhibit a sequence $\alpha[n]$ converging to α as $n \to +\infty$, such that

(a) the sequence of linearizing maps $\phi_{\alpha[n]}$ converges uniformly on every compact susbet of $\mathbb{D}(0, r_1)$ to ϕ_α;
(b) $P_{\alpha[n]}$ has a periodic cycle of period q_n arbitrarily close to $\phi_\alpha(\partial\mathbb{D}(0, r_1))$ for the Hausdorff topology on compact subsets of $\mathbb{C}$.

The proof of (a) consits in estimating the size of the Siegel disk of a map which is close to a rotation as done in [C] part 2, and is given in sections 3 and 4. The proof of (b) is based on the work performed in [C] part 1, and is given in section 5.

We may therefore find an angle $\alpha(1) = \alpha[n]$ with n very large, so that $P_{\alpha(1)}$ has a periodic cycle close to $\phi_\alpha(\partial\mathbb{D}(0, r_1))$ and so that the restriction of $\phi_{\alpha(1)}$ to the circle $\partial\mathbb{D}(0, r)$ is $\varepsilon/2$ close to the restriction of ϕ_α to $\partial\mathbb{D}(0, r)$ for some C^{ℓ_0} norm.

We then choose a radius $r_2 \in]r, r_{\alpha(1)}[$ and we perturb $\alpha(1)$ in order to get $\alpha(2)$ so that $P_{\alpha(2)}$ has a periodic cycle very close to $\phi_{\alpha(1)}(\partial\mathbb{D}(0,r_2))$ and so that the restriction of $\phi_{\alpha(2)}$ to the circle $\partial\mathbb{D}(0,r)$ is $\varepsilon/4$ close to the restriction of $\phi_{\alpha(1)}$ to $\partial\mathbb{D}(0,r)$ for some C^{ℓ_0+1} norm. Iterating the construction, we can produce a sequence $\alpha(j)$ converging to a limit α', so that each $P_{\alpha(j+1)}$ has a periodic cycle very close to $\phi_{\alpha(j)}(\partial\mathbb{D}(0,r_{j+1}))$ and so that the restriction of $\phi_{\alpha(j+1)}$ to the circle $\partial\mathbb{D}(0,r)$ is $\varepsilon/2^{j+1}$ close to the restriction of $\phi_{\alpha(j)}$ to $\partial\mathbb{D}(0,r)$ for some C^{ℓ_0+j} norm.

It follows that the sequence $(\phi_{\alpha(j)}|_{\partial\mathbb{D}(0,r)})_{j\geq 0}$ is a Cauchy sequence for all norms C^ℓ, $\ell \geq 0$. The limit map $\phi_{\alpha'} : \partial\mathbb{D}(0,r) \to \mathbb{C}$ is C^∞ and ε close to the map $\phi_\alpha : \partial\mathbb{D}(0,r) \to \mathbb{C}$ for the norm C^{ℓ_0}. Each curve $\phi_{\alpha(j)}(\partial\mathbb{D}(0,r))$ is invariant by $P_{\alpha(j)}$, and so, the curve $\phi_{\alpha'}(\partial\mathbb{D}(0,r))$ is invariant by $P_{\alpha'}$. We will see that we may choose the sequence $\alpha(j)$ so that this curve is accumulated by small cycles of $P_{\alpha'}$. Thus, it is the boundary of the Siegel disk of $P_{\alpha'}$. This will complete the proof of our main theorem.

The purpose of Figure 2 is to illustrate this construction. We have drawn the boundary of three quadratic Siegel disks, for

$$\alpha = (\sqrt{5}+1)/2 = [1,1,1,1,\ldots], \quad \alpha(1) = [1,1,1,1,1,1,1,25,1,1,1,\ldots]$$

and

$$\alpha(2) = [1,1,1,1,1,1,1,25,10^5,1,1,1,\ldots].$$

For $\alpha(1)$, there is a periodic cycle of period 8 that forces the boundary of the Siegel disk to oscillate slightly. For $\alpha(2)$, there is an additional cycle (of period 205) that forces the boundary to oscillate much more. We have not been able to produce a picture for a possible choice of $\alpha(3)$. The number of iterates of the critical point required to get a relevant picture was much too large (more than 100 billions).

We would like to make an observation. In the same way one produces C^∞ functions which are nowhere analytic with lacunary Fourier series, our Siegel disks are produced with rotation numbers whose continued fractions have large coefficients (in a certain sense) which are more and more spaced out. This is not completely surprising. Indeed, if $\phi : \mathbb{D} \to \Delta$ is a linearizing map, then the coefficients b_k of the power series of ϕ are the Fourier coefficients of the angular parametrization of the boundary of Δ. Those coefficient are also related to the arithmetic nature of α. Indeed, they are defined by the recursive formula

$$b_1 = r_\alpha \quad \text{and} \quad b_{n+1} = \frac{1}{e^{2i\pi\alpha}(e^{2i\pi n\alpha}-1)} \sum_{i=1}^{n} b_i b_{n+1-i}.$$

If the entry $k+1$ in the continued fraction of α is large, $e^{2i\pi q_k \alpha} - 1$ is close to 0 and b_{q_k+1} is large.

2. Arithmetical preliminaries.

If $(a_k)_{k\geq 0}$ are integers, we use the notation $[a_0, a_1, \ldots, a_k, \ldots]$ for the continued fraction

$$[a_0, a_1, \ldots, a_k, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \ddots}}}.$$

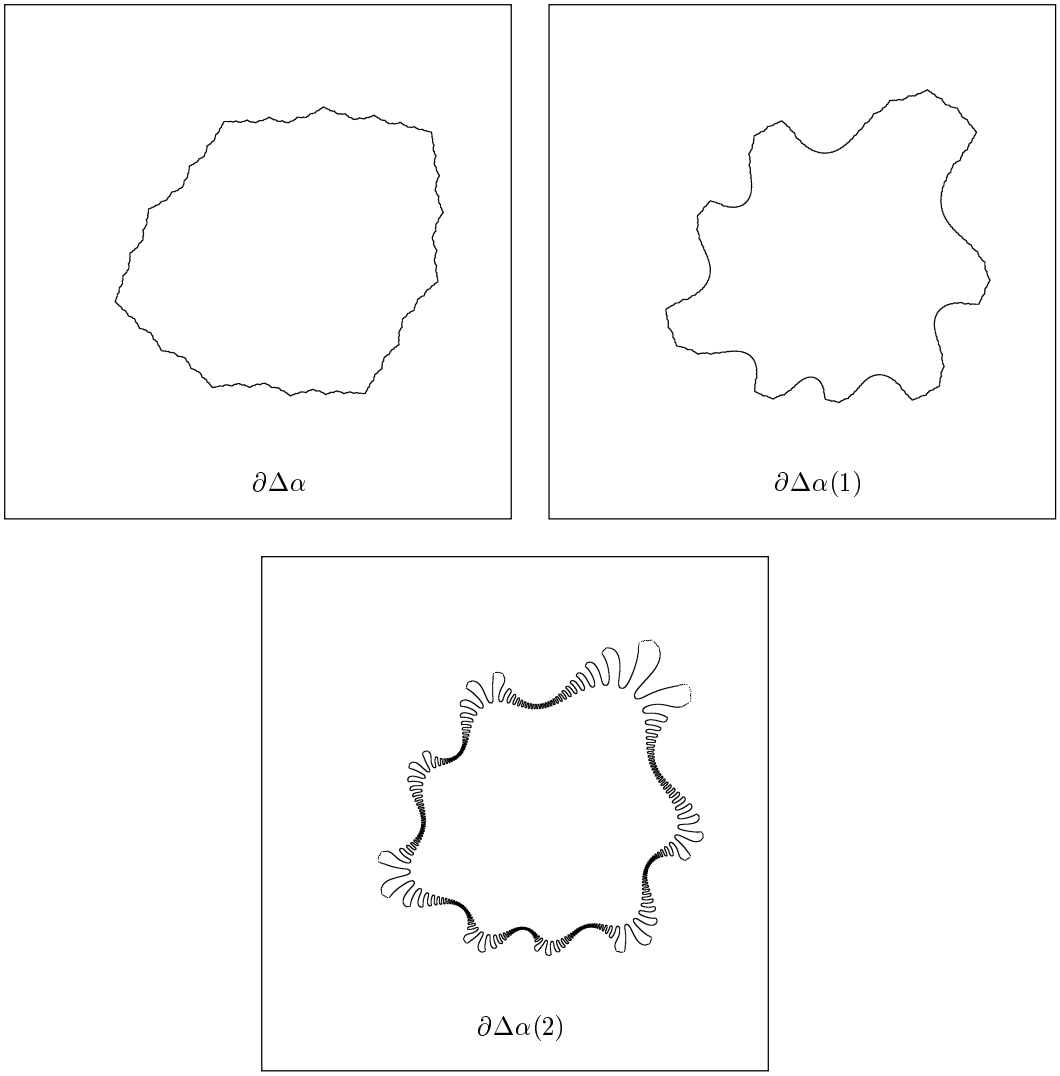

FIGURE 2. The first steps in the construction of a Siegel disk with smooth boundary.

We will say that a_0 is entry 0 of the continued fraction, a_1 is entry 1 of the continued fraction, and a_k is the entry k of the continued fraction. We define two sequences $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ recursively by

$$\begin{array}{llll} p_{-1}=1, & p_0=a_0, & p_k=a_k p_{k-1}+p_{k-2}, \\ q_{-1}=0, & q_0=1, & q_k=a_k q_{k-1}+q_{k-2}. \end{array}$$

The numbers p_k, q_k satisfy

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

In particular, p_k and q_k are coprime. Moreover, if $a_1, a_2, \ldots$ are positive integers, then for all $k \geq 0$, we have

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k].$$

For any irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we denote by $\lfloor \alpha \rfloor \in \mathbb{Z}$ the integer part of α, i.e., the largest integer $n \leq \alpha$, by $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ the fractional part of α, and we define two sequences $(a_k)_{k\geq 0}$ and $(\alpha_k)_{k\geq 0}$ recursively by setting

$$a_0 = \lfloor \alpha \rfloor, \quad \alpha_0 = \{\alpha\}, \quad a_{k+1} = \left\lfloor \frac{1}{\alpha_k} \right\rfloor \quad \text{and} \quad \alpha_{k+1} = \left\{ \frac{1}{\alpha_k} \right\},$$

so that

$$\frac{1}{\alpha_k} = a_{k+1} + \alpha_{k+1}.$$

We then set $\beta_{-1} = 1$ and $\beta_k = \alpha_0 \alpha_1 \ldots \alpha_k$.

It is well known that

$$\alpha = [a_0, a_1, \ldots, a_k, \ldots].$$

More precisely, we have the following formulas.

Proposition 1. *Let α be an irrational number and define the sequences $(a_k)_{k\geq 0}$, $(\alpha_k)_{k\geq 0}$, $(\beta_k)_{k\geq -1}$, $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ as above, so that*

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k].$$

Then, we have the formulas

$$\alpha = \frac{p_k + p_{k-1}\alpha_k}{q_k + q_{k-1}\alpha_k}, \quad q_k \alpha - p_k = (-1)^k \beta_k,$$

$$q_{k+1}\beta_k + q_k \beta_{k+1} = 1 \quad \text{and} \quad \frac{1}{q_{k+1} + q_k} < \beta_k < \frac{1}{q_{k+1}}.$$

3. The Yoccoz function.

Definition 2. (The Bruno function). *If α is an irrational number, we set*

$$B(\alpha) = \sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k}.$$

If α is a rational number we set $B(\alpha) = +\infty$. The Bruno numbers are those α for which $B(\alpha) < +\infty$.

Remark. The set of Bruno numbers has full measure in $\mathbb{R}$. It contains the set of all diophantine numbers, i.e., numbers for which $\log q_{k+1} = \mathcal{O}(\log q_k)$.

We can give an equivalent definition of Bruno numbers via the Yoccoz function.

Definition 3. (The Yoccoz function). *If α is an irrational number, we set*

$$\Phi(\alpha) = \sum_{k=0}^{+\infty} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

If α is a rational number we set $\Phi(\alpha) = +\infty$. The Bruno numbers are those α for which $\Phi(\alpha) < +\infty$.

Remark. Observe that for any $k_0 \geq 1$, we have

$$\Phi(\alpha) = \beta_{k_0-1}\Phi(\alpha_{k_0}) + \sum_{k=0}^{k_0-1} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

In [Y], Yoccoz uses a modified version of continued fractions but we won't need those modifications. The function Φ we will use is not exactly the same as the one introduced by Yoccoz, but the difference between the two functions is bounded (see [Y], page 14).

The reason why the two definitions of Bruno numbers are equivalent is contained in the following proposition. The proof is essentially given in [Y], page 14.

Proposition 2. *There exists a universal constant C such that for all $k_0 \geq 0$, we have*

$$\left| \sum_{k=0}^{k_0} \beta_{k-1} \log \frac{1}{\alpha_k} - \sum_{k=0}^{k_0} \frac{\log q_{k+1}}{q_k} \right| < C.$$

The Yoccoz function is highly discontinuous since it takes the value $+\infty$ on a dense set containing $\mathbb{Q}$ and takes finite values on a set of full measure. Let us now see what we can control.

Proposition 3. *The Yoccoz function is lower semi-continuous.*

Proof. Let us first assume that α is an integer. On the one hand, as $\alpha' \to \alpha$ with $\alpha' > \alpha$, we have $\alpha'_0 \to 0$. Thus,

$$\Phi(\alpha') > \log \frac{1}{\alpha'_0} \to +\infty.$$

On the other hand, as $\alpha' \to \alpha$ with $\alpha' > \alpha$, we have $\alpha'_0 \to 1$ with $\alpha'_0 < 1$, and $\alpha'_1 = \{1/\alpha'_0\} \to 0$. Thus,

$$\Phi(\alpha') > \log \frac{1}{\alpha'_0} + \alpha'_0 \log \frac{1}{\alpha'_1} \to +\infty.$$

Now, if α belongs to $\mathbb{Q} \setminus \mathbb{Z}$, we may define $\alpha_0 = \{\alpha\}$, $\alpha_1 = \{1/\alpha_0\}$ and so on, until we get to $\alpha_{k_0+1} = \{1/\alpha_{k_0}\} = 0$. Then, we cannot define the sequence α_k further. When $\alpha' \to \alpha$, we may also define $\alpha'_0 = \{\alpha'\}$, $\alpha'_1 = \{1/\alpha'_0\}$ up to at least $\alpha'_{k_0+1} = \{1/\alpha'_{k_0}\}$. Then, $\alpha'_0 \to \alpha_0$, $\alpha'_1 \to \alpha_1$, and so on, until we get to $\alpha'_{k_0} \to \alpha_{k_0}$. Depending on whether $\alpha'_{k_0} < \alpha_{k_0}$ or $\alpha'_{k_0} > \alpha_{k_0}$ we have $\alpha'_{k_0+1} \to 1$ or $\alpha'_{k_0+1} \to 0$. In both cases, $\Phi(\alpha'_{k_0+1}) \to +\infty$ and

$$\Phi(\alpha') > \alpha'_0 \cdots \alpha'_{k_0} \Phi(\alpha'_{k_0+1}) \to +\infty.$$

Now, assume α is irrational. We must show that

$$\liminf_{\alpha' \to \alpha} \Phi(\alpha') \geq \Phi(\alpha).$$

Without loss of generality, we may assume that α' is irrational, so that the sequence α'_k is well defined for all $k \geq 0$. Since $\alpha' \to \alpha$, it follows that $\alpha'_k \to \alpha_k$ and $\beta'_k \to \beta_k$. As a consequence, for all $k \geq 0$, we have

$$\beta'_{k-1} \log \frac{1}{\alpha'_k} \underset{\alpha' \to \alpha}{\longrightarrow} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

Since those terms are positive, we get (Fatou's lemma)

$$\Phi(\alpha) = \sum_{k=0}^{+\infty} \liminf_{\alpha' \to \alpha} \left(\beta'_{k-1} \log \frac{1}{\alpha'_k} \right) \leq \liminf_{\alpha' \to \alpha} \left(\sum_{k=0}^{+\infty} \beta'_{k-1} \log \frac{1}{\alpha'_k} \right) = \liminf_{\alpha' \to \alpha} \Phi(\alpha').$$

■

We will now show that the closure of the graph of Φ contains all the points (α, t) with $t \geq \Phi(\alpha)$. For this purpose, we will have to approximate α by sequences of irrational numbers. In order to avoid the confusion between such a sequence and the sequence $(\alpha_k)_{k \geq 0}$ introduced previously, we will denote the new sequence by $(\alpha[n])_{n \geq 0}$.

Proposition 4. *Let $\alpha = [a_0, a_1, \ldots] \in \mathbb{R}$ be a Bruno number. Let $p_k/q_k = [a_0, a_1, \ldots, a_k]$ be the approximants to α. Let $(A_k)_{k \geq 0}$ be a sequence of positive integers such that $(A_k)^{1/q_k}$ has a limit $A > 1$ as $k \to +\infty$. Finally, for each integer $n \geq 0$, set*

$$\alpha[n] = [a_0, a_1, \ldots, a_n, A_n, 1, 1, \ldots]$$

Then,

$$\Phi(\alpha[n]) \underset{n \to +\infty}{\longrightarrow} \Phi(\alpha) + \log A.$$

Proof. For each integer $n \geq 0$, let us denote by $(\alpha_k[n])_{k \geq 0}$ and $(\beta_k[n])_{k \geq -1}$ the sequences associated to $\alpha[n]$. For each fixed k, we have

$$\lim_{n \to +\infty} \alpha_k[n] = \alpha_k \quad \text{and} \quad \lim_{n \to +\infty} \beta_k[n] = \beta_k.$$

In particular,

$$\lim_{n \to +\infty} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} = \beta_{k-1} \log \frac{1}{\alpha_k}.$$

Observe that for $k \leq n$, the approximants of $\alpha[n]$ and α are the same: p_k/q_k. Hence, if $k \leq n-1$, we have

$$\beta_{k-1}[n] < \frac{1}{q_k} \quad \text{and} \quad \frac{1}{\alpha_k[n]} < \frac{1}{\beta_k[n]} < 2q_{k+1}.$$

It follows that when $k \leq n-1$, we have

$$\beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} < \frac{\log 2}{q_k} + \frac{\log q_{k+1}}{q_k}.$$

The right terms form a convergent series since α is a Bruno number. Thus, we have

$$\sum_{k=0}^{n-1} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} \underset{n \to +\infty}{\longrightarrow} \sum_{k=0}^{+\infty} \beta_{k-1} \log \frac{1}{\alpha_k} = \Phi(\alpha). \tag{1}$$

For example, we may apply Lebesgue dominated convergence theorem, to the sequence of functions $f_n : (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) \to \mathbb{R}$, where μ is the counting measure and $f_n(k) = \beta_{k-1}[n] \log(1/\alpha_k[n])$.

We will now estimate the term $\beta_{n-1}[n] \log(1/\alpha_n[n])$ in the Yoccoz function. First, observe that

$$\frac{1}{\alpha_n[n]} = A_n + \frac{1}{\theta},$$

where $\theta = (\sqrt{5}+1)/2$ is the golden mean. As $n \to +\infty$, we have $\log A_n \sim q_n \log A$ and thus,

$$\beta_{n-1}[n] \log \frac{1}{\alpha_n[n]} \underset{n\to+\infty}{\sim} \beta_{n-1}[n] q_n \log A.$$

We know that $\beta_{n-1} q_n \in]1/2, 1[$ and we would like to prove that in our case, this sequence tends to 1. Observe that

$$\beta_{n-1}[n] q_n = (1 - \beta_n[n] q_{n-1}).$$

Moreover,

$$\begin{aligned}
\alpha[n] &= \frac{p_n \left(A_n + \frac{1}{\theta}\right) + p_{n-1}}{q_n \left(A_n + \frac{1}{\theta}\right) + q_{n-1}} \\
&= \frac{p_n}{q_n}\left(1 + \frac{p_{n-1}}{p_n}\frac{1}{A_n} - \frac{q_{n-1}}{q_n}\frac{1}{A_n} + \mathcal{O}\left(\frac{1}{A_n^2}\right)\right) \\
&= \frac{p_n}{q_n} + \frac{(-1)^n}{q_n^2 A_n} + \mathcal{O}\left(\frac{1}{A_n^2}\right).
\end{aligned}$$

It follows that

$$\beta_n[n] = (-1)^n (q_n \alpha[n] - p_n) = \frac{1}{q_n A_n} + \mathcal{O}\left(\frac{q_n}{A_n^2}\right).$$

Since $A_n^{1/q_n} \to A > 1$, we see that for n large enough, $A_n^{1/q_n} > A^{1/2}$, and thus, $A_n > A^{q_n/2}$. So, $q_n^2 = o(A_n)$ and thus, ←

$$\beta_n[n] \sim \frac{1}{q_n A_n},$$

and since $q_{n-1} < q_n \to +\infty$ as $n \to +\infty$, we get $\beta_n[n] q_{n-1} \to 0$ as $n \to +\infty$. As a consequence,

$$\beta_{n-1}[n] \log \frac{1}{\alpha_n[n]} \underset{n\to+\infty}{\longrightarrow} \log A. \tag{2}$$

Finally, we have $\alpha_{n+1}[n] = 1/\theta$ and thus

$$\sum_{k=n+1}^{+\infty} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} = \beta_n[n] \cdot \Phi\left(\frac{1}{\theta}\right) \underset{n\to+\infty}{\longrightarrow} 0. \tag{3}$$

Combining inequalities (1), (2) and (3) gives the required result. ■

Remark. Instead of continuing the fraction of α_n with only 1's, we could have taken $\alpha[n] = [a_0, \ldots, a_n, A_n, \theta] = [a_0, \ldots, a_n, A_n, t_0, t_1, \ldots]$ where $\theta = [t_0, t_1, \ldots]$ is any Bruno number greater than 1. We could even have taken any sequence $\alpha[n] = [a_0, \ldots, a_n, A_n, \theta(n)]$, where $\theta(n) > 1$ are Bruno numbers such that $\Phi(1/\theta(n)) = o(q_n A_n)$.

4. Semi-continuity for Siegel disks.

In this section we will study perturbations of rotations on the unit disk. We will generalize the following two theorems due to Yoccoz [Y] for the first and independently to Risler [R] and Chéritat [C] for the second.

For any real number α, we denote by $\mathcal{D}_\alpha$ the set of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ which satisfy $f(0) = 0$ and $f'(0) = e^{2i\pi\alpha}$ and we denote by $\mathcal{S}_\alpha$ the set of functions $f \in \mathcal{D}_\alpha$ which are univalent. Given a map $f \in \mathcal{D}_\alpha$, we can define K_f to be the set of points $z \in \mathbb{D}$ whose orbit under iteration of f is well defined and does not escape from $\mathbb{D}$. It is not difficult to show that the connected component of $\overset{\circ}{K}_f$ which contains 0 is a Siegel disk for f (if it is not empty).

Theorem 1. (Yoccoz) *There exists a universal constant C_0 such that for any Bruno number α, any function $f \in \mathcal{S}_\alpha$ has a Siegel disk containing the disk centered at 0 with radius $\exp(-\Phi(\alpha) - C_0)$.*

Remark. As mentioned earlier, the function Φ defined by Yoccoz in [Y] is not exactly the same as the one we defined in this article, but the difference of the two functions is bounded by a universal constant, so that the result is true as stated here.

Theorem 2. (Risler-Chéritat) *Let α be a Bruno number and let $f_n \in \mathcal{D}_\alpha$ be a sequence of functions which converge uniformly to the rotation $R_\alpha : z \mapsto e^{2i\pi\alpha}$ on every compact subset of $\mathbb{D}$. Then, for any $\eta > 0$, if n is sufficiently large, f_n has a Siegel disk containing the disk centered at 0 with radius $1 - \eta$.*

Remark. The maps f_n are not required to be univalent, but since $f_n \to R_\alpha$ uniformly on every compact subset of $\mathbb{D}$, the maps f_n are automatically univalent on subdisks whose radii tend to 1 as n tends to $+\infty$.

Our goal is to generalize those results as follows.

Theorem 3. *Assume $(\alpha[n])_{n\geq 0}$ is a sequence of Bruno numbers converging to a Bruno number α as $n \to +\infty$. Moreover, assume*

$$\limsup_{n\to+\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C$$

for some constant $C \geq 0$. Let $f_n \in \mathcal{D}_{\alpha[n]}$ be a sequence of functions which converge uniformly to the rotation $R_\alpha : z \mapsto e^{2i\pi\alpha}$ on every compact subset of $\mathbb{D}$. Then, for any $\eta > 0$, if n is sufficiently large, f_n has a Siegel disk containing the disk centered at 0 with radius $e^{-C-\eta}$.

Proof. The basic construction in Yoccoz's proof of theorem 1 is a "renormalization operator" $\mathcal{R}$ which maps $\mathcal{S}_\alpha$ to $\mathcal{S}_{1/\alpha}$. The operator $\mathcal{R}$ was first considered by Douady and Ghys in the course of proving that the set of numbers α such that every element of $\mathcal{S}_\alpha$ is linearizable is invariant under $\mathrm{PSL}_2\mathbb{Z}$. The construction goes as follows.

Consider a map $f \in \mathcal{S}_\alpha$ with $0 < \alpha < 1$. There exists a unique lift $F : \mathbb{H} \to \mathbb{C}$ of f such that $e^{2i\pi F(Z)} = f(e^{2i\pi Z})$ and $F(Z) = Z + \alpha + u(Z)$, where u is $\mathbb{Z}$-periodic and $|u(Z)| \to 0$ as $\mathrm{Im}(Z) \to \infty$. Then, for each $\varepsilon > 0$, there exists a smallest height $c_{f,\varepsilon}$ such that $|u(Z)| < \varepsilon\alpha$ and $|u'(Z)| < \varepsilon$ whenever $\mathrm{Im}(Z) > c_{f,\varepsilon}$.

Let us call L_0 the half line $L_0 = \{it \mid t > c_{f,\varepsilon}\}$ and $L_0' = F(L_0)$. If ε is sufficiently small, then L_0' is very close to the vertical line $\alpha + L_0$. More precisely, by choosing ε sufficiently small, we may assume that for all $Z \in L_0'$, $\mathrm{Re}(Z) < 1$ (this is possible

because $\alpha < 1$), that the union $L_0 \cup [ic_{f,\varepsilon}, F(ic_{f,\varepsilon})] \cup L_0' \cup \{\infty\}$ forms a Jordan curve in the Riemann sphere $\mathbb{P}^1$ bounding a region $\mathcal{U}$, that for every $Y > c_{f,\varepsilon}$, the segment $[iY, F(iY)]$ is contained in $\overline{\mathcal{U}}$, and that $F^{-1}(\mathcal{U})$ is a left neighborhood of L_0 (see [C] page 70 for details). If we glue the sides L_0 and L_0' of $\overline{\mathcal{U}}$ via F, we obtain a topological surface $\mathcal{V}$. We denote by $\iota : \overline{\mathcal{U}} \to \mathcal{V}$ the canonical projection. The space $\mathcal{V}$ is a topological surface with boundary, whose boundary $\iota([ic_{f,\varepsilon}, F(ic_{f,\varepsilon})])$ is denoted $\partial\mathcal{V}$. We set $\overset{\circ}{\mathcal{V}} = \mathcal{V} \setminus \partial\mathcal{V}$. Since the gluing map F is analytic, the surface $\overset{\circ}{\mathcal{V}}$ has a canonical analytic structure induced by the one of $\mathcal{U}$. More precisely, the complex structure on $\overset{\circ}{\mathcal{V}}$ is obtained from the one on $\mathcal{U} \cup L_0 \cup F^{-1}(\mathcal{U})$ by identifying Z and $F(Z)$ whenever $Z \in F^{-1}(\mathcal{U})$.

Let us now see that the Riemann surface $\overset{\circ}{\mathcal{V}}$ is isomorphic to $\mathbb{D}^*$. Denote by B_0 the half-strip $B_0 = \{Z \in \mathbb{H} \mid 0 < \mathrm{Re}(Z) < 1\}$ and consider the map $H : \overline{B_0} \to \overline{\mathcal{U}}$ defined by

$$H(X + iY) = (1 - X)iW + XF(iW), \quad \text{with} \quad W = \alpha Y + c_{f,\varepsilon}.$$

Then,

$$\frac{\partial H}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial H}{\partial X} - i\frac{\partial H}{\partial Y}\right) = \frac{1}{2}\big(F(iW) - iW - \alpha - \alpha X(F'(iW) - 1)\big)$$

and

$$\frac{\partial H}{\partial Z} = \frac{1}{2}\left(\frac{\partial H}{\partial X} + i\frac{\partial H}{\partial Y}\right) = \frac{1}{2}\big(2\alpha + F(iW) - iW - \alpha + \alpha X(F'(iW) - 1)\big).$$

It follows from the choice of $c_{f,\varepsilon}$ that

$$\left|\frac{\partial H}{\partial \overline{Z}}\right| < \varepsilon\alpha \quad \text{and} \quad \left|\frac{\partial H}{\partial Z}\right| > (1 - \varepsilon)\alpha.$$

In particular, when $\varepsilon < 1/2$, H is a K_ε-quasiconformal homeomorphism between $\overline{B_0}$ and $\overline{\mathcal{U}}$, with $K_\varepsilon = 1/(1 - 2\varepsilon)$. When $Z \in i\mathbb{R}^+$, $H(Z + 1) = F(H(Z))$, and so, the homeomorphism $H : \overline{B}_0 \to \overline{\mathcal{U}}$ induces a homeomorphism between the half cylinder $\mathbb{H}/\mathbb{Z}$ and the Riemann surface $\overset{\circ}{\mathcal{V}}$. This homeomorphism is clearly quasiconformal on the image of B_0 in $\mathbb{H}/\mathbb{Z}$, i.e., outside an $\mathbb{R}$-analytic curve. It is therefore quasiconformal in the whole half cylinder ($\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms). Therefore, there exists an isomorphism between $\overset{\circ}{\mathcal{V}}$ and $\mathbb{D}^* = \mathbb{H}/\mathbb{Z}$, which, by a theorem of Carathéodory, extends to a homeomorphism between $\partial\mathcal{V}$ and $\partial\mathbb{D} = \mathbb{R}/\mathbb{Z}$. Let $\phi : \overline{\mathcal{U}} \to \overline{\mathbb{D}}^*$ be the composition of $\iota : \overline{\mathcal{U}} \to \mathcal{V}$ and such an isomorphism and let $K : \overline{\mathcal{U}} \to \overline{\mathbb{H}}$ be a lift of ϕ. The map K is unique up to post-composition with a real translation. We choose K such that $K(ic_{f,\varepsilon}) = 0$. By construction, if $Z \in L_0$, then $K(F(Z)) = K(Z) + 1$.

Now, set

$$\mathcal{U}' = \{Z \in \mathcal{U} \mid \mathrm{Im}(Z) > 3\varepsilon + c_{f,\varepsilon}\}.$$

If $Z \in \mathcal{U}'$, then $|F(Z) - Z - \alpha| < \varepsilon\alpha$, and the iterates $F^{\circ m}(Z)$ stay above the line starting at Z and going down with a slope $\tan(\arcsin(\varepsilon))$ ($< 2\varepsilon$ when $\varepsilon < 1/2$), as long as $\mathrm{Im}(F^{\circ m}(Z)) > c_{f,\varepsilon}$. It follows that there is a least integer $m_Z \geq 1$ for which $\mathrm{Re}(F^{\circ m_Z}(Z)) \geq 1$, and then, $\mathrm{Im}(F^{\circ m_Z}(Z)) \geq i\varepsilon + ic_{f,\varepsilon}$. As a consequence, $F^{\circ m_Z}(Z) - 1 \in \mathcal{U}$. We can therefore consider the mapping $G : \mathcal{U}' \to \mathcal{U}$ given by

$$G(Z) = F^{\circ m_Z}(Z) - 1.$$

Figure 3 shows the construction of the map G.

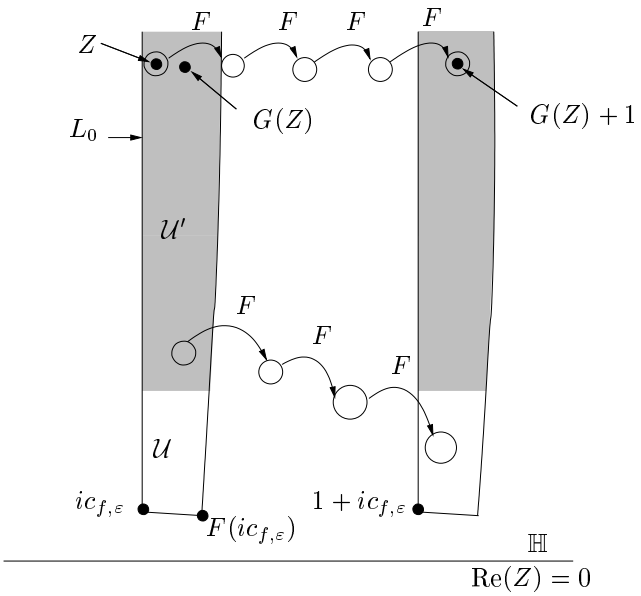

FIGURE 3. The regions $\mathcal{U}$, $\mathcal{U}'$ and the map $G : \mathcal{U}' \to \mathcal{U}$.

The mapping G induces a univalent map $g : \phi(\mathcal{U}') \to \mathbb{D}^*$ such that $g \circ \phi = \phi \circ G$. The fact that g is univalent, and in particular holomorphic, is not completely obvious (see [Y] for details). The renormalized map is $\mathcal{R}(f) = \overline{g(r_{f,\varepsilon}\overline{z})/r_{f,\varepsilon}}$, where $r_{f,\varepsilon}$ is the radius of the largest disk centered at 0 and contained in $\phi(\mathcal{U}')$. By the removable singularity theorem, this map extends holomorphically to the origin by $\mathcal{R}(f)(0) = 0$, and it is possible to show that $[\mathcal{R}(f)]'(0) = e^{2i\pi/\alpha}$ (again, see [Y] for details). Observe that this construction depends on the choice of ε, which has to be sufficiently small. This completes the description of the renormalization operator.

We now come back to the proof of the theorem. Let us now assume that $f_n \in \mathcal{D}_{\alpha[n]}$ is a sequence of functions which converge uniformly to the rotation R_α on every compact subset of $\mathbb{D}$, and assume by contradiction that the theorem is false. Then, extracting a subsequence if necessary, we may assume that there is a radius $r_0 < e^{-C}$, and a sequence of points $z_n \in \mathbb{D}(0, r_0)$ such that the orbit of z_n under iteration of f_n escapes from $\mathbb{D}$. Without loss of generality, conjugating with scaling maps whose ratio tends to 1 as n tends to $+\infty$, we may assume that $f_n \in \mathcal{S}_{\alpha[n]}$, and conjugating with a rotation if necessary, we may assume that $z_n \in]0, r_0[$. For each fixed ε, the constants $c_{f_n,\varepsilon}$ converge to 0 as $n \to +\infty$ (because $f_n \to R_\alpha$). Thus, one can choose a sequence $\varepsilon_n \to 0$ so that $c_{f_n,\varepsilon_n} \to 0$ and do the renormalization described above. The rotation number $\alpha[n]$ belongs to $\mathbb{R}/\mathbb{Z}$ and we have a choice of representative. We choose $\alpha_0[n] = \{\alpha[n]\}$. Then, the lifts F_n converge to the translation $Z \mapsto Z + \alpha_0$ with $\alpha_0 = \{\alpha\}$. It follows that the K_{ε_n}-quasiconformal homeomorphisms $H_n : \overline{B_0} \to \overline{\mathcal{U}_n}$ converge to the map $Z \mapsto \alpha_0 Z$ uniformly on $\overline{B_0}$. Moreover, $K \circ H : \overline{B_0} \to K(\overline{\mathcal{U}_n})$ is a K_{ε_n}-quasiconformal homeomorphism which satisfies $K \circ H(Z + 1) = K \circ H(Z) + 1$ for $Z \in i\mathbb{R}^+$ and sends 0 to 0. Therefore,

it extends by periodicity to a K_{ε_n}-quasiconformal automorphism of $\overline{\mathbb{H}}$ fixing 0, 1 and ∞ (we know that the extension is quasiconformal outside $\mathbb{Z}+i\mathbb{R}^+$, and thus, it is quasiconformal on $\mathbb{H}$ since $\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms). Since $K_{\varepsilon_n} = 1/(1-2\varepsilon_n) \to 1$ as $n \to +\infty$, we see that $K \circ H$ converges uniformly on $\overline{\mathbb{H}}$ to the identity as $n \to \infty$. It follows that the maps $\phi_n : \overline{\mathcal{U}} \to \overline{\mathbb{D}}^*$ converge to $Z \mapsto e^{2i\pi Z/\alpha_0}$ uniformly on every compact subset of $B_{\alpha_0} = \{Z \in \mathbb{H} \mid 0 \leq \mathrm{Re}(Z) < \alpha_0\}$. In particular, we have $r_{f_n,\varepsilon_n} \to 1$ as $n \to +\infty$.

Let us consider the points $Z_n \in i\mathbb{R}^+$ such that $e^{2i\pi Z_n} = z_n$. Then, $\mathrm{Im}(Z_n) > -\log(r)/2\pi$ and for n large enough, $Z_n \in U_n$ and $z_n' = \phi_n(Z_n)$ is close to $e^{2i\pi Z_n/\alpha_0}$. In particular, given any radius $r_1 > r_0$, we see that for n large enough, $|z_n'| < r_1^{1/\alpha_0}$. Moreover, since the orbit of z_n escapes from $\mathbb{D}$ under iteration of f_n, the orbit of Z_n under iteration of F_n escapes from $\mathbb{H}$ and thus the orbit of z_n' under iteration of $\mathcal{R}(f_n)$ escapes from $\mathbb{D}$.

Now, we can renormalize one more time. The map $\mathcal{R}(f_n)$ belongs to $\mathcal{S}_{1/\alpha_0[n]} = \mathcal{S}_{\alpha_1[n]}$. We may therefore choose lifts of $\mathcal{R}(f_n)$ which converge to the translation $Z \mapsto Z + \alpha_1$ as n tends to $+\infty$. It follows that given any radius $r_2 > r_1$ and for n large enough, we find points z_n'' with $|z_n''| < r_2^{1/(\alpha_0\alpha_1)}$, whose orbit under iteration of $\mathcal{R}^{\circ 2}(f_n)$ escapes from $\mathbb{D}$. Renormalizing similarly k times, we see that for any radius $r > r_0$ and for n sufficiently large, there are points w_n with $|w_n| < r^{1/\beta_{k-1}}$, whose orbit under iteration of $\mathcal{R}^{\circ k}(f_n)$ escapes from $\mathbb{D}$.

We may now apply Yoccoz's result (theorem 1). It asserts that the Siegel disk of $r > r_0$ and for n sufficiently large, $\mathcal{R}^{\circ k}(f_n) \in \mathcal{S}_{\alpha_k[n]}$ contains a disk centered at 0 with radius $\exp(-\Phi(\alpha_k[n]) - C_0)$. As a consequence, we see that for any

$$\frac{1}{\beta_{k-1}} \log r > -\Phi(\alpha_k[n]) - C_0.$$

Observe that

$$\beta_{k-1}[n]\Phi(\alpha_k[n]) = \Phi(\alpha[n]) - \sum_{j=0}^{k-1} \beta_{j-1}[n] \log \frac{1}{\alpha_j[n]},$$

and thus,

$$\limsup_{n\to+\infty} \Big(\beta_{k-1}[n]\Phi(\alpha_k[n])\Big) \leq \Phi(\alpha) + C - \sum_{j=0}^{k-1} \beta_{j-1} \log \frac{1}{\alpha_j}.$$

We are now done: take $\eta > 0$ sufficiently small so that $r = e^{-C-\eta} > r_0$; take k sufficiently large so that

$$\beta_{k-1}C_0 < \frac{\eta}{3} \quad \text{and} \quad \Phi(\alpha) - \sum_{j=0}^{k-1} \beta_{j-1} \log \frac{1}{\alpha_j} < \frac{\eta}{3};$$

finally take n sufficiently large so that

$$\beta_{k-1}\Phi(\alpha_k[n]) < \Phi(\alpha) + C - \sum_{j=0}^{k-1} \beta_{j-1} \log \frac{1}{\alpha_j} + \frac{\eta}{3}$$

and so that there are points w_n with $|w_n| < r^{1/\beta_{k-1}}$, whose orbit under iteration of $\mathcal{R}^{\circ k}(f_n)$ escapes from $\mathbb{D}$. We then get

$$-C-\eta = \log r > -\beta_{k-1}\Phi(\alpha_k[n]) - \beta_{k-1}C_0 > -C-\eta,$$

which is a contradiction. ■

Let us now see how to apply this result to the family of quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$. Remember that P_α has a Siegel disk Δ_α if and only if α is a Bruno number. In that case, there is a unique real number $r_\alpha > 0$ such that there exists a conformal isomorphism $\phi_\alpha : \mathbb{D}(0, r_\alpha) \to \Delta_\alpha$ sending 0 to 0 with derivative $\phi'_\alpha(0) = 1$. It is well known that

$$\phi_\alpha(e^{2i\pi\alpha}z) = P_\alpha \circ \phi_\alpha(z).$$

The number r_α is called the *conformal radius* of Δ_α at 0 and the map $\phi_\alpha : \mathbb{D}(0, r_\alpha) \to \Delta_\alpha$ is called a *linearizing parametrization*.

Proposition 5. *Assume α is a Bruno number and $\alpha[n]$ is a sequence of Bruno numbers such that*

$$\limsup_{n\to+\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C,$$

for some constant C. Then,

$$\liminf_{n\to+\infty} r_{\alpha[n]} \geq r_\alpha e^{-C},$$

and the linearizing parametrizations $\phi_{\alpha[n]}$ converge uniformly to ϕ_α on every compact subset of $\mathbb{D}(0, r_\alpha e^{-C})$.

Proof. Since $\alpha[n] \to \alpha$, we have $P_{\alpha[n]} \to P_\alpha$ uniformly on every compact subset of $\mathbb{C}$. Let us consider the maps

$$f_n = \phi_\alpha^{-1} \circ P_{\alpha[n]} \circ \phi_\alpha.$$

The map f_n is not defined on the whole disk $\mathbb{D}(0, r_\alpha)$, but for any $r < r_\alpha$, f_n is defined on $\mathbb{D}(0, r)$ for n sufficiently large. Moreover, as $n \to +\infty$, the sequence f_n converges uniformly on $\mathbb{D}(0, r)$ to the rotation R_α. So, for any $\eta > 0$, the maps

$$g_n : z \mapsto \frac{1}{r_\alpha e^{-\eta}} f_n(r_\alpha e^{-\eta} z)$$

belong to $\mathcal{D}_{\alpha[n]}$ for n large enough, and as $n \to +\infty$, we have $g_n \to R_\alpha$ uniformly on $\mathbb{D}$. It therefore follows from theorem 3 that if n is sufficiently large, the Siegel disk of g_n contains the disk $\mathbb{D}(0, e^{-C-\eta})$. As a consequence, for any $\eta > 0$, if n is sufficiently large, the Siegel disk of f_n contains the disk $\mathbb{D}(0, r_\alpha e^{-C-2\eta})$.

Thus, for n sufficiently large, $\Delta_{\alpha[n]}$ contains $\phi_\alpha(\mathbb{D}(0, r_\alpha e^{-C-2\eta}))$. It follows that the map $\phi_{\alpha[n]}^{-1} \circ \phi_\alpha$ is defined on the whole disk $\mathbb{D}(0, r_\alpha e^{-C-2\eta})$ and takes its values in the disk $\mathbb{D}(0, r_{\alpha[n]})$. Since $[\phi_{\alpha[n]}^{-1} \circ \phi_\alpha]'(0) = 1$, it follows from Schwarz's lemma that for n large enough, $r_{\alpha[n]} \geq r_\alpha e^{-C-2\eta}$, and thus

$$\liminf_{n\to+\infty} r_{\alpha[n]} \geq r_\alpha e^{-C-2\eta}.$$

Letting η tend to 0 shows that $\liminf r_{\alpha[n]} \geq r_\alpha e^{-C}$.

Let us now deal with the linearizing parametrizations. The map $\phi_{\alpha[n]}$ is defined on the disk $\mathbb{D}(0, r_{\alpha[n]})$. Since $\liminf r_{\alpha[n]} \geq r_\alpha e^{-C}$, we see that for any $r < r_\alpha e^{-C}$, the map $\phi_{\alpha[n]}$ is defined on $\mathbb{D}(0, r)$ as soon as n is greater than some integer n_r. Since for $n \geq n_r$, the map $\phi_{\alpha[n]} : \mathbb{D}(0, r) \to \mathbb{C}$ takes its values in the filled-in Julia set $K(P_{\alpha[n]}) \subset \overline{\mathbb{D}(0, 2)}$ and so, the family $\{\phi_{\alpha[n]} : \mathbb{D}(0, r) \to \mathbb{C}\}_{n \geq n_r}$ is normal. We must show that any limit value $\phi : \mathbb{D}(0, r) \to \mathbb{C}$ coincides with $\phi_\alpha : \mathbb{D}(0, r) \to \mathbb{C}$. Since $\phi(0) = 0$ and $\phi'(0) = 1$, it is sufficient, by uniqueness of the linearizing

parametrization, to show that ϕ linearizes P_α. This is immediate by passing to the limit, as $n \to +\infty$, on the equation

$$\phi_{\alpha[n]} \circ R_{\alpha[n]} \circ \phi_{\alpha[n]}^{-1} = P_{\alpha[n]}.$$

■

Corollary 3. *Let α and θ be Bruno numbers, and let $\alpha[n]$ be a sequence of Bruno numbers defined as in proposition 4. Then,*

$$\liminf_{n\to+\infty} r_{\alpha[n]} \geq r_\alpha/A,$$

and the linearizing parametrizations $\phi_{\alpha[n]}$ converge uniformly to ϕ_α on every compact subset of $\mathbb{D}(0, r_\alpha/A)$.

Remark. In the next section, we will see that in fact, $r_{\alpha[n]} \to r_\alpha/A$ as $n \to +\infty$.

Figure 4 shows the boundary of the Siegel disks for $\alpha = (\sqrt{5}+1)/2 = [1, 1, 1, \ldots]$, and $\alpha[n]$, $n = 4..7$ with $A_n = \lfloor 1.5^{q_n} \rfloor$. The reader should try and convince himself that as n grows this boundary oscillates more and more between $\partial\Delta_\alpha$ and $\phi_\alpha\left(\frac{2}{3}\partial\mathbb{D}(0, r_\alpha)\right)$ which appear in the last frame.

5. Parabolic explosion.

In this section, every time we use the notation $\alpha = p/q$, we mean that α is rational and that p and q are coprime with $q > 0$.

Let us fix a rational number $\alpha_0 = p/q \in \mathbb{Q}$. Then, 0 is a parabolic fixed point of the quadratic polynomial $P_{\alpha_0} : z \mapsto e^{2i\pi\alpha_0} z + z^2$. It is known (see [DH], chapter IX) that there exists a complex number $A \in \mathbb{C}^*$ such that

$$P_{\alpha_0}^{\circ q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}).$$

Following [C], we introduce the notion of asymptotic size.

Definition 4. *If $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ such that $f(z) = z + Az^{r+1} + \mathcal{O}(z^{r+2})$, then the asymptotic size of f at 0 is the positive number*

$$L_a(f, 0) = \frac{1}{|rA|^{1/r}}.$$

Remark. This quantity measures how fast orbits attracted by 0 under iteration of f will converge to 0:

$$\forall z \text{ such that } f^{\circ n}(z) \underset{\neq}{\longrightarrow} 0, \quad |f^{\circ n}(z)| \underset{n\to+\infty}{\sim} \frac{L_a(f, 0)}{n^{1/r}}.$$

It follows immediately that if $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ are conjugate by a germ $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, i.e., $g = \phi \circ f \circ \phi^{-1}$ and $\phi'(0) \neq 0$, then

$$L_a(g, 0) = |\phi'(0)| L_a(f, 0).$$

Definition 5. *For each $\alpha_0 = p/q \in \mathbb{Q}$, we denote by $L_a(\alpha_0)$ the asymptotic size of $P_{\alpha_0}^{\circ q}$ at 0.*

When $\alpha \neq \alpha_0$ is a small perturbation of $\alpha_0 = p/q$, 0 becomes a simple fixed point of P_α and $P_\alpha^{\circ q}$ has q fixed points close to 0. The dependence of those fixed points on α is studied in [C].

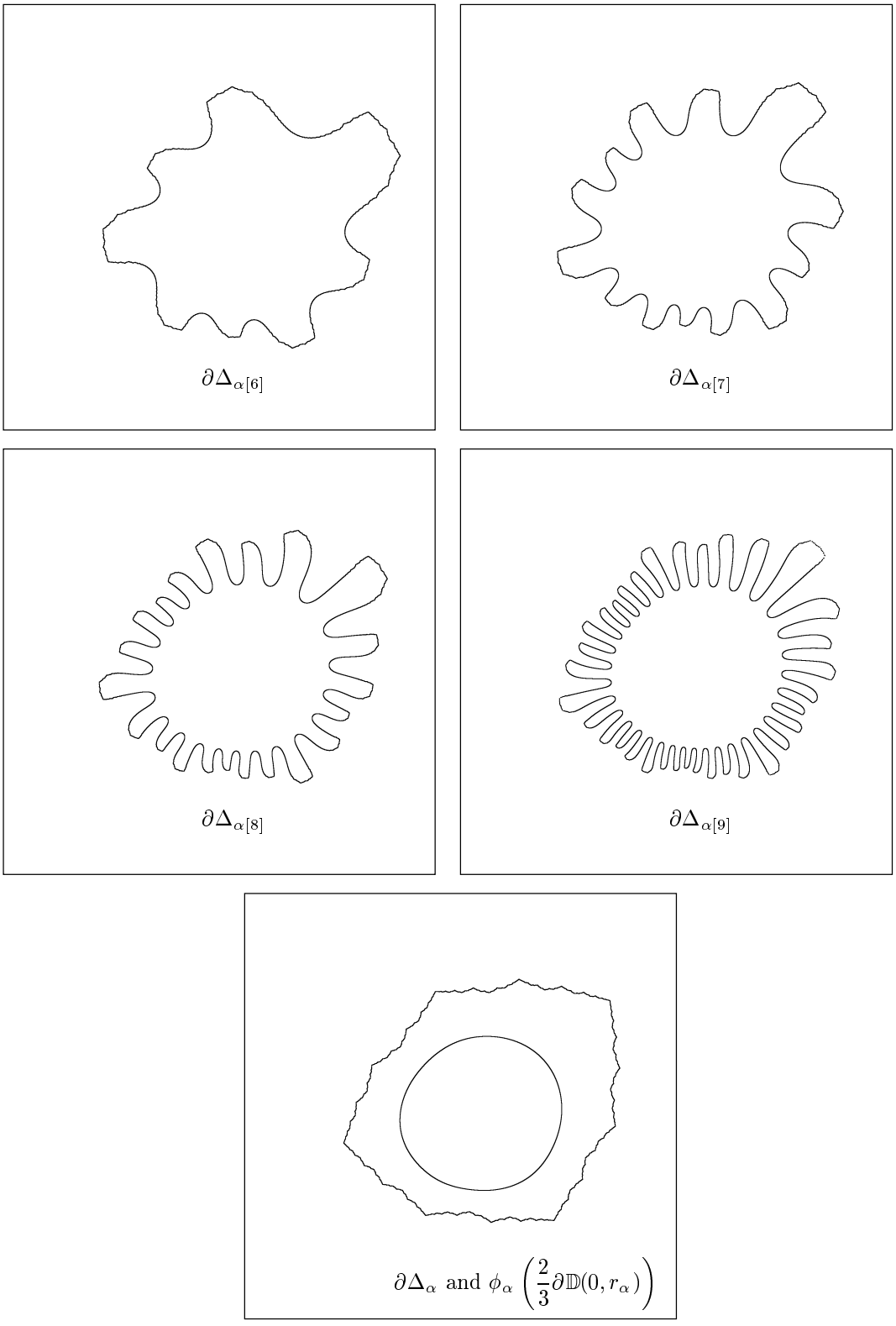

FIGURE 4. Some boundaries of Siegel disks for a sequence $\alpha[n]$.

Definition 6. *Let $\mathcal{P}_q$ be the set of parameters $\alpha \in \mathbb{C}$ such that $P_\alpha^{\circ q}$ has a parabolic fixed point with multiplier 1 and for each $\alpha_0 = p/q$, set*

$$R_{\alpha_0} = d(\alpha_0, \mathcal{P}_q \setminus \{\alpha_0\}).$$

The following proposition corresponds to proposition 2.2 in [C].

Proposition 6. *For each rational number $\alpha_0 = p/q$, there exists a holomorphic function $\chi : \mathbb{D} \to \mathbb{C}$ such that*

- $\chi(0) = 0$,
- $|\chi'(0)| = (2\pi q^2 R_{\alpha_0})^{1/q} L_a(\alpha_0)$ *and*
- *for any $\delta \in \mathbb{D}$,*

$$\left\langle \chi(\delta), \chi(e^{2i\pi\alpha_0}\delta), \chi(e^{2i\pi 2\alpha_0}\delta), \dots, \chi(e^{2i\pi(q-1)\alpha_0}\delta) \right\rangle$$

forms a cycle of period q of P_α with $\alpha = \alpha_0 + R_{\alpha_0}\delta^q$.

Remark. Note that we consider complex perturbations of α_0. Also, we use a different function χ than the one defined in [C] since we prefer to have a domain of definition that does not depend on p/q. The function χ we define here, and the function χ_C defined in [C] are related by $\chi_C(z) = \chi([R_{p/q}]^{1/q} z)$.

Proof. Let us consider the polynomial of two variables $F : \mathbb{C}^2 \to \mathbb{C}$ defined by

$$F(\alpha, z) = \begin{cases} \dfrac{P_\alpha^{\circ q}(z) - z}{z} & \text{if } z \neq 0 \\ e^{2i\pi\alpha} - 1 & \text{if } z = 0. \end{cases}$$

Then, define the set

$$X = \{(\alpha, z) \in \mathbb{D}(\alpha_0, R_{\alpha_0}) \times \mathbb{C} \mid F(\alpha, z) = 0\}.$$

Finally, let $\pi_1 : X \to \mathbb{D}(\alpha_0, R_{\alpha_0})$ and $\pi_2 : X \to \mathbb{C}$ be the projections

$$\pi_1(\alpha, z) = \alpha \quad \text{and} \quad \pi_2(\alpha, z) = z.$$

If $(\alpha, z) \in X$ and $\alpha \neq \alpha_0$, then $P_\alpha^{\circ q}(z) = z$ and $z \neq 0$. By assumption on R_{α_0}, $[P_\alpha^{\circ q}]'(z) \neq 1$. An elementary computation gives

$$\frac{\partial F}{\partial z}(\alpha, z) = \frac{[P_\alpha^{\circ q}]'(z) - 1}{z} \neq 0.$$

Thus, X is smooth at (α, z). If $(\alpha_0, z) \in X$ and $z \neq 0$, then z is a simple root of $P_{\alpha_0}^{\circ q}$. Indeed, P_{α_0} has only one parabolic cycle. As above, we conclude that X is smooth at (α_0, z). Finally,

$$F(\alpha_0 + \varepsilon, z) = 2i\pi q\varepsilon + Az^q + \mathcal{O}(\varepsilon z) + \mathcal{O}(z^{q+1}).$$

In particular, $F(\alpha_0, 0) = 0$ and $\partial F/\partial\alpha(\alpha_0, 0) = 2i\pi q$. Thus, $(\alpha_0, 0) \in X$ and X is smooth at $(\alpha_0, 0)$. As a consequence, we see that X is smooth, and so, X is a Riemann surface. We denote by Y the connected component of X that contains $(\alpha_0, 0)$.

Now, if $|z| > 1 + |e^{2i\pi\alpha}|$, then $|P_\alpha(z)| = |z| \cdot |e^{2i\pi\alpha} + z| > |z|$, and so, z is not a periodic point of P_α. Therefore, $\pi_1 : Y \to \mathbb{D}(\alpha_0, R_{\alpha_0})$ is a proper mapping, i.e., a ramified covering. When $(\alpha, z) \in Y$ and $(\alpha, z) \neq (\alpha_0, 0)$, z is a simple root of the equation $P_\alpha^{\circ q}(z) - z$ and by the implicit function theorem, π_1 is unramified at (α, z). Thus, the ramified covering $\pi_1 : Y \to \mathbb{D}(\alpha_0, R_{\alpha_0})$ is ramified only at $(\alpha_0, 0)$. So, Y is simply connected and this ramified covering is analytically isomorphic to $\delta \mapsto \alpha_0 + R_{\alpha_0}\delta^n$ from $\mathbb{D}$ to $\mathbb{D}(\alpha_0, R_{\alpha_0})$ for some integer n, as a ramified covering

above $\mathbb{D}(\alpha_0, R_{\alpha_0})$. In other words, there exists an isomorphism $\phi : \mathbb{D} \to Y$ such that $\phi(0) = (\alpha_0, 0)$ and $\pi_1 \circ \phi(\delta) = \alpha_0 + R_{\alpha_0}\delta^n$.

We may now define $\chi = \pi_2 \circ \phi : \mathbb{D} \to \mathbb{C}$. Then $\chi(0) = \pi_2(\alpha_0, 0) = 0$ and $\chi(\delta) = \chi'(0)\delta + \mathcal{O}(\delta)$. Therefore, we have

$$F \circ \phi(\delta) = 0 = 2i\pi q R_{\alpha_0}\delta^n + A\big(\chi'(0)\delta\big)^q + \mathcal{O}(\delta^{n+1}) + \mathcal{O}(\delta^{q+1}).$$

It follows that $n = q$ and $|\chi'(0)| = (2\pi q R_{\alpha_0})^{1/q} L_a(\alpha_0)$ (note that when $q = 1$, the covering $\pi_1 : Y \to \mathbb{D}(\alpha_0, R_{\alpha_0})$ is in fact unramified).

To see that

$$\Big\langle \chi(\delta), \chi(e^{2i\pi\alpha_0}\delta), \chi(e^{2i\pi 2\alpha_0}\delta), \dots, \chi(e^{2i\pi(q-1)\alpha_0}\delta) \Big\rangle$$

forms a cycle of period q of P_α with $\alpha = \alpha_0 + R_{\alpha_0}\delta^q$, we must show that

$$\chi(e^{2i\pi\alpha_0}\delta) = P_{\alpha_0 + R_{\alpha_0}\delta^q}(\chi(\delta)).$$

The right side is a holomorphic function of δ. It coincides with a non-zero fixed point of $P^{\circ q}_{\alpha_0 + R_{\alpha_0}\delta^q}$ which tends to 0 as δ tends to 0. Thus, it must be one of the point $\chi(e^{2i\pi k\alpha_0}\delta)$ for some $k = 0, \dots, q-1$. Computing the derivatives with respect to δ at $\delta = 0$ gives $k = 1$. ∎

Remark. The isomorphism $\phi : \mathbb{D} \to Y$ such that $\phi(0) = (\alpha_0, 0)$ and $\pi_1 \circ \phi(\delta) = \alpha_0 + R_{\alpha_0}\delta^n$ is unique up to precomposition with an n-th root of unity. Thus, the function χ is unique up to precomposition with a rotation of a q-th root of unity.

The function χ obviously depends on $\alpha_0 = p/q$ and from now on, we will denote it by $\chi_{p/q}$. Our goal is to understand how the maps $\chi_{p/q}$ behave when p/q tends to some irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In order to be able to say something, we must first understand how the numbers $R_{p/q}$ behave as $q \to \infty$. This is the object of the following proposition.

Proposition 7. *There exists a constant C such that for every rational number $\alpha_0 = p/q$, we have*

$$R_{\alpha_0} \geq \frac{C}{q^3}.$$

Proof. The proof relies on Douady's landing theorem and the Yoccoz inequality (see [Hu] or [P1]).

Let us choose a rational number $\alpha_0 = p/q$, and assume that $\alpha \in \mathcal{P}_q \setminus \{\alpha_0\}$. In other words, assume $\alpha \neq \alpha_0$ and $P_\alpha^{\circ q}$ has a multiple fixed point z_0. Then, P_α has a parabolic cycle $\langle z_0, z_1, \dots, z_{q_1-1}\rangle$ of period q_1 dividing q, and the immediate basin of this parabolic cycle contains the critical point $\omega_0 = -e^{2i\pi\alpha}/2$ of P_α. As a consequence, the Julia set $J(P_\alpha)$ is connected and all other periodic cycles of P_α are repelling.

It follows that either 0 is parabolic, i.e., $\alpha = p'/q'$ with q' dividing q. Then, the distance between α and α_0 is trivially bounded from below by $1/q$.

Or 0 is repelling, i.e., α belongs to the lower half-plane $\{\mathrm{Im}(\alpha) < 0\}$. Since the Julia set is connected, Douady's landing theorem asserts that there are finitely many rays landing at 0, let's say q'. Those q' rays can be ordered cyclically by their arguments. They are permuted by P_α and each ray is mapped to the one which is p' further counter-clockwise for some $p' < q'$. Then, the Yoccoz inequality implies that α belongs to the closed disk of radius $(\log 2)/(2\pi q')$ tangent to the real axis at p'/q' (see for example [Hu]). A key result that we will not prove here is

that we necessarily have $q > q'$ (see for example [C], subsection 6.2 pages 54-55). Pythagorus theorem then gives

$$|\alpha - \alpha_0| \geq \frac{1}{2\pi}\sqrt{\left(2\pi\left(\frac{p'}{q'} - \frac{p}{q}\right)\right)^2 + \left(\frac{\log 2}{q'}\right)^2} - \frac{\log 2}{q'}.$$

Since $q' < q$, an elementary computation gives

$$|\alpha - \alpha_0| \geq \frac{2\pi}{q^3 \log 2 \left(\sqrt{\left(\frac{2\pi}{q \log 2}\right)^2 + 1} + 1\right)}.$$

As q tends to ∞, this last term is equivalent to $\pi/(q^3 \log 2)$, which proves the proposition. ∎

We may now study the asymptotic behaviour of the functions $\chi_{p/q} : \mathbb{D} \to \mathbb{C}$ as p/q tends to some real number $\alpha \in \mathbb{R}$.

Proposition 8. *The family $\{\chi_{p/q} : \mathbb{D} \to \mathbb{C}\}_{p/q \in \mathbb{Q}}$ is normal. If $p/q \to \alpha \in \mathbb{R}$ then*

- *either P_α has no Siegel disk and $\chi_{p/q} \to 0$ uniformly on every compact subset of $\mathbb{D}$,*
- *or P_α has a Siegel disk Δ and any limit value of $\chi_{p/q}$ is a map $\chi : \mathbb{D} \to \Delta$ which takes its values in Δ and fixes 0.*

Moreover, if $p_k/q_k \to \alpha$ are the approximants to α given by the continued fraction, then any limit value of χ_{p_k/q_k} is an isomorphism $\chi : \mathbb{D} \to \Delta$ that fixes 0.

Proof. Let us decompose the proof in several lemmas.

Lemma 1. *The family $\{\chi_{p/q} : \mathbb{D} \to \mathbb{C}\}_{p/q \in \mathbb{Q}}$ is normal.*

Proof. We will show that the family is uniformly bounded. For each $\delta \in \mathbb{D}$, $\chi_{p/q}(\delta)$ is a periodic point of P_α with $\alpha = p/q + R_{p/q}\delta^q$. Observe that $R_{p/q} \leq 1$ since $\mathbb{Z} \subset \mathcal{P}_q$ for every q. Hence, we have $\mathrm{Im}(\alpha) > -1$, and thus, $P_\alpha(z) = \lambda z + z^2$, with $|\lambda| < e^{2\pi}$. It follows that the periodic points of P_α are in the disk $\mathbb{D}(0, 1+e^{2\pi})$ since outside this disk $|\lambda z + z^2| = |z| \cdot |\lambda + z| > |z|$. Thus, $|\chi_{p/q}(\delta)| < 1 + e^{2\pi}$ for every rational number p/q and every $\delta \in \mathbb{D}$. □

Lemma 2. *If $p/q \to \alpha \in \mathbb{R}$ then any limit value of $\chi_{p/q}$ takes its values in the filled-in Julia set of P_α.*

Proof. We have to show that for any $\delta \in \mathbb{D}$, if $\chi_{p/q}(\delta) \to z$ as $p/q \to \alpha$, then z belongs to the filled-in Julia set $K(P_\alpha)$. Let us proceed by contradiction. If z does not belong to the filled-in Julia set of P_α, then, there exists an integer n such that $|P_\alpha^{\circ n}(z)| > 1 + e^{2\pi}$. Since $|\delta| < 1$ and $R_{p/q} < 1$, we see that $p/q + R_{p/q}\delta^q \to \alpha$ as $p/q \to \alpha$. It follows that when p/q is sufficiently close to α,

$$\left|P^{\circ n}_{p/q + R_{p/q}\delta^q}(\chi(\delta))\right| > 1 + e^{2\pi}.$$

This gives a contradiction since we have seen in the previous lemma that the cycle

$$\left\langle \chi(\delta), \chi(e^{2i\pi p/q}\delta), \ldots, \chi(e^{2i\pi(q-1)p/q}\delta) \right\rangle$$

belongs to the disk $\mathbb{D}(0, 1 + e^{2\pi})$. □

Observe that a limit value $\chi : \mathbb{D} \to K(P_\alpha)$ is either constant or open. Thus, it is either constant or takes its values in the interior of $K(P_\alpha)$. Since $\chi_{p/q}(0) = 0$

for any $p/q \in \mathbb{Q}$, we have $\chi(0) = 0$ and so, either P_α has no Siegel disk, 0 is in the boundary of $K(P_\alpha)$ and $\chi \equiv 0$ is constant, or P_α has a Siegel disk Δ and χ takes its values in Δ.

The following lemma appears in Jellouli's thesis [J].

Lemma 3. *Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is chosen so that P_α has a Siegel disk Δ and let $p_k/q_k \to \alpha$ be the approximants to α given by the continued fraction. Then, $P_{p_k/q_k}^{\circ q_k}$ converges uniformly to the identity on every compact subset of Δ.*

Proof. Let $\phi : \mathbb{D} \to \Delta$ be an isomorphism that fixes 0 and let $r < 1$ be an arbitrary radius (in the end, we will let r tend to 1). For each $\alpha' \in \mathbb{C}$, consider the map $f_{\alpha'} = \phi^{-1} \circ P_{\alpha'} \circ \phi$ which is defined on a neighborhood of 0 in $\mathbb{D}$. As $\alpha' \to \alpha$, $P_{\alpha'} \to P_\alpha$. Let us choose $\varepsilon > 0$ sufficiently small so that

$$\forall \alpha' \in \mathbb{D}(\alpha, \varepsilon), \quad f_{\alpha'} \text{ is defined on } \mathbb{D}(0, r) \quad \text{and} \quad \left| \frac{f_{\alpha'}(z)}{e^{2i\pi\alpha} z} - 1 \right| < 1.$$

Under this condition, we may define a function u analytic on $\mathbb{D}(\alpha, \varepsilon) \times \mathbb{D}(0, r)$ by

$$u(\alpha', Z) = \log \left(\frac{f_{\alpha'}(z)}{e^{2i\pi\alpha'} z} \right),$$

where $\log : 1 + \mathbb{D} \to \mathbb{C}$ is the branch of logarithm that sends 1 to 0. Note that the function $Z \mapsto u(\alpha', e^{2i\pi Z})$ is $\mathbb{Z}$-periodic.

Let us now consider the upper half-plane

$$\mathbb{H}_r = \left\{ Z \in \mathbb{C} \mid \operatorname{Im}(Z) > \frac{1}{2\pi} \log \frac{1}{r} \right\}.$$

We then define a function $F_{\alpha'} : \mathbb{H}_r \to \mathbb{H}$ by

$$F_{\alpha'}(Z) = Z + \alpha' + u(\alpha', e^{2i\pi Z}).$$

We have

$$f_{\alpha'}(e^{2i\pi Z}) = e^{2i\pi F_{\alpha'}(Z)}.$$

Thus, we have lifted $f_{\alpha'} : \mathbb{D}(0, r) \to \mathbb{D}$ to a map $F_{\alpha'} : \mathbb{H}_r \to \mathbb{H}$. Since $P_{\alpha'} \to P_\alpha$ as $\alpha' \to \alpha$, we have $u(\alpha, z) = 0$ for every $z \in \mathbb{D}(0, r)$. Without loss of generality, we may assume that u is analytic on a neighborhood of $\overline{\mathbb{D}(\alpha, \varepsilon)} \times \overline{\mathbb{D}(0, r)}$, and so, that $\partial u / \partial \alpha'$ is uniformly bounded by a constant K on $\mathbb{D}(\alpha, \varepsilon) \times \mathbb{D}(0, r)$. It follows that

$$\forall \alpha' \in \mathbb{D}(\alpha, \varepsilon), \quad \forall z \in \mathbb{D}(0, r), \quad |u(\alpha', z)| \leq K |\alpha' - \alpha|.$$

We will now use a fundamental property of the approximants p_k/q_k of α:

$$\left| \alpha - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k^2}.$$

If $p_k/q_k \in \mathbb{D}(\alpha, \varepsilon)$, we therefore have

$$\left| F_{p_k/q_k}(Z) - Z - \alpha' \right| \leq K \left| \frac{p_k}{q_k} - \alpha \right| \leq \frac{K}{q_k^2}.$$

It follows by induction that if $p_k/q_k \in \mathbb{D}(\alpha, \varepsilon)$ and $\operatorname{Im}(Z) > \dfrac{1}{2\pi} \log \dfrac{1}{r} + \dfrac{K}{q_k}$, then for any $m = 0, \ldots, q_k$, we have

$$\left| F_{p_k/q_k}^{\circ m}(Z) - Z - m\alpha' \right| \leq m \frac{K}{q_k^2}.$$

As a consequence, if $p_k/q_k \in \mathbb{D}(\alpha, \varepsilon)$ and $\mathrm{Im}(Z) > \frac{1}{2\pi}\log\frac{1}{r} + \frac{K}{q_k}$, then

$$\left|F_{p_k/q_k}^{\circ q_k}(Z) - Z - q_k\alpha'\right| \leq \frac{K}{q_k},$$

and when k is sufficiently large, $f_{p_k/q_k}^{\circ q_k}$ is uniformly close to the identity on a disk centered at 0 of radius arbitrarily close to r. The lemma follows since r can be chosen arbitrarily close to 1. □

Lemma 4. *Assume P_α has a Siegel disk Δ and let $\phi : \mathbb{D} \to \Delta$ be an isomorphism that fixes 0. Then,*

$$\liminf_{n\to+\infty} L_a\left(\frac{p_k}{q_k}\right) \geq |\phi'(0)|,$$

where $p_k/q_k \to \alpha$ are the approximants to α given by the continued fraction.

Proof. As in the previous lemma, we may consider the maps $f_\alpha = \phi^{-1} \circ P_\alpha \circ \phi$. Then,

$$f_{p_k/q_k}^{\circ q_k} = z + A_k z^{q_k+1} + \mathcal{O}(|z|^{q_k+2}),$$

for some complex number $A_k \neq 0$ and

$$L_a\left(\frac{p_k}{q_k}\right) = \frac{|\phi'(0)|}{|q_k A_k|^{1/q_k}}.$$

The previous lemma asserts that $f_{p_k/q_k}^{\circ q_k}$ converges uniformly to the identity on every compact subset of $\mathbb{D}$. For any radius $r < 1$, we may find an integer N so that $f_{p_k/q_k}^{\circ q_k}$ is defined on $\mathbb{D}_r$ for $n \geq N$. Since $f_{p_k/q_k}^{\circ q_k}$ takes its values in $\mathbb{D}$, we have

$$|A_k| = \frac{1}{2\pi}\left|\int_{\{|z|=r\}} \frac{f_{p_k/q_k}^{\circ q_k}(z)}{z^{q_k+2}} dz\right| \leq \frac{1}{r^{q_k+1}}.$$

Therefore,

$$|q_k A_k|^{1/q_k} \leq \frac{q_k^{1/q_k}}{r \cdot r^{1/q_k}} \underset{n\to\infty}{\longrightarrow} \frac{1}{r}.$$

It follows that

$$\liminf_{n\to\infty} L_a\left(\frac{p_k}{q_k}\right) \geq r|\phi'(0)|,$$

and the result follows by letting $r \to 1$. □

Now, let $\chi : \mathbb{D} \to \Delta$ be a limit value of the sequence $\chi_{p_k/q_k} : \mathbb{D} \to \mathbb{C}$. Remember that

$$|\chi'_{p/q}(0)| = (2\pi q^2 R_{p/q})^{1/q} L_a(p/q),$$

where $L_a(p/q)$ is the asymptotic length of $P_{p/q}$ at 0. Since we have the estimate $C/q^3 \leq R_{p/q} \leq 1$ for some universal constant C, we see that $|\chi'_{p/q}(0)|$ and $L_a(p/q)$ are equivalent as $q \to \infty$. Thus, the previous lemma shows that $|\chi'(0)| \geq |\phi'(0)|$. The third part of the proposition then follows from Schwarz's lemma. Indeed, the map $\phi^{-1} \circ \chi : \mathbb{D} \to \mathbb{D}$ fixes 0 and its derivative at 0 is of modulus greater or equal to 1 by the previous lemma. ■

Corollary 4. *Assume α is a Bruno number and let p_k/q_k be the approximants to α given by the continued fraction. Then*

$$\lim_{k\to+\infty} \left| L_a\left(\frac{p_k}{q_k}\right)\right| = r_\alpha,$$

where r_α is the conformal radius at 0 of the Siegel disk of P_α.

Proof. As $q \to +\infty$, the sequences $|\chi'_{p/q}(0)|$ and $L_a(p/q)$ are equivalent. If $\chi : \mathbb{D} \to \Delta$ is a limit value of χ_{p_k/q_k}, then it is an isomorphism that fixes 0, and thus, $|\chi'(0)| = r_\alpha$. ∎

Proposition 9. *Assume α is a Bruno number and let p_k/q_k be the approximants to α defined by the continued fraction. Assume $\alpha[n]$ is a sequence of real numbers such that*

$$\left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \underset{n\to+\infty}{\longrightarrow} \delta < 1.$$

Then, the conformal radii $r_{\alpha[n]}$ of the Siegel disks of the quadratic polynomials $P_{\alpha[n]}$ satisfy:

$$\limsup r_{\alpha[n]} \leq r_\alpha \delta.$$

Proof. For each n, let δ_n be a complex number such that

$$\frac{p_n}{q_n} + R_{p_n/q_n}\delta_n^{q_n} = \alpha[n].$$

As $n \to +\infty$, $|\delta_n| \to \delta$ (because $(R_{p_n/q_n})^{1/q_n} \to 1$) and any limit value of the sequence $\chi_{p_n/q_n} : \mathbb{D} \to \mathbb{C}$ coincides with $z \mapsto \phi_\alpha(r_\alpha z)$ up to precomposition with a rotation. Therefore, the compact sets

$$C_n = \chi_{p_n/q_n}\{\delta_n e^{2i\pi k/q_n} \mid k = 1..q_n\}$$

converge to $\phi_\alpha(\partial\mathbb{D}(0, \delta r_\alpha))$ for the Hausdorff topology on compact subsets of $\mathbb{C}$.

Let us assume that r is a limit value of a subsequence $r_{\alpha(n_k)}$. Then, for any $r' < r$, if n is sufficiently large, $\phi_{\alpha(n_k)}$ is defined on the disk $\mathbb{D}(0, r')$. The maps $\phi_{\alpha(n_k)} : \mathbb{D}(0, r') \to \mathbb{C}$ are univalent, fix 0 and have derivative 1 at the origin. Therefore, extracting a further subsequence if necessary, we may assume that the sequence $\phi_{\alpha(n_k)} : \mathbb{D}(0, r') \to \Delta_{\alpha(n_k)}$ converges to a limit $\phi : \mathbb{D}(0, r') \to \mathbb{C}$. The map $\phi_{\alpha(n_k)}$ take its values in the Siegel disk $\Delta_{\alpha(n_k)}$, and so, it omits the periodic cycle C_n of $P_{\alpha[n]}$. As a consequence, the limit map ϕ must omit $\phi_\alpha(\partial\mathbb{D}(0, \delta r_\alpha))$.

Therefore, the map $\phi_\alpha^{-1} \circ \phi$ sends $\mathbb{D}(0, r')$ into $\mathbb{D}(0, \delta r_\alpha)$, it fixes 0 and has derivative 1 at 0. Thus, by Schwarz's lemma, $r' \leq \delta r_\alpha$. Letting $r' \to r$ shows that any limit value of a subsequence $r_{\alpha(n_k)}$ is less or equal than δr_α, q.e.d. ∎

Corollary 5. *Let α and θ be Bruno numbers, and let $\alpha[n]$ be a sequence of Bruno numbers defined as in proposition 4. Then,*

$$\lim_{n\to+\infty} r_{\alpha[n]} = r_\alpha/A.$$

Proof. We have already seen (corollary 3) that $\liminf r_{\alpha[n]} \geq r_\alpha/A$. We have also seen that

$$\left|\alpha[n] - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2 A_n} + \mathcal{O}\left(\frac{1}{A_n^2}\right).$$

The result follows by applying proposition 9 with $\delta = 1/A$, since

$$\left(\frac{1}{q_n^2 A_n}\right)^{1/q_n} \underset{n\to+\infty}{\longrightarrow} \frac{1}{A}$$

by choice of the sequence A_n. ∎

6. Boundaries of Siegel disks.

Proof of the Main Theorem. Let α be an arbitrary Bruno number, let r be an arbitrary real number in $]0, r_\alpha[$, let $\eta > 0$ and $\varepsilon > 0$ be arbitrarily small real numbers and let $\ell_0 \geq 1$ be an arbitrarily large integer. Let us first exhibit a Bruno number α' such that $|\alpha' - \alpha| < \eta$, the conformal radius of the Siegel disk $\Delta_{\alpha'}$ of $P_{\alpha'}$ is $r_{\alpha'} = r$, and the linearizing parameterization $\phi_{\alpha'} : \mathbb{D}(0,r) \to \Delta_{\alpha'}$ has a continuous extension to the boundary, such that the $\mathbb{Z}$-periodic function $u_{\alpha'} : \mathbb{R} \to \mathbb{C}$ defined by $u_{\alpha'}(t) = \lim_{z \to re^{2i\pi t}} \phi_{\alpha'}(z)$ is C^∞ and satisfies

$$\sup_{j=0..\ell_0} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u_{\alpha'}}{\partial t^j}(t) - \frac{\partial^j u_\alpha}{\partial t^j}(t) \right| < \varepsilon,$$

where $u_\alpha : \mathbb{R} \to \mathbb{C}$ is the $\mathbb{Z}$-periodic analytic curve $t \mapsto \phi_\alpha(re^{2i\pi t})$.

In order to prove this result, let us introduce the following definition. If $\phi : \partial\mathbb{D}(0,r) \to \mathbb{C}$ is a C^∞ function and $\ell \geq 0$ is an integer, we denote by

$$\|\phi\|_{C^\ell} = \sup_{j=0..\ell} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u}{\partial t^j}(t) \right|,$$

where $u : \mathbb{R} \to \mathbb{C}$ is the function $t \mapsto \phi(re^{2i\pi t})$. Note that if a sequence of functions ϕ_j, holomorphic in a common neighborhood of $\overline{\mathbb{D}(0,r)}$, converge uniformly on this neighborhood to a limit ϕ, then, for all $\ell > 0$, we have

$$\|\phi_j - \phi\|_{C^\ell} \underset{n\to+\infty}{\longrightarrow} 0.$$

Note also that when a sequence of functions ϕ_j, holomorphic in a neighborhood of $\overline{\mathbb{D}(0,r)}$, is a Cauchy sequence for the norm $\|.\|_{C^0}$ we just defined, then by the maximum modulus principle, it is a Cauchy sequence for the norm $\|.\|_\infty$ as a sequence of functions from $\overline{\mathbb{D}(0,r)} \to \mathbb{C}$ and so, it has a limit $\phi : \overline{\mathbb{D}(0,r)} \to \mathbb{C}$ which is holomorphic in $\mathbb{D}(0,r)$ and continuous in $\overline{\mathbb{D}(0,r)}$.

Now, set $\alpha(0) = \alpha$, let $[a_0, a_1, a_2, \ldots]$ be the continued fraction of α and let $p_k(0)/q_k(0)$ be the approximants to $\alpha(0)$ defined by its continued fraction. Moreover, let $r_0 = r_{\alpha(0)}$ be the conformal radius at 0 of the Siegel disk of P_α.

Choose an arbitrary radius $r_1 \in]r, r_0[$ and define a sequence

$$\alpha(0)[n] = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n, A_n, 1, 1, 1, \ldots],$$

where A_n is the integer part of $(r_0/r_1)^{q_n(0)}$ and is the entry $n+1$ of the continued fraction. As $n \to \infty$, we have $r_{\alpha(0)[n]} \to r_1$ by corollary 5, $\phi_{\alpha(0)[n]} \to \phi_{\alpha(0)}$ uniformly on every compact subset of $\mathbb{D}(0,r_1)$ by corollary 3, and $L_a(p_n(0)/q_n(0)) \to r_0$ by corollary 4. We choose n_0 sufficiently large, so that

$$\left|r_{\alpha(0)[n_0]} - r_1\right| < \frac{\varepsilon}{2}, \quad \left\|\phi_{\alpha(0)[n_0]} - \phi_{\alpha(0)}\right\|_{C^{\ell_0}} < \frac{\varepsilon}{2} \quad \text{and} \quad \left| L_a\left(\frac{p_{n_0}(0)}{q_{n_0}(0)}\right) - r_0 \right| < \frac{\varepsilon}{2}.$$

We then define $A'_1 = A_{n_0}$, $\alpha_1 = \alpha(0)[n_0]$ and $r'_1 = r_{\alpha(0)[n_0]}$, and we let $p_n(1)/q_n(1)$ be the approximants to $\alpha(1)$. Note that for $n \leq n_0$, we have $p_n(1) = p_n(0)$ and $q_n(1) = q_n(0)$.

We now almost repeat the process. First, we choose an arbitrary radius $r_2 \in]r, r'_1[$ and we define a sequence

$$\alpha(1)[n] = [a_0, a_1, \ldots, a_{n_0-1}, a_{n_0}, A'_1, 1, 1, \ldots, 1, 1, A_n, 1, 1, 1, \ldots],$$

where A_n is the integer part of $(r'_1/r_2)^{q_n(1)}$ and is the entry $n+1$ of the continued fraction. As $n \to \infty$, we have $r_{\alpha(1)[n]} \to r_2$ by corollary 5, $\phi_{\alpha(1)[n]} \to \phi_{\alpha(1)}$ uniformly on every compact subset of $\mathbb{D}(0, r_2)$ by corollary 3, and $L_a(p_n(1)/q_n(1)) \to r'_1$ by corollary 4. Thus, we may choose n_1 sufficiently large, so that

$$\left|r_{\alpha(1)[n_1]} - r_2\right| < \frac{\varepsilon}{4}, \quad \left\|\phi_{\alpha(1)[n_1]} - \phi_{\alpha(1)}\right\|_{C^{\ell_0+1}} < \frac{\varepsilon}{4}$$

and

$$\left|L_a\left(\frac{p_{n_1}(1)}{q_{n_1}(1)}\right) - r'_1\right| < \frac{\varepsilon}{4}.$$

We then define $A'_2 = A_{n_1}$, $\alpha(2) = \alpha(1)[n_1]$ and $r'_2 = r_{\alpha(1)(n_1)}$, and we let $p_n(2)/q_n(2)$ be the approximants to $\alpha(2)$.

Inductively, we construct a sequence $\alpha(j)$. At each step, we choose a radius $r_{j+1} \in]r, r'_j[$ and we define a sequence

$$\alpha(j)[n] = [a_0, a_1, \ldots, a_{n_0}, A_{n_0}, 1, 1, \ldots, 1, 1, A_n, 1, 1, 1 \ldots],$$

where A_n is the integer part of $(r'_j/r_{j+1})^{q_n(j)}$. We then choose n_j sufficiently large so that

$$\left|r_{\alpha(j)[n_j]} - r_{j+1}\right| < \frac{\varepsilon}{2^{j+1}}, \quad \left\|\phi_{\alpha(j)[n_j]} - \phi_{\alpha(j)}\right\|_{C^{\ell_0+j}} < \frac{\varepsilon}{2^{j+1}}$$

and

$$\left|L_a\left(\frac{p_{n_j}(j)}{q_{n_j}(j)}\right) - r'_j\right| < \frac{\varepsilon}{2^{j+1}}.$$

We finally define $A'_{j+1} = A_{n_j}$, $\alpha(j+1) = \alpha(j)[n_j]$ and $r'_{j+1} = r_{\alpha(j)[n_j]}$, and we let $p_n(j+1)/q_n(j+1)$ be the approximants to $\alpha(j+1)$.

Now, let α' be the limit of the sequence $\alpha(j)$ and let p_k/q_k be its approximants. The continued fraction expansion of α' is

$$\alpha' = [a_0, \ldots, a_{n_0}, A'_1, 1, 1, \ldots, 1, 1, A'_2, 1, 1, \ldots, 1, 1, A'_3, 1, 1, \ldots]$$

where A'_{j+1} is the integer part of $(r'_j/r_{j+1})^{q_{n_j}}$ and is the entry n_j+1 of the continued fraction. By construction, the sequence $\phi_{\alpha(j)} : \overline{\mathbb{D}(0,r)} \to \mathbb{C}$ is a Cauchy sequence for all the norms $\|.\|_{C^\ell}$, $\ell \geq 0$, we defined at the beginning of the proof. In particular, it has a limit $\phi : \overline{\mathbb{D}(0,r)} \to \mathbb{C}$ which is holomorphic in $\mathbb{D}(0,r)$ and continuous in $\overline{\mathbb{D}(0,r)}$. Moreover, the function $u : t \mapsto \phi(re^{2i\pi t})$ is C^∞. By construction and as required, we have

$$\|\phi - \phi_\alpha\|_{C^{\ell_0}} = \sup_{j=0..\ell_0} \sup_{t\in\mathbb{R}} \left|\frac{\partial^j u}{\partial t^j}(t) - \frac{\partial^j u_\alpha}{\partial t^j}(t)\right| < \varepsilon.$$

Since the map $\phi : \mathbb{D}(0,r) \to \mathbb{C}$ is the limit of the linearizing parametrizations $\phi_{\alpha(j)}$ for $P_{\alpha(j)}$, it is a linearizing parametrization for $P_{\alpha'}$. Thus, $P_{\alpha'}$ has a Siegel disk $\Delta_{\alpha'}$ which contains $\phi(\mathbb{D}(0,r))$.

Since the choice of $r_{j+1} \in]r, r'_j[$ is arbitrary, we may assume that the sequence $(r_j)_{j\geq 0}$ converges to r as $j \to +\infty$. We claim that in that case, $\phi(\partial\mathbb{D}(0,r))$ is

actually the boundary of the Siegel disk, or equivalently, that the conformal radius $r_{\alpha'}$ is equal to r. By construction, for $k \leq n_j$, we have $p_k = p_k(j)$ and $q_k = q_k(j)$. As a consequence,

$$L_a\left(\frac{p_{n_j}}{q_{n_j}}\right) = L_a\left(\frac{p_{n_j}(j)}{q_{n_j}(j)}\right).$$

Thus,

$$\begin{aligned}\left|L_a\left(\frac{p_{n_j}}{q_{n_j}}\right) - r\right| &\leq \left|L_a\left(\frac{p_{n_j}(j)}{q_{n_j}(j)}\right) - r'_j\right| + |r'_j - r_{j+1}| + |r_{j+1} - r| \\ &\leq \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{2^{j+1}} + |r_{j+1} - r| \underset{j\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore,

$$L_a\left(\frac{p_{n_j}}{q_{n_j}}\right) \underset{j\to+\infty}{\longrightarrow} r,$$

and by corollary 4, we have $r_{\alpha'} = r$.

Now, if choose n_0 sufficiently large, then any irrational number α' whose continued fraction has the same $n_0 + 1$ first entries as α is at distance less than η from α. ■

Since $\phi_{\alpha'}$ is a homeomorphism between $\mathbb{D}(0,r)$ to $\Delta_{\alpha'}$, and since it extends continuously to the closure $\overline{\mathbb{D}(0,r)}$ which is compact, it follows that $\phi(\partial\mathbb{D}(0,r)) = \partial\Delta_{\alpha'}$. Moreover, by a theorem of Carathéodory, the boundary of the Siegel disk is locally connected. It follows that it must be a Jordan curve (see [Mi], Lemma 18.7). Since the derivative of the $\mathbb{Z}$-periodic map $u_\alpha : \mathbb{R} \to \mathbb{C}$ does not vanish, its modulus reaches a minimum. Choosing ε smaller than this minimum guarantees that the derivative of $u : \mathbb{R} \to \mathbb{C}$ does not vanish. So, when ε is small enough, the map $u : \mathbb{R} \to \mathbb{C}$ is injective with non-vanishing derivative and the boundary of the Siegel disk $\Delta_{\alpha'}$ is a smooth Jordan curve.

Let us finally prove that we may choose α' so that $\phi(\partial\mathbb{D}(0,r))$ is accumulated by small cycles, which provides an alternate way of proving that $\Delta_{\alpha'} = \phi(\mathbb{D}(0,r))$. We do the same proof, with the following modifications. First, we may choose r_1 sufficiently close to r so that the Jordan curve $\phi_\alpha(\partial\mathbb{D}(0,r_1))$ is contained in an $\varepsilon/3$ neighborhood of the Jordan curve $\phi_\alpha(\partial\mathbb{D}(0,r))$. By proposition 8, we know that by choosing n_0 sufficiently large, we may assume that P_{α_1} has a periodic cycle $\mathcal{C}'_0$ of period q_{n_0} in an $\varepsilon/3$ neighborhood of the Jordan curve $\phi_\alpha(\partial\mathbb{D}(0,r_1))$. Now, as α' varies in a sufficiently small neighborhood of α_1, this cycle varies holomorphically with α'. Thus, by choosing n_1 sufficiently large, we may assume that the final polynomial $P_{\alpha'}$ has a periodic cycle $\mathcal{C}_0$ of period q_{n_0} at distance less than $\varepsilon/3$ from the cycle $\mathcal{C}'_1$. This cycle is contained in an ε neighborhood of the Jordan curve $\phi_\alpha(\partial\mathbb{D}(0,r))$, which, by construction, is itself contained in an ε neighborhood of the boundary of the Siegel disk $\Delta_{\alpha'}$.

In summary, by choosing n_0 and n_1 sufficiently large, we may assume that $P_{\alpha'}$ has a cycle $\mathcal{C}_0$ of period q_{n_0} in a 2ε neighborhood of the boundary of $\Delta_{\alpha'}$. Similarly, by choosing n_1 and n_2 sufficiently large, we may assume that $P_{\alpha'}$ has a periodic cycle $\mathcal{C}_1$ of period q_{n_1} in an ε neighborhood of the boundary of $\Delta_{\alpha'}$, and by induction, we see that we can choose the n_i's so that for each $j \geq 0$, $P_{\alpha'}$ has a cycle $\mathcal{C}_j$ in an $\varepsilon/2^{j-1}$ neighborhood of the boundary of $\Delta_{\alpha'}$. This precisely shows that the boundary of $\Delta_{\alpha'}$ is accumulated by small cycles. ■

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