SIEGEL DISKS WITH SMOOTH BOUNDARIES

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Abstract. We show the existence of angles $\alpha \in \mathbb{R}/\mathbb{Z}$ such that the quadratic polynomial $P_\alpha(z) = e^{2i\pi\alpha}z + z^2$ has a Siegel disk with C^∞-smooth boundary. This result was first announced by R. Pérez-Marco in 1993.

Introduction

Assume U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a holomorphic map which satisfies $f(0) = 0$ and $f'(0) = e^{2i\pi\alpha}$, $\alpha \in \mathbb{R}/\mathbb{Z}$. We say that f is *linearizable* at 0 if it is topologically conjugate to the rotation $R_\alpha : z \mapsto e^{2i\pi\alpha}z$ in a neighborhood of 0. If $f : U \to \mathbb{C}$ is linearizable, there is a largest f-invariant domain $\Delta \subset U$ containing 0 on which f is conjugate to the rotation R_α. This domain is simply connected and is called the *Siegel disk* of f. A basic but remarkable fact is that the conjugacy can be taken holomorphic.

In this article, we are mainly concerned with the dynamics of the quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. They have $z = 0$ as an indifferent fixed point.

For every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, there exists a unique formal power series $\phi_\alpha(z) = z + b_2z^2 + b_3z^3 + \ldots$ such that

$$\phi_\alpha \circ R_\alpha = P_\alpha \circ \phi_\alpha.$$

We denote by $r_\alpha \geq 0$ the radius of convergence of the series ϕ_α. It is known (see [Y1] for example) that $r_\alpha > 0$ for Lebesgue almost every $\alpha \in \mathbb{R}$. More precisely, $r_\alpha > 0$ if and only if α satisfies the Bruno condition (see definition 2 below).

From now on, we assume that $r_\alpha > 0$. In that case, the map $\phi_\alpha : B(0, r_\alpha) \to \mathbb{C}$ is univalent, and it is well known that its image Δ_α coincides with the Siegel disk of P_α associated to the point 0. The number r_α is called the *conformal radius* of the Siegel disk. The Siegel disk is also the connected component of $\mathbb{C}\setminus J(P_\alpha)$ which contains 0, where $J(P_\alpha)$ is the Julia set of P_α, i.e., the closure of the set of repelling periodic points. Figure 1 shows the Julia sets of the quadratic polynomials P_α, for $\alpha = \sqrt{2}$ and $\alpha = \sqrt{10}$. Both polynomials have a Siegel disk colored grey.

In this article, we investigate the structure of the boundary of the Siegel disk. It is known since Fatou that this boundary is contained in the closure of the forward orbit of the critical point $\omega_\alpha = -e^{2i\pi\alpha}/2$ (for example, see [Mi] Theorem 11.17 or [Mi] Corollary 14.4). By plotting a large number of points in the forward orbit of ω_α, we should therefore get a good idea of what those boundaries look like. In practice, that works only when α is sufficiently well-behaved, the number of iterations needed being otherwise enormous.

In 1983, Herman [He1] proved that when α satisfies the Herman condition, the critical point actually belongs to the boundary of the Siegel disk. (Recall that Herman's condition is the optimal arithmetical condition to ensure that every analytic circle diffeomorphism with rotation number α is analytically linearizable near the

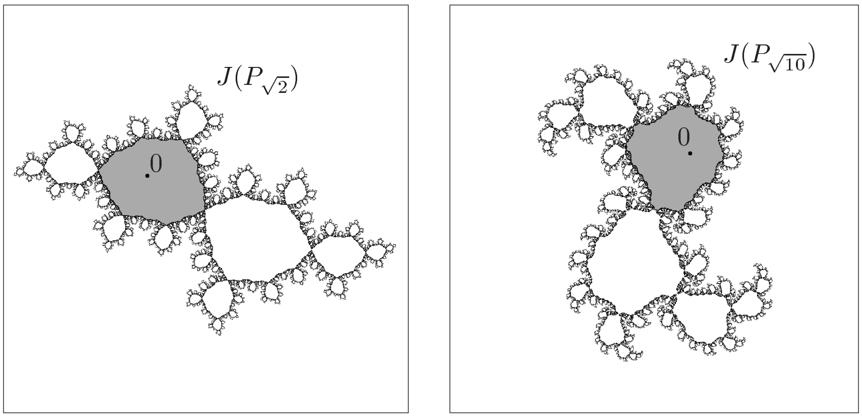

FIGURE 1. Left: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{2}}z + z^2$. Right: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{10}}z + z^2$. In both cases, there is a Siegel disk.

circle. We will not give a precise description here. See [Y2] for more details.) Using a construction due to Ghys, Herman [He2] also proved the existence of quadratic polynomials P_α for which the boundary of the Siegel disk is a quasi-circle which does not contain the critical point. Later, following an idea of Douady [D] and using work of Światek [S] (see also [P2]), he proved that when α is Diophantine of exponent 2, the boundary of the Siegel disk is a quasi-circle containing the critical point. In [McM1], McMullen showed that the corresponding Julia sets have Hausdorff dimension less than 2 and that when α is a quadratic irrational, the boundary of the Siegel disk is self-similar about the critical point. More recently, Petersen and Zakeri [PZ] proved that for Lebesgue almost every $\alpha \in \mathbb{R}/\mathbb{Z}$, the boundary is a Jordan curve containing the critical point. Moreover, when α is not Diophantine of exponent 2, this Jordan curve is not a quasi-circle (see [PZ]).

In [PM], Pérez-Marco proves that there exist univalent maps in $\mathbb{D}$ having Siegel disks compactly contained in $\mathbb{D}$ whose boundaries are C^∞-smooth Jordan curves. This result is very surprising and very few people suspected that such a result could be true. The boundary cannot be an analytic Jordan curve since in that case the linearizing map would extend across it by Schwarz reflection. Pérez-Marco even produces examples where an uncountable number of intrinsic rotations extend univalently to a neighborhood of the closure of the Siegel disk. Pérez-Marco's results have several nice corollaries (see [PM]). For example, it follows that there exist analytic circle diffeomorphisms which are C^∞ linearizable but not analytically linearizable. This answers a question asked by Katok in 1970.

In a 1993 seminar at Orsay, Pérez-Marco announced the existence of quadratic polynomials having Siegel disks with smooth boundaries. According to Pérez-Marco, his proof is rather technical. In 2001, the second and third authors [BC1] found a different approach to the existence of such quadratic polynomials. In [A], the first author considerably simplified the proof.

Definition 1. *We say that the boundary of a Siegel disk Δ_α is accumulated by cycles if every neighborhood of $\overline{\Delta}_\alpha$ contains a (whole) periodic orbit of P_α.*

Main Theorem. *Assume $\alpha \in \mathbb{R}$ is a Bruno number and $r \in (0, r_\alpha)$ and $\varepsilon > 0$ are real numbers. Let $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be the function $t \mapsto \phi_\alpha(re^{2i\pi t})$. Then, there exists a Bruno number α' with the following properties:*

- $|\alpha' - \alpha| < \varepsilon$,
- $r_{\alpha'} = r$,
- *the linearizing map $\phi_{\alpha'} : B(0,r) \to \Delta_{\alpha'}$ extends continuously to a function $\phi_{\alpha'} : \overline{B(0,r)} \to \overline{\Delta_{\alpha'}}$,*[1]
- *the function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ defined by $v(t) = \phi_{\alpha'}(re^{2i\pi t})$ is a C^∞ embedding (thus the boundary of the Siegel disk is a smooth Jordan curve), and*
- *the functions u and v are ε-close in the Fréchet space $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.*

Additional Information. *We may choose α' so that the boundary of the Siegel disk $\Delta_{\alpha'}$ is accumulated by cycles.*[2]

Remark. When the polynomial P_α is not linearizable, i.e., $r_\alpha = 0$, it is known that 0 is accumulated by cycles (see [Y1]). It may be the case that the boundary of the Siegel disk of a quadratic polynomial is always accumulated by cycles.

Corollary 1. *There exist quadratic polynomials with Siegel disks whose boundaries do not contain the critical point.*

First Proof. Let α be any Bruno number, choose $r \in (0, r_\alpha)$ sufficiently small so that $\phi_\alpha(\partial B(0,r)) \subset B(0, 1/10)$. Then, for ε small enough, the boundary of the Siegel disk $\Delta_{\alpha'}$ given by the Main Theorem is contained in $B(0, 1/5)$. Therefore, the critical point $\omega_{\alpha'} = -e^{2i\pi\alpha'}/2$ cannot belong to the boundary of the Siegel disk $\Delta_{\alpha'}$. ■

Second Proof. The Main Theorem gives quadratic polynomials with Siegel disks whose boundaries are smooth Jordan curves. But an invariant Jordan curve cannot be smooth at both the critical point and the critical value. ■

Note that our proof of the existence of quadratic Siegel disks whose boundaries do not contain critical points is completely different from Herman's proof.

Corollary 2. *The set $\mathcal{S} \subset \mathbb{R}$ of real numbers α for which P_α has a Siegel disk with smooth boundary is dense in $\mathbb{R}$ and has uncountable intersection with any open subset of $\mathbb{R}$.*

Remark. By [He1] or [PZ], the set $\mathcal{S}$ has Lebesgue measure zero.

Proof. Given any Bruno number α and any $\eta > 0$, the conformal radius $r_{\alpha'}$ (for the α' provided by the Main Theorem) can take any value in the interval $(0, r_\alpha)$ and so the intersection of $\mathcal{S}$ with the interval $(\alpha - \eta, \alpha + \eta)$ is uncountable. The proof is completed since the set of Bruno numbers is dense in $\mathbb{R}$. ■

It follows from a theorem of Mañé that the boundary of a Siegel disk of any rational map is contained in the accumulation set of some recurrent critical point (see for example [ST]). Thus, for a quadratic polynomial, a critical point with orbit

[1] It automatically maps the boundary of $B(0,r)$ to the boundary of $\Delta_{\alpha'}$

[2] In that case, the critical point of α' is not accessible through the basin of infinity (see for example [K] or [Z])

falling on the boundary of a fixed Siegel disk must belong itself to this boundary. As a consequence, if P_α has a Siegel disk Δ_α with smooth boundary, the orbit of the critical point avoids $\overline{\Delta_\alpha}$ and thus all the preimages of the Siegel disk also have smooth boundaries.

The main tool in the proof of the Main Theorem is a perturbation lemma.

Main Lemma. *Given any Bruno number α and any radius r_1 such that $0 < r_1 < r_\alpha$, there exists a sequence of Bruno numbers $\alpha[n] \longrightarrow \alpha$ such that $r_{\alpha[n]} \to r_1$.*

We shall also need the following standard fact: if $\theta_n \to \theta$ and $r_{\theta_n} \geq r$ then $r_\theta \geq r$ (thus the conformal radius is upper semicontinuous) and $\phi_{\theta_n} \to \phi_\theta$ uniformly on compact subsets of $B(0,r)$. Indeed, the linearizing maps $\phi_{\theta_n}|B(0,r)$ are univalent with $\Phi_{\theta_n}(0) = 0$, $\phi'_{\theta_n}(0) = 1$, and thus form a normal family. Passing to the limit in the equation

$$\phi_{\theta_n}(e^{2i\pi\theta_n} z) = P_{\theta_n}(\phi_{\theta_n}(z)),$$

we see that any subsequence limit of $\{\phi_{\theta_n}$ linearizes P_θ and thus coincides with ϕ_θ on $B(0,r)$ by uniqueness of the linearizing map.

Proof of the Main Theorem assuming the Main Lemma. We define sequences $\alpha(n)$ and ε_n inductively as follows. Let r_n be a decreasing sequence converging to r with $r_0 = r_\alpha$. Take $\alpha(0) = \alpha$, $\varepsilon_0 = \varepsilon/10$. Assuming that $\alpha(n)$ and ε_n are defined, let $\varepsilon_{n+1} < \varepsilon_n/10$ be such that $r_\theta < r_{\alpha(n)} + \varepsilon_n$ whenever $|\theta - \alpha(n)| < \varepsilon_{n+1}$ (this is possible by upper semicontinuity). With the help of the Main Lemma, choose $\alpha(n+1)$ such that $|\alpha(n+1) - \alpha(n)| < \varepsilon_{n+1}/10$, $r_{n+1} < r_{\alpha(n+1)} < r_n$, and the real-analytic functions $u_{n+1} : t \mapsto \phi_{\alpha(n+1)}(re^{2i\pi t})$ and $u_n : t \mapsto \phi_{\alpha(n)}(re^{2i\pi t})$ are ε_{n+1}-close in the Fréchet space $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

Let $\alpha' = \lim \alpha(n)$. By the construction, $|\alpha' - \alpha(n)| < \varepsilon_{n+1}$ for $n \geq 0$. By the definition of ε_{n+1}, this implies $r_{\alpha'} < r_{\alpha(n)} + \varepsilon_n$. Since $\varepsilon_n \to 0$ and $r_{\alpha(n)} \to r$, we have $r_{\alpha'} \leq r$. On the other hand, by upper semicontinuity, we have $r_{\alpha'} \geq \lim r_{\alpha(n)}$, so $r_{\alpha'} = r$. The functions u_n converge to a C^∞ function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ which is ε-close to $u = u_0$ in the Fréchet space $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. In particular (by taking ε smaller) this implies that v is an embedding. Since $\phi_{\alpha'} = \lim \phi_{\alpha(n)}|_{B(0,r)}$, it follows that $\phi_{\alpha'}$ has a continuous (actually C^∞) extension to the boundary of $B(0,r)$ given by $\phi_{\alpha'}(re^{2i\pi t}) = v(t)$. This completes the proof of the Main Theorem. ■

The purpose of Figure 2 is to illustrate this construction. We have drawn the boundary of three quadratic Siegel disks, for $\alpha = (\sqrt{5}+1)/2$, $\alpha(1)$ which is close to α and $\alpha(2)$ which is much closer to $\alpha(1)$. For $\alpha(1)$, there is a cycle of period 8 that forces the boundary of the Siegel disk to oscillate slightly. For $\alpha(2)$, there is an additional cycle (of period 205) that forces the boundary to oscillate much more. We have not been able to produce a picture for a possible choice of $\alpha(3)$. The number of iterates of the critical point required to get a relevant picture was much too large.

In this article, we present two independent proofs of the Main Lemma. The second and third authors found a proof that goes as follows. We first give a lower bound for the size of the Siegel disk of a map which is close to a rotation as done in [C] part 2 (see Section 5). We then use the techniques of parabolic explosions in the quadratic family introduced in [C] part 1 in order to control the conformal radius from above (see Section 6). A proof of the Main Lemma follows (see section

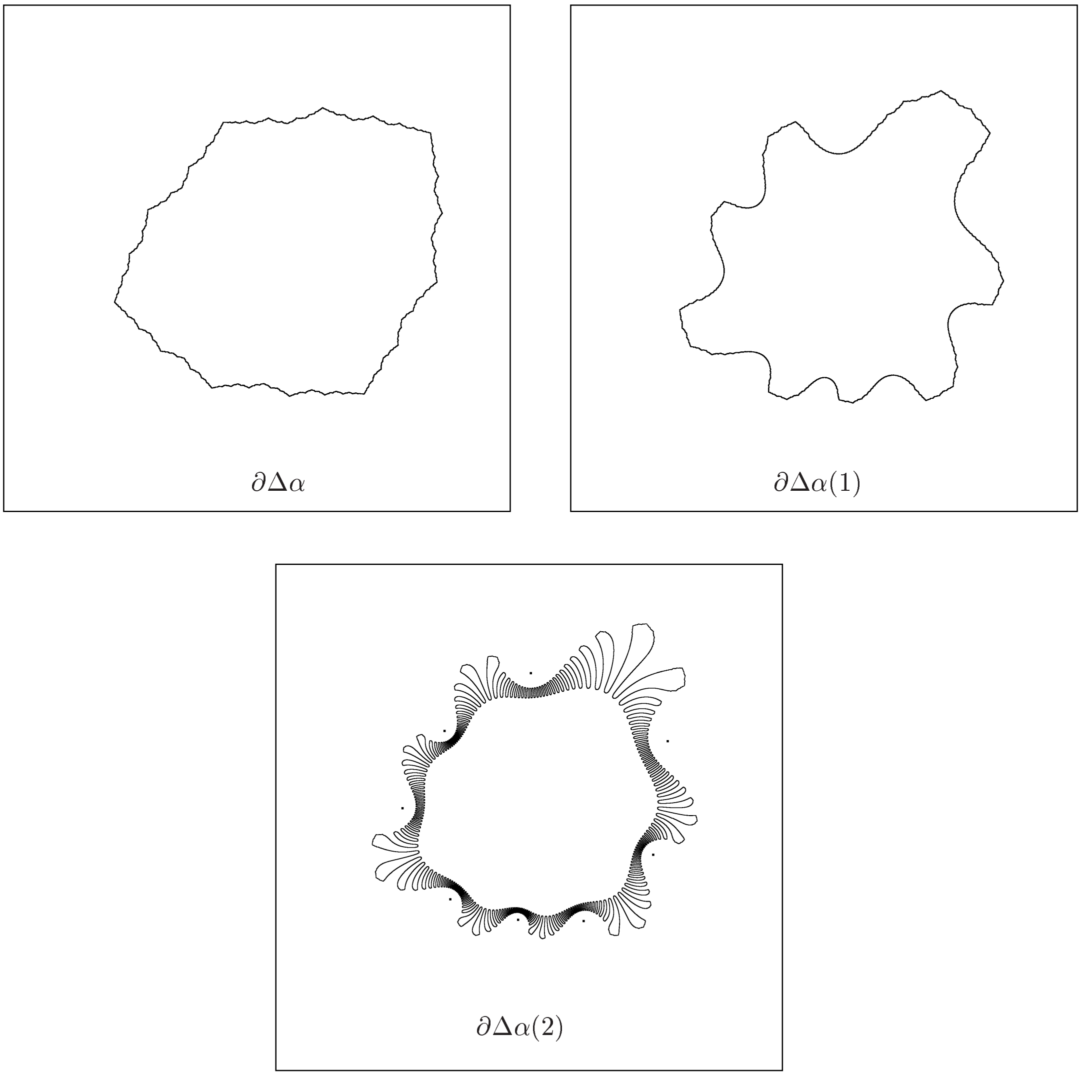

FIGURE 2. The first steps in the construction of a Siegel disk with smooth boundary. In the last frame, we plotted the period 8 cycle that creates the first order oscillation. The period 205 cycle that creates the stronger oscillation is too close to $\partial\Delta\alpha(2)$ to be clearly represented here.

3). This approach has the advantage of showing that one can find smooth Siegel disks accumulated by cycles (see section 7).

The first author simplified this proof (see section 4), replacing the technique of parabolic explosion by Yoccoz's Theorem on the optimality of the Bruno condition for the linearization problem in the quadratic family. The argument automatically applies to other families where this optimality is known to hold, as the examples of Geyer [G].

We would like to end this introduction with an observation. In the same way one uses lacunary Fourier series to produce C^∞ functions which are nowhere analytic, our Siegel disks can be produced with rotation numbers whose continued fractions have large coefficients (in a certain sense) which are more and more spaced out. The two phenomena are not completely unlinked. Indeed, if $\phi : \mathbb{D} \to \Delta$ is the normalized linearizing map, then the coefficients b_k of the power series of ϕ are the Fourier coefficients of the angular parametrization of the boundary of Δ. These coefficients also depend on the arithmetic nature of α. Indeed, they are defined by the recursive formula

$$b_1 = r_\alpha \quad \text{and} \quad b_{n+1} = \frac{1}{e^{2i\pi\alpha}(e^{2i\pi n\alpha}-1)} \sum_{i=1}^{n} b_i b_{n+1-i}.$$

If the $(k+1)$-st entry in the continued fraction of α (see the definition below) is large, $e^{2i\pi q_k \alpha} - 1$ is close to 0 and b_{1+q_k} is large (p_k/q_k is the k-th convergent of α, see the definition below).

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1. Arithmetical preliminaries

This section gives a short account of a very classical theory. See for instance [HW] or [Mi].

If $(a_k)_{k\geq 0}$ are integers, we use the notation $[a_0, a_1, \ldots, a_k, \ldots]$ for the continued fraction

$$[a_0, a_1, \ldots, a_k, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \ddots}}}.$$

We call a_k the k-th entry of the continued fraction. The 0-th entry may be any integer in $\mathbb{Z}$, but we require the others to be positive. Then the sequence of finite fractions converges, and the notation refers to its limit. We define two sequences $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ recursively by

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_k = a_k p_{k-1} + p_{k-2}, \\ q_{-1} &= 0, \quad q_0 = 1, \quad\ \ q_k = a_k q_{k-1} + q_{k-2}. \end{aligned}$$

The numbers p_k, q_k satisfy

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

In particular, p_k and q_k are coprime. Moreover, if $a_1, a_2, \ldots$ are positive integers, then for all $k \geq 0$, we have

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k].$$

The number p_k/q_k is called the k-th *convergent* of α.

For any irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we denote by $\lfloor \alpha \rfloor \in \mathbb{Z}$ the integer part of α, i.e., the largest integer $\leq \alpha$, by $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ the fractional part of α, and we define two sequences $(a_k)_{k\geq 0}$ and $(\alpha_k)_{k\geq 0}$ recursively by setting

$$a_0 = \lfloor \alpha \rfloor, \quad \alpha_0 = \{\alpha\}, \quad a_{k+1} = \left\lfloor \frac{1}{\alpha_k} \right\rfloor \quad \text{and} \quad \alpha_{k+1} = \left\{ \frac{1}{\alpha_k} \right\},$$

so that

$$\frac{1}{\alpha_k} = a_{k+1} + \alpha_{k+1}.$$

We then set $\beta_{-1} = 1$ and $\beta_k = \alpha_0\alpha_1 \ldots \alpha_k$.

It is well known that

$$\alpha = [a_0, a_1, \ldots, a_k, \ldots].$$

More precisely, we have the following formulas.

Proposition 1. *Let α be an irrational number and define the sequences $(a_k)_{k\geq 0}$, $(\alpha_k)_{k\geq 0}$, $(\beta_k)_{k\geq -1}$, $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ as above, so that*

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k].$$

Then, for $k \geq 0$, we have the formulas

$$\alpha = \frac{p_k + p_{k-1}\alpha_k}{q_k + q_{k-1}\alpha_k}, \quad q_k\alpha - p_k = (-1)^k\beta_k,$$

$$q_{k+1}\beta_k + q_k\beta_{k+1} = 1 \quad \textit{and} \quad \frac{1}{q_{k+1} + q_k} < \beta_k < \frac{1}{q_{k+1}}.$$

The last inequalities imply, for $k \geq 0$,

$$\frac{1}{2q_kq_{k+1}} < \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_kq_{k+1}}.$$

Moreover, for all $k \geq 0$,

$$\alpha_k = [0, a_{k+1}, a_{k+2}, \ldots]$$

2. The Yoccoz function

Definition 2 (The Yoccoz function and Bruno numbers)**.** *If α is an irrational number, we set*

$$\Phi(\alpha) = \sum_{k=0}^{+\infty} \beta_{k-1} \log \frac{1}{\alpha_k},$$

where α_k and β_k are defined as in section 1. If α is a rational number we set $\Phi(\alpha) = +\infty$. We say $\alpha \in \mathbb{R}$ is a Bruno number if $\Phi(\alpha) < +\infty$.

Remark. Observe that for any $k_0 \geq 0$, and all α irrational, we have

$$\Phi(\alpha) = \sum_{k=0}^{k_0-1} \beta_{k-1} \log \frac{1}{\alpha_k} + \beta_{k_0-1}\Phi(\alpha_{k_0}). \tag{1}$$

In [Y1], Yoccoz uses a modified version of continued fractions but we will not need that modification. The function Φ we will use is not exactly the same as the one introduced by Yoccoz, but the difference between the two functions is bounded (see [Y1], page 14).

For the next proposition, we will have to approximate α by sequences of irrational numbers. In order to avoid the confusion between such a sequence and the sequence $(\alpha_k)_{k\geq 0}$ introduced previously, we will denote the new sequence by $(\alpha[n])_{n\geq 0}$. One corollary of the following proposition is that the graph of Φ contains all the points (α, t) with $t \geq \Phi(\alpha)$.

Definition 3. *Given any Bruno number $\alpha = [a_0, a_1, \ldots]$, any real number $A \geq 1$ and any integer $n \geq 0$, we set*

$$\mathcal{T}(\alpha, A, n) = [a_0, a_1, \ldots, a_n, A_n, 1, 1, \ldots]$$

where $A_n = \lfloor A^{q_n} \rfloor$ is the integer part of A^{q_n}.

Proposition 2. *Let $\alpha \in \mathbb{R}$ be a Bruno number and $A \geq 1$ be a real number. For each integer $n \geq 0$, set $\alpha[n] = \mathcal{T}(\alpha, A, n)$. Then, $\alpha[n] \longrightarrow \alpha$ and*

$$\Phi(\alpha[n]) \underset{n\to\infty}{\longrightarrow} \Phi(\alpha) + \log A.$$

Proof. That $\alpha[n] \to \alpha$ is clear, since convergence of the entries in the continued fraction ensures convergence of the numbers themselves. For each integer $n \geq 0$, let us denote by $(\alpha_k[n])_{k\geq 0}$ and $(\beta_k[n])_{k\geq -1}$ the sequences associated to $\alpha[n]$. For each fixed k, we have

$$\lim_{n\to\infty} \alpha_k[n] = \alpha_k \quad \text{and} \quad \lim_{n\to\infty} \beta_k[n] = \beta_k.$$

In particular,

$$\lim_{n\to\infty} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} = \beta_{k-1} \log \frac{1}{\alpha_k}.$$

Observe that for $k \leq n$, the convergents of $\alpha[n]$ and α are the same, namely p_k/q_k. Hence, if $0 < k \leq n-1$, we have by Proposition 1

$$\beta_{k-1}[n] < \frac{1}{q_k} \quad \text{and} \quad \frac{1}{\alpha_k[n]} \leq \frac{1}{\beta_k[n]} < 2q_{k+1}.$$

It follows that when $0 < k \leq n-1$, we have

$$\beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} < \frac{\log 2}{q_k} + \frac{\log q_{k+1}}{q_k}.$$

The right terms form a convergent series since α is a Bruno number. Thus, as a function of k, the pointwise convergence with respect to n of the summand $\beta_{k-1}[n] \log 1/\alpha_k[n]$ is dominated. Therefore we have

$$\sum_{k=0}^{n-1} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} \underset{n\to\infty}{\longrightarrow} \sum_{k=0}^{+\infty} \beta_{k-1} \log \frac{1}{\alpha_k} = \Phi(\alpha). \tag{2}$$

We will now estimate the term $\beta_{n-1}[n] \log(1/\alpha_n[n])$ in the Yoccoz function. First, observe that

$$\frac{1}{\alpha_n[n]} = A_n + \frac{1}{\theta},$$

where $\theta = [1, 1, 1, \ldots] = (\sqrt{5}+1)/2$ is the golden mean. If $A = 1$, then $A_n = 1$ and we trivially get

$$\beta_{n-1}[n] \log \frac{1}{\alpha_n[n]} \underset{n\to\infty}{\to} 0.$$

Let us now assume that $A > 1$. As $n \to \infty$, we have $\log A_n \sim q_n \log A$ and thus,

$$\beta_{n-1}[n] \log \frac{1}{\alpha_n[n]} \underset{n\to\infty}{\sim} \beta_{n-1}[n] q_n \log A.$$

We know that $\beta_{n-1}[n]q_n \in (1/2, 1)$ and we would like to prove that in our case, this sequence tends to 1. Observe that

$$\beta_{n-1}[n]q_n = 1 - \beta_n[n]q_{n-1} = 1 - \alpha_n[n] \cdot \frac{q_{n-1}}{q_n} \cdot \beta_{n-1}[n]q_n,$$

so

$$\beta_{n-1}[n]q_n = \frac{q_n}{q_n + \alpha_n[n]q_{n-1}},$$

which clearly tends to 1 as $n \to \infty$. As a consequence, $\beta_{n-1}[n]q_n \longrightarrow 1$ and thus,

$$\beta_{n-1}[n] \log \frac{1}{\alpha_n[n]} \underset{n\to\infty}{\longrightarrow} \log A. \tag{3}$$

Finally, we have $\alpha_{n+1}[n] = 1/\theta$ and thus by (1)

$$\sum_{k=n+1}^{+\infty} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} = \beta_n[n] \cdot \Phi\left(\frac{1}{\theta}\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{4}$$

Combining the limits (2), (3) and (4) gives the required result. ∎

Remark. The above proof shows that instead of using a sequence of the form $\alpha[n] = \mathcal{T}(\alpha, A, n)$, we could have taken any sequence $\alpha[n] = [a_0, \ldots, a_n, A_n, \theta_n]$, where A_n are positive integers such that

$$A_n^{1/q_n} \longrightarrow A$$

and $\theta_n > 1$ are Bruno numbers such that

$$\Phi(1/\theta_n) = o(q_n A_n).$$

3. A first proof of the Main Lemma

In this section, we give a first proof of the Main Lemma based on Corollaries 3 and 5 below.

Let α be a Bruno number and choose $r_1 < r_\alpha$. For all $n \geq 1$, set $\alpha[n] = \mathcal{T}(\alpha, r/r_1, n)$ (see Definition 3). Then,

$$\Phi(\alpha[n]) \longrightarrow \Phi(\alpha) + \log \frac{r}{r_1} \quad \text{and} \quad \left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \longrightarrow \frac{r_1}{r}.$$

The first limit is proved in Proposition 2, the second follows from

$$\left|\alpha[n] - \frac{p_n}{q_n}\right| = \frac{|\beta_n|}{q_n} = \frac{\alpha_n[n]}{q_n^2} \cdot \beta_{n-1}[n]q_n \sim \frac{1}{q_n^2 A_n}$$

with $A_n = \lfloor (r/r_1)^{q_n} \rfloor$.

According to Corollary 3, we have $\liminf\limits_{n\to\infty} r_{\alpha[n]} \geq r_1$ and according to corollary 5, we have $\limsup\limits_{n\to\infty} r_{\alpha[n]} \leq r_1$. This proves the Main Lemma.

Figure 3 shows the boundary of the Siegel disks for $\alpha = (\sqrt{5}+1)/2 = [1, 1, 1, \ldots]$, and $\alpha[n]$, $n = 5, \ldots, 8$ with $A_n = \lfloor 1.5^{q_n} \rfloor$. The reader should try to convince himself that as n grows, this boundary oscillates more and more between $\partial\Delta_\alpha$ and $\phi_\alpha(\partial B(0, 2r_\alpha/3))$ both of which appear in the last frame.

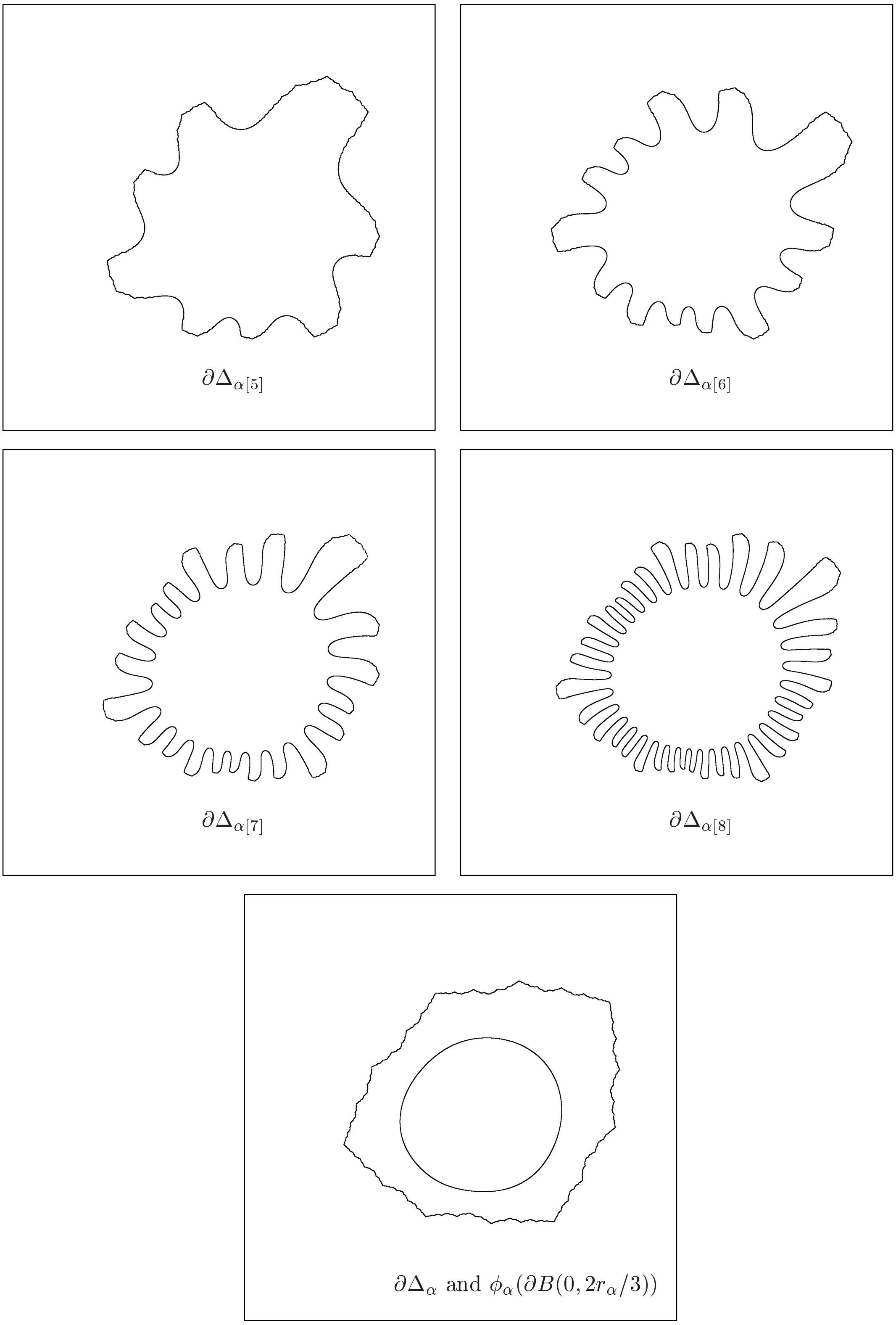

FIGURE 3. Some boundaries of Siegel disks for a sequence $\alpha[n]$.

4. A second proof of the Main Lemma

In this section, we give a second proof of the main lemma based on Corollary 4 (it is possible to extract a proof from Proposition 10 of [R]) and on Yoccoz's Theorem on the optimality of the Bruno condition for the linearization problem in the quadratic family [Y1].

Let $p_n/q_n \in \mathbb{Q}$ be an increasing sequence converging to α. Let

$$\alpha[n] = \inf\{\theta \in (p_n/q_n, \alpha] \setminus \mathbb{Q} \text{ such that } r_\theta \geq r_1\} \in [p_n/q_n, \alpha].$$

Notice that $\alpha[n] \to \alpha$ and $r_{\alpha[n]} \geq r_1$ (see the discussion after the statement of the Main Lemma). In particular, $P_{\alpha[n]}$ is linearizable. In order to prove the Main Lemma, it is enough to show that $r_{\alpha[n]} \leq r_1$ for every n.

By Yoccoz's Theorem on the optimality of the Bruno condition for the linearization problem in the quadratic family, we know that $\alpha[n]$ is actually a Bruno number. Let $\{\theta_m\}$ be an increasing sequence of Bruno numbers in $(p_n/q_n, \alpha[n])$ converging to $\alpha[n]$ and satisfying $\lim \Phi(\theta_m) = \Phi(\alpha[n])$ (the existence of such a sequence follows from Proposition 2 with $A = 1$, see also Proposition 1 of [R]). By the definition of $\alpha[n]$, we have $r_{\theta_m} < r_1$, and by Corollary 4, $\lim\limits_{m\to\infty} r_{\theta_m} = r_{\alpha[n]}$, so $r_{\alpha[n]} \leq r_1$.

Remark. Lukas Geyer has given an alternative argument for the key estimate $\limsup\limits_{\theta\to\alpha[n]-} r_\theta \geq r_{\alpha[n]}$, which is the "hard" property of the conformal radius of quadratic polynomials exploited above, as opposed to the "soft" property of upper semicontinuity. It is based on the fact that the function $\alpha \mapsto \log r_\alpha \in [-\infty, \infty)$ (extended as $-\infty$ to $\mathbb{Q}$) is the boundary value of a harmonic function defined on the upper half plane which is bounded from above (see [Y1]).

5. Semi-continuity with loss for Siegel disks

5.1. **Normalized statements.** We will bound from below the size of Siegel disks of perturbations of rotations on the unit disk. We will use a theorem due to Yoccoz [Y1] and generalize a theorem independently due to Risler [R] and Chéritat [C].

Definition 4. *For any irrational $\alpha \in (0,1)$, let $\mathcal{O}_\alpha$ be the set of holomorphic functions f defined in an open subset of $\mathbb{D}$ containing 0, which satisfy $f(0) = 0$ and $f'(0) = e^{2i\pi\alpha}$. We define $\mathcal{S}_\alpha$ as the set of functions $f \in \mathcal{O}_\alpha$ which are defined and univalent on $\mathbb{D}$.*

Given $f \in \mathcal{O}_\alpha$, consider the set K_f of points in $\mathbb{D}$ whose infinite forward orbit under iteration of f is defined. The map f is linearizable at 0 if and only if 0 belongs to the interior of K_f. In that case, the connected component of the interior of K_f which contains 0 is the Siegel disk Δ_f for f (as defined at the beginning of the introduction). We denote by $\mathrm{inrad}(\Delta_f)$ the radius of the largest disk centered at 0 and contained in Δ_f.

Theorem 1 (Yoccoz)**.** *There exists a universal constant C_0 such that for any Bruno number α and any function $f \in \mathcal{S}_\alpha$,*

$$\mathrm{inrad}(\Delta_f) \geq \exp(-\Phi(\alpha) - C_0).$$

Remark. The function Φ defined by Yoccoz in [Y1] is not exactly the same as the one we defined in this article, but the difference between the two functions is bounded by a universal constant, so that Theorem 1 holds as stated here.

In the following, when we say that a sequence of functions f_n converges uniformly on compact subsets of $\mathbb{D}$ to a function f, we do not require the f_n to be defined on $\mathbb{D}$. We only ask that any compact set $K \subset \mathbb{D}$ be contained in the domain of f_n for n large enough. In this case, we write $f_n \rightrightarrows f$ on $\mathbb{D}$.

Theorem 2 (Risler-Chéritat)**.** *Assume α is a Bruno number, $f_n \in \mathcal{O}_\alpha$ and $f_n \rightrightarrows R_\alpha$ on $\mathbb{D}$. Then,*

$$\lim_{n\to\infty} \operatorname{inrad}(\Delta_{f_n}) = 1.$$

Our goal is to generalize this result as follows.

Theorem 3. *Assume $(\alpha[n])_{n\geq 0}$ is a sequence of Bruno numbers converging to a Bruno number α such that*

$$\limsup_{n\to\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C$$

for some constant $C \geq 0$. Assume $f_n \in \mathcal{O}_{\alpha[n]}$ with $f_n \rightrightarrows R_\alpha$ on $\mathbb{D}$. Then,

$$\liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n}) \geq e^{-C}.$$

The proof will be given in section 5.3.

Corollary 3. *Under the same assumption on α and $(\alpha[n])_{n\geq 0}$ as in Theorem 3, we have*

$$\liminf_{n\to\infty} r_{\alpha[n]} \geq r_\alpha e^{-C}.$$

Proof. Since $\alpha[n] \to \alpha$, we have $P_{\alpha[n]} \to P_\alpha$ uniformly on compact subsets of $\mathbb{C}$. Let us consider the maps

$$f_n(z) = \frac{1}{r_\alpha}\phi_\alpha^{-1} \circ P_{\alpha[n]} \circ \phi_\alpha(r_\alpha z).$$

Then, $f_n \in \mathcal{O}_{\alpha[n]}$ and $f_n \rightrightarrows R_\alpha$ on $\mathbb{D}$. We can now apply Theorem 3. ∎

Corollary 4. *Assume α is a Bruno number and $\alpha[n]$ is a sequence of Bruno numbers such that $\Phi(\alpha[n]) \to \Phi(\alpha)$. Then, $r_{\alpha[n]} \to r_\alpha$.*

Proof. By Corollary 3 with $C = 0$, we know that if $\Phi(\alpha[n]) \to \Phi(\alpha)$, then

$$\liminf_{n\to\infty} r_{\alpha[n]} \geq r_\alpha.$$

As mentioned in the introduction after the statement of the Main Lemma, the conformal radius depends upper semicontinuously on α, and so, $r_{\alpha[n]} \to r_\alpha$. ∎

5.2. **The Douady-Ghys renormalization.** In this section, we describe a renormalization construction introduced by Douady [D] and Ghys. This construction is at the heart of Yoccoz's proof of Theorem 1. We adapt this construction to our setting, i.e, to maps which are univalent on $\mathbb{D}$ and close to a rotation.

Step 1. Construction of a Riemann surface. Consider a map $f \in \mathcal{S}_\alpha$. Let $\mathbb{H}$ be the upper half plane. There exists a unique lift $F : \mathbb{H} \to \mathbb{C}$ of f such that

$$e^{2i\pi F(Z)} = f(e^{2i\pi Z}) \quad \text{and} \quad F(Z) = Z + \alpha + u(Z),$$

where u is holomorphic, $\mathbb{Z}$-periodic and $u(Z) \to 0$ as $\operatorname{Im}(Z) \to \infty$.

Definition 5. *For $\delta > 0$, and $0 < \alpha < 1$ we define $\mathcal{S}_\alpha^\delta$ as the set of functions $f \in \mathcal{S}_\alpha$ such that for all $Z \in \mathbb{H}$,*

$$|u(Z)| < \delta\alpha \quad \text{and} \quad |u'(Z)| < \delta.$$

Remark. a) If $\delta < 1$, the condition $|u'(Z)| < \delta$ implies that F has a continuous and injective extension to $\overline{\mathbb{H}}$, and so, f has a continuous and injective extension to $\overline{\mathbb{D}}$.

b) Given $\alpha \in (0,1)$ and $\delta \in (0,1/2)$, if $f \in \mathcal{S}_\alpha$, and if $|f(z) - e^{2i\pi\alpha}z| < \delta\alpha$ and $|f'(z) - e^{2i\pi\alpha}| < \delta/4$ on $\mathbb{D}$, then $f \in \mathcal{S}_\alpha^\delta$.

We now assume that $\delta \in (0,1/2)$ and $f \in \mathcal{S}_\alpha^\delta$. Set $L_0 = i\mathbb{R}^+$ and $L_0' = F(L_0)$. Note that for all $Z \in \mathbb{H}$, $F(Z)$ belongs to the disk centered at $Z+\alpha$ with radius $\delta\alpha$. It follows that the angle between the horizontal and the segment $[Z, F(Z)]$ is less than $\arcsin(\delta) < \pi/6$. Moreover, for all $Z \in \mathbb{H}$, we have $|\arg(F'(Z))| < \arcsin(\delta)$. So the tangents to the smooth curve L_0' make an angle of less than $\pi/6$ with the vertical. This implies that the union $L_0 \cup [0, F(0)] \cup L_0' \cup \{\infty\}$ forms a Jordan curve in the Riemann sphere bounding a region U such that for $Y > 0$, the segment $[iY, F(iY)]$ is contained in $\overline{U}$. We set $\mathcal{U} = U \cup L_0$.

Denote by B_0 the half-strip

$$B_0 = \{Z \in \mathbb{H} \mid 0 < \mathrm{Re}(Z) < 1\}$$

and consider the map $H : \overline{B_0} \to \overline{\mathcal{U}}$ defined by

$$H(Z) = (1-X)i\alpha Y + XF(i\alpha Y) = \alpha Z + Xu(i\alpha Y)$$

where $Z = X + iY$, $(X,Y) \in [0,1] \times [0,+\infty)$. Then,

$$\frac{\partial H}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial H}{\partial X} + i\frac{\partial H}{\partial Y}\right) = \frac{1}{2}\big(u(i\alpha Y) - \alpha X u'(i\alpha Y)\big)$$

and

$$\frac{\partial H}{\partial Z} = \frac{1}{2}\left(\frac{\partial H}{\partial X} - i\frac{\partial H}{\partial Y}\right) = \alpha + \frac{1}{2}\big(u(i\alpha Y) + \alpha X u'(i\alpha Y)\big).$$

It follows that

$$\left|\frac{\partial H}{\partial \overline{Z}}\right| < \alpha\delta \quad \text{and} \quad \left|\frac{\partial H}{\partial Z}\right| > \alpha(1-\delta)$$

and since $\delta < 1/2$, H is a K_δ-quasiconformal homeomorphism between B_0 and $\mathcal{U}$, with $K_\delta = 1/(1-2\delta)$.

If we glue the sides L_0 and L_0' of $\overline{\mathcal{U}}$ via F, we obtain a topological surface $\overline{\mathcal{V}}$. We denote by $\iota : \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ the canonical projection. The space $\overline{\mathcal{V}}$ is a topological surface homeomorphic to a closed 2-cell with a puncture with the boundary $\partial\mathcal{V} = \iota([0, F(0)])$. We set $\mathcal{V} = \overline{\mathcal{V}} \setminus \partial\mathcal{V}$. Since the gluing map F is analytic, the surface $\mathcal{V}$ has a canonical analytic structure induced by that of $\mathcal{U}$ (see [C] page 70 or [Y1] for details).

When $Z \in L_0$, $H(Z+1) = F(H(Z))$, and so the homeomorphism $H : \overline{B}_0 \to \overline{\mathcal{U}}$ induces a homeomorphism between the half cylinder $\mathbb{H}/\mathbb{Z}$ and the Riemann surface $\mathcal{V}$. This homeomorphism is clearly quasiconformal on the image of B_0 in $\mathbb{H}/\mathbb{Z}$, i.e., outside an $\mathbb{R}$-analytic curve. It is therefore quasiconformal in the whole half cylinder ($\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms). Therefore, there exists an analytic isomorphism between $\mathcal{V}$ and $\mathbb{D}^*$, which, by a theorem of Carathéodory, extends to a homeomorphism between $\partial\mathcal{V}$ and $\partial\mathbb{D}$. Let $\phi : \overline{\mathcal{V}} \to \overline{\mathbb{D}}^*$ be such an isomorphism and let $\mathcal{K} : \overline{\mathcal{U}} \to \overline{\mathbb{H}}$ be a lift of $\phi \circ \iota$ by the exponential map $Z \mapsto \exp(2i\pi Z) : \overline{\mathbb{H}} \to \overline{\mathbb{D}}^*$. The map $\mathcal{K}$ is unique up to post-composition with a real translation. We choose ϕ and $\mathcal{K}$ such that $\mathcal{K}(0) = 0$. By construction, if $Z \in L_0$, then

$$\mathcal{K}(F(Z)) = \mathcal{K}(Z) + 1.$$

Step 2. The renormalized map. Let us now set

$$\mathcal{U}' = \{Z \in \mathcal{U} \mid \operatorname{Im}(Z) > 5\delta\}, \quad \overline{\mathcal{V}'} = \iota(\overline{\mathcal{U}'})$$

and let $\mathcal{V}'$ be the interior of $\overline{\mathcal{V}'}$.

Let us consider a point $Z \in \mathcal{U}'$. The segment $[Z-1, Z]$ intersects neither L_0' nor $[0, F(0)]$. So either $Z-1 \in \mathcal{U}$ or $\operatorname{Re}(Z-1) < 0$. For $m \geq 0$, the iterates

$$Z_m \stackrel{\text{def}}{=} F^{\circ m}(Z-1)$$

stay above the line starting at $Z-1$ and going down with a slope $\tan(\arcsin(\delta))$ ($< 2\delta$ when $\delta < 1/2$), as long as $Z_m \in \mathbb{H}$. Since $\operatorname{Re}(Z-1) \geq -1$ and $\operatorname{Im}(Z-1) > 5\delta$, there exists a least integer $n \geq 0$ such that Z_n is defined and $\operatorname{Re}(Z_n) \geq 0$.

Let us show that $Z_n \in \mathcal{U}$. If $Z-1 \in \mathcal{U}$, then $n = 0$ and there is nothing to prove. Otherwise, $n \geq 1$ and $\operatorname{Re}(Z_{n-1}) < 0$. Since Z_{n-1} is above the line starting at $Z-1$ and going down with a slope 2δ, we have $\operatorname{Im}(Z_{n-1}) > 3\delta$. Consider the horizontal segment I joining Z_{n-1} to L_0. Let J be its image under F. Since $|F'(Z) - 1| < \delta < 1/2$, J is a curve whose tangents make an angle less than $\pi/6$ with the horizontal. Thus, J is to the right of Z_n and in particular, to the right of L_0. Moreover, the tangents of L_0' make an angle less than $\pi/6$ with the vertical. So, J joins Z_n to L_0' and remains to the left of L_0'. Finally, points in I have imaginary part greater than 3δ, and since $|F(Z) - Z - \alpha| < \delta\alpha < \delta$, points in J have imaginary part greater than 2δ. Thus, J does not hit the segment $[0, F(0)]$. It follows that $Z_n \in \mathcal{U}$. Now define a "first-return map" $G : \mathcal{U}' \to \mathcal{U}$ by setting $G(Z) = Z_n$. Note that G is a priori discontinuous since it is a priori defined by part, using different iterates. Figure 4 shows the construction of the map G.

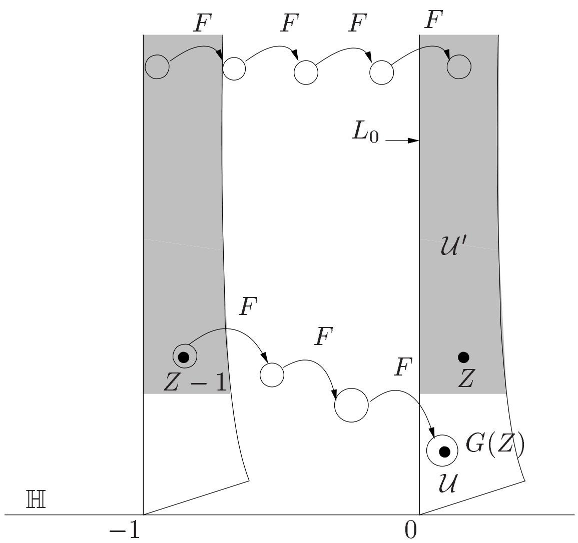

FIGURE 4. The regions $\mathcal{U}$, $\mathcal{U}'$ and the map $G : \mathcal{U}' \to \mathcal{U}$.

The map $G : \mathcal{U}' \to \mathcal{U}$ induces a univalent map $g : \phi(\mathcal{V}') \to \mathbb{D}^*$ such that $g \circ \phi \circ \iota = \phi \circ \iota \circ G$. (The fact that g is univalent is not completely obvious; see [Y1] for details). We define the renormalization of f by

$$\mathcal{R}(f) : z \mapsto \overline{g(\overline{z})}.$$

By the removable singularity theorem, this map extends holomorphically to the origin once we set $\mathcal{R}(f)(0) = 0$, and it is possible to show that $[\mathcal{R}(f)]'(0) = e^{2i\pi/\alpha}$ (again, see [Y1] for details). Thus, $\mathcal{R}(f) \in \mathcal{O}_{\alpha_1}$, where α_1 denotes the fractional part of $1/\alpha$. This completes the description of the renormalization operator.

5.3. **The proof of Theorem 3.** Let us now assume that $(\alpha[n])_{n\geq 0}$ is a sequence of Bruno numbers converging to a Bruno number α such that

$$\limsup_{n\to\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C$$

for some constant $C \geq 0$. We define the sequences $(\alpha_k)_{k\geq 0}$, $(\beta_k)_{k\geq -1}$, $(\alpha_k[n])_{k\geq 0}$ and $(\beta_k[n])_{k\geq -1}$ as in section 1.

Lemma 1. *For all $k \geq 0$, we have*

$$\limsup_{n\to\infty} \Phi(\alpha_k[n]) \leq \Phi(\alpha_k) + \frac{C}{\beta_{k-1}}.$$

Proof. We have by (1)

$$\begin{aligned}\Phi(\alpha[n]) - \Phi(\alpha) \quad &= \quad \sum_{j=0}^{k-1} \left(\beta_{j-1}[n] \log \frac{1}{\alpha_j[n]} - \beta_{j-1} \log \frac{1}{\alpha_j} \right) \\ &\quad\quad + \beta_{k-1}[n]\Phi(\alpha_k[n]) - \beta_{k-1}\Phi(\alpha_k).\end{aligned}$$

For each fixed $j \geq 0$, we have $\alpha_j[n] \to \alpha_j$ and $\beta_j[n] \to \beta_j$ as $n \to \infty$, and so,

$$\lim_{n\to\infty} \beta_{j-1}[n] \log \frac{1}{\alpha_j[n]} = \beta_{j-1} \log \frac{1}{\alpha_j}.$$

Thus,

$$\begin{aligned} C \quad &\geq \quad \limsup_{n\to\infty} \Phi(\alpha[n]) - \Phi(\alpha) \\ &= \quad \limsup_{n\to\infty} \beta_{k-1}[n]\Phi(\alpha_k[n]) - \beta_{k-1}\Phi(\alpha_k) \\ &= \quad \beta_{k-1} \left(\limsup_{n\to\infty} \Phi(\alpha_k[n]) - \Phi(\alpha_k) \right). \end{aligned}$$

∎

Now, for all $k \geq 0$, we set

$$\rho_k = \inf \left\{ \liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n}) \right\},$$

where the infimum is taken over all sequences $\big(f_n \in \mathcal{O}_{\alpha_k[n]}\big)_{n\geq 0}$ such that $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$. Similarly, we set

$$\rho'_k = \inf \left\{ \liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n}) \right\},$$

where the infimum is taken over all sequences $\left(f_n \in \mathcal{S}^{\delta_n}_{\alpha_k[n]}\right)_{n\geq 0}$ such that $\delta_n \to 0$ (note that this implies $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$). It is easy to check that each infimum is realized for some sequence f_n. We will show that

$$\log \rho_0 \geq -C,$$

which is a restatement of Theorem 3.

Lemma 2. *For all $k \geq 0$, we have $\rho_k = \rho'_k$.*

Proof. We clearly have $\rho'_k \geq \rho_k$ since $\mathcal{S}^{\delta_n}_{\alpha_k[n]} \subset \mathcal{O}_{\alpha_k[n]}$. Now, assume $(\delta_n)_{n\geq 0}$ and $(f_n \in \mathcal{O}_{\alpha_k[n]})_{n\geq 0}$ are sequences such that $\delta_n \to 0$ and $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$. Then, we can find a sequence of real numbers $\lambda_n < 1$ such that $\lambda_n \to 1$ and

$$g_n : z \mapsto \frac{1}{\lambda_n} f_n(\lambda_n z)$$

belongs to $\mathcal{S}^{\delta_n}_{\alpha_k[n]}$. The Siegel disk Δ_{f_n} contains $\lambda_n \Delta_{g_n}$. Therefore

$$\liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n}) \geq \liminf_{n\to\infty} \lambda_n \operatorname{inrad}(\Delta_{g_n}) \geq \rho'_k.$$

This shows that $\rho_k \geq \rho'_k$. ■

Lemma 3. *For all $k \geq 0$, we have*

$$\log \rho_k \geq -\Phi(\alpha_k) - \frac{C}{\beta_{k-1}} - C_0,$$

where C_0 is the universal constant provided by Theorem 1. In particular, $\rho_k > 0$.

Proof. Indeed, Theorem 1 implies that when $f_n \in \mathcal{S}_{\alpha_k[n]}$, then

$$\log \operatorname{inrad}(\Delta_{f_n}) \geq -\Phi(\alpha_k[n]) - C_0.$$

Since by Lemma 1

$$\limsup_{n\to\infty} \Phi(\alpha_k[n]) \leq \Phi(\alpha_k) + \frac{C}{\beta_{k-1}},$$

the lemma follows. ■

Let us now fix some $k \geq 0$. Assume that $\left(f_n \in \mathcal{S}^{\delta_n}_{\alpha_k[n]}\right)_{n\geq 0}$ is a sequence of functions such that $\delta_n \to 0$. Then, $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$ and for large n, $\delta_n < 1/2$. So, we can perform the Douady-Ghys renormalization. We lift $f_n : \mathbb{D} \to \mathbb{C}$ to a map $F_n : \mathbb{H} \to \mathbb{C}$ via $\pi : Z \mapsto \exp(2i\pi Z)$:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "F_n"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{D}^* \arrow[r, "f_n"'] & \mathbb{C}^* \end{tikzcd}$$

We similarly define $\mathcal{U}_n$, $\mathcal{U}'_n$, $\mathcal{V}_n$, $\iota_n : \overline{\mathcal{U}_n} \to \overline{\mathcal{V}_n}$, $H_n : \overline{B_0} \to \overline{\mathcal{U}_n}$, $\phi_n : \overline{\mathcal{V}_n} \to \overline{\mathbb{D}}^*$ and $\mathcal{K}_n : \overline{\mathcal{U}_n} \to \overline{\mathbb{H}}$. Recall that H_n conjugates the translation $T_1 : Z \mapsto Z+1$ (from the left boundary of B_0 to the right boundary of B_0) to F_n (from the left boundary of $\mathcal{U}_n$ to the right boundary of $\mathcal{U}_n$).

$$\begin{tikzcd} (B_0, T_1) \arrow[d, "H_n"'] & \\ (\mathcal{U}_n, F_n) \arrow[r, "\mathcal{K}_n"] \arrow[d, "\iota_n"'] & (\mathbb{H}, T_1) \arrow[d, "\pi"] \\ (\mathcal{V}_n, \mathrm{Id}) \arrow[r, "\phi_n"'] & (\mathbb{D}^*, \mathrm{Id}). \end{tikzcd}$$

Then, we define a "first-return map" $G_n : \mathcal{U}'_n \to \mathcal{U}_n$ which induces a univalent map g_n such that $g_n \circ \phi_n \circ \iota_n = \phi_n \circ \iota_n \circ G_n$ defined on D^*_n, the interior of $\phi_n \circ \iota_n(\overline{\mathcal{U}'_n})$.

$$\begin{tikzcd} \mathcal{U}'_n \arrow[r, "G_n"] \arrow[d, "\phi_n\circ\iota_n"'] & \mathcal{U}_n \arrow[d, "\phi_n\circ\iota_n"] \\ D^*_n \arrow[r, "g_n"'] & \mathbb{D}^*. \end{tikzcd}$$

The renormalized map is

$$\mathcal{R}(f_n) : z \mapsto \overline{g_n(\overline{z})}.$$

Note that $\mathcal{R}(f_n)$ belongs to $\mathcal{O}_{\alpha_{k+1}[n]}$ and not to $\mathcal{S}_{\alpha_{k+1}[n]}$.

Lemma 4. *The maps $\phi_n \circ \iota_n : \mathcal{U}_n \to \overline{\mathbb{D}}^*$ converge to $Z \mapsto e^{2i\pi Z/\alpha_k}$ uniformly on compact subsets of $B_{\alpha_k} = \{Z \in \mathbb{H} \mid 0 \le \mathrm{Re}(Z) < \alpha_k\}$.*

Proof. The lifts F_n converge to the translation $Z \mapsto Z + \alpha_k$. It follows that the K_{δ_n}-quasiconformal homeomorphisms $H_n : \overline{B_0} \to \overline{\mathcal{U}_n}$ converge to the scaling map $Z \mapsto \alpha_k Z$ uniformly on $\overline{B_0}$. Moreover, $\mathcal{K}_n \circ H_n : \overline{B_0} \to \mathcal{K}_n(\overline{\mathcal{U}_n})$ is a K_{δ_n}-quasiconformal homeomorphism which satisfies $\mathcal{K}_n \circ H_n(Z+1) = \mathcal{K}_n \circ H_n(Z) + 1$ for $Z \in i\mathbb{R}^+$ and sends 0 to 0. Therefore, it extends by periodicity to a K_{δ_n}-quasiconformal automorphism of $\overline{\mathbb{H}}$ fixing 0, 1 and ∞ (the extension is quasiconformal outside $\mathbb{Z} + i\mathbb{R}^+$, and thus it is quasiconformal on $\mathbb{H}$ since $\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms). Since $K_{\delta_n} = 1/(1-2\delta_n) \to 1$ as $n \to \infty$, we see that $\mathcal{K}_n \circ H_n$ converges uniformly on compact subsets of $\overline{\mathbb{H}}$ to the identity as $n \to \infty$. As a consequence, the maps $\mathcal{K}_n$ converge to $Z \mapsto Z/\alpha_k$ uniformly on compact subsets of B_{α_k}. So, the maps $\phi_n \circ \iota_n : \mathcal{U}_n \to \overline{\mathbb{D}}^*$ converge to $Z \mapsto e^{2i\pi Z/\alpha_k}$ uniformly on compact subsets of B_{α_k}. ■

Lemma 5. *For all $k \ge 0$, we have*

$$\log \rho_k \ge \alpha_k \log \rho_{k+1}.$$

Proof. Let us assume that $\rho_k < 1$ since otherwise, the result is obvious. Let us choose a sequence $\delta_n \to 0$ and a sequence of functions $f_n \in \mathcal{S}^{\delta_n}_{\alpha_k[n]}$ which converge to the rotation R_{α_k} and such that

$$\rho_k = \lim_{n\to\infty} \mathrm{inrad}(\Delta_{f_n}).$$

Then, we can find a sequence of points $z_n \in \mathbb{D}$ such that $|z_n| \to \rho_k$ and the orbit of z_n under iteration of f_n escapes from $\mathbb{D}$. By conjugating f_n with a rotation fixing 0 if necessary, we may assume that $z_n \in (0,1)$. Let us consider the points $Z_n \in i\mathbb{R}^+$ such that $e^{2i\pi Z_n} = z_n$. Then, $\mathrm{Im}(Z_n) \to -\log(\rho_k)/2\pi$. Since $\delta_n \to 0$, it follows that for n large enough, $Z_n \in \mathcal{U}'_n$.

Recall that by Lemma 3, $\rho_k > 0$. So Z_n remains in a compact subset of $B_{\alpha_k} = \{Z \in \mathbb{H} \mid 0 \le \mathrm{Re}(Z) < \alpha_k\}$. Thus, Lemma 4 implies that for n large enough, the point $z'_n = \phi_n \circ \iota_n(Z_n)$ is close to $e^{2i\pi Z_n/\alpha_k}$. In particular, we see that for n large enough, $|z'_n| \to \rho_k^{1/\alpha_k}$. Moreover, since the orbit of z_n escapes from $\mathbb{D}$ under iteration of f_n, the orbit of Z_n under iteration of F_n escapes from $\mathbb{H}$ and thus the orbit of z'_n under iteration of $\mathcal{R}(f_n)$ escapes from $\mathbb{D}$. It follows that

$$\log \rho_k = \lim_{n\to\infty} \log |z_n| = \alpha_k \lim_{n\to\infty} \log |z'_n| \ge \alpha_k \liminf_{n\to\infty} \log \mathrm{inrad}\left(\Delta_{\mathcal{R}(f_n)}\right).$$

But Lemma 4 also implies that the sequence $(\mathcal{R}(f_n))_{n\geq 0}$ converges to the rotation $R_{\alpha_{k+1}}$ uniformly on compact subsets of $\mathbb{D}$. The definition of ρ_{k+1} implies that

$$\liminf_{n\to\infty} \log \operatorname{inrad}(\Delta_{\mathcal{R}(f_n)}) \geq \log \rho_{k+1},$$

and this completes the proof. ■

The proof of Theorem 3 is now completed easily. Indeed, we see by induction that for all $k \geq 0$, we have

$$\log \rho_0 \geq \alpha_0 \cdots \alpha_k \log \rho_{k+1} = \beta_k \log \rho_{k+1}.$$

And by Lemma 3, we get

$$\log \rho_0 \geq -\beta_k \Phi(\alpha_{k+1}) - C - \beta_k C_0.$$

We clearly have

$$\lim_{k\to\infty} \beta_k C_0 = 0.$$

Moreover, the first term on the right side is the tail of the series defining $\Phi(\alpha)$ (see 1). This series converges, and so,

$$\lim_{k\to\infty} \beta_k \Phi(\alpha_{k+1}) = 0.$$

6. Parabolic explosion for quadratic polynomials

From now on, in the notation p/q for a rational number, we imply that p and q are coprime with $q > 0$.

Let us fix a rational number p/q. Then, 0 is a parabolic fixed point of the quadratic polynomial $P_{p/q} : z \mapsto e^{2i\pi p/q} z + z^2$. It is known (see [DH1], chapter IX) that there exists a complex number $A \in \mathbb{C}^*$ such that

$$P_{p/q}^{\circ q}(z) = z + A z^{q+1} + \mathcal{O}(z^{q+2}).$$

This number should not be mistaken for the formal invariant of the parabolic germ, i.e., the residue of the 1-form $\dfrac{dz}{z - P_{p/q}^{\circ q}(z)}$ at 0.

Definition 6. *For each rational number p/q, let us denote by $A(p/q)$ the coefficient of z^{q+1} in the power series at 0 of $P_{p/q}^{\circ q}$.*

Definition 7. *Let $\mathcal{P}_q$ be the set of parameters $\alpha \in \mathbb{C}$ such that $P_\alpha^{\circ q}$ has a parabolic fixed point with multiplier 1. For each rational number p/q, set*

$$R_{p/q} = dist\left(\frac{p}{q}, \mathcal{P}_q \setminus \left\{\frac{p}{q}\right\}\right).$$

Remark. Note that we consider complex perturbations of p/q: $P_\alpha : z \mapsto e^{2i\pi\alpha} z + z^2$.

Proposition 2 in [BC2] (see also proposition 2.3 part 1 in [C]) asserts that for all rational numbers p/q,

$$R_{p/q} \geq \frac{1}{q^3}. \tag{5}$$

When $\alpha \neq p/q$ is a small perturbation of p/q, 0 becomes a simple fixed point of P_α and $P_\alpha^{\circ q}$ has q other fixed points close to 0. The dependence of these fixed points on α is locally holomorphic when α is not in $\mathcal{P}_q$. If we add p/q, we get a holomorphic dependence on the q-th root of the perturbation $\alpha - p/q$. The following proposition corresponds to proposition 2.2 part 1 in [C] (compare with [BC2] Proposition 1).

Proposition 3. *For each rational number p/q, there exists a holomorphic function $\chi : B = B(0, R_{p/q}^{1/q}) \to \mathbb{C}$ with the following properties:*

(1) $\chi(0) = 0$,
(2) $\chi'(0)^q = -\dfrac{2\pi i q}{A(p/q)} \neq 0$,
(3) *for every $\delta \in B \setminus \{0\}$, $\left\langle \chi(\delta), \chi(\zeta\delta), \dots, \chi(\zeta^{q-1}\delta)\right\rangle$ forms a cycle of period q of P_α with $\zeta = e^{2i\pi p/q}$ and $\alpha = \dfrac{p}{q} + \delta^q$. In other words,*

$$\chi(\zeta\delta) = P_\alpha(\chi(\delta)) \quad \textit{for every } \delta \in B.$$

Moreover, any function satisfying the above conditions is of the form $\delta \mapsto \chi(\zeta^k\delta)$ for some $k \in \{0, \dots, q-1\}$.

In this article, we prefer to normalize χ differently. We will use the symbol ψ for the new function, and define it by $\psi(\delta) = \chi(\delta/\chi'(0))$ wherever it is defined. This amounts to replacing the relation $\alpha = \frac{p}{q} + \delta^q$ by

$$\alpha = \frac{p}{q} - \frac{A(p/q)}{2i\pi q}\delta^q.$$

There are two advantages in doing this. First, this function does not depend on the choice of χ among the q possibilities. Second, it makes the statement of proposition 6 look nicer. Let

$$\rho_{p/q} = \left|\frac{2\pi q R_{p/q}}{A(p/q)}\right|^{1/q}, \tag{6}$$

and let us give the version of proposition 3 that we will use here.

Proposition 4. *For each rational number p/q, there exists a unique holomorphic function $\psi = \psi_{p/q} : B(0, \rho_{p/q}) \to \mathbb{C}$ such that*

(1) $\psi(0) = 0$,
(2) $\psi'(0) = 1$,
(3) *for every $\delta \in B(0, \rho_{p/q}) \setminus \{0\}$,*

$$\left\langle \psi(\delta), \psi(\zeta\delta), \psi(\zeta^2\delta), \dots, \psi(\zeta^{q-1}\delta)\right\rangle$$

forms a cycle of period q of P_α with $\zeta = e^{2i\pi p/q}$ and

$$\alpha = \frac{p}{q} - \frac{A(p/q)}{2i\pi q}\delta^q.$$

In particular,

$$\psi(\zeta\delta) = P_\alpha(\psi(\delta)) \quad \textit{for every } \delta \in B(0, \rho_{p/q}) \ .$$

We will now make use of the following lemma which appears in Jellouli's thesis [J1] (compare with [J2] Theorem 1).

Lemma 6. *Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is chosen so that P_α has a Siegel disk Δ_α and let $p_k/q_k \to \alpha$ be the convergents of α given by the continued fraction. Then, $P_{p_k/q_k}^{\circ q_k}$ converges uniformly to the identity on every compact subset of Δ_α.*

Proposition 5. *Assume α is an irrational number such that P_α has a Siegel disk and p_k/q_k are the convergents to α. Then,*

$$\liminf_{k\to\infty} \rho_{p_k/q_k} \geq r_\alpha.$$

Proof.
Let $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ be the linearizing map which fixes 0 and has derivative 1 there. For each $k \geq 0$, set

$$g_k = \phi_\alpha^{-1} \circ P_{p_k/q_k} \circ \phi_\alpha.$$

Then, since $\phi_\alpha'(0) = 1$, an elementary computation gives

$$g_k^{\circ q_k} = z + A(p_k/q_k) z^{1+q_k} + \mathcal{O}(z^{2+q_k}).$$

The previous lemma implies that $g_k^{\circ q_k}$ converges to the identity uniformly on compact subsets of $B(0,r_\alpha)$. For any radius $r < r_\alpha$, we may find an integer N so that $g_k^{\circ q_k}$ is defined on $B(0,r)$ for $n \geq N$. Since $g_k^{\circ q_k}$ takes its values in $B(0,r_\alpha)$, we have

$$|A(p_k/q_k)| = \frac{1}{2\pi}\left|\int_{\partial B(0,r)} \frac{g_k^{\circ q_k}(z)}{z^{2+q_k}} dz\right| \leq \frac{r_\alpha}{r^{1+q_k}}.$$

This, combined with (5) and (6) gives

$$\liminf_{k\to\infty} \rho_{p_k/q_k} \geq \liminf_{k\to\infty} \left(\frac{2\pi}{q_k^2}\right)^{1/q_k} \cdot \left(\frac{r}{r_\alpha}\right)^{1/q_k} \cdot r = r.$$

The result follows by letting $r \to r_\alpha$. ■

We may now study the asymptotic behavior of the functions ψ_{p_k/q_k} as $k \to \infty$.

Proposition 6. *Assume $\alpha \in \mathbb{R}$ is an irrational number such that P_α has a Siegel disk Δ_α and p_k/q_k are the convergents of α. Then,*

- $\lim_{k\to\infty} \rho_{p_k/q_k} = r_\alpha$ *and*
- *the sequence of functions $\psi_{p_k/q_k} : B(0,\rho_{p_k/q_k}) \to \mathbb{C}$ converges uniformly on compact subsets of $B(0,r_\alpha)$ to the linearization $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ which fixes 0 with derivative 1.*

Proof. We have just seen that

$$\liminf_{k\to\infty} \rho_{p_k/q_k} \geq r_\alpha.$$

Therefore, given any radius $r < r_\alpha$, the function ψ_{p_k/q_k} is defined on the disk $B(0,r)$ for large enough k. If $\alpha \in B(p/q, R_{p/q})$ and z is a periodic point of P_α, then $|z| \leq 1 + e^{2\pi}$.[3] Therefore, the functions ψ_{p_k/q_k} all take their values in the disk $B(0, 1+e^{2\pi})$. It follows that the sequence of functions

$$\psi_{p_k/q_k} : B(0,r) \to B(0, 1+e^{2\pi})$$

is normal. Let $\psi : B(0,r) \to \mathbb{C}$ be a subsequence limit. We have

$$\psi_{p_k/q_k}(e^{2i\pi p_k/q_k}\delta) = P_{\alpha[k]} \circ \psi_{p_k/q_k}(\delta),$$

[3]Since $R_{p/q} \leq 1$, if $\alpha \in B(p/q, R_{p/q})$, then $\mathrm{Im}(\alpha) > -1$ and thus $|e^{2i\pi\alpha}| < e^{2\pi}$. So, if $\alpha \in B(p/q, R_{p/q})$ and $|z| > 1 + e^{2\pi}$, then $|P_\alpha(z)| = |z| \cdot |z + e^{2i\pi\alpha}| > |z|$ and z cannot be a periodic point of P_α.

where
$$\alpha[k] = \frac{p_k}{q_k} - \frac{A(p_k/q_k)}{2i\pi q_k}\delta^{q_k}.$$
Since
$$|A(p_k/q_k)| \leq \frac{r_\alpha}{r^{1+q_k}}$$
by the proof of Proposition 5, we have
$$\left|\frac{A(p_k/q_k)}{2i\pi q_k}\delta^{q_k}\right| \leq \frac{r_\alpha}{2\pi r q_k}\left(\frac{\delta}{r}\right)^{q_k} \underset{k\to\infty}{\longrightarrow} 0.$$
It follows that $\alpha[k] \to \alpha$ as $k \to \infty$. Hence, for any $\delta \in B(0,r)$, we have
$$\psi(e^{2i\pi\alpha}\delta) = P_\alpha \circ \psi(\delta).$$
Since $\psi'(0) = 1$, ψ is non-constant and so it coincides with the linearizing parametrization $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$. As a consequence, the whole sequence $\{\psi_{p_k/q_k}\}$ converges on compact subsets of $B(0,r_\alpha)$ to the isomorphism ϕ_α.

Now, let r be defined by
$$r = \limsup_{k\to\infty} \rho_{p_k/q_k}.$$
By passing to a subsequence if necessary, we may assume that the sequence ρ_{p_k/q_k} converges to r. Then, the same argument as above shows that the extracted subsequence ψ_{p_k/q_k} converges on compact subsets of $B(0,r)$ to a holomorphic map $\phi : B(0,r) \to \mathbb{C}$ which fixes 0 with derivative 1 and linearizes P_α. In particular, the linearizing parametrization $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ is holomorphic on the disk of radius r and so, $r \leq r_\alpha$. ■

Corollary 5. *Assume α is a Bruno number and let p_k/q_k be the convergents of α defined by its continued fraction. Assume $\alpha[n]$ is a sequence of Bruno numbers such that*
$$\left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \underset{n\to\infty}{\longrightarrow} \lambda < 1.$$
For each n, let δ_n be a complex number which satisfies
$$\frac{p_n}{q_n} - \frac{A(p_n/q_n)}{2i\pi q_n}\delta_n^{q_n} = \alpha[n].$$
Then, the set
$$O_n = \psi_{p_n/q_n}\left\{\delta_n e^{2i\pi k/q_n} \;\middle|\; k = 1,\ldots,q_n\right\}$$
is a periodic orbit of $P_\alpha[n]$ of period q_n which converges to the analytic curve $\phi_\alpha(\partial B(0,\lambda r_\alpha))$ in the Hausdorff topology on compact subsets of $\mathbb{C}$. As a result, the conformal radius $r_{\alpha[n]}$ of the Siegel disk of the quadratic polynomial $P_{\alpha[n]}$ satisfies
$$\limsup_{n\to\infty} r_{\alpha[n]} \leq \lambda r_\alpha.$$

Proof. As $n \to \infty$,
$$|\delta_n| = \left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \cdot \left|\frac{2\pi q_n}{A(p_n/q_n)}\right|^{1/q_n} \to \lambda r_\alpha.$$
Moreover, the sequence ψ_{p_n/q_n} converges to the linearizing parametrization $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$. Therefore, the sequence of compact sets O_n converges to $\phi_\alpha(\partial B(0,\lambda r_\alpha))$ for the Hausdorff topology on compact subsets of $\mathbb{C}$.

Let us assume that r is the limit of a subsequence $r_{\alpha[n_k]}$. Then, for any $r' < r$, if k is sufficiently large, $\phi_{\alpha[n_k]}$ is defined on the disk $B(0,r')$. The maps $\phi_{\alpha[n_k]} : B(0,r') \to \mathbb{C}$ are univalent, fix 0 and have derivative 1 at the origin. Therefore, extracting a further subsequence if necessary, we may assume that the sequence $\phi_{\alpha[n_k]} : B(0,r') \to \Delta_{\alpha[n_k]}$ converges to a non-constant limit $\phi : B(0,r') \to \mathbb{C}$. The map $\phi_{\alpha[n_k]}$ take its values in the Siegel disk $\Delta_{\alpha[n_k]}$, and so it omits the periodic orbit O_n of $P_{\alpha[n]}$. As a consequence, the limit map ϕ must omit $\phi_\alpha(\partial B(0,\lambda r_\alpha))$.

Therefore, the map $\phi_\alpha^{-1} \circ \phi$ sends $B(0,r')$ into $B(0,\lambda r_\alpha)$, fixes 0 and has derivative 1 at 0. Thus, by Schwarz's lemma, $r' \le \lambda r_\alpha$. Letting $r' \to r$ shows that $\limsup_{n\to\infty} r_{\alpha[n]}$ is less than or equal to λr_α. ■

7. Accumulation by cycles

Let us explain how to modify the proof of the Main Theorem in order to obtain the existence of a Siegel whose boundary is smooth and accumulated by periodic cycles.

We define sequences of Bruno numbers $\alpha(n)$, positive numbers ε_n and r_n and a sequence of finite sets C_n – which will be repelling cycles for $P_{\alpha(n)}$ – as follows.

Take $\alpha(0) = \alpha$, $\varepsilon_0 = \varepsilon/10$, $r_0 = r_\alpha$ and let C_0 be the repelling fixed point of P_α. Assuming that $\alpha(n)$, ε_n, r_n and C_n are defined, let $\varepsilon_{n+1} < \varepsilon_n/10$ be such that $r_\theta < r_{\alpha(n)} + \varepsilon_n$ and P_θ has a repelling cycle ε_n-close to C_n whenever $|\theta - \alpha(n)| < \varepsilon_{n+1}$ (this is possible since repelling cycles move holomorphically).

Next, choose $r_{n+1} \in (r, r_{\alpha(n)})$ sufficiently close to r so that $r_{n+1} - r < \varepsilon_{n+1}$ and $\phi_{\alpha(n)}(\partial B(0,r_{n+1}))$ is ε_{n+1}-close to $\phi_{\alpha(n)}(\partial B(0,r))$ in the Hausdorff metric. Finally, choose $\alpha(n+1)$ such that

- $|\alpha(n+1) - \alpha(n)| < \varepsilon_{n+1}/10$,
- $r_{\alpha(n+1)} > r$,
- the real-analytic functions $u_{n+1} : t \mapsto \phi_{\alpha(n+1)}(re^{2i\pi t})$ and $u_n : t \mapsto \phi_{\alpha(n)}(re^{2i\pi t})$ are ε_{n+1}-close in the Fréchet space $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C})$,
- $u_{n+1}(\mathbb{R}/\mathbb{Z})$ and $u_n(\mathbb{R}/\mathbb{Z})$ are ε_{n+1}-close in the Hausdorff metric, and
- $P_{\alpha(n+1)}$ has a repelling cycle C_{n+1} which is ε_{n+1}-close to $\phi_{\alpha(n)}(\partial B(0,r_{n+1}))$ – and so, $2\varepsilon_{n+1}$-close to $\phi_{\alpha(n)}(\partial B(0,r))$ – in the Hausdorff metric (this is possible by Corollary 5).

Let $\alpha' = \lim \alpha(n)$. Since for $n \ge 1$, $r_n - r < \varepsilon_n$ we see that r_n is a decreasing sequence converging to r. Thus, as in the proof of the Main Theorem, we have $r_{\alpha'} = r$ and the functions u_n converge to a C^∞ embedding $v : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ which parametrizes the boundary of the Siegel disk $\Delta_{\alpha'}$. By the construction, for each $n \ge 1$ $|\alpha' - \alpha(n)| < \varepsilon_{n+1}$, so $P_{\alpha'}$ has a cycle C'_n which is ε_n-close to C_n. Since C_n is $2\varepsilon_n$-close to $u_{n-1}(\mathbb{R}/\mathbb{Z})$ and $v(\mathbb{R}/\mathbb{Z})$ is $2\varepsilon_n$-close to $u_{n-1}(\mathbb{R}/\mathbb{Z})$ in the Hausdorff metric, we see that C'_n is ε_{n-1}-close to the boundary of the Siegel disk $\Delta_{\alpha'}$.

8. Conclusion

As mentioned in the introduction, Petersen and Zakeri proved that there exists quadratic Siegel disks whose boundaries are Jordan curves containing the critical point but are not quasicircles. They even give an arithmetical condition for this to hold: when $\alpha = [a_0, a_1, a_2, \dots]$ with $\{a_n\}$ unbounded but $\log a_n = \mathcal{O}(\sqrt{n})$ as $n \to \infty$.

The quadratic Siegel disks constructed by Herman which do not contain the critical point in their boundaries are quasidisks. The authors do not know if one can control the regularity of the boundary with Herman's methods.

The techniques we developed in this article are very flexible. We can apply them in order to prove the existence of Siegel disks whose boundaries are Jordan curves avoiding the critical point but are not quasicircles or, for each integer $k \geq 0$, the existence of Siegel disks whose boundaries are C^k but not C^{k+1} (see [BC3]).

One can also ask about the Hausdorff dimension of the boundaries of Siegel disks. It is known that when $\alpha = [a_0, a_1, a_2, \ldots]$ with $\{a_n\}$ bounded, the Hausdorff dimension is greater then 1 (Graczyk-Jones [GJ]) and less than 2 (because it is a quasi-circle). In the case of Siegel disks with smooth boundaries, the Hausdorff dimension is obviously equal to 1. This naturally leads to the following.

Problem 2. Does there exist a quadratic Siegel disk whose boundary is a Jordan curve with Hausdorff dimension 2?

We believe we can produce a quadratic Siegel disk whose boundary does not contain the critical point and has packing dimension 2 and Hausdorff dimension 1. The problem of producing a Siegel disk whose boundary has Hausdorff dimension 2 seems more tricky.

Next, the quadratic Siegel disks we produce are accumulated by cycles. This is the way we control that the Siegel disk is not larger than expected. Pérez-Marco has produced maps which are univalent in the unit disk, have Siegel disks with smooth boundaries that are not accumulated by small cycles.

Problem 3. Does there exist a quadratic polynomial having a Siegel disk whose boundary is not accumulated by cycles?

Finally, it is known that when α satisfies the Herman condition (see the introduction), the critical point is on the boundary of the Siegel disk. It would be interesting to quantify the construction we give in this article.

Problem 4. Give an arithmetical condition which ensures that the critical point is not on the boundary of the Siegel disk. Or, give an arithmetical condition which ensures that the boundary of the Siegel disk is smooth.

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