THE BRJUNO FUNCTION CONTINUOUSLY ESTIMATES THE SIZE OF QUADRATIC SIEGEL DISKS.

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Abstract. If α is an irrational number, we define Yoccoz's Brjuno function Φ by

$$\Phi(\alpha) = \sum_{n\geq 0} \alpha_0\alpha_1\cdots\alpha_{n-1}\log\frac{1}{\alpha_n},$$

where α_0 is the fractional part of α and α_{n+1} is the fractional part of $1/\alpha_n$. The numbers α such that $\Phi(\alpha)<\infty$ are called the Brjuno numbers.

The quadratic polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$ has an indifferent fixed point at the origin. If P_α is linearizable, we let $r(\alpha)$ be the conformal radius of the Siegel disk and we set $r(\alpha) = 0$ otherwise.

Yoccoz [Y] proved that $\Phi(\alpha) = \infty$ if and only if $r(\alpha) = 0$ and that the restriction of $\alpha \mapsto \Phi(\alpha)+\log r(\alpha)$ to the set of Brjuno numbers is bounded from below by a universal constant. In [BC2], we proved that it is also bounded from above by a universal constant. In fact, Marmi, Moussa and Yoccoz [MMY] conjecture that this function extends to $\mathbb{R}$ as a Hölder function of exponent 1/2. In this article, we prove that there is a continuous extension to $\mathbb{R}$.

1. Introduction.

For any irrational number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, we denote by $(p_n/q_n)_{n\geq 0}$ the approximants to α given by its continued fraction expansion (by convention, $p_0 = \lfloor\alpha\rfloor$ is the integer part of α and $q_0 = 1$).

Remark. Every time we use the notation p/q for a rational number, we mean that $q > 0$ and p and q are coprime.

We denote by $\lfloor\alpha\rfloor \in \mathbb{Z}$ the integer part of α, i.e., the largest integer $n \leq \alpha$, by $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$ the fractional part of α, and we define $(\alpha_n)_{n\geq 0}$ recursively by setting $\alpha_0 = \{\alpha\}$ and $\alpha_{n+1} = \{1/\alpha_n\}$. We then define $\beta_{-1} = 1$ and $\beta_n = \alpha_0\alpha_1\cdots\alpha_n$.

Definition 1. (The Yoccoz function). *If α is an irrational number, we define*

$$\Phi(\alpha) = \sum_{n=0}^{+\infty} \beta_{n-1}\log\frac{1}{\alpha_n}.$$

If α is a rational number we define $\Phi(\alpha) = +\infty$. Irrational numbers for which $\Phi(\alpha) < \infty$ are called Brjuno numbers. Other irrational numbers are called Cremer numbers.

Remark. The set $\mathcal{B}$ of Brjuno numbers has full measure in $\mathbb{R}$. It contains the set of all Diophantine numbers, i.e., numbers for which $\log q_{n+1} = \mathcal{O}(\log q_n)$.

We study the quadratic polynomials

$$P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$$

for $\alpha \in \mathbb{R}$. It is known that such P_α is linearizable – and so, has a Siegel disk – if and only if α is a Brjuno number.

Definition 2. *If $U \subsetneq \mathbb{C}$ is a simply connected domain containing* 0*, we denote by* $\mathrm{rad}(U)$ *the conformal radius of U at* 0*, i.e.,* $\mathrm{rad}(U) = |\phi'(0)|$ *where* $\phi : (\mathbb{D}, 0) \to (U, 0)$ *is any conformal representation.*

Definition 3. *For any Brjuno number $\alpha \in \mathcal{B}$, we denote by $r(\alpha)$ the conformal radius at 0 of the Siegel disk of the quadratic polynomial P_α. If $\alpha \in \mathbb{R} \setminus \mathcal{B}$, we define* $r(\alpha) = 0$.

It is known that there exists a constant C_0 such that for any Brjuno number $\alpha \in \mathcal{B}$ and any univalent map $f : \mathbb{D} \to 0$ which fixes 0 with derivative $e^{2i\pi\alpha}$, f has a Siegel disk which contains $B(0, r)$ with $\Phi(\alpha) + \log r \geq C_0$. In particular, for all $\alpha \in \mathcal{B}$, we have

$$\Phi(\alpha) + \log r(\alpha) \geq C_0. \tag{1}$$

Remark. The existence of Δ_f is due to Brjuno [Brj]. The lower bound (1) is due to Yoccoz [Y].

In [BC2], we prove that there exists a universal constant C_1 such that for all $\alpha \in \mathcal{B}$, we have

$$\Phi(\alpha) + \log r(\alpha) \leq C_1. \tag{2}$$

Inequalities (1) and (2) imply that $\Phi(\alpha) + \log r(\alpha)$ is uniformly bounded on $\mathcal{B}$:

$$(\exists C \in \mathbb{R}),\ (\forall \alpha \in \mathcal{B}),\quad |\Phi(\alpha) + \log r(\alpha)| \leq C. \tag{3}$$

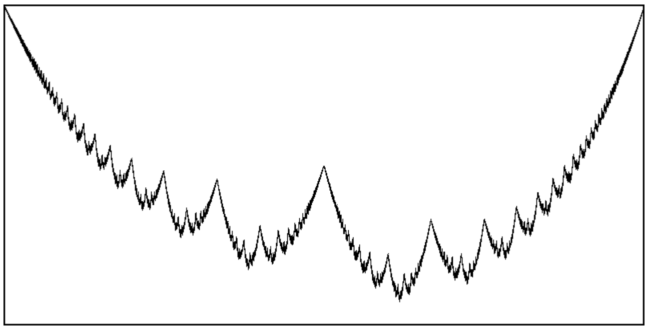

FIGURE 1. The graph of the function $\alpha \mapsto \Phi(\alpha) + \log r(\alpha)$ with $\alpha \in [0, 1]$. The range is $[0, \log(2\pi)]$.

In this article we prove the following result which was conjectured by Marmi [Ma].

Theorem 1. *The function $\alpha \mapsto \Phi(\alpha) + \log r(\alpha)$ extends to $\mathbb{R}$ as a continuous function.*

In fact, Marmi, Moussa and Yoccoz made the following stronger conjecture ([MMY] and [Ca]).

Conjecture 1. *The function $\alpha \mapsto \Phi(\alpha) + \log r(\alpha)$ —which is well-defined on $\mathcal{B}$— is Hölder of exponent* $1/2$.

Remark. In [Y], Yoccoz uses a modified version of continued fractions. He defines a sequence $\tilde{\alpha}_n$ defined by $\tilde{\alpha}_0 = d(\alpha, \mathbb{Z})$ and $\tilde{\alpha}_{n+1} = d(1/\tilde{\alpha}_n, \mathbb{Z})$. The corresponding function $\widetilde{\Phi}$ defined by

$$\widetilde{\Phi}(\alpha) = \sum_{n \geq 0} \tilde{\alpha}_0 \cdots \tilde{\alpha}_{n-1} \log \frac{1}{\tilde{\alpha}_n}$$

has the additional property that $\widetilde{\Phi}(1-\alpha) = \widetilde{\Phi}(\alpha)$. Figure 2 shows the graph of the function $\alpha \mapsto \widetilde{\Phi}(\alpha) + \log r(\alpha)$. Theorem 4.6 in [MMY] asserts that the restriction of $\Phi - \widetilde{\Phi}$ to $\mathcal{B}$ extends to $\mathbb{R}$ as a 1/2-Hölder continuous periodic function with period one. It follows from this result and theorem 1 that the function $\alpha \mapsto \widetilde{\Phi}(\alpha) + \log r(\alpha)$ extends to $\mathbb{R}$ as a continuous function (and that the Marmi-Moussa-Yoccoz conjecture is equivalent with Φ replaced by $\widetilde{\Phi}$).

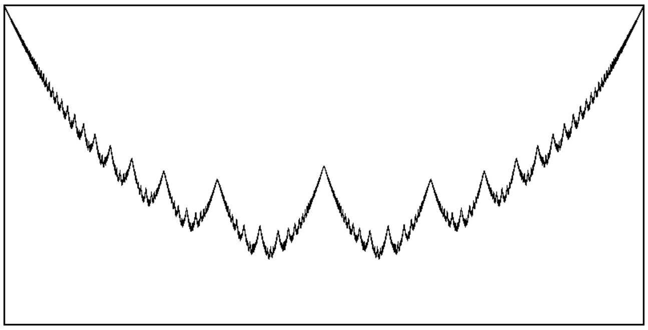

FIGURE 2. The graph of the function $\alpha \mapsto \widetilde{\Phi}(\alpha) + \log r(\alpha)$ with $\alpha \in [0,1]$. The range is $[0, \log(2\pi)]$.

2. Statement of results.

In this section, we will define a function $\Upsilon : \mathbb{R} \to \mathbb{R}$ and in the rest of the article, we will show that for all $\alpha \in \mathbb{R}$,

$$\lim_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') = \Upsilon(\alpha).$$

It is an easy exercise to prove that Υ is then continuous.

Remark. For $\alpha \in \mathbb{Q}$, we give a computable formula of $\Upsilon(\alpha)$.

The value of Υ at Brjuno numbers is obvious.

Definition 4. *For $\alpha \in \mathcal{B}$, we set*

$$\Upsilon(\alpha) = \Phi(\alpha) + \log r(\alpha).$$

2.1. **Strategy of the proof.** The strategy for proving that for all $\alpha \in \mathbb{R}$,

$$\lim_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') = \Upsilon(\alpha)$$

consists in bounding $\Phi(\alpha') + \log r(\alpha')$ from above and from below as $\alpha' \in \mathcal{B}$ tends to α. The upper bound follows from techniques of parabolic explosion developed in [Ch] and [BC2]. Those techniques are presented in section 3.

The lower bound essentially follows from techniques of renormalization introduced by Yoccoz in [Y]. He uses estimates which are valid for all maps which are univalent in $\mathbb{D}$ and fix 0 with derivative of modulus 1. In our case, we will need to improve those estimates for maps which are close to rotations and maps which have at most one fixed point in $\mathbb{D}^*$ (see section 5).

The rest of this section is devoted to the definition of Υ at rational and Cremer numbers.

2.2. **The value of Υ at rational numbers.** A rational number $\alpha = p/q \in \mathbb{Q}$ has two finite continued fraction expansions, corresponding to two sequences of approximants p_n/q_n, two sequences α_n, and two sequences β_n. One of the sequences α_n is provided by the usual algorithm: $\alpha_0 = \{\alpha\}$ and $\alpha_{n+1} = \{1/\alpha_n\}$, which eventually gives $\alpha_m = 0$ for some $m \in \mathbb{N}$, after which the sequence is not defined any more. The other has the same α_k for $k < m$, its $\alpha_m = 1$, and has one more term, $\alpha_{m+1} = 0$. [1]

In both cases, the sequence β is defined by $\beta_n = \alpha_0 \cdots \alpha_n$. Let $n_0 = m$ or $m+1$ be the last index of the sequence α_n of p/q that we chose. We have $\alpha_{n_0} = 0$. We can form the finite sum

$$\Phi_{\mathrm{trunc}}(p/q) = \sum_{n=0}^{n_0-1} \beta_{n-1} \log \frac{1}{\alpha_n}$$

(with the convention that a sum $\sum_{n=0}^{n=-1} \cdots$ is equal to 0). It turns out to be independent of the choice between the two values of n_0, as can easily be checked.

The following two definitions and their relations with the conformal radii of Siegel disks appear in [Ch].

Definition 5. *Assume $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ having a multiple fixed point at the origin whose Taylor expansion is*

$$f(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}), \quad \textit{with} \quad A \in \mathbb{C}^*.$$

The asymptotic size of f at 0 is defined by

$$L_a(f, 0) = \left| \frac{1}{qA} \right|^{1/q}.$$

Definition 6. *Assume $p/q \in \mathbb{Q}$ is a rational number. Then, we define*

$$L_a(p/q) = L_a(P_{p/q}^{\circ q}, 0).$$

Definition 7. *For all rational number p/q, we define*

$$\Upsilon\left(\frac{p}{q}\right) = \Phi_{\mathrm{trunc}}\left(\frac{p}{q}\right) + \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q}.$$

2.3. **The value of Υ at Cremer numbers.**

Definition 8. *For all irrational number α and all integer $n \geq 0$, we define*

$$\Phi_n(\alpha) = \sum_{k=0}^{n} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

[1] A number α' tending to p/q has its α'_k that tends to the α_k of p/q for all $k < m$. According to whether α' tends to p/q by the left or the right, α'_m tends to one of the two values defined above, that is 0 or 1, the correspondence depending on the parity of m. Moreover, if it is 1, then α'_{m+1} tends to 0. This motivates the two definitions we made.

Definition 9. *If $U \subset \mathbb{C}$ is a hyperbolic connected domain containing 0, we denote by $\mathrm{rad}(U)$ the conformal radius of U at 0, i.e., $\mathrm{rad}(U) = |\pi'(0)|$ where $\pi : (\mathbb{D}, 0) \to (U, 0)$ is any universal covering.*

Remark. This definition of conformal radius coincides with the one given in the introduction in the case of simply connected domains.

Definition 10. *For all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and all integer $n \geq 0$, we define*

$$X_n(\alpha) = \{z \in \mathbb{C}^* \mid z \textit{ is a periodic point of } P_\alpha \textit{ of period } \leq q_n\}$$

where p_n/q_n are the approximants to α,

$$r_n(\alpha) = \mathrm{rad}(\mathbb{C} \setminus X_n(\alpha)) \quad \textit{and} \quad d_n(\alpha) = d(0, X_n(\alpha)).$$

Remark. If $n \geq 2$, then $q_n \geq 2$, $X_n(\alpha)$ contains at least two points and $r_n(\alpha) \in]0, \infty[$. Moreover, for $n \geq 2$, the function $\alpha \mapsto \log r_n(\alpha)$ is well-defined and continuous in a neighborhood of every point $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

For all irrational number α, the sequence $(r_n(\alpha))_{n \geq 0}$ is decreasing and converges to $r(\alpha)$ as $n \to \infty$. Indeed, if 0 is not linearizable, it is accumulated by periodic points of P_α.[2] If 0 is linearizable, the Siegel disk Δ_α is contained in $\mathbb{C} \setminus X_n(\alpha)$ for all $n \geq 0$ and the boundary of Δ_α is accumulated by periodic points of P_α.[3] Since P_α is tangent the rotation of angle α and α is irrational, if 0 is not linearizable, then

$$r_n(\alpha) \underset{n \to +\infty}{\sim} d_n(\alpha).$$

If α is a Brjuno number, then

$$\lim_{n \to \infty} \Phi_n(\alpha) + \log r_n(\alpha) = \Upsilon(\alpha).$$

Yoccoz's work [Y] implies that there exists a constant C_0 such that for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and all $n \geq 0$,

$$\Phi_n(\alpha) + \log r_n(\alpha) \geq C_0.$$

Thus, the following definition makes sense.

Definition 11. *For all Cremer number α, we define*

$$\Upsilon(\alpha) = \liminf_{n \to \infty} \Phi_n(\alpha) + \log r_n(\alpha).$$

In fact, we will see (section 3) that this is an actual limit.

Theorem 2. *For all Cremer number α,*

$$\Upsilon(\alpha) = \lim_{n \to +\infty} \Phi_n(\alpha) + \log r_n(\alpha).$$

Corollary 1. *For all Cremer number α,*

$$\Upsilon(\alpha) = \lim_{n \to +\infty} \Phi_n(\alpha) + \log d_n(\alpha).$$

[2] In fact, Yoccoz proved that 0 is accumulated by whole cycles.

[3] It is not known whether $\partial\Delta_\alpha$ is always accumulated by whole cycles.

Our goal is to prove that for all $\alpha \in \mathbb{R}$, the value of $\Upsilon(\alpha)$ defined previously (see definitions 4, 7 and 11) is the limit of $\Phi(\alpha') + \log r(\alpha')$ as $\alpha' \in \mathcal{B}$ tends to α. In section 3 we introduce the techniques of parabolic explosion and in section 4 we show that for all $\alpha \in \mathbb{R}$,

$$\limsup_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \leq \Upsilon(\alpha). \tag{4}$$

In section 5, we refine Yoccoz's estimates for renormalization of univalent maps $f : \mathbb{D} \to \mathbb{C}$ which fix 0 with derivative of modulus 1, and in sections 6 and 7 we show that for all $\alpha \in \mathbb{R}$,

$$\liminf_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Upsilon(\alpha). \tag{5}$$

Let us mention that inequality (4) without inequality (5) (respectively inequality (5) without inequality (4)) is not sufficient to conclude that Υ is upper semi-continuous (respectively lower semi-continuous) since we only consider approximating α with sequences of Brjuno numbers.

3. Parabolic explosion.

In this section, we first present the techniques of parabolic explosion. We then apply those techniques in order to prove theorem 2.

3.1. Definitions. Assume $p/q \in \mathbb{Q}$ is a rational number. The origin is a parabolic fixed point for the quadratic polynomial $P_{p/q}$. It is known (see [DH], chapter IX) that there exists a complex number $A \in \mathbb{C}^*$ such that

$$P_{p/q}^{\circ q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}).$$

Thus, $P_{p/q}^{\circ q}$ has a fixed point of multiplicity $q+1$ at the origin. By Rouché's theorem, when α is close to p/q, the polynomial $P_\alpha^{\circ q}$ has $q+1$ fixed points close to 0. One coincides with 0. The others form a cycle of period q for P_α. More precisely, we have the following (see [Ch] or [BC2] proposition 1 for a proof).

Proposition 1. *Let p/q be a rational number, and $\zeta = e^{2i\pi p/q}$. There exists an analytic function $\chi : B(0, 1/q^{3/q}) \to \mathbb{C}$ such that $\chi(0) = 0$ and for any $\delta \in B(0, 1/q^{3/q}) \setminus \{0\}$, $\chi(\delta) \neq 0$ and the set*

$$\left\langle \chi(\delta), \chi(\zeta\delta), \chi(\zeta^2\delta), \ldots, \chi(\zeta^{q-1}\delta) \right\rangle$$

forms a cycle of period q of $P_{p/q+\delta^q}$. We will note $\chi = \chi_{p/q}$, since it depends on p/q.

Remark. Observe that $\delta \in B(0, 1/q^{3/q})$ if and only if $\alpha = p/q + \delta^q \in B(p/q, 1/q^3)$.

In the following definition, note that α is a **complex** number.

Definition 12. *For all $p/q \in \mathbb{Q}$ and all $\alpha \in B(p/q, 1/q^3)$, we define*

$$\mathcal{C}_{p/q}(\alpha) = \chi_{p/q}\left\{ \sqrt[q]{\alpha - p/q} \right\},$$

where $\sqrt[q]{z}$ denotes the set *of complex q-th roots of z.*

The set $\mathcal{C}_{p/q}(\alpha)$ is a cycle of period q for P_α, except when $\alpha = p/q$, in which case it is reduced to $\{0\}$. In particular, if α is irrational, $p/q = p_n/q_n$ is an approximant to α and $|\alpha - p_n/q_n| < 1/q_n^3$, then $\mathcal{C}_{p_n/q_n}(\alpha) \subset X_n(\alpha)$. Note that when $|\alpha_0 - p/q| < 1/2q^3$, the cycle $\mathcal{C}_{p/q}(\alpha)$ is defined for all $\alpha \in B(\alpha_0, 1/2q^3)$, and not reduced to $\{0\}$.

3.2. A preliminary lemma.

Lemma 1. *Assume $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n be an approximant to α_0 with $q_n \geq 2$. Assume $\alpha \in \mathbb{C}$, $\alpha \neq p_n/q_n$, $q \leq q_n$ and $P_\alpha^{\circ q}$ has a multiple fixed point. Then,*

$$|\alpha_0 - \alpha| \geq \frac{1}{2q_n^3}.$$

Proof. Either $\alpha = p/q$ for some integer p. Within the disk $B(\alpha_0, 1/2q_n^3)$, the only possibility is $p/q = p_n/q_n$. Or α belongs to a Yoccoz disk of radius $\log 2/(2\pi q') < 1/8q'$ tangent to the real axis at p'/q' for some rational number p'/q' with $q' < q \leq q_n$. By a well-known property of approximants, we have

$$|q'\alpha_0 - p'| \geq |q_{n-1}\alpha_0 + p_{n-1}| \geq \frac{1}{q_n + q_{n-1}} \geq \frac{1}{2q_n}.$$

Moreover, by Pythagoras' theorem,

$$\begin{aligned}
|\alpha - \alpha_0| &\geq \frac{1}{q'}\left(\sqrt{(q'\alpha_0 - p')^2 + (1/8)^2} - 1/8\right)\\
&\geq \frac{1}{q_n}\left(\sqrt{1/(2q_n)^2 + 1/8^2} - 1/8\right)\\
&= \frac{1/(2q_n)^2}{q_n\left(\sqrt{1/(2q_n)^2 + 1/8^2} + 1/8^2\right)}\\
&\geq \frac{1}{2q_n^3} \cdot \frac{1}{2\left(\sqrt{1/4^2 + 1/8^2} + 1/8^2\right)} \quad \geq \frac{1}{2q_n^3}.
\end{aligned}$$

■

Corollary 2. *Assume $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n be an approximant to α_0 with $q_n \geq 2$. The set*

$$X(\alpha) = \{z \in \mathbb{C}^* \mid z \text{ is a periodic point of } P_\alpha \text{ of period } \leq q_n\}$$

moves holomorphically with respect to $\alpha \in B(\alpha_0, 1/2q_{n+1}^3)$.

Proof. If the set $X(\alpha)$ fails to move holomorphically at a point $\alpha \in \mathbb{C}$, then, for some integer $q \leq q_n$, $P_\alpha^{\circ q}$ has a multiple fixed point. Either $\alpha = p_n/q_n$, and (according to a property of approximants) $|\alpha - \alpha_0| \geq 1/(2q_n q_{n+1}) > 1/2q_{n+1}^3$. Or $\alpha \neq p_n/q_n$, and by the previous lemma $|\alpha - \alpha_0| \geq 1/2q_n^3 > 1/2q_{n+1}^3$. ■

3.3. A technical lemma.

Lemma 2. *There exists $C \in \mathbb{R}$ such that for all $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ and all $p/q \in \mathbb{Q}$ with $q \geq 2$, the following holds. Assume $V(\alpha) \ni 0$ is an open set that moves holomorphically with respect to $\alpha \in B(\alpha_0, 1/2q^3)$.*

- *If $|\alpha_0 - p/q| \geq 1/2q^3$, set $V'(\alpha_0) = V(\alpha_0)$.*

- *If* $|\alpha_0 - p/q| < 1/2q^3$, *assume* $\mathcal{C}_{p/q}(\alpha) \subset V(\alpha)$ *for all* $\alpha \in B(\alpha_0, 1/2q^3)$ *and set* $V'(\alpha_0) = V(\alpha_0) \setminus \mathcal{C}_{p/q}(\alpha_0)$.

Then,

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \leq \frac{\log|\alpha_0 - p/q|}{q} + C\frac{\log q}{q}.$$

Proof. Let us first assume that $|\alpha_0 - p/q| \geq 1/2q^4 \geq 1/q^5$ (this comprises the case $V'(\alpha_0) = V(\alpha_0)$). Then,

$$\log|\alpha_0 - p/q| + 5\log q \geq 0$$

and the lemma follows trivially with $C = 5$ since

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \leq 0.$$

So, let us assume that $|\alpha_0 - p/q| < 1/2q^4$. Then,

$$B \overset{\mathrm{def}}{=} B(p/q, 1/2q^4) \subset B(\alpha_0, 1/q^4) \subset B(\alpha_0, 1/2q^3).$$

We set

$$U = \{\delta \in \mathbb{C} \mid p/q + \delta^q \in B\} \quad \text{and} \quad S = \{\delta \in U \mid p/q + \delta^q = \alpha_0\}.$$

Note that $\chi_{p/q}(S) = \mathcal{C}_{p/q}(\alpha_0)$.

The radius of the disk U is $1/(2q^4)^{1/q}$ and the set S consists in q points equidistributed on a circle of radius $|\alpha_0 - p/q|^{1/q}$. So, according to proposition 11 (see the appendix A), we have

$$\log \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)} < \log \frac{|\alpha_0 - p/q|^{1/q}}{1/(2q^4)^{1/q}} + \frac{C}{q}$$

for some universal constant C.

According to proposition 12 (see the appendix A), there exists for $\alpha \in B(\alpha_0, 1/2q^3)$ an analytic family of universal coverings $\pi_\alpha : \widetilde{V}(\alpha) \to V(\alpha)$, where $\widetilde{V}(\alpha)$ are open subsets of $B(0,4)$, and $\widetilde{V}(\alpha_0) = \mathbb{D}$. The set $V(\alpha)$ moves holomorphically with $\alpha \in B(\alpha_0, 1/2q^3)$ and when $\delta \in U$, $\alpha(\delta) = p/q + \delta^q$ belongs to $B \subset B(\alpha_0, 1/q^4)$. For $\alpha \in B$, the sets $\widetilde{V}(\alpha)$ are all contained in some ball $B(0,\rho)$ with

$$\log \rho = \frac{2\log 4}{1 + \dfrac{1/2q^3}{1/q^4}} = \frac{\log 16}{1 + q/2}.$$

The map $\chi_{p/q}$ "lifts" to a map $\phi : U \to B(0,\rho)$ such that $\phi(\delta) \in \widetilde{V}(\alpha(\delta))$. It follows from the definitions that,

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} = \log \frac{\mathrm{rad}(V(\alpha_0) \setminus \mathcal{C}_{p/q}(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} = \log \frac{\mathrm{rad}\left(\widetilde{V}(\alpha_0) \setminus \pi_{\alpha_0}^{-1}\left(\chi_{p/q}(S)\right)\right)}{\mathrm{rad}(\widetilde{V}(\alpha_0))}.$$

Now $\widetilde{V}(\alpha_0) = \mathbb{D}$ and $\phi(S) \subset \pi_{\alpha_0}^{-1}(\chi_{p/q}(S))$, thus

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \leq \log \mathrm{rad}(\mathbb{D} \setminus \phi(S)) \leq \log \mathrm{rad}(B(0,\rho) \setminus \phi(S)).$$

The range of the function ϕ needs not to be a subset of $\mathbb{D}$, but we know proposition 10 (see the appendix A),

$$\begin{aligned}\log \operatorname{rad}(B(0,\rho)\setminus\phi(S)) &\leq \log\frac{\operatorname{rad}(U\setminus S)}{\operatorname{rad}(U)}+\log\rho\\ &\leq \frac{\log|\alpha_0-p/q|}{q}+4\frac{\log q}{q}+\frac{\log 2}{q}+\frac{C}{q}+\frac{\log 16}{1+q/2}\\ &\leq \frac{\log|\alpha_0-p/q|}{q}+C'\frac{\log q}{q}\end{aligned}$$

for some universal constant C'. ■

3.4. **A short remark.** Let F_n be the smallest possible value of q_n over all irrationals α, where p_n/q_n is the n-th approximant to α. Then F_n is the Fibonacci sequence defined by

$$F_{-1}=0,\ F_0=1,\ F_{n+1}=F_n+F_{n-1}.$$

The first terms are

$$F_{-1}=0,\ F_0=1,\ F_1=1,\ F_2=2,\ F_3=3,\ F_4=5,\ \ldots$$

The function $x\mapsto x/\log x$ is decreasing on $[e,+\infty[$, thus

$$\forall n\geq 3,\ \frac{\log q_n}{q_n}\leq\frac{\log F_n}{F_n}.$$

For $n=1$ and 2, the biggest possible value of $\log(q_n)/q_n$ is $\log(3)/3$.

3.5. **An important corollary.** The next proposition tells us that for all irrational α, the sequence $\Phi_n(\alpha)+\log r_n(\alpha)$ is essentially decreasing, in the sense that it can not increase too fast.

Proposition 2. *There exists a constant $C\in\mathbb{R}$ such that for all $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and all $n\geq 1$ such that $q_n\geq 2$ (with p_n/q_n the approximants to α), we have*

$$\Big(\Phi_{n+1}(\alpha)+\log r_{n+1}(\alpha)\Big)-\Big(\Phi_n(\alpha)+\log r_n(\alpha)\Big)\leq C\frac{\log q_{n+1}}{q_{n+1}}.$$

Proof. Let us fix $\alpha_0\in\mathbb{R}\setminus\mathbb{Q}$ and choose n so that $q_n\geq 2$. We want to apply lemma 2 with $p/q=p_{n+1}/q_{n+1}$ and

$$V(\alpha)=\mathbb{C}\setminus\{z\in\mathbb{C}^*\mid z \text{ is a periodic point of } P_\alpha \text{ of period } \leq q_n\}.$$

By definition, $0\in V(\alpha)$ and by corollary 2, the set $V(\alpha)$ moves holomorphically with respect to $\alpha\in B(\alpha_0,1/2q_{n+1}^3)$. Also, $V(\alpha)$ contains the periodic cycles of P_α of period q_{n+1} and so, if $|\alpha_0-p/q|<1/2q^3$, then $\mathcal{C}_{p/q}(\alpha)\subset V(\alpha)$ for all $\alpha\in B(\alpha_0,1/2q^3)$. As in lemma 2, if $|\alpha_0-p/q|\geq 1/2q^3$, we set $V'(\alpha_0)=V(\alpha_0)$ and otherwise, we set $V'(\alpha_0)=V(\alpha_0)\setminus\mathcal{C}_{p/q}(\alpha_0)$. Then,

$$r_n(\alpha_0)=\operatorname{rad}(V(\alpha_0))\quad\text{and}\quad r_{n+1}(\alpha_0)\leq\operatorname{rad}(V'(\alpha_0)).$$

So, lemma 2 implies that

$$\begin{aligned}\log r_{n+1}(\alpha_0)-\log r_n(\alpha_0) &\leq \frac{\log|\alpha_0-p_{n+1}/q_{n+1}|}{q_{n+1}}+C\frac{\log q_{n+1}}{q_{n+1}}\\ &= \frac{\log\beta_{n+1}}{q_{n+1}}+(C-1)\frac{\log q_{n+1}}{q_{n+1}}.\end{aligned}$$

Since $\beta_{n+1} \le \alpha_{n+1}$ and $1/q_{n+1} \ge \beta_n$:

$$\begin{aligned}\log r_{n+1}(\alpha_0) - \log r_n(\alpha_0) &\le -\beta_n \log \frac{1}{\alpha_{n+1}} + (C-1)\frac{\log q_{n+1}}{q_{n+1}} \\ &= -\Phi_{n+1}(\alpha_0) + \Phi_n(\alpha_0) + (C-1)\frac{\log q_{n+1}}{q_{n+1}}\end{aligned}$$

for some universal constant C. ■

The bound we gave depends on α, but for each n, the supremum over all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is exponentially decreasing.

3.6. **Application to the proof of theorem 2.** Assume α is a Cremer number, define $u_n = \Phi_n(\alpha) + \log r_n(\alpha)$ and let us recall that by definition, $\Upsilon(\alpha) = \liminf_{n\to\infty} u_n$. The sequence u_n is not decreasing, but it is "essentially decreasing", in the sense that proposition 2 gives us

$$u_{n+1} - u_n \le C\frac{\log q_{n+1}}{q_{n+1}}$$

and $(\log q_{n+1})/q_{n+1}$ decreases exponentially fast. Therefore the sequence u_n converges: indeed, if we choose n_0 large enough so that

$$\sum_{n\ge n_0} C\frac{\log q_{n+1}}{q_{n+1}} \le \varepsilon \quad \text{and} \quad u_{n_0} \le \Upsilon(\alpha) + \varepsilon,$$

then $\Upsilon(\alpha) \le u_n \le \Upsilon(\alpha) + 2\varepsilon$ for all $n \ge n_0$.

4. Proof of inequality (4).

4.1. **Irrational numbers.** We will now show that for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$,

$$\limsup_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \le \Upsilon(\alpha).$$

Let us fix $\varepsilon > 0$. We must show that for $\alpha' \in \mathcal{B}$ sufficiently close to α, $\Phi(\alpha') + \log r(\alpha') \le \Upsilon(\alpha) + \varepsilon$. Remember that as $n \to \infty$, $\Phi_n(\alpha) + \log r_n(\alpha) \to \Upsilon(\alpha)$. So, let us choose n_0 large enough so that

$$\Phi_{n_0}(\alpha) + \log r_{n_0}(\alpha) \le \Upsilon(\alpha) + \varepsilon/3.$$

Increasing n_0 if necessary, we may also assume that $n_0 \ge 2$ and

$$\sum_{n\ge n_0} C\frac{\log F_{n+1}}{F_{n+1}} \le \varepsilon/3,$$

where C is the constant in proposition 2. In a neighborhood of α, the functions Φ_{n_0} and $\log r_{n_0}$ are continuous. So, if α' is sufficiently close to α,

$$\Phi_{n_0}(\alpha') + \log r_{n_0}(\alpha') \le \Phi_{n_0}(\alpha) + \log r_{n_0}(\alpha) + \varepsilon/3$$

and summing the inequality of proposition 2 from $n = n_0$ to $n = +\infty$ yields

$$\Phi(\alpha') + \log r(\alpha') \le \Upsilon(\alpha) + \varepsilon.$$

■

4.2. **Rational numbers.** We will show that

$$\limsup_{\alpha'\to p/q,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \le \Upsilon(p/q).$$

In the whole section, we will use the notation

$$\varepsilon = \alpha' - p/q.$$

For $\alpha' \in \mathbb{C}$ and $\theta \in \mathbb{R}$, we will also denote $R_{\alpha'}(\theta)$ the external ray of argument θ of $P_{\alpha'}$. The external rays for the Mandelbrot set will be noted $R_M(\theta)$.

The polynomial P_α is conjugate to the quadratic polynomial $z \mapsto z^2 + c$ with $c = e^{2i\pi\alpha}/2 - e^{4i\pi\alpha}/4$. When $\mathrm{Im}(\alpha) \longrightarrow -\infty$ and $\mathrm{Re}(\alpha) \longrightarrow \widetilde{\theta}$, then $|c| \longrightarrow +\infty$ and $\arg c \longrightarrow 2\widetilde{\theta} + \frac{1}{2} \bmod 1$. Given $\widetilde{\theta} \in \mathbb{R}$, we will denote by $\mathcal{R}(\widetilde{\theta})$ the connected component of the preimage of $R_M(2\widetilde{\theta} + 1/2)$ by $\alpha \mapsto c$, whose real part tends to $\widetilde{\theta}$.

When α is real, the parameter c is on the boundary of the main cardioid of the Mandelbrot set. If $\alpha = p/q \notin \mathbb{Z}$, $c \neq 1/4$ and there are two external rays of M landing at c. We denote by $\theta^- < \theta^+$ their arguments in $]0,1[$. The arguments θ^+ and θ^- are periodic of period q under multiplication by 2 modulo 1. They belong to the same orbit Θ. In the dynamical plane of $P_{p/q}$, the rays $R_{p/q}(\theta)$, $\theta \in \Theta$, form a periodic cycle of rays which land at 0. If $p/q \in \mathbb{Z}$, the dynamical ray of argument 0 is fixed and lands at 0. We set $\theta^- = \theta^+ = 0$ and $\Theta = \{0\}$.

Let us recall the following rule: the ray $R_{\alpha'}(\theta)$ moves holomorphically with α' as long as c does not belong to the closure of the union of the $R_M(2^k\theta)$ for $k \in \mathbb{N}^*$.

Definition 13. *When $\alpha' \in \mathbb{R}$ is close to p/q, the rays $R_{\alpha'}(\theta)$, $\theta \in \Theta$, form a cycle of rays which land on the cycle $\mathcal{C}_{p/q}(\alpha')$. We denote by $Y(\alpha')$ the union of $\mathcal{C}_{p/q}(\alpha')$ and this cycle of rays.*

Figure 3 shows the rays of argument 1/7, 2/7 and 4/7 and the boundary of the Siegel disk for the polynomial $P_{(1/3)+\varepsilon}$ for $\varepsilon = \sqrt{2}/1000$ and $\varepsilon = \sqrt{2}/10000$.

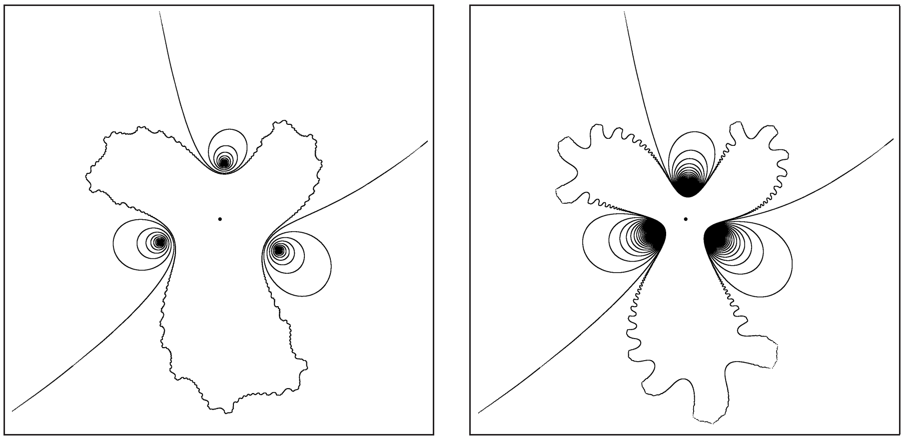

FIGURE 3. The rays of argument 1/7, 2/7 and 4/7 and the boundary of the Siegel disk for the polynomial $P_{(1/3)+\varepsilon}$: left for $\varepsilon = \sqrt{2}/1000$ and right for $\varepsilon = \sqrt{2}/10000$.

If ε is irrational and is close enough to 0, then p/q is an approximant p'_{n_0}/q'_{n_0} to α', and its index n_0 is the same number as in section 2.2 and depends on the sign of ε. As $\alpha' \to p/q$, $\log \mathrm{rad}(\mathbb{C}\setminus Y(\alpha')) \to -\infty$ and $\beta'_{n_0-1}\log(1/\alpha'_{n_0}) \to -\infty$. We postpone the proof of the following lemma to section 4.3.

Lemma 3. *We have*

$$\limsup_{\alpha'\to p/q,\ \alpha'\in\mathbb{R}\setminus\mathbb{Q}} \log \mathrm{rad}(\mathbb{C}\setminus Y(\alpha')) + \beta'_{n_0-1}\log\frac{1}{\alpha'_{n_0}} \le \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q}.$$

When α' is close to p/q but not necessarily real, the dynamical rays of argument $\theta\in\Theta$ may bifurcate. In a neighborhood of p/q, this precisely occurs when $c' = e^{2i\pi\alpha'}/2 - e^{4i\pi\alpha'}/4$ belongs to $\mathcal{R}_M(\theta^+)$ or $\mathcal{R}_M(\theta^-)$.

Lemma 4. *There exists a constant $c\in]0,1]$, which depends on p/q, such that the following holds. Assume $\alpha'\in\mathbb{R}\setminus\mathbb{Q}$ and p/q is an approximant to α'. Let n_0 be its index. Let p'_{n_0+1}/q'_{n_0+1} be α''s next approximant. Then, for all $\alpha''\in B(\alpha', c/(q'_{n_0+1})^2)$, the dynamical rays of argument $\theta\in\Theta$ do not bifurcate. In particular, $Y(\alpha'')$ moves holomorphically with respect to $\alpha''\in B(\alpha', c/(q'_{n_0+1})^2)$.*

Proof. There is exactly one pair $\widetilde{\theta}^- < \widetilde{\theta}^+$, with $2\widetilde{\theta}^+ + 1/2 = \theta^+$ and $2\widetilde{\theta}^- + 1/2 = \theta^-$ such that $\mathcal{R}(\widetilde{\theta}^+)$ and $\mathcal{R}(\widetilde{\theta}^-)$ land on p/q. The rays $\mathcal{R}(\widetilde{\theta}^+)$ and $\mathcal{R}(\widetilde{\theta}^-)$ are separated from the upper half plane (that corresponds to the cardioid by $\alpha\mapsto c$), by a smooth curve having a contact of order 2 with the real line, at p/q. Also, the other external rays $R_M(\theta')$ for $\theta'\in\Theta\setminus\{\theta^+,\theta^-\}$ do not land on the cardioid. Therefore, there exists a constant $c'>0$ such that the dynamical rays of argument $\theta\in\Theta$ do not bifurcate when $\alpha''\in B(\alpha', c'|\alpha'-p/q|^2)$. The result follows since

$$\left|\alpha' - \frac{p}{q}\right|^2 \ge \left(\frac{1}{2q'_{n_0}q'_{n_0+1}}\right)^2 = \frac{1}{4q^2(q'_{n_0+1})^2}.$$

□

Let us choose c as in lemma 4 and $\alpha'\in\mathcal{B}$ sufficiently close to p/q so that $q'_{n_0+1} > 1/2c$ (we denote by p'_n/q'_n the approximants to α'). Then, the set $Y(\alpha'')$ moves holomorphically with respect to $\alpha''\in B(\alpha', 1/2(q'_{n_0+1})^3)$. Let us also assume that $q'_{n_0+1}\ge 2$

Lemma 5. *Under the assumptions above, we have*

$$\Phi(\alpha') + \log r(\alpha') \le \Phi_{n_0}(\alpha') + \log \mathrm{rad}(\mathbb{C}\setminus Y(\alpha')) + (C-1)\sum_{n\ge n_0+1}\frac{\log q'_n}{q'_n},$$

where C is the constant provided by lemma 2.

Proof. For $\alpha''\in B(\alpha', 1/2(q'_{n_0+1})^3)$, let us define $V_{n_0}(\alpha'') = \mathbb{C}\setminus Y(\alpha'')$ and by induction, for $n\ge n_0+1$ and $\alpha''\in B(\alpha', 1/2(q'_{n+1})^3)$, let us define

- $V_n(\alpha'') = V_{n-1}(\alpha'')\setminus\mathcal{C}_{p'_n/q'_n}(\alpha'')$ if $|\alpha' - p'_n/q'_n| < 1/2(q'_n)^3$ and
- $V_n(\alpha'') = V_{n-1}(\alpha'')$ otherwise.

Then, the hypotheses of lemma 2 are satisfied and (as in proposition 2), we have

$$\begin{aligned}\log \mathrm{rad}(V_n(\alpha')) - \log \mathrm{rad}(V_{n-1}(\alpha')) &\le \frac{\log|\alpha' - p'_n/q'_n|}{q'_n} + C\frac{\log q'_n}{q'_n}\\ &\le -\Phi_n(\alpha') + \Phi_{n-1}(\alpha') + (C-1)\frac{\log q'_n}{q'_n},\end{aligned}$$

where C is the constant provided by lemma 2. The Siegel disk $\Delta_{\alpha'}$ is contained in the intersection of the sets $V_n(\alpha')$, and so,

$$\log r(\alpha') - \log \operatorname{rad}(V_{n_0}(\alpha')) \leq -\Phi(\alpha') + \Phi_{n_0}(\alpha') + (C-1) \sum_{n \geq n_0+1} \frac{\log q'_n}{q'_n}.$$

□

As α' tends to p/q, each q'_{n_0+k} (for $k \geq 1$) tends to ∞, thus the n_0+k-th summand tends to 0. Since the sum is dominated by a summable sequence $(\log(F_n)/F_n)$, this yields

$$\sum_{n \geq n_0+1} \frac{\log q'_n}{q'_n} \to 0.$$

Moreover, $\Phi_{n_0-1}(\alpha')$ converges to $\Phi_{\mathrm{trunc}}(p/q)$ and by lemma 3,

$$\limsup_{\alpha' \to p/q,\ \alpha' \in \mathbb{R} \setminus \mathbb{Q}} \Phi_{n_0}(\alpha') + \log \operatorname{rad}(\mathbb{C} \setminus Y(\alpha')) \leq \Upsilon(p/q).$$

This completes the proof of inequality (4).

4.3. **Proof of lemma 3.** We recall that $\alpha' = p/q + \varepsilon$ is real, and that n_0 depends on the sign of ε.

Lemma 6. *For $\varepsilon \in \mathbb{R}^*$ small enough, let z_ε be a periodic point of $P_{\alpha'}$ in the cycle $\mathcal{C}_{p/q}(\alpha')$. Then,*

$$\log |z_\varepsilon| + \beta'_{n_0-1} \log \frac{1}{\alpha'_{n_0}} = \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q} + \mathcal{O}(\varepsilon^{1/q}).$$

Proof. By definition of the asymptotic size, we have

$$L_a(p/q) = \left|\frac{1}{qA}\right|^{1/q} \quad \text{with} \quad P^{\circ q}_{p/q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}).$$

Moreover, $P^{\circ q}_{p/q+\varepsilon}(0) = 0$ and $(P^{\circ q}_{p/q+\varepsilon})'(0) = e^{2i\pi q\varepsilon}$. So

$$P^{\circ q}_{p/q+\varepsilon}(z) = e^{2i\pi q\varepsilon} z + Az^{q+1} + \mathcal{O}(\varepsilon z^2).$$

We know that $z_\varepsilon \longrightarrow 0$ and that $P^{\circ q}_{p/q+\varepsilon}(z_\varepsilon) = z_\varepsilon$. Therefore, we have

$$z_\varepsilon^q = \frac{1 - e^{2i\pi q\varepsilon}}{A}\left(1 + \mathcal{O}(z_\varepsilon)\right) = \frac{-2i\pi q\varepsilon}{A}\left(1 + \mathcal{O}(z_\varepsilon) + \mathcal{O}(\varepsilon)\right).$$

Thus, $z_\varepsilon = \mathcal{O}(\varepsilon^{1/q})$ and

$$\log |z_\varepsilon| = \frac{1}{q} \log \left|\frac{2\pi q\varepsilon}{A}\right| + \mathcal{O}(\varepsilon^{1/q}).$$

Observe that

$$\frac{1}{q} \log \left|\frac{2\pi q\varepsilon}{A}\right| = \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q} + \frac{1}{q} \log q^2 |\varepsilon|.$$

Now, if α' is sufficiently close to p/q, then the n_0-th approximant p'_{n_0}/q'_{n_0} to α' is p/q, and therefore when ε's sign is fixed, n_0 is fixed, and the numbers q'_{n_0} and

q'_{n_0-1} are constants. We have

$$\begin{aligned}
\beta'_{n_0-1} &= |q'_{n_0-1}\alpha' - p'_{n_0-1}| = \left|q'_{n_0-1}\Big(\frac{p_{n_0}}{q_{n_0}}+\varepsilon\Big) - p'_{n_0-1}\right| \\
&= \left|\frac{1}{q'_{n_0}} \pm q'_{n_0-1}\varepsilon\right| = \frac{1}{q'_{n_0}} + \mathcal{O}(\varepsilon), \quad \text{and} \\
\beta'_{n_0} &= q'_{n_0}|\varepsilon|, \quad \text{thus} \\
\alpha'_{n_0} &= \frac{\beta'_{n_0}}{\beta'_{n_0-1}} = (q'_{n_0})^2|\varepsilon|(1+\mathcal{O}(\varepsilon)).
\end{aligned}$$

Thus, we have

$$\beta'_{n_0-1}\log|\alpha'_{n_0}| = \Big(\frac{1}{q}+\mathcal{O}(\varepsilon)\Big)\log\big(q^2|\varepsilon|(1+\mathcal{O}(\varepsilon))\big) = \frac{1}{q}\log q^2|\varepsilon| + \mathcal{O}(\varepsilon\log|\varepsilon|).$$

■

Let us now study the dynamical behaviour of $P_{p/q+\varepsilon}$ at the scale of z_ε. For this purpose, we rescale the dynamical plane. More precisely, we introduce the conjugate polynomial

$$Q_\varepsilon : w \mapsto \frac{1}{z_\varepsilon}P_{p/q+\varepsilon}(z_\varepsilon w).$$

This polynomial is conjugate to $P_{p/q+\varepsilon}$. It fixes 0 with derivative $e^{2i\pi(p/q+\varepsilon)}$ and has a cycle of period q containing 1.

As $\varepsilon \to 0$, Q_ε converges uniformly on every compact subset of $\mathbb{C}$ to the rotation $w \mapsto e^{2i\pi p/q}w$. Hence, $Q_\varepsilon^{\circ q}$ converges uniformly on every compact subset of $\mathbb{C}$ to the identity. However, the limit of the dynamics of Q_ε is richer than the dynamics of the identity. In some sense, it contains the real flow of the vector field $2i\pi q w(1-w^q)\frac{\partial}{\partial w}$.

Lemma 7. *We have*

$$Q_\varepsilon^{\circ q}(w) = w + 2i\pi q\varepsilon w(1-w^q) + \varepsilon R_\varepsilon(w),$$

with $R_\varepsilon \to 0$ uniformly on every compact subset of $\mathbb{C}$ as $\varepsilon \to 0$.

Proof. Since

$$P^{\circ q}_{p/q+\varepsilon}(z) = e^{2i\pi q\varepsilon}z + Az^{q+1} + \mathcal{O}(\varepsilon z^2),$$

we have

$$\begin{aligned}
\frac{1}{z_\varepsilon}P^{\circ q}_{p/q+\varepsilon}(z_\varepsilon w) &= e^{2i\pi q\varepsilon}w + Az_\varepsilon^q w^{q+1} + \mathcal{O}(\varepsilon z_\varepsilon w^2) \\
&= w + 2i\pi q\varepsilon(w - w^{q+1}) + \mathcal{O}(\varepsilon^{1+1/q}w^2) + \mathcal{O}(\varepsilon^2 w).
\end{aligned}$$

■

Figure 4 shows some trajectories of the real flow of the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$ for $q=3$. The origin is a center and its basin Ω is colored light grey.

Let us now define

$$Y_\varepsilon = \frac{1}{z_\varepsilon}Y\left(\frac{p}{q}+\varepsilon\right).$$

The set Y_ε contains 1 and we have

$$\log\operatorname{rad}(\mathbb{C}\setminus Y(p/q+\varepsilon)) = \log\operatorname{rad}(Y_\varepsilon) + \log|z_\varepsilon|.$$

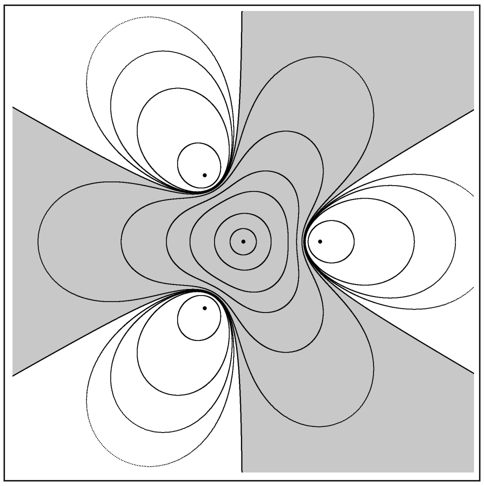

FIGURE 4. Some trajectories of the real flow of the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$ for $q=3$.

Thus, we must show that

$$\limsup_{\varepsilon\to 0,\ \varepsilon\in\mathbb{R}} \log \operatorname{rad}(\mathbb{C}\setminus Y_\varepsilon) \le 0.$$

Set $\overline{Y_\varepsilon} = Y_\varepsilon \cup \{\infty\}$. This set is compact in $\mathbb{P}^1$. Without loss of generality, extracting a subsequence if necessary, we may assume that it converges for the Hausdorff topology on compact subsets of $\mathbb{P}^1$ to some limit $\overline{Y_0}$ as $\varepsilon \to 0$. We define $Y_0 = \overline{Y}_0 \setminus \{\infty\}$. Each $\overline{Y_\varepsilon}$ is connected and contains 1 and ∞. Passing to the limit, we see that $\overline{Y_0}$ is also connected and contains 1 and ∞. Moreover, Q_ε converges uniformly on compact subsets of $\mathbb{C}$ to the rotation $w\mapsto e^{2i\pi p/q}w$. Since $Q_\varepsilon(Y_\varepsilon)=Y_\varepsilon$, we see that Y_0 is invariant under this rotation. Note that $Q_\varepsilon^{\circ q}(Y_\varepsilon)\subset Y_\varepsilon$ and

$$Q_\varepsilon^{\circ q}(w) = w + 2i\pi q\varepsilon w(1-w^q) + \varepsilon R_\varepsilon(w)$$

with $R_\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $\varepsilon\to 0$. It follows that Y_0 is forward invariant under the real flow of the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$. Consider the map $\phi : w \mapsto \zeta = w^q/(w^q-1)$. It is the composition of $w\mapsto w^q$, (which identifies the quotient of $\mathbb{P}^1$ under the rotation of angle $1/q$ with $\mathbb{P}^1$), with a Möbius transformation fixing 0, sending 1 to ∞, and ∞ to 1. It sends the above vector field to the circular vector field $(2\pi q^2)i\zeta\frac{\partial}{\partial\zeta}$. It follows that Y_0 contains the set $\phi^{-1}(\mathbb{C}\setminus\mathbb{D})$. Thus, we have

$$\limsup_{\varepsilon\to 0,\ \varepsilon\in\mathbb{R}} \log \operatorname{rad}(\mathbb{C}\setminus Y_\varepsilon) \le \log \operatorname{rad}(\phi^{-1}(\mathbb{D})) = 0.$$

The proof of lemma 3 is completed.

5. Yoccoz's renormalization techniques.

In this section, we present the techniques of renormalization developed by Yoccoz [Y]. We will follow the presentation given by Pérez-Marco [PM].

Remark. There will be many constants in the discussion. Their sharp value is not important for the application we will make here, so we did not try to optimize them. Moreover, in many estimates where $C\delta$ appears, it can be weakened to $\varepsilon(\delta)$, where $\varepsilon(x) \underset{x\to 0}{\longrightarrow} 0$, while still applying to our proof.

5.1. **Renormalization principle.** Here, we recall what Pérez-Marco writes in [PM] section III, adapting it to the setting of maps which are close to translations.

We denote by T the translation $Z \mapsto Z+1$, by $S(\alpha)$ the space of univalent mappings $F : \mathbb{H} \to \mathbb{C}$ such that $F \circ T = T \circ F$ and such that $F(Z) - Z \to \alpha$ as $\mathrm{Im}(Z) \to +\infty$. This space is compact for the topology of uniform convergence on compact subsets of $\mathbb{H}$.

Given $\delta > 0$, we denote by $S_\delta(\alpha)$ the space of maps $F \in S(\alpha)$ such that

$$(\forall Z \in \mathbb{H}) \qquad |F(Z) - Z - \alpha| \leq \delta\alpha \quad \text{and} \quad |F'(Z) - 1| \leq \delta. \tag{6}$$

Such a function F extends continuously to $\mathbb{H} \cup \mathbb{R}$.

Step 1. Assume $F \in S_\delta(\alpha)$ and define $l = i\mathbb{R}$ and $l' = [0, F(0)]$. If δ is sufficiently small (for example $\delta < 1/10$), $l \cup l' \cup F(l)$ bounds an open strip $\mathcal{U}$ in $\mathbb{C}$. Gluing the curves l and $F(l)$ in the boundary of $\overline{\mathcal{U}}$ via F, we obtain a surface $\mathcal{V}$, whose remaining boundary corresponds to the segment l'. Its interior is a Riemann surface for the complex structure inherited from $\overline{\mathcal{U}}$ (the gluing is analytic). It is biholomorphic to the punctured disk $\mathbb{D}^*$. Lifting via $Z \mapsto z = e^{2i\pi Z}$, we get an injective holomorphic map $L : \mathcal{U} \to \mathbb{H}$ which extends continuously to $\overline{\mathcal{U}}$ and such that

$$(\forall Z \in l) \qquad L(F(Z)) = L(Z) + 1.$$

We normalize L by requiring $L(0) = 0$.

Proposition 3. *For all $\delta \in]0, 1/10[$, all $\alpha \in]0, 1[$, all $F \in S_\delta(\alpha)$, and all $Z \in \overline{\mathcal{U}}$,*

$$\mathrm{Im}(Z) - 2\delta < \alpha \mathrm{Im}(L(Z)) < \mathrm{Im}(Z) + 2\delta. \tag{7}$$

Proposition 4. *Under the same assumptions, the map L extends to a univalent map on*

$$\mathcal{W} = \overline{\mathcal{U}} \cup \{Z \in \mathbb{C}\ ;\ -1 \leq \mathrm{Re}(Z) \leq 0 \text{ and } \mathrm{Im}(Z) \geq 4\delta\},$$

From now on, L will refer to this extension. The definition of $\mathcal{W}$ is so that any point $Z \in W$ is eventually mapped to $\mathcal{U}$ under iteration of F: $F^k(Z) = Z' \in \mathcal{U}$ for some $k \in \mathbb{N}$. Then, one defines $L(Z) = L(Z') - k$. In particular, L conjugates F to the translation T.

Step 2. Given $\delta \in]0, 1/10[$ and $F \in S(\alpha)$, we can define inductively a sequence of univalent maps $(F_n)_{n\geq 0}$ such that $F_n \in S(\alpha_n)$. The construction depends on the choice at each step of some real number $t_n > 0$. We start with $F_0 = F - a_0$ (where $a_0 = \lfloor \alpha \rfloor$) and we assume that F_n is constructed. We choose t_n such that the fundamental estimates (6) hold for $\mathrm{Im}(Z) \geq t_n$ (which is always possible). It follows that $G_n : Z \mapsto F_n(Z + it_n) - it_n$ belongs to $S_\delta(\alpha_n)$. For G_n, we construct $\mathcal{U}_n$, $\mathcal{W}_n$ and L_n as above. Let H_n be defined on $L_n\{Z \in \overline{\mathcal{U}}\ ;\ \mathrm{Im}(z) > 4\delta\}$ by $H_n(z) = L_n \circ T^{-1} \circ L_n^{-1}$. Note that, by proposition 3, if $\mathrm{Im}(Z) > 6\delta/\alpha_n$, there exists an integer k such that $Z - k$ belongs to D, the domain of definition of H_n.

Then, $D + \mathbb{Z}$ contains the half plane "$\mathrm{Im}(Z) > 6\delta/\alpha_n$". Moreover, the map H_n commutes with the translation T on the set of points in $L_n(i[0, +\infty[)$ whose imaginary part is $> 6\delta/\alpha_n$. This set being analytically removable, this implies H_n extends univalently to the upper half-plane $\{Z \in \mathbb{C} \mid \mathrm{Im}(Z) > 6\delta/\alpha_n\}$. Moreover, as $\mathrm{Im}(Z) \to +\infty$, $H_n(Z) - Z \to -1/\alpha_n = -a_{n+1} - \alpha_{n+1}$.

We set

$$\mathcal{W}'_n = \mathcal{W}_n + it_n$$

and we define $K_n : \mathcal{W}'_n \to \mathbb{C}$ by

$$K_n(Z) = s \circ L_n(Z - it_n) - i\frac{6\delta}{\alpha_n}$$

where $s(x+iy) = -x+iy$, and $F_{n+1} \in S(\alpha_{n+1})$ defined on $\mathbb{H}$ by

$$F_{n+1} = K_n \circ T^{-1} \circ K_n^{-1} - a_{n+1}.$$

Note that on $\mathcal{W}'_n \cap F_n^{-1}(\mathcal{W}'_n)$, K_n conjugates F_n to T^{-1}.

Step 3. Next, to a point $Z \in \mathbb{H}$, we associate a sequence $(Z_n)_{n\geq 0}$ as follows. We define $Z_0 = Z$. If $d_n = \mathrm{Im}(Z_n) \geq 4\delta + t_n$, we choose Z'_n such that $Z_n - Z'_n \in \mathbb{Z}$ and $-1 \leq \mathrm{Re}(Z'_n) < 0$, and we define

$$Z_{n+1} = K_n(Z'_n).$$

The sequence $(Z_n)_{n\geq 0}$ may be finite or infinite. The estimates of proposition 3 imply that for $n \geq 0$ such that Z_{n+1} is defined,

$$\mathrm{Im}(Z_n) - t_n - 8\delta \leq \alpha_n \mathrm{Im}(Z_{n+1}) \leq \mathrm{Im}(Z_n) - t_n - 4\delta.$$

For $n_0 \geq 0$:

$$\sum_{n=0}^{n_0-1} \beta_{n-1}(t_n + 4\delta) \leq d_0 - \beta_{n_0-1} d_{n_0} \leq \sum_{n=0}^{n_0-1} \beta_{n-1}(t_n + 8\delta) \tag{8}$$

Which implies

$$\sum_{n=0}^{n_0-1} \beta_{n-1} t_n \leq d_0 - \beta_{n_0-1} d_{n_0} \leq 32\delta + \sum_{n=0}^{n_0-1} \beta_{n-1} t_n \tag{9}$$

Indeed, $1 + \beta_0 + \cdots + \beta_{n-2} \leq 4$ since $\beta_{-1} = 1$, $\beta_0 = \alpha_0 \leq 1$ and, $\beta_{n+2} \leq \beta_n/2$.

Proposition 5. *If $Z \in \mathbb{H}$ and if there exists $m \geq 0$ such that $F^{\circ m}(Z) \notin \mathbb{H}$, then the sequence $(Z_n)_{n\geq 0}$ is finite.*

Proof. Let H_n be the half plane defined by "$\mathrm{Im} Z > t_n$". If Z_n is defined, let $1+k_n$ (with $k_n \geq 0$) be the rank of the first iterate of Z_n under $F_n : \mathbb{H} \to \mathbb{C}$ that leaves H_n. Note that if $k_n = 0$, then Z_{n+1} is not defined. Now, if Z_{n+1} is defined and $k_{n+1} > 0$, this means that $Z_n - k_{n+1}$ is eventually mapped back to $\mathcal{U}_n$ by iteration of F_n, without leaving H_n. Therefore (since $|F_n(Z) - (Z + \alpha_n)| < \alpha_n/10$ on H_n),

$$k_{n+1} \leq \frac{11}{10}\alpha_n k_n.$$

Since $\alpha_n \alpha_{n+1} \leq 1/2$ this implies $k_{n+2} \leq \frac{121}{200} k_n$ whenever defined, from which the proposition follows. ∎

We can now reformulate Theorem III.1.1 in [PM] as follows.

Proposition 6. *Assume we can choose the sequence $(t_n)_{n\geq 0}$ so that the n-th renormalization F_n satisfies the fundamental estimates (6) when $\mathrm{Im}(Z) > t_n$ and so that*

$$\Phi = \sum_{n=0}^{+\infty} \beta_{n-1} t_n < +\infty.$$

Then F is linearizable and its Siegel disk contains the following upper half-plane:

$$\{Z \in \mathbb{C} \mid \mathrm{Im}(Z) > \Phi + 32\delta\}\,.$$

Proof. It is enough to prove that all point Z in the half plane has infinite orbit. By proposition 5, this follows from the sequence Z_n being infinite. Indeed, assume Z_n is defined. According to the previous computations,

$$\begin{aligned}
\beta_{n-1}d_n &\geq d_0 - \sum_{k=0}^{n-1}\beta_{k-1}t_k - (1+\cdots+\beta_{n-2})8\delta \\
&= (d_0 - \Phi - 32\delta) + \beta_{n-1}t_n + \sum_{k=n+1}^{+\infty}\beta_{k-1}t_k + \\
&\quad \big(4-(1+\cdots+\beta_{n-1})\big)8\delta + \beta_{n-1}8\delta \\
&\geq \beta_{n-1}(t_n+8\delta)
\end{aligned}$$

Therefore, $d_n \geq t_n + 8\delta$. Since $8 > 4$, this implies Z_{n+1} is defined. ■

Also, there is a correspondence between periodic orbits for F and for F_n. Given a map $F : \mathbb{H} \to \mathbb{C}$ that commutes with T, we will say that $Z \in \mathbb{C}$ is periodic with rotation number p/q when $F^q(Z) = Z + p$. In this case, p and q need not to be coprime.

Proposition 7. *Let $n_0 \geq 0$. If F_{n_0} has a fixed point with rotation number $0/1$ and imaginary part h_{n_0}, then F has a periodic orbit with rotation number p_{n_0}/q_{n_0} contained in the strip*

$$\{Z \in \mathbb{C};\ H \leq \mathrm{Im}(Z) \leq H + 32\delta\} \quad \textit{with} \quad H = \beta_{n_0-1}h_{n_0} + \sum_{n=0}^{n_0-1}\beta_{n-1}t_n.$$

Reciprocally, if F has a periodic orbit with rotation number p_{n_0}/q_{n_0} whose imaginary part h_0 satisfies $h_0 > \sum_{n=0}^{n_0-1}\beta_{n-1}t_n + 32\delta$, then F_{n_0} has a fixed point of rotation number $0/1$, and height h_{n_0} satisfying

$$h_0 - 32\delta \leq \beta_{n_0-1}h_{n_0} + \sum_{n=0}^{n_0-1}\beta_{n-1}t_n \leq h_0.$$

Proof. Same as in [PM] annex 2.e. ■

In the previous proposition, the reader should be aware that $F_{n_0}(Z) = z + k$ with $k \in \mathbb{Z}^*$ is not considered as a fixed point with rotation number $0/1$.

5.2. **Proof of proposition 3.** To obtain inequality (7) we will control of the distortion of quasiconformal maps as follows. Since $F(Z) - Z - \alpha$ is periodic of period 1, we have

$$|F(Z) - Z - \alpha| \leq \delta\alpha e^{-2\pi \mathrm{Im}(Z)} \quad \text{and} \quad |F'(Z) - 1| \leq \delta e^{-2\pi \mathrm{Im}(Z)}.$$

Let B be the half-band $\{Z \in H \mid 0 < \mathrm{Re}(Z) < 1\}$. Let $H : \overline{B} \to \overline{\mathcal{U}}$ be the map defined by

$$H(X+iY) = i\alpha Y + X\big[F(i\alpha Y) - i\alpha Y\big]. \tag{10}$$

An elementary computation shows that $\|\overline{\partial}H/\partial H\|_\infty < 1$ and if we set

$$K_H = \frac{1+|\overline{\partial}H/\partial H|}{1-|\overline{\partial}H/\partial H|},$$

One computes that

$$\begin{array}{rcl} \left|\partial H - \alpha\right| & \leq & \alpha\delta e^{-2\pi\alpha Y} \\ \left|\overline{\partial}H\right| & \leq & \alpha\delta e^{-2\pi\alpha Y} \end{array}$$

And therefore[4]

$$K_H(X+iY) \leq \frac{1}{1-2\delta e^{-2\pi\alpha Y}}.$$

Then, using $\delta < 1/10$, we have the inequality

$$K_H(X+iY) \leq 1 + \frac{5}{2}\delta e^{-2\pi\alpha Y}.$$

In particular, H is a $(1+\frac{5}{2}\delta)$-quasiconformal homeomorphism. Moreover, by definition

$$\mathrm{Im}(H(Z)) - \alpha\delta \leq \alpha\mathrm{Im}(Z) \leq \mathrm{Im}(H(Z)) + \alpha\delta,$$

and thus for all $Z \in \overline{\mathcal{U}}$, since $\alpha < 1$:

$$\mathrm{Im}(Z) - \delta \leq \alpha\mathrm{Im}(H^{-1}(Z)) \leq \mathrm{Im}(Z) + \delta.$$

Since L is conformal, the map $G = L\circ H$ is quasiconformal with the same dilatation as H. Moreover, $G(iY+1) = G(iY)+1$ and so, since the imaginary axis is quasiconformally removable, G extends to a quasiconformal homeomorphism $\mathbb{H} \to \mathbb{H}$. We will show that for all $Z \in \mathbb{H}$, we have

$$\alpha\mathrm{Im}(Z) - \delta \leq \alpha\mathrm{Im}(G(Z)) \leq \alpha\mathrm{Im}(Z) + \delta.$$

It follows that

$$\mathrm{Im}(Z) - 2\delta \leq \alpha\mathrm{Im}(H^{-1}(Z)) - \delta \leq \alpha\mathrm{Im}(L(Z)) \leq \alpha\mathrm{Im}(H^{-1}(Z)) + \delta \leq \mathrm{Im}(Z) + 2\delta.$$

Lemma 8. *Assume $\psi : (\mathbb{D},0) \to (\mathbb{D},0)$ is a K-quasiconformal homeomorphism. Then, for all $z \in \mathbb{D}$,*

$$4^{1-K}|z|^K \leq |\psi(z)| \leq 4^{1-1/K}|z|^{1/K}.$$

Proof. To prove the upper bound, note that ψ sends the annulus $\mathbb{D}\setminus[0,z]$ to an annulus separating 0 and $\psi(z)$ from S^1. The modulus is divided by at most K. So,

$$|\psi(z)| \leq \mu^{-1}\left(\frac{\mu(|z|)}{K}\right),$$

where, for $r \in]0,1[$, $\mu(r)$ is the modulus of the annulus $\mathbb{D}\setminus[0,r]$ (it is a decreasing function). The estimate

$$\mu^{-1}\left(\frac{\mu(r)}{K}\right) \leq 4^{1-1/K}r^{1/K}$$

can be found in [AVV] corollary 5.44.

The lower bound is obtained by applying the upper bound to ψ^{-1} which is K-quasiconformal. ∎

[4] A quick majoration yields a 4, having a 2 requires more care.

Lemma 9. *If $\Psi : \mathbb{H} \to \mathbb{H}$ is a K-quasiconformal homeomorphism such that $\Psi \circ T = T \circ \Psi$, then*

$$\frac{1}{K}\mathrm{Im}(Z) - \frac{K-1}{2\pi K}\log 4 \le \mathrm{Im}(\Psi(Z)) \le K\mathrm{Im}(Z) + \frac{K-1}{2\pi}\log 4.$$

Proof. Ψ is the lift, via $Z \mapsto z = e^{2i\pi Z}$, of a K-quasiconformal homeomorphism $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ as in the previous lemma. ■

We now come to the control of the quasiconformal homeomorphism G.

Lemma 10. *Let ε and η be any two positive real numbers. Assume $G : \mathbb{H} \to \mathbb{H}$ is a $(1+\varepsilon)$-quasiconformal homeomorphism such that $G \circ T = T \circ G$ and*

$$K_G(X + iY) \le 1 + \varepsilon e^{-\eta Y}.$$

Then,

$$\mathrm{Im}(Z) - \frac{\varepsilon}{\eta} - \frac{\varepsilon}{2\pi(1+\varepsilon)}\log 4 \le \mathrm{Im}(G(Z)) \le \mathrm{Im}(Z) + \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi}\log 4,$$

which yields

$$\big|\mathrm{Im}(G(Z)) - \mathrm{Im}(Z)\big| \le \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi}\log 4.$$

Proof. We can write $G = G_2 \circ G_1$ with

$$G_1(X + iY) = X + i\frac{1}{1+\varepsilon}\left(Y - \frac{\varepsilon}{\eta}e^{-\eta Y} + \frac{\varepsilon}{\eta}\right).$$

An elementary computation shows that

$$K_{G_1}(X + iY) = \frac{1+\varepsilon}{1+\varepsilon e^{-\eta Y}} \quad \text{and} \quad \mathrm{Im}(G_1(Z)) \le \frac{1}{1+\varepsilon}\left(\mathrm{Im}(Z) + \frac{\varepsilon}{\eta}\right).$$

So, we can apply the previous lemma to G_2 with $K = 1+\varepsilon$, which yields the upper bound for $\mathrm{Im}(G(Z))$.

To get the lower bound, we use the same argument, writing $G = G_4 \circ G_3$ with

$$G_3(X + iY) = X + i(1+\varepsilon)\left(Y + \frac{1}{\eta}\log\frac{1+\varepsilon e^{-\eta Y}}{1+\varepsilon}\right).$$

We have

$$K_{G_3}(X + iY) = \frac{1+\varepsilon}{1+\varepsilon e^{-\eta Y}} \quad \text{and} \quad \mathrm{Im}(G_3(Z)) \ge (1+\varepsilon)\left(\mathrm{Im}(Z) - \frac{\varepsilon}{\eta}\right).$$

■

To conclude the proof of the proposition, we apply the previous lemma to $\varepsilon = \frac{5}{2}\delta$ and $\eta = 2\pi\alpha$. Using $\alpha < 1$, we have

$$\frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi}\log 4 = \frac{5\delta}{4\pi\alpha}\big(1 + \alpha\log 4\big) \le \frac{\delta}{\alpha}.$$

5.3. **Controlling the height of renormalization.** In this section, we determine an upper bound for the height t above which the fundamental estimates (6) are satisfied. The first result is due to Yoccoz (it easily follows from the compactness of $S(0)$, but the interested reader can find sharper bounds in [Y], in the lemma of section 3.2, page 26).

Proposition 8. *For all $\delta \in]0, 1/10[$, there exists a constant C_δ such that for all $F \in S(\alpha)$,*

$$\mathrm{Im}(Z) \geq C_\delta \quad \Longrightarrow \quad |F'(Z) - 1| \leq \delta$$

and

$$\mathrm{Im}(Z) \geq \frac{1}{2\pi} \log \frac{1}{\alpha} + C_\delta \quad \Longrightarrow \quad |F(Z) - Z - \alpha| \leq \delta\alpha.$$

(Of course, $C_\delta \longrightarrow +\infty$ when $\delta \longrightarrow 0$.)

Remark. In particular, F can not have fixed points above $\frac{1}{2\pi} \log \frac{1}{\alpha}$ plus some universal constant.

The next result is a slight generalization of a result of Pérez-Marco.

Proposition 9. *For all $\delta \in]0, 1/10[$, there exists a constant C_δ such that the following holds. Assume $\mathrm{Im}(Z_0) \in \mathbb{H}$, $\alpha \in]0, 1[$ and $F \in S(\alpha)$ has no fixed point except possibly Z_0 and its translates by an integer. If*

$$\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0) + \frac{1}{2\pi} \Big(\log \log \frac{e}{\alpha} - \log(1 + 2\pi \mathrm{Im}(Z_0)) \Big) + C_\delta$$

then

$$|F(Z) - Z - \alpha| \leq \delta\alpha.$$

One can rewrite

$$\log \log \frac{e}{\alpha} - \log(1 + 2\pi \mathrm{Im}(Z_0)) = \log \frac{1 + \log(\alpha^{-1})}{1 + 2\pi \mathrm{Im}(Z_0)}.$$

Thus for $\mathrm{Im}(Z_0) < \log(\alpha^{-1})/2\pi$, this number is positive. From this, and the remark following proposition 8, it follows that we can take the same constants C_δ in propositions 8 and 9.

Remark. It follows that if F has no fixed point, the fundamental estimates (6) are satisfied as soon as

$$\mathrm{Im}(Z) \geq \frac{1}{2\pi} \log \log \frac{e}{\alpha} + C_\delta.$$

This result is due to Pérez-Marco [PM]. This is the form we will use in section 6.

Remark. If $\mathrm{Im}(Z_0) \geq \frac{1}{2} \cdot \frac{1}{2\pi} \log \frac{1}{\alpha}$, it follows from the two propositions and an elementary computation that the fundamental estimates (6) are satisfied as soon as

$$\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0) + 1 + C_\delta.$$

This is the form[5] we will use in section 7.

Proof of proposition 9. Without loss of generality, we may assume that

$$\mathrm{Im}(Z_0) < \frac{1}{2\pi} \log \frac{1}{\alpha}$$

[5]The assumption $\mathrm{Im}(Z_0) \geq \frac{1}{2} \cdot \frac{1}{2\pi} \log \frac{1}{\alpha}$ can be replaced by $\mathrm{Im}(Z_0) \geq \mu \cdot \frac{1}{2\pi} \log \frac{1}{\alpha}$ with $\mu \in]0, 1[$, giving the condition $\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0) + \log(\mu^{-1})/2\pi + C_\delta$.

since otherwise, the result follows from proposition 8. Let us set $r = e^{-2\pi \mathrm{Im}(Z_0)}$ if F has a fixed point at Z_0 and $r = 1$ if F has no fixed point. Then, $\alpha < r$.

Let us now define $u(Z) = F(Z) - Z$. Since u is $\mathbb{Z}$-periodic, there exists a function $g : \mathbb{D}^* \to \mathbb{C}$ such that $u(Z) = g(e^{2i\pi Z})$. The map g extends holomorphically at 0 by $g(0) = \alpha$. We need now to find an upper bound on $|z|$ which ensures that $|g(z) - \alpha| < \alpha\delta$. By compactness of $S(0)$, we can find a (universal) radius $r_0 < 1$ such that on $B(0, r_0)$, g takes its values in $B(0, e)$. Moreover, if F has a fixed point at Z_0, we define $\zeta_0 = e^{2i\pi Z_0}$. Then $g(\zeta_0) = 0$ and g does not vanish in $\mathbb{D} \setminus \{\zeta_0\}$. If F has no fixed point, g does not vanish in $\mathbb{D}$. In both cases, the map $g : B(0, r_0) \setminus \{\zeta_0\} \to B(0, e) \setminus \{0\}$ is contracting for the hyperbolic metrics.

The coefficient of the hyperbolic metrics of $B(0, e) \setminus \{0\}$ at the point α is equal to $1/(\alpha \log(e/\alpha))$, so at first approximation, points at hyperbolic distance of order $\delta/\log(e/\alpha)$ should be at Euclidean distance of order $\delta\alpha$. The lemma below makes a rigorous statement.

Lemma 11. *$(\forall\delta \in]0, 1/10[)$, $(\forall\alpha \in]0, 1[)$,*

$$d_{B(0,e)\setminus\{0\}}(\alpha, z) \le \frac{\delta}{2\log e/\alpha} \quad \Longrightarrow \quad |z - \alpha| \le \delta\alpha.$$

Proof. For $x < \alpha$, let $\rho(x)$ be the infimum of the coefficient of the hyperbolic metric on the Euclidean circle of center α and radius x. If $|z - \alpha| > \delta\alpha$, then the hyperbolic geodesic in $B(0, e) \setminus \{0\}$ from α to z is longer than

$$\int_0^{\delta\alpha} \rho(x)dx.$$

Let us introduce the function

$$\begin{cases} f(x) = \frac{1}{x \log e/x} & 0 < x \le 1 \\ f(x) = 1 & 1 \le x < e \end{cases}$$

Then f is decreasing, and $\rho(x) = f(x + \alpha)$. Moreover, f is C^1 and convex, and therefore above its tangents. Therefore

$$\begin{aligned} d_{B(0,e)\setminus\{0\}}(\alpha, z) &\ge \int_\alpha^{\alpha+\delta\alpha} f(x)dx \\ &\ge \int_\alpha^{\alpha+\delta\alpha} \big(f(\alpha) + (x-\alpha)f'(\alpha)\big)dx \\ &= \frac{\delta}{\log e/\alpha}\left(1 - \frac{\delta}{2}\Big(1 - \frac{1}{\log(e/\alpha)}\Big)\right) \ge c\frac{\delta}{\log e/a} \end{aligned}$$

with $c = 19/20 > 1/2$. □

The next lemma is also motivated by a hyperbolic metrics coefficient computation.

Lemma 12. *$(\forall r_0 < 1)$, $(\exists\gamma > 0)$, $(\forall\delta \in]0, 1/10[)$, if $0 < \alpha < r \le 1$, then*

$$|z| \le \gamma\delta r \frac{\log e/r}{\log e/\alpha} \quad \Longrightarrow \quad d_{B(0,r_0)\setminus\{r\}}(0, z) \le \frac{\delta}{2\log e/\alpha}.$$

Proof. First case: $r \geq r_0/2$.
When $|z| \leq \delta r_0$, then

$$d_{B(0,r_0)\setminus\{r\}}(0,z) \leq d_{B(0,r_0/2)}(0,z) = \log \frac{1+2|z|/r_0}{1-2|z|/r_0} \leq \frac{5|z|}{r_0}.$$

Thus, when $r \geq r_0/2$, we can take any γ such that

$$\gamma \leq \min_{r\in[r_0/2,1]} \frac{r_0}{10r\log e/r} = \frac{1}{10\log e/r_0}.$$

Second case: $r < r_0/2$.
We first solve the problem when $r_0 = 1$. Let $\rho(z)|dz|$ be the element of hyperbolic metric on $\mathbb{D}\setminus\{r\}$. A computation gives

$$\rho(z) = \frac{1-r^2}{|1-rz|\cdot|z-r|\cdot\log\left(\frac{|1-rz|}{|z-r|}\right)}.$$

A majoration gives, for $|z| < r/10$, $\rho(z) < 10/(9r\log|s|^{-1})$ with $s = (z-r)/(1-rz)$. Then, $|s| < 11r/(10+r^2) < 11r/10$. Thus

$$\forall r \in]0,1/2[,\ \forall z \text{ with } |z| \leq \frac{r}{10},\ \rho(z) \leq \frac{12}{r\log e/r}.$$

Therefore, for $r_0 = 1$, we can take $\gamma = \gamma_1$, with

$$\gamma_1 = 12.$$

For $r_0 \in]0,1[$, we rescale the problem by the factor $1/r_0$, and according to what we did above, a sufficient condition on z is that

$$\left|\frac{z}{r_0}\right| < \gamma_1\delta\frac{r}{r_0}\frac{\log er_0/r}{\log e/\alpha},$$

Then, using $r < r_0/2$, we can take

$$\gamma \leq \gamma_1\frac{\log 2e}{\log 2e + \log r_0^{-1}}.$$

$\square$

The two previous lemmas show that there exists $\gamma > 0$ such that for all $\delta \in]0,1/10[$,

$$|z| \leq \gamma\delta r\frac{\log e/r}{\log e/\alpha} \quad\Longrightarrow\quad |g(z)-\alpha| \leq \delta\alpha.$$

As a consequence,

$$\mathrm{Im}(Z) \geq \frac{1}{2\pi}\left(\log\frac{1}{\gamma\delta} + \log\frac{1}{r} + \log\frac{\log e/\alpha}{\log e/r}\right) \quad\Longrightarrow\quad |F(Z)-Z-\alpha| \leq \delta\alpha.$$

$\blacksquare$

6. Proof of inequality (5) in most cases.

We will use the following fact several times. Assume $\alpha' \in]0,1[$ tends to $\alpha \in]0,1[$ and $F_{\alpha'} \in S(\alpha')$ tends to the translation $T_\alpha : Z \mapsto Z+\alpha$ uniformly on every compact subset of $\mathbb{H}$. Then, the convergence is uniform on every upper half-plane of the form "$\mathrm{Im}(Z) \geq t > 0$", and $F'_{\alpha'} \longrightarrow 1$ uniformly on these half-planes. Therefore, given $\delta \in]0,1/10[$ and $t > 0$, if $F_{\alpha'}$ is sufficienlty close to T_α, the map $G_{\alpha'} : Z \mapsto F_{\alpha'}(Z+it)-it$ belongs to $S_\delta(\alpha')$ (it is important that $\alpha \neq 0$). For $G_{\alpha'}$ we can construct $\mathcal{U}_{\alpha'}$, $\mathcal{W}_{\alpha'}$ and $L_{\alpha'}$ as in section 5.1. We then define $\mathcal{W}'_{\alpha'} = \mathcal{W}_{\alpha'} + it$, $K_{\alpha'} : \mathcal{W}'_{\alpha'} \to \mathbb{C}$ by

$$K_{\alpha'}(Z) = s \circ L_{\alpha'}(Z - it) - i\frac{6\delta}{\alpha'}$$

and $F_{\alpha',1} \in S(\alpha'_1)$ by

$$F_{\alpha',1} = K_{\alpha'} \circ T^{-1} \circ K_{\alpha'}^{-1} - \left\lfloor \frac{1}{\alpha'} \right\rfloor,$$

where $s(Z) = -\overline{Z}$.

As $F_{\alpha'}$ tends to T_α, $L_{\alpha'}$ tends to $Z \mapsto Z/\alpha$ uniformly on every compact subset of $\mathcal{W}$. Indeed, as in section 5.2, we can write $L = G \circ H^{-1}$ where H is defined by equation (10). Then, H converges to $Z \mapsto \alpha Z$ uniformly on $\overline{B}$ and $G : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is a K-quasiconformal homeomorphism such that $G(0) = 0$ and $G \circ T = T \circ G$. Moreover, $K \longrightarrow 1$ as $F_{\alpha'} \longrightarrow T_\alpha$. Thus G converges to the identity uniformly on every compact subset of $\overline{\mathbb{H}}$.

It follows that $K_{\alpha'}$ tends to $Z \mapsto (s(Z) - it - i6\delta)/\alpha$ uniformly on every compact subset of $\mathcal{W}'_{\alpha'}$ and $F_{\alpha',1}$ tends to the translation $Z \mapsto Z + \alpha_1$ uniformly on every compact subset of $\mathbb{H}$.

6.1. **Brjuno numbers.** Assume $\alpha \in]0,1[$ is a Brjuno number and let $\phi_\alpha : \mathbb{D} \to \Delta_\alpha$ be a linearizing parameterization. Note that $|\phi'_\alpha(0)| = r(\alpha)$. For α' close to α, let us define

$$f_{\alpha'} = \phi_\alpha^{-1} \circ P_{\alpha'} \circ \phi_\alpha$$

on $\phi_\alpha^{-1}(\Delta_\alpha \cap P_{\alpha'}^{-1}(\Delta_\alpha))$. Since $P_\alpha(\Delta_\alpha) = \Delta_\alpha$ and $P_{\alpha'} \longrightarrow P_\alpha$ as $\alpha' \longrightarrow \alpha$, we see that when $\alpha' \longrightarrow \alpha$, f_α converges uniformly on every compact subset of $\mathbb{D}$ to the rotation of angle α. Note that when α' is a Brjuno number, $f_{\alpha'}$ has a Siegel disk of radius $\rho(\alpha') \leq r(\alpha')/r(\alpha)$. Indeed, the image of this Siegel disk by ϕ_α is contained in the Siegel disk of $P_{\alpha'}$. Finally, let $F_{\alpha'}$ be the lift of $f_{\alpha'}$ via $Z \mapsto e^{2i\pi Z}$ which satisfies $|F(Z) - Z - \alpha'| \longrightarrow 0$ when $\mathrm{Im}(Z) \longrightarrow +\infty$.

Let us now fix $\eta > 0$, $\delta \in]0,1/10[$ and $n_0 \geq 1$. For $n \geq 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\alpha',n+1} \in S(\alpha'_{n+1})$ as in section 5.1.

According to the fact mentioned at the beginning of section 6, and using induction on n_0, we know that provided $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$ is sufficiently close to α, we can take

$$t'_0 = \ldots = t'_{n_0} = \eta/(n_0+1).$$

By proposition 8, for $n \geq n_0 + 1$, we can take

$$t'_n = \frac{1}{2\pi} \log \frac{1}{\alpha'_n} + C_\delta$$

for some constant C_δ which only depends on δ.

It follows from proposition 6 that if $\alpha' \in \mathcal{B}$ is sufficiently close to α, we have

$$\begin{aligned}\log\frac{r(\alpha)}{r(\alpha')} \quad\le\quad \log\frac{1}{\rho(\alpha')} \quad&\le\quad 2\pi\left(\sum_{n=0}^{\infty}\beta'_{n-1}t'_n + 32\delta\right)\\ &\le\quad \Phi(\alpha') - \Phi_{n_0}(\alpha') + 2\pi(\eta + 4\beta'_{n_0}C_\delta + 32\delta)\end{aligned}$$

(we used $\beta'_{n_0} + \beta'_{n_0+1} + \ldots \le 4\beta'_{n_0}$ which follows from $\beta'_{k+1} \le \beta'_k$ and $\beta'_{k+2} \le \beta'_k/2$). Let us rewrite it

$$\Phi(\alpha') + \log r(\alpha') \ge \Phi_{n_0}(\alpha') + \log r(\alpha) - 2\pi(\eta + 4\beta'_{n_0}C_\delta + 32\delta).$$

Letting $\alpha' \longrightarrow \alpha$ and using $\Phi_{n_0}(\alpha') \longrightarrow \Phi_{n_0}(\alpha)$ and $\beta'_{n_0} \longrightarrow \beta_{n_0}$,

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \ge \Phi_{n_0}(\alpha) + \log r(\alpha) - 2\pi(\eta + 4\beta_{n_0}C_\delta + 32\delta).$$

Now, as $n_0 \longrightarrow +\infty$, $\Phi_{n_0}(\alpha) \longrightarrow \Phi(\alpha)$ and $\beta_{n_0} \longrightarrow 0$. Thus

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \ge \Phi(\alpha) + \log r(\alpha) - 2\pi(\eta + 32\delta).$$

Since this is valid for all $\eta > 0$ and $\delta \in]0, 1/10[$, it implies

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \ge \Phi(\alpha) + \log r(\alpha) = \Upsilon(\alpha).$$

6.2. **Rational numbers.** We consider a rational number $\alpha = p/q$ and a Brjuno number α' close to p/q. Let us note α'_n and β'_n the sequences associated to α'. According to the sign of $\varepsilon = \alpha' - p/q$, we associated in section 2.2 to $\alpha = p/q$ an integer $n_0 \in \mathbb{N}$, and finite sequences $\alpha_0, \alpha_1, \ldots, \alpha_{n_0} = 0$, and $p_0/q_0, p_1/q_1, \ldots, p_{n_0}/q_{n_0} = p/q$ such that for all $k \le n_0$, $\alpha'_k \longrightarrow \alpha_k$, $p'_k \longrightarrow p_k$ and $q'_k \longrightarrow q_k$ when $\alpha' \longrightarrow \alpha$ on one side.

We will use the notations of section 4.2. Let z_ε be a point of the cycle $\mathcal{C}_{p/q}(\alpha')$. To study the dynamics of $P_{p/q+\varepsilon}$ at the scale of z_ε, we defined

$$Q_\varepsilon : w \mapsto \frac{1}{z_\varepsilon} P_{p/q+\varepsilon}(z_\varepsilon w).$$

Lemma 7 asserts that

$$Q_\varepsilon^{\circ q}(w) = w + 2i\pi q\varepsilon w(1 - w^q) + \varepsilon R_\varepsilon(w), \tag{11}$$

with $R_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\mathbb{C}$ as $\varepsilon \longrightarrow 0$.

Set $\phi(w) = \omega^q/(1-\omega^q)$ and $\Omega = \phi^{-1}(\mathbb{D})$. It is the preimage by $w \mapsto w^q$ of the half plane "Re$(z) < 1/2$" and is illustrated as a gray set for $q = 3$ in figure 4 page 15. Let $\psi : \Omega \to \mathbb{D}$ be a holomorphic map satisfying $\psi(w)^q = \phi(w)$. Then, $\psi(0) = 0$, $|\psi'(0)| = 1$ and ψ is a conformal representation between Ω and $\mathbb{D}$. It sends the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$ to the vector field $2i\pi q\zeta\frac{\partial}{\partial\zeta}$. We define

$$f_\varepsilon = \psi \circ Q_\varepsilon \circ \psi^{-1}$$

on $\psi(\Omega \cap Q_\varepsilon^{-1}(\Omega))$. As $\varepsilon \longrightarrow 0$, f_ε converges uniformly on every compact subset of $\mathbb{D}$ to the rotation of angle p/q. Moreover by (11) we see that when $\varepsilon \longrightarrow 0$,

$$f_\varepsilon^{\circ q}(z) = z + 2i\pi q\varepsilon z + \varepsilon g_\varepsilon(z),$$

with $g_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\overline{\mathbb{D}}$. Note that when $\alpha' = p/q+\varepsilon$ is a Brjuno number, f_ε has a Siegel disk of conformal radius

$$\rho(\varepsilon) \le r(\alpha')/|z_\varepsilon|.$$

Let F_ε be the lift of f_ε via $Z \mapsto e^{2i\pi Z}$ which satisfies $|F_\varepsilon(Z) - Z - \alpha'| \longrightarrow 0$ when $\mathrm{Im}(Z) \longrightarrow +\infty$. When $\varepsilon \longrightarrow 0$,

$$F_\varepsilon^{\circ q} \circ T^{-p}(Z) = Z + q\varepsilon + \varepsilon G_\varepsilon(Z)$$

with $G_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$.

Let us fix $\delta \in]0, 1/10[$ and $\eta > 0$. For $n \geq 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\varepsilon,n+1} \in S(\alpha'_{n+1})$.

As ε tends to 0, F_ε converges uniformly to the translation by p/q on the upper half-plane $\{Z \in \mathbb{C} \mid \mathrm{Im}(Z) \geq \eta/(n_0+1)\}$. Moreover, for $n \leq n_0 - 1$, as $\varepsilon \longrightarrow 0$, $\alpha'_n \longrightarrow \alpha_n \neq 0$. Thus, if ε is sufficiently close to 0, we can take

$$t'_0 = t'_1 = \ldots = t'_{n_0-1} = t \overset{\text{def}}{=} \eta/(n_0+1).$$

We will call $\mathcal{W}'_{\varepsilon,n}$ and $K_{\varepsilon,n} : \mathcal{W}'_{\varepsilon,n} \to \mathbb{C}$ the objects corresponding to W'_n and K_n defined in section 5.1. When $\varepsilon \longrightarrow 0$, the interior of $\mathcal{W}'_{\varepsilon,n}$ tends to the interior of a set $\mathcal{W}'_{0,n}$ which is the union of two half strips "$-1 \leq \mathrm{Re}(Z) \leq 0$ and $\mathrm{Im}(Z) \geq 4\delta + t$" and "$0 \leq \mathrm{Re}(Z) \leq \alpha_n$ and $\mathrm{Im}(Z) \geq t$". For $n \leq n_0 - 1$, as ε tends to 0, $K_{\varepsilon,n}$ tends to $Z \mapsto (s(Z) - it - i6\delta)/\alpha_n$ uniformly on every compact subset of $\mathcal{W}'_{0,n}$, where $s(Z) = -\overline{Z}$.

Now, when $\varepsilon \longrightarrow 0$, F_{ε,n_0} converges uniformly to the translation $Z \mapsto Z + \alpha_{n_0} = Z + 0$, i.e., to the identity.

Lemma 13. *If ε is small enough, we can take $t'_{n_0} = \eta/(n_0+1)$.*

Proof. Let us now consider the map

$$\Psi_\varepsilon = K_{\varepsilon,n_0-1} \circ \ldots \circ K_{\varepsilon,0}.$$

Its set of definition eventually contains every compact subset of the interior of

$$\mathcal{W}'' = \big\{Z \in \mathbb{C}\ ;\ -\beta_{n_0-1} \leq (-1)^{n_0}\mathrm{Re}(Z) \leq \beta_{n_0-2} \text{ and } \mathrm{Im}(Z) \geq t' - 2\delta\beta_{n_0-2}\big\},$$

with $t' = (t + 6\delta)(1 + \beta_1 + \ldots + \beta_{n_0-2})$. On every of these compact subsets, Ψ_ε eventually conjugates $F_\varepsilon^{\circ q} \circ T^{-p}$ to F_{ε,n_0}.

As ε tends to 0, Ψ_ε converges to $Z \mapsto (s^{n_0}(Z) - it')/\beta_{n_0-1}$, uniformly on every compact subset of the interior of $\mathcal{W}''$. Thus, since $s^{n_0} \circ \Psi_\varepsilon$ is holomorphic, the derivative of $s^{n_0} \circ \Psi_\varepsilon$ converges to $1/\beta_{n_0-1}$, uniformly on every compact subset of the interior of $\mathcal{W}''$. Therefore

$$F_{\varepsilon,n_0}(Z) = Z + \frac{q|\varepsilon|}{\beta_{n_0-1}} + \varepsilon H_\varepsilon(Z)$$

with $H_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$. Since $\alpha'_{n_0} = q|\varepsilon|/\beta'_{n_0-1} = q|\varepsilon|/\beta_{n_0-1} + \mathcal{O}(\varepsilon^2)$, $|F_{\varepsilon,n_0}(Z) - Z - \alpha'_{n_0}| = \alpha'_{n_0} I_\varepsilon(Z)$ with $I_\varepsilon(Z) \longrightarrow 0$ uniformly on every compact subset of $\Psi_0(\mathcal{W}'')$. This set contains "$-1 < \mathrm{Re}(Z) < 1$ and $\mathrm{Im}(Z) > 0$". Since F_{ε,n_0} commutes with T, this implies that $|F_{\varepsilon,n_0}(Z) - Z - \alpha'_{n_0}| = \alpha'_{n_0} I_\varepsilon(Z)$ with $I_\varepsilon(Z) \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$. As a consequence $|F'_{\varepsilon,n_0}(Z) - 1| \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$. ∎

Finally, for $n \geq n_0 + 1$, we can take

$$t'_n = \frac{1}{2\pi} \log \frac{1}{\alpha'_n} + C_\delta$$

where C_δ is the constant in proposition 8. So, if ε is sufficiently small, we have

$$\log \frac{|z_\varepsilon|}{r(\alpha')} \leq 2\pi \left(\sum_{n=0}^{\infty} \beta'_{n-1} t'_n + 32\delta \right) \leq \Phi(\alpha') - \Phi_{n_0}(\alpha') + 2\pi(\eta + 4\beta'_{n_0} C_\delta + 32\delta).$$

Reordering the terms, we obtain

$$\Phi(\alpha') + \log r(\alpha') \geq \log |z_\varepsilon| + \Phi_{n_0}(\alpha') - 2\pi(\eta + 4\beta'_{n_0} C_\delta + 32\delta).$$

As $\varepsilon \longrightarrow 0$, $\log |z_\varepsilon| + \Phi_{n_0}(\alpha')$ tends to $\Upsilon(p/q)$ and β'_{n_0} tends to 0. We therefore have (see lemma 6)

$$\liminf_{\alpha' \to p/q,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Upsilon\left(\frac{p}{q}\right) - 2\pi(\eta + 32\delta)$$

and the proof of inequality (5) at rational numbers is completed since η and δ can be chosen arbitrarily small.

6.3. **Cremer numbers whose Pérez-Marco sum converges.** It is possible to give a proof that works for all Cremer numbers at the same time, but for clarity, we prefer to study two cases (which overlap) separately. Here, we will assume α is a Cremer number such that

$$\sum_{n=0}^{\infty} \beta_{n-1} \log\log \frac{e}{\alpha_n} < \infty.$$

We will call this sum the Pérez-Marco sum, since it was introduced by Pérez-Marco in [PM]. There, he proves that, under this condition, every germ that fixes 0 with derivative $e^{2i\pi\alpha}$ is linearizable or has small cycles.

Let us fix $\eta > 0$, $\delta \in]0, 1/10[$ and $n_0 \geq 1$. For $n_1 \geq n_0$, we set

$$d_{n_1}(\alpha') = d(0, X_{n_1}(\alpha'))$$

(see definition 10 for X_n). Since a Cremer point of a polynomial is accumulated by periodic points, and because we defined $X_{n_1}(\alpha)$ as the set of all periodic points of period $\leq q_{n_1}$ except 0, we have $d_{n_1}(\alpha) \longrightarrow 0$ when $n_1 \longrightarrow +\infty$. Thus, provided n_1 is big enough, we see that for all α' close enough to α, $F_{\alpha'}$ is injective on $B(0, d_{n_1}(\alpha'))$. Let $F_{\alpha'} \in S(\alpha')$ be the lift of $P_{\alpha'}$ via $Z \mapsto d_{n_1}(\alpha') e^{2i\pi Z}$. This amounts to restrict the polynomial $P_{\alpha'}$ to the disk $B(0, d_{n_1}(\alpha'))$ where there are no periodic cycle of period less than or equal to q_{n_1}, except 0. Note that when α' is a Brjuno number, this restriction has a Siegel disk of conformal radius $\leq r(\alpha')$.

For $n \geq 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\alpha',n+1} \in S(\alpha'_{n+1})$.

Lemma 14. *If n_1 is sufficiently large and α' is sufficiently close to α, we can take*

$$t'_0 = t'_1 = \ldots = t'_{n_0} = \eta/(n_0 + 1).$$

Proof. Let us choose ε sufficiently small so that $\alpha'_0 \neq 0, \ldots, \alpha'_{n_0} \neq 0$ for all $\alpha' \in [\alpha - \varepsilon, \alpha + \varepsilon]$. As $n_1 \longrightarrow \infty$, $(\alpha', Z) \mapsto F_{\alpha'}(Z) - Z - \alpha'$ converges uniformly to 0 on $[\alpha - \varepsilon, \alpha + \varepsilon] \times \{Z \in \mathbb{C} \mid \mathrm{Im}(Z) \geq \eta/(n_0 + 1)\}$. If n_1 is sufficiently large, we can therefore take $t'_0 = t'_1 = \ldots = t'_{n_0} = \eta/(n_0 + 1)$. ■

By construction, the maps $F_{\alpha'}$ have no periodic cycle of period less than or equal to q_{n_1}. So, by proposition 7, for $n \leq n_1$, the renormalizations $F_{\alpha',n}$ have no fixed point in $\mathbb{H}$. Thus, by proposition 9, we can take

$$t'_{n_0+1} = \frac{1}{2\pi}\log\log\frac{e}{\alpha_{n_0+1}} + C_\delta \quad \dots \quad t'_{n_1} = \frac{1}{2\pi}\log\log\frac{e}{\alpha_{n_1}} + C_\delta$$

for some constant C_δ which only depends on δ. Finally, by proposition 8, for $n \geq n_1 + 1$, we can take

$$t'_n = \frac{1}{2\pi}\log\frac{1}{\alpha'_n} + C_\delta.$$

Now, proposition 6 yields

$$\frac{1}{2\pi}\log\frac{d_{n_1}(\alpha')}{r(\alpha')} \leq \sum_{n=0}^{\infty}\beta'_{n-1}t'_n + 32\delta.$$

Using the value of t'_n chosen above, we get

$$\begin{aligned}\Phi(\alpha') + \log r(\alpha') \quad &\geq \quad \Phi_{n_1}(\alpha') + \log d_{n_1}(\alpha') - \sum_{n=n_0+1}^{n_1}\beta'_{n-1}\log\log\frac{e}{\alpha'_n}\\ &\qquad -2\pi(\eta + 4\beta'_{n_0}C_\delta + 32\delta).\end{aligned}$$

Let α' tend to α:

$$\begin{aligned}\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}}\Phi(\alpha') + \log r(\alpha') \quad &\geq \quad \Phi_{n_1}(\alpha) + \log d_{n_1}(\alpha) - \sum_{n=n_0+1}^{n_1}\beta_{n-1}\log\log\frac{e}{\alpha_n}\\ &\qquad -2\pi(\eta + 4\beta_{n_0}C_\delta + 32\delta).\end{aligned}$$

Let n_1 tend to $+\infty$. Remind that $d_{n_1}(\alpha) \sim r_{n_1}(\alpha)$, and by definition $\Upsilon(\alpha) = \liminf_{n_1\longrightarrow+\infty}\Phi_{n_1}(\alpha) + r_{n_1}(\alpha)$. Thus,

$$\begin{aligned}\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}}\Phi(\alpha') + \log r(\alpha') \quad &\geq \quad \Upsilon(\alpha) - \sum_{n=n_0+1}^{+\infty}\beta_{n-1}\log\log\frac{e}{\alpha_n}\\ &\qquad -2\pi(\eta + 4\beta_{n_0}C_\delta + 32\delta).\end{aligned}$$

Let n_0 tend to $+\infty$. Since $\beta_{n_0} \longrightarrow 0$ and the Pérez-Marco sum of α was assumed to be convergent, we have

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}}\Phi(\alpha') + \log r(\alpha') \quad \geq \quad \Upsilon(\alpha) - 2\pi(\eta + 32\delta).$$

Since this is valid for arbitrarily small η and δ, this concludes the proof for the case when the Pérez-Marco sum of α converges.

7. Proof of inequality (5) when the Pérez-Marco sum diverges.

In this section, we assume that α is a Cremer number such that

$$\sup_n \beta_{n-1}\log\frac{1}{\alpha_n} = \infty.$$

To deal with this case, we will have to combine techniques of parabolic explosion and techniques of renormalization.

Note that if $\beta_{n-1}\log 1/\alpha_n \leq C < \infty$ for all $n \geq 0$, then $\beta_{n-1}\log\log(e/\alpha_n) \leq \beta_{n-1}\log(1 + C/\beta_{n-1})$ decreases exponentially fast, and α belongs to the set of Cremer numbers studied in section 6.3.

7.1. **Parabolic explosion.** The techniques of parabolic explosion are used to have a precise control on the position of some periodic points of $P_{\alpha'}$ for α' close to α. The maps $P_{\alpha'}$, for α' real, are injective on $B(0,1/2)$. We let $F_{\alpha'} \in S(\alpha')$ be the lift of $P_{\alpha'}$ via $Z \mapsto \frac{1}{2}e^{2i\pi Z}$. Let us recall that we called a periodic point of a map F that commutes with T, a point Z such that $F^q(Z) = p$ for integers $q \in \mathbb{N}^*$ and $p \in \mathbb{Z}$ (p and q need not be coprime). Then q is called the period, and p/q the rotation number.

Lemma 15. *There exists a constant $B_\alpha > 0$ such that for all Brjuno number α' sufficiently close to α and all integer $n \geq 2$,*

a) *if $\frac{1}{2\pi}\Phi_n(\alpha') - B_\alpha > 0$, then $F_{\alpha'}$ has a periodic point with period $\leq q_n$ and imaginary part $h'_0 \geq \frac{1}{2\pi}\Phi_n(\alpha') - B_\alpha$*
b) *in the upper half-plane*

$$\left\{ Z \in \mathbb{C} \mid \mathrm{Im}(Z) \geq \frac{1}{2\pi}\Phi_{n-1}(\alpha') + B_\alpha \right\}$$

the periodic points Z of $F_{\alpha'}$ of period less than or equal to q_n come from $\mathcal{C}_{p_n/q_n}(\alpha')$ (in the sense that $\frac{1}{2}e^{2i\pi Z} \in \mathcal{C}_{p_n/q_n}(\alpha')$).

Proof. For $n \geq 2$ and for $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$, let us define $X_n^*(\alpha') = X_n(\alpha') \setminus \mathcal{C}_{p_n/q_n}(\alpha')$ $r_n^*(\alpha') = \mathrm{rad}(X_n^*(\alpha'))$, $d_n(\alpha') = d(0, X_n(\alpha'))$ and $d_n^*(\alpha') = d(0, X_n^*(\alpha'))$.

By proposition 2 (since $q_2 \geq 2$), we have for α' close enough to α,

$$\Phi_n(\alpha') + \log r_n(\alpha') \leq \Phi_2(\alpha') + \log r_2(\alpha') + C\sum_{k=3}^{n} \frac{\log q_k}{q_k}.$$

As $\alpha' \longrightarrow \alpha$, the right hand term is bounded independently of n. So, there exists a constant C_α such that for all $n \geq 2$ and all $\alpha' \in \mathcal{B}$ sufficiently close to α,

$$\Phi_n(\alpha') + \log d_n(\alpha') \leq \Phi_n(\alpha') + \log r_n(\alpha') \leq 2\pi C_\alpha.$$

Thus, if α' is sufficiently close to α, $F_{\alpha'}$ has a periodic point with imaginary part $h'_0 \geq \dfrac{1}{2\pi}\Phi_n(\alpha') - C_\alpha - \dfrac{\log 2}{2\pi}$ when the right hand is positive. This proves part a).

By lemma 1, in $B(\alpha', 1/2q_n^3)$, the only cycle of period less than or equal to q_n that does not move holomorphically is the cycle $\mathcal{C}_{p_n/q_n}(\alpha')$. So, as in lemma 5, for all $n \geq 2$, we have

$$\Phi(\alpha') + \log r(\alpha') \leq \Phi_{n-1}(\alpha') + \log r_n^*(\alpha') + (C-1)\sum_{k \geq n} \frac{\log q'_k}{q'_k},$$

where C is the constant provided by lemma 2. By inequality (1), $\Phi(\alpha') + \log r(\alpha')$ is universally bounded from below. So, there exists a constant C' such that for all $n \geq 2$ and all α' sufficiently close to α,

$$\Phi_{n-1}(\alpha') + \log r_n^*(\alpha') \geq -C'.$$

Finally, we claim that there exists a constant C'_α such that for all $n \geq 2$ and all α' sufficiently close to α, we have

$$\log d_n^*(\alpha') \geq \log r_n^*(\alpha') - C'_\alpha.$$

Part b) follows easily. To prove the claim, let $\rho' = e^{2i\pi\alpha'}$ and $\rho = e^{2i\pi\alpha}$. Let n_0 be such that $d_{n_0}^*(\alpha) < |\rho - 1|/4$ (this is possible since α is a Cremer number). For α' close enough to α, $d_{n_0}^*(\alpha') < |\rho' - 1|/2$. For each fixed value of $n < n_0$,

$\log d_n^*(\alpha') - \log r_n^*(\alpha') \longrightarrow \log d_n^*(\alpha) - \log r_n^*(\alpha)$ when $\alpha' \longrightarrow \alpha$. For $n \geq n_0$, let $z \in X_n^*(\alpha')$ be a point that realizes the distance $d_n^*(\alpha')$ and set $w = P_{\alpha'}(z) = \rho' z + z^2$. Then, $|z| = d_n^*(\alpha') \leq d_{n_0}^*(\alpha') < |\rho' - 1|/2$ and

$$r_n^*(\alpha') \leq \mathrm{rad}(\mathbb{C} \setminus \{z, w\}) = d_n^*(\alpha') \cdot \mathrm{rad}\left(\mathbb{C} \setminus \{1, w/z\}\right).$$

As α' tends to α, $w/z = \rho' + z$ remains in a compact subset of $\mathbb{C} \setminus \{1\}$ and so, $\mathrm{rad}(\mathbb{C} \setminus \{1, w/z\})$ is bounded. ■

7.2. **Renormalization.** Let us now fix $\delta \in]0, 1/10[$. For $n \geq 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\alpha',n} \in S(\alpha'_n)$ as in section 5.1.

Let us set

$$C' = 2\pi(B_\alpha + 4C_\delta + 32\delta),$$

where B_α is the constant in lemma 15.

Now, let us choose n_0 so that $\beta_{n_0-1} \log 1/\alpha_{n_0} > 4C'$ (this is possible because $\sup \beta_{n-1} \log 1/\alpha_n = \infty$). If α' is sufficiently close to α, we have

$$\beta'_{n_0-1} \log 1/\alpha'_{n_0} > 4C'.$$

By proposition 8, we can take

$$t'_0 = \frac{1}{2\pi} \log \frac{1}{\alpha'_0} + C_\delta \quad \ldots \quad t'_{n_0-1} = \frac{1}{2\pi} \log \frac{1}{\alpha'_{n_0-1}} + C_\delta.$$

By lemma 15 part a), $F_{\alpha'}$ has a periodic point Z'_0 with period $\leq q_{n_0}$ satisfying $\mathrm{Im}(Z'_0) = h'_0 \geq \frac{1}{2\pi}\Phi_{n_0}(\alpha') - B_\alpha$. Note that

$$\frac{1}{2\pi}\Phi_{n_0}(\alpha') - B_\alpha \geq \frac{1}{2\pi}\Phi_{n_0-1}(\alpha') + \frac{4C'}{2\pi} - B_\alpha \geq \frac{1}{2\pi}\Phi_{n_0-1}(\alpha') + B_\alpha.$$

By lemma 15 part b), this periodic point comes from $\mathcal{C}_{p_{n_0}/q_{n_0}}(\alpha')$, and thus has rotation number p_{n_0}/q_{n_0}. By proposition 7, F_{α',n_0} has a fixed point Z'_{n_0} with $\mathrm{Im}(Z'_{n_0}) = h'_{n_0}$ satisfying

$$\beta'_{n_0-1} h'_{n_0} + \sum_{n=0}^{n_0-1} \beta'_{n-1} t'_n + 32\delta > h'_0$$

(see inequality (9)). So,

$$h'_{n_0} > \frac{1}{2\pi} \log \frac{1}{\alpha'_{n_0}} - \frac{B_\alpha + 4C_\delta + 32\delta}{\beta'_{n_0-1}} > \frac{3}{4} \cdot \frac{1}{2\pi} \log \frac{1}{\alpha'_{n_0}}.$$

If $Z \neq Z'_{n_0}$ is another fixed point of F_{α',n_0}, then proposition 7 and lemma 15 imply that

$$\beta'_{n_0-1}\mathrm{Im}(Z) + \sum_{n=0}^{n_0-1} \beta'_{n-1} t'_n < \frac{1}{2\pi} \sum_{n=0}^{n_0-1} \beta'_{n-1} \log \frac{1}{\alpha'_n} + B_\alpha.$$

Thus,

$$\mathrm{Im}(Z) < \frac{B_\alpha}{\beta'_{n_0-1}} < \frac{1}{4} \cdot \frac{1}{2\pi} \log \frac{1}{\alpha'_{n_0}}.$$

So, there is a gap of height greater than $\frac{1}{2} \cdot \frac{1}{2\pi} \log \frac{1}{a'_{n_0}}$ that separates the fixed point Z'_{n_0} of F_{α',n_0} from the other fixed points of F_{α',n_0}. According to the second remark after proposition 9, we can therefore take

$$t'_{n_0} = h'_{n_0} + 1 + C_\delta.$$

Finally, for $n \geq n_0 + 1$, we can take

$$t'_n = \frac{1}{2\pi} \log \frac{1}{\alpha'_n} + C_\delta.$$

As in the previous section, proposition 6 we have

$$\begin{aligned}
\log \frac{1}{2r(\alpha')} &\leq 2\pi \left(\sum_{n=0}^{\infty} \beta'_{n-1} t'_n + 32\delta \right) \\
&\leq 2\pi \left(\sum_{n=0}^{n_0-1} \beta'_{n-1} t'_n + \beta'_{n_0-1} h'_{n_0} \right) + \sum_{n=n_0+1}^{\infty} \beta'_{n-1} \log \frac{1}{\alpha'_n} \\
&\quad + 2\pi(\beta'_{n_0-1}(4C_\delta + 1) + 32\delta) \\
&\leq 2\pi h'_0 + \Phi(\alpha') - \Phi_{n_0}(\alpha') + 2\pi(\beta'_{n_0-1}(4C_\delta + 1) + 32\delta).
\end{aligned}$$

Note that $2\pi h'_0 \leq -\log(2d_{n_0}(\alpha'))$ where $d_{n_0}(\alpha') = d(0, X_{n_0}(\alpha'))$. So, reordering the terms and simplifying by $\log 2$, we get

$$\Phi(\alpha') + \log r(\alpha') \geq \Phi_{n_0}(\alpha') + \log d_{n_0}(\alpha') - 2\pi(\beta'_{n_0-1}(4C_\delta + 1) + 32\delta).$$

We can now conclude as in section 6.3.

Appendix A. Extracts from [BC2]

The following proposition is proposition 10 from [BC2].

Proposition 10. *Assume $U, V \subset \mathbb{C}$ are two hyperbolic domains containing 0 and $\chi : U \to V$ is a holomorphic map fixing 0. Let S be a finite subset of U avoiding 0, such that $\chi(S)$ avoids 0. Then,*

$$\frac{\mathrm{rad}(V \setminus \chi(S))}{\mathrm{rad}(V)} \leq \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)}.$$

Given an integer $q \geq 1$, set

$$\mathbb{U}_q = \left\{ e^{2i\pi k/q} \mid k = 0, \ldots, q-1 \right\}.$$

The following proposition is proposition 12 from [BC2].

Proposition 11. *There exists a constant $C > 0$ such that for $q \geq 2$ and $r < 1$, we have*

$$\log \mathrm{rad}(\mathbb{D} \setminus r\mathbb{U}_q) \leq \log r + \frac{C}{q}.$$

one can take $C = \log 4 + 2\log(1 + \sqrt{2})$.

Let V_λ be hyperbolic subdomains of $\mathbb{C}$ which contain 0 and move holomorphically with respect to $\lambda \in \mathbb{D}$. The following proposition is proposition 13 from [BC2].

Proposition 12. *There exists a family of simply connected open sets $\widetilde{V}_\lambda$ and of universal coverings $\pi_\lambda : \widetilde{V}_\lambda \to V_\lambda$ such that $\widetilde{V}_0 = \mathbb{D}$, the set*

$$\widetilde{\mathcal{V}} = \left\{ (\lambda, z) \in \mathbb{D} \times \mathbb{C} \,\middle|\, z \in \widetilde{V}_\lambda \right\}$$

is open, and $\Pi : (\lambda, z) \in \widetilde{\mathcal{V}} \mapsto \pi_\lambda(z)$ is analytic.
For all $\lambda \in \mathbb{D}$,

$$\widetilde{V}_\lambda \subset B(0, \rho) \text{ with } \log \rho = \frac{2 \log 4}{1 + |\lambda|^{-1}}.$$

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