SYSTOLES IN TRANSLATION SURFACES

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Abstract. For a translation surface, we define the systole to be the length of the shortest saddle connection. We give a characterization of the maxima of the systole function on a stratum, and give a family of examples providing local but nonglobal maxima on each stratum of genus at least three. We further study the relation between (locally) maximal values of the systole function and the number of shortest saddle connections.

1. Introduction

This paper deals with flat metric defined by Abelian differentials on compact Riemann surfaces (*translation surfaces*). A sequence of area one translation surfaces in a stratum leaves any compact set if and only if the length of the shortest saddle connection tends to zero. The set of translation surfaces with short saddle connections and compactification issues of strata are related to dynamics and counting problems on translation surfaces and have been widely studied in the last 30 years (see for instance [6, 3, 2]).

In this paper, we are interested in the opposite problem: we study surfaces that are as far as possible from the boundary and that would represent the "core" of a stratum. For a translation surface, we define the *systole* $\mathrm{Sys}(S)$ to be the length of the shortest saddle connection of S. Our primary goal is to study global and local maxima of the function Sys when restricted to area one translation surface.

This kind of question appears also in other contexts. Maxima of the systole function for moduli spaces of hyperbolic surfaces, where the systole is the length of the shortest closed geodesic, has been studied by various authors, for instance Bavard [1], Schmutz Schaller [9, 10], or more recently Fanoni and Parlier [4]. This is also closely related to the maximal number of geodesics realizing the systole, the so called kissing number.

In the context of area one translation surfaces, while the characterization of global maxima for Sys seems to have been known for some time in the mathematic community, the existence of local maxima was unknown. We provide explicit examples of local maxima that are not global in each stratum with no marked points and genus $g \geq 3$. We also study the relation between the (locally) maximal values of the function Sys and the big number of shortest saddle connections.

The paper is organized as follows. In Section 2, we give some general background on translation surfaces.

In Section 3, we study global maxima of the function Sys for area one translation surfaces. We prove the following theorem (see Theorem 3.3):

Theorem. *Let S be a translation surface of area one in $\mathcal{H}(k_1, \ldots, k_r)$. Then*

$$\mathrm{Sys}(S) \leq \left(\frac{\sqrt{3}}{2}(2g - 2 + r) \right)^{-\frac{1}{2}}.$$

The equality is obtained if and only if S is built with equilateral triangles with sides saddle connections of length $\mathrm{Sys}(S)$. *Such surface exists in any connected component of any stratum*

This result was independently proven recently by Judge and Parlier [5] for the strata $\mathcal{H}(2g-2)$: the authors are interested in shortest closed curves but their proof should work in any strata in our context.

In Section 4, we study local maxima of the function Sys that are not global. With the help of explicit examples we prove the following result which is Theorem 4.8 in the text.

Theorem. *Each stratum of area one surfaces with no marked points and genus $g \geq 3$ contains local maxima of the function* Sys *that are not global.*

The examples are obtained by considering surfaces that decompose into equilateral triangles and regular hexagons, with some further conditions (see Theorem 4.1 for a precise statement).

In the last section, we study the relation between (locally) maximal values of the function Sys and the big number of shortest saddle connections. We call a surface *rigid* if it corresponds to a local maximum of the number of shortest saddle connections. While the connection is clear for global maxima (see Proposition 5.1), the situation is more complex for the local maxima. The examples that we provide for local maxima are rigid. Even more, a surface that is a local maximum and that decomposes into equilateral triangles and regular hexagons must be rigid (Proposition 5.2). However, rigid surfaces are not necessarily local maxima (see Proposition 5.3).

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2. Background

A *translation surface* is a (real, compact, connected) genus g surface S with a translation atlas *i.e.* a triple $(S,\mathcal{U},\Sigma)$ such that Σ is a finite subset of S (whose elements are called *singularities*) and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $S \setminus \Sigma$ whose transition maps are translations. We will require that for each $s \in \Sigma$, there is a neighborhood of s isometric to a Euclidean cone whose total angle is a multiple of 2π. One can show that the holomorphic structure on $S \setminus \Sigma$ extends to S and that the holomorphic 1-form $\omega = dz_i$ extends to a holomorphic $1-$form on X where Σ corresponds to the zeroes of ω and maybe some marked points. We usually call ω an *Abelian differential*. A zero of ω of order k corresponds to a singularity of angle $(k+1)2\pi$. A saddle connection is a geodesic segment joining two singularities (possibly the same) and with no singularity at in its interior. Integrating ω along the saddle connection we get a complex number. Considered as a planar vector, this complex number represents the affine holonomy vector of the saddle connection. In particular, its Euclidean length is the modulus of its holonomy vector.

For $g \geq 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the moduli space of pairs (X,ω) where X is a genus g (compact, connected) Riemann surface and ω non-zero holomorphic $1-$form defined on X. The term moduli space means that we identify the points (X,ω) and (X',ω') if there exists an analytic isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$. The group $\mathrm{SL}(2,\mathbb{R})$ naturally acts on the moduli space of translation surfaces by post composition on the charts defining the translation structures.

One can also see a translation surface obtained as a polygon (or a finite union of polygons) whose sides come by pairs, and for each pair, the corresponding segments are parallel and of the same length. These parallel sides are glued together by translation and we assume that this identification preserves the natural orientation of the polygons. In this context, two translation surfaces are identified in the moduli space of Abelian differentials if and only if the corresponding polygons can be obtained from each other by cutting and gluing and preserving the identifications. Also, the $\mathrm{SL}(2,\mathbb{R})$ action in this representation is just the natural linear action on the polygons.

The moduli space of Abelian differentials is stratified by the combinatorics of the zeroes; we will denote by $\mathcal{H}(k_1,\dots,k_r)$ the stratum of $\mathcal{H}_g$ consisting of (classes of) pairs (X,ω) such that ω has exactly r zeroes, of order $k_1,\dots,k_r$. It is well known that this space is (Hausdorff) complex analytic. We often restrict to the subset $\mathcal{H}_1(k_1,\dots,k_r)$ of *area* *one* surfaces. Local coordinates for a stratum of Abelian differentials are obtained by integrating the holomorphic 1–form along a basis of the relative homology $H_1(S, \Sigma; \mathbb{Z})$, where Σ denotes the set of conical singularities of S.

3. Maximal systole

We recall that the systole $\mathrm{Sys}(S)$ of a translation surface S is the length of the shortest saddle connection of S. The aim of this section is to prove Theorem 3.3 which characterizes translation surfaces of area one with maximal systole. One key tool are Delaunay triangulations.

Let S be a translation surface. A *Delaunay triangulation* S is a triangulation of S such that the vertices are singularities, the 1-cells (the sides of the triangles) are saddle connections and, for a 2-cell (triangle) T of the triangulation, the circumcircle of any representative $\tilde{T}$ of the universal covering does not have any singularity in its interior.

In Section 4 of [8] Masur and Smillie prove the existence of Delaunay triangulations for every translation surface S.

Lemma 3.1. *All shortest saddle connections of S are 1-cells in every Delaunay triangulation of S.*

Proof. Let σ be a saddle connection that is not included in a Delaunay triangulation $\mathcal{T}$. Denote by P, Q the extremities of σ. Let $T \in \mathcal{T}$ be the triangle in $\mathcal{T}$ with P as a vertex and containing a subsegment of σ. Let P', P'' be the other vertices of T.

Consider the circumcircle c of T, and the open arc of $P'P''$ that does not contain P. Each chord of c joining P to an element of this arc is of length strictly greater than $\min(d(P, P'), d(P, P'')) \geq \mathrm{Sys}(S)$. One of these chords is in the direction of σ and since there is no singularity in the interior of c, this chord is a subsegment of σ. Therefore, σ is not a shortest saddle connection. $\square$

The first statement of the following lemma is needed for the proof of the next theorem. The second statement will be useful for Theorem 4.8.

Lemma 3.2. *Let $\mathcal{C} \subset \mathcal{H}(k_1, \dots, k_r)$ be a connected component of a stratum of abelian differentials with $k_1, \dots, k_r \geq 0$.*

1. *There exists in $\mathcal{C}$ a surface S that decomposes into equilateral triangles with sides saddle connections.*
2. *Furthermore, for each $i \neq j$ we can find such a surface with a side of an equilateral triangle being a saddle connection joining a singularity of degree k_i to a singularity of degree k_j.*

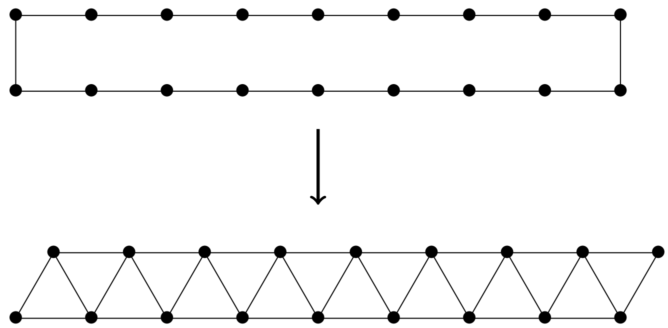


FIGURE 1. Surface with a equilateral triangle decomposition

Proof. We first prove (1). By Lemma 4 in [7] there exists in each connected component of each stratum a surface with a horizontal one cylinder decomposition. Such surface can be described as a rectangle with the two vertical sides identified that correspond to a saddle connection, and each horizontal side decomposes into horizontal saddle connections (each one appearing on the top and on the bottom). We can freely change the lengths of these saddle connections hence we can assume they are all of length one, and get a square tiled surface with singularities in each corner of the squares. Now we rotate the vertical one until it makes an angle of $\pi/3$ with the horizontal ones (see Figure 1), this gives the surface S required.

The proof of (2) is a small variation of the above proof: observe first that each singularity appears both on the top line and on the bottom line of the cylinder. Recall that $\mathrm{SL}(2,\mathbb{R})$ acts on the connected component of the stratum by linear action on the polygons. Then applying the matrix $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and suitably cutting and pasting we obtain a new rectangle. For a suitable n there is a vertical length one saddle connection joining the singularity of degree k_i to the singularity of degree k_j, and the above argument finishes the proof. □

Theorem 3.3. *Let S be a translation surface in $\mathcal{H}_1(k_1,\dots,k_r)$. Then*

$$\mathrm{Sys}(S) \leq \left(\frac{\sqrt{3}}{2}(2g-2+r)\right)^{-\frac{1}{2}}.$$

The equality is obtained if and only if S is built with equilateral triangles with sides saddle connections of length $\mathrm{Sys}(S)$*. Such surface exists in any connected component of any stratum.*

Proof. For simplicity, instead of looking at a translation surface of area one and trying to determine the longest systole possible, we suppose that S has a systole of length 1 and we try to minimize the area $\mathcal{A}(S)$.

We consider a Delaunay triangulation of S given by saddle connections. By Lemma 3.1 all shortest saddle connections of S are 1-cells in this triangulation. Note that some triangles in the Delaunay triangulation might have small area.

We consider the Voronoi diagram of S. This is a partitioning of S into cells. Each cell contains exactly one singularity and is the set of points of S that are closer to that singularity than to any other. The boundary of each cell consists of points that are equidistant to at least two singularities in the sense that there are at least two different distance realizing geodesics of equal length connecting the point with a singularity.

The boundaries of the cells of the Voronoi diagram are parts of the orthogonal bisectors of the saddle connection in the Delaunay triangulation. Even though the triangulation is not unique, the Voronoi diagram is unique.

We can compute $\mathcal{A}(S)$ as the sum of the areas of the triangles with one of the vertices a singularity and its opposite side a side of the Voronoi cell containing the singularity. The height of such a triangle is a half of a saddle connection and hence its length is greater than or equal to $\frac{1}{2}$. Therefore $\mathcal{A}(S)$ is greater or equal to one half of the sum of the lengths of all the sides of the Voronoi cells.

For each triangle T in the triangulation we consider the sum $\sigma(T)$ of the signed distances from the circumcenter of T to its sides. The sum of the lengths of all the sides of the Voronoi cells equals the sum of $\sigma(T)$ of all T in the triangulation. We want to bound from below $\sigma(T)$ for each triangle T in the Delaunay triangulation.

By Carnot's theorem[1] $\sigma(T)$ is equal to the sum of the inradius r and the circumradius R. The aim now is to show that $R + r \geq \frac{\sqrt{3}}{2}$ with equality exactly for T equilateral with side 1.

First we note that when we shrink T we decrease the sum $R + r$. So without loss of generality, we can assume that at least one of the sides of T is of length 1. So for the triangle $\triangle ABC$ with $1 = AB \leq BC \leq AC$ we take a point D on the side BC so that $BD = AB$. Note that $AD \geq 1$. For the inradius $\tilde{r}$ and the circumradius $\tilde{R}$ of the isosceles $\triangle ABD$ we can see that $\tilde{r} \leq r$ and $\tilde{R} \leq R$. Indeed, the circumcenter of $\triangle ABD$ is nearer to AB than the circumcenter of $\triangle ABC$ and therefore

[1]Lazare Carnot 1753-1823.

$\tilde{R} \leq R$. And to obtain that $\tilde{r} \leq r$, we note that the incenter of $\triangle ABD$ is nearer to B than the incenter of $\triangle ABC$.

For a triangle with sides $1, 1$ and x, we can find the inradius and the circumraduis with the help of the lengths of the sides:

$$\tilde{R}(x) = \sqrt{\frac{1}{4-x^2}}, \qquad \tilde{r}(x) = \frac{x}{2}\sqrt{\frac{2-x}{2+x}},$$

with $x \in [1, 2)$. For the sum $(\tilde{R} + \tilde{r})(x)$ and its derivative we obtain

$$(\tilde{R} + \tilde{r})(x) = \frac{2 + 2x - x^2}{2\sqrt{4-x^2}}, \quad (\tilde{R} + \tilde{r})'(x) = \frac{8 - 6x + x^3}{2\sqrt{(4-x^2)^3}}.$$

Since $8 - 6x + x^3 = x(1-x)^2 + 2(2-x)^2 + x > 0$ for $x \in [1, 2)$, we have that $(\tilde{R} + \tilde{r})(x)$ is strictly increasing in the interval $[1, 2)$ and hence obtains its minimum for $x = 1$. Therefore $R + r \geq \tilde{R} + \tilde{r} \geq \frac{\sqrt{3}}{2}$ with equality exactly when the triangle T is equilateral with side 1.

The number of triangles in the triangulation is $2(2g - 2 + r)$. Hence $\mathcal{A}(S) \geq \frac{\sqrt{3}}{2}(2g - 2 + r)$ if the systole is of length 1. Thus for a translation surface with area one, we have that the systole is at most $\left(\frac{\sqrt{3}}{2}(2g-2+r)\right)^{-\frac{1}{2}}$ and can be obtained only if S is built with equilateral triangles with sides saddle connections of length $\mathrm{Sys}(S)$.

We conclude by using the first statement of Lemma 3.2. $\square$

4. Locally maximal systole

The question is if there exist local but not global maxima in any given stratum $\mathcal{H}_1(k_1, \ldots, k_r)$ of translation surfaces of area one. Note that such maxima is never strict since rotating a translation surface preserves the systole. We denote by $\mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$ the moduli space of translation surfaces in $\mathcal{H}(k_1, \ldots, k_r)$ up to rotation and scaling. The systole function is well defined in $\mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$: for $[S] \in \mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$, we define $\mathrm{Sys}([S])$ to be $\mathrm{Sys}(S)$, where S is any area one representative of $[S]$.

In this section, we show examples of local maxima of the function Sys that are not global and prove that such examples are realized in all but a finite number of strata.

We need first, for technical reasons, to define a distance around a point in $\mathcal{H}(k_1, \ldots, k_r)$ and in $\mathbb{P}\mathcal{H}(k_1, \ldots, k_r)$. Let $S_0 \in \mathcal{H}(k_1, \ldots, k_r)$. Fix a basis of the relative homology, given by saddle connections, that determines local coordinates $(v_1, \ldots, v_k)$ around S_0. Then for S in a sufficiently small neighborhood of S_0, we define $d(S, S_0) = \max_i\{|v_i - v_{i_0}|\}$.

We will identify a sufficiently small neighborhood of an element $[S_0] \in \mathbb{P}\mathcal{H}(k_1, \dots, k_r)$, with the subset of representatives in $\mathcal{H}(k_1, \dots, k_r)$ normalized in the following way:

(1) the first coordinate v_1 is in $]0, +\infty[$,
(2) the length of the shortest saddle connection is 1.

Then, the distance to $[S_0]$ is the distance in $\mathcal{H}(k_1, \dots, k_r)$ following this identification.

Theorem 4.1. *Let S_{reg} be a translation surface in $\mathcal{H}_1(k_1, \dots, k_r)$ such that when cut along its saddle connections of length* $\mathrm{Sys}(S_{reg})$*, it decomposes to equilateral triangles and regular hexagons so that:*

- *the set of the equilateral triangles without the vertices is connected,*
- *the boundary of each polygon is contained in the boundary of the set of triangles.*

Then $\mathrm{Sys}(S_{reg})$ *is a local maximum in $\mathcal{H}_1(k_1, \dots, k_r)$ and even a strict local maximum in $\mathbb{P}\mathcal{H}(k_1, \dots, k_r)$.*

Remark 4.2. The second condition of the above statement is equivalent to having the hexagons neither adjacent not self-adjacent.

The idea of the proof is the following: when deforming a little $[S_{reg}]$ following the normalization described above, the area of each triangle does not decrease, the area of each hexagon might decrease, but this will be compensated by an increase coming from at least one triangle.

The next lemma is an elementary lemma if Euclidean geometry needed for Lemma 4.4 and 4.5.

Lemma 4.3. *Let $\triangle ABC$ be such that its sides AC and BC are of lengths greater or equal to 1, the side AB is of length stricly less than 2 and the angles $\measuredangle BAC$ and $\measuredangle ABC$ are less or equal than $\frac{\pi}{2}$. Further let C' be such that AC' and BC' are of length 1. Then* $\mathrm{Area}(\triangle ABC) \geq \mathrm{Area}(\triangle ABC')$.

Proof. Without loss of generality we can assume that C and C' are in the same half-plane determined by the line of AB and that $d(A, C) \leq d(B, C)$. We have $\measuredangle BAC' \leq \measuredangle BAC \leq \pi/2$. Hence $d(C, AB) \geq d(C', AB)$ and therefore we obtain $\mathrm{Area}(\triangle ABC) \geq \mathrm{Area}(\triangle ABC')$. □

The next two lemmas are estimations of the variation of areas of hexagons and triangles that are deformed in our context.

Lemma 4.4. *Let H_{reg} be the regular hexagon of sides of length* 1*. There exists a positive constant c such that for every $\varepsilon > 0$ small enough and* *every convex hexagon* $H = A_1A_2\dots A_6$ *with sides of lengths in the interval* $[1, 1+\varepsilon]$ *and diagonals* A_1A_3*,* A_3A_5 *and* A_5A_1 *of lengths in the interval* $[\sqrt{3}-\varepsilon, \sqrt{3}+\varepsilon]$*, we have* $\mathrm{Area}(H) \geq \mathrm{Area}(H_{reg}) - c\varepsilon^2$*.*

Proof. We consider the convex hexagon $H' = A_1A_2'A_3A_4'A_5A_6'$ such that all of its sides are of length 1. By Lemma 4.3, we see that $\mathrm{Area}(H) \geq \mathrm{Area}(H')$.

We note the lengths of the diagonals A_1A_3, A_3A_5 and A_5A_1 by d_1, d_2 and d_3 respectively. The area of the hexagon H' is given by

$$\mathrm{Area}(H') = F(d_1, d_2, d_3) = \sum_{i=1}^{3} \frac{1}{4}\sqrt{d_i^2(d_i+2)(2-d_i)} +$$
$$+ \frac{1}{4}\sqrt{(d_1+d_2+d_3)(-d_1+d_2+d_3)(d_1-d_2+d_3)(d_1+d_2-d_3)}$$

The function F is differentiable in the point $(\sqrt{3}, \sqrt{3}, \sqrt{3})$ and the partial derivatives $\partial_i F(\sqrt{3}, \sqrt{3}, \sqrt{3})$, $i \in \{1, 2, 3\}$, are all equal to 0. Therefore by the Taylor-Young formula we obtain

$$F(d_1, d_2, d_3) = F(\sqrt{3}, \sqrt{3}, \sqrt{3}) + o(||(d_1 - \sqrt{3}, d_2 - \sqrt{3}, d_3 - \sqrt{3})||^2).$$

Since for $i \in \{1, 2, 3\}$ we have $d_i \in [\sqrt{3}-\varepsilon, \sqrt{3}+\varepsilon]$ and $\mathrm{Area}(H_{reg}) = F(\sqrt{3}, \sqrt{3}, \sqrt{3})$, there exists a constant $c \in \mathbb{R}$ such that

$$\mathrm{Area}(H) \geq \mathrm{Area}(H_{reg}) - c\varepsilon^2.$$

□

Lemma 4.5. *Let* T_{reg} *be the equilateral triangle of sides of length 1. There exists a positive constant* $c \in \mathbb{R}$ *such that for every* $\varepsilon > 0$ *small enough and every triangle* T *with one of its sides of length* $1+\varepsilon$ *and the other sides of lengths in the interval* $[1, 1+\varepsilon]$*, we have that* $\mathrm{Area}(T) > \mathrm{Area}(T_{reg}) + c\varepsilon$*.*

Proof. Let $T = \triangle ABC$ and $d(A, B) = 1 + \varepsilon$. By Lemma 4.3, we have $\mathrm{Area}(\triangle ABC) > \mathrm{Area}(\triangle ABC')$ where C' is such that $d(A, C') = d(B, C') = 1$.

For the area of $\triangle ABC'$ we obtain:

$$\begin{aligned}\text{Area}(\triangle ABC') &= \frac{1}{4}\sqrt{(3+\varepsilon)(1-\varepsilon)(1+\varepsilon)^2} = \\ &= \frac{1}{4}\sqrt{3+4\varepsilon+o(\varepsilon)} = \frac{\sqrt{3}}{4}\sqrt{1+\frac{4}{3}\varepsilon+o(\varepsilon)} = \\ &= \frac{\sqrt{3}}{4}(1+\frac{1}{2}(\frac{4}{3}\varepsilon+o(\varepsilon))+o(\frac{4}{3}\varepsilon+o(\varepsilon))) = \\ &= \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{6}\varepsilon+o(\varepsilon)\end{aligned}$$

Therefore there exists a constant $c>0$ such that for all ε small enough we have $\text{Area}(T) > \text{Area}(T_{reg}) + c\varepsilon$. □

Lemma 4.6. *Let ABC be an equilateral triangle of side of length 1. Further let $A'B'C'$ be a triangle with sides of lengths in the interval $[1, 1+\varepsilon]$ and such that that $d(A,A') \leq \varepsilon$, $d(B,B') \leq \varepsilon$ and C and C' are in the same half-plane determined by AB. Then $d(C,C') \leq 10\varepsilon$.*

Proof. We consider first the translation τ of $\mathbb{R}^2$ of direction $\overrightarrow{A'A}$. We remark that $\tau(A') = A$ and $d(B', \tau(B')) < 2\varepsilon$. Then we consider the rotation ρ with center A and of angle $\measuredangle BA\tau(B')$. We note $X'' = \rho(\tau(X'))$ where $X \in \{A', B', C'\}$. We remark that A, B and B'' are on the same line and that

$$d(\tau(C'), C'') = \frac{d(\tau(A'), \tau(C'))}{d(\tau(A'), \tau(B'))} d(\tau(B'), B'').$$

Since $d(\tau(B'), B'') \leq d(\tau(B'), B) + d(B, B'') < 2\varepsilon + \varepsilon$, we obtain that (assuming $\varepsilon < \frac{1}{2}$)

$$d(\tau(C'), C'') < (1+\varepsilon)(3\varepsilon) < 5\varepsilon.$$

We want to bound $x := d(C, C'')$. We consider the different possibilities for C'' (see Figure 2).

It is not possible for C'' to be in the white zone because otherwise $d(A'', C'') < 1$ or $d(B'', C'') < 1$.

If C'' is in the light gray zone, then $\measuredangle ACC'' \geq \frac{2\pi}{3}$. From the cosine formula for $\triangle ACC''$ we obtain

$$d(A, C'')^2 \geq d(A,C)^2 + x^2 + x d(A,C) > d(A,C)^2 + x d(A,C).$$

Therefore

$$x < \frac{d(A,C'')^2 - d(A,C)^2}{d(A,C)}.$$

Since $d(A,C)$ and $d(A,C'')$ are in the interval $[1, 1+\varepsilon]$ and ε is small enough (e.g. less than $\frac{1}{2}$), we obtain that $x < 3\varepsilon$.

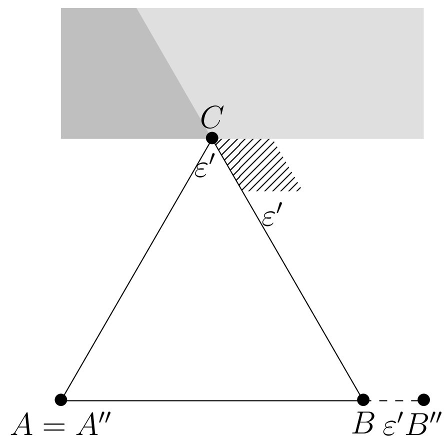


FIGURE 2. Possible regions for C''

If C'' is in the dark gray zone, then $\measuredangle B''CC'' \geq \frac{2\pi}{3}$. As in the previous case, and since $d(B'', C) \geq 1$, we conclude that $x < 3\varepsilon$.

And finally, if C'' is in the rombus of side $\varepsilon' = d(B, B'') \leq 2\varepsilon$, then $x < 4\varepsilon$.

We conclude by

$$d(C, C') \leq d(C, C'') + d(C'', \tau(C')) + d(\tau(C'), C') < 4\varepsilon + 5\varepsilon + \varepsilon = 10\varepsilon.$$

□

Proof of Theorem 4.1. We show directly that $\mathrm{Sys}([S_{reg}])$ is a strict local maximum in the projective stratum, and replace S_{reg} by a surface, still denoted S_{reg} with shortest saddle connections of length one.

First, we remark that removing all shortest saddle connections of S_{reg} gives a union of topological disks. Hence we can find a basis of the relative homology that consists of shortest saddle connections $(\gamma_1, \dots, \gamma_r)$ and we can assume that γ_1 is horizontal and oriented from left to right. We use this basis to fix local coordinates of the stratum $\mathcal{H}(k_1, \dots, k_r)$, and define a distance in a neighborhood of S_{reg}. Recall that we identify a neighborhood of element $[S_{reg}] \in \mathbb{P}\mathcal{H}(k_1, \dots, k_r)$ with a subset $\mathcal{U}$ of $\mathcal{H}(k_1, \dots, k_r)$ satisfying the following conditions: the shortest saddle connection is of length 1 and γ_1 stays horizontal. For $S \in \mathcal{U}$, we call *short saddle connection* any saddle connection that corresponds to a shortest saddle connection of S_{reg}.

Let $\varepsilon > 0$ be small enough and $S \in \mathcal{U}$ be such that $\varepsilon = d(S, S_{reg})$. We observe that since any short saddle connection γ is a linear combination of $\{\gamma_1, \dots, \gamma_r\}$ in the relative homology group, then its corresponding affine holonomy v_γ satisfies $|v_\gamma - v_{\gamma,reg}| \leq K_1\varepsilon$. Since there are only a finite number of short saddle connections, K_1 can be made universal all short saddle connections.

Claim: There is a constant K_2 such that for ε small enough, there is a short saddle connection γ_{long} on S of length at least $1 + K_2\varepsilon$ (in other words: lengths of short saddle connections control the distance from S to S_{reg}).

Assuming the claim, we have the following observations:

(1) For any equilateral triangle T_{reg} in the decomposition of S_{reg} and T the corresponding triangle in S, we have $Area(T) \geq Area(T_{reg})$. Indeed T has sides of lengths in $[1, 1 + K_1\varepsilon]$ and we apply Lemma 4.5 where the length of the longest side of T is $1 + \varepsilon'$.

(2) There exists at least one equilateral triangle T_{reg} in the decomposition of S_{reg}, such that for the corresponding triangle T in S we have $Area(T) \geq Area(T_{reg}) + C\varepsilon$ where C is a constant. Indeed, by the claim, there exists a saddle connection γ_{long} in S of length in $[1+K_2\varepsilon, 1+K_1\varepsilon]$. The saddle connection γ_{long} is on the boundary of a triangle T. We apply Lemma 4.5 to this triangle.

(3) For any hexagon H_{reg} in the decomposition of S_{reg} and H the corresponding hexagon in S, we have $Area(H) \geq Area(H_{reg}) - C'\varepsilon^2$, where C' is a constant. Indeed, this comes from Lemma 4.4 applied to H with $\varepsilon' = 2K_1\varepsilon$.

Summing up all contributions, we see that the area of S is greater than the area of S_{reg} for $\varepsilon > 0$ small enough. Hence S_{reg} is a local maximum of Sys which is nonglobal since the surface S_{reg} is not built with equilateral triangles of sides saddle connections.

Now we prove the claim. We prove that there exists a constant $D > 1$ such that if all short saddle connections in S have lengths less than $1 + \delta$ with δ sufficiently small and if we assume that γ_1 does not change direction then $d(S, S_{reg}) < D\delta$. This clearly implies the desired result with $K_2 = 1/D$.

Let $\gamma \in \{\gamma_2, \ldots, \gamma_k\}$ be a saddle connection in the fixed basis. By hypothesis, there is a sequence of pairwise distinct equilateral triangles $T_1, \ldots, T_l$ (whose sides are length one saddle connections) that form a "path" from γ_1 to γ, *i.e.* such that

(1) γ_1 is a side of T_1,
(2) for each $i \in \{1, \ldots, l-1\}$, T_i and T_{i+1} are adjacent,
(3) γ is a side of T_l.

Observe that l is bounded from above by the total number N of triangles in the decomposition of S_{reg}. Denote by v_{reg} the affine holonomy of γ in S_{reg} and by v the afffine holonomy of $\gamma \in S$. We will use Lemma 4.6 to bound $|v - v_{reg}|$.

Using the developing map, we can view the triangles $(T_i)_i$ as a sequence of adjacent equilateral triangles of the plane although in this case the triangles might intersect. We deform the surface S_{reg} to obtain the surface S. The triangles $(T_i)_i$ persist but are not necessarily equilateral any more. Again, we can view them as a sequence of adjacent triangles $(T'_i)_i$ in the plane.

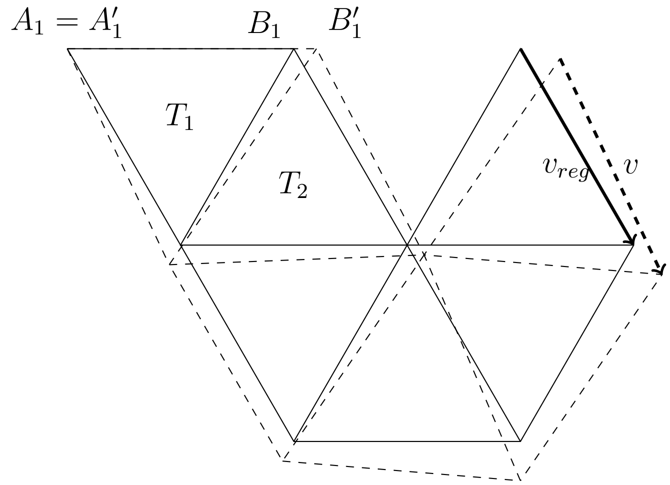


FIGURE 3. A sequence of adjacent triangles and the perturbed ones

Denote by $T_1 = A_1B_1C_1$ and $T'_1 = A'_1B'_1C'_1$. We can assume that $A_1 = A'_1$ is the vertex neither in T_2 nor in T'_2, and B_2, B'_2 are such that the segments A_1B_1 and $A'_1B'_1$ are horizontal (see Figure 3). More generally for $i > 1$, denote the triangle T_i by $A_iB_iC_i$ in such a way that A_iB_i is a side of previous triangle and that B_iC_i is a side of the next triangle, and we denote analogously the vertices of T'_i. Using Lemma 4.5 we see that $d(C_1, C'_1) < 10\delta$. Since $d(B_1, B'_1) < \delta < 10\delta$ we can apply Lemma 4.5 to the triangles T_2 and T'_2 for the constant 10δ and we get $d(C_2, C'_2) < 10^2\delta$. Finally, since l is bounded from above by N and δ can be chosen arbitrarily small, we get $d(C_l, C'_l) < 10^l\delta$ and $d(B_l, B'_l) < 10^{l-1}\delta$. Finally, observe that v is given by the difference of the coordinates of B'_l and C'_l, and therefore:

$$|v - v_{reg}| < (10^l + 10^{l-1})\delta < 2.10^N\delta.$$

This concludes the proof of the claim and of the theorem.

□

Example 4.7. *The surfaces given in Figure 4 are examples (with one hexagon) of local maxima that are nonglobal in the strata* $\mathcal{H}(4)$, $\mathcal{H}(2,2)$, $\mathcal{H}(2,0)$, $\mathcal{H}(1,1,0,0)$ *and* $H(1,1,0)$.

The above examples will be used in the next theorem in order to build examples in each stratum with no marked points.

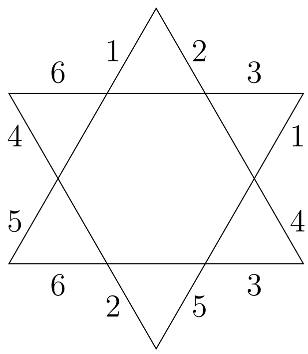
Theorem 4.8. *Each stratum of area 1 surfaces with no marked points and genus* $g \geq 3$ *contains local maxima of the function* Sys *that are not global.*

Proof. Recall that elements in $\mathcal{H}(4)$ and $\mathcal{H}(2,2)$ have already been constructed.

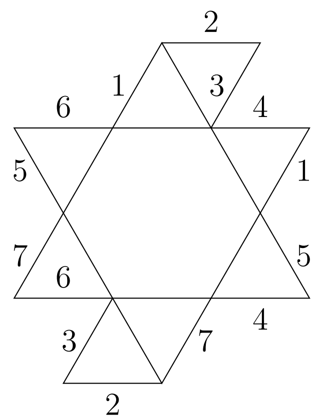
We first build local maxima in $\mathcal{H}(6+2n)$ for $n \geq 0$. We start from the example S_4 in $\mathcal{H}(4)$, and a global maximum $S_{2n,0}$ in $\mathcal{H}(2n,0)$. Recall that the global maximum decomposes along the shortest saddle connection into equilateral triangles. There is necessarily a shortest saddle connection γ_1 in S_4 joining the singularity to itself, and a shortest saddle connection γ_2 in $S_{2n,0}$ joining the marked point to the singularity of degree $2n$. We can assume the two are vertical and of the same length. Now we glue the two surfaces by the following classical surgery: cut the two surfaces along γ_1 and γ_2, and glue the left side of γ_1 with the right side of γ_2 and the right side of γ_1 with the right side of γ_2. We get a surface in $\mathcal{H}(2n+6)$ that satisfies the hypothesis of Theorem 4.1 and hence is a local but nonglobal maximum.

Now we construct surfaces that provide a local maximum in any stratum of the form $\mathcal{H}(p,q,n_1,\dots,n_k)$ with $p \geq 3$, $q \geq 1$, $k \geq 0$ and $p+q+\sum_i n_i = 2g-2$. We start from the example $S_{2,0}$ in $\mathcal{H}(2,0)$ and a global maximum in $\mathcal{H}(p-3,q-1,n_1,\dots,n_r)$. By Lemma 3.2, there is such global maximum with a shortest saddle connection γ_2 joining the singularity of degree $p-3$ to the singularity of degree $q-1$. There is a shortest saddle connection γ_1 in $S_{2,0}$ joining the the two singularities. Now a similar surgery as above gives a surface in $\mathcal{H}(p,q,n_1,\dots,n_k)$ which is a local but nonglobal maximum.

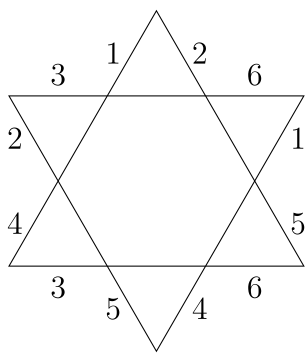
It remains to construct the strata of surfaces with $p \geq 0$ singularities of degree 1 and $q \geq 0$ singularities of degree 2. We note such strata has $\mathcal{H}(1^p,2^q)$. Remark that we cannot build with these constructions local maxima in $\mathcal{H}(2)$ and in $\mathcal{H}(1,1)$. Indeed, for $\mathcal{H}(2)$ we need one hexagon and two triangles and there is only one possibility that provides a surface in $\mathcal{H}(2)$. But in this case the hexagon is self-adjacent (see next section for a proof that it not a local maximum). For $\mathcal{H}(1,1)$, we need



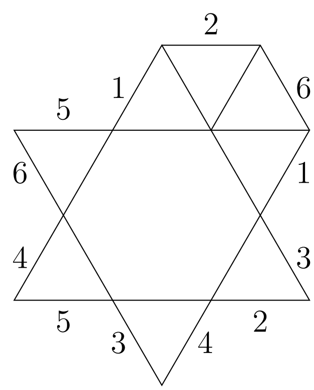
$S_4 \in \mathcal{H}(4)$



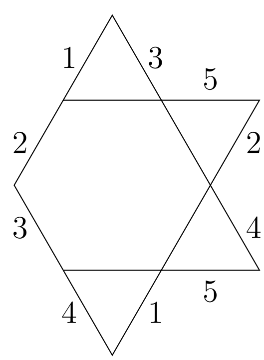
$S_{2,2} \in \mathcal{H}(2,2)$



$S_{1,1,0} \in \mathcal{H}(1,1,0)$



$S_{1,1,0,0} \in \mathcal{H}(1,1,0,0)$



$S_{2,0} \in \mathcal{H}(2,0)$

FIGURE 4. Examples of local but nonglobal maxima.

one hexagon and four triangles, and by checking all the possibilities we see that we cannot built the required example.

Now we build local maxima in any stratum of this form, with $p, q \geq 1$ and $p+q \geq 3$. We start from the surface $S_{1,1,0} \in \mathcal{H}(1,1,0)$ and a global maximum $S \in \mathcal{H}(0, n, n_1, \dots, n_k)$ with $n \in \{0,1\}$ and $n_1, \dots, n_k \in \{1,2\}$ with $k \geq 0$ such that $n + \sum_i n_i = 2g-2 \geq 0$. By Lemma 3.2, we can assume again that there is a saddle connection joining the singularity of degree 0 to the singularity of degree n. Note that there is in $S_{1,1,0}$ a saddle connection joining a singularity of degree 1 to the singularity of degree 0 (see the example above). Then the same surgery as above gives a local but nonglobal maximum in $\mathcal{H}(1, 2, n+1, n_1, \dots, n_k)$.

For $q = 0$ and $p \geq 4$. We construct in the same way from the example $S_{1,1,0,0}$ and a global maximum in $\mathcal{H}(0,0,1^{p-4})$, a local but nonglobal maximum in $\mathcal{H}(1^p)$.

There remains to construct examples in $\mathcal{H}(2^q)$, for $q \geq 3$. We start from $S_{1,1,0} \in \mathcal{H}(1,1,0)$. By the same surgery with a global maximum $S_{0,0} \in \mathcal{H}(0,0)$, we get a surface in $\mathcal{H}(2,1,1)$. There is a saddle connection joining the two singularities of degree 1. We glue this surface with a global maximum in $\mathcal{H}(0,0,2^{q-3})$. We get a local but nonglobal maximum in $\mathcal{H}(2^q)$.

□

5. Number of shortest saddle connections

It seems that a related concept to the (locally) maximal values of the function Sys is the big number of short saddle connections. In this section, we explore the relations between the two concepts.

5.1. **Maximal number.** In the case of global maxima, the relation is clear as shown in the next proposition.

Proposition 5.1. *The greatest number of shortest saddle connections of a surface in $\mathcal{H}(k_1, \dots, k_r)$ is equal to $\sum_{i=1}^{r} 3(k_i+1)$ and this number is realized if and only if the surface is a global maximum for the function* Sys *in $\mathbb{P}\mathcal{H}(k_1, \dots, k_r)$.*

Proof. Let S be a surface in $\mathcal{H}(k_1, \dots, k_r)$. We consider two saddle connections γ_1 and γ_2 in S starting at the same singularity.

Let us assume that the conical angle between γ_1 and γ_2 is less than $\frac{\pi}{3}$. Then

- either the not common ends of γ_1 and γ_2 can be connected by a saddle connection and as consequence this saddle connection is shorter than γ_1 and γ_2,

- or there is a saddle connection between γ_1 and γ_2 (starting at the same singularity) that is shorter than them.

In both cases we have a contradiction and hence the maximal number of saddle connections starting at a singularity of order k_i is $6(k_i+1)$. This gives us that the total number of shortest saddle connections cannot exceed $\sum_{i=1}^r 3(k_i+1)$.

This number is the number of 1-cells in the Delaunay triangulation. Hence, by Lemma 3.1, the surface has this number of shortest saddle connections if and only if its Delaunay triangulation is given by equilateral triangles. By Theorem 3.3 this situation corresponds precisely to global maxima of the function Sys. □

5.2. **Locally maximal number: rigid surfaces.** For a given translation surface, one would like to find a path joining this surface to a global maximum for the function Sys. Following the above proposition, a greedy algorithm could be to try to increase the number of shortest saddle connections until we reach a surface with the maximal number. Unfortunately, this algorithm does not always work.

We call a surface S in $\mathcal{H}(k_1,\dots,k_r)$ *rigid* if there exists a punctured neighbourhood of $[S]\in\mathbb{P}\mathcal{H}(k_1,\dots,k_r)$ where all surfaces have a strictly smaller number of shortest saddle connections. As explained above, the global maxima of the systole function are rigid surfaces.

An example of a rigid surface is every surface S that, when cut along its shortest saddle connections, decomposes into equilateral triangles and polygons with no singularities in the interior satisfying the following conditions:

- the set of the equilateral triangles without the vertices is connected,
- the boundary of each polygon is contained in the boundary of the set of triangles.

Indeed, when deforming such a surface in a way that the initial shortest saddle connections stay of the same length, the set of triangles is isometrically preserved and therefore the set of polygons. In particular, the examples of Theorem 4.1 are rigid surfaces.

We give another family of examples: consider a surface S as above, but instead of having one, it has 2 or 3 connected components of triangles. We further assume that there is a polygon $\mathcal{P}$ such that the sum of the affine holonomy of the set of saddle connections of its boundary associated to each component of triangles is nonzero when orienting the saddle connections accordingly to the natural orientation of the $\partial\mathcal{P}$. Indeed as above, when deforming such a surface in a way that the initial shortest saddle connections stay of the same length, then each connected component of triangles is isometrically preserved, and the condition on the holonomy implies that the boundary $\mathcal{P}$ is unchanged, which rigidifies the whole surface. If further the polygons are regular hexagons, we can adapt the proof of Theorem 4.1 to show that these are also local but nonglobal maxima.

The examples given in Figure 5 show that it is not sufficient to be decomposed into equilateral triangles and regular hexagons in order to be a local maximum: in this figure, the shortest saddle connections remain of length one and hence the area of the triangles does not change, but the hexagon is deformed and therefore its area decreases. The first example has one connected component of triangles but the hexagon is self-adjacent. The second one has two connected components of triangles. Note that the example in $\mathcal{H}(0,0,0)$ can be easily modified to give a surface with true singularities (see Remark 5.4).

More generally, we have the following proposition:

Proposition 5.2. *Let S be a translation surface such that, when cut along its saddle connections of shortest length, it decomposes into equilateral triangles and regular hexagons. If the function* Sys *admits a local maximum at $[S] \in \mathbb{P}\mathcal{H}(k_1, \dots, k_r)$, then S is rigid.*

Proof. We assume that S is nonrigid, and deform the surface so that we keep all shortest saddle connections of the same length 1. This deformation does not change the metric on each triangle. Therefore, it must change the metric on at least one hexagon, otherwise the metric would be globally unchanged and the transformation would be just a rotation. In particular, the area of the deformed hexagons must strictly decrease, while the area of the triangles (and the unchanged hexagons) remains the same. Hence the area of the surface decreases and thus $\mathrm{Sys}([S])$ increases. □

An interesting question is if the converse of the above proposition is true. We can also ask if, in general, any local maximum for Sys comes from a rigid surface. Note that in general, rigid surfaces do not necessarily give local maxima, as shown in the following example.

Proposition 5.3. *The translation surface given by Figure 6 is rigid but it is not a local maximum for the function* Sys *in $\mathbb{P}\mathcal{H}$ for $n \geq 3$.*

Remark 5.4. Note that the translation surface given in Figure 6 contains marked points in the set of singularities. We can easily make them true singularities by surgeries analogous to the ones described in the proof of Theorem 4.8.

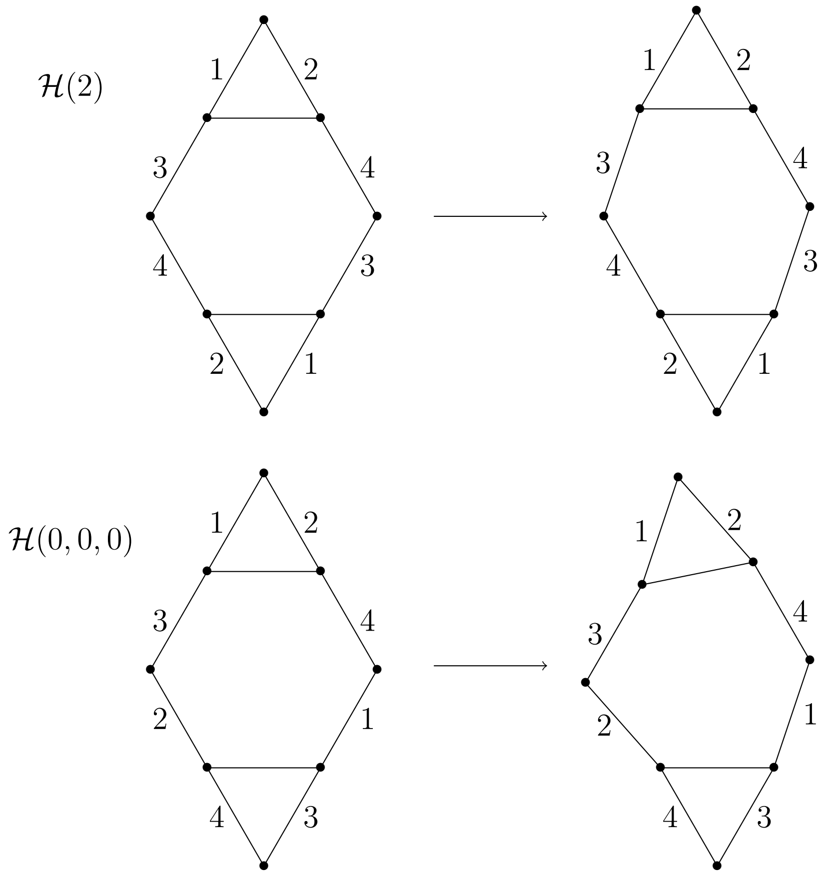


FIGURE 5. Examples of nonrigid surfaces in $\mathcal{H}(0,0,0)$ and in $\mathcal{H}(2)$.

Proof. The fact that the surface is rigid is clear: when cut along shortest saddle connections it decomposes into equilateral triangles and a non self-adjacent polygon with no singularities in the interior in such a way that the set of triangles is connected.

Now, we deform the surface as shown in the figure: the only short saddle connections that change are the horizontal ones in the parallelograms drawn with fat sides (see the labels "1" and "$n-1$") and their diagonals. The affine holonomy of the saddle connection corresponding to the label "1" is changed by adding $-i\varepsilon$ and similarly, we add $i\varepsilon$ to the one corresponding to the label "$n-1$".

Since all short saddle connections keep to be of length at least one, we need to check that the area of the surface decreases.

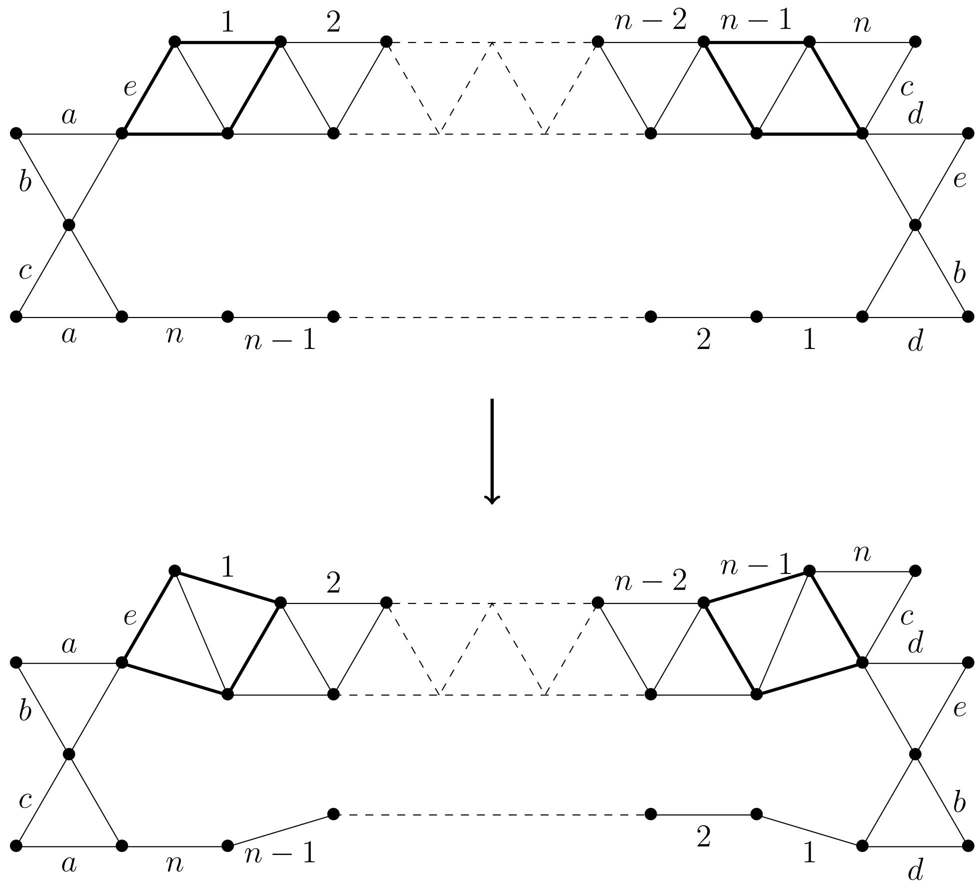


FIGURE 6. Example of a rigid surface that is not a local maximum

(1) The area of each fat parallelogram increases exactly by the area of the gray parallelogram in Figure 7, which is less that ε, and the two fat parallelograms in Figure 6 are disjoints for $n \geq 3$.

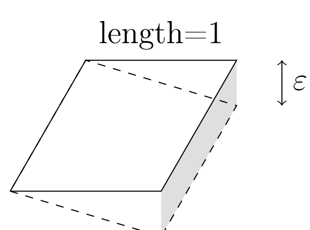


FIGURE 7. Comparing the area of the two parallelograms

(2) The area of the polygon decreases by $(n-1)\varepsilon + (n-2)\varepsilon = (2n-3)\varepsilon$.

Hence the total area decreases if $n \geq 3$. □

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