LENGTHS SPECTRUM OF HYPERELLIPTIC COMPONENTS

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To the memory of Jean-Christophe Yoccoz

ABSTRACT. We propose a general framework for studying pseudo-Anosov homeomorphisms on translation surfaces. This new approach, among other consequences, allows us to compute the systole of the Teichmüller geodesic flow restricted to the hyperelliptic connected components, settling a question of [Far06]. We stress that all proofs and computations are performed without the help of a computer. As a byproduct, our methods give a way to describe the bottom of the lengths spectrum of the hyperelliptic components.

1. INTRODUCTION

Every affine pseudo-Anosov map ϕ on a half-translation surface S has an expansion factor $\lambda(\phi) \in \mathbb{R}$ recording the exponential growth rate of the lengths of the curves under iteration of ϕ. The set of the logarithms of all expansion factors (when fixing the genus g of the surfaces) is a discrete subset of $\mathbb{R}$: this is the lengths spectrum of the moduli space $\mathcal{M}_g$.

One can also consider other lengths spectrum $\mathrm{spec}(H), \mathrm{spec}(\mathcal{C}) \subset \mathrm{spec}(\mathrm{Mod}_g)$ for various subgroups $H < \mathrm{Mod}(g)$ or for various connected components of strata $\mathcal{C}$ of the moduli spaces of quadratic differentials.

These objects have been the subject of many investigations recently (we refer to the recent work of McMullen, Farb–Leininger–Margalit, Leininger, Agol–Leininger–Margalit).

Describing $\mathrm{spec}(\mathcal{C})$ is a difficult problem and, so far, only bounds on the systole $L(\mathrm{spec}(\mathcal{C}))$ are known for various cases.

In this paper we present a general framework for studying pseudo-Anosov homeomorphisms. As a consequence, we provide a complete description of $L(\mathrm{spec}(\mathcal{C}))$ when $\mathcal{C} = C^{\mathrm{hyp}}$ is a *hyperelliptic connected component* of the moduli space of Abelian differentials. This is the very first instance of an explicit computation of the systole for an infinite family of strata.

Theorem A. *The minimum value of the expansion factor $\lambda(\phi)$ over all affine pseudo-Anosov maps ϕ on a translation surface $S \in \mathcal{C}^{\mathrm{hyp}}$ is given by the largest root of the polynomial*

$$
\begin{array}{ll}
X^{2g+1} - 2X^{2g-1} - 2X^2 + 1 & \text{if } S \in \mathcal{H}^{hyp}(2g-2), \\
X^{2g+2} - 2X^{2g} - 2X^{g+1} - 2X^2 + 1 & \text{if } S \in \mathcal{H}^{hyp}(g-1, g-1), \ g \text{ is even,} \\
X^{2g+2} - 2X^{2g} - 4X^{g+2} + 4X^g + 2X^2 - 1 & \text{if } S \in \mathcal{H}^{hyp}(g-1, g-1), \ g \text{ is odd.}
\end{array}
$$

Moreover the conjugacy mapping class realizing the minimum is unique.

This theorem settles a question of Farb [Far06, Problem 7.5] for an infinite family of connected component of strata of the moduli space of Abelian differentials.

With more efforts, our method gives a way to also compute the other elements of the spectrum $\mathrm{Spec}(\mathcal{C}^{\mathrm{hyp}})$ where $\mathcal{C}^{\mathrm{hyp}}$ ranges over all hyperelliptic connected components, for any genus. As an instance we will also prove (see Theorem 2.3):

Theorem B. *For any g even, $g \not\equiv 2 \mod 3$, $g \geq 9$, the second least dilatation of an affine pseudo-Anosov maps ϕ on a translation surface $S \in \mathcal{H}^{hyp}(2g-2)$ is given by the largest root of the polynomial*

$$X^{2g+1} - 2X^{2g-1} - 2X^{\lceil 4g/3 \rceil} - 2X^{\lfloor 2g/3 \rfloor+1} - 2X^2 + 1.$$

Moreover the conjugacy mapping class realizing the minimum is unique.

It is proved in [BL12] that $L(\mathrm{Spec}(\mathcal{C}^{\mathrm{hyp}})) \in\ ]\sqrt{2}, \sqrt{2} + \frac{1}{2^{g-1}}[$.
For $\mathcal{H}^{hyp}(2)$ and $\mathcal{H}^{hyp}(1,1)$, the corresponding systoles are the largest root of the polynomials $X^5 - 2X^3 - 2X^2 + 1 = (X+1)(X^4 - X^3 - X^2 - X + 1)$ and $X^6 - 2X^4 - 2X^3 - 2X^2 + 1 = (X+1)^2(X^4 - 2X^3 + X^2 - 2X + 1)$ respectively, recovering previous result of [LT10].

For small values of g, as an illustration of our construction, we are able to produce a complete description of the bottom of the spectrum of $\mathcal{C}^{\mathrm{hyp}}$.

Theorem C. *For $g \leq 10$, the lengths l of the closed Teichmüller geodesics on $\mathcal{H}^{hyp}(2g-2)$ satisfying $l < 2$ are recorded in the table below. For genus between 4 and 10, we only indicate the number of geodesic lengths.*

g	lengths of closed Teichmüller geodesics on $\mathcal{H}^{hyp}(2g-2)$ less that 2
2	*Perron root of $X^5 - 2X^3 - 2X^2 + 1 \sim 1.72208380573904$*
3	*Perron root of $X^7 - 2X^5 - 2X^2 + 1 \sim 1.55603019132268$*
	Perron root of $X^7 - 2X^5 - X^4 - X^3 - 2X^2 + 1 \sim 1.78164359860800$
	Perron root of $X^7 - 3X^5 - 3X^2 + 1 \sim 1.85118903363607$
	Perron root of $X^7 - 2X^5 - 2X^4 - 2X^3 - 2X^2 + 1 \sim 1.94685626827188$
4	*11 geodesic lengths*
5	*22 geodesic lengths*
6	*79 geodesic lengths*
7	*142 geodesic lengths*
8	*452 geodesic lengths*
9	*1688 geodesic lengths*
10	*4887 geodesic lengths*

Obviously our techniques also provide a way to investigate the bottom of the spectrum for the stratum $\mathcal{H}^{hyp}(g-1, g-1)$ for various g and various bound (not necessarily 2). This will appear in the forthcoming paper.

2. Overall strategy

Our strategy is to convert the computation of mapping classes and their expanding factors into a finite combinatorial problem. This is classical in pseudo-Anosov theory: the Rauzy–Veech theory and the train track theory are now well established. However a major difficulty comes from the very complicated underlying combinatorics.

2.1. **Hyperelliptic connected components.** In the sequel, for any integer $n \geq 4$, we will consider the hyperelliptic Rauzy diagram $\mathcal{D}_n$ of size $2^{n-1} - 1$ containing the permutation

$$\pi_n = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$$

and by $\mathcal{C}_n^{\mathrm{hyp}}$ the associated connected component. If $n = 2g$ is even then $\mathcal{C}_n^{\mathrm{hyp}} = \mathcal{H}^{hyp}(2g-2)$ and if $n = 2g + 1$ is odd then $\mathcal{C}_n^{\mathrm{hyp}} = \mathcal{H}^{hyp}(g-1, g-1)$. The precise description of these diagrams was given by Rauzy [Rau79].
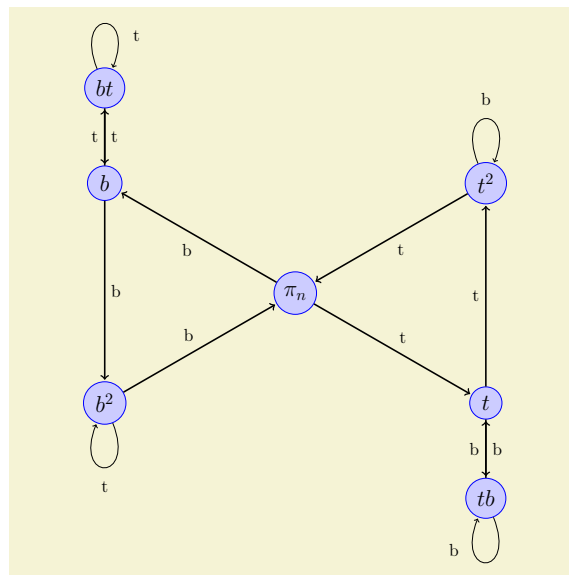

FIGURE 1. Rauzy diagram for $n = 4$

Equipped with this and the Kontsevich-Zorich [KZ03] classification, one can convert our Main Theorem into Theorem 2.1 below:

Theorem 2.1. *For any $n \geq 4$, the minimum value of the expansion factor $\lambda(\phi)$ over all affine pseudo-Anosov map ϕ on a translation surface $S \in \mathcal{C}_n^{\mathrm{hyp}}$ is given by the largest root of the polynomial*

$$
\begin{array}{ll}
X^{n+1} - 2X^{n-1} - 2X^2 + 1 & \text{if } n \text{ is even,} \\
X^{n+1} - 2X^{n-1} - 2X^{(n-1)/2+1} - 2X^2 + 1 & \text{if } n \equiv 1 \mod 4, \\
X^{n+1} - 2X^{n-1} - 4X^{(n-1)/2+2} + 4X^{(n-1)/2} + 2X^2 - 1 & \text{if } n \equiv 3 \mod 4.
\end{array}
$$

Moreover the conjugacy mapping class realizing the minimum is unique.

The same applies to the theorem for the second least expanding factor :

Theorem 2.2. *For any $n \geq 18$, if $n \not\equiv 4 \mod 6$ is even then the second least element of $\mathrm{Spec}(\mathcal{C}_n^{\mathrm{hyp}})$ is given by the largest root of the polynomial*

$$
X^{2g+1} - 2X^{2g-1} - 2X^{\lceil 4g/3 \rceil} - 2X^{\lfloor 2g/3 \rfloor + 1} - 2X^2 + 1.
$$

Moreover the conjugacy mapping class realizing the minimum is unique.

As mentioned above, one possible way to tackle Theorem 2.1 is by the use of the Rauzy–Veech induction. It is now well established that closed loops in the Rauzy diagram $\mathcal{D}_n$ furnishes pseudo-Anosov maps *fixing a separatrix* (see Section 3). Hence the "usual" construction is not sufficient to capture all relevant maps. In [BL12], it is shown that the square of *any* pseudo-Anosov maps fixes a separatrix, up to adding a regular point. The cost to pay is that this produces very complicated Rauzy diagrams and the computation of the precise systole seems out of reach with this method.

We will propose a new construction in order to solve these two difficulties at the same time. The proof of our Main Theorem is divided into three parts of different nature: geometric, dynamical, combinatoric.

2.2. **Geometric part.** In this paper we propose a new construction of pseudo-Anosov map, denoted the *Symmetric Rauzy-Veech construction*. One key of this construction is the following definition.

Definition 2.1. *A pseudo Anosov homeomorphism is positive (respectively, negative) if it fixes (respectively, reverses) the orientation of the invariant measured foliations.*

The classical Rauzy–Veech construction associates to any closed path γ in the Rauzy diagram a non negative matrix $V(\gamma)$. If this matrix is *primitive* (we will also say Perron–Frobenius), *i.e.* a power of $V(\gamma)$ has all its coefficients greater than zero, we obtain a positive pseudo-Anosov map $\phi(\gamma)$ with expansion factor $\rho(V(\gamma))$.

In Section 4 we will associate to any non-closed path γ in a given Rauzy diagram $\mathcal{D}$ connecting a permutation π to its symmetric $s(\pi)$ (see Section 4), a matrix $V(\gamma)$. If the matrix is *primitive* we obtain a negative pseudo-Anosov map $\phi(\gamma)$ with expansion factor $\rho(V(\gamma))$.

The converse holds in the following sense (see Theorem 4.3 and Proposition 4.4).

Theorem (Geometric Statement)**.**

   (1) *Any affine negative pseudo-Anosov map ϕ on a translation surface S that fixes a point is obtained by the Symmetric Rauzy-Veech construction.*

(2) *Any affine pseudo-Anosov ϕ on a translation surface $S \in \mathcal{C}_n^{hyp}$ is obtained as above, up to replacing ϕ by $\tau \circ \phi$, where τ is the hyperelliptic involution.*

2.3. **Dynamical part.** From now on, we assume that the underlying connected component is hyperelliptic. The theorem above reduces our problem to a combinatorial problem on graphs. However the new problem is still very complicated since the complexity of paths is too large. To bypass this difficulty we introduce a renormalization process (ZRL for Zorich Right Left induction), analogously to the Veech–Zorich and Marmi–Moussa–Yoccoz acceleration of the Rauzy induction, see Section 5. This allows us to reduce considerably the range of paths to analyze in the Rauzy diagram $\mathcal{D}_n$.

The particular shape of $\mathcal{D}_n$ will be strongly used. We will usually refer to the permutation π_n as the *central permutation*. For any permutation $\pi \in \mathcal{D}_n$, there is a unique shortest path from π_n to π, that can be expressed as a word w with letters 't' and 'b'. We will often write this permutation as $\pi_n.w$. The loop in $\mathcal{D}_n$ composed by the vertices $\{\pi_n.t^k\}_{k=0,\ldots,n-2}$ will be referred to as the *central loop*.

In the next, an *admissible path* is a path in $\mathcal{D}_n$ from a permutation π to $s(\pi)$ whose corresponding matrix $V(\gamma)$ is primitive. The path is *pure* if the corresponding pseudo-Anosov map is not obtained by the usual Rauzy–Veech construction. This definition is motivated by Proposition 3.3 and the main result of [BL12]: pseudo-Anosov homeomorphism in hyperelliptic connected components with small expansion factors are *not* obtained by the usual Rauzy–Veech construction.

We show (see Theorem 5.1):

Theorem (Dynamical Statement). *There exists a map (denoted by ZRL) defined on the set of pure admissible paths on $\mathcal{D}_n$ satisfying the following property: The paths γ and $ZRL(\gamma)$ define the same pseudo-Anosov homeomorphism up to conjugacy, and there exists an iterate $m \geq 0$ such that $\gamma' = ZRL^{(m)}(\gamma)$ is an admissible path starting from a permutation π that belongs to the central loop. In addition the first step of γ' is of type 'b'.*

This results allows us to restrict the computation to paths γ starting from the central loop. As a byproduct one can actually compute *all* the expansion factors less than two *i.e.* one can compute explicitly the finite set

$$\{\lambda(\phi); \ \phi : S \to S, \ S \in \mathcal{H}_g^{\mathrm{hyp}} \text{ and } \lambda(\phi) < 2\}$$

for small values of g up to 20 (see Theorem 1).

2.4. **Combinatoric part.** The last part of the proof deals with the estimates of the spectral radius of all the paths γ starting from the central loop. We first show that this problem reduces to a problem on a finite number of paths (for a given $n \geq 4$). Then we show how to compare spectral radius $\rho(V(\gamma))$ for various γ starting from the permutation $\pi := \pi_n.t^k$ for some k. At this aim, for a given $n \geq 4$, we introduce several notations.

Notation. *For $n \in \mathbb{N}$ we set $K_n = \lfloor \frac{n}{2} \rfloor - 1$ and $L_n = n - 2 - K_n$.*
For any $k \in \{1, \ldots, K_n\}$ we define the path

$$\gamma_{n,k} : \quad \pi \longrightarrow s(\pi) : \quad b^{n-1-k}t^{n-1-2k}$$

For any $k \in \{1, \ldots, K_n\}$ and $l \in \{1, \ldots, 2n - 2 - 3k\}$ we define the path

$$\gamma_{n,k,l} : \pi \longrightarrow s(\pi) : \begin{cases} b^l t^{n-1-k-l} b^{n-1-k-l} t^{n-1-2k} & \text{if} \quad 1 \le l \le n - 2 - k \\ b^{n-1-k} t^{l-(n-1-k)} b^{2(n-1-k)-l} t^{2n-2-3k-l} & \text{if} \quad n - 2 - k < l \le 2n - 2 - 3k \end{cases}$$

Loosely speaking $\gamma_{n,k}$ is the shortest path from π to $s(\pi)$ starting with a label 'b', i.e. it is the concatenation of the small loop attached to π labelled by 'b' followed by the shortest path from π to $s(\pi)$. Similarly, $\gamma_{n,k,l}$ is a path joining π to $s(\pi)$ obtained from $\gamma_{n,k}$ by adding a loop: if $l \le n - 2 - k$ then it is a 't' loop based at $\pi_n.t^k b^l$, if $l \ge n - 1 - k$ then it is a 'b' loop based at $\pi_n.t^{l-(n-1-2k)}$ (see Figure 2).
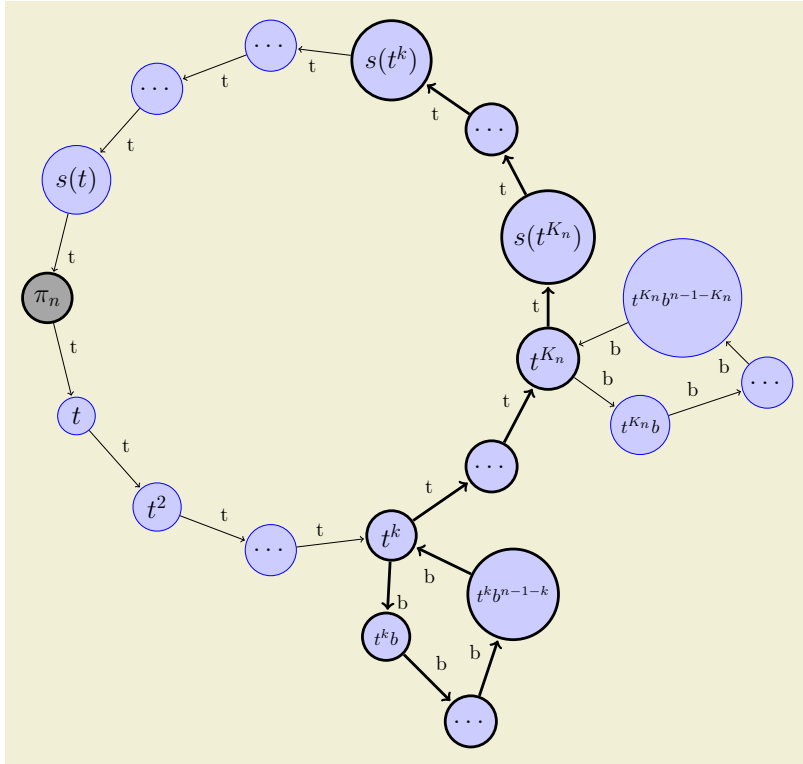

FIGURE 2. Half of the Rauzy diagram. In bold the path $\gamma_{n,k}$.

For any path γ in the Rauzy diagram, we will denote by $\theta(\gamma)$ the maximal real eigenvalue of the matrix $V(\gamma)$. We will also use the notations $\theta_{n,k}$ (respectively, $\theta_{n,k,l}$) for $\theta(\gamma_{n,k})$ (respectively, $\theta(\gamma_{n,k,l})$). Observe that the matrix $V(\gamma)$ is not necessarily primitive, but the spectral radius is still an eigenvalue by the Perron–Frobenius theorem. We show (see Proposition 6.1, Lemma A.1 and Proposition A.4):

Theorem (Combinatoric Statement)**.** *Let $n \ge 4$ be any integer. The followings hold.*

(1) *Let γ be an admissible path starting from $\pi = \pi_n.t^k$, for some $k \le n - 1$, such that the first step is 'b'. We assume that $\theta(\gamma) < 2$. Then, $k \le K_n$ and $\theta(\gamma) \ge \theta_{n,k}$.*

Furthermore, if $\gamma \neq \gamma_{n,k}$ then there exists $l \in \{1, \ldots, 2n - 2 - 3k\}$ such that

$$\theta(\gamma) \geq \theta_{n,k,l}.$$

If $V(\gamma_{n,k})$ is not primitive, we can also assume that $l \neq n - 1 - k$ in the previous statement.

(2) *For any $k \in \{1, \ldots, K_n\}$, let $d = \gcd(n - 1, k)$. If $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$ then the matrix $V_{n',k'}$ is primitive and $\theta_{n,k} = \theta_{n',k'}$.*

(3) *Let $n \equiv 3 \mod 4$ and $l \in \{1, \ldots, L_n + 3\}$. The matrix $V_{n,K_n,l}$ is primitive if and only if l is odd. Moreover if l is even then $\theta_{n,K_n,l} = \theta_{n',K_{n'},l'}$ with $n' = (n + 1)/2$, $l' = l/2$ and $K_{n'} = K_n/2$.*

2.5. **Proof of the Main Theorem.** Theorems 2.1 and 2.2 will follow from Theorem 2.3, Theorem 2.5 and Theorem 2.4 below.

Theorem 2.3. *The following holds:*

(1) *If $n \geq 4$ is even then $L(\mathrm{Spec}(\mathcal{C}_n^{hyp})) = \log(\theta_{n,K_n})$.*

(2) *If $n \not\equiv 4 \mod 6$ and $n \geq 18$ is even then the second least element of $\mathrm{Spec}(\mathcal{C}_n^{hyp})$ is $\log(\theta_{n,K_n-1})$.*

Moreover the conjugacy mapping class realizing the minimum is unique.

Letting $P_{n,k}$ the characteristic polynomial of $V(\gamma_{n,k})$ *multiplied* by $X + 1$, by definition $\theta_{n,k}$ is the maximal real root of $P_{n,k}$. By Lemma A.2 we have, when n is even:

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^2 + 1$$

and when $n \not\equiv 4 \mod 6$ and $\gcd(n - 1, K_n - 1) = 1$:

$$P_{n,K_n-1} = X^{n+1} - 2X^{n-1} - 2X^{\lceil 2n/3 \rceil} - 2X^{\lfloor n/3 \rfloor+1} - 2X^2 + 1$$

that are the desired formulas.

Proof of Theorem 2.3. We will show that $L(\mathrm{Spec}(\mathcal{C}_n^{hyp})) = \log(\theta_{n,K_n})$. Observe that $K_n = n/2 - 1$ thus $\gcd(n - 1, K_n) = 1$ and the matrix V_{n,K_n} is primitive by the Combinatoric Statement. This shows $L(\mathrm{Spec}(\mathcal{C}_n^{hyp})) \leq \log(\theta_{n,K_n})$. A simple computation shows $\theta_{n,K_n} < 2$ (see Lemma A.2).

Now by the Dynamical Statement, $L(\mathrm{Spec}(\mathcal{C}_n^{hyp})) = \log(\theta(\gamma))$ for some path γ starting from $\pi = \pi_n.t^k$ with first step of type 'b'. By the Combinatoric Statement, $\theta(\gamma) \geq \theta_{n,k}$ for some $k \in \{1, \ldots, K_n\}$. Thus $\theta_{n,k} \leq \theta_{n,K_n}$. We need to show that $k = K_n$. Let us assume that $k < K_n$.

(1) If $\gcd(k, n - 1) = 1$ then Lemma B.3 implies $\theta_{n,k} > \theta_{n,K_n}$ that is a contradiction.

(2) If $\gcd(k, n - 1) = d > 1$ then by the second point of the Combinatoric Statement, $\theta_{n,k} = \theta_{n',k'}$ where $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$. Since n' is even, $k' \neq K_{n'}$ and $\gcd(k', n' - 1) = 1$, the previous step applies and $\theta_{n',k'} > \theta_{n',K_{n'}}$. By Lemma B.2 the sequence $(\theta_{2n,K_{2n}})_n$ is a decreasing sequence, hence $\theta_{n',K_{n'}} > \theta_{n,K_n}$. Again we run into a contradiction.

In conclusion $k = K_n$ and $L(\text{Spec}(\mathcal{C}_n^{\text{hyp}})) = \log(\theta_{n,K_n})$. By construction and since all inequality above are strict, the conjugacy mapping class realizing this minimum is unique.

We now finish the proof of the theorem with the second least dilatation. The assumption $n \not\equiv 4 \mod 6$ implies $\gcd(K_n - 1, n - 1) = 1$. Hence V_{n,K_n-1} is primitive and the second least dilatation is less than θ_{n,K_n-1}. As before, a simple computation shows that $\theta_{n,K_n-1} < 2$ (see Lemma A.2).

Conversely, the second least dilatation equals $\theta(\gamma)$ for some admissible path γ starting from $\pi = \pi_n.t^k$, for some $k \in \{1, \ldots, K_n\}$, with a 'b' as the first step. If $k = K_n$, since $\gamma \neq \gamma_{n,K_n}$, the Combinatoric statements implies that $\theta(\gamma) \geq \theta_{n,K_n,l}$ for some $l \in \{1, \ldots, 2n - 2 - 3K_n\} = \{1, \ldots, L_n+2\}$. Again the Combinatoric statements shows that if $k \leq K_n - 1$ then $\theta(\gamma) \geq \theta_{n,k}$.

The theorem will follow from the following two assertions, that are proven in Proposition B.8.

- For any $k = 1, \ldots, K_n - 2$ one has $\theta_{n,k} > \theta_{n,K_n-1}$.
- For any $l = 1, \ldots, L_n + 2$ one has $\theta_{n,K_n,l} > \theta_{n,K_n-1}$.

This ends the proof of Theorem 2.3.   □

Theorem 2.4. *If $n \geq 5$ and $n \equiv 1 \mod 4$ then $L(\mathcal{C}_n^{\text{hyp}}) = \log(\theta_{n,K_n})$.*

By Lemma A.2, when $n = 1 \mod 4$, we have that θ_{n,K_n} is the maximal real root of

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^{\frac{n+1}{2}} - 2X^2 + 1.$$

Proof of Theorem 2.4. We follow the same strategy than that of the proof of the previous theorem. Namely $L(\mathcal{C}_n^{\text{hyp}}) \leq \log(\theta_{n,K_n})$ since V_{n,K_n} is irreducible (by the Combinatoric statement).

Conversely $L(\mathcal{C}_n^{\text{hyp}}) = \log(\theta(\gamma))$, where γ is an admissible path starting from $\pi_n.t^k$ for some $k \in \{1, \ldots, K_n\}$, and the path starts by a 'b'. Recall that we have shown $\theta(\gamma) \leq \theta_{n,K_n}$. We need to show that for any $k \leq K_n - 1$, $\theta_{n,k} > \theta_{n,K_n}$. Again by the Combinatoric statement, $\theta_{n,k} = \theta_{n',k'}$, where $n' = \frac{n-1}{d} + 1$, $k' = \frac{k}{d}$ and $d = \gcd(n-1, k)$. By Proposition B.9

$$\theta_{n',k'} > \theta_{n,K_n}.$$

This finishes the proof of Theorem 2.4.   □

Theorem 2.5. *If $n \geq 7$ and $n \equiv 3 \mod 4$ then $L(\mathcal{C}_n^{\text{hyp}}) = \log(\theta_{n,K_n,L_n})$.*

Proposition A.4 gives that θ_{n,K_n,L_n} is the maximal real root of

$$P_{n,K_n,L_n} = X^{n+1} - 2X^{n-1} - 4X^{(n-1)/2+2} + 4X^{(n-1)/2} + 2X^2 - 1$$

that gives the desired formula.

Proof of Theorem 2.5. We follow the strategy used in the two previous proofs. Namely $L(\mathcal{C}_n^{\text{hyp}}) \leq \log(\theta_{n,K_n,L_n})$ since V_{n,K_n,L_n} is irreducible (see the Combinatoric statement where $n \equiv 3 \mod 4$ and L_n is odd).

Conversely $L(\mathcal{C}_n^{\text{hyp}}) = \log(\theta(\gamma))$, where γ is an admissible path starting from $\pi_n.t^k$ for some $k \in \{1, \ldots, K_n\}$, and the path starts by a 'b' (recall that $\theta(\gamma) \leq \theta_{n,K_n,L_n}$).

Assume $k \leq K_n - 1$. Then the Combinatoric statement shows that $\theta(\gamma) \geq \theta_{n,k}$. Letting $n' = (n-1)/d + 1$ and $k' = k/d$ where $d = \gcd(n-1, k)$ we have $\theta_{n,k} = \theta_{n',k'}$. By Proposition B.6

$$\theta_{n',k'} > \theta_{n,K_n,L_n}.$$

This is a contradiction with $\theta(\gamma) \leq \theta_{n,K_n,L_n}$. Hence we necessarily have $k = K_n$.

Now, the matrix V_{n,K_n} is *not* primitive, hence $\gamma \neq \gamma_{n,K_n}$. Thus the Combinatoric statement implies that $\theta(\gamma) \geq \theta_{n,K_n,l}$ for some $l \in \{1, \ldots, 2n - 2 - 3K_n\} = \{1, \ldots, L_n + 3\}$ with $l \neq n - K_n - 1 = L_n + 1$. We will discuss two different cases depending on the parity of l (recall that L_n is odd).

Case 1. l is even. By the Combinatoric statement, one has $\theta_{n,K_n,l} = \theta_{n',K_{n'},l'}$ with $n' = (n+1)/2$, $l' = l/2$ and $K_{n'} = K_n/2$. If $l < L_n$ then $l' < L_{n'}$ and Lemma B.7 applies (since $n' \geq 4$ is even):

$$\theta_{n',K_{n'},l'} > \theta_{n',K_{n'},L_{n'}}.$$

If $l > L_n$ then $l = L_n + 3$. Again the Combinatoric Statement gives $\theta_{n,K_n,L_n+3} = \theta_{n',K_{n'},L_{n'}+2}$. In all cases, Proposition B.6 applies and:

$$\theta_{n',K_{n'},L_{n'}} > \theta_{n,K_n,L_n}$$

running into a contradiction.

Case 2. l is odd. If $l < L_n$ then by Proposition B.5 we have $\theta_{n,K_n,l} > \theta_{n,K_n,L_n}$. This is again a contradiction. The case $l > L_n$, namely $l = L_n + 2$, is ruled out by Lemma B.4: $\theta_{n,K_n,L_n+2} > 2 > \theta_{n,K_n,L_n}$.

In conclusion $l = L_n$ and $\gamma = \gamma_{n,K_n,L_n}$. The Main Theorem is proved. $\square$

3. Rauzy–Veech induction and pseudo-Anosov homeomorphism

In this section, we briefly recall the notions of interval exchange transformations, suspension data, Rauzy–Veech induction, and the associated construction of pseudo-Anosov homeomorphisms. We also provide a slight generalization of these standard notions.

3.1. **Interval exchange transformation.** Let $I \subset \mathbb{R}$ be an open interval and let us choose a finite partition of I into $d \geq 2$ open subintervals $\{I_j, \ j = 1, \ldots, d\}$. An interval exchange transformation is a one-to-one map T from I to itself that permutes, by translation, the subintervals I_j. It is easy to see that T is precisely determined by the following data: a permutation that encodes how the intervals are exchanged, and a vector with positive entries that encodes the lengths of the intervals.

We use the representation introduced first by Kerckhoff [Ker85] and formalised later by Bufetov [Buf06] and Marmi, Moussa & Yoccoz [MMY05].

We will attribute a name to each interval I_j. In this case, we will speak of *labeled* interval exchange maps. One gets a pair of one-to-one maps (π_t, π_b) (t for "top" and b for "bottom") from a finite alphabet $\mathcal{A}$ to $\{1, \ldots, d\}$ in the following way. In the partition of I into intervals, we denote the kth interval, when counted from the left to the right, by $I_{\pi_t^{-1}(k)}$. Once the intervals are exchanged, the interval number k is $I_{\pi_b^{-1}(k)}$. Then with this convention, the

permutation encoding the map T is $\pi_b \circ \pi_t^{-1}$. We will denote the length of the intervals by a vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

We will call the pair (π_t, π_b) a *labeled* permutation, and $\pi_b \circ \pi_t^{-1}$ a permutation (or *reduced* permutation). One usually represents labeled permutations $\pi = (\pi_t, \pi_b)$ by a table:

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \dots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \dots & \pi_b^{-1}(d) \end{pmatrix}.$$

3.2. **Suspension data and weak suspension data.** The next construction provides a link between interval exchange transformations and translation surfaces. A suspension datum for $T = (\pi, \lambda)$ is a vector $(\tau_\alpha)_{\alpha \in \mathcal{A}}$ such that

- $\forall 1 \leq k \leq d-1, \; \sum_{\pi_t(\alpha) \leq k} \tau_\alpha > 0$,
- $\forall 1 \leq k \leq d-1, \; \sum_{\pi_b(\alpha) \leq k} \tau_\alpha < 0$.

We will often use the notation $\zeta = (\lambda, \tau)$. To each suspension datum τ, we can associate a translation surface $(X, \omega) = X(\pi, \zeta)$ in the following way.

Consider the broken line L_t on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. Similarly, we consider the broken line L_b defined by concatenation of the vectors $\zeta_{\pi_b^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. If the lines L_t and L_b have no intersections other than the endpoints, we can construct a translation surface X by identifying each side ζ_j on L_t with the side ζ_j on L_b by a translation. The resulting surface is a translation surface endowed with the form dz^2. Note that the lines L_t and L_b might have some other intersection points. But in this case, one can still define a translation surface by using the *zippered rectangle construction*, due to Veech [Vee82].

We will need to extend a little the definition of a suspension datum.

Definition 3.1. *Let $T = (\pi, \lambda)$ an interval exchange map. A* weak suspension data *for T is an element $\tau \in \mathbb{R}^{\mathcal{A}}$, such that there exists $h \in \mathbb{R}$ such that:*

i- $\forall 1 \leq k \leq d-1, \; h + \sum_{\pi_t(\alpha) \leq k} \tau_\alpha > 0$,

ii- $\forall 1 \leq k \leq d-1, \; h + \sum_{\pi_b(\alpha) \leq k} \tau_\alpha < 0$.

iii- *If $\pi_t^{-1}(1) = \pi_b^{-1}(d)$, then $\sum_{\pi_t(\alpha) \neq 1} \tau_\alpha < 0$*

iv- *If $\pi_t^{-1}(d) = \pi_b^{-1}(1)$, then $\sum_{\pi_b(\alpha) \neq 1} \tau_\alpha > 0$*

The parameter h above will be called *height* of the weak suspension datum τ. Note that a usual suspension datum corresponds to the case $h = 0$: in this case, the first two conditions imply the two others.

Observe that, in a very similar way as for the case of suspension data, one can associate to a pair (π, λ, τ, h) a translation surface $X = X(\pi, \lambda, \tau)$ and a horizontal interval $I_h \subset X$. One use a small variation of the zippered rectangle construction: let I_h be a horizontal interval of length $\sum_\alpha \lambda_\alpha$. For $\alpha \in \mathcal{A}$, we consider the rectangle R_α of width λ_α and of height

$$h_\alpha = \sum_{\pi_t(\beta) \leq \pi_t(\alpha)} \tau_\beta - \sum_{\pi_b(\beta) \leq \pi_b(\alpha)} \tau_\beta$$
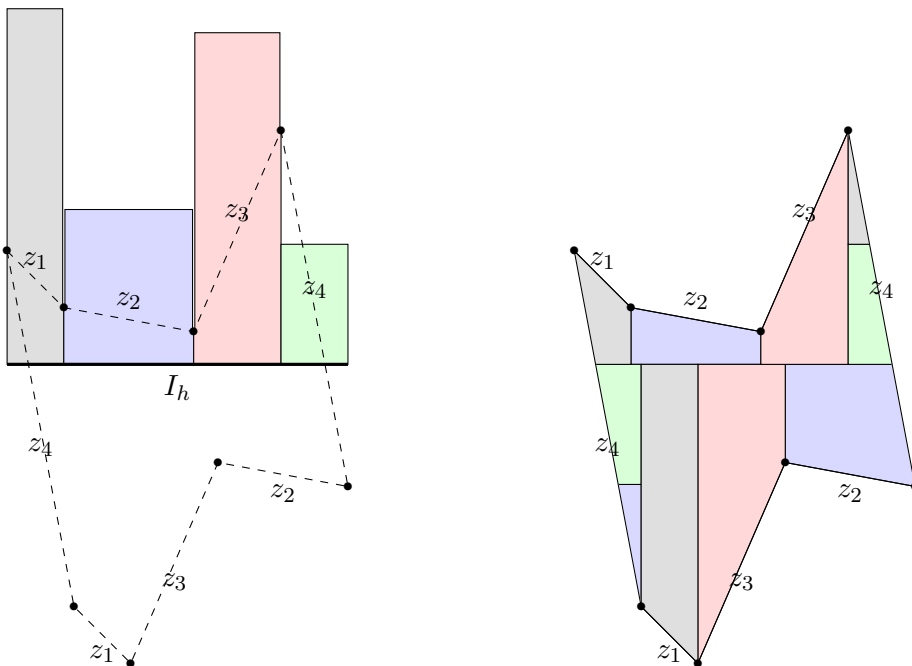
FIGURE 3. Zippered rectangle construction for a weak suspension datum

By definition of weak suspension datum, $h_\alpha > 0$ for each α. Note that, for instance, Condition *iii-* is needed to insure that $h_a > 0$ for a permutation π of the form $\left(\begin{smallmatrix} a & *** & * \\ * & ** & a \end{smallmatrix} \right)$). Then, we glue the rectangles $(R_\alpha)_\alpha$ in a similar way as for the usual zippered rectangle construction. Two values h_1, h_2 give canonically isometric surfaces $X(\pi, \lambda, \tau)$, where the intervals I_{h_1} and I_{h_2} differs by a vertical translation (of length $h_2 - h_1$). In other words, there is a immersed Euclidean rectangle of height $|h_2 - h_1|$ whose horizontal sides are I_{h_1} and I_{h_2}. Also, vertical leaves starting from the endpoints of I_h satisfy the "classical conditions", *i.e.* each leaf will hit a singularity before intersecting I_h (in the positive or negative direction depending on h and the suspension data).

Conversely, let X be a translation surface and $I \subset X$ be a horizontal interval with the same classical conditions as above. We assume that there are no vertical saddle connections. In a similar way to the classical case, there exists a (unique) weak suspension datum (π, λ, τ, h) such that $(X, I) = (X(\pi, \lambda, \tau), I_h)$. The datum (π, λ) is given by considering the interval exchange T defined by the first return map of the vertical flow on I, h is the time (positive or negative) for which the vertical geodesic starting from the left end hits a singularity. The parameters τ_α are obtained by considering vertical geodesics starting from the discontinuities of T and the time where they hit singularities: the corresponding time t_k for the k-th discontinuity is $h + \sum_{\pi_t(\alpha) \le k} \tau_\alpha$. Also, if $\pi_t^{-1}(1) = \pi_b^{-1}(d)$ then Condition *iii-* corresponds to the fact that the vertical geodesics starting from the right end of I hits a singularity before intersecting I again (and similarly for Condition *iv-*).

3.3. **Rauzy-Veech induction and other Rauzy operations.** The Rauzy-Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to a certain subinterval J of I (see [Rau79, MMY05] for details).

We recall very briefly the construction. Following [MMY05] we define the *type* of T by t if $\lambda_{\pi_t^{-1}(d)} > \lambda_{\pi_b^{-1}(d)}$ and b if $\lambda_{\pi_t^{-1}(d)} < \lambda_{\pi_b^{-1}(d)}$. When T is of type t (respectively, b) we will say that the label $\pi_t^{-1}(d)$ (respectively, $\pi_b^{-1}(d)$) is the winner and that $\pi_b^{-1}(d)$ (respectively, $\pi_t^{-1}(d)$) is the loser. We define a subinterval J of I by

$$J = \begin{cases} I \backslash T(I_{\pi_b^{-1}(d)}) & \text{if } T \text{ is of type t;} \\ I \backslash I_{\pi_t^{-1}(d)} & \text{if } T \text{ is of type b.} \end{cases}$$

The image of T by the Rauzy-Veech induction $\mathcal{R}$ is defined as the first return map of T to the subinterval J. This is again an interval exchange transformation, defined on d letters (see e.g. [Rau79]). The data of $\mathcal{R}(T)$ are very easy to express in term of those of T.

There are two cases to distinguish depending whether T is of type t or b; the labeled permutations of $\mathcal{R}(T)$ only depends on π and on the type of T. If $\varepsilon \in \{t, b\}$ is the type of T, this defines two maps $\mathcal{R}_t$ and $\mathcal{R}_b$ by $\mathcal{R}(T) = (\mathcal{R}_\varepsilon(\pi), \lambda')$. We will often make use of the following notation: if $\varepsilon \in \{t, b\}$ we denote by $1 - \varepsilon$ the other element of $\{t, b\}$.

(1) T has type t. Let $k \in \{1, \ldots, d-1\}$ such that $\pi_b^{-1}(k) = \pi_t^{-1}(d)$. Then $\mathcal{R}_t(\pi_t, \pi_b) = (\pi_t', \pi_b')$ where $\pi_t = \pi_t'$ and

$$\pi_b'^{-1}(j) = \begin{cases} \pi_b^{-1}(j) & \text{if } j \leq k \\ \pi_b^{-1}(d) & \text{if } j = k+1 \\ \pi_b^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(2) T has type b. Let $k \in \{1, \ldots, d-1\}$ such that $\pi_t^{-1}(k) = \pi_b^{-1}(d)$. Then $\mathcal{R}_b(\pi_t, \pi_b) = (\pi_t', \pi_b')$ where $\pi_b = \pi_b'$ and

$$\pi_t'^{-1}(j) = \begin{cases} \pi_t^{-1}(j) & \text{if } j \leq k \\ \pi_t^{-1}(d) & \text{if } j = k+1 \\ \pi_t^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(3) Let us denote by $E_{\alpha\beta}$ the $d \times d$ matrix of which the α, β-th element is equal to 1, all others to 0. If T is of type t then let $(\alpha, \beta) = (\pi_t^{-1}(d), \pi_b^{-1}(d))$ otherwise let $(\alpha, \beta) = (\pi_b^{-1}(d), \pi_t^{-1}(d))$. Then $V_{\alpha\beta}\lambda' = \lambda$, where $V_{\alpha\beta}$ is the transvection matrix $I + E_{\alpha\beta}$.

If τ is a suspension data over (π, λ) then we define $\mathcal{R}(\pi, \lambda, \tau)$ by

$$\mathcal{R}(\pi, \lambda, \tau) = (\mathcal{R}_\varepsilon(\pi), V^{-1}\lambda, V^{-1}\tau),$$

where ε is the type of $T = (\pi, \lambda)$ and V is the corresponding transition matrix. In other terms $V_{\alpha\beta}\zeta' = \zeta$ where $\zeta = (\lambda, \tau)$.

Remark 3.1. *By construction the two translation surfaces $X(\pi, \zeta)$ and $X(\pi', \zeta')$ are naturally isometric (as translation surfaces).*

Remark 3.2. *We can extend the Rauzy–Veech operation on the space of weak suspension data by using the same formulas. We easily see that $V^{-1}\zeta$ is a weak suspension data for $\mathcal{R}_\varepsilon(\pi)$. Also, $X(\mathcal{R}(\pi, \zeta))$ and $X(\pi, \zeta)$ are also naturally isometric as translation surfaces.*

Now if we iterate the Rauzy induction, we get a sequence (α_k, β_k) of winners/losers. Denoting $\mathcal{R}^{(n)}(\pi, \lambda) = (\pi^{(n)}, \lambda^{(n)})$, the transition matrix that relates $\lambda^{(n)}$ to λ is the product of the transition matrices:

$$\left(\prod_{k=1}^{n} V_{\alpha_k \beta_k}\right) \lambda^{(n)} = \lambda. \tag{1}$$

Now, we define other Rauzy moves that will be used later. Let π be a labeled permutation.

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \dots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \dots & \pi_b^{-1}(d) \end{pmatrix}.$$

We define the *symmetric* of π, denoted by $s(\pi)$, the following labeled permutation.

$$s(\pi) = \begin{pmatrix} \pi_b^{-1}(d) & \pi_b^{-1}(d-1) & \dots & \pi_b^{-1}(1) \\ \pi_t^{-1}(d) & \pi_t^{-1}(d-1) & \dots & \pi_t^{-1}(1) \end{pmatrix}.$$

Observe that if τ is a weak suspension datum for (π, λ), then τ is also a weak suspension datum for $(s(\pi), \lambda)$ (it is not necessarily true for usual suspension data). For simplicity, we define $s(\pi, \lambda, \tau) = (s(\pi), \lambda, \tau)$. Also, the translation surfaces $X(\pi, \lambda, \tau)$ and $X(s(\pi, \lambda, \tau))$ are related by the element $-I \in \mathrm{SL}(2, \mathbb{R})$.

Left Rauzy induction can be defined analogously as the Rauzy induction, by "cutting" the interval on the left. It can also be defined by $\mathcal{R}_L = s \circ \mathcal{R} \circ s$. From the above study, we see that $\mathcal{R}_L$ preserves weak suspension data, and that $X((\pi, \lambda, \tau))$ and $X(\mathcal{R}_L(\pi, \lambda, \tau))$ are naturally isometric as translation surfaces.

3.4. **Labeled Rauzy diagrams.** For a labeled permutation π, we call the *labeled Rauzy diagram*, denoted by $\mathcal{D}(\pi)$, the graph whose vertices are all labeled permutations that we can obtained from π by the combinatorial Rauzy moves. From each vertices, there are two edges labeled t and b (the type) corresponding to the two combinatorial Rauzy moves. We will denote by $\pi \xrightarrow{\alpha, \beta} \pi'$ for the edge corresponding to $\mathcal{R}_\varepsilon(\pi) = \pi'$ where $\varepsilon \in \{t, b\}$ and α/β is the winner/loser. To each path γ of this diagram, there is thus a sequence of winners/losers. We will denote by $\widetilde{V}(\gamma)$ the product of the transition matrices in Equation (1).

Similarly, one can define the reduced Rauzy diagram $\mathcal{D}_{red}(\pi)$ by considering (reduced) permutations as vertices. There is clearly a canonical map $\mathcal{D}(\pi) \to \mathcal{D}_{red}(\pi)$.

3.5. **Coordinates on the hyperelliptic Rauzy diagrams.** The hyperelliptic Rauzy diagram $\mathcal{D}_n$ of size $2^{n-1} - 1$ is the one that contain the permutation

$$\pi_n = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}.$$

When n is even, it corresponds to the hyperelliptic connected component $\mathcal{H}^{hyp}(n-2)$ and when n is odd, it corresponds to the hyperelliptic component $\mathcal{H}^{hyp}(\frac{n-1}{2}, \frac{n-1}{2})$.

The precise description of these diagrams was given by Rauzy [Rau79]. See also [BL12, Section 3.7.1]. An easy corollary is the following proposition.

Proposition 3.1. *Let $n \geq 2$. For any $\pi \in \mathcal{D}_n$ there exists a unique shortest path joining π_n to π.*

For a permutation π in $\mathcal{D}_n$ as above. We can write the path of the proposition as a unique sequence of $n_1 > 0$ Rauzy moves of type $\varepsilon \in \{t, b\}$, then $n_2 > 0$ Rauzy moves of type $1 - \varepsilon$, etc. The sequence of non negative integers $n_1, \dots, n_{k-1}$ defines the permutation π once ε is chosen. Observe that replacing the starting Rauzy type ε by $1 - \varepsilon$ changes $\pi = (\pi_t, \pi_b)$ by (π_b, π_t) up to renaming. For our purpose, we wont need to distinguish these two cases. Observe also that we necessarily have $\sum_{i=1,\dots,k-1} n_i < n - 1$.

Definition 3.2. *Let π be a permutation in $\mathcal{D}_n$ and let $n_1, \dots, n_{k-1}$ be as above, the coordinates of π are*

$$(n_1, \dots, n_{k-1}, n - 1 - \sum_{i=1,\dots,k-1} n_i)$$

This definition is motivated by the following easy proposition.

Proposition 3.2. *Let π be a permutation in $\mathcal{D}_n$. The followings hold:*
- *If π has coordinates $(n_1, \dots, n_k)$ then $s(\pi)$ has coordinates $(n_k, \dots, n_1)$.*
- *If k is even then π and $s(\pi)$ belong to the same component of $\mathcal{D}_n \backslash \{\pi_n\}$.*
- *If k is odd then π and $s(\pi)$ belong to a different component of $\mathcal{D}_n \backslash \{\pi_n\}$.*

Proof of Proposition 3.2. The proof is straightforward. To see that the last two assertions hold, we can remark that π and $s(\pi)$ belong to the same connected component of $\mathcal{D}_n \backslash \{\pi_n\}$ if and only if the minimal paths (from π_n) to π and $s(\pi)$ have the same starting Rauzy type. $\square$

We end this section by recalling the construction of Veech on pseudo-Anosov homeomorphisms.

3.6. Pseudo-Anosov homeomorphism and Rauzy-Veech induction.

We follow Veech's work [Vee82]. To any path γ in the labeled Rauzy diagram, whose image in the reduced Rauzy diagram is closed, one can associate a matrix V as follows. We denote by (π_t, π_b) and (π'_t, π'_b) the labeled permutations corresponding to the endpoints of γ. By definition of γ, they both define the same underlying permutation. We associate to it a matrix $\widetilde{V}$ as before. Let P be the permutation matrix defined by permuting the columns of the $d \times d$ identity matrix according to the maps π_t, π'_t, *i.e.* $P = [p_{\alpha\beta}]_{\alpha,\beta \in \mathcal{A}^2}$, with $p_{\alpha\beta} = 1$ if $\beta = \pi_t^{-1}(\pi'_t(\alpha))$ and 0 otherwise. The transition matrix associated to the path γ is then:

$$V = \widetilde{V} \cdot P. \tag{2}$$

Observe that V is obtained from $\widetilde{C}$ by replacing, for each $k \in \{1, \dots, d\}$, the column $\pi_t^{-1}(k)$ by the column $\pi_t'^{-1}(k)$ or for each letter α, the column labelled α by the column labelled $\pi_t'^{-1} \circ \pi_t(\alpha)$.

Assume now that the matrix V is a primitive (*i.e.* it as a power with only positive entries). Let $\theta > 1$ be its Perron-Frobenius eigenvalue. We choose a positive eigenvector λ for θ. It can

be shown that V is symplectic [Vee82], thus let us choose an eigenvector τ for the eigenvalue θ^{-1} with $\tau_{\pi_t^{-1}(d)} > 0$. It turns out that τ defines a suspension data over $T = (\pi, \lambda)$. Indeed, the set of suspension data is an open cone, that is preserved by V^{-1}. Since the matrix V^{-1} has a dominant eigenvalue θ (for the eigenvector τ), the vector τ must belong to this cone. If $\zeta = (\lambda, \tau)$, one has, for some integer k

$$\mathcal{R}^k(\pi, \zeta) = (\pi, V^{-1}\zeta) = (\pi, V^{-1}\lambda, V^{-1}\tau) = (\pi, \theta^{-1}\lambda, \theta\tau) =$$
$$= g_t(\pi, \lambda, \tau), \qquad \text{where} \qquad t = \log(\theta) > 0.$$

Hence the two surfaces $X(\pi, \zeta)$ and $g_t X(\pi, \zeta)$ differ by some element of the mapping class group (see Remark 3.1). In other words there exists a pseudo-Anosov homeomorphism ϕ affine with respect to the translation surface $X(\pi, \zeta)$ and such that $D\phi = g_t$. The action of ϕ on the relative homology of (X, ω) is $V(\gamma)$ thus the expanding factor of ϕ is θ.

By construction ϕ fixes the zero on the left of the interval I and thus it fixes a horizontal separatrix adjacent to this zero (namely, the oriented half line corresponding to the interval I). Conversely:

Theorem (Veech). *Any pseudo-Anosov homeomorphism (preserving a orientable foliation, and) fixing a horizontal separatrix is obtained by the above construction.*

Recall that the main result of [BL12] was based on the following key proposition:

Proposition 3.3 ([BL12], Propositions 4.3 & 4.4). *For any $n \geq 2$ and any pseudo-Anosov homeomorphism ϕ that is affine with respect to $(X, \omega) \in \mathcal{C}_n^{hyp}$, if ϕ fixes a horizontal separatrix attached to a zero of ω, or a marked point, then its expanding factor is bounded from below by 2.*

4. Obtaining all "small" pseudo-Anosov homeomorphisms

The proof of Theorem 2.1 uses a generalization of the Rauzy–Veech construction of pseudo-Anosov homeomorphisms. We explain in Appendix C why a naive generalization of the Rauzy induction does not work.

4.1. Construction of negative pseudo-Anosov maps.

The usual Rauzy–Veech construction naturally produces pseudo-Anosov maps that preserve the orientation of the stable and unstable foliation. The proposed generalization produces maps that reverse the unstable foliation.

We consider a non closed path γ in the labeled Rauzy diagram such that its starting point π and its ending point π' satisfy $s(\pi) = \pi'$ up to a relabelling. By a slight abuse of language we say that γ is a path from π to $s(\pi)$.

As above, we associate to such path a matrix V by multiplying the corresponding product of the transition matrices by a suitable permutation matrix that corresponds to the relabelling between $s(\pi')$ and π. As before, V is symplectic, thus let us choose an eigenvector λ for the eigenvalue θ and τ for the eigenvalue θ^{-1}. It turns out that τ is not necessarily a suspension datum, but it is a weak suspension datum, as we show in the next Proposition.

Proposition 4.1. *Let γ by a path in a Rauzy diagram from π to $s(\pi)$, and let $V := V(\gamma)$ be the corresponding matrix. We assume that V is primitive. If λ, τ are the eigenvectors as above then*

- *up to multiplying τ by -1, (λ, τ) is a weak suspension data for π.*
- *the constructed surface $S(\pi, \lambda, \tau)$ admits an affine map ϕ whose derivative is*

$$\begin{pmatrix} -\theta & 0 \\ 0 & -1/\theta \end{pmatrix}$$

  where θ is the maximal eigenvalue of M.
- *Furthermore, ϕ admits a regular fixed point (with positive index).*

Proof of Proposition 4.1. Let $C(h)$ be the cone of weak suspension data of heights h over π. We have a map $V^{-1} : \mathbb{P}C(h) \longrightarrow \mathbb{P}C(h')$. Observe that $C(h)$ is open and the union $C = \bigcup_p C(p)$ is an open convex cone. Hence there is an element $\tau \in C$ such that $[\tau] \in \mathbb{P}\overline{C}$ is fixed by V^{-1}. Hence there is $\theta \in \mathbb{R}$ such that $V^{-1}\tau = \theta\tau$. If $\tau \in \partial C$ then it still defines a nice translation surface since there is no vertical saddle connection (see [BL09, Figure 14] for a detailed argument). In this case the corresponding horizontal segment contains some singularities and thus $[\tau]$ cannot be fixed by V^{-1} that is a contraction. Hence $V^{-1}\tau = \theta\tau \in \mathring{C}$. Furthermore, θ is the greatest eigenvalue of V^{-1}, hence of V. We construct the surface $(X, \omega) = X(\pi, \lambda, \tau)$. If $\zeta = (\lambda, \tau)$, one has, for some integer k

$$\mathcal{R}^k(\pi, \zeta) = (s(\pi), V^{-1}\zeta) = (s(\pi), \theta^{-1}\lambda + i\theta\tau)$$

Now, there is a natural map from f_1 from $S_1 = X(\pi, \zeta) = X(s(\pi), \theta^{-1}\lambda + i\theta\tau)$ to $S_2 = X(s(\pi), \zeta)$ with $Df_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and a natural map f_2 from $S_2 = X(s(\pi), \zeta)$ to $S_1 = X(s(\pi), \theta^{-1}\lambda + i\theta\tau)$ with $Df_2 = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}$ and the composition $\phi = f_2 \circ f_1$ gives a pseudo-Anosov map affine on S_1 with θ as expansion factor and that reverse vertical and horizontal foliations.

Now we prove that ϕ has a regular fixed point. Let h be a height of (π, λ, τ), and I_h the corresponding interval. Then, Rauzy–Veech induction gives a subinterval I_h' of I_h, and the image of I_h by ϕ gives a interval I_{h_2}' defining the same weak suspension datum, with a different height (see Figure 4). Hence, there is a map from $\phi(I_h)$ to $I_{h_1}' \subset I_h$ with derivative $-\theta^{-1}$. It as a fixed point x, hence x and $\phi(x)$ are the endpoint a vertical segment J_x that do not contain a singularity. There is a fixed point of ϕ in J_x, which concludes the proof. $\qquad\square$

4.2. Converse of the Symmetric Rauzy–Veech construction. As for the usual Rauzy–Veech construction, we have a converse.

Definition 4.1. *Let ϕ be an affine pseudo-Anosov on S, that that reverse the orientation of the foliations, and with a regular fixed point (with positive index). A curve L is suitable for ϕ if*

(1) *it is made by a horizontal segment, starting from p, and then followed by a vertical segment, ending at a singular point. We do not allow L to have self-intersections.*
(2) *L and $\phi(L)$ do not have intersections in their interior.*

Given a suitable curve, we call base segment the horizontal part of $L \cup \phi(L)$
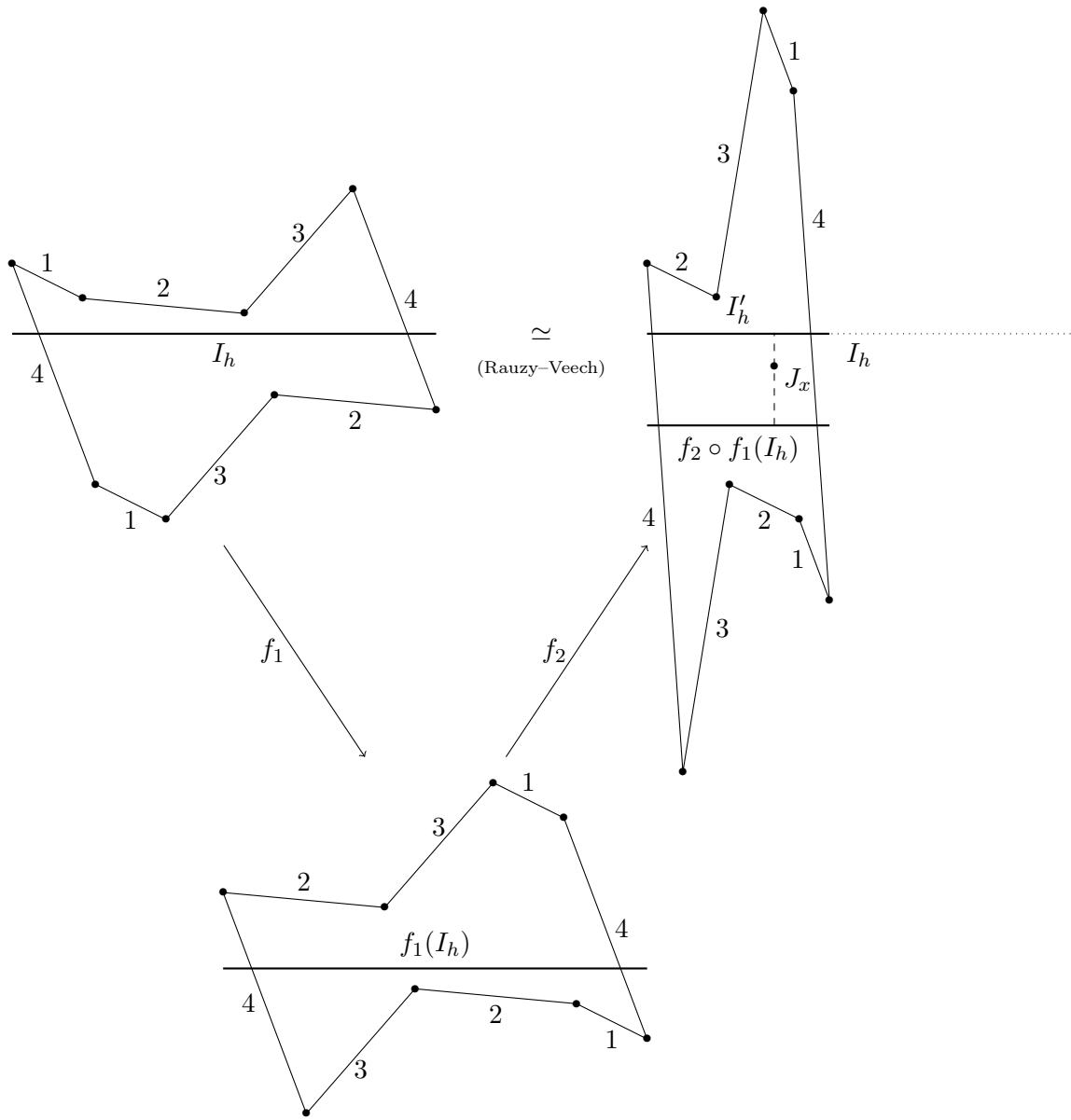
FIGURE 4. The Symmetric Rauzy–Veech construction

Proposition 4.2. *A base segment defines, by considering first return map of the vertical flow, an interval exchange transformation and weak suspension datum that can be obtained by the above construction.*

Proof. The fact that the base segment $I =]a, b[$ determines an interval exchange transformation $T = (\pi, \lambda)$ and a weak suspension datum τ, h for T is similar to the case for classical suspension

data, except that the segment is not attached on the left to a singularity. This is left to the reader.

We only need to check that T defines a Rauzy path from π to $s(\pi)$ such that λ, τ are the corresponding eigenvectors.

The key remark is the following: the horizontal part of $\phi(L) \cup \phi^2(L)$ is a segment $I' \subset I$ that has the same left end as I. Observe that $\phi(L)$ and $\phi^2(L)$ do not have intersection on their interior hence, by a classical argument, the interval exchange transformation T' associated to I' is obtained from T by applying a finite number of time the Rauzy induction on the right to T. Similarly, for the weak suspension datum, we get $(\pi', \lambda', \tau') = (\pi^{(n)}, \lambda^{(n)}, \tau^{(n)})$.

Rotating by $180°$ the picture we have, up to relabelling, $s(\pi^{(n)}) = \pi$, $\lambda^{(n)} = 1/\theta\lambda$ and $\tau^{(n)} = \theta\tau$. This ends the proof. $\square$

Now we prove the first part of the Geometric Statement.

Theorem 4.3. *Let ϕ be an affine pseudo-Anosov on S having a regular fixed point P and that reverse horizontal and vertical foliations. Then S is obtained by the above construction.*

Proof of Theorem 4.3. By Proposition 4.2, all we need to show is to produce a suitable curve L for ϕ.

We start from any oriented curve L in X made by a finite horizontal segment, starting from P, and then followed by a finite vertical segment, ending at a singular point. We do not allow L to have self-intersections. Such a curve always exists (*e.g.* by using the Veech's polygonal representation of translation surfaces). We denote by L_x and L_y the (oriented) components of L: they bound a rectangle R whose opposite corners are P and a zero σ of ω. We denote by $\{c(L)\} = L_x \cap L_y$ and by h the length of L_y.

Now if $\phi(L) \cap \overset{\circ}{L} = \emptyset$ we are done. Otherwise one of the following intersections is non empty (possibly the two):

$$\phi(L_x) \cap \overset{\circ}{L}_y \neq \emptyset \qquad \text{or} \qquad \phi(L_y) \cap \overset{\circ}{L}_x \neq \emptyset.$$

We will perform several operations on L, in order to obtain the required condition. The strategy is the following:

- 1st Step: arrange that L bounds an immersed euclidean rectangle $i(R)$.
- 2nd Step: arrange that $\phi(L_x) \cap \overset{\circ}{L}_y = \emptyset$.
- 3nd Step: change the fixed point in order to get suitable curve.

1st Step. We assume that h is *minimal* in the following sense: for each $x \in L_x$, the unit speed vertical geodesic starting from x (in the same direction as L_y), does not hit a singularity at a time less than h. Now, let $R \subset \mathbb{R}^2$ be the open rectangle of width $|L_x|$ and of height h. There is a natural translation map $i : R \to S$ that sends the bottom side of R to L_x. By the above hypothesis, $i(R)$ does not contain any singularity. Note that i might not be an embedding but just an immersion. However, P is not in $i(R)$, otherwise one easily see that the interior of L_x intersects the interior of L_y (see also Figure 5).

Assume that h is not minimal, then there is a $x_0 \in L_x$ whose corresponding vertical geodesic hits a singularity for a time h_0 minimal. We then consider the new oriented curve L_0, starting from P such that L_{0x} is the segment joining P to x_0 and L_{0y} is the vertical segment of length

h_0. Note that L_0 still satisfies the no self intersection hypothesis otherwise we would find an element $x_1 \in L_x$ with $h_1 < h_0$, contradicting the minimality hypothesis.

Note also that if $\phi(L_x) \cap \overset{\circ}{L}_y = \emptyset$, then it is still also the case for L_0. Indeed, the rectangle R_0 of width $|L_x|$ and of height h_0 immerses in S (in a similar way as R). If $\phi(L_{0x}) \cap \overset{\circ}{L}_{0y} \neq \emptyset$, then we find P in $i(R_0)$, which is not possible for the same reason as above (see Figure 5).
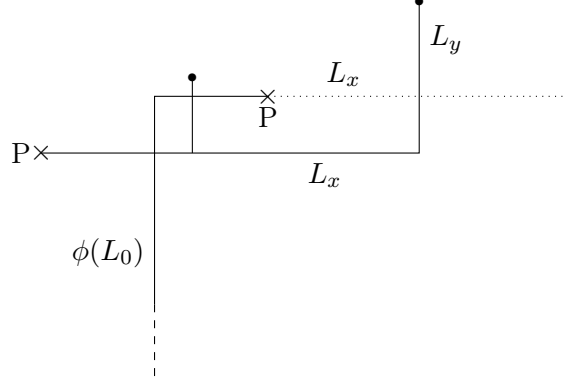

FIGURE 5. Step 1: If $\phi(L_x) \cap L_y = \emptyset$, and h is not minimal

2d Step. Now we assume $\phi(L_x) \cap \overset{\circ}{L}_y \neq \emptyset$. We first show that $\phi(L_y) \cap \overset{\circ}{L}_x \neq \emptyset$. Let Q be the point in the intersection $\phi(L_x) \cap \overset{\circ}{L}_y$ such that the vertical distance from $c(L)$ to Q is minimal. Since $|\phi(L_x)| = \lambda^{-1}|L_x| < |L_x|$ one has $\phi(c(L)) \in i(R)$. If $\phi(L_y) \cap \overset{\circ}{L}_x = \emptyset$ then the vertical segment $\phi(L_y)$ is contained in $i(R)$, in particular $\phi(\sigma) \in i(R)$: this contradicts the 1st Step since there is no singularity inside $i(R)$.

Now we replace L by L' as follows: We choose Q' in $\phi(L_y) \cap \overset{\circ}{L}_x$ such that the horizontal distance from P to Q' is minimal. Then, we define L' by considering the horizontal segment, starting from P and ending at Q', and the vertical segment from Q' and ending at $\phi(\sigma)$. Since $\phi(L'_x) \cap L'_y \subset \phi(L_x) \cap \phi(L_y) = \{\phi(c(L))\}$, one has $\phi(L'_x) \cap \overset{\circ}{L}'_y = \emptyset$ as required. Now, up to shortening L' as in the first step, we can assume that L'_x and $\overset{\circ}{L}'_y$ bound an immerse rectangle R' and we still have $\phi(L'_x) \cap \overset{\circ}{L}'_y = \emptyset$.

3d Step. Let $\widetilde{S}$ be the universal covering of S. Choose $\widetilde{P}$ a preimage of P, $\widetilde{L}$ a preimage of L attached to $\widetilde{P}$. Now the rectangle R, as defined in the 1st Step, embeds as a rectangle $\widetilde{R}$ in $\widetilde{S}$ with $\widetilde{L}$ as bottom and right sides. For any lift $\widetilde{\phi}$ of ϕ, such that $\widetilde{\phi}(\widetilde{L_y})$ intersects $\widetilde{R}$, $\widetilde{\phi}(\widetilde{L_y})$ intersects the interior of $\widetilde{L_x}$ (in a unique point Q, since we are working on the universal cover). Now, we choose a lift $\widetilde{\phi}$ of ϕ that minimize the length d of the vertical segment joining Q to the singular point that is the end of $\widetilde{\phi}(\widetilde{L_y})$. Now, we easily see that $\widetilde{\phi}(\widetilde{R})$ intersects $\widetilde{R}$ as

in Figure 6. As in the proof of Proposition 4.1, we find $x \in L_x$ such that the corresponding vertical leaf is fixed by $\widetilde{\phi}$. Then, we find in $\widetilde{R}$ a fixed point $\widetilde{P'}$ for ϕ.
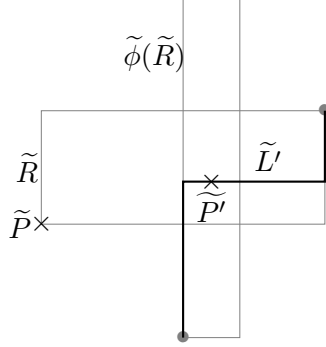

FIGURE 6. Step 3: changing the fixed point

Now we consider $\widetilde{L'}$ obtained as follows: take the horizontal segment with left end $\widetilde{P'}$ and whose right end is in $\widehat{L_x}$, then consider the vertical segment that ends in the same singularity as $\widetilde{L}$ (see Figure 6). We claim that $\widetilde{L'}$ projects in S into a suitable curve L'. Indeed, otherwise $L'_x \cup \phi(L'_x)$ either intersects the interior of L'_y or the interior of $\phi(L'_y)$. In both cases, we find another intersection point between L_x and $\phi(L'_y)$ which contradicts the minimality of d. Therefore, we have found a suitable curve, the theorem is proven. $\qquad\square$

The next proposition implies the second part of the Geometric Statement.

Proposition 4.4. *Let ϕ be an affine pseudo-Anosov map on a surface S in a hyperelliptic component, and τ be the hyperelliptic involution. Then, $\tau \circ \phi$ is also an affine pseudo-Anosov map on S. Denote $\{\phi, \tau \circ \phi\} = \{\phi^+, \phi^-\}$ such that ϕ^+ preserves the orientation of the vertical and horizontal foliations. We have the following:*

- *ϕ^- is obtained by the symmetric Rauzy–Veech construction.*
- *If ϕ^+ is not obtained by the usual Rauzy–Veech construction, then there are exactly two regular fixed points for ϕ^-, that are interchanged by the hyperelliptic involution.*

Proof. We prove the first part. From the previous theorem, all we need to show is that ϕ^- has a regular fixed point. For a homeomorphism ϕ, we denote by ϕ_* the linear action of ϕ on the homology $H_1(S, \mathbb{R})$. We recall the Lefschetz formula:

$$2 - Tr(\phi_*) = \sum_{\phi(x)=x} Ind(\phi, x).$$

When ϕ is of type pseudo-Anosov, and x a fixed point, we can show that:

- $Ind(\phi, x) < 0$ if there is a fixed separatrix.
- $Ind(\phi, x) = 1$ otherwise.

Assume that $\phi = \phi^+$ and the underlying translation surface is in $\mathcal{H}(2g-2)$. Then the unique singularity P is necessarily fixed and has index ≤ 1. All the other fixed point (possibly zero) are regular point so have negative index. Hence $2 - Tr(\phi_*) \leq 1$. Since $Tr(\tau \circ \phi) = -Tr(\phi)$, we conclude that $2 - Tr(\phi_*^-) \geq 3$. Therefore, there must be at least 2 regular fixed points for ϕ^-.

Now assume that the underlying translation surface is in $\mathcal{H}(g-1, g-1)$. The two singularities P_1, P_2 are either fixed or interchanged by ϕ^+.

- If P_1, P_2 are fixed. Then, as before, $2 - Tr(\phi_*^+) \leq 2$ hence $2 - Tr(\phi_*^-) \geq 2$. So there are a least two fixed points. But P_1, P_2 are interchanged by ϕ^-, hence the two fixed points are regular.
- If P_1, P_2 are interchanged by ϕ^+, then $2 - Tr(\phi_*^+) \leq 0$, hence $2 - Tr(\phi_*^-) \geq 4$, there are at least four fixed points so, at least 2 regular fixed points.

Now, we see that in the above proof, the case where ϕ^+ is not obtained by the usual Rauzy–Veech construction (*i.e.* when ϕ^+ does not have negative index fixed point) corresponds to the equality case. There is exactly one pair of regular fixed points $\{Q_1, Q_2\}$. Since ϕ_- and τ commute, we see that $\tau(Q_1)$ is a fixed point, hence Q_1 or Q_2. It cannot be Q_1 otherwise $\tau \circ \phi_- = \phi^+$ has a regular fixed point, contradicting the hypothesis. Hence, $\tau(Q_1) = Q_2$, this concludes the proof. $\qquad\square$

We end this section with the following useful proposition.

Proposition 4.5. *If an admissible path from π to $s(\pi)$ passes through the central permutation then $\tau \circ \phi$ is obtained by the usual Rauzy–Veech construction, where τ is the hyperelliptic involution.*

Proof of Proposition 4.5. The map $\psi = \tau \circ \phi$ preserves the orientation of the vertical and horizontal leafs. Hence, all we need to show is that there is a fixed separatrix.

Let (π, λ, τ) be the weak suspension data associated to the admissible path. Let h be a height, and $I = I_h \subset X(\pi, \lambda, \tau)$.

We claim that there is an immersed Euclidean rectangle with $\psi(I)$ as one horizontal side, and the other one is a subinterval of I. Assuming the claim, there is an isometry f from $\psi(I)$ to I obtained by following a vertical leaf. The map $f \circ \psi$ is therefore a contracting map from I to itself (its derivative is θ^{-1}), hence has a fixed point. It means that there is an element x in I whose image by ψ is in the vertical leaf l passing through x. Thus, this vertical leaf l is preserved by ψ. Since ψ, restricted to l as derivative $\theta \neq 1$, there is a fixed point of ψ on l. This fixed point is either a conical singularity or a regular point. In any case, ψ fixes a vertical separatrix. Hence ψ fixes also a horizontal separatrix. It is therefore obtained by the usual Rauzy–Veech construction.

Now we prove the claim. By construction of ϕ, there is base segment I such that $I' = \phi(I) \subset I$. This subinterval I' is obtained from I after doing Rauzy–Veech induction until the end of the defining path in the Rauzy diagram. By hypothesis, there is a step of the form $(\pi_n, \lambda'', \tau'')$, where π_n is the central permutation, and that corresponds to a interval I'', with $I' \subset I'' \subset I$. Now, we easily see that $\tau(I'')$ and I'' are the horizontal sides of a immersed rectangle. We now conclude as in the proof of Proposition 4.1 that there is a fixed separatrix. $\qquad\square$

5. Renormalization and changing the base permutation

The aim of this section is to reduce our analysis to the set of paths of $\mathcal{D}_n$ starting from the central loop. First, recall that, from [BL12], pseudo-Anosov homeomorphisms in hyperelliptic connected components obtained by the Rauzy–Veech construction have expansion factors at least 2, and are not the minimal ones. Recall that an admissible path is pure if the corresponding map is not obtained by the usual Rauzy–Veech construction.

Theorem 5.1. *Let γ be a pure admissible path in $\mathcal{D}_n$ joining a permutation π to its symmetric $s(\pi)$, and let ϕ be the corresponding pseudo-Anosov map. Then, ϕ can also be obtained by a path γ' starting from the central loop. In addition the first step of γ' is on a secondary loop (i.e. of type 'b').*

The construction of γ' in terms of γ is not obvious. To prove the theorem, we appeal to the dynamics of the induction.

5.1. **Renormalization: The ZRL acceleration.** In this section we will consider only pure admissible paths from π to $s(\pi)$. If we denote by $(n_1, \ldots, n_k)$ the coordinates of π then k is even (otherwise Proposition 3.2 implies that γ passes through the central permutation and Proposition 4.5 is a contradiction with the path being pure).

Convention 1. *We will always assume that the first Rauzy move is 't', i.e. π is obtained by applying the sequence $t^{n_1} b^{n_2} \ldots t^{n_{k-1}}$ to π_n.*

Proposition 5.2. *Let $\alpha = \pi_b^{-1}(n)$ and $\beta = \pi_t^{-1}(1)$. Then $\phi(\zeta_\alpha) = \zeta_\beta$.*
In particular there exists a canonical base segment I' obtained from the previous one as follows: We apply a right induction until α is is loser (i.e. Rauzy path of type $b^l t$, for some $l \geq 0$) followed by a left induction until β is loser (i.e. left Rauzy path of type $\bar{t}^m \bar{b}$, for some $m \geq 0$). This segment I' defines a new Rauzy path γ' starting point π' of γ' satisfying Convention 1 (up to permuting top and bottom).

Definition 5.1. *The map ZRL from the space of pure admissible paths satisfying Convention 1 to itself is defined by $ZRL(\gamma) = \gamma'$, where γ, γ' are as in Proposition 5.2.*

Proof of Proposition 5.2. By hypothesis, π is in the connected component of $\mathcal{D}_n \backslash \{\pi_n\}$ where $\beta = \pi_t^{-1}(1)$ is never winner. Since γ does not pass through the central permutation, α is never winner, hence the parameter ζ_β is unchanged during the whole Rauzy path. In particular, by definition of ϕ (see Figure 4), the segment corresponding to ζ_α is sent by ϕ to the segment corresponding to ζ_α. Hence, the curve L' as in Figure 7 is admissible and defines a new admissible path γ' from a permutation π' to $s(\pi')$. The horizontal part of $L' \cup \phi(L')$ defines a new base segment I' and parameters (π', ζ') as described in Section 3.2. We obtain (π', ζ') by the sequence of right and left Rauzy move described in the statement of the proposition. If π' does not satisfy Convention 1, we interchange the two lines, that is equivalent to interchanging "up" and "down" on the surface, and therefore conjugating ϕ with the surface orientation changing homeomorphism. □

Remark 5.1. *ZRL stands for Zorich acceleration of Right-Left induction. By construction the two surfaces $S(\gamma)$ and $S(\gamma')$ are in the same Teichmüller orbit (perhaps up to an affine*
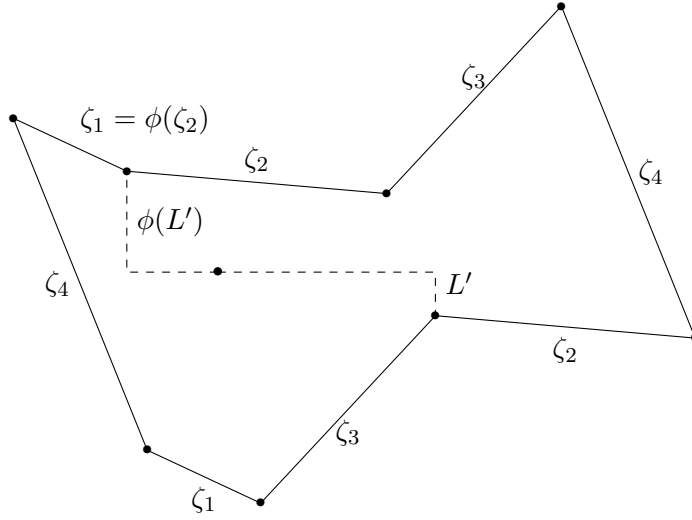
FIGURE 7. Finding a new suitable curve: $\phi(\zeta_2) = \zeta_1$.

isometry with derivative map $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). It is difficult to express the path γ' in a simple way in terms of the path γ. However the coordinates for the permutations (introduced in Section 3.5) allow us to express the new starting point π' in an easier way, only in terms of the old starting point (see Proposition 5.4 below).

In view of considering iterates of the map ZRL, we will use the following lemma.

Lemma 5.3. *Assume that γ is a pure admissible path. Then the ZRL orbit of γ is infinite and all the letters are winner and loser infinitely often.*

Proof. By Proposition 5.2, the new path γ' is a pure admissible path. Hence, the ZRL orbit is infinite. In this proof, we do not interchange top and bottom line in order to follow Convention 1, but consider a sequence of base segment $(I_n)_{n \in \mathbb{N}}$ on the same underlying surface S, with $I_{n+1} \subset I_n$. Note that the vertical flow on S is minimal (and uniquely ergodic) since S carries an affine pseudo-Anosov homeomorphism. We identify those segments as subsets of the real line, $I_n =]b_n, a_n[$, with (b_n) increasing and (a_n) decreasing. As for the usual Rauzy induction, all letter are winner infinitely often if and only if $|I_n| = a_n - b_n$ tends to zero. Let us assume that it is not the case. Then $a_n \to a$, $b_n \to b$ with $\phi(a) = b$. We consider the sequence (λ_n, τ_n) of corresponding suspension data. Define t_n to be the time when the vertical unit speed geodesic starting from a_n reach a singularity, note that t_n can be positive or negative, observe that $|t_n| \to \infty$ otherwise the set of singularities in the surface S would not be discrete. Without loss of generalities, we can assume that there is a subsequence (n_k) such that $t_{n_k} > 0$. There are two cases.

(1) If the geodesic in the positive direction starting from a, denoted as g_a, is infinite, then it follows the (finite) one starting from a_{n_k} for a longer and longer time. By density there is a time t such that $g_b(t)$ intersects $]b, a[$, hence for k large enough, the

geodesics starting from b_{n_k} intersects $]b, a[$, hence $I_{n,k}$ before reaching a singularity. Contradiction.

(2) If the geodesic g_a is finite in the positive direction, then it is necessarily infinite in the negative direction. If there is a subsequence $t_{m_k} < 0$, the same argument as above gives a contradiction. Hence, for n large enough $t_n > 0$. The same argument also works for an non accelerated Right-Left induction. Hence for n large enough, the 'right' part of the ZRL move is necessarily b. Similarly, the 'left' part of the ZRL move is necessarily $\bar{t}$. But such move cannot hold for an infinite number of step. Contradiction.

The lemma is proved. $\qquad\square$

5.2. **Action of ZLR on the starting point.** As promised we now explain how ZRL acts on the starting point, in terms of the coding introduced in Section 3.5.

Proposition 5.4. *Let γ be an admissible path, starting from a permutation π. Let $(n_1, \ldots, n_k)$, $k \geq 4$ be the coding of π. The coding of the starting point of $ZRL(\gamma)$ is obtained by the following rules.*

- *we replace (n_{k-1}, n_k) by* $\begin{cases} (n_{k-1} + 1, n_k - 1), & \text{or} \\ (n_{k-1}, n_k - 1, 1), & \text{or} \\ (n_{k-1}, \bar{l}, 1, n_k - 1 - \bar{l}). \end{cases}$
- *we replace (n_1, n_2) by* $\begin{cases} (n_1 - 1, n_2 + 1), & \text{or} \\ (1, n_1 - 1, n_2), & \text{or} \\ (n_1 - 1 - \bar{m}, 1, \bar{m}, n_2). \end{cases}$

where l, m are positive integers with $l < n_k - 1$, $m < n_1 - 1$, and $\bar{l}$, respectively $\bar{m}$, is the remainder of the Euclidean division of l, respectively m, by n_k, respectively n_1.

Proof of Proposition 5.4. By convention, the starting permutation π is of the form $t^{n_1} b^{n_2} \ldots t^{n_{k-1}}$. The map ZRL acts on π by a sequence of Rauzy moves of the form $b^l t \bar{t}^m \bar{b}$ and then followed (perhaps) by a permutations of the lines (which does not change the coding).

The Rauzy moves $b^l t$ acts on the coding as:

(1) $(n_1, \ldots, n_k) \mapsto (n_1, \ldots, n_{k-1} + 1, n_k - 1)$ if $l = 0 \mod (n_k)$.
(2) $(n_1, \ldots, n_k) \mapsto (n_1, \ldots, n_{k-1}, n_k - 1, 1)$ if $l = -1 \mod (n_k)$.
(3) $(n_1, \ldots, n_k) \mapsto (n_1, \ldots, n_{k-1}, \bar{l}, 1, n_k - 1 - \bar{l})$ otherwise.

(where $\bar{l}$ is the remainder of the Euclidean division of l by n_k).
This gives the first part of the proposition.

The remaining part is obtained similarly: we must act on the left by the moves $\bar{t}^m \bar{b}$, which is equivalent to the moves $s.b^m.t.s$. This proves the proposition. $\qquad\square$

5.3. **Proof Theorem 5.1.** We are now ready to prove the theorem announced at the beginning of this section.

Proof Theorem 5.1. Let γ be an admissible path, with corresponding expansion factor $\lambda < 2$. We need to show that there exists an iterate of ZRL which starts from a permutation π starting from the central loop, *i.e.* satisfying $\pi_t^{-1}(n) = \pi_b^{-1}(1)$.

We first observe the following: let $(l, \ldots, l')$ be a coordinate of π, then $l \leq l'$. Indeed, if $l > l'$, then a path joining π to $s(\pi)$ must pass through the central permutation.

Now, let $(n_1, , n_2, \ldots, n_{k-1}, n_k) = (l, x, \ldots, x', l')$ be the coding of a permutation, with $k \geq 4$ even. We prove by induction the following property $\mathcal{P}(l)$:

(1) $l = l'$
(2) After applying a finite sequence of ZRL, we reach the permutation coded by $(l + x, \ldots, x' + l')$, and during this sequence, the letters "between the blocks corresponding to x and x'", i.e. $\pi_\varepsilon^{-1}(j)$ for $j \in \{l + x + 2, \ldots, n - 1 - l' - x'\}$ are constant along the ZRL-orbit and non-winner. Note that if $k \geq 4$, this set of letters is nonempty.

Initialisation corresponds to the case $(1, x, \ldots, x', l')$. Assume that $l' > 1$, then after one step of ZRL, the left part must be $(1 + x, \ldots)$, but in this case, in order to preserve the parity of the numbers of blocks, we must have:

$$(x + 1, \ldots, x', l' - 1, 1)$$

with $x + 1 > 1$, which contradicts the initial observation. So, we have $l' = 1$, and ZRL maps $(1, x, \ldots, x', 1)$ to $(x + 1, \ldots, x' + 1)$, and we see directly that the second condition is fulfilled.

Now, let $1 < l \leq l'$ and let the initial permutation be coded by $(l, x, \ldots, x', l')$. Assume $\mathcal{P}(l'')$ for any $l'' < l$. By Proposition 5.4 and observing again that the parity of the number of blocks is constant, after one step of ZRL the coding is one of the following:

(1) $(1, l - 1, x, \ldots, x', l' - 1, 1)$.
(2) $(l - 1, x + 1, \ldots, x' + 1, l' - 1)$.
(3) $(l - 1, x + 1, \ldots, x', l'_2, 1, l'_1)$, for some l'_1, l'_2 satisfying $l'_1 + l'_2 + 1 = l'$.
(4) $(l_1, 1, l_2, x, \ldots, x' + 1, l' - 1)$, for some l_1, l_2 satisfying $l_1 + l_2 + 1 = l$.
(5) $(l_1, 1, l_2, x, \ldots, x', l'_2, 1, l'_1)$, for l_1, l_2, l'_1, l'_2 as above.

Now, we study these different cases:

(1) The following step is necessarily $(l, x, \ldots, x', l')$.
(2) By the induction hypothesis, $l - 1 = l' - 1$, and after some steps, we obtain $(l + x, \ldots, l' + x')$.
(3) By induction hypothesis, we have $l - 1 = l'_1$, and after some steps of ZRL we have $(l + x, \ldots, x', l'_2, l)$, which again contradicts the first observation.
(4) By induction hypothesis, we have $l_1 = l' - 1$. But, $l_1 = l - 1 - l'_2 < l - 1 \leq l' - 1$. Contradiction.
(5) By induction hypothesis, we have $l'_1 = l_1$ and after some steps, we have $(l_1 + 1, l_2, x, \ldots, x', l'_2, l_1 + 1)$, and again after some steps, we have $(l, x, \ldots, x', l')$. During these two sequences of ZRL, all the letters between the blocks corresponding to l_2 and l'_2 are unchanged and non-winner, and this set is nonempty.

Note that it is impossible to repeat infinitely the steps (1) or (5) by Lemma 5.3, hence we will eventually get Step (2), proving $\mathcal{P}(l)$.

Hence, after a finite number of ZRL steps, we obtain a permutation with a coding with two blocks (k=2), which corresponds to a starting point in the central loop. This proves the first part of Theorem 5.1.

We now turn into the proof of the second part: the first step of the corresponding admissible Rauzy path leaves the central loop, i.e. is of type b. In this case, ZRL acts on the starting

point π by $t\bar{b}$. Write π as:

$$\begin{pmatrix} a & *** & b \\ b & *** & c \end{pmatrix}$$

We easily see that $t\bar{b}$ preserves π, and λ_b becomes $\lambda_b - \lambda_a - \lambda_c$. Iterating ZRL, after a finite number of steps, b is not winner any more. Theorem 5.1 is proved. $\qquad\square$

6. Reducing to a finite number of paths

In view of Section 4, for a given n, one needs to control the spectral radii of matrices $V(\gamma)$ for all paths γ in $\mathcal{D}_n$. Section 5 shows that it is enough to consider paths γ starting from the central loop of $\mathcal{D}_n$. However this still produces an infinite number of paths to control. The proposition below shows how one can restrict our problem to a *finite* number of paths.

Proposition 6.1. *Let γ be an admissible path starting from a permutation $\pi = \pi_n.t^k$ in the central loop. We assume that the first step goes in the secondary loop.*

- *(1) If $k > K_n$, then $\theta(\gamma) > 2$.*
- *(2) If $k \le K_n$ then $\theta(\gamma) \ge \theta_{n,k}$.*
- *(3) If $V(\gamma_{n,k})$ is not primitive, then either $\theta(\gamma) > 2$ or there exists $l \in \{1, \ldots, 2n-2-3k\}$, $l \ne n-1-k$, such that $\theta(\gamma) \ge \theta_{n,k,l}$.*

For two paths γ, γ' we will write $\gamma' \le \gamma$ if the path γ' is a subset of the graph γ (viewed as an ordered collection of edges). For a real matrix $A \in M_n(\mathbb{R})$, we will write $A \ge 0$ (respectively, $A \gg 0$) to mean that $A_{ij} \ge 0$ (respectively, $A_{ij} > 0$) for all indices $1 \le i, j \le n$, and similarly for vectors $v \in \mathbb{R}^n$. The notation $A \ge B$ means $A - B \ge 0$. Before proving Proposition 6.1, we will use the following:

Proposition 6.2. *Let $\gamma' \le \gamma$ i.e. the path γ is obtained from the path γ' by adding (possibly zero) closed loops. Then $V(\gamma') \le V(\gamma)$ and $\theta(\gamma') \le \theta(\gamma)$. Moreover if $V(\gamma)$ is primitive and $\gamma \ne \gamma'$ then $\theta(\gamma') < \theta(\gamma)$.*

Proof of Proposition 6.2. If $V(\gamma') = V_1 \cdot V_2 \cdot \ldots \cdot V_l \cdot P'$ is the matrix associated to the path γ', where V_i are the elementary Rauzy–Veech matrices and P' is a permutation matrix, then the matrix associated to γ has the form

$$V(\gamma) = V_1 \cdot N_1 \cdot V_2 \cdot N_2 \cdot \ldots \cdot V_l \cdot N_l \cdot P,$$

where:

- $N_1, \ldots, N_l$ are products of (possibly empty) elementary Rauzy–Veech matrices, hence of the type $I + N_i'$, where I is the identity matrix and N_i' is a matrix with nonnegative coefficients.
- P is the permutation matrix corresponding to the end point of γ.

Since the labeled Rauzy diagram and the reduced Rauzy diagram coincide, the endpoints of γ and γ' also coincide in the labeled Rauzy diagram, hence $P = P'$. From these facts we deduce that $V(\gamma') \le V(\gamma)$. Let us show that $\theta(\gamma') \le \theta(\gamma)$.

Recall that $V(\gamma')$ is not necessarily primitive. However there is a permutation matrix P_σ such that $P_\sigma V(\gamma') P_{\sigma^{-1}} = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ * & A_2 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ * & \dots & * & A_s \end{pmatrix}$, where the matrices A_i are primitive matrices. Up to a change of basis one can assume that the spectral radius of A_s is achieved by $\theta(\gamma')$. Thus there is a non-negative vector w such that $A_s w = \theta(\gamma') w$. Now $v' := P_{\sigma^{-1}} \begin{pmatrix} 0 \dots 0 \ w \end{pmatrix}^T$ is a non-negative right eigenvector of $V(\gamma')$ for the eigenvalue $\theta(\gamma')$.

Let v be a positive left eigenvector of $V(\gamma)$: $vV(\gamma) = \theta(\gamma)v$ and $v > 0$. From $V(\gamma') \leq V(\gamma)$ one has
$$vV(\gamma)v' \geq vV(\gamma')v'.$$
Hence $\theta(\gamma)vv' \geq \theta(\gamma')vv'$ and since $vv' > 0$ we draw $\theta(\gamma) \geq \theta(\gamma')$ as desired.

We now prove that last claim: assume $V(\gamma') \leq V(\gamma)$ and $V(\gamma) \neq V(\gamma')$. Then there exists $k \in \mathbb{Z}_{>0}$ such that $V(\gamma)^k \gg V(\gamma')^k$. In particular we can find $\alpha > 1$ such that $V(\gamma)^k \geq \alpha V(\gamma')^k$. Thus $\rho(\gamma)^k \geq \alpha\rho(\gamma')^k$ proving the proposition. $\square$

Proof of Proposition 6.1. We prove the first assertion. Let k be any integer satisfying $k > (n-1)/2$. Then $n - 1 - k < k$, hence $s(\pi) = t^{n-1-k}\pi_n$ is "before" π in the central loop. Since γ is admissible, γ is passes thought the central permutation. Now if $k = (n-1)/2$ then, up to relabeling, $\pi = s(\pi)$. By using the alphabet $\mathcal{A} = \{1, \dots, n\}$, we have:
$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ n & \frac{n-1}{2} & \dots & \frac{n+1}{2} \end{pmatrix}.$$
Hence the relabeling does not changes the letter n. In particular it must be winner at least once. This imply that γ contains the step $\pi \to t.\pi$, hence passes through the central permutation. In both situation $\theta > 2$.

We now turn to the proof of the second part of the proposition. The Rauzy diagram that we consider have the particular property that removing any vertex disconnects it (except the the particular ones that satisfies $t.\pi = \pi$ or $b.\pi = \pi$). This implies that the path γ is obtained from the path $\gamma_{n,k}$ by adding (possibly zero) closed loops, namely $\gamma_{n,k} \leq \gamma$. Hence we are in a position to apply Proposition 6.2: this finishes the proof of this part.

The proof of the last statement is similar, once we remark that if $V_{n,k}$ is not primitive, then the path γ is obtained from $\gamma_{n,k}$ by adding at least one closed loop. Assuming that $\theta(\gamma) < 2$ this gives by definition a path of the form $\gamma_{n,k,l}$ for some $l \in \{1, \dots, 2n - 2 - 3k\}$. Thus $\gamma_{n,k,l} \leq \gamma$ and Proposition 6.2 again applies. If $l = n - 1 - k$ then $\gamma_{n,k,l}$ is obtained from $\gamma_{n,k}$ by adding twice the same loop. Hence $V_{n,k,l}$ is *not* primitive and $\gamma \neq \gamma_{n,k,l}$. The same argument above applies and γ must contain $\gamma_{n,k,l}$ for some $l \neq n - 1 - k$. We conclude with Proposition 6.2. $\square$

APPENDIX A. MATRIX COMPUTATIONS

The aim of this section is the computation of the different Rauzy–Veech matrices in $M_n(\mathbb{Z})$ and especially their characteristic polynomials. The rome technique (see below) reduces these computations to matrices in $M_2(\mathbb{Z}[X])$ and $M_3(\mathbb{Z}[X])$ making them possible.

In the sequel we will denote by $P_{n,k}$ (respectively, $P_{n,k,l}$) the characteristic polynomial of the matrices $V(\gamma_{n,k})$ (respectively, $V(\gamma_{n,k,l})$) *multiplied* by $(X+1)$. Their maximal real root are $\theta_{n,k}$ and $\theta_{n,k,l}$, respectively.

A.1. **The rome technique (after** [BGMY1980]**).** To compute the characteristic polynomial of matrices we will use the rome method, developed in [BGMY1980]. To this end it is helpful to represent a matrix V into the form of a combinatorial graph which amounts to draw all paths of length 1 associated to V.

Given a $n \times n$ matrix $V = (v_{ij})$, a path $\eta = (\eta_i)_{i=0}^l$ of width $w(\eta)$ and length l is a sequence of elements of $\{1, 2, \ldots, n\}$ such that $w(\eta) = \prod_{j=1}^l v_{\eta_{j-1}\eta_j} \neq 0$. If $\eta_l = \eta_0$ we say that η is a loop.

A subset $R \subset \{1, 2, \ldots, k\}$ is called a "rome" if there is no loop outside R. Given $r_i, r_j \in R$, a path from r_i to r_j is a "first return path" is it does not intersect R, except at its starting and ending points. This allows us to define an $r \times r$ matrix-valued real function $V_R(X)$, where r is the size of R, by setting $V_R(X) = (a_{ij}(X))$, where $a_{ij}(X) = \sum_\eta w(\eta) \cdot X^{-l(\eta)}$, where the summation is over all first return paths beginning at r_i and ending at r_j.

Theorem (Theorem 1.7 of [BGMY1980])**.** *If R is a rome of cardinality r of a $n \times n$ matrix V then the characteristic polynomial $\chi_V(X)$ of V is equal to*

$$(-1)^{n-r} X^n \det(V_R(X) - \mathrm{Id}_r).$$

Remark A.1. *The matrices $V = V_{n,k}$ or $V = V_{n,k,l}$ can be seen as the action of homeomorphisms on absolute homology if n is even and relative homology otherwise. Thus their characteristic polynomials are reciprocal polynomials (resp. anti-reciprocal polynomials) i.e. $\chi_V(X) = X^n \chi_V(X^{-1})$ (resp. $\chi_V(X) = -X^n \chi_V(X^{-1})$). Thus*

$$\chi_V = (-1)^r \det(V_R(X^{-1}) - \mathrm{Id}_r)$$

A.2. **The paths $\gamma_{n,k}$.** We briefly recall Notation 2.4 (see also Figure 2).
For any $k = 1, \ldots, K_n$ and any $l = 1, \ldots, n-2-k$ we define the path

$$\gamma_{n,k}: \quad \pi := \pi_n.t^k \longrightarrow s(\pi): \quad b^{n-1-k}t^{n-1-2k}$$

Lemma A.1. *Let $n \geq 4$ and $1 \leq k \leq K_n$. Set $d = \gcd(n-1, k)$. We denote by $n' = \frac{n-1}{d} + 1$ and $k' = \frac{k}{d}$. Then the matrix $V_{n',k'}$ is primitive and $\theta_{n,k} = \theta_{n',k'}$.*

Proof of Lemma A.1. We first assume that k and $n-1$ are relatively prime. We compute the matrix $V_{n,k}$ associated to the path $\gamma_{n,k}$. For the sequel, in order to be able to compare the top eigenvalues of the matrices $V_{n,k}$ we will compute them with a labelling depending on k, with alphabet $\mathcal{A} = \{1, \ldots, n\}$. To do this, we start from the central permutation, with the following labelling

$$\pi_n = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & n \\ n & \alpha_{n-1} & \ldots & \alpha_2 & \alpha_1 \end{pmatrix}$$

where we have for each $i \leq n-1$ $\alpha_i k = i - 1 \mod (n-1)$ and $\alpha_i \in \{1, \ldots, n-1\}$. This is well defined since k and $n-1$ are relatively prime. In particular $1 = \alpha_{k+1}$, $2 = \alpha_{2k+1}$, $\alpha_1 = n - 1$,

and more generally, $\alpha_{i+k} = \alpha_i + 1 \mod n - 1$. The starting point of $\gamma_{k,n}$ is

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \dots & \alpha_{n-1} & n \\ n & \dots & \alpha_1 & \alpha_{n-1} & \dots & \alpha_{k+1} \end{pmatrix}$$

The path $\gamma_{n,k}$ consists of the Rauzy moves $b^{n-1-k}t^{n-1-2k}$, hence we have the following sequence of winners/losers:

- $1 = \alpha_{k+1}$ is successively winner against $\alpha_{k+2}, \dots, \alpha_{n-1}, n$
- then, n is successively winner against $\alpha_{k+1}, \dots, \alpha_{n-1-k}$. Note that $k + 1 < n - 1 - k$.

also, the first line of the labeled permutation $s(\pi')$, where π' is the endpoint of $\gamma_{n,k}$ is

$$(\alpha_1', \dots, \alpha_{n-1}', n) = (\alpha_{n-k}, \alpha_{n-k+1}, \dots, \alpha_{n-1}, \alpha_0, \dots, \alpha_{n-1-k}, n)$$

i.e. $\alpha_i' = \alpha_{i-k}$ or $\alpha_i' = \alpha_{i+n-1-k}'$ depending which of $i - k$ or $i + n - 1 - k$ is in $\{1, \dots, n-1\}$. In any case, we obtain, $\alpha_i' = \alpha_i - 1$. Hence the matrix $V_{n,k}$ is obtained from the product of elementary Rauzy–Veech matrices by translating cyclically the first $n - 1$ columns by 1 on the right. Finally, we have

$$V_{n,k} = \begin{pmatrix} a_{n-1} & 2 & a_2 & \dots & \dots & a_{n-2} & 1 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 & 0 \\ 1 & 0 & & \dots & & 0 & 0 \\ b_{n-1} & 1 & b_2 & \dots & \dots & b_{n-2} & 1 \end{pmatrix}$$

Where for $i \in \{2, \dots, n-2\}$, we have:

$$a_i = \begin{cases} 2 & \text{if} & i \in \{\alpha_{k+2}, \dots, \alpha_{n-1-k}\} \\ 1 & \text{if} & i \in \{\alpha_{n-k}, \dots, \alpha_{n-1}\} \\ 0 & \text{if} & i \in \{\alpha_1 \dots \alpha_k\} \end{cases}$$

and $b_i = 1$ if and only if $a_i = 2$, and $b_i = 0$ otherwise. Note that $a_{n-1} = 0$, hence $b_{n-1} = 0$. The matrix $V_{n,k}$ is clearly irreducible (see Figure 8) and thus primitive since there is a non zero diagonal element.

Now if $d = \gcd(n - 1, k) > 2$, we define α in the following way:

- $\alpha_1 = n - 1$, and $\alpha_{[1+ik]} = i$ for $i \leq \frac{n-1}{d}$, where $[1 + ik]$ is the representative modulo $n - 1$ of $1 + ik$, which is in $\{1, \dots, n-1\}$.
- The other α_i are chosen in any way.

The matrix $V_{n,k}$, with this labelling, is

$$\left(\begin{array}{c|c} V_{\frac{n-1}{d}+1, \frac{k}{d}} & * \\ \hline 0 & * \end{array} \right)$$

where the bottom right corner is a permutation matrix. This ends the proof of the lemma. $\square$

Lemma A.2. *Let $n \geq 4$ and $1 \leq k \leq K_n$. If $\gcd(n-1, k) = 1$ then*

$$P_{n,k} = X^{n+1} - 2X^{n-1} - 2\sum_{j \in J_{n,k}} X^j - 2X^2 + 1$$

where $J_{n,k} = \{3, \ldots, n-2\} \setminus \left\{ \lceil \frac{i(n-1)}{k} \rceil + \varepsilon \mid i = 1, \ldots, k-1 \text{ and } \varepsilon = 0, 1 \right\}$. In particular:

(1) *For even n*

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^2 + 1.$$

and if $n \not\equiv 4 \mod 6$, $\gcd(n-1, K_n - 1) = 1$ and

$$P_{n,K_n-1} = X^{n+1} - 2X^{n-1} - 2X^{\lceil 2n/3 \rceil} - 2X^{\lfloor n/3 \rfloor + 1} - 2X^2 + 1.$$

(2) *For $n = 1 \mod 4$, we have $\gcd(n-1, K_n) = 1$ and*

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^{\frac{n+1}{2}} - 2X^2 + 1$$

(3) *For $n = 3 \mod 4$, we have $\gcd(n-1, K_n - 1) = 1$ and*

$$P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^{n-2\lfloor \frac{n-5}{8} \rfloor - 2} - 2X^{\frac{n+1}{2}} - 2X^{2\lfloor \frac{n-5}{8} \rfloor + 3} - 2X^2 + 1$$

(4) *More generally, for n even, and $k \leq K_n - 1$, the highest nonzero monomial (except X^{n+1} and $-2X^{n-2}$) has degree at least $\lceil \frac{2n}{3} \rceil$.*

Proof of Lemma A.2. We will use the rome method as explained in Section A.1. We use the notation of the proof of Lemma A.1. We will write $V_{n,k}$ as $V_{n,k} = A_n - B_{n,k}$ where

$$A_n = \begin{pmatrix} 0 & 2 & 2 & \ldots & \ldots 2 & 1 & 1 \\ \vdots & \ddots & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & 1 & 0 \\ 1 & 0 & \ldots & & & & 0 \\ 0 & 1 & 1 & \ldots & \ldots 1 & 0 & 1 \end{pmatrix},$$

and the only non zero entries of $B_{n,k}$ are as follows:

For any $i = 1, \ldots, k-1$, set $l := \lfloor \frac{i(n-1)}{k} \rfloor + 1$. Then $\begin{cases} b_{1,l} = 1 \\ b_{1,l+1} = 2 \\ b_{n,l} = b_{n,l+1} = 1 \end{cases}$

Observe that $1 \leq k \leq n/2 - 1$, hence for $i \in \{1, \ldots, k-1\}$

$$2 < 2i\frac{(n-1)}{n-2} \leq \frac{i(n-1)}{k} \leq \frac{(k-1)(n-1)}{k} = n - 1 - \frac{n-1}{k} < n - 3.$$

In particular, all integers of the form $\lceil i(n-1)/k \rceil + \varepsilon$ for $i \in \{1, \ldots, k-1\}$ and $\varepsilon \in \{0, 1\}$ are mutually disjoint and in $\{3, \ldots, n-2\}$.

Clearly the set $R = \{1, n\}$ is a rome for A_n. Thus it is also a rome for $V_{n,k}$ (since we pass from A_n to $V_{n,k}$ by removing some paths). The 2×2 matrix $(A_n)_R$ is easily obtained as

$$(A_n)_R = \begin{pmatrix} X^2 + 2S_n & X \\ S_n & X \end{pmatrix} \text{ where } S_n = \sum_{i=3}^{n-1} X^i.$$
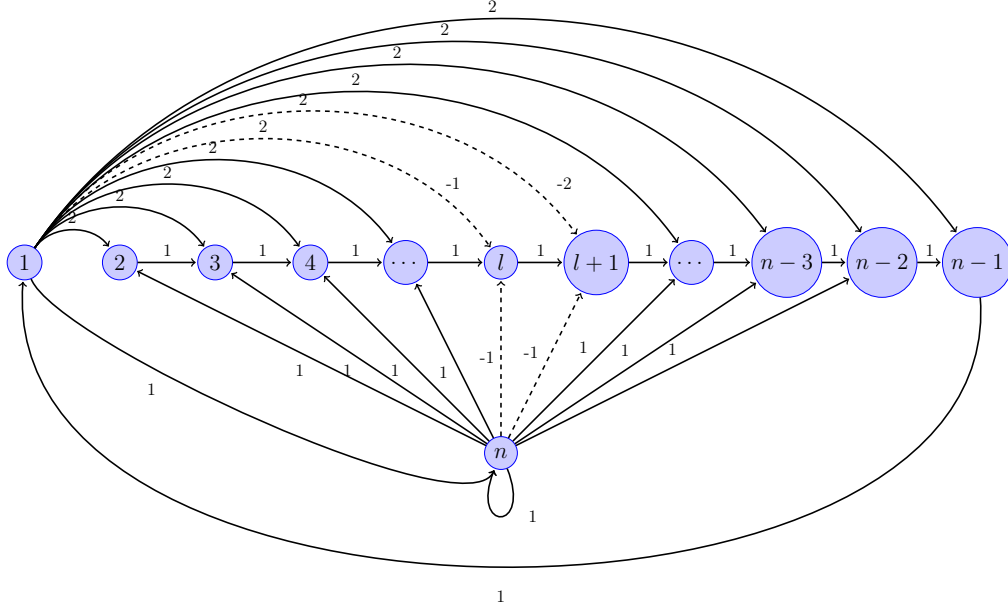
FIGURE 8. The graph associated to $V_{n,k}$. In dashed line we have represented arrow that need to be removed from the graph in blue. Multiplicity are also indicated. To be more more precise, there is one arrow from vertex labelled 1 to the vertex labelled l and no arrow from vertex labelled 1 to the vertex labelled l+1. In the graph $l = \lfloor \frac{i(n-1)}{k} \rfloor + 1$ for any $i = 1, \ldots, k-1$. Obviously the graph associated to A_n is drawn in blue colour

To obtain the matrix $(V_{n,k})_R$ one has to subtract the polynomial corresponding to the paths passing through arrows in dashed line (passing through vertices l and $l+1$ where $l = \lfloor \frac{i(n-1)}{k} \rfloor + 1$ for some $i = 1, \ldots, k-1$). At this aim we define

$$\begin{cases} T_{n,k} = \sum_{i=1}^{k-1} X^{n-1-(\lfloor \frac{i(n-1)}{k} \rfloor + 1)} \\ Q_{n,k} = \sum_{i=1}^{k-1} X^{n-1-\lfloor \frac{i(n-1)}{k} \rfloor} = \sum_{i=1}^{k-1} X^{\lceil i(n-1)/k \rceil}. \end{cases}$$

The polynomial $T_{n,k}$ (respectively, $2Q_{n,k}$) takes into account all simple paths from the vertex 1 to the vertex 1 passing through an arrows in dashed line connecting 1 to l (respectively, to $l+1$). Hence

$$(V_{n,k})_R = \begin{pmatrix} X^2 + 2S_n - T_{n,k} - 2Q_{n,k} & X \\ S_n - T_{n,k} - Q_{n,k} & X \end{pmatrix}$$

Note that $T_{n,k} = XQ_{n,k}$. By using [BGMY1980], a straightforward computation gives:

$$\chi_{V_{n,k}} = X^3 - X^2 - X + 1 + S_n \cdot (X - 2) + 2Q_{n,k}$$

Hence,

$$P_{n,k} = X^{n+1} - 2\sum_{i=2}^{n-1} X^i + 1 + 2\sum_{i=1}^{k-1}\left(X^{\lceil i(n-1)/k\rceil} + X^{\lceil i(n-1)/k\rceil + 1}\right)$$

which implies the first statement of the lemma.

We give a little more information on the set $J_{n,k}$. We write $J_{n,k} = \{j_1, \ldots, j_r\}$ with $j_1 < \cdots < j_r$ (possibly, $r = 0$). We have $\lceil \frac{i(n-1)}{k}\rceil = 2i + \lceil\frac{i(n-1-2k)}{k}\rceil$. In particular, we see that $j_s = 2i_s + s$, where $i_s \geq 0$ is the smallest integer such that $\frac{i_s(n-1-2k)}{k} > s$, i.e $i_s = \lfloor\frac{sk}{n-1-2k}\rfloor + 1$. Note also that $P_{n,k}$ must be reciprocal, hence $j \in J_{n,k}$ if and only if $n+1-j \in J_{n,k}$.

Now we compute particular cases.

(1) If n is even, $K_n = n/2 - 1$. We have $n - 1 - 2K_n = 1$, $2i_1 + 1 = n - 1 > n - 2$, hence $J_{n,K_n} = \emptyset$. For $k = K_n - 1 = n/2 - 2$ and n even, $n \neq 4 \mod 6$ we have

$$2i_1 + 1 \quad = \quad 2\lfloor(n-4)/6\rfloor + 3 = \lfloor n/3\rfloor + 1 = j_1$$
$$2i_2 + 2 \quad = \quad 2\lfloor(n-4)/3\rfloor + 4 = \lceil 2n/3\rceil = j_2 = n + 1 - j_1$$

Hence $J_{n,K_n-1} = \{\lfloor n/3\rfloor + 1, \lceil 2n/3\rceil\}$.

(2) If $n = 1 \mod 4$, $K_n = (n-1)/2 - 1 = \frac{n-3}{2}$, we have $n - 1 - 2K_n = 2$, hence:

$$2i_1 + 1 \quad = \quad 2\lfloor(n-3)/4\rfloor + 3 = \frac{n+1}{2} = j_1$$

So, $J_{n,K_n} = \{\frac{n+1}{2}\}$.

(3) If $n = 3 \mod 4$, $k = K_n - 1 = \frac{n-5}{2}$, $n - 1 - 2k = 4$, hence

$$2i_1 + 1 \quad = \quad 2\lfloor(n-5)/8\rfloor + 3 = j_1$$
$$2i_2 + 2 \quad = \quad 2\lfloor(n-5)/4\rfloor + 4 = \frac{n+1}{2} = j_2$$
$$2i_3 + 3 \quad = \quad 2\lfloor 3(n-5)/8\rfloor + 5 = n + 1 - j_1$$

So, $J_{n,K_n} = \{2\lfloor(n-5)/8\rfloor + 3, \frac{n+1}{2}, n - 2\lfloor(n-5)/8\rfloor - 2\}$

(4) More generally, for n even and $k \leq K_n - 1$ we evaluate j_1.

$$2i_1 + 1 = 2\lfloor\frac{n-1-2k}{k}\rfloor + 3 \leq 2\lfloor\frac{n-1-2(K_n-1)}{K_n-1}\rfloor + 3 = \lfloor n/3\rfloor + 1$$

since $k \mapsto \frac{n-1-2k}{k}$ is increasing, and we conclude by using that the polynomial is reciprocal.

This finishes the proof.   □

A.3. **The paths** $\gamma_{n,K_n,l}$. We recall Notation 2.4 (see also Figure 2) when $k = K_n$. For any $l \in \{1, \ldots, 2n - 2 - 3K_n\}$ we define the path

$$\gamma_{n,k,l} : \pi \longrightarrow s(\pi) : \begin{cases} b^l t^{L_n+1-l} b^{L_n+1-l} t^{n-1-2K_n} & \text{if } 1 \leq l \leq L_n \\ b^{L_n+1} t^{l-(L_n+1)} b^{2(L_n+1)-l} t^{2n-2-3K_n-l} & \text{if } L_n < l \leq 2n - 2 - 3K_n \end{cases}$$

Note that when n is even $2n - 2 - 3K_n = L_n + 2$ otherwise $2n - 2 - 3K_n = L_n + 3$.

Proposition A.3. *Let $n \geq 4$ be an even integer.*

(1) If $l \in \{1, \ldots, L_n\}$ then

$$P_{n,K_n,l} = P_{n,K_n} - \frac{X^{n-2l+2} + X^{n-2l+4} + 2X^{n-1} - X^{2l+1} - X^{2l-1} - 2X^4}{(X+1)(X-1)}$$

(2) For $l = L_n$ one has

$$P_{n,K_n,L_n} = X^{n+1} - 2X^{n-1} - X^{n-3} - X^4 - 2X^2 + 1$$

(3) For $l = L_n + 2$ $V_{n,K_n,L_n+2} = V_{n,K_n} + B_n$ where $B_n = (b_{i,j})$ and

$$\begin{cases} b_{1,2j} = b_{n-2,2j} = 1, \ for \ 1 \leq j \leq K_n - 1, \\ b_{1,n} = b_{n-2,n} = 1, \\ b_{i,j} = 0 \ otherwise. \end{cases}$$

Proof. Again we appeal to the rome method in oder to compute the characteristic polynomial. Since n is even, the matrix V_{n,K_n} is primitive. Thus Lemma A.1 applies and we have a nice expression for the matrix V_{n,K_n} (see the proof of Lemma A.1). Let $l \in \{1, \ldots, L_n\}$. By construction, $\gamma_{n,K_n,l}$ is a path obtained from γ_{n,K_n} by adding a closed loop (of type 't' and of length $n - 1 - K_n - l$) at $b^l t^{K_n} \pi_n$. Hence the matrices V_{n,K_n} and $V_{n,K_n,l}$ differ by a non negative matrix, namely $V_{n,K_n,l} - V_{n,K_n} = C_{n,l} = (c_{i,j})$ where

$$\begin{cases} c_{2l,2} = c_{2l,2l+1} = 1, \\ c_{2l,2j+3} = 2 \ for \ l \leq j \leq K_n - 1, \\ c_{i,j} = 0 \ otherwise. \end{cases}$$

On the way, the same argument applied to the matrix V_{n,K_n,L_n+2} applies, proving the last statement.

We are now in a position to compute $P_{n,K_n,l}$. The graph associated to the matrix V_{n,K_n} is already presented in the proof of Lemma A.2. For readability purposes, we reproduce this graph below, where we have added the edges corresponding to the graph associated to $C_{n,l}$.

Clearly the set $R = \{1, 2l, n\}$ is a rome for $V_{n,K_n,l}$, and thus for V_{n,K_n} (since we pass from $V_{n,K_n,l}$ to V_{n,K_n} by removing some paths). The 3×3 matrix $(V_{n,K_n})_R$ is easily obtained as

$$(V_{n,K_n})_R = \begin{pmatrix} \sum_{\substack{i=2 \\ i \ even}}^{n-2l} X^i & 2X^{2l-1} + \sum_{\substack{i=2 \\ i \ even}}^{2l-2} X^i & X \\ \sum_{\substack{i=2 \\ i \ even}}^{n-2l} X^i & X^{2l-1} & 0 \\ 0 & X^{2l-1} & X \end{pmatrix}$$

where $\sum_{\substack{i=2 \\ i \ even}}^{2l-2} X^i = 0$ if $l = 1$.

Adding the matrix $C_{n,l}$ consists of adding two arrows form the vertex labelled by $2l$ to the vertices $2, 2l+1$ (with multiplicity 1) and $K_n - l$ arrows to the vertices $2l+3, \ldots, n-3, n-1$ (with multiplicity 2). In this situation R is still a rome. To compute the matrix $(V_{n,K_n,l})_R$

we need to consider all paths passing through a dashed edge in the graph in Figure 9. Thus $(V_{n,K_n,l})_R(X) = (V_{n,K_n})_R(X) + C(X)$ where

$$C(X) = \begin{pmatrix} 0 & 0 & 0 \\ 2\sum_{\substack{i=2 \\ i \ even}}^{n-2l-2} X^i + X^{n-2l} & X^{2l-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence by [BGMY1980] we draw $\chi_{V_{n,K_n,l}} = -\det((V_{n,K_n})_R(X) + C(X) - \mathrm{Id}_3)$. By multilinearity, one has:

$$P_{n,K_n,l} = P_{n,K_n} - (1+X) \cdot \det(W_{n,l})$$

where

$$W_{n,l} = \begin{pmatrix} \sum_{\substack{i=2 \\ i \ even}}^{n-2l} X^i - 1 & 2X^{2l-1} + \sum_{\substack{i=2 \\ i \ even}}^{2l-2} X^i & X \\ 2\sum_{\substack{i=2 \\ i \ even}}^{n-2l-2} X^i + X^{n-2l} & X^{2l-1} & 0 \\ 0 & X^{2l-1} & X-1 \end{pmatrix}$$

A direct computation gives

$$\det(W_{n,l}) = \frac{X^{n-2l+2} + X^{n-2l+4} + 2X^{n-1} - X^{2l+1} - X^{2l-1} - 2X^4}{(X+1)^2(X-1)}$$

that is the desired result.

The second assertion comes for free since $2L_n = n - 2$ and by Lemma A.2 one has $P_{n,K_n} = X^{n+1} - 2X^{n-1} - 2X^2 + 1$. Proposition A.3 is proved. $\square$

Proposition A.4. *Let $n \geq 4$ be an integer with $n \equiv 3 \mod 4$. Fix $l \in \{1, \ldots, L_n + 3\}$:*

(1) *If l is even then $V_{n,K_n,l}$ is reducible and $\theta_{n,K_n,l} = \theta_{n',K_{n'},l'}$ with $n' = (n+1)/2$, $l' = l/2$ and $K_{n'} = K_n/2$.*

(2) *If $l \leq L_n$ is odd then $V_{n,K_n,l}$ is primitive and*

$$P_{n,K_n,l} = \frac{S_n(X) + 2\left(X^{\frac{n+7}{2}-l} - X^l + X^{l+\frac{n-1}{2}} - X^{n+3-l}\right)}{(X-1)(X+1)}$$

where $S_n = 1 - 3X^2 - 2X^{\frac{n-1}{2}} + 8X^{\frac{n+3}{2}} - 2X^{\frac{n+7}{2}} - 3X^{n+1} + X^{n+3}$.

(3) *If $l = L_n + 2$ then $V_{n,K_n,l} = \begin{pmatrix} 0_{K_n \times K_n} & \mathrm{Id}_{K_n \times K_n} & 0_{K_n \times 3} \\ \hline 2 \cdots 2 & 0 \cdots 0 & 2\ 3\ 2 \\ \hline 1 \cdots 1 & 0 \cdots 0 & 0\ 2\ 1 \\ \hline \mathrm{Id}_{K_n \times K_n} & 0_{K_n \times K_n} & 0_{K_n \times 3} \\ \hline 0 \cdots 0 & 0 \cdots 0 & 1\ 1\ 1 \end{pmatrix}$*

Proof of Proposition A.4. We follow the strategy of the proof of the previous proposition. The first point is clear. In the sequel, let $m = K_n + 1 = (n-1)/2$. Again we appeal to the rome

method in oder to compute the characteristic polynomial. We have $V_{n,K_n,l} = A_n + B_{n,l}$ where

$$A_n = \left(\begin{array}{ccc|ccc|ccc} 0_{K_n \times K_n} & Id_{K_n \times K_n} & 0_{K_n \times 3} \\ \hline 1 \cdots 1 & 0 \cdots 0 & 2\ 2\ 1 \\ \hline 0 \cdots 0 & 0 \cdots 0 & 0\ 1\ 0 \\ \hline Id_{K_n \times K_n} & 0_{K_n \times K_n} & 0_{K_n \times 3} \\ \hline 0 \cdots 0 & 0 \cdots 0 & 1\ 1\ 1 \end{array} \right)$$

and the only non zero entries of $B_{n,l} = (b_{i,j})$ are

$$\begin{cases} b_{n-l,i} = 2 \text{ for } i = 1, \dots, m-l \\ b_{n-l,l} = 1 \\ b_{n-l,n-1} = 2 \\ b_{n-l,n-2} = 1 \end{cases}$$

It is helpful to represent the matrices in form of a combinatorial graph which amounts to draw all paths.

The graph associated to A_n is rather simple. Clearly a rome is made of the subsets labelled $R = \{n, n-1, m\}$. The matrix $A_R(X)$ in this label is

$$A_R(X) = \left(\begin{array}{ccc} X & X & X^m \\ 0 & X^m & 0 \\ X & R & S \end{array} \right)$$

$$R = 2X + \sum_{i=3, i \text{ odd}}^{m} X^i \quad = \quad 2X + X^3 \cdot \frac{1 - X^{m-1}}{1 - X^2}$$
$$S = 2X^m + \sum_{i=2, i \text{ even}}^{m-1} X^i \quad = \quad 2X^m + X^2 \cdot \frac{1 - X^{m-1}}{1 - X^2}$$

Adding the matrix $B_{n,l}$ consists of adding arrows form the vertex labelled by $n-l$ to the vertices $1, \dots, m-l, n-2$ with multiplicity 1 and to the vertex $n-1$ with multiplicity 2. In this situation R is still a rome. To compute the matrix $(V_{n,K_n,l})_R$ we need to consider all paths passing through a dashed edge on the graph in Figure 9. Thus $(V_{n,K_n,l})_R(X) = A_R(X) + C(X)$ where

$$C(X) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & P & Q \\ 0 & \sum_{i=0}^{(l-1)/2-1} \frac{P}{X^{2i}} & \sum_{i=0}^{(l-1)/2-1} \frac{Q}{X^{2i}} \end{array} \right)$$

and

$$P = X^m + 2 \sum_{i=l, i \text{ i odd}}^{m-2} X^i = X^m + 2X^l \cdot \frac{1 - X^{m-l}}{1 - X^2},$$

$$Q = X^{m+l-1} + 2 \sum_{i=l+1, i \text{ i even}}^{m-1} X^i = X^{m+l-1} + 2X^{l+1} \cdot \frac{1 - X^{m-l}}{1 - X^2}.$$

Hence by [BGMY1980] we draw $\chi_{V_{n,K_n,l}} = -\det(A_R(X) + C(X) - Id_3)$. By using the fact that $C(X) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & P & Q \\ 0 & \frac{1 - X^{1-l}}{1 - X^{-2}} P & \frac{1 - X^{1-l}}{1 - X^{-2}} Q \end{array} \right)$ and $Q = X^m(X^{l-1} - X) + XP$ we easily obtained the desired equality.
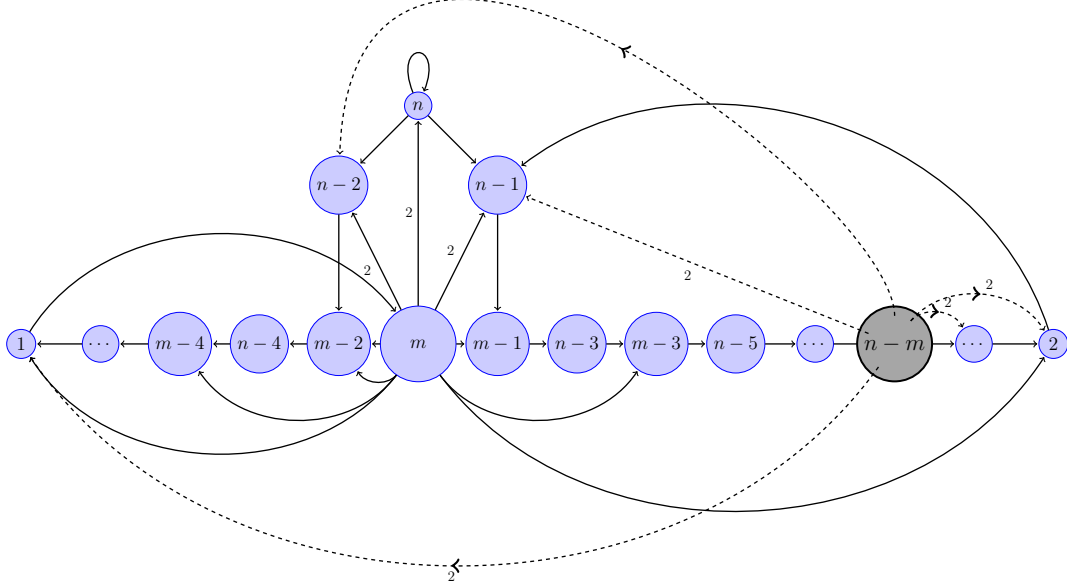
FIGURE 9. The graph associated to $V_{n,K_n,l}$. In dashed line we have represented arrow coming from the matrix $B_{n,l}$. The multiplicity is indicated only when it is 2, otherwise it is 1.

The last assertion is also easy to derive from the fact that γ_{n,K_n,L_n+2} is obtained from γ_{n,K_n} by adding a 'b' loop at $t^{K_n+1}\pi_n$ and from the shape of A_n. This ends the proof of Proposition A.4. $\square$

APPENDIX B. COMPARING ROOTS OF POLYNOMIALS

This section is devoted to comparing $\theta_{n,k}$ and $\theta_{n,k,l}$ (the maximal real roots of the polynomials $P_{n,k}$ and $P_{n,k,l}$ introduced in Appendix A) for various n,k,l. Observe that by construction $\theta_{n,k} > \sqrt{2}$ and $\theta_{n,k,l} > \sqrt{2}$.

One key ingredient for comparing maximal real roots of these polynomials is the easy

Lemma B.1. *Let P_1, P_2 be two unitary polynomials of degree at least one such that for $x > \sqrt{2}$, $P_1(x) - P_2(x) > 0$. We assume that P_1 has a root $\theta_1 > \sqrt{2}$. Then P_2 has a root $\theta_2 > \theta_1$.*

Proof. The assumption implies $P_2(\theta_1) < 0$. Since P_2 is unitary, the result follows by the mean value theorem. $\qquad\square$

Lemma B.2. *The followings hold:*

(1) *The sequence $(\theta_{2n,K_n})_n$ is a decreasing sequence.*
(2) *The sequence $(\theta_{1+4n,K_n})_n$ is a decreasing sequence.*
(3) *The sequence $(\theta_{3+4n,K_n,L_n})_n$ is a decreasing sequence.*

Proof of Lemma B.2. We establish the lemma case by case.
Case 1. By Lemma A.2, $\theta_{2n,K_{2n}}$ is the largest root of

$$P_{2n} = X^{2n+1} - 2X^{2n-1} - 2X^2 + 1.$$

Observe that $P_{2n+2} - X^2 P_{2n} = 2X^4 - X^2 + 1$. Thus $P_{2n+2}(x) - x^2 P_{2n}(x) > 0$ for $x > \sqrt{2}$ and Lemma B.1 gives that $\theta_{2n,K_{2n}} < \theta_{2n+2,K_{2n+2}}$.

Case 2. We observe that $\theta_{1+4n,K_{1+4n}}$ is the largest root of

$$P_{1+4n,K_{1+4n}} = X^{4n+2} - 2X^{4n} - 2X^{2n+1} - 2X^2 + 1.$$

Again a simple computation establishes $P_{5+4n,K_{5+4n}} - X^4 P_{1+4n,K_{1+4n}} = 2X^{2n+5} - 2X^{2n+3} + 2X^6 - 2X^2 - X^4 + 1$. Hence $P_{5+4n,K_{5+4n}}(x) - x^4 P_{1+4n,K_{1+4n}}(x) > 0$ for $x > \sqrt{2}$ and by Lemma B.1:

$$\theta_{5+4n,K} < \theta_{1+4n,K}.$$

Case 3. Similarly

$$P_{3+4n,K,L} = X^{4n+4} - 2X^{4n+2} - 4X^{2n+3} + 4X^{2n+1} + 2X^2 - 1.$$

Hence, for $x > \sqrt{2}$:

$$P_{7+4n,K,L} - x^4 P_{3+4n,K,L} = (x^2-1)(4x^{2n+5} - 4x^{2n+3} - 2x^4 - x^2 + 1) > 4x^{2n+3} - 2x^4 - x^2 + 1 > 0.$$

Hence Lemma B.1 applies and $\theta_{7+4n,K,L} < \theta_{3+4n,K,L}$. The lemma is proved. $\qquad\square$

Lemma B.3. *Let $n \geq 4$ and $1 \leq k < k' \leq K_n$. If $\gcd(n-1,k) = \gcd(n-1,k') = 1$ then*

$$\theta_{n,k'} < \theta_{n,k}.$$

Proof of Lemma B.3. From the proof of Lemma A.2, we have $P_{n,k'} - P_{n,k} = (X+1)\left(Q_{n,k'} - Q_{n,k}\right)$, where $Q_{n,k} = \sum_{i=1}^{k-1} X^{\lceil i(n-1)/k \rceil}$. From Lemma B.1, we need to show that $P_{n,k'}(x) - P_{n,k}(x) > 0$ for $x > \sqrt{2}$. First we observe that

$$Q_{n,k'} - Q_{n,k} = \sum_{i=k}^{k'-1} X^{\lceil \frac{i(n-1)}{k'} \rceil} + \sum_{p=1}^{k-1} X^{\lceil \frac{(k'-p)(n-1)}{k'} \rceil} - X^{\lceil \frac{(k-p)(n-1)}{k} \rceil}.$$

Now for any $p \in \{1, \ldots, k-1\}$

$$\lceil \frac{(k-p)(n-1)}{k} \rceil \leq \lceil \frac{(k'-p)(n-1)}{k'} \rceil.$$

So, for any $x > 1$, $P_{n,k'}(x) - P_{n,k}(x) > 0$, proving the lemma. $\qquad\square$

Before comparing roots using polynomials, we end this subsection with a simple lemma:

Lemma B.4. *Let $n \geq 7$ be an integer satisfying $n \equiv 3 \mod 4$. Then $\theta_{n,K_n,L_n+2} > 2$.*
Let $n \geq 4$ be an even integer. Then $\theta_{n,K_n,L_n+1} > 3^{\frac{1}{2}}$.
Let $n \geq 6$ be an even integer. Then $\theta_{n,K_n,L_n+2} > 6^{\frac{1}{4}}$.

Proof of Lemma B.4. We will use the following classical inequality for the Perron root $\rho(A)$ of a non negative primitive matrix $A = (a_{ij})_{i,j=1,\ldots,n}$: $\rho(A) > \delta(A)$ where $\delta(A) = \min_{j=1}^{n} \sum_{i=1}^{n} a_{ij}$ (see *e.g.* [BL12, Proposition 4.2.]).

We prove the first assertion. The matrix $V(\gamma_{n,K_n,L_n+2})$ is primitive (by Proposition A.4) and θ_{n,K_n,L_n+2}^2 is the Perron root of V_{n,K_n,L_n+2}^2. It suffices to show that $\delta(V_{n,K_n,L_n+2}^2) > 4$. By Proposition A.4 one has

$$
V_{n,K_n,L_n+2} = \left(\begin{array}{ccc|ccc|ccc}
\multicolumn{3}{c|}{0_{K_n \times K_n}} & \multicolumn{3}{c|}{\mathrm{Id}_{K_n \times K_n}} & \multicolumn{3}{c}{0_{K_n \times 3}} \\
\hline
\multicolumn{3}{c|}{2 \cdots 2} & \multicolumn{3}{c|}{0 \cdots 0} & 2 & 3 & 2 \\
\hline
\multicolumn{3}{c|}{1 \cdots 1} & \multicolumn{3}{c|}{0 \cdots 0} & 0 & 2 & 1 \\
\hline
\multicolumn{3}{c|}{\mathrm{Id}_{K_n \times K_n}} & \multicolumn{3}{c|}{0_{K_n \times K_n}} & \multicolumn{3}{c}{0_{K_n \times 3}} \\
\hline
\multicolumn{3}{c|}{0 \cdots 0} & \multicolumn{3}{c|}{0 \cdots 0} & 1 & 1 & 1
\end{array}\right)
$$

The result then follows from an easy matrix computation.

For the second claim, The matrix $V(\gamma_{n,K_n,L_n+2})$ is primitive (by Lemma A.1). By Proposition A.3 and a matrix computation, we draw $\delta(V_{n,K_n,L_n+2}^4) = 6$ for any $n \geq 6$. Hence $\theta_{n,K_n,L_n+2} > 6^{\frac{1}{4}}$. Lemma B.4 is proved. □

B.1. Comparing $\theta_{n,k,l}$ when $n \equiv 3 \mod 4$.

Lemma B.5. *Let $n \geq 7$ such that $n \equiv 3 \mod 4$. If $l, l' \in \{1, \ldots, L_n\}$ are odd and $l < l'$ then*

$$\theta_{n,K_n,l} > \theta_{n,K_n,l'}.$$

Proof of Lemma B.5. We follow the notation of the proof of Proposition A.4. A simple computation shows:

$$
P_{n,K_n,l} - P_{n,K_n,l'} = \frac{2(X^l - X^{l'})(X^m - 1)(X^{l+l'} + X^{4+m})}{X^{l+l'}(X-1)(X+1)}.
$$

In particular $P_{n,K_n,l}(x) - P_{n,K_n,l'}(x) < 0$ for $x > \sqrt{2}$. Lemma B.1 gives $\theta_{n,K_n,l} > \theta_{n,K_n,l'}$. □

Proposition B.6. *Let $n \geq 7$ be an integer satisfying $n \equiv 3 \mod 4$.*

(1) If $n' = \frac{n+1}{2}$ then $\theta_{n',K_{n'},L_{n'}} > \theta_{n,K_n,L_n}$.
(2) If $n' = \frac{n+1}{2}$ then $\theta_{n',K_{n'},L_{n'}+2} > \theta_{n,K_n,L_n}$.
(3) Let $1 \leq k \leq K_n - 1$. If $d = \gcd(k, n-1)$, $n' = \frac{n+1}{d}$ and $k' = k/d$ then $\theta_{n,K_n,L} < \theta_{n',k'}$.

Proof of Proposition B.6. **Case (1).** We start with the first statement. Using Proposition A.3 and Proposition A.4, we have:

$$P_{n',K_{n'},L_{n'}} = x^{n'+1} - 2x^{n'-1} - x^{n'-3} - x^4 - 2x^2 + 1$$

$$P_{n,K_n,L_n} = x^{n+1} - 2x^{n-1} - 4x^{(n+3)/2} + 4x^{(n-1)/2} + 2x^2 - 1$$

Noticing that $n - n' = \frac{n-1}{2}$, hence $n - n' + 2 = \frac{n+3}{2}$, we have:

$$P_{n,K_n,L_n} - x^{n-n'} P_{n',K_{n'},L_{n'}} = x^{n-3} + x^{4+n-n'} - 2x^{2+n-n'} + 3x^{n-n'} + 2x^2 - 1$$

This polynomial clearly takes only positive values for $x > \sqrt{2}$, which proves the required inequality.

Case (2). Now we come to the second statement. Assume $n \geq 11$ (for $n = 7$ we directly prove the inequality). In this situation $n' \geq 4$ and Lemma B.4 gives

$$\theta_{n',K_{n'},L_{n'}+2} > 6^{\frac{1}{4}}$$

Let $\theta = \theta_{n,K_n,L_n}$ for simplicity. By Proposition A.4, we have:

$$\theta^{n-1}(\theta^2 - 2) = 4\theta^{(n+3)/2} - 4\theta^{(n-1)/2} - 2\theta^2 + 1$$

Hence

$$\theta - \sqrt{2} = \frac{1}{\theta + \sqrt{2}} \frac{4\theta^{(n-1)/2}(\theta^2 - 1) + 1 - 2\theta^2}{\theta^{n-1}} < \frac{4}{\theta^{(n-1)/2}} < \frac{1}{\sqrt{2}^{(n-5)/2}}$$

Obviously $\frac{1}{\sqrt{2}^{(n-5)/2}} < 6^{\frac{1}{4}} - \sqrt{2}$ for $n \geq 23$. Hence $\theta_{n,K_n,L_n} < 6^{\frac{1}{4}} < \theta_{n',K_{n'},L_{n'}+2} >$. For $n < 23$ we check directly the that this inequality holds.

Case (3). Finally we prove the last statement. Assume first that $\gcd(n-1,k) = 1$, then $n' = n$ and $k' = k$. Note that $\gcd(n-1, K_{n-1} - 1) = 1$. From Lemma B.3 $\theta_{n,k} \geq \theta_{n,K_{n-1}}$ for any $k = 1, \ldots, K_n - 1$. Since L is odd we have, by Proposition A.4 and Lemma A.2:

$$P_{n,K_n-1} = x^{n+1} - 2x^{n-1} - 2x^{n-2\lfloor \frac{n-5}{8} \rfloor - 2} - 2x^{\frac{n+1}{2}} - 2x^{2\lfloor \frac{n-5}{8} \rfloor + 3} - 2x^2 + 1$$
$$P_{n,K_n,L} = x^{n+1} - 2x^{n-1} - 4x^{(n+3)/2} + 4x^{(n-1)/2} + 2x^2 - 1$$

Hence, for $x > \sqrt{2}$

$$P_{n,K_n,L} - P_{n,K_n-1} = 2x^{n-2\lfloor \frac{n-5}{8} \rfloor - 2} - 4x^{(n+3)/2} + 2x^{\frac{n+1}{2}} + 4x^{(n-1)/2} + 2x^{2\lfloor \frac{n-5}{8} \rfloor + 3} + 4x^2 - 2$$
$$> 2x^{n-2\lfloor \frac{n-5}{8} \rfloor - 2} - 4x^{(n+3)/2}$$

For $n \geq 11$, $n - 2\lceil \frac{n-5}{8} \rceil - 2 \geq \frac{n+3}{2} + 2$, hence $P_{n,K_n,L} - P_{n,K_n-1} > 0$. For $n = 7$, we compute directly the roots: we have $\theta_{n,K_n-1} \approx 1,96$ and $\theta_{n,K_n,L} \approx 1,84$.

Now we assume that $\gcd(k, n-1) = d > 1$.

(1) If n' is odd (thus $n' \equiv 3 \mod 4$) then by the above case $\theta_{n',k'} > \theta_{n',K_{n'},L_{n'}}$. We conclude with Lemma B.2.

(2) If n' is even then we need to show directly that $\theta_{n',k'} > \theta_{n,K_n,L}$. Note n' even implies that d is an odd multiple of 2. There are two cases:

- $d \geq 6$. In this case, $\theta_{n',k'} \geq \theta_{n',K_{n'}}$ ($k' = K_{n'}$ is possible). We have

$$P_{n,K_n,L} - x^{n-n'}P_{n',K_{n'}} = 2x^{2+n-n'} - x^{n-n'} - 4x^{\frac{n+3}{2}} + 4x^{\frac{n-1}{2}} + 2x^2 - 1$$
$$> 2x^{n-n'} - 4x^{\frac{n+3}{2}}$$

We necessarily have $n > 10$ hence $n - n' \geq \frac{n+3}{2} + 2$ hence the above polynomial is positive for $x > \sqrt{2}$.

- $d = 2$. In this case $\theta_{n',K_{n'}} = \theta_{n,K_n} < \theta_{n,K_n,L}$, but $k' < K_{n'}$. Hence the previous strategy does not work. Since n' and $K_{n'} - 1$ are not necessarily relatively prime, we compare directly $\theta_{n',k'}$ with $\theta_{n,K_n,L}$ by using Statement 4 of Lemma A.2. We have

$$P_{n,K_n,L} - x^{n-n'} P_{n',k'} \geq 2x^{n-n'+\lceil \frac{2n'}{3} \rceil} - 4x^{\frac{n+3}{2}}$$

  For $n \geq 8$, we have $n - n' + \lceil \frac{2n}{3} \rceil \geq \frac{n+3}{2} + 2$, implying the desired inequality. For $n = 7$, we necessarily have $d = 1$.

This ends the proof of the proposition. $\square$

B.2. Comparing $\theta_{n,k,l}$ when $n \equiv 0 \mod 2$.

Lemma B.7. *Let $n \geq 4$ be an even integer. If $l, l' \in \{1, \ldots, L_n\}$ satisfy $l < l'$ then*

$$\theta_{n,K_n,l} > \theta_{n,K_n,l'}.$$

Proof of Lemma B.7. Proposition A.3 and a simple computation show

$$P_{n,K_n,l} - P_{n,K_n,l'} = \frac{(X^l - X^{l'})(X^2 + 1)(X^{2l+2l'} + X^{n+3})(X^l + X^{l'})}{X^{2l+2l'+1}(X+1)(X-1)}.$$

In particular $P_{n,K_n,l}(x) - P_{n,K_n,l'}(x) < 0$ for $x > \sqrt{2}$. Lemma B.1 gives $\theta_{n,K_n,l} > \theta_{n,K_n,l'}$. $\square$

Proposition B.8. *Let $n \geq 18$ be an even integer satisfying $n \not\equiv 4 \mod 6$. Then, $\gcd(n - 1, K_n - 1) = 1$ and the followings hold:*
  - *(1) For any $k = 1, \ldots, K_n - 2$ one has $\theta_{n,k} > \theta_{n,K_n-1}$.*
  - *(2) For any $l = 1, \ldots, L_n$ one has $\theta_{n,K_n,l} > \theta_{n,K_n-1}$.*
  - *(3) For $l = L_n + 1$, one has $\theta_{n,K_n,L_n+1} > \theta_{n,K_n-1}$.*
  - *(4) For $l = L_n + 2$, one has $\theta_{n,K_n,L_n+2} > \theta_{n,K_n-1}$.*

Proof. Let us consider the first claim and set $d = \gcd(k, n - 1)$. If $d = 1$ then $\theta_{n,k} > \theta_{n,K_n-1}$ by Lemma B.3. Otherwise let $n' = \frac{n-1}{d} + 1 < n$ and $k' = \frac{k}{d}$. Note that $\gcd(k', n' - 1) = 1$ and $\theta_{n,k} = \theta_{n',k'}$. By Lemma B.3 $\theta_{n',k'} > \theta_{n',K_{n'}}$. It suffices to show $\theta_{n',K_{n'}} > \theta_{n,K_n-1}$. We have, for $x > \sqrt{2}$,

$$P_{n,K_n-1} - x^{n-n'} P_{n',K_{n'}} = 2x^{2+n-n'} - x^{n-n'} - 2x^{\lceil 2n/3 \rceil} - 2x^{\lfloor n/3 \rfloor+1} - 2x^2 + 1$$
$$> 2x^{n-n'} - 2x^{\lceil 2n/3 \rceil} + x^{n-n'} - 2x^{\lfloor n/3 \rfloor+1} - 2x^2 + 1$$

Note that $n - 1$ is odd and not a multiple of 3, hence $d \geq 5$. Since n is large enough, we have

$$n - n' - \frac{2n}{3} = (n-1)\left(1 - \frac{1}{d}\right) - \frac{2n}{3} \geq \frac{4}{5}(n-1) - \frac{2n}{3} \geq 0$$

Hence, $\lfloor n - n' - \frac{2n}{3} \rfloor = n - n' - \lceil \frac{2n}{3} \rceil \geq 0$. Similarly, $n - n' \geq 4 + \lfloor n/3 \rfloor + 1$. Hence

$$P_{n,K_n-1} - x^{n-n'} P_{n',K_{n'}} > 2x^{\lfloor n/3 \rfloor+1} - 2x^2 + 1 > 0.$$

The inequality $\theta_{n',K_{n'}} > \theta_{n,K_n-1}$ follows by Lemma B.1.

We now prove the second claim for $l = 1, \ldots, L_n$. Since $\theta_{n,K_n,l} > \theta_{n,K_n,L}$ by Lemma B.7, it suffices to show $\theta_{n,K_n,L} > \theta_{n,K_n-1}$. We have:

$$P_{n,K_n-1} - P_{n,K_n,L} = x^{n-3} - 2x^{\lceil 2n/3 \rceil} - 2x^{\lfloor n/3 \rfloor+1} + x^4$$

If $n \geq 24$, we get $n - 3 \geq 4 + \lceil 2n/3 \rceil$, hence:

$$P_{n,K_n-1} - P_{n,K_n,L} > 2x^{\lceil 2n/3 \rceil} - 2x^{\lfloor n/3 \rfloor+1} + x^4 > 0$$

For $n \in \{18, 20\}$, we check directly that $P_{n,K_n-1} - P_{n,K_n,L} > 0$. The inequality $\theta_{n,K_n,L} > \theta_{n,K_n-1}$ follows by Lemma B.1.

Next we prove the third claim for $l = L_n + 1$ (in this case, the computation is different). By Lemma B.4 we have $\theta_{n,K_n,L_n+1} > 3^{\frac{1}{2}}$. For simplicity let $\theta = \theta_{n,K_n-1}$. By Lemma A.2:

$$\theta^{n-1}(\theta^2 - 2) = 2\theta^{\lceil 2n/3 \rceil} + 2\theta^{\lfloor n/3 \rfloor+1} + 2\theta^2 - 1 < 2\theta^{\lceil 2n/3 \rceil} + 2\theta^{\lfloor n/3 \rfloor+1} + 2\theta^2$$

Thus

$$\theta - \sqrt{2} < \frac{2}{\theta + \sqrt{2}} \left(\frac{1}{\theta^{n-1-\lceil 2n/3 \rceil}} + \frac{1}{\theta^{n-2-\lfloor n/3 \rfloor}} + \frac{1}{\theta^{n-3}} \right) < \frac{3}{\sqrt{2}^{n/3-1}}$$

since $\theta > \sqrt{2}$ and $n > 3$. Clearly $\frac{3}{\sqrt{2}^{n/3-1}} < 3^{\frac{1}{2}} - \sqrt{2}$ for $n > 22$ hence $\theta < 3^{\frac{1}{2}} < \theta_{n,K_n,L_n+1}$ that is the desired inequality. For $n = 20, 22$ we directly check the inequality.

Finally we prove the last claim for $l = L_n + 2$. By Lemma B.4 we have $\theta_{n,K_n,L_n+2} > 6^{\frac{1}{4}}$. We can check that $\frac{3}{\sqrt{2}^{n/3-1}} < 6^{\frac{1}{4}} - \sqrt{2}$ for $n > 28$ hence $\theta < 6^{\frac{1}{4}} < \theta_{n,K_n,L_n+2}$ that is the desired inequality (for $n < 30$ we directly check the inequality). The proposition is proved. $\square$

B.3. Case $n \equiv 1 \mod 4$.

Proposition B.9. *Let $n \geq 5$ such that $n \equiv 1 \mod 4$. For any $1 \leq k \leq K_n - 1$ we define $d = \gcd(k, n-1)$ and $n' = \frac{n-1}{d} + 1$, $k' = k/d$. Then $\theta_{n',k'} > \theta_{n,K_n}$.*

Proof of Proposition B.9. Let $k \in \{1, \ldots, K_n - 1\}$. If $\gcd(k, n-1) = 1$ then Lemma B.3 implies that $\theta_{n,k} > \theta_{n,K_n}$ as desired.

If $\gcd(k, n-1) = d > 1$ there are three cases depending the value of $n' \mod 4$.

  (1) If $n' \equiv 1 \mod 4$ then the previous argument shows that $\theta_{n',k'} > \theta_{n',K_{n'}}$. By Lemma B.2, the sequence $(\theta_{n,K_n})_n$ is decreasing for $n \equiv 1 \mod 4$, so we have $\theta_{n',K_{n'}} > \theta_{n,K_n}$ as desired.
  (2) If n' is even then Lemma B.3 implies $\theta_{n',k'} > \theta_{n',K_{n'}}$. By Lemma A.2, for any $x > \sqrt{2}$:

$$\begin{aligned} P_{n,K_n} - x^{n-n'}P_{n',K_{n'}} &= 2x^{2+n-n'} - x^{n-n'} - 2x^{\frac{n+1}{2}} - 2x^2 + 1 \\ &> 3x^{n-n'} - 2x^{\frac{n+1}{2}} - 2x^2 > 0 \end{aligned}$$

  (the last inequality comes from $d \geq 4$ and $n \geq 5$). By Lemma B.1 $\theta_{n',K_{n'}} > \theta_{n,K_n}$.
  (3) If $n' \equiv 3 \mod 4$ then Proposition B.6 implies $\theta_{n',k'} > \theta_{n',K_{n'},L_{n'}}$. For $x > \sqrt{2}$:

$$\begin{aligned} P_{n,K_n} - x^{n-n'}P_{n',K_{n'},L_{n'}} &= 4x^{n-n'+\frac{n'+3}{2}} - 4x^{n-n'+\frac{n'-1}{2}} - 2x^{2+n-n'} + x^{n-n'} - 2x^{\frac{n+1}{2}} - 2x^2 + 1 \\ &> 4x^{n-n'+\frac{n'-1}{2}} - 2x^{2+n-n'} - 2x^{\frac{n+1}{2}} + x^{n-n'} - 2x^2 \end{aligned}$$

  Assumption on n, n' implies that $n \neq 5, 9$, hence $n \geq 13$ and $n' \geq 7$. This implies that $n - n' + \frac{n'-1}{2} = (n-1)(1 - \frac{1}{2d}) \geq \frac{n+1}{2}$, $n - n' + \frac{n'-1}{2} \geq 2 + n - n'$ and $n - n' \geq 4$. Thus $P_{n,K_n} - x^{n-n'}P_{n',K_{n'},L_{n'}} > 0$ and Lemma B.1 implies $\theta_{n,K_n} < \theta_{n',K_{n'},L_{n'}}$.

Proof of Proposition B.9 is complete. $\square$

APPENDIX C. A NAIVE ATTEMPT TO GENERALIZE THE RAUZY–VEECH CONSTRUCTION

The classical construction of pseudo-Anosov homeomorphism by Rauzy induction necessarily produces maps that preserves a singularity, and a horizontal separatrix. This clearly comes from the fact that only right Rauzy induction is used. So, it is natural to expect to produce pseudo-Anosov homeomorphism that do not fix a separatrix by combining right and left induction.

For instance, we consider a path γ in the labeled (extended) Rauzy diagram such that.

- the image of γ in the reduced extended Rauzy diagram is closed.
- γ is the concatenation of a path γ_1 that consists only of right Rauzy moves, and a path γ_2 that consists only of left Rauzy moves.

As above, we associate to such path a matrix V by multiplying the corresponding product of the transition matrices by a suitable permutation matrix. Assume now that the matrix V is a primitive. Let $\theta > 1$ be its Perron-Frobenius eigenvalue. We choose a positive eigenvector λ for θ. As before, V is symplectic, thus let us choose an eigenvector τ for the eigenvalue θ^{-1}. It turns out that τ is not necessarily a suspension datum, but it is a weak suspension datum (up to replacing it by its opposite). Indeed, the set of weak suspension data is a open cone W, which is by construction invariant by V^{-1}, and we conclude as previously (the proof is the same as in Proposition 4.1).

Proposition C.1. *A pseudo-Anosov homeomorphism affine on a translation surface, and constructed as above fixes a vertical separatrix. In particular it is obtained by the usual Rauzy–Veech construction.*

Proof. Let (π, λ, τ) be the weak suspension datum defined as above and h be a height. We will denote the associated surface by $X = X(\pi, \lambda, \tau)$ and by $I = I_h$ the corresponding horizontal interval.

After the prescribed the prescribed sequence of right and left Rauzy induction, we obtain the suspension datum $(\pi, \lambda', \tau') = (\pi, \frac{1}{\theta}\lambda, \theta\tau)$ defining the same surface X, with corresponding interval $I'_{h'} = I' \subset I$ (recall the Rauzy–Veech induction corresponds to cutting the interval on the right, or on the left). Also, θh is an obvious height for $(\pi, \frac{1}{\theta}\lambda, \theta\tau)$, and the corresponding interval $I'_{\theta h}$ is the image by ϕ of the interval I_h.

Hence there is an isometry f from $I'_{\theta h}$ to $I'_{h'}$ obtained by following a vertical leaf (see Section 3.2). The map $f \circ \phi$ is therefore a contracting map from I_h to itself (its derivative is θ^{-1}), hence has a fixed point. It means that there is an element x in I_h whose image by ϕ is in the vertical leaf l passing through x. Thus, this vertical leaf l is preserved by ϕ. Since ϕ, restricted to l as derivative $\theta \neq 1$, there is a fixed point of ϕ on l. This fixed point is either a conical singularity or a regular point. In any case, ϕ fixes a vertical separatrix. Hence ϕ fixes also a horizontal separatrix. It is therefore obtained by the usual Rauzy–Veech construction. □

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