PSEUDO-ANOSOV HOMEOMORPHISMS ON TRANSLATION SURFACES IN HYPERELLIPTIC COMPONENTS HAVE LARGE ENTROPY

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ABSTRACT. We prove that the dilatation of any pseudo-Anosov homeomorphism on a translation surface that belongs to a hyperelliptic component is bounded from below uniformly by $\sqrt{2}$. This is in contrast to Penner's asymptotic. Penner proved that the logarithm of the least dilatation of *any* pseudo-Anosov homeomorphism on a surface of genus g tends to zero at rate $1/g$ (as g goes to infinity).

We also show that our uniform lower bound $\sqrt{2}$ is sharp. More precisely, the least dilatation of a pseudo-Anosov on a genus $g > 1$ translation surface in a hyperelliptic component belongs to the interval $\left] \sqrt{2}, \sqrt{2} + 2^{1-g} \right[$. The proof uses the Rauzy-Veech induction.

1. INTRODUCTION

Pseudo-Anosov homeomorphisms play an important role in Teichmüller theory. Calculating problems for their dilatations have a long history in differential geometry. The unit cotangent bundle of the moduli space of compact genus g Riemann surfaces $\mathcal{M}_g$ can be viewed as the moduli space of holomorphic quadratic differentials $Q_g \to \mathcal{M}_g$. This space is naturally stratified by strata of quadratic differentials with singularities of prescribed multiplicities. The Teichmüller geodesic flow acts naturally on these strata, and closed loops of length $\log(\theta) > 0$ for this flow correspond to conjugacy classes of Pseudo-Anosov homeomorphisms with dilatation $\theta > 1$. An important problem concerns the asymptotic behavior of the smallest dilatations.

The strata are not necessarily connected (Kontsevich & Zorich [KZ03], Lanneau [Lan08]). In this paper we use a discretization of the Teichmüller geodesic flow, i.e. the Rauzy-Veech induction, in order to tackle the minimization problem for *hyperelliptic components*. This is the first instance of dilatation asymptotics for components of the moduli spaces. We shall prove

Theorem. *Let $g \geq 2$. Let δ_g^{hyp} be the least dilatation of pseudo-Anosov homeomorphisms that are affine on translation surfaces belonging to some hyperelliptic connected component of the moduli space Q_g. Then*

$$\delta_g^{hyp} \in \left] \sqrt{2}, \sqrt{2} + \frac{1}{2^{g-1}} \right[.$$

We will give a precise statement later.

Mapping class group. The mapping class group $\mathrm{Mod}(S)$ of a closed orientable surface S of genus $g \geq 1$ is defined to be the group of homotopy classes of orientation preserving homeomorphisms of S. An irreducible mapping class is an isotopy class of homeomorphisms such that no powers preserve a nontrivial subsurface of S. By the Thurston-Nielsen classification [Thu88], irreducible mapping classes are either periodic (analogous to roots of unity) or are of a type called pseudo-Anosov [FLP79]. To each pseudo-Anosov mapping class $[\phi] \in \mathrm{Mod}(S)$ one can attach a dilatation factor $\theta(\phi) > 1$. The logarithm of $\theta(\phi)$ can be viewed as the minimal topological entropy of any element in the homotopy class of ϕ (uniquely realized by some element, ϕ).

Thurston proved that this number is an algebraic integer and even a Perron number. $\theta(\phi) > 1$ is also the exponential growth rate of lengths of curves under iteration of ϕ (in any metric on S). These numbers appear naturally as the length spectrum of the moduli space of genus g Riemann surfaces.

It is an open question to characterize the set of dilatations of pseudo-Anosov homeomorphisms. Thurston has conjectured that pseudo-Anosov dilatations (ignoring genus) are precisely the algebraic units that are Perron and also larger than the Galois conjugates of their inverses.

Minimization problem. The set of dilatations for fixed genus is discrete as a subset of $\mathbb{R}$ (see [AY81, Iva88]. Hence the least dilatation δ_g is well defined. We know very little about the values of the constants δ_g. The precise value of δ_2 has been recently calculated (see Cho & Ham [CH08] and Lanneau & Thiffeault [LT10]) but the values of δ_g for $g \geq 3$ are still unknown.

Upper bounds are not hard to derive from examples and there are a lot of results in that direction (see e.g. Penner, McMullen, Hironaka, Kin, Minakawa). So far the best general upper bound for $g \log(\delta_g)$ is the one given by Hironaka [Hir09] and Kin & Takasawa [KT10]. But again very little is known about *lower* bounds. Penner [Pen91] proved that $\log(\delta_g) \geq \frac{\log(2)}{6g-6}$, using general properties of the Perron-Frobenius matrices. There is also a result of Tsai [Tsa09] for pseudo-Anosov on *punctured* surfaces.

In general lower bounds are much subtle to obtain than upper bounds. In contrast to our understanding of the asymptotic of $\log(\delta_g)$, we still do not know the answer to the following question, posed by McMullen [McM00, Section 10]:

$$\text{Does } \lim_{g \to +\infty} g \log(\delta_g) \text{ exist? What is its value?}$$

Subgroups of the modular group and strata. In his book [Far06], Farb proposed two natural refinements of the minimization problem.

The first one is related to subgroups of the modular group. More precisely, for a subgroup $H \subseteq \mathrm{Mod}(S)$, the problem is to understand the least dilatation $\delta(H)$ of pseudo-Anosov classes $[\phi] \in H$.

Leininger [Lei04] and then Farb, Leininger & Margalit [FLM08] tackled the minimization problem for the subgroups of $Mod(S)$ given by the Thurston's construction and by the Torelli group, respectively. They provide evidence for the principle that algebraic complexity implies dynamical complexity.

In order to state the second problem, let us recall the definition of what we will refer to as strata here (see Section 2 for precise statements). A pseudo-Anosov homeomorphism ϕ on a compact surface S defines uniquely a pair of transverse measured foliations on S. It is well known [HM79] that these

data are equivalent to a pair (X, q) where X is a Riemann surface homeomorphic to S and q is a non zero holomorphic quadratic differential. Moreover $q = \omega^2$ (where ω is an holomorphic $1-$form) if and only if the foliations are orientable; In this case we will call the pair (X, ω) a translation surface (see Section 2.1). These data endow X with an Euclidean structure for which ϕ is affine.

A stratum is the moduli space of quadratic differentials with prescribed multiplicities of the zeroes of the differentials. Given $(k_1, \ldots, k_n)$ satisfying the Gauss-Bonnet equality $\sum_{i=1}^n k_i = 2g - 2$ we define

$$\delta^+(k_1, \ldots, k_n) := \inf \left\{ \log(\theta(\phi)) : \begin{array}{l} \text{pseudo-Anosov } \phi \text{ whose corresponding } (X, \omega) \\ \text{is in the stratum defined by } (k_1, \ldots, k_n) \end{array} \right\}.$$

Similarly we define $\delta^-(k_1, \ldots, k_n)$ for non-orientable measured foliations. These quantities are well defined (up to three exceptions) by a result of Masur & Smillie [MS93]. Obviously one has

$$\delta_g = \min\{\delta^+(k_1, \ldots, k_n), \ \delta^-(k_1, \ldots, k_n)\},$$

where the min is taken over all k_i that correspond to genus g. It is thus natural to study the minimal dilatation in a given stratum as proposed by Farb [Far06].

It turns out that strata are not connected in general [KZ03, Lan08]. We can thus consider the natural following refinement of Farb's question: give the minimal dilatation of a pseudo-Anosov homeomorphism on a surface in a given connected component of a stratum.

From now on we will restrict to the case where the foliations are orientable. The components are distinguished by two invariants: the *parity of the spin structure* and the *hyperellipticity*. The latter are defined as follows.

Recall that a Riemann surface X of genus $g \geq 2$ is hyperelliptic if there exists an holomorphic involution τ with $2g + 2$ fixed points.

Convention. *In all of this paper, we will use the following convention: a translation surface (X, ω) is hyperelliptic if the underlying Riemann surface is hyperelliptic. The fixed points of τ are usually called the Weierstrass points.*

Definition 1.1. *A surface (X, ω) is in a hyperelliptic connected component if and only if the following hold*

    (1) *(X, ω) is hyperelliptic,*
    (2) *ω has at most two zeroes,*
    (3) *τ permutes the two zeroes (in case of two zeroes).*

Remark 1.1.

    (1) *If (X, ω) is hyperelliptic but does not satisfy the condition (2) or (3) then the property of being hyperelliptic is destroyed by some small perturbations inside the ambient stratum (see [KZ03] for more details).*
    (2) *Condition (3) in Definition 1.1 can be replaced by "the two zeroes are not Weierstrass points". Of course if ω has a single zero then it is at a Weierstrass point.*

In this paper we will investigate the second problem and answer a question of Farb [Far06, Problem 7.5]. We shall prove

Theorem 1.1. *Let $g \geq 2$. Let ϕ be a pseudo-Anosov homeomorphism affine on a genus g translation surface (X, ω) in a hyperelliptic connected component. Then*

$$\theta(\phi) > \sqrt{2}.$$

This theorem provides further evidence for the principle that algebraic complexity implies dynamical complexity.

Recall that Penner [Pen91] proved that, as the genus increases, there are pseudo-Anosov homeomorphisms with dilatations arbitrarily close to 1.

This uniform lower bound is sharp as shows Theorem 1.2.

Theorem 1.2. *Let $g \geq 2$. Let $\delta^{hyp}(2g-2)$ (respectively, $\delta^{hyp}(g-1, g-1)$) be the least dilatation of pseudo-Anosov homeomorphisms affine on a genus g translation surface (X, ω) in the hyperelliptic connected component corresponding to one zero (respectively, two zeroes). Then*

$$\sqrt{2} < \delta^{hyp}(2g-2) < \sqrt{2} + \tfrac{1}{2^{g-1}}, \text{ and}$$

$$\sqrt{2} < \delta^{hyp}(g-1, g-1) < \sqrt{2} + \tfrac{4}{\sqrt{2^g}}.$$

Remark 1.2. *Let us remark that all the inequalities in the above statements are strict, since the bounds we give are never a dilation.*

Non-hyperelliptic components. The reader may wonder why we impose restrictions on the action of the hyperelliptic involution on the zeroes in the definition of hyperelliptic connected component. It turns out that if we relax the condition (3) in Definition 1.1 then the asymptotic behavior may be very different if we consider different connected components.

As for example one can construct (see Appendix B) a sequence $(\varphi_g)_{g \geq 3}$ (with g odd) of pseudo-Anosov homeomorphisms on a hyperelliptic translation surface (X_g, ω_g) of genus g having two zeroes of degree $g-1, g-1$, and such that the dilatation of φ_g is the Perron root of the polynomial

$$X^{2g} - X^{2g-1} - 4X^g - X + 1.$$

In particular

$$\lim_{k \to +\infty} \theta(\varphi_{2k+1}) = 1.$$

Of course in that case the hyperelliptic involution fixes the two zeroes. In the above case, the (*non-hyperelliptic*) components are distinguished by a parity of the spin structure (see [KZ03] and Section 2.2). Let C_g^{odd} be the (*non-hyperelliptic*) odd component of the stratum with two zeroes in genus g. Then we shall prove

Theorem 1.3. *Let $g \geq 2$, be an odd integer. Let $\delta(C_g^{odd})$ be the least dilatation of pseudo-Anosov homeomorphisms affine on a genus g translation surface $(X, \omega) \in C_g^{odd}$. Then*

$$1 < \delta(C_g^{odd}) < 1 + \frac{1}{g}.$$

Remark 1.3. *E. Hironaka noticed that the dilatations of the above examples coincide with the dilatations found in [HK06]. It could be interesting to know whether or not these examples are the same.*

Quadratic differentials. A quadratic differential (X, q) is strictly quadratic if it is not the square of any holomorphic 1-form. In Section 2.3 we derive from our theorem some results for some strata of strict quadratic differentials, namely we will prove Theorem 2.2, page 8.

Rauzy-Veech induction. If C is a connected component of some strata then there is a finite ramified covering $\widehat{C} \to C$ consisting of marking a zero and a separatrix on surfaces in C. The Rauzy-Veech induction provides a discrete representation (symbolic coding) of the Teichmüller flow on $\widehat{C}$ [Vee82] (see Section 3 for precise definitions). Periodic orbits in C correspond to conjugacy classes of pseudo-Anosov on $(X, \omega) \in C$.

As shown by Veech, to each periodic orbit of the Teichmüller geodesic flow $\gamma \subset \widehat{C}$ there corresponds a closed loop in some graph (called a Rauzy diagram) $\mathcal{D}_r(C)$ and a renormalization matrix $V(\gamma) \in SL(h, \mathbb{Z})$, where $h = 2g + n - 1$. This matrix corresponds to the action of the corresponding pseudo-Anosov homeomorphism in relative homology of the underlying surface with respect to the singularities of the Abelian differential. Hence the spectral radius of $V(\gamma)$ is the dilatation of the pseudo-Anosov on $(X, \omega) \in C$.

Two crucial points in this paper are the following

- we show that "enough" pseudo-Anosov are obtained by the Rauzy-Veech induction;
- we carefully analyze the *global* geometry of the Rauzy diagrams. This approach was already used by Avila & Viana [AV07] for the dynamical properties of the Teichmüller geodesic flow.

Outline of a proof of our main result. The Rauzy-Veech induction allows one to relate pseudo-Anosov homeomorphisms and closed loops in Rauzy diagrams. We conclude by sketching its use in the proof of Theorem 1.1.

1. Let ϕ be a pseudo-Anosov homeomorphism affine with respect to a translation surface (X, ω) in a hyperelliptic component C^{hyp}. We prove that ϕ commutes with the hyperelliptic involution (Lemma 2.3). Thus ϕ induces a pseudo-Anosov homeomorphism on the sphere.
2. Using the Brouwer fixed point theorem, we show that ϕ^2 fixes a separatrix of the horizontal measured foliation on X (Proposition 4.1).
3. Hence ϕ^2 is also affine with respect to a surface $(X, \omega) \in \widehat{C^{hyp}}$ (with a marked separatrix). Thus ϕ^2 is obtained by taking a suitable closed loop γ in the Rauzy diagram $\mathcal{D}_r^{hyp}$ corresponding to $\widehat{C^{hyp}}$.
4. We use the representation introduced by Kerckhoff [Ker85] and formalized later by Bufetov [Buf06] and Marmi, Moussa & Yoccoz [MMY05], which furnishes a finite covering $\mathcal{D}^{hyp} \to \mathcal{D}_r^{hyp}$. Lift of γ in $\mathcal{D}^{hyp}$, denoted $\hat{\gamma}$, is easier to describe than γ in $\mathcal{D}_r^{hyp}$.
5. Analyzing carefully the combinatorics of these Rauzy diagrams we show that the spectral radius of $V(\hat{\gamma})$ is greater than 2 implying $\theta(\phi^2) \geq 2$ (see Section 3.7.2 and Section 4). Since $\theta(\phi^2) = \theta(\phi)^2$ one gets the desired result.
6. In Appendix A we prove that our uniform bound is sharp by exhibiting a suitable sequence of pseudo-Anosov homeomorphisms. In Appendix B we show that the action of the hyperelliptic involution on the zeroes is crucial.

Further results and notes. Fehrenbach & Los [FL87] proved the following inequality for pseudo-Anosov homeomorphisms ϕ having a periodic orbit of length $n \geq 3$: $\log(\phi) \geq \frac{1}{n}\log(1+\sqrt{2})$. However this does not easily imply a uniform lower bound. Indeed for our examples in Appendix A we checked (for $g \leq 10$) that

$$\min\{n \geq 3, \text{ there exists a periodic orbit of length } n\} = 2g + 1.$$

Our examples also give the systole $\delta^{hyp}(2g-2)$ and $\delta^{hyp}(g-1,g-1)$ for $g \leq 4$ [LT10]. Also we have $\delta_2 = \delta^{hyp}(2) = \delta^-(1,1,2)$. See also [LT10] for a presentation of the examples in term of the braid group.

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2. BACKGROUND

We review basic notions and results concerning Abelian differentials, translation surfaces, pseudo-Anosov homeomorphisms and moduli spaces. For general references see say [MT02, Vee82, Rau79, MS93, MMY05].

2.1. **Flat surfaces and pseudo-Anosov homeomorphisms.** A *flat surface* S is a (real, compact, connected) genus g surface equipped with a flat atlas i.e. a triple $(S, \mathcal{U}, \Sigma)$ such that Σ is a finite subset of S (whose elements are called *singularities*) and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $S \setminus \Sigma$ with transition maps $z \mapsto \pm z +$ constant. We will require that for each $s \in \Sigma$, there is a neighborhood of s isometric to a Euclidean cone. Therefore we get a *quadratic differential* defined locally in the coordinates z_i by the formula $q = dz_i^2$. This form extends to the points of Σ to zeroes, simple poles or marked points (we will usually call the zeroes and poles *singular points* or simply *singularities*).

If there exists a sub-atlas such that all transition functions are translations then the quadratic differential q is the global square of an Abelian differential $\omega \in H^1(X, \mathbb{C})$. We will then say that (X, ω) is a translation surface.

A homeomorphism $f : X \to X$ is an *affine homeomorphism* if f restricts to a diffeomorphism of $X \setminus \Sigma$ of constant derivative. It is equivalent to say that f restricts to an isomorphism of $X \setminus \Sigma$ which preserves the induced affine structure given by q.

Lemma 2.1. *If (X, ω) is a hyperelliptic translation surface then the hyperelliptic involution is affine.*

Proof of the lemma. The hyperelliptic involution τ induces on $H^1(X, \mathbb{C})$ a linear involution, which splits this space as a direct sum between an invariant and an anti-invariant subspace. Since the invariant part is isomorphic to $H^1(\mathbb{P}^1(\mathbb{C}), \mathbb{C})$, hence it is trivial. Therefore, $\tau^* \omega = -\omega$. $\square$

There is a standard classification of elements of $SL_2(\mathbb{R})$ into three types: elliptic, parabolic and hyperbolic. This induces a classification of affine diffeomorphisms. An affine diffeomorphism is parabolic, or elliptic, or pseudo-Anosov, respectively, if $|\text{trace}(Df)| = 2$, $|\text{trace}(Df)| < 2$, or $|\text{trace}(Df)| > 2$, respectively. If ϕ is pseudo-Anosov, in the coordinates of the stable and unstable measured foliations determined by ϕ, one has $D\phi = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}$ where $|\theta| > 1$. The number $|\theta|$ is called the *dilatation* of ϕ. From now all flat surfaces considered will be translation surfaces, except in Section 2.3 and Appendix B.

For $g \geq 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the moduli space of pairs (X, ω) where X is a genus g (compact, connected) Riemann surface and $\omega \in \Omega(X)$ a non-zero holomorphic $1-$form defined on X. The term moduli space means that we identify the points (X, ω) and (X', ω') if there exists an analytic isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$. The group $\mathrm{SL}_2(\mathbb{R})$ naturally acts on the moduli space of flat surfaces by post composition on the charts.

One can also see a translation surface obtained as a polygon, or a finite union of polygons, whose sides come in pairs such that the two sides in each pair are parallel with equal lengths. To get the surface, these parallel sides are glued together by translation and we assume that this identification preserves the natural orientation of the polygons. In this context, two translation surfaces are identified in the moduli space of Abelian differentials if and only if the corresponding polygons can be obtained from each other by "cutting" and "gluing" and preserving the identifications (i.e. the two surfaces represent the same point in the moduli space). Also, the $\mathrm{SL}_2(\mathbb{R})$ action in this representation is just the natural linear action on the polygons.

Veech showed that an affine homeomorphism with a derivative map which is not the identity is not isotopic to the identity. Hence a homeomorphism f is an affine homeomorphism on the flat surface (X, ω), with derivative map $Df = A$, if and only if the matrix A stabilizes the surface (X, ω). That is (X, ω) can be obtained from $A \cdot (X, \omega)$ by "cuttings" and "gluings" on the corresponding polygons.

2.2. Connected components of the strata.
The moduli space of Abelian differentials is stratified by the combinatorics of the zeroes; We will denote by $\mathcal{H}(k_1, \ldots, k_r)$ the stratum of $\mathcal{H}_g$ consisting of (classes of) pairs (X, ω) such that ω possesses exactly r zeroes on X with multiplicities $(k_1, \ldots, k_r)$.

It is a well known part of the Teichmüller theory that these spaces are (Hausdorff) complex analytic, and in fact algebraic, orbifolds.

These strata are non-connected in general but each stratum has at most three connected components (see [KZ03] for a complete classification). In particular for $g \geq 4$ the stratum with a single zero, $\mathcal{H}(2g-2)$, has three connected components. The stratum $\mathcal{H}(g-1, g-1)$ has two or three connected components depending whether g is even or odd, respectively.

2.2.1. *Hyperelliptic component.* This component contains precisely pairs (X, ω) where X is a hyperelliptic surface and ω is a one-form whose zeroes (if there are two) are interchanged by the hyperelliptic involution. An equivalent formulation is to require that there exists a ramified double cover $\pi : X \to \mathbb{P}^1$ over the sphere and a quadratic differential q on $\mathbb{P}^1$ having only one zero and simples poles such that $\omega^2 = \pi^* q$. We will denote these components by $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$.

2.2.2. *Other components.* The other (*non-hyperelliptic*) components are distinguished by a parity of the spin structure. There are two ways to compute the parity of the spin structure of a translation surface X. The first way is to use the Arf formula on a symplectic basis (see [KZ03]). The second possibility applies if X comes from a quadratic differential, i.e. if X possesses an involution such that the quotient produces a half-translation surface [Lan04bis]. We will apply this in Appendix B.

2.3. Application of Theorem 1.1 to quadratic differentials.
In this section, we extend Theorem 1.1 to some other strata in the moduli space of quadratic differentials. This part is independent from the rest of the paper and can be skipped for a first reading. However, Lemma 2.3 will be needed later.

As for Abelian differentials, strata of the moduli space of quadratic differentials are not connected in general (see [Lan08] for a complete classification). We can deduce from Theorem 1.1 results on some hyperelliptic components in the quadratic case.

We denote by $Q(k_1, \ldots, k_n)$ strata of the moduli space of half translation surfaces where the vector $(k_1, \ldots, k_n)$ agrees with the Gauss-Bonnet formula $\sum_{i=1}^n k_i = 4g - 4$. For $g \geq 2$, let us consider the two strata $Q(-1, -1, 2g-1, 2g-1)$ and $Q(-1, -1, 4g-2)$ of the moduli space of genus g. Their hyperelliptic component can be defined as follows (see [Lan04]):

$$C_1^{hyp} := \left\{ (X, q) : \begin{array}{l} X \text{ is hyperelliptic, } \tau \text{ is affine for } q, (X, q) \in Q(-1, -1, 2g-1, 2g-1), \\ \text{and the involution permutes the two zeroes and the two poles} \end{array} \right\}.$$

$$C_2^{hyp} := \left\{ (X, q) : \begin{array}{l} X \text{ is hyperelliptic, } \tau \text{ is affine for } q, (X, q) \in Q(-1, -1, 4g-2), \\ \text{and the involution permutes the two poles} \end{array} \right\}.$$

Theorem 2.2. *Let $g \geq 2$. Let C be C_1^{hyp} or C_2^{hyp} be defined as above. Let ϕ be a pseudo-Anosov homeomorphism affine on a half translation surface $(X, q) \in C$. Then the dilatation $\theta(\phi)$ satisfies*

$$\theta(\phi) > \sqrt{2}.$$

We will use the following result.

Lemma 2.3. *Let f be an affine homeomorphism on hyperelliptic flat surface of genus greater than or equal to two. The map f commutes with the hyperelliptic involution τ.*

Proof of Lemma 2.3. Since the derivative map Df of f is constant in the affine charts, the conjugate $f^{-1}\tau f$ is conformal and is an involution with $2g + 2$ fixed points. It is thus the hyperelliptic involution τ [FK92]. This proves the result. $\square$

Proof of Theorem 2.2. Let $(X, q) \in Q(-1, -1, 2g-1, 2g-1)$ be a half translation surface in the hyperelliptic component. Let ϕ be a pseudo-Anosov homeomorphism on X and τ the hyperelliptic involution.

Passing to the quotient we get a meromorphic quadratic differential q' on the projective line $\mathbb{P}^1$ with $2g + 3$ simple poles and a single zero of degree $2g - 1$. Taking the standard orientating cover $\pi : Y \to \mathbb{P}^1$ over $\mathbb{P}^1$ having ramification points precisely over odd degree singularities (namely the poles and the zero) we obtain a translation surface (Y, ω) where $\omega^2 = \pi^* q'$. By construction (see Subsection 2.2.1) (Y, ω) belongs to the hyperelliptic component $\mathcal{H}^{hyp}(2g)$. Now by Lemma 2.3 ϕ commutes with τ on Y. Thus ϕ induces a pseudo-Anosov homeomorphism φ on $\mathbb{P}^1$ with the same dilatation. Since φ preserves the set of ramification points of π, φ lifts to a new pseudo-Anosov homeomorphism, say $\hat{\varphi}$, on Y.

$$
\begin{array}{ccccc}
X & \xrightarrow{\phi} & X & \quad Y & \xrightarrow{\hat{\varphi}} Y \\
\downarrow & & \downarrow & & \\
\mathbb{P}^1 & \xrightarrow[\varphi]{} & \mathbb{P}^1 & &
\end{array}
$$

Now $\theta(\phi) = \theta(\varphi) = \theta(\hat{\varphi})$. By Theorem 1.1 we get that $\theta(\phi) > \sqrt{2}$.

The second case is similar and left to the reader. $\square$

3. RAUZY-VEECH INDUCTION AND PSEUDO-ANOSOV HOMEOMORPHISMS

In this section we recall the basic construction of pseudo-Anosov homeomorphisms using the Rauzy-Veech induction (for details see [Vee82], §8, and [Rau79, MMY05]). We first review the link between interval exchange maps and translation surfaces.

3.1. **Interval exchange transformations.** Let $I \subset \mathbb{R}$ be an open interval and let us choose a finite partition of I into $d \geq 2$ open subintervals $\{I_j, \ j = 1, \dots, d\}$. An interval exchange transformation is a one-to-one map T from I to itself that permutes, by translation, the subintervals I_j. It is easy to see that T is precisely determined by the following data: a permutation that encodes how the intervals are exchanged, and a vector with positive entries that encodes the lengths of the intervals.

We use the representation introduced first by Kerckhoff [Ker85] and formalized later by Bufetov [Buf06] and Marmi, Moussa &Yoccoz [MMY05]. This will simplify the description of the induction which will be very useful for the proof of our result.

We will attribute a name to each interval I_j. In this case, we will speak of *labeled* interval exchange maps. One gets a pair of one-to-one maps (π_t, π_b) (t for "top" and b for "bottom") from a finite alphabet $\mathcal{A}$ to $\{1, \dots, d\}$ in the following way. In the partition of I into intervals, we denote the kth interval, when counted from the left to the right, by $I_{\pi_t^{-1}(k)}$. Once the intervals are exchanged, the interval number k is $I_{\pi_b^{-1}(k)}$. Then with this convention, the permutation encoding the map T is $\pi_b \circ \pi_t^{-1}$. We will denote the length of the intervals by a vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

Definition 3.1. *We will call the pair (π_t, π_b) a* labeled *permutation, and $\pi_b \circ \pi_t^{-1}$ a permutation (or* reduced *permutation). If it is clear from the context, then we will just use the term permutation. We will also usually write a reduced permutation as a labeled one with $\mathcal{A} = \{1, \dots, d\}$ and $\pi_t = Id$.*

One usually represents labeled permutations $\pi = (\pi_t, \pi_b)$ by a table:

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \dots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \dots & \pi_b^{-1}(d) \end{pmatrix}.$$

3.2. **Suspension data.** The next construction provides a link between interval exchange transformations and translation surfaces. A suspension datum for $T = (\pi, \lambda)$ is a collection of vectors $\{\tau_\alpha\}_{\alpha \in \mathcal{A}}$ such that

- $\forall 1 \leq k \leq d-1, \ \sum_{\pi_t(\alpha) \leq k} \tau_\alpha > 0$,
- $\forall 1 \leq k \leq d-1, \ \sum_{\pi_b(\alpha) \leq k} \tau_\alpha < 0$.

We will often use the notation $\zeta = (\lambda, \tau)$. To each suspension datum τ, we can associate a translation surface $(X, \omega) = X(\pi, \zeta)$ in the following way.

Consider the broken line L_t on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. Similarly, we consider the broken line L_b defined by concatenation of the vectors $\zeta_{\pi_b^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. If the lines L_t and L_b have no intersections other than the endpoints, we can construct a translation surface X by identifying each side ζ_j on L_t with the side ζ_j on L_b by a translation. The resulting surface is a translation surface endowed with the
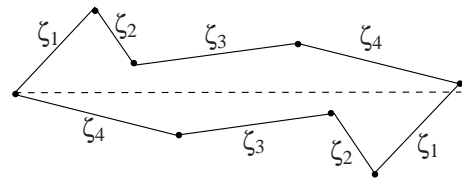
form dz^2. Note that the lines L_t and L_b might have some other intersection points. But in this case, one can still define a translation surface by using the *zippered rectangle construction*, due to Veech ([Vee82]).

Let $I \subset X$ be the horizontal interval defined by $I = (0, \sum_\alpha \lambda_\alpha) \times \{0\}$. Then the interval exchange transformation T is precisely the one defined by the first return map of the vertical flow on X to I.

Each (π, λ) with π irreducible (i.e. $\pi_t^{-1}(\{1, \ldots, k\}) \neq \pi_b^{-1}(\{1, \ldots, k\})$ for all $1 \leq k < d$) admits suspension data.

3.3. **Rauzy-Veech induction.** The Rauzy-Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to a certain subinterval J of I (see [Rau79, MMY05] for details).

We recall very briefly the construction. Following [AGY06] we define the *type* of T by t if $\lambda_{\pi_t^{-1}(d)} > \lambda_{\pi_b^{-1}(d)}$ and b if $\lambda_{\pi_t^{-1}(d)} < \lambda_{\pi_b^{-1}(d)}$. When T is of type t (respectively, b) we will say that the label $\pi_t^{-1}(d)$ (respectively, $\pi_b^{-1}(d)$) is the winner and that $\pi_b^{-1}(d)$ (respectively, $\pi_t^{-1}(d)$) is the looser. We define a subinterval J of I by

$$J = \begin{cases} I \setminus T(I_{\pi_b^{-1}(d)}) & \text{if } T \text{ is of type t;} \\ I \setminus I_{\pi_t^{-1}(d)} & \text{if } T \text{ is of type b.} \end{cases}$$

The image of T by the Rauzy-Veech induction $\mathcal{R}$ is defined as the first return map of T to the subinterval J. This is again an interval exchange transformation, defined on d letters (see e.g. [Rau79]). The data of $\mathcal{R}(T)$ are very easy to express in term of those of T.

There are two cases to distinguish depending whether T is of type t or b; the labeled permutations of $\mathcal{R}(T)$ only depends on π and on the type of T. If $\varepsilon \in \{t, b\}$ is the type of T, this defines two maps $\mathcal{R}_t$ and $\mathcal{R}_b$ by $\mathcal{R}(T) = (\mathcal{R}_\varepsilon(\pi), \lambda')$. We will often make use of the following notation: if $\varepsilon \in \{t, b\}$ we denote by $1 - \varepsilon$ the other element of $\{t, b\}$.

(1) T has type t. Let $k \in \{1, \ldots, d-1\}$ such that $\pi_b^{-1}(k) = \pi_t^{-1}(d)$. Then $\mathcal{R}_t(\pi_t, \pi_b) = (\pi'_t, \pi'_b)$ where $\pi_t = \pi'_t$ and

$$\pi_b'^{-1}(j) = \begin{cases} \pi_b^{-1}(j) & \text{if } j \leq k \\ \pi_b^{-1}(d) & \text{if } j = k+1 \\ \pi_b^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(2) T has type b. Let $k \in \{1, \ldots, d-1\}$ such that $\pi_t^{-1}(k) = \pi_b^{-1}(d)$. Then $\mathcal{R}_b(\pi_t, \pi_b) = (\pi'_t, \pi'_b)$ where $\pi_b = \pi'_b$ and

$$\pi_t'^{-1}(j) = \begin{cases} \pi_t^{-1}(j) & \text{if } j \leq k \\ \pi_t^{-1}(d) & \text{if } j = k+1 \\ \pi_t^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(3) Let us denote by $E_{\alpha\beta}$ the $d \times d$ matrix of which the α, β-th element is equal to 1, all others to 0. If T is of type t then let $(\alpha, \beta) = (\pi_t^{-1}(d), \pi_b^{-1}(d))$ otherwise let $(\alpha, \beta) = (\pi_b^{-1}(d), \pi_t^{-1}(d))$. Then $V_{\alpha\beta}\lambda' = \lambda$, where $V_{\alpha\beta}$ is the transvection matrix $I + E_{\alpha\beta}$.

Remark 3.1. *In the Veech's original construction, the matrices used to obtain λ' in terms of λ were more complicated: of the form $P + E_{\alpha\beta}$ where P is a permutation matrix. Indeed, after the Rauzy induction "bottom", we usually have $\pi'_t \neq Id$, and we must "renumber" the intervals.*

This construction is due to Rauzy. This induction is called the Rauzy-Veech induction since Veech observed that one can actually define the induction on the suspension data in the following way. If τ is a suspension data over (π, λ) then we define $\mathcal{R}(\pi, \lambda, \tau)$ by

$$\mathcal{R}(\pi, \lambda, \tau) = (\mathcal{R}_\varepsilon(\pi), V^{-1}\lambda, V^{-1}\tau),$$

where ε is the type of $T = (\pi, \lambda)$ and V is the corresponding transition matrix. In other terms $V_{\alpha\beta}\zeta' = \zeta$ where $\zeta = (\lambda, \tau)$.

Remark 3.2. *By construction the two translation surfaces $X(\pi, \zeta)$ and $X(\pi', \zeta')$ are naturally isometric (as translation surfaces).*

Now if we iterate the Rauzy induction, we get a sequence (α_k, β_k) of winners/losers. Denoting $\mathcal{R}^{(n)}(\pi, \lambda) = (\pi^{(n)}, \lambda^{(n)})$, the transition matrix that relates $\lambda^{(n)}$ to λ is the product of the transition matrices:

$$(1) \qquad \left(\prod_{k=1}^{n} V_{\alpha_k \beta_k}\right) \lambda^{(n)} = \lambda.$$

3.4. Rauzy diagrams.

3.4.1. *Labeled Rauzy diagrams.* For a labeled permutation π, we call the *labeled Rauzy diagram*, denoted by $\mathcal{D}(\pi)$, the graph whose vertices are all labeled permutations that we can obtained from π by the combinatorial Rauzy moves. From each vertices, there are two edges labeled t and b (the type) corresponding to the two combinatorial Rauzy moves. We will denote by $\pi \xrightarrow{\alpha,\beta} \pi'$ for the edge corresponding to $\mathcal{R}_\varepsilon(\pi) = \pi'$ where $\varepsilon \in \{t, b\}$ and α/β is the winner/looser. To each path γ of this diagram, there is thus a sequence of winners/losers. We will denote by $V(\gamma)$ the product of the transition matrices in Equation (1). The next lemma is clear from the definition.

Lemma 3.1. *Let $\gamma_n = \pi_1 \ldots \pi_n$ be a path in the labeled Rauzy diagram, and let V_n be the matrix associated to the path γ_n. Let α, β be the winner/looser associated to the edge $\pi_{n-1} \xrightarrow{\alpha,\beta} \pi_n$. Then V_n is obtained from V_{n-1} by adding the column α to the column β.*

Definition 3.2. *A closed path in the labeled Rauzy diagram is said to be primitive if the associated matrix V is primitive, i.e. if there exists a power of V such that all the entries are positive. We will also say that a path contains the letter α as winner (respectively, looser) if it contains the edge $\cdot \xrightarrow{\alpha,\beta} \cdot$ (respectively, $\cdot \xrightarrow{\beta,\alpha} \cdot$), for some β.*

We have the following proposition (see [MMY05], proposition in section 1.2.3),

Proposition 3.2 (Marmi, Moussa & Yoccoz). *A closed path γ in a labeled Rauzy diagram is primitive if and only if γ contains all the letters as winner at least once.*

3.4.2. *Reduced Rauzy diagrams.* We have previously defined Rauzy induction and Rauzy diagrams for labeled interval exchange transformations. One can also define the same for reduced interval exchange transformations, as it was first, for which the corresponding labeled permutation is just a permutation of $\{1, \ldots, d\}$ (see [Vee82]). These are obtained after identifying (π_t, π_b) with (π'_t, π'_b) if $\pi_b \circ \pi_t^{-1} = \pi'_b \circ \pi'^{-1}_t$. In the next we will use the notation $\mathcal{D}_r(\pi)$ to denote the reduced Rauzy diagram associated to the permutation π.

Note that the labeled Rauzy diagram is naturally a covering of the reduced Rauzy diagram, and they are usually not isomorphic.

Convention. *Let π be a permutation. We will denote by $\mathcal{D}(\pi)$ the labeled Rauzy diagram of π and by $\mathcal{D}_r(\pi)$ the reduced one.*

Given a closed path γ in the reduced Rauzy diagram, as previously, one can associate a matrix V as follows: we take (π_t, π_b) the labeled permutation corresponding to the endpoint of γ so that $\pi_t = Id$. Then we consider $\hat{\gamma}$ a lift of γ in the labeled Rauzy diagram. The path $\hat{\gamma}$ is not necessarily closed and it ends at a permutation (π'_t, π'_b). We can associate to it a matrix $\widehat{V}$ as before. Let P be the permutation matrix defined by permuting the columns of the $d \times d$ identity matrix according to the permutation π'_t, *i.e.* the $P = [p_{ij}]$, with $p_{ij} = 1$ if $j = \pi'_t(i)$ and 0 otherwise. The transition matrix associated to the path is then:

$$(2) \qquad\qquad\qquad\qquad V = \widehat{V} \cdot P.$$

As before, a closed path in the reduced Rauzy diagram is called *primitive* if V is primitive. A standard reference for the next two sections is [Vee82].

3.5. Construction of pseudo-Anosov homeomorphisms.
There is a natural $\mathrm{SL}_2(\mathbb{R})$-action on the strata. In particular, the one-parameter subgroup $g_t = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ is called the Teichmüller geodesic flow. It can be shown that conjugacy classes of pseudo-Anosov homeomorphisms are one-to-one with closed geodesics of the Teichmüller geodesic flow on strata. There is a very nice construction of pseudo-Anosov homeomorphisms using the Rauzy-Veech induction, and we recall now this construction.

Let π be an irreducible permutation and let γ be a closed loop in the reduced Rauzy diagram associated to π. One can associate to γ a matrix $V(\gamma)$ (see section above). Let us assume that V is *primitive* and let $\theta > 1$ be its Perron-Frobenius eigenvalue. We choose a positive eigenvector λ for θ. It can be shown that V is symplectic [Vee82], thus let us choose an eigenvector τ for the eigenvalue θ^{-1} with $\tau_{\pi_0^{-1}(d)} > 0$. It turns out that τ defines a suspension data over $T = (\pi, \lambda)$. Indeed, the set of suspension data is an open cone, that is preserved by V^{-1}. Since the matrix V^{-1} has a dominant eigenvalue θ (for the eigenvector τ), the vector τ must belong to this cone. If $\zeta = (\lambda, \tau)$, one has

$$\mathcal{R}(\pi, \zeta) = (\pi, V^{-1}\zeta) = (\pi, V^{-1}\lambda, V^{-1}\tau) = (\pi, \theta^{-1}\lambda, \theta\tau) =$$
$$= g_t(\pi, \lambda, \tau), \qquad \text{where} \qquad t = \log(\theta) > 0.$$

Hence the two surfaces $X(\pi, \zeta)$ and $g_t X(\pi, \zeta)$ differ by some element of the mapping class group (see Remark 3.2). In other words there exists a pseudo-Anosov homeomorphism ϕ affine with respect to the translation surface $X(\pi, \zeta)$ and such that $D\phi = g_t$. The action of ϕ on the relative homology of (X, ω) is $V(\gamma)$ thus the dilatation of ϕ is θ. Note that by construction ϕ fixes the zero on the left of the interval I and also a horizontal separatrix adjacent to this zero (namely, the oriented half line corresponding to the interval I).

It turns out that this construction is very general as we will see in the coming section.

3.6. **Discrete representation of the geodesic flow.** Let us fix an irreducible permutation π defined over d letters. If $\mathcal{C}$ is the connected component of some stratum, let $\widehat{\mathcal{C}}$ be the ramified cover over $\mathcal{C}$ obtained by considering the set of triplets (X, ω, l) where $(X, \omega) \in \mathcal{C}$ and l is a horizontal separatrix adjacent to a zero of ω.

Clearly the set of (λ, τ) such that τ is a suspension data over π is a connected space and the map $(\lambda, \tau) = \zeta \mapsto X(\pi, \zeta)$ is continuous. Thus all surfaces obtained by this construction belong to the same connected component of some strata, say $\mathcal{C}(\pi) \subset \mathcal{H}(\sigma)$. Moreover σ can be computed easily in terms of π. We also define $\widehat{\mathcal{C}}(\pi)$ to be the set of $(X(\pi, \zeta), l)$ where l is the horizontal separatrix that corresponds to $\mathbb{R}^+ \times \{0\}$ in the Veech's construction. We define

$$\mathcal{T}(\widehat{\mathcal{C}}) = \left\{ (\pi, \zeta); \ \widehat{\mathcal{C}}(\pi) = \widehat{\mathcal{C}}, \text{ and } \zeta \text{ is a suspension datum for } \pi \right\}.$$

The Rauzy-Veech induction is (almost everywhere) well defined and one-to-one on $\mathcal{T}(\widehat{\mathcal{C}})$. Hence let $\mathcal{H}(\widehat{\mathcal{C}})$ be the quotient of $\mathcal{T}(\widehat{\mathcal{C}})$ by the induction.

The Veech zippered rectangle's construction provides (almost everywhere) a one-to-one map $Z : \mathcal{H}(\widehat{\mathcal{C}}) \to \widehat{\mathcal{C}}$ (see [Boi09] for details).

$$
\begin{array}{ccc}
\mathcal{H}(\widehat{\mathcal{C}}) & \xrightarrow{Z} & \widehat{\mathcal{C}} \\
 & & \downarrow \\
 & \mathcal{C} & \subset \quad \mathcal{H}(\sigma)
\end{array}
$$

One can define the Teichmüller geodesic flow on $\mathcal{H}(\widehat{\mathcal{C}})$ by $g_t(\pi, \zeta) = (\pi, g_t \zeta)$. The Teichmüller flow on $\mathcal{C}$ lifts to a flow g_t on $\widehat{\mathcal{C}}$. It is easy to check that g_t is equivariant with Z i.e. $g_t Z = Z g_t$.

By construction, periodic orbits of g_t on $\mathcal{C}$ corresponding to pseudo-Anosov homeomorphisms that fix the separatrix I lift to periodic orbits on $\widehat{\mathcal{C}}$ for g_t. Thus they produce periodic orbits for g_t acting on the level of suspensions $\mathcal{H}(\widehat{\mathcal{C}})$.

In fact Veech proved that all pseudo-Anosov homeomorphisms fixing a separatrix arise in this way. To see that one can show that the subset of $\mathcal{T}(\widehat{\mathcal{C}})$ defined by

$$\left\{ (\pi, \zeta) \in \mathcal{T}(\widehat{\mathcal{C}}); \ 1 \le |Re(\zeta)| \le 1 + \min\left(Re(\zeta_{\pi_t^{-1}(d)}), Re(\zeta_{\pi_b^{-1}(d)})\right) \right\}.$$

is a fundamental domain of $\mathcal{T}(\widehat{\mathcal{C}})$ for the quotient map $\mathcal{T}(\widehat{\mathcal{C}}) \to \mathcal{H}(\widehat{\mathcal{C}})$, and the Poincaré map of the Teichmüller flow on the section

$$\mathcal{S} = \{(\pi, \zeta); \ \pi \text{ irreducible}, \ |Re(\zeta)| = 1\}/\sim$$

is precisely the *renormalized* Rauzy-Veech induction on suspensions:

$$\widehat{\mathcal{R}}(\pi, \lambda) := (\mathcal{R}_\varepsilon(\pi), V^{-1}\lambda/|V^{-1}\lambda|).$$

We can summarize the above discussion by the following theorem.

Theorem (Veech)**.** *Let γ be a closed loop, based at π, in a reduced Rauzy diagram $\mathcal{D}_r(\pi)$ and let $V = V(\gamma)$ be the product of the associated transition matrices. Let us assume that V is primitive. Let λ be a positive eigenvector for the Perron-Frobenius eigenvalue θ of V and let τ be an eigenvector for the eigenvalue θ^{-1} of V with $\tau_{\pi_t^{-1}(1)} > 0$. Then*

    (1) $\zeta = (\lambda, \tau)$ *is a suspension datum for $T = (\pi, \lambda)$;*

(2) *The matrix $A = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}$ is the derivative map of an affine pseudo-Anosov homeomorphism ϕ on $X(\pi, \zeta)$; The action on relative homology of ϕ is given by the matrix V (in a suitable basis).*

(3) *The dilatation of ϕ is θ;*

(4) *All pseudo-Anosov homeomorphisms that fix a separatrix are obtained in this way.*

We will use this theorem in order to prove our main result. Veech's construction can only build a pseudo-Anosov homeomorphism fixing a separatrix. But there can be many pseudo-Anosov homeomorphism that do not fix any separatrix [Los09, Lan10]. In our case, Proposition 4.1 allows us to restrict to the setting of Veech's theorem.

3.7. Examples of labeled and reduced Rauzy diagrams.

We detail here two examples of Rauzy diagrams (hyperelliptic connected component and hyperelliptic connected component with a marked point) that will correspond to the two cases in Proposition 4.1.

3.7.1. *Hyperelliptic connected components.* Let $n \geq 2$. A representative labeled permutation for the connected component $\mathcal{H}^{hyp}(2g-2)$ (respectively, $\mathcal{H}^{hyp}(g-1, g-1)$) is

$$(3) \qquad \tau_n = \begin{pmatrix} 0 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & \dots & 3 & 2 & 0 \end{pmatrix},$$

where $n = 2g$ (respectively, $n = 2g+1$). We use these "non standards" labels in order to simplify the notations in the next sections. It turns out that these labeled and reduced Rauzy diagrams are isomorphic. The precise description of the diagrams was given by Rauzy [Rau79]. Let us recall his result here.

If $\pi = (\pi_t, \pi_b)$ is a labeled permutation, for $\varepsilon \in \{t, b\}$ we define $G^{\varepsilon}(\pi)$ to be the subdiagram of the Rauzy diagram of π whose vertices are obtained from π by a simple path, and whose first step is the map $\mathcal{R}_{\varepsilon}$. Recall that for $\varepsilon \in \{t, b\}$, we denote by $1 - \varepsilon$ the other element of $\{t, b\}$.

Proposition (Rauzy). *Let τ_n be the labeled permutation defined in Equation (3). Then*

(1) *The vertices of the Rauzy diagram of τ_n is the disjoint union of the vertices of $G^0(\tau_n)$, $G^1(\tau_n)$, and $\{\tau_n\}$.*

(2) *Let $\tau_{n,k} = \mathcal{R}_{\varepsilon}^k(\tau_n)$, for some $\varepsilon \in \{t, b\}$ and $k \in \{1, \dots, n-1\}$. Then $G^{1-\varepsilon}(\tau_{n,k})$ is naturally isomorphic to $G^{1-\varepsilon}(\tau_{n-k})$.*

(3) *The cardinality of the Rauzy diagram is $2^{n-1} - 1$.*

Using this result, one can show that the labeled and reduced diagrams are isomorphic. We will denote by $\mathcal{D}^{hyp} = \mathcal{D}(\tau_n) \approx \mathcal{D}_r(\tau_n)$ this Rauzy diagram. A consequence of the previous proposition is the following:

Corollary 3.3. *Let γ be a closed (oriented) path in $\mathcal{D}^{hyp}$. Assume that it contains the step $\pi \xrightarrow{\alpha, \beta} \pi'$ with $\pi \neq \pi'$, then there exists β' such that it contains the step $\pi' \xrightarrow{\alpha, \beta'} \pi''$.*

Proof. A consequence of the Rauzy description of $\mathcal{D}^{hyp}$ is that $\mathcal{D}^{hyp} \setminus \{\pi'\}$ is not connected any more. Assume that, in the step following $\pi \xrightarrow{\alpha, \beta} \pi'$, the symbol α is looser, then the new permutation and π belong to two different connected components of $\mathcal{D}^{hyp} \setminus \pi'$. Looking at the sequence of permutations that appear, one must come back to the connected component that contains π, and hence, the step $\pi' \xrightarrow{\alpha, \beta'} \pi''$ eventually appears. $\qquad \square$
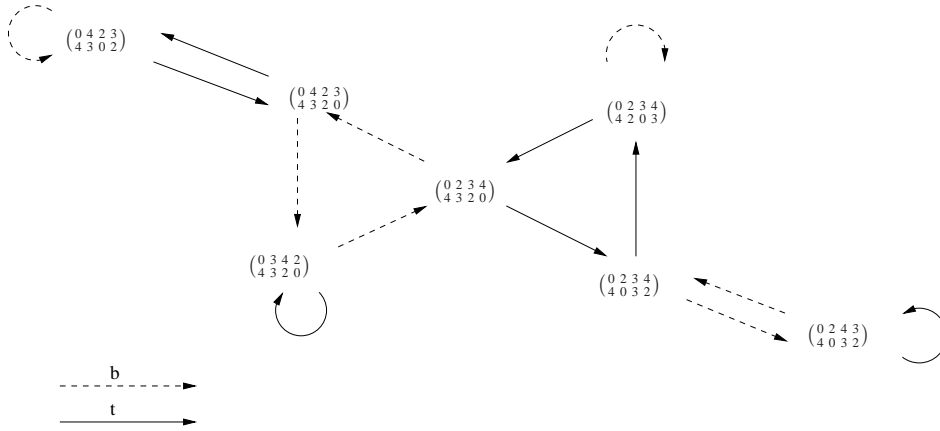
FIGURE 1. The diagram $\mathcal{D}^{hyp} = \mathcal{D}(\tau_n) \approx \mathcal{D}_r(\tau_n)$ for $n = 4$.

3.7.2. *Hyperelliptic connected components with a marked point.* Here we present a second family of Rauzy diagrams that will be useful for our proof. There description is a little bit more technical, but will be needed later. We start with a very informal description of the labeled Rauzy diagram. The precise description can be skipped in a first reading.

The labeled Rauzy diagram is a covering of the reduced one. The cardinalities of these labeled Rauzy diagrams have been calculated (see Delecroix [Del10]). The degree of the covering is also known [Boi10].

A fundamental domain of this covering can be roughly seen as a copy of the hyperelliptic Rauzy diagram described previously, with some added permutations (see Figure 2). This fundamental domain (a *leaf*) is composed at first glance by two principal loops that intersect in a *central permutation*, and from the other vertices of these loops start a secondary loop. The whole diagram is obtained by taking several copies of this "leaf". The different leaves are joined together by the *transition permutations*: each secondary loop contains a unique such transition permutation, and a "k-th secondary loop" of a leaf is attached to a "k-th secondary loop" of another leaf (see Figure 3, and compare it with Figure 2).

We now give the precise description of this diagram. A representative permutation for the connected component $\mathcal{H}^{hyp}(0, 2g-2)$ (respectively, $\mathcal{H}^{hyp}(0, g-1, g-1)$), with a regular marked point, is

(4)
$$\pi_n = \begin{pmatrix} 0 & 2 & 3 & ... & n-1 & 1 & n \\ n & n-1 & ... & 3 & 2 & 1 & 0 \end{pmatrix},$$

where $n = 2g$ (respectively, $n = 2g+1$). Contrary to the previous case, the labeled diagram and the reduced diagram are not isomorphic anymore. As we will see the cardinality of the reduced diagram is $2^{n-1} - 1 + n$ and the cardinality of the labeled diagram is $(2^{n-1} - 1 + n)(n-1)$.

Our next goal is to describe the reduced Rauzy diagrams $\mathcal{D}(\pi_n)$ and $\mathcal{D}_r(\pi_n)$ associated to the above permutation π_n. The key point is to observe that if we forbid the letter 1 to be winner or looser in the construction of the diagram $\mathcal{D}(\pi_n)$ starting from π, then one gets more or less the diagram $\mathcal{D}^{hyp}$ (compare with Avila & Viana [AV07]). Remark that with our choice of permutation π_n, when n is
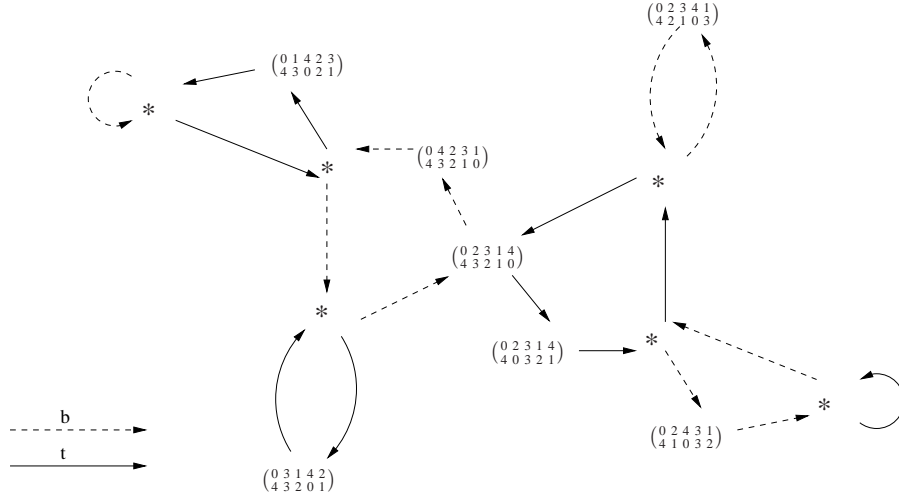
FIGURE 2. The augmented hyperelliptic diagram, and added permutations for $n = 4$.

the winner it has to be on top and similarly 0 when winner is on bottom. Thus one can obtain a subdiagram of $\mathcal{D}(\pi_n)$ starting from $\mathcal{D}^{hyp}$ as follows.

(1) To each permutation $\pi \in \mathcal{D}^{hyp}$ construct a new permutation $\tilde{\pi} \in \mathcal{D}(\pi_n)$ by adding the letter 1 to the left of the letter n on the top and to the left of the letter 0 on the bottom. We will refer to the *augmented hyperelliptic diagram*.

(2) Replace each edge $\pi \xrightarrow{k,n} \pi'$ in $\mathcal{D}^{hyp}$ by two edges $\tilde{\pi} \xrightarrow{k,n} \pi'' \xrightarrow{k,1} \tilde{\pi}'$ in $\mathcal{D}(\pi_n)$, where the permutation π'' has the letter 1 for the last letter on the top.

(3) Replace each edge $\pi \xrightarrow{k,0} \pi'$ in $\mathcal{D}^{hyp}$ by two edges $\tilde{\pi} \xrightarrow{k,0} \pi'' \xrightarrow{k,1} \tilde{\pi}'$ in $\mathcal{D}(\pi_n)$, where the permutation π'' has the letter 1 for the last letter on the bottom.

(4) We will denote by $\mathcal{A}_n \subset \mathcal{D}(\pi_n)$ the added permutations of the operations (2) and (3). All the edges are built except the cases when $\pi'' \in \mathcal{A}_n$ and where 1 is winner.

Observe that operation (2) (respectively, (3)) arises exactly when the letter n (respectively, 0) is looser in the diagram $\mathcal{D}^{hyp}$. Hence the description given in the previous section of this hyperelliptic diagram leads to the following.

Lemma 3.4. *In $\mathcal{D}^{hyp}$, the edges where n (respectively, 0) is looser are*

$$\mathcal{R}_t^k(\tau_n) \xrightarrow{*,n} \mathcal{R}_b \mathcal{R}_t^k(\tau_n) \qquad (\text{respectively, } \mathcal{R}_b^k(\tau_n) \xrightarrow{*,0} \mathcal{R}_t \mathcal{R}_b^k(\tau_n)),$$

for $k \in \{0, \dots, n-2\}$. So, there are $n-1$ edges where n is looser and $n-1$ edges where 0 is looser. Therefore, the added permutations $\mathcal{A}_n$ in $\mathcal{D}(\pi_n)$ by the operations (2) and (3) are:

$$(5) \qquad \mathcal{R}_b \mathcal{R}_t^k(\pi_n) = \begin{pmatrix} 0 & 2 & \dots & k-1 & k & n & k+1 & \dots & n-1 & 1 \\ n & k-1 & \dots & 2 & 1 & 0 & n-1 & \dots & k+1 & k \end{pmatrix},$$

$$(6) \qquad \mathcal{R}_t \mathcal{R}_b^k(\pi_n) = \begin{pmatrix} 0 & n-k+2 & \dots & n-1 & 1 & n & 2 & \dots & n-k+1 \\ n & n-1 & \dots & n-k+2 & n-k+1 & 0 & n-k & \dots & 1 \end{pmatrix},$$

for $k \in \{2, \ldots, n-1\}$, and

$$(7) \qquad\qquad \mathcal{R}_t(\pi_n) = \begin{pmatrix} 0 & 2 & 3 & \ldots & n-1 & 1 & n \\ n & 0 & n-1 & \ldots & 3 & 2 & 1 \end{pmatrix}$$

$$(8) \qquad\qquad \mathcal{R}_b(\pi_n) = \begin{pmatrix} 0 & n & 2 & 3 & \ldots & n-1 & 1 \\ n & n-1 & \ldots & 3 & 2 & 1 & 0 \end{pmatrix}$$

In addition for all $k = 2, \ldots, n-1$ the two new permutations in the labeled diagram $\mathcal{D}(\pi_n)$ defined by (5) and (6) correspond to the same permutation in the reduced diagram $\mathcal{D}_r(\pi_n)$; the renumbering corresponds to σ^{-k} with σ the cyclic permutation $(1, 2, \ldots, n-1)$. The two new permutations corresponding to $\mathcal{R}_t(\pi_n)$ and $\mathcal{R}_b(\pi_n)$ are different in $\mathcal{D}_r(\pi_n)$.

Proof of Lemma 3.4. The proof is obtained by straightforward computation and is left to the reader.

$\square$

Now in order to finish the construction of $\mathcal{D}(\pi_n)$ and $\mathcal{D}_r(\pi_n)$, one has to consider from the added permutations $\mathcal{A}_n$ the operations top/bottom where the letter 1 is winner. In fact it turns out that the reduced diagram $\mathcal{D}_r(\pi_n)$ is already constructed. Namely,

Corollary 3.5. *The diagram $\mathcal{D}_r(\pi_n)$ corresponds to adding the new permutations defined in above Lemma 3.4, namely $\mathcal{T}_n$ to the augmented hyperelliptic diagram, up to renumbering. In particular the cardinality of this diagram is $2^{n-1} - 1 + n$.*

Proof of Corollary 3.5. Let $\pi' \in \mathcal{A}_n$ be an added permutation. Assume that it is given by (5), that is $\pi' = \mathcal{R}_b \mathcal{R}_t^k(\pi_n)$ for some k. The edge $\pi' \to \mathcal{R}_b(\pi')$ was constructed when defining $\mathcal{A}_n$. By Lemma 3.4, we also have that $\pi' = \mathcal{R}_t \mathcal{R}_b^k(\pi_n)$ in the reduced Rauzy diagram, hence the edge $\pi' \to \mathcal{R}_t(\pi')$ was also already constructed. This corresponds to $n-2$ added permutations.

The remaining cases are when π' is given by (7) or (8). Then it is clear that the array corresponding to 1 being winner are arrays from π' to itself. This corresponds to 2 permutations.

Hence, the diagram $\mathcal{D}_r(\pi_n)$ is completely built. Since $\#\mathcal{D}^{hyp} = 2^{n-1} - 1$ one has $\#\mathcal{D}_r(\pi_n) = 2^{n-1} - 1 + n$.

$\square$

Lemma 3.4 and Corollary 3.5 imply the following description of the diagram $\mathcal{D}(\pi_n)$.

Proposition 3.6. *The augmented hyperelliptic diagram together with the permutations $\mathcal{T}_n$ form a fundamental domain for the covering map $\mathcal{D}(\pi_n) \longrightarrow \mathcal{D}_r(\pi_n)$. A fiber for this covering consists of labeled permutations of the kind $\{\pi, \pi \circ \sigma, \ldots, \pi \circ \sigma^{n-2}\}$, where $\pi \circ \sigma^k$ means the labeled permutation π after renumbering with σ^k. In particular the cardinality of this diagram is $(2^n - 1 + n)(n - 1)$.*

The following technical lemma is similar to Corollary 3.3. It is a very important lemma since it will allow us to give information about *all* irreducible paths.

Lemma 3.7. *Let γ be a closed oriented path in $\mathcal{D}(\pi_n)$: Let $k \in \{2, \ldots, n-2\}$ and let $k' \in \{2, n-k\}$ and let $\varepsilon \in \{t, b\}$.*

    a) *If γ contains the step $\mathcal{R}_\varepsilon^{k'-1} \mathcal{R}_{1-\varepsilon}^k(\pi_n) \to \mathcal{R}_\varepsilon^{k'} \mathcal{R}_{1-\varepsilon}^k(\pi_n)$, then it also contains the following step $\mathcal{R}_\varepsilon^{k'} \mathcal{R}_{1-\varepsilon}^k(\pi_n) \to \mathcal{R}_\varepsilon^{k'+1} \mathcal{R}_{1-\varepsilon}^k(\pi_n)$.*

    b) *If γ contains the step $\mathcal{R}_\varepsilon^{k-1}(\pi_n) \to \mathcal{R}_\varepsilon^k(\pi_n)$, then it also contains the step $\mathcal{R}_\varepsilon^k(\pi_n) \to \mathcal{R}_\varepsilon^{k+1}(\pi_n)$, or the step $\mathcal{R}_{1-\varepsilon}^k(\pi_n \circ \sigma^i) \to \mathcal{R}_{1-\varepsilon}^{k+1}(\pi_n \circ \sigma^i)$, for some $i \in \{0, \ldots, n-2\}$.*
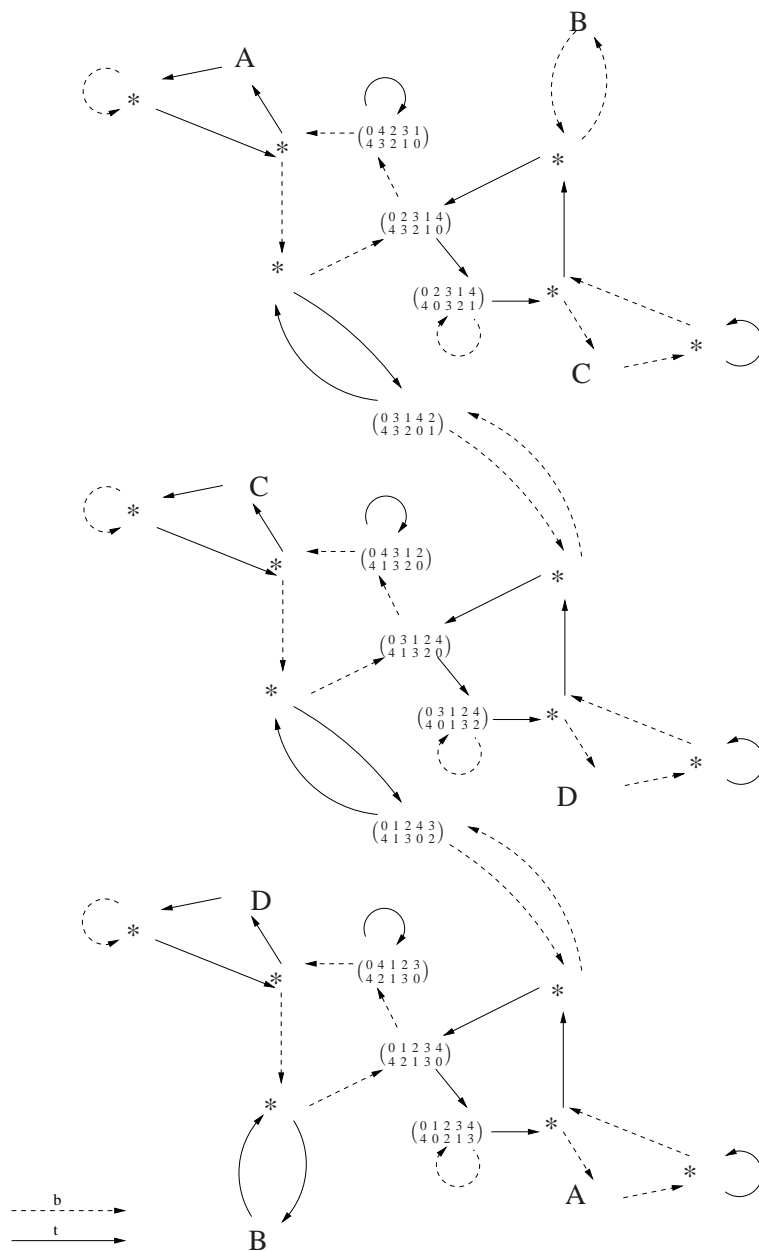
FIGURE 3. The complete diagram $\mathcal{D}(\pi_n)$ for $n = 4$. The vertices labeled by letters A, B, C and D should be identified.

This property is also true if γ is a nonclosed path that starts and ends in the set $\{\pi_n, \pi_n \circ \sigma, \dots, \pi_n \circ \sigma^{n-2}\}$.

Proof. a) Using the construction of $\mathcal{D}(\pi_n)$ from $\mathcal{D}^{hyp}$, we see that, as in Corollary 3.3, the diagram $\mathcal{D}(\pi_n)\backslash\{\mathcal{R}_\varepsilon^{k'}\mathcal{R}_{1-\varepsilon}^k(\pi_n)\}$ is nonconnected.

b) Here, we have that $\mathcal{D}(\pi_n)\backslash\{\mathcal{R}_\varepsilon^k(\pi_n),\ \mathcal{R}_{1-\varepsilon}^k(\pi_n\circ\sigma^i)\}$ is nonconnected, for the parameter i such that $\mathcal{R}_{1-\varepsilon}\mathcal{R}_\varepsilon^k(\pi_n)=\mathcal{R}_\varepsilon\mathcal{R}_{1-\varepsilon}^k(\pi_n\circ\sigma^i)$. $\qquad\square$

4. MINIMAL DILATATIONS IN A HYPERELLIPTIC CONNECTED COMPONENT

4.1. **A key proposition.** Veech's construction can only build a pseudo-Anosov homeomorphism fixing a separatrix. But there can be many pseudo-Anosov homeomorphism that do not fix any separatrix [Los09, Lan10]. In our case, we are saved by the following key proposition.

Proposition 4.1. *Let $g\geq 1$ and let (X,ω) be a flat surface in some hyperelliptic connected component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$. Then for any pseudo-Anosov homeomorphism ϕ affine with respect to (X,ω), ϕ^2 has a fixed point of negative index i.e. ϕ^2 fixes a point and a outgoing separatrix issued from that point.*

Proof of Proposition 4.1. Let τ be the hyperelliptic involution on X. Firstly by Lemma 2.3, the involution τ commutes with ϕ. Thus ϕ descends to a pseudo-Anosov homeomorphism f on the sphere. By assumption, the quadratic differential associated to f has a single zero (of order $2g-3$ or $2g-2$ depending the case). Thus f fixes this zero and induces a disc pseudo-Anosov homeomorphism, say g. Remark that g can be defined as a homeomorphism on the closed disc by extending it to the rays emanating from the zero. Now by Brouwer's theorem, g has a fixed point on the disc. Either this point is inside the disc (thus this is a regular point) or on the boundary. If the last case occur, then f fixes the separatrices issued from the singularity. It is then not hard to see that either ϕ^2 fixes a regular point and the outgoing separatrix, or a singular point and the outgoing separatrices. Proposition 4.1 is proven. $\qquad\square$

Remark 4.1. *Another Brouwer's theorem states that a homeomorphism on the plane with a periodic orbit has a fixed point. With this theorem, it is easy to show that there actually exists a regular fixed point for ϕ^2. However, we believe that the case where the separatrix is attached to a singularity is useful to present the main ideas without the technical difficulties of the other case.*

The next two sections analyze the two cases depending whether the fixed point is singular or regular.

4.2. **Case when the fixed separatrix is adjacent to the singularity.** In this section, we study the case when the pseudo-Anosov homeomorphism fixes a horizontal separatrix starting from a singular point. We thus have to consider the corresponding diagrams $\mathcal{D}^{hyp}$. We first prove a sufficient condition for a primitive path to have a transition matrix with a spectral radius greater than 2.

Proposition 4.2. *Let γ be a primitive closed path in some reduced Rauzy diagram. Let us assume that there is a lift $\hat{\gamma}$ of γ in the labeled Rauzy diagram that contains all the letters as losers, or all the letters as winners. Then the Perron-Frobenius eigenvalue of the matrix $V(\gamma)$ is bounded from below by 2.*

Proof of the proposition. If $\hat{\gamma}$ contains all the looser (respectively, all the winner) then Lemma 3.1 implies that the minimum of the sums of the columns (respectively, the lines) of the corresponding matrix $\widehat{V}$, before remuneration, is at least 2. Since $V(\gamma)=\widehat{V}\cdot P$ where P is a permutation matrix (see Equation 2) the same property on the sums of the columns (respectively, the lines) holds for $V(\gamma)$.

By assumption, the matrix $V = V(\gamma)$ is primitive. Up to replacing V by ${}^tV$, which does not change its eigenvalues, we can assume that the sum of the coefficients on each lines is at least 2. Let x be a Perron-Frobenius eigenvector with positive entries associated to the Perron-Frobenius eigenvalue $\rho(V)$ of V. Let i_0 be such that $\min\limits_{j=1,\dots,d} x_j = x_{i_0} > 0$. Then

$$\rho(V)x_{i_0} = \sum_{j=1}^{d} v_{i_0 j} x_j \geq x_{i_0} \sum_{j=1}^{d} v_{i_0 j}.$$

Since $x_{i_0} > 0$, we see that there exists i_0 such that

$$\rho(V) \geq \sum_{j=1}^{d} v_{i_0 j} \geq 2.$$

$\square$

Proposition 4.3. *Let ϕ be a pseudo-Anosov homeomorphism affine on a translation surface $(X, \omega) \in \mathcal{H}^{hyp}(2g-2)$ or $(X, \omega) \in \mathcal{H}^{hyp}(g-1, g-1)$. If ϕ fixes a horizontal separatrix emanating from a singular point of ω then the dilatation of ϕ is bounded from below by* 2.

Proof. By Veech's Theorem, ϕ is obtained by taking a closed loop in a Rauzy diagram. By a Lemma of Rauzy (see [Rau79]), any Rauzy diagram contains a vertex of the kind:

$$\nu = \begin{pmatrix} \alpha & \dots & \beta \\ \beta & \dots & \alpha \end{pmatrix}.$$

Since the underlying flat surface is in the hyperelliptic connected component, it is easy to see that $\nu = \tau_n$ and therefore this Rauzy diagram is necessarily $\mathcal{D}^{hyp}$ (one can also use the main theorem of [Boi09]). As we have seen, the labeled and reduced diagrams coincide. Hence, for any closed path in the reduced diagram whose associated matrix is primitive, the corresponding lift contains all the letters as winner (see Proposition 3.2). Hence by Proposition 4.2 the associated dilatation is at least 2. $\square$

4.3. **Case when the fixed separatrix is adjacent to a regular point.** One needs to prove the following proposition.

Proposition 4.4. *Let ϕ be a pseudo-Anosov homeomorphism on a surface in either $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$. If ϕ fixes a separatrix adjacent to a regular point, then the dilatation of ϕ is bounded from below by* 2.

The idea of the proof is similar to the one of Proposition 4.3: we must consider pseudo-Anosov homeomorphisms that are obtained by the Veech construction using the diagram $\mathcal{D}_r(\pi_n)$, were $\pi_n = \begin{pmatrix} 0 & 2 & 3 & \dots & n-1 & 1 & n \\ n & n-1 & \dots & 3 & 2 & 1 & 0 \end{pmatrix}$. In this case, the reduced diagram is different from the labeled diagram (see section 3.7.2). But we will show that we can still apply Proposition 4.2.

Remark 4.2. *Our case is more subtle than the previous case. Indeed, start with the permutation $\begin{pmatrix} 1 & 3 & 0 & 2 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \in \mathcal{D}(\pi_4)$. If we consider the path $b - t - b - t - t - t$ then the letter 3 is never winner nor looser. But the corresponding path in $\mathcal{D}_r(\pi_4)$ is closed and primitive. Nevertheless the lift based at the permutation $\begin{pmatrix} 1 & 2 & 3 & 0 & 4 \\ 4 & 3 & 2 & 0 & 1 \end{pmatrix}$ is $t - t - t - b - t - b$, and all the letters are looser.*

We will call the permutations in the diagram $\mathcal{D}(\pi_n)$ that are in $\{\pi_n, \pi_n \circ \sigma, \dots, \pi_n \circ \sigma^{n-2}\}$ the *central permutations*. We will call permutations in the diagram $\mathcal{D}(\pi_n)$ that corresponds to $\mathcal{A}_n$ up to renumbering with σ^k the *transition permutations*.

We have the following technical lemmas.

Lemma 4.5. *Let γ be a primitive closed path in $\mathcal{D}_r(\pi_n)$. There is a lift $\hat{\gamma}$ of γ in $\mathcal{D}(\pi_n)$, not necessarily closed, that starts and ends at central permutations.*

Lemma 4.6. *Let $\hat{\gamma}$ be a path in $\mathcal{D}_r(\pi_n)$ that starts and ends at central permutations. Then $\hat{\gamma}$ contains all the letters $0, 1, 2, \dots, n$ as looser.*

We first prove the proposition assuming Lemma 4.5 and Lemma 4.6.

Proof of Proposition 4.4. Let ϕ be a pseudo-Anosov homeomorphism that fixes a separatrix adjacent to a regular point. By Veech's theorem ϕ is obtain by taking a closed loop in some Rauzy diagram. Using Rauzy Lemma stated in the proof of Proposition 4.3, we can show that the Rauzy diagram is necessarily $\mathcal{D}_r(\pi_n)$ (one can also use the main theorem of [Boi09]). Let γ be the corresponding closed primitive path in $\mathcal{D}_r(\pi_n)$. By Lemma 4.5, there is a lift of γ that starts and ends at central permutations. This lift contains all the letters as losers by Lemma 4.6, and by Proposition 4.2, the Perron-Frobenius eigenvalue of the corresponding matrix, and hence the dilatation of ϕ is at least 2. $\qquad\square$

The proof of the lemmas is strongly related to the geometry of the diagram $\mathcal{D}(\pi_n)$. Before giving a formal proof, we first present a very informal proof that uses the informal description of the diagram given in the beginning of Section 3.7.2.

(1) For the first lemma it is enough to prove that a primitive closed path in the labeled Rauzy diagram must pass through a central permutation. Such path must pass through a principal loop where the letter 0 or the letter n is winner. If it enters in a k-th secondary loop (attached to the k-th vertex of a principal loop), the geometry of the diagram imposes that it either leaves the secondary loop from the same vertex (and therefore joins the (k+1)-th vertex of the principal loop) or joins a k-th secondary loop in another leaf and escape it at the k-th vertex of another principal loop, and therefore joins the (k+1)-th vertex of this loop. Iterating this argument, it eventually joins a central permutation.

(2) The key observation for the second lemma is that a path joining two central permutations will either pass through all the vertices of a principal loop, or will pass from one principal loop to another one through a transition permutation and the corresponding pair of secondary loops. In any cases all the letters will appear as losers.

We now give a proof of the lemmas.

Proof of Lemma 4.5. Let γ be a primitive closed path in $\mathcal{D}_r(\pi_n)$, and let $\hat{\gamma}$ be a lift of γ in $\mathcal{D}(\pi_n)$. The path $\hat{\gamma}$ is not necessarily closed, but a power of γ admits a lift η which is closed and primitive. Furthermore, η consists of a concatenation of lifts of γ. By Proposition 3.2 the path η contains all the letter $0, \dots, n$ as winner, in particular the letter 0 as winner.

The point is that the letter 0 appears as winner only on the steps of the form $\mathcal{R}_b^{k-1}(\pi_n \circ \sigma^i) \to \mathcal{R}_b^k(\pi_n \circ \sigma^i)$. One can assume that $i = 0$. By Lemma 3.7, there exists in η the step $\mathcal{R}_b^k(\pi_n) \xrightarrow{0,*}$

$\mathcal{R}_b^{k+1}(\pi_n)$ or the step $\mathcal{R}_t^{k}(\pi_n \circ \sigma^j) \xrightarrow{n,*} \mathcal{R}_t^{k+1}(\pi_n \circ \sigma^j)$, where j corresponds to the index such that $\mathcal{R}_b \mathcal{R}_t^{k}(\pi_n \circ \sigma^j) = \mathcal{R}_t \mathcal{R}_b^{k}(\pi_n)$.

Since, 0 and n play a symmetric role, we can iterate the argument and therefore, we must reach $k = n$, which corresponds to a central permutation. Hence η contains a central permutation. Since η is a concatenation of lifts of γ, this path passes through the reduced permutation π_n. Since γ is closed, we can assume that it starts and ends at π_n. Hence, any lift of γ has endpoints that are in the preimage of π_n, which are the central permutations. Lemma 4.5 is proven. □

Proof of Lemma 4.6. Let $\hat{\gamma}$ be a path in $\mathcal{D}(\pi_n)$ connecting two central permutations. For simplicity, assume that $\hat{\gamma}(1)$ is π_n. Again, by symmetry, one can assume that the first arrow is given by the map $\mathcal{R}_t$, *i.e.* $\hat{\gamma}(1) \xrightarrow{n,0} \hat{\gamma}(2) = \mathcal{R}_t(\pi_n)$. By Lemma 3.7 there exists $i_2, \ldots, i_{n-1}$ such that the path $\hat{\gamma}$ contains the steps $\mathcal{R}_{\varepsilon_2}^{2}(\pi_n \circ \sigma^{i_2}) \to \mathcal{R}_{\varepsilon_2}^{3}(\pi_n \circ \sigma^{i_2}), \ldots, \mathcal{R}_{\varepsilon_{n-1}}^{n-1}(\pi_n \circ \sigma^{i_{n-1}}) \to \mathcal{R}_{\varepsilon_{n-1}}^{n}(\pi_n \circ \sigma^{i_n})$.

The first possibility is that $\hat{\gamma}$ does not change of leaf, *i.e.* for all $k \in \{2, \ldots, n-1\}$, $i_k = 0$. Then the steps previously written form a subpath that can be rewritten as $\hat{\gamma}(1) \xrightarrow{n,0} \hat{\gamma}(j_2) \xrightarrow{n,1} \hat{\gamma}(j_3) \xrightarrow{n,2} \ldots \xrightarrow{n,n-1} \hat{\gamma}(1)$. Then all the letters $0, \ldots, n-1$ are losers. The path $\hat{\gamma}$ is closed, so n cannot always be winner. Hence it is also looser.

The second possibility is that $\hat{\gamma}$ changes a leaf, *i.e.* there exists a smallest k such that $i_k > 0$. Then by Lemma 3.7, a subpath of γ is obtained by the following way (compare with Figure 3):

- We start from π_n.
- We apply k times the Rauzy move $\mathcal{R}_t$ (these are the moves $\hat{\gamma}(1) \xrightarrow{n,0} \hat{\gamma}(j_2) \xrightarrow{n,1} \ldots \xrightarrow{n,k-1} \hat{\gamma}(j_{k+1}) = \mathcal{R}_t^{k}(\pi_n))$.
- We apply once the Rauzy move $\mathcal{R}_b$ (we reach the permutation $\mathcal{R}_b \mathcal{R}_t^{k}(\pi_n) = \mathcal{R}_t \mathcal{R}_b^{k}(\pi_n \circ \sigma^{i_k})$). This is the move $\gamma(j_{k+1}) \xrightarrow{k,n} \hat{\gamma}(j_{k+2})$.
- We apply $k' = n - k$ times the Rauzy move $\mathcal{R}_t$ (until we get the permutation $\mathcal{R}_b^{k}(\pi_n \circ \sigma^{i_k})$). These are the moves $\hat{\gamma}(j_{k+2}) \xrightarrow{1,k} \hat{\gamma}(j_{k+3}) \xrightarrow{1,k+1} \ldots \xrightarrow{1,n-1} \hat{\gamma}(j_{n+2})$.

We see that all the letters are losers. This proves Lemma 4.6. □

APPENDIX A. EXAMPLES IN HYPERELLIPTIC COMPONENTS

In this appendix we show that the uniform lower bound on dilatations of pseudo-Anosov homeomorphisms in Theorem 1.1 is sharp by constructing suitable examples. This will thus give a proof of Theorem 1.2.

A.1. Hyperelliptic connected component $\mathcal{H}^{hyp}(2g - 2)$.

Proposition A.1. *Let $g \geq 2$. There exists a pseudo-Anosov homeomorphism ϕ_g affine on a translation surface in $\mathcal{H}^{hyp}(2g-2)$ whose dilatation is the Perron root of the polynomial $X^{2g+1} - 2X^{2g-1} - 2X^2 + 1$. This dilatation satisfies*

$$0 < \theta(\phi_g) - \sqrt{2} < \frac{1}{2^{g-1}}.$$

Lemma A.2. *The polynomial $P_g = X^{2g+1} - 2X^{2g-1} - 2X^2 + 1$ admits a unique real root α greater than $\sqrt{2}$.*

Proof. $P_g(\sqrt{2}) = -4 + 1 < 0$ and $P_g'(x) > 0$ for $x > \sqrt{2}$. □

Lemma A.3. *Let $M = [m_{ij}]$ be the matrix of dimension $2g$ defined by:*

- *for all $j \in \{g+1, \ldots, 2g\}$, $m_{1,j} = 1$, and $m_{1,g} = 2$,*
- *for all $i \in \{2, \ldots, g\}$, $m_{i,i+g-1} = 1$,*
- *for all $j \in \{g+1, \ldots, 2g-1\}$, $m_{i,i-g} = 1$,*
- *$m_{2g,g} = m_{2g,2g} = 1$.*
- *all the other elements are zero.*

$$
M_g = \left(
\begin{array}{c|ccccc|c}
 & 2 & 1 & \cdots & \cdots & 1 & 1 \\
 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0_{g \times g-1} & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
 & \vdots & & \ddots & \ddots & 0 & \vdots \\
 & 0 & \cdots & \cdots & 0 & 1 & 0 \\
\hline
 & & & & & & 0 \\
I_{g-1 \times g-1} & & 0_{g-1 \times g} & & & & \vdots \\
 & & & & & & \vdots \\
 & & & & & & 0 \\
\hline
0 & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}
\right)
$$

Then M_g has α and $1/\alpha$ as eigenvalues where α is defined by the previous lemma.

Proof. We must compute the characteristic polynomial $\chi_g = \det(M_g - xI_{2g \times 2g})$. Since P_g is reciprocal, *i.e.* $x^{\deg(P_g)} P_g(x^{-1}) = P_g(x)$, and $\alpha \neq -1$, we just need to show that $(x+1)\chi_g = P_g$ for all $g \geq 2$. We denote by L_i the $i-th$ line of $M_g - xI_{2g \times 2g}$. For each $i \in \{1, \ldots, g\}$ we replace L_i by $L_i + xL_{i+g}$. Then we develop the determinant g times along the first column, to get:

$$
\begin{aligned}
\det(M_g - xI) &= (-1)^{(g+2)(g-1)} \left(2(1-x) + (-1)^{g+2} \left((-1)^{g+1}.1 + x(1-x)D_{g-1} \right) \right), \\
&= 2(1-x) + (-1) + (-1)^g x(1-x)D_{g-1},
\end{aligned}
$$

were D_{g-1} is the $(g-1) \times (g-1)$ determinant:

$$
D_{g-1} = \begin{vmatrix}
1-x^2 & 1 & 1 & \cdots & \cdots & 1 \\
1 & -x^2 & 0 & \cdots & \cdots & 0 \\
0 & 1 & -x^2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & 0 & 1 & -x^2 & 0 \\
0 & \cdots & \cdots & 0 & 1 & -x^2
\end{vmatrix}.
$$

Developing D_g on the last column, we obtain $D_g = -x^2 D_{g-1} + (-1)^{g+1}$. Then using this last expression, we see that:

$$
(x+1)\left(\chi_{g+1} - x^2\chi_g\right) = (x+1)(1 - x - 2x^2 + 2x^3) = (2x^4 - 3x^2 + 1) = P_{g+1} - x^2 P_g
$$

Since $(1+x)\chi_2 = P_2$, we deduce that $(1+x)\chi_g = P_g$ for all $g \geq 2$, which proves the proposition. $\quad\square$

Lemma A.4. *Let λ (respectively, τ) be an eigenvector of M for the eigenvalue α (respectively, $1/\alpha$). Then, up to replacing λ or τ by its opposite, (λ, τ) defines a suspension datum for the permutations π and π', with:*

$$\pi = \begin{pmatrix} 1 & 2 & \dots & \dots & \dots & 2g-1 & 2g \\ 2g & g & \dots & 1 & 2g-1 & \dots & g+1 \end{pmatrix} \quad and \quad \pi' = \begin{pmatrix} g+1 & \dots & \dots & 2g-1 & 1 & \dots & g & 2g \\ 2g & 2g-1 & \dots & \dots & \dots & \dots & 2 & 1 \end{pmatrix}$$

Moreover (π, λ, τ) and (π', λ, τ) define the same two surfaces in the moduli space.

Proof. We have $M_g \lambda = \alpha \lambda$ and $M_g \tau = \frac{1}{\alpha} \tau$. This gives:

(9) $$\forall i \in \{1, \dots, 2g-2\} \setminus \{g\} \quad \lambda_i = \alpha^2 \lambda_{i+1} \quad and \quad \tau_i = \frac{1}{\alpha^2} \tau_{i+1}$$

(10) $$\lambda_{2g-1} = \alpha \lambda_g \quad and \quad \tau_{2g-1} = \frac{1}{\alpha} \tau_g$$

(11) $$\lambda_g + \lambda_{2g} = \alpha \lambda_{2g} \quad and \quad \tau_g + \tau_{2g} = \frac{1}{\alpha} \tau_{2g}$$

Since $\alpha > 1$, Equation (11) implies that λ_g is of the same sign as λ_{2g}. The other equations imply that all the λ_i are of the same sign. Hence all the λ_i are positive if we choose $\lambda_{2g} > 0$.

By a similar argument, for τ, we see that we can choose $\tau_{2g} < 0$ and have $\tau_i > 0$ for all $i < 2g$. Furthermore, the label "2g" is the last of the first line of π (respectively, π') and the first of the second line of π (respectively, π'). Then in order to prove that (λ, τ) is a suspension data for π and π', it is enough to show that $s = \sum_{i=1}^{2g} \tau_i$ is negative. By Equations (9) and (10), we have

$$s = \tau_{2g} + \sum_{k=0}^{2g-2} \frac{1}{\alpha^k} \tau_g = \frac{1}{\alpha} \tau_{2g} + \sum_{k=1}^{2g-2} \frac{1}{\alpha^k} \tau_g = \dots = \frac{1}{\alpha^{2g-1}} \tau_{2g} < 0.$$

Hence we have proven that (π, λ, τ) and (π', λ, τ) define translation surfaces by the Veech's construction (see Section 3.2). Now we remark that, if P_1 is the polygon associated to (π, λ, τ), and P_2 is the polygon associated to (π', λ, τ), then we obtain P_2 from P_1 by a $180°$ rotation. Since these flat surfaces are hyperelliptic we conclude that the two surfaces defined by (π, λ, τ) and (π', λ, τ) are isometric. $\square$

Lemma A.5. *There is a pseudo-Anosov homeomorphism on the surface defined by the data (π, λ, τ) with dilatation α.*

Proof. We start from π', and consider the path γ in the corresponding Rauzy graph obtained by applying g times the map $\mathcal{R}_b$ to π' and once the map $\mathcal{R}_t$. We obtain the permutation

$$\pi'' = \begin{pmatrix} g+1 & \dots & \dots & 2g-1 & 1 & \dots & g & 2g \\ 2g & 1 & 2g-1 & \dots & \dots & \dots & \dots & 2 \end{pmatrix}.$$

The sequence of winners/losers is $(1,2g),(1,g),\ldots,(1,2),(2g,1)$. And therefore, the corresponding transition matrix is:

$$M'_g = \left(\begin{array}{ccccc|c|c}
2 & 1 & \cdots & \cdots & 1 & & 1 \\
0 & 1 & 0 & \cdots & 0 & & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & 0_{g\times g-1} & \vdots \\
\vdots & & \ddots & \ddots & 0 & & \vdots \\
0 & \cdots & \cdots & 0 & 1 & & 0 \\
\hline
& & & & & & 0 \\
& & & & & & \vdots \\
& & 0_{g-1\times g} & & & I_{g-1\times g-1} & \vdots \\
& & & & & & \vdots \\
& & & & & & 0 \\
\hline
1 & 0 & \cdots & \cdots & 0 & 0 & 1
\end{array} \right)$$

The permutation π'' is obtained from π after renumbering. Multiplying M' by the corresponding permutation matrix, one get precisely the matrix M_g. Let us apply $g+1$ times the Rauzy-Veech induction on (π',λ,τ) to obtain after renumbering $(\pi,\frac{1}{\alpha}\lambda,\alpha\tau)$. But the two surfaces $X(\pi,\frac{1}{\alpha}\lambda,\alpha\tau)$ and $X(\pi',\frac{1}{\alpha}\lambda,\alpha\tau)$ are the same surface by above lemma. Therefore $X(\pi',\frac{1}{\alpha}\lambda,\alpha\tau)$ and $X(\pi',\lambda,\tau)$ define also the same surface in the moduli space, hence $X(\pi',\lambda,\tau)$ admits a pseudo-Anosov homeomorphism with dilatation α. $\qquad\square$

Proof of the proposition. The existence of the pseudo-Anosov homeomorphism ϕ_g is clear from the previous lemmas. We have $\alpha = \theta(\phi_g)$, and:

$$\alpha^{2g+1} - 2\alpha^{2g-1} - 2\alpha^2 + 1 = \alpha^2\left(\alpha^{2g-1} - 2\alpha^{2g-3} - 2\right) + 1 = 0,$$

thus

$$\alpha^{2g-1} - 2\alpha^{2g-3} - 2 = \alpha^{2g-3}\left(\alpha^2 - 2\right) - 2 < 0.$$

Since $\alpha > \sqrt{2}$ we obtain $0 < \alpha - \sqrt{2} < \frac{2}{(\alpha+\sqrt{2})(\alpha^{2g-3})} < \frac{1}{2^{g-1}}$.

$\qquad\square$

A.2. Hyperelliptic connected component $\mathcal{H}^{hyp}(g-1,g-1)$.

Proposition A.6. *Let $g \geq 2$. There exists a pseudo-Anosov homeomorphism on a translation surface in $\mathcal{H}^{hyp}(g-1,g-1)$ with dilatation the Perron root of the polynomial:*

$$\begin{array}{ll}
X^{2g+2} - 2X^{2g} - 2X^{g+1} - 2X^2 + 1, & \text{if g is even,} \\
X^{2g+2} - 2X^{2g} - 4X^{g+2} + 4X^g + 2X^2 - 1, & \text{if g is odd.}
\end{array}$$

The dilatation satisfies

$$0 < \theta(\phi_g) - \sqrt{2} < \frac{4}{\sqrt{2}^g}$$

Proof. The idea of the proof is very similar to the previous one. We just present here the corresponding paths in the Rauzy diagram.

- If g is even, we start from the permutation $\pi = \begin{pmatrix} 1 & \cdots & g+1 & g+2 & \cdots & 2g & 2g+1 \\ 2g+1 & g+1 & \cdots & 1 & 2g & \cdots & g+2 \end{pmatrix}$. As previously, we consider the "reverse" permutation $\pi' = \begin{pmatrix} g+2 & \cdots & 2g & 1 & \cdots & g+1 & 2g+1 \\ 2g+1 & 2g & \cdots & g+2 & g+1 & \cdots & 1 \end{pmatrix}$. Then the path in the Rauzy diagram starting from π' and defined by $(g+1)$ times b, two times t. One gets the permutation $\pi'' = \begin{pmatrix} g+2 & \cdots & 2g & 1 & 2 & 3\cdots & g+1 & 2g+1 \\ 2g+1 & 1 & 2 & 2g & \cdots & g+2 & g+1 & \cdots & 3 \end{pmatrix}$, which is a renumbering of π, and the associated matrix is:

$$N_g = \left(\begin{array}{c|ccccc|c} & 2 & 2 & 1 & \cdots & 1 & 1 \\ & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0_{g+1 \times g-1} & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & \vdots & & \ddots & \ddots & 0 & \vdots \\ & 0 & \cdots & \cdots & 0 & 1 & 0 \\ \hline & & & & & & 0 \\ & & & & & & \vdots \\ I_{g-1 \times g-1} & & 0_{g-1 \times g+1} & & & \vdots \\ & & & & & & 0 \\ \hline 0 & 1 & 1 & 0 & \cdots & \cdots & 0 & 1 \end{array} \right)$$

One can check that $(X+1)\chi_{N_g}(X) = X^{2g+2} - 2X^{2g} - 2X^{g+1} - 2X^2 + 1$, and that, for α the Perron root of $\chi_{N_g}(X)$, the eigenvectors of N_g corresponding to α and $\frac{1}{\alpha}$ define a suspension data for π (and π'). Then the corresponding translation surface admits the required pseudo-Anosov homeomorphism by construction.

Also, we have

$$\alpha^{2g}(\alpha^2 - 2) < 2\alpha^{g+1} + 2\alpha^2$$

Thus,

$$\alpha - \sqrt{2} < \frac{2}{\alpha + \sqrt{2}} \left(\frac{1}{\alpha^{g-1}} + \frac{1}{\alpha^{2g-2}} \right) < \frac{2}{\sqrt{2}^g},$$

since $\alpha > \sqrt{2}$.

- If g is odd, we consider π as previously and we take the path defined by g times b, then $t - b - t - t$. We obtain the required pseudo-Anosov with the same construction we presented have above. Also, we have:

$$\alpha - \sqrt{2} = \frac{1}{\alpha + \sqrt{2}} \frac{4\alpha^g(\alpha^2 - 1) + 1 - 2\alpha^2}{\alpha^{2g}} < \frac{4}{\alpha^g} < \frac{4}{\sqrt{2}^g}$$

$\square$

APPENDIX B. EXAMPLES IN NON HYPERELLIPTIC COMPONENTS

In this appendix we motivate assumptions of Theorem 1.1. We show that if we relax the condition on the hyperelliptic involution, one can construct pseudo-Anosov homeomorphisms in the non hyperelliptic connected components whose dilatations tend to 1 when the genus tends to infinity. This will thus give a proof of Theorem 1.3.

We will construct a sequence of pseudo-Anosov homeomorphisms φ_g on the flat sphere i.e. φ_g is affine with respect to a quadratic differential q_g on the sphere. The construction involves the Rauzy-Veech induction for quadratic differentials [BL09], using *generalized permutations*.

Proposition B.1. *For each $g \geq 3$, there exists a pseudo-Anosov homeomorphism φ_g on the flat sphere with $2g$ poles and two zeroes of order $g - 2$, fixing a pole, and having for dilatations $\theta(\varphi_g)$ the Perron root of the polynomial*

$$P = X^{2g} - X^{2g-1} - 4X^g - X + 1.$$

Proof of the proposition. We consider the Rauzy diagram corresponding to the stratum of quadratic differentials on the sphere having the given singularity data. It is sufficient to give a closed path in this diagram and to check that the renormalization matrix is irreducible and has θ_g for eigenvalue, where θ_g is the Perron root of P.

Let us consider the following generalized permutation (on $2g + 1$ letters)

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 & \cdots & g+1 & g+1 \\ g+2 & g+3 & g+3 & \cdots & 2g+1 & 2g+1 & g+2 & 1 \end{pmatrix}.$$

For instance for $g = 3$ this gives

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 5 & 6 & 6 & 7 & 7 & 5 & 1 \end{pmatrix}.$$

The Rauzy path we will consider is $t - b - t - b - t - b$. A simple calculation shows that the renormalization matrix $\widehat{V}$ (for the labeled permutation) and permutation matrix P are, respectively

$$\widehat{V} = \begin{pmatrix} 2 & & & & & & 1 \\ & 1 & & & & & \\ & & \ddots & & & & \\ 2 & & & 1 & 1 & & 2 \\ & & & 1 & 2 & & \\ & & & & & \ddots & \\ 1 & & & & 1 & & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & & & \\ \vdots & & I_{2g \times 2g} & \\ 0 & & & \\ \hline 1 & 0 & \cdots & 0 \end{pmatrix},$$

where the "2" in the diagonal of the matrix $\widehat{V}$ appears at the position $g + 2$. All the other entries in $\widehat{V}$ are zeroes. Thus the renormalization matrix (for the reduced permutations) is $V = \widehat{V} \cdot P$. One can show that the matrix V is irreducible and its characteristic polynomial is

$$(X - 1)(X^{2g} - X^{2g-1} - 4X^g - X + 1) = (X - 1)P(X).$$

Then as in the previous section, we see that some eigenvectors of V define suspension data, and the corresponding flat surface has a pseudo-Anosov homeomorphism whose dilatation is the Perron root of P. □

Finally we have the following corollary, which justifies the assumptions of Theorem 1.1 on the hyperelliptic involution

Corollary B.2. *For $g \geq 3$ odd, there exists an affine pseudo-Anosov homeomorphism ϕ_g on a translation surface (X, ω) with the following properties:*

(1) $(X, \omega) \in \mathcal{H}^{odd}(g-1, g-1)$,
(2) (X, ω) *is hyperelliptic,*
(3) ϕ_g *fixes a separatrix on* (X, ω) *(issued from a Weierstrass point),*
(4) *the dilatation of* ϕ_g *is the Perron root of the polynomial*

$$X^{2g} - X^{2g-1} - 4X^g - X + 1.$$

In particular

$$\lim_{g \to +\infty} \theta(\phi_g) = 1.$$

Remark B.1. *Obviously the hyperelliptic involution on X fixes the two zeroes of* ω, *and* (X, ω) *is not in the hyperelliptic connected component of the corresponding stratum.*

Proof of the corollary. For g odd, the examples given by Proposition B.1 lift to pseudo-Anosov homeomorphisms ϕ_g on the orientating cover $(X, \omega) \in \mathcal{H}(g-1, g-1)$. By construction the surface is hyperelliptic and since the poles are ramification points, there is one lift that fixe a regular (Weierstrass) point and the separatrix issued from that point.

To identify the connected component, we use the formula in [Lan04bis]. Since $(X, \omega) \to (\mathbb{P}^1, q)$ is the orientating cover, the spin structure determined by (X, ω) is

$$\left[\frac{|n_{+1} - n_{-1}|}{4} \right] \qquad \mod 2,$$

where n_{+1} is the number of singularities of q of degrees $1 \mod 4$ and n_{-1} is the number of singularities of q of degrees $-1 \mod 4$.

(1) If $g = 1 \mod 4$: then $g - 2 = -1$ modulo 4, thus $n_{+1} = 0$ and $n_{-1} = 2g + 2$. Hence the parity of the spin structure on (X, ω) is (with $g = 1 + 4k$)

$$\left[\frac{|n_{+1} - n_{-1}|}{4} \right] = \frac{2g+2}{4} = 1 + 2k = 1 \mod 2.$$

(2) If $g = -1 \mod 4$: then $g - 2 = 1$ modulo 4, thus $n_{+1} = 2$ and $n_{-1} = 2g$. Hence the parity of the spin structure on (X, ω) is (with $g = -1 + 4k$)

$$\left[\frac{|n_{+1} - n_{-1}|}{4} \right] = \frac{2g-2}{4} = -1 + 2k = 1 \mod 2.$$

$\square$

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