MODULI SPACE OF MEROMORPHIC DIFFERENTIALS WITH MARKED HORIZONTAL SEPARATRICES

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Abstract. We study framed translation surfaces corresponding to meromorphic differentials on compact Riemann surfaces, for which a horizontal separatrix is marked for each pole or zero. Such geometric structures naturally appear when studying flat geometry surfaces "near" the Deligne-Mumford boundary.

We compute the number of connected components of the corresponding strata, and give a simple topological invariant that distinguishes them. In particular we see that for $g > 0$, there are at most two such components, except in the hyperelliptic case.

1. Introduction

A nonzero holomorphic one-form (Abelian differential) on a compact Riemann surface naturally defines a flat metric with conical singularities on this surface. Geometry and dynamics on such flat surfaces, in relation to geometry and dynamics on the corresponding moduli space of Abelian differentials is a very rich topic and has been widely studied in the last 30 years. It is related to interval exchange transformations, billards in polygons, Teichmüller dynamics.

A non-compact translation surface corresponds to a one-form on a non-compact Riemann surface. The dynamics and geometry on some special cases of non-compact translation surfaces have been studied more recently.

In [3], we have investigated the case of translation surfaces that come from meromorphic differentials defined on compact Riemann surfaces. In this case, we obtain non-compact translation surfaces with infinite

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area. Such structures naturally appear when studying compactifications of strata of the moduli space of Abelian differentials. For instance, Eskin, Kontsevich and Zorich [4], based on results of Rafi [9], showed that when a sequence of Abelian differentials (X_i, ω_i) converges to a boundary point in the Deligne-Mumford compactification, then subsets $(Y_{i,j}, \omega_{i,j})$ corresponding to thick components of the X_i, after suitable rescaling converge to meromorphic differentials (see [4], Theorem 10). Similar results were independently proved by Grushevsky and Krichever [5], by Koch and Hubbard [6] and by Smillie. See also [1].

In this paper, a meromorphic differential on a compact Riemann surface will be called *translation surface with poles*, or simply translation surface when there is no confusion with the usual (compact) translation surfaces.

This work was suggested to the author by Smillie, as a step in a project of constructing a geometric compactification of the strata of the moduli space of Abelian differentials by using only flat geometry. A (compact) translation surface "near" the boundary, should be seen as a collection of translation surfaces with poles, glued together suitably after cutting out a neighborhood of a collection of singularities (including all the poles, in order to obtain in the end a compact translation surface). However, the gluing operation requires some extra combinatorial data, that can be expressed in terms of a "frame" on the translation surfaces with poles.

As in [2], a framed translation surface is a translation surface with a choice, for each singularity of a horizontal separatrix (see Section 3 for a precise definition). When the singularity is a conical singularity (*i.e.*, a zero of the corresponding one-form), it corresponds to a horizontal separatrix. When the singularity corresponds to a non-simple pole, it corresponds to an equivalence class of horizontal geodesics going to infinity for the flat metric. A singularity of degree $n \in \mathbb{Z}$ will have $|n+1|$ possible choices of horizontal separatrices. Such a framed translation surface will be also called a translation surface with marked horizontal separatrices.

The number of connected components of the moduli space of framed (compact) translation surfaces was computed by the author in [2]. In this paper, we answer the same question for the moduli space of framed translation surfaces with poles.

The first theorem deals with the case of nonhyperelliptic connected components in genus at least 1.

Theorem 1.1. *Let $g \geq 1$. Let $\mathcal{H}$ be a stratum of the moduli space of genus g meromorphic differentials, and $\mathcal{C} \subset \mathcal{H}$ be a nonhyperelliptic connected component. Let $\mathcal{H}_{\mathcal{C}}^{hor}$ be the moduli space of translation surfaces in $\mathcal{C}$ with marked horizontal separatrices. We assume that the set of poles does not consists of a pair of simple poles. We have:*

- *If there exists a simple pole, or if there are only even degree singularities, then $\mathcal{H}_{\mathcal{C}}^{hor}$ is connected.*
- *Otherwise, $\mathcal{H}_{\mathcal{C}}^{hor}$ has two connected components that are distinguished by the invariant Sp defined in Section 5.1.*

When the set of poles consists of a pair of simple poles, we have the following result.

- *If there are only even degree zeroes, then $\mathcal{H}_{\mathcal{C}}^{hor}$ is connected.*
- *Otherwise, $\mathcal{H}_{\mathcal{C}}^{hor}$ has two connected components that are distinguished by the invariant Sp defined in Section 5.1.*

The topological invariant Sp that distinguishes the connected components $\mathcal{H}_{\mathcal{C}}^{hor}$ is a variation of the classical Arf invariant for moduli space of Abelian differentials, and is therefore easily computable in terms of the flat structure.

The case of hyperelliptic connected components is easy and studied in Section 5.3. In this case, there are more connected components for $\mathcal{H}_{\mathcal{C}}^{hor}$ due to the extra symmetry of the surfaces.

The genus zero case is particular: there might be many more components, as described in the following theorem.

Theorem 1.2. *Let $\mathcal{H} = \mathcal{H}(n_1, \dots n_r)$ be a stratum of genus zero translation surfaces. Let $\mathcal{H}^{hor}$ be the moduli space of translation surfaces in $\mathcal{H}$ with marked horizontal separatrices. Let*

$$N = \prod_{i,j} \gcd\left(\{n_k\}_{k \notin \{i,j\}} \cup \{n_i + 1, n_j + 1\}\right)$$

- *If there exists $i \in \{1, \dots, r\}$ such that $n_i = -1$, then $\mathcal{H}^{hor}$ is connected.*
- *If all n_i are different from -1 and if there are at most two odd degree singularities, then there are N connected components of $\mathcal{H}^{hor}$ that are distinguished by the invariant Φ defined in Section 6.*
- *Otherwise, there are $2N$ connected components of $\mathcal{H}^{hor}$ that are distinguished by the invariant (Φ, Sp).*

The topological invariant Φ in the above theorem is easily computable in terms of the flat structure. The idea is to look at indices of the Gauss map modulo relevant integers for a certain collection of paths.

Structure of the paper. The paper is organized as follows:

- Section 2 is devoted to generalities and background about translation surfaces with poles. The classification theorem of the connected components of moduli space of meromorphic differentials by the author is recalled, and few important statements about the structure of these connected components. We end with the proof of a preliminary result about the existence, in each connected component, of a surface with a pole of prescribed degree and zero residue.
- Section 3 gives the precise definition of the moduli space of framed meromorphic differentials, and reduces the problem to the computation of the index of a subgroup H of a product of cyclic groups.
- Section 4 describes paths in the underlying stratum that produces some particular elements in H that will be ultimately proven to be the generators of H. One key step there is to show that these elements exist for each connected component of each stratum.
- Section 5 defines first a topological invariant for the positive genus case, then proves Theorem 1.1.
- Section 6 defines a topological invariant for the zero genus case, then proves Theorem 1.2.

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2. Preliminaries

2.1. **Holomorphic one-forms and flat structures.** Let X be a Riemann surface and let ω be a holomorphic one-form. For each $z_0 \in X$ such that $\omega(z_0) \neq 0$, integrating ω in a neighborhood of z_0 gives local coordinates whose corresponding transition functions are translations, and therefore X inherits a flat metric, on $X \backslash \Sigma$, where Σ is the set of zeroes of ω.

In a neighborhood of an element of Σ, such metric admits a conical singularity of angle $(k+1)2\pi$, where k is the order of the corresponding zero of ω. Indeed, a zero of order k is given locally, in suitable coordinates by $\omega = (k+1)z^k dz$. This form is precisely the pre-image of the constant form dz by the ramified covering $z \to z^{k+1}$. In terms of the flat metric, it means that the flat metric defined locally by a zero of order k appears as a connected covering of order $k+1$ over a flat disk, ramified at zero.

When X is compact, the pair (X, ω), seen as a smooth surface with such translation atlas and conical singularities, is usually called a *translation surface*.

If ω is a meromorphic differential on a compact Riemann surface $\overline{X}$, we can consider the translation atlas defined by ω on $X = \overline{X} \backslash \Sigma'$, where Σ' is the set of poles of ω. We obtain a translation surface with infinite area. We will call such a surface a *translation surface with poles*, or simply a *translation surface*.

Convention 2.1. *When speaking of a translation surface with poles $S = (X, \omega)$: the surface S equipped with the flat metric is noncompact; the underlying Riemann surface X is a punctured surface and ω is a holomorphic one-form on X; the corresponding closed Riemann surface is denoted by $\overline{X}$, and ω extends to a meromorphic differential on $\overline{X}$ whose set of poles is precisely $\overline{X} \backslash X$.*

As in the case of Abelian differentials, a *saddle connection* is a geodesic segment that joins two conical singularities (or a conical singularity to itself) with no conical singularities on its interior.

We fix some terminology, that we will use during this paper.

- The order, or degree of a zero of ω is defined as usual. The cone angle at a zero of degree n is $2\pi(n+1)$.
- The order of a pole of ω is defined as usual. It is a *positive* integer.
- A singularity of (X, ω) is a zero or a pole of ω. By convention, the *degree* of the singularity P will correspond to its order if P is a zero, or the opposite of its order if P is a pole. For instance, a pole of order 2 corresponds to a singularity of degree -2. We denote by $\deg(P) \in \mathbb{Z}$ the degree of P.

With the above convention, we recall that it is well known that $\sum_{i=1}^{r} n_i = 2g - 2$, where $\{n_1, \dots, n_r\}$ is the set (with multiplicities) of the degree of the singularities of (X, ω).

2.2. **Local model for poles.** The neighborhood of a pole in $\overline{X}$ of order one is an infinite cylinder with one end. Indeed, up to rescaling, the pole is given in local coordinates by $\omega = \frac{1}{z} dz$. Writing $z = e^{z'}$, we have $\omega = dz'$, and z' is in an infinite cylinder.

Now we describe the flat metric in a neighborhood of a pole in $\overline{X}$ of order $k \geq 2$ (see also [10, 3]). First, consider the meromorphic 1-form on $\mathbb{C} \cup \{\infty\}$ defined on $\mathbb{C}$ by $\omega = z^k dz$. Changing coordinates

$w = 1/z$, we see that this form has a pole P of order $k+2$ at ∞, with zero residue. In terms of the translation structure, a neighborhood of the pole is obtained by taking an infinite cone of angle $(k+1)2\pi$ and removing a compact neighborhood of the conical singularity. Since the residue is the only local invariant for a pole of order k, this gives a local model for a pole with zero residue.

Now, define $U_R = \{z \in \mathbb{C} \,|\, |z| > R\}$ equipped with the standard flat metric. Let V_R be the Riemann surface obtained after removing from U_R the π–neighborhood of the real half line $\mathbb{R}^-$, and identifying by the translation $z \to z + \imath 2\pi$ the lines $-\imath\pi + \mathbb{R}^-$ and $\imath\pi + \mathbb{R}^-$. The surface V_R is naturally equipped with a holomorphic one-form ω coming from dz on V_R. We will show that this one-form has a pole of order 2 at infinity and residue -1. Start from the one-form on $U_{R'}$ defined by $(1+1/z)dz$ and integrate it. Choosing the usual determination of $\ln(z)$ on $\mathbb{C}\backslash\mathbb{R}^-$, one gets the map $z \to z + \ln(z)$ from $U_{R'}\backslash\mathbb{R}^-$ to $\mathbb{C}$, which extends to an injective holomorphic map f from $U_{R'}$ to V_R, if R' is large enough. We claim that f is also surjective around infinity, *i.e.* for $Z \in V_R$ with large enough modulus, there exists $z \in U_{R'}$ with $f(z) = Z$. Indeed, considering Z as an element of $\mathbb{C}$, we consider the map $g(z) = Z - \ln(z)$ which is contracting. Choosing $\rho = 2\ln(|Z|)$ we consider the ball B of center Z and radius ρ. If B intersects $\mathbb{R}^-$, we consider $B^+ = B \cap \{Im(z) > 0\}$ if $Im(Z) > 0$ (or $B^- = B \cap \{Im(z) < 0\}$ if $Im(Z) < 0$). Then, if $|Z|$ is greater than a constant large enough, we see that $g(B) \subset B$ ($g(B^\pm) \subset B^\pm$ in the relevant cases), hence there is a fixed point of g in $\overline{B}$ and therefore an element z satisfying $z + \ln(z) = Z$.

Furthermore, the pullback by f of the form ω on V_R gives $\omega' = (1+1/z)dz$. Then, the change of coordinate $w = 1/z$ gives us that (U_R, ω') has a pole of order two at infinity with residue -1. Hence it is also the case for (V_R, ω).

Let $k \geq 2$. The pullback of the form $(1+1/z)dz$ by the map $z \to z^{k-1}$ gives $((k-1)z^{k-2} + (k-1)/z)dz$, *i.e.* we get at infinity a pole of order k with residue $-(k-1)$. In terms of the flat metric, a neighborhood of a pole of order k and residue $-(k-1)$ is just the natural cyclic $(k-1)$–covering of V_R. Then, suitable rotation and rescaling gives the local model for a pole of order k with a nonzero residue.

2.3. **Moduli space.** If (X, ω) and (X', ω') are such that there is a biholomorphism $f : X \to X'$ with $f^*\omega' = \omega$, then f is an isometry for the metrics defined by ω and ω'. Even more, for the local coordinates defined by ω, ω', the map f is in fact a translation.

As in the case of Abelian differentials, we consider the moduli space of meromorphic differentials, where $(X,\omega) \sim (X',\omega')$ if there is a biholomorphism $f : X \to X'$ such that $f^*\omega' = \omega$. A stratum corresponds to prescribed degree of zeroes and poles. We denote by $\mathcal{H}(n_1^{\alpha_1},\dots,n_r^{\alpha_r})$ the *stratum* that corresponds to meromorphic differentials with α_i singularities of degree n_i. Such stratum is nonempty if and only if $\sum_{i=1}^{r} \alpha_i n_i = 2g-2$ for some integer $g \geq 0$ and if there is not just one simple pole.

We define the topology on this space in the following way: a small neighborhood of S, with conical singularities Σ, is defined to be the equivalence classes of surfaces S' for which there is a differentiable injective map $f : S\backslash V(\Sigma) \to S'$ such that $V(\Sigma)$ is a (small) neighborhood of Σ, Df is close the identity in the translation charts, and the complement of the image of f is a union of disks. One can easily check that this topology is Hausdorff.

2.4. **Connected components of the moduli space of meromorphic differentials.** The connected components of the moduli space of meromorphic differentials were classified by the author in [3]. Here we recall this classification, and state some technical facts that appear in the proof, and that are necessary for this paper. First, recall the well known fact that any stratum of genus zero meromorphic differentials is connected since it corresponds more or less to a moduli space of marked points on the sphere.

Let γ be a simple closed curve parametrized by the arc length on a translation surface that avoids the singularities. Then $t \to \gamma'(t)$ defines a map from $\mathbb{S}^1$ to $\mathbb{S}^1$. We denote by $Ind(\gamma)$ the index of this map.

Assume that the surface has genus one. Let (a,b) be a pair of closed curves representing a symplectic basis of the homology of S, then we define the *rotation number* of S as

$$rot(S) = \gcd(Ind(a), Ind(b), n_1, \dots n_r, p_1, \dots, p_s)$$

where $n_1,\dots,n_r$ are the order of zeroes of S and $p_1,\dots,p_s$ are the degree of poles of S. We can show that it does not depend on the choice of (a,b) and hence is an invariant of connected components. We have the following result.

Theorem 2.2. *Let $\mathcal{H}(n_1,\dots,n_r,p_1,\dots,p_s)$, with $n_i > 0$, $p_j < 0$ and $\sum_j p_j < -1$ be a stratum of genus one meromorphic differentials. Let d be a positive divisor of $N = \gcd(n_1,\dots,n_r,p_1,\dots,p_s)$. There is a unique connected component of $\mathcal{H}(n_1,\dots,n_r,p_1,\dots,p_s)$ with rotation number d, except when $r = s = 1$ and $d = N$, in which case such a component does not exists.*

A translation surface $S = (X, \omega)$ is *hyperelliptic* if the underlying Riemann surface is hyperelliptic, *i.e.* there is an involution i such that X/i is the Riemann sphere, and if ω satisfies $i^*\omega = -\omega$.

Assume that the translation surface S has only even degree singularities $S \in \mathcal{H}(2n_1, \dots, 2n_r, 2p_1, \dots, 2p_s)$. Let $(a_i, b_i)_{i\in\{1,\dots,g\}}$ be a collection of simple closed curves representing a symplectic basis of the homology of S. We define the *spin structure* of S as

$$\sum_{i=1}^{g}(ind(a_i)+1)(ind(b_i)+1) \mod 2.$$

It is an invariant of connected components of the moduli space of meromorphic differentials. When the surface S has only a pair of poles that are simple, and with even degree zeroes, *i.e.* S is in the stratum $\mathcal{H}(2n_1, \dots, 2n_r, -1, -1)$, it is also possible to define a "spin structure" invariant by considering a surface in $\mathcal{H}(2n_1, \dots, 2n_r)$ obtained after cutting the ends of the two infinite cylinders, and gluing them together (see [3]).

Note that an elementary computation shows that, when a surface of genus one has only even degree singularities, then it has an even spin structure if and only if its rotation number is odd.

In the next theorem, we say that the set of poles and zeroes is:

- of *hyperelliptic type* if the degree of the zeroes are of the kind $\{2n\}$ or $\{n, n\}$, for some positive integer n, and if the degree of the poles are of the kind $\{2p\}$ or $\{p, p\}$, for some negative integer p.
- of *even type* if the degrees of zeroes are all even, and if the degrees of the poles are either all even, or are $\{-1, -1\}$.

Theorem 2.3. *Let $\mathcal{H} = \mathcal{H}(n_1, \dots, n_r, p_1, \dots, p_s)$, with $n_i > 0$, $p_j < 0$ be a stratum of genus $g \geq 2$ meromorphic differentials. We have the following.*

(1) *If $\sum_i p_i$ is odd and smaller than -2, then $\mathcal{H}$ is nonempty and connected.*

(2) *If $\sum_i p_i = -2$ and $g = 2$, then:*
   - *if the set of poles and zeroes is of hyperelliptic type, then there are two connected components, one hyperelliptic, the other not (in this case, these two components are also distinguished by the parity of the spin structure).*
   - *otherwise, the stratum is connected.*

(3) *If $\sum_i p_i < -2$ or if $g > 2$, then:*
   - *if the set of poles and zeroes is of hyperelliptic type, there is exactly one hyperelliptic connected component, and one or*

two nonhyperelliptic components that are described below. Otherwise, there is no hyperelliptic component.

- *if the set of poles and zeroes is of even type, then $\mathcal{H}$ contains exactly two nonhyperelliptic connected components that are distinguished by the parity of the spin structure. Otherwise $\mathcal{H}$ contains exactly one nonhyperelliptic component.*

The proof of these theorems involve some constructions, introduced first by Kontsevich and Zorich in [8]. These constructions are called *breaking up a zero* and *bubbling a handle*. We do not give a precise definition here since we will generalize them in Section 4.1, but we summarize the important properties.

- *Breaking up a zero* is a local surgery in a neighborhood of a singularity of order $n \geq 0$ (the metric is unchanged outside that neighborhood), that replaces that singularity by a pair of singularities of order $n_1, n_2 \geq 0$, with $n_1+n_2 = n$. We can show (see [3]) that each connected component of the moduli space of meromorphic differentials can be obtained from a connected component of a stratum of the form $\mathcal{H}(n, p_1, \dots, p_s)$ (with $n \geq 0$ and $p_1, \dots, p_s < 0$) after successive use of that surgery. In the case that either n_1 or n_2 is zero, we just add a marked point and the metric is unchanged.
- *Bubbling a handle* is a local surgery in a neighborhood of a singularity of order $n \geq 0$, that replaces that singularity by a singularity of order $n + 2$. The genus of the surface increases by one. We can show (see [3]) that each minimal connected component can be obtained starting from a genus zero stratum, by using this surgery repeatedly.

2.5. **Poles with zero residues.** The geometric constructions involved in Section 4 often require the use of a pole with zero residue. Here we give a necessary and sufficient condition for a connected component of stratum to contain a surface with a pole of a given order with zero residue.

The following lemma lists some well known cases where all poles necessarily have non-zero residues.

Lemma 2.4. *Let ω be a meromorphic one-form on a closed Riemann surface S and P be a (non-simple) pole. Then, P has necessarily nonzero residue in the following two cases.*

- *$S = \mathbb{CP}^1$ and ω has exactly two poles and a zero.*
- *There exists exactly one other pole, which is simple.*

Proof. For the first case: let p and q be the degree of the poles. We identify $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$, and can assume that $P = 0$, the other pole is 1, and the zero of ω is at ∞. Then, up to a multiple constant, $\omega = z^p(1-z)^q dz$, and we easily check that the residue at 0 is nonzero.

For the second case, the residue of a simple pole is nonzero and if P is the only other pole, it has opposite residue since by Stokes theorem the sum of residues of poles is zero. □

Proposition 2.5. *Let $\mathcal{C} \subset \mathcal{H}(n_1, \dots, n_r, p_1, \dots, p_s)$, with $n_i > 0$, $p_j < 0$ be a connected component of the moduli space of meromorphic differentials. We assume that there exists $p \in \{p_1, \dots, p_s\}$, such that $p < -1$. We assume that we are not in the case of the previous lemma. Then, there exists in $\mathcal{C}$ a flat surface with a pole of degree p with zero residue.*

Proof. The case is trivial when there is only one pole. In this proof, we will assume first that there are exactly two poles of degree p and q, (by assumption, we must have $p, q < -1$). This leads to the study of three cases, depending on the genus. Then, we will deal with the case of at least three poles.

Case 1: two poles, genus zero: Since we are not in the case of the previous lemma, there are necessarily at least two zeroes. We start from $(\mathbb{CP}^1, z^{-p-2}dz)$, $(\mathbb{CP}^1, z^{-q-2}dz)$, then break the zero P of the first one (resp. Q of the second one) into a pair of zeroes P_1, P_2 of order $p_1, p_2 \geq 0$ with $p_1 + p_2 = -p - 2$ (resp. Q_1, Q_2 of order $q_1, q_2 \geq 0$ with $q_1 + q_2 = -q - 2$), so that there is a vertical saddle connection γ_1 (resp. γ_2) of length ε joining the two singularities. We obtain two surfaces S_1 and S_2. Then, cut γ_1, γ_2, and paste the left part of γ_1 (resp. γ_2) to the right part of γ_2 (resp. γ_1). This defines the segment a (resp. b) in Figure 1. We obtain a flat surface in $\mathcal{H}(p, q, p_1 + q_1 + 1, p_2 + q_2 + 1)$. Choosing suitably p_i, q_i, we can obtain any stratum with two zeroes. Examples in the other strata are obtained from these examples by breaking up zeros. Since each stratum in genus zero is connected, the case is proven.

Case 2: two poles, genus one: We first build a suitable surface in any component of the stratum $\mathcal{H}(p, q, -p-q)$ of genus one surfaces. We start from $S_0 = (\mathbb{CP}^1, z^{-p-2}dz)$ and S_2 as previously. The surface S_0 has a zero P of order $-p-2$, and the surface S_2 has a pair of zeroes Q_1, Q_2 of orders q_1, q_2 with $q_1 + q_2 = -q - 2$.

Consider a metric segment $[P_2, P_3]$ on S_0, with P on its middle, and such that one of the angular sectors at P defined by this segment has angle π (see Case a) of Figure 2). Similarly, we consider a segment

FIGURE 1. Surface of genus zero with two poles and no residue.

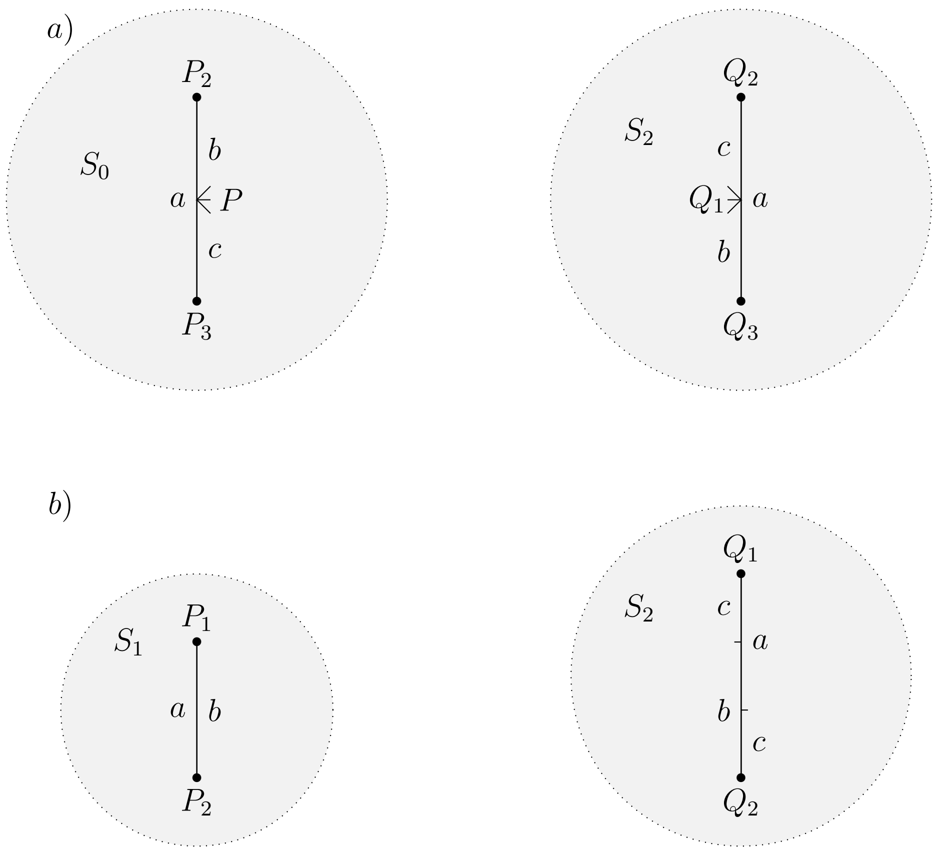

FIGURE 2. Surfaces in $\mathcal{H}(-p,-q,p+q)$.

$[Q_2, Q_3]$ on S_2, with Q_1 on its middle and the same condition on the angular sector at Q_1. We remark that such segment exists, since Q_1, Q_2 are obtained after breaking up a singularity, and in this case, there is by construction (see [8]) a segment joining Q_1 to Q_2 that we can assume to be arbitrarily small. We can assume that the two segments are vertical, isometric, and with opposite orientation. Then, cutting the surfaces along these segments, and gluing them according to Figure 2, one gets a surface S in $\mathcal{H}(p,q,-p-q)$ (note that the midpoint of a is not a singularity). We must check that all connected components of this stratum are obtained. We first consider a basis for the homology of S: consider the saddle connection γ_b corresponding to b, joining the unique conical singularity to itself. At the singularity, it defines a sector of angle $2\pi(1+q_2+1+(-p-2))+\pi=(-2p+2q_2+1)\pi$. We take a smooth path η_b homotopic to γ_b that avoids the singularity. It has index $-p+q_2$ (or $-p-q-(-p+q_2)$). Similarly, we have a smooth path η_c, homotopic to the saddle connection corresponding to the segment c with index $-p$, and η_b,η_c define a symplectic basis of S. So the rotation number of the surface is $\gcd(q_2,-p,-q)$, with q_2 that can be any integer in $\{0,\dots,-q-2\}$. If $-q>-p$, we can clearly obtain any divisor of $\gcd(p,q,-p-q)$, so we obtain any connected component. When $p=q$, one cannot obtain in this way the component with rotation number $-p-1$. But since the rotation number must divide p, we are in the case $p=q=-2$. In this case, we glue two Euclidean planes as in Figure 2, b). Here, paths η_a and η_c define a symplectic basis of S, and we see that the rotation number is 1, since the index of η_a is 1. Finally, once obtained any connected component of $\mathcal{H}(p,q,-p-q)$, breaking up the zero in a suitable way gives any component of any stratum of genus one with two non-simple poles.

Case 3: two poles, higher genus: Here, suitably bubbling handles from genus one surfaces leads to any minimal connected component in higher genus, and breaking up the zero leads to any connected component of the moduli space of meromorphic differentials.

Case 4: at least three poles:

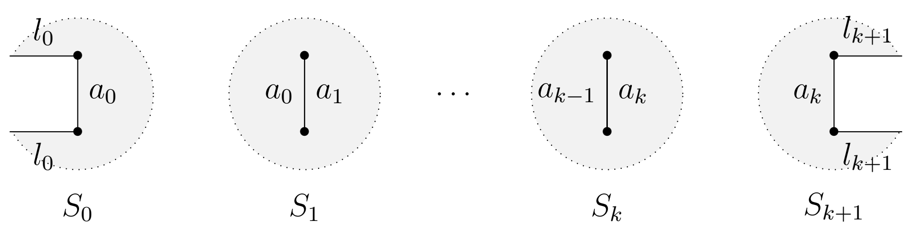

FIGURE 3. Surface of genus zero in $\mathcal{H}(n,p_0,\dots,p_{k+1})$ with k poles of zero residues and two poles of nonzero residues.

Let $k\geq 1$, we first build a genus zero surface with $2+k$ poles of degree $p_0,\dots,p_{k+1}$ respectively. We assume that $p_0,\dots,p_{k+1}<-1$. We start from spheres $S_0,\dots,S_{k+1}$ with exactly one pole of degree

$p_0, \ldots, p_{k+1}$ respectively and one zero (of degree $-p_0-2, \ldots, -p_{k+1}-2$ respectively). Consider on S_0 an infinite horizontal segment l_0 joining the zero to the pole P_0 (l_0 is chosen so that it identifies by a translation map to the half-line $]-\infty, 0[$), then consider the half-infinite horizontal band of width 1 with bottom side l_0. Cut this band, and glue together by translation the two horizontal sides. One obtains a surface, still denoted S_0 with a small vertical boundary component of length 1, and the pole P_0 has now a nonzero residue. We do the same for S_{k+1}, but starting from an half-line l_{k+1} that identifies to $]0, \infty[$. Then, on each S_i, we cut along a vertical segment of length 1, that is attached to a singularity. Then, as in Figure 3, we glue by translation a vertical boundary segment of S_i to the corresponding one of S_{i-1}, and the other vertical boundary segment of S_i to the corresponding one of S_{i+1}. This defines a (closed) flat sphere with poles $P_1, \ldots, P_k$ of zero residue, two poles P_0, P_{k+1} of nonzero residue, and a single singularity of positive degree. The surfaces S_0, S_{k+1} can be replaced without difficulties by half-infinite cylinders, hence we can have P_0 or P_{k+1} that are simple poles.

If we want to have more simple poles, we start from a horizontal half-line joining P_0 to the singularity of positive degree, then cut along this line, consider a half-infinite horizontal band, and glue together each infinite horizontal side of the band to the corresponding half-line on the surface. We obtain a translation surface with a boundary component which is a vertical segment. We glue on this segment a half-infinite cylinder so that we obtain a translation surface with no boundary component. This procedure adds a simple pole, modifies the residue of P_0, and leaves invariant the other residues. Repeating this procedure, we obtain as many simple poles as we want, and we only change the residue of P_0.

Now, suitably bubbling handles and breaking up zeroes, we obtain any connected component with at least three poles, and this does not change the residues of $P_1, \ldots, P_k$. □

3. Moduli spaces of framed meromorphic differentials

As in [2], a *frame* on a translation surface S is a map F_S from a finite alphabet $\mathcal{A}$ to a discrete combinatorial data of S.

For a suitable collection of frames on translation surfaces in a stratum $\mathcal{H}(n_1^{\alpha_1}, \ldots, n_r^{\alpha_r})$, we define the corresponding moduli space of framed surfaces by identifying (S, F_S) and $(S', F_{S'})$ if there is a translation mapping from S to S' that is consistent with the frames.

We are interested in two cases. The first case is when the alphabet $\mathcal{A}$ admits a partition $\sqcup_{i=1}^{r}\mathcal{A}_i$ with $|\mathcal{A}_i| = \alpha_i$ and the collection of frames we consider are all possible one-to-one maps F_S from $\mathcal{A}$ to the set of singularities of S such that, for all i, for all $P \in \mathcal{A}_i$, $F_S(a)$ is a singularity of S of degree n_i. We obtain the moduli space of translation surface with named singularities $\mathcal{H}^{sing}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$.

The following proposition will be useful.

Proposition 3.1. *The connected components of $\mathcal{H}^{sing}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$ are in one-to-one correspondence with the connected components of $\mathcal{H}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$.*

Proof. This is clearly the case for genus zero stratum. Otherwise, we use the fact that each connected component of $\mathcal{H}^{sing}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$ is adjacent to the minimal stratum obtained by coalescing all singularities of positive degree. Then the proof is similar to the proof of Proposition 7.2 of [3]. See also the connected sum construction of this paper in Section 4.1. □

Now we define a more specific combinatorial datum for a singularity.

Definition 3.2. Let P be a singularity of S which is not a simple pole. A (right) *horizontal separatrix* for P is an equivalence class of horizontal geodesics $\gamma :]a, b[\to S$, satisfying $\gamma' = 1$, $\lim_{t\to a} \gamma(t) = P$ with the following conditions:

- if $\deg(P) > 0$: $a = 0$ and $\gamma_1 \sim \gamma_2$ if they coincide on a subinterval of the form $]0, \varepsilon[$.
- if $\deg(P) < -1$: $a = -\infty$, and $\gamma_1 \sim \gamma_2$ if the distance for the euclidean metric between $\gamma_1(t)$ and $\gamma_2(t)$ is bounded as t tends to $-\infty$.

Remark 3.3. For a singularity P (pole or zero), the number of possible choices of horizontal separatrices is $|\deg(P) + 1|$.

Recall that two translation surfaces S_1 and S_2 define the same element in the moduli space if and only if there is a one-to-one map $f : S_1 \to S_2$ which is a translation. In particular such map sends a horizontal separatrix attached to $P \in S_1$ to a horizontal separatrix attached to $f(P) \in S_2$.

Now we define the second moduli space of framed meromorphic differentials. It corresponds to choosing a horizontal separatrix for each singularity. More precisely, we still assume that $\mathcal{A}$ admits a partition $\sqcup_{i=1}^{r}\mathcal{A}_i$ with $|\mathcal{A}_i| = \alpha_i$ and the collection of frames we consider are all possible maps $\tilde{F}_S$ such that:

- if $n_i \neq -1$ then for all $P \in \mathcal{A}_i$, $\tilde{F}_S(P)$ is a horizontal separatrix for a singularity of degree n_i.
- if $n_i = -1$ then for all $P \in \mathcal{A}_i$, $\tilde{F}_S(P)$ is a singularity of degree -1.
- if $P \neq Q$, then the singularity corresponding to $\tilde{F}_S(P)$ is different from the singularity corresponding to $\tilde{F}_S(Q)$.

In particular, two framed surfaces $(S_1, \tilde{F}_1)$ and $(S_2, \tilde{F}_2)$ represent the same element in the moduli space if:

- there is an one-to-one map $f : S_1 \to S_2$ that is a translation.
- $f \circ \tilde{F}_{S_1} = \tilde{F}_{S_2}$, *i.e.* for each $P \in \mathcal{A}$,
  - the image of the singularity labeled P of S_1 is the singularity labeled P of S_2,
  - the image by f of the marked horizontal separatrix of the singularity labeled P of S_1 is corresponding horizontal separatrix labelled P of S_2 (when the singularity is not a simple pole).

We obtain the moduli space of translation surfaces with marked horizontal separatrices $\mathcal{H}^{hor}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$.

Denote by $\pi_h : \mathcal{H}^{hor}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r}) \to \mathcal{H}^{sing}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$ and $\pi_s : \mathcal{H}^{sing}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r}) \to \mathcal{H}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$ the coverings obtained by forgetting the horizontal separatrices, and the names of the singularities, respectively.

Let $\mathcal{C}$ be a connected component of $\mathcal{H}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$. From Proposition 3.1, $\mathcal{C}^{sing} = \pi_s^{-1}(\mathcal{C})$ is connected. We define $\mathcal{H}_{\mathcal{C}}^{hor}$ the subset of $\mathcal{H}^{hor}(n_1^{\alpha_1}, \dots, n_r^{\alpha_r})$ whose underlying translation surfaces are in $\mathcal{C}$. We have $\mathcal{H}_{\mathcal{C}}^{hor} = (\pi_h)^{-1}\mathcal{C}^{sing}$. We want to compute the number of connected components of $\mathcal{H}_{\mathcal{C}}^{hor}$.

We choose S_b^{hor} a base element of $\mathcal{H}_{\mathcal{C}}^{hor}$ and $S_b^{sing} = \pi_h(S_b^{hor})$ the corresponding flat surface (with marked singularities). By definition, for each element S of $\pi_h^{-1}(S_b^{sing})$, and each element $P \in \mathcal{A}$, $\tilde{F}_S(P)$ is a horizontal separatrix attached to the same singularity as $\tilde{F}_{S_b^{hor}}(P)$.

For each singularity P of S_b^{sing} which is not a simple pole, the set of horizontal separatrices, once fixed the corresponding one of S_b^{hor}, is identified to the cyclic group $\mathbb{Z}/(\deg(P)+1)\mathbb{Z}$ in the following way:

(1) if P is a conical singularity, we go from a separatrix to the next one by considering a small counterclockwise arc of angle 2π around the singularity. Hence, the identification is just given by the cyclic (counterclockwise) order around P.
(2) if P is an nonsimple pole, we go from a separatrix to the next one by considering a large counterclockwise arc of angle 2π. In

particular, from the point of view of the pole P, it corresponds to the cyclic *clockwise* order around P.

We consider the action of the monodromy group of the covering π_h, on the fiber $\pi_h^{-1}(S_b^{sing})$, *i.e.* all the possible frames on the surface S_b^{sing}.

Definition 3.4. Let Hor be the group $\prod_P \mathbb{Z}/(\deg(P)+1)\mathbb{Z}$, where the product is taken over all singularities of degree different from -1. From above, Hor is identified with the fiber $\pi_h^{-1}(S_b^{sing})$. We define Mon to be the image of the element $(0,\dots,0) \in Hor$ for the monodromy action.

Lemma 3.5. *The action of the monodromy group of the covering π_h on Hor corresponds to the addition of the corresponding elements of Mon. In particular, Mon is a subgroup of Hor.*

Proof. Let $(l_P)_P \in Hor$ and let $\gamma = (S_t)_{t\in[0,1]}$ be a closed path in $\mathcal{C}^{sing}$ with $S_0 = S_b^{sing}$, and that defines a element $(x_P)_P \in Mon$. For each P, the angle between two separatrices of P is preserved along γ, hence the image of the action of γ on $(l_P)_P$ is $(l_P + x_P)_P$. □

Since $\mathcal{C}^{sing}$ is connected, we can therefore identify connected components of $\mathcal{H}_{\mathcal{C}}^{hor}$ with cosets of Mon in Hor. This proves the following:

Corollary 3.6. *The number of connected component of $\mathcal{H}_{\mathcal{C}}^{hor}$ is the index of the subgroup Mon of Hor.*

Definition 3.7. Let Hor be the group defined above, and let P be a singularity of the surface S_b^{hor}. We denote by δ_P the element in Hor which is 1 on the factor corresponding to P, and zero everywhere else. If P is a simple pole, then $\delta_P = 0$. For any singularity, we have $(\deg(P)+1)\delta_P = 0$.

The goal of the next section is to prove the following two propositions, which give a collection of elements that are in Mon.

Proposition 3.8. *Let S_b^{hor} be a framed genus zero translation surface. Let P, Q be a pair of singularities of S_b^{hor}. We have*

$$\tau_{P,Q} := \deg(Q)\delta_P + \deg(P)\delta_Q \in Mon.$$

Note that $\tau_{P,Q} = (\deg(P) + \deg(Q) + 1)(\delta_P + \delta_Q)$.

Proposition 3.9. *Let S_b^{hor} be a framed translation surface of genus $g \geq 1$, such that the underlying translation surface is neither in a hyperelliptic connected component, nor in a stratum of the kind $\mathcal{H}(-1,-1,n_1,\dots,n_r)$, with $n_i > 0$.*

(1) *Let P, Q be a pair of singularities of S_b^{hor}. We assume that neither P nor Q is the only pole of S_b^{hor}. We have*

$$\tau_{P,Q} := \deg(Q)\delta_P + \deg(P)\delta_Q \in Mon.$$

(2) *Let P be a singularity of S_b^{hor}. We have*

$$\sigma_P := 2\delta_P \in Mon.$$

If the underlying translation surface is in a nonhyperelliptic connected component of a stratum of the kind $\mathcal{H}(-1,-1,n_1,\dots,n_r)$, with $n_i > 0$, then the previous statement is still true if we assume that neither P nor Q are poles.

4. Some elementary moves

4.1. **Connected sums.** Let S, S' be translation surfaces. Let $N \in S$, be a singularity of degree $n \geq 0$ and let $N' \in S'$ be a singularity of degree $n' = -2-n < 0$. We assume that N' has zero residue. A pointed neighborhood $V\backslash\{N'\}$ of N' is isometric to the complement of a metric disk centered in 0, in the cone defined by the form $z^{-2-n'}dz = z^n dz$. Hence, after scaling (shrinking) appropriately the surface S' so that this metric disk is isometric (as a translation surface) to a neighborhood U of N, we can glue together $S\backslash U$ and $S'\backslash V$ along their boundaries to get a translation surface. This surgery is a flat version of the topological connected sum of two surfaces. If $n \leq -2$, $n' \geq 0$, the construction is the same by reversing the roles of S, S'. If $n = -1$, then $n' = -1$, we must assume that the two simple poles have opposite residues, we obtain two half infinite cylinders with isometric waist curves. Cutting and pasting along such waist curves gives the required surface.

We are interested in some particular cases for S', where it generalizes the two surgeries introduced by Kontsevich and Zorich in [8], *breaking up a singularity* and *bubbling a handle*.

If S' is a sphere with three singularities, *i.e.* $S' \in H(-n-2, n_1, n_2)$, the above construction, when possible, replaces the singularity of degree n by a pair of singularities of degree n_1, n_2 with $n = n_1 + n_2$.

- If $n \geq 0$ and $n_1, n_2 \geq 0$, the construction is always possible and is precisely "breaking up a singularity" (see [8]).
- If $n \leq -1$, the construction is always possible, and breaks up the pole of degree n into a pair of singularities of degree n_1 and n_2.
- If $n \geq 0$ and either n_1 or n_2 is negative. Say $n_1 < 0$ and $n_2 \geq 0$. The above construction is not possible since, S' is a sphere with two poles (of respective degree $-n-2$ and n_1) and a zero, and

in the case, the pole of degree $-n-2$ would have zero residue which would contradict Lemma 2.5.

If S' is a torus in $\mathcal{H}(-n-2, n+2)$, then the surgery adds a handle to the surface S, and the singularity of degree n is replaced by a singularity of degree $n+2$.

- For $n \geq 0$, the construction is always possible, and if we choose S' that contains a cylinder, this is precisely the surgery "bubbling a handle" (see [8]).
- For $n = -3$ or $n = -1$, $\mathcal{H}(-n-2, n+2) = \mathcal{H}(-1, 1) = \emptyset$, so the construction does not make sense.
- If $n \leq -4$ or $n = -2$, the construction works well as soon as the pole of degree n has zero residue.

Remark 4.1. When $n' < 0$ and N' has nonzero residue, the above construction is not possible since the boundary of a pointed neighborhood of N' is never isometric to the boundary of a neighborhood of N. However, once a proper rescaling (shrinking) of the surface S' is done, it is possible to perform a surgery on S that creates a geodesic boundary component ("hole") adjacent to the singularity N, so that the boundary of a neighborhood of N becomes isometric to the boundary of a neighborhood of N', making the construction doable, see Section 4.3. Note that if S has no poles, then due to Stokes theorem, this necessarily creates on S at least one other boundary component. This idea has been continued in [1].

4.2. **A realization of $\tau_{P,Q}$, local case.** Consider a translation surface S^{hor} with labeled horizontal separatrices. Let P and Q be two singularities of degree p and q respectively. Assume that P, Q are obtained after the surgery "breaking up a singularity" above, with either $p, q \geq 0$, or $p + q \leq -2$, in the zero residue case. By construction, the singularities P, Q are on a metric disc whose boundary is a covering of Euclidean circle. Cutting the surface along the circle and rotating the disc by an angle θ, one gets a family of surfaces (S_θ). For $\theta = 2\pi(p+q+1)$, one has $S_\theta = S$. Keeping track of the marked horizontal separatrices, we see at the end that the marked horizontal separatrices for P, Q have changed by an angle $2\pi(p+q+1)$, and the horizontal separatrices of the other singularities have not changed.

Now assume that S_b^{hor} is in the same connected component as a translation surface S^{hor} as above. Then, conjugating the above transformation with a path joining S_b^{hor} to S^{hor} gives the element $(p+q+1)(\delta_P + \delta_Q) = q\delta_P + p\delta_Q = \tau_{P,Q}$ in Mon (recall that $(p+1)\delta_P = (q+1)\delta_Q = 0$).

4.3. **A realization of $\tau_{P,Q}$, nonlocal case.** The above transformation fails if $p+q \geq 0$ and either p or q is negative.

Here we describe a (nonlocal) surgery that produces the same effect on the set of separatrices. We must first describe a way to do a connected sum with a surface in $\mathcal{H}(p,q,-2-p-q)$. The idea is to make a "hole" (*i.e.* a geodesic boundary component) adjacent to the singularity of degree $p+q$. The transformation is then obtained by continuously rotating the hole by an angle $2\pi(p+q+1)$.

We start from a surface S_0 in the stratum obtained by coalescing P and Q. We assume that this is not a stratum of holomorphic differentials. We can assume that S_0 does not have any vertical saddle connections. Then, it is obtained by the *infinite zippered rectangle construction*. We refer to [3], Section 3.3 for a precise construction, and present an example here (see Figure 4). Note that in this figure, the parameters $z_1, \dots, z_4$ have positive real part and may take any value satisfying this condition, and $l_1, \dots, l_4$ are horizontal half-lines.

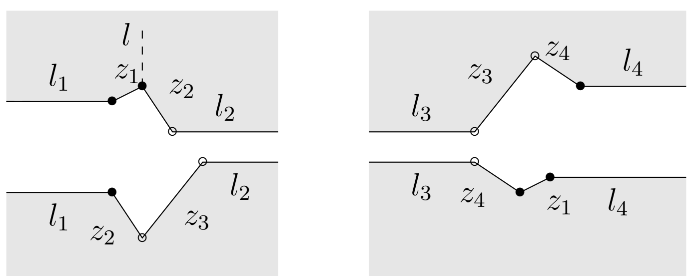

FIGURE 4. Infinite zippered rectangle representation of a surface in $\mathcal{H}(-2,-2,2,2)$.

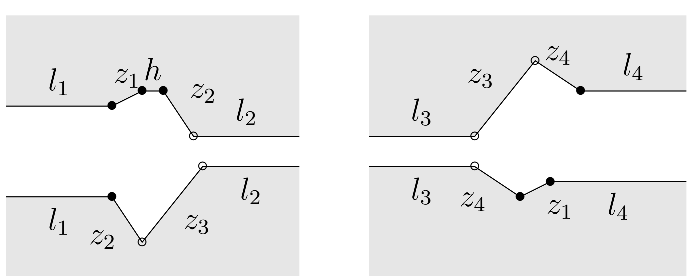

FIGURE 5. Same surface as before, after creating a hole.

We choose a vertical separatrix l adjacent to a singularity of degree $p+q$ (the dashed line in the above picture). Then, insert an horizontal saddle connection as in Figure 5. This creates a hole on the surface orthogonal to the direction l. This resulting surface, for a suitable rescaling and a parameter h small enough, can be glued as in Section 4.1 to a flat sphere S_1 in $\mathcal{H}(p, q, -2-p-q)$, where the pole of degree $-2-p-q$ has suitable residue.

Now, we can continuously rotate the segment h by an angle π and obtain the surface with a hole that would be obtained from the separatrix l' obtained after rotating l by π. This operation is described in Figure 6.

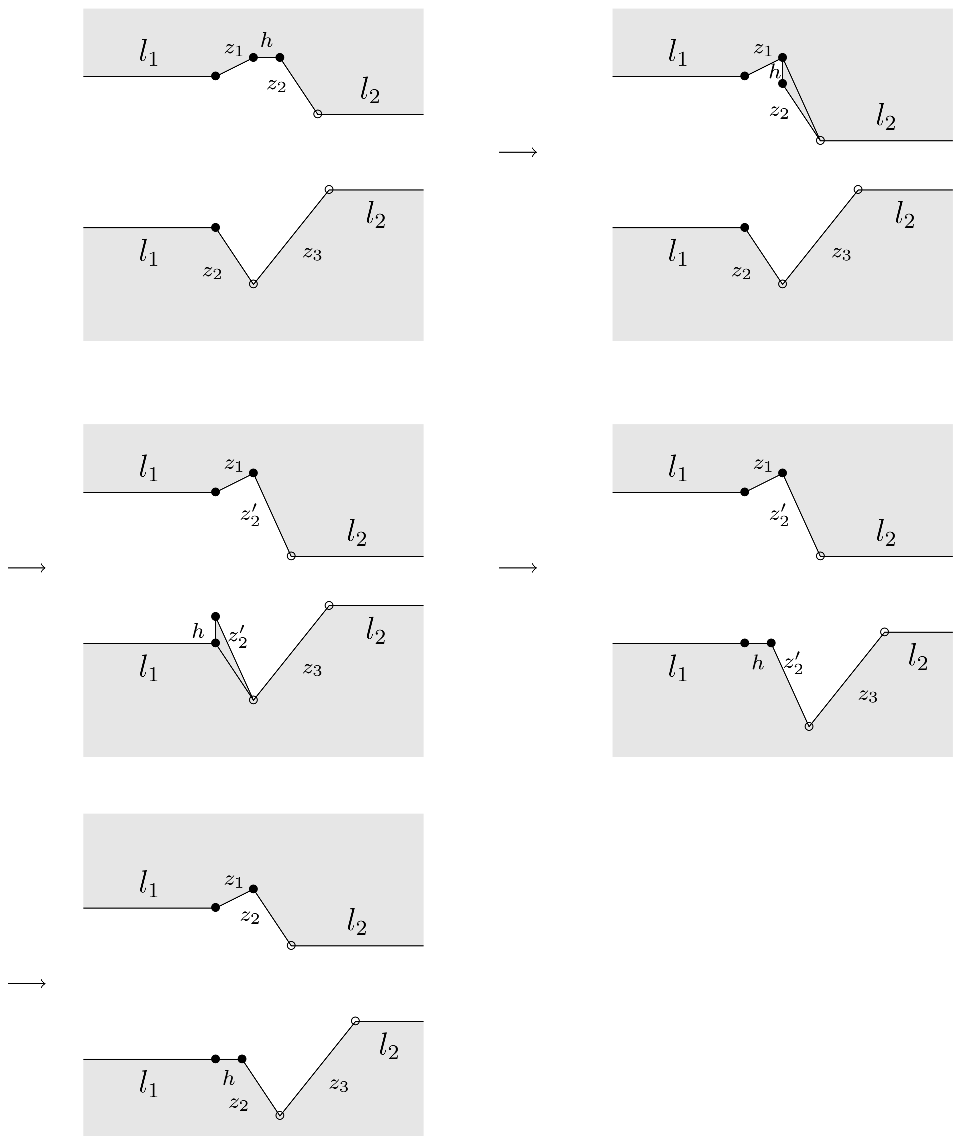

FIGURE 6. Rotating (clockwise) the hole by π.

Repeating this operation until the separatrix l rotates by the angle $(p+q+1)2\pi$ we get a continuous family $S_{0,\theta}$, that we glue as in Section 4.1 to the surface $r_\theta.S_1$. For $\theta=(p+q+1)2\pi$, we have $r_\theta.S_1=S_1$, and by construction $S_{0,\theta}=S_{0,0}$. Hence, we get a closed path in the moduli space of meromorphic differentials. The corresponding transformation on the marked horizontal separatrices is $\tau_{P,Q}$.

4.4. **A realization of σ_P.** As in Section 4.2, let P be a singularity of degree $p\neq\pm 1$, obtained after bubbling a handle as above, *i.e.* we started from a singularity P' of degree $p-2$ (with zero residue if $p<0$), and attached to it a torus in $\mathcal{H}(-p,p)$. A metric circle $\mathcal{C}$ around P' is preserved by the construction and becomes a metric circle around the handle. Now we cut S along $\mathcal{C}$ and rotate the disc by an angle θ. We get a family of surfaces (S_θ). Since $\mathcal{C}$ is a $|p-2+1|$ covering of the Euclidean circle, for $\theta=2\pi(|p-2+1|)$, $S_\theta=S$. Keeping track of the marked horizontal separatrices, we see that the marked horizontal separatrix for P has changed by an angle $-\theta=\pm 2\pi(p-1)$, hence $\pm 4\pi$ since the total angle at P is $2\pi(p+1)$. As before, all the other horizontal separatrices are unchanged. Similarly, if S_b^{hor} is in the same connected component as a surface where the singularity P is obtained after the bubbling a handle, one gets the element $2\delta_P=\sigma_P$ of Hor.

4.5. **Existence of the elementary moves.** The goal of this section is to prove Proposition 3.8 and Proposition 3.9. We first describe the cases when the hypothesis needed for the transformations given in Sections 4.2, 4.3 and 4.4 are satisfied.

Lemma 4.2. *Let S_b^{hor} be a framed surface of genus g such that the underlying translation surface is in a nonhyperelliptic connected component $\mathcal{C}$ of a stratum $\mathcal{H}=\mathcal{H}(p,q,n_1,\dots,n_r,m_1,\dots,m_s)$, with $r,s\geq 0$, $n_1,\dots,n_r>0$ and $m_1,\dots,m_s<0$. Let P,Q be singularities of S_b^{hor} of degree p,q respectively. We assume that the conditions (S) and (CC) below are both realized:*

- (S) *One of the following condition is fulfilled.*
  - i) $p,q\geq 0$,
  - ii) $p+q\leq -2$, *and* $\sum_j m_j\neq -1$ *and* ($g\neq 0$ *or* $r\neq 1$ *or* $s\neq 1$).
  - iii) $p+q\geq 0$, $p<0$ *and* $\sum_j m_j<-1$.
- (CC) *If $g=1$, we are not in the following case: $Rot(S_b)=d$ and $\mathcal{H}(p+q,n_1,\dots,n_r,m_1,\dots,m_s)=\mathcal{H}(-d,d)$ for some $d\geq 2$.*

Then, $\tau_{P,Q}\in Mon$.

Proof. We denote by $\mathcal{H}_0$ the stratum $\mathcal{H}(p+q,n_1,\dots,n_r,m_1,\dots,m_s)$. Observe first that if $p,q\geq 0$ or $p+q\leq -2$, then the stratum $\mathcal{H}_0$ is not empty. If $p+q\geq 0$ and $p<0$, then the stratum $\mathcal{H}_0$ is empty if and only if $\sum_i m_i=1$.

Also, remark that by Proposition 2.5, Case ii) corresponds to $p+q\leq -2$ and there exists in each connected component of $\mathcal{H}_0$ a flat surface with a pole of degree $p+q$ and zero residue.

Hence, the hypothesis (S) implies that $\mathcal{H}_0$ is nonempty and we can break a singularity of degree $p+q$ of a surface $S_0\in\mathcal{H}_0$ and obtain an element $S\in\mathcal{H}$. We can also choose the connected component of $\mathcal{H}_0$ were S_0 is. From Sections 4.2 and 4.3, $\tau_{P,Q}\in Mon$ as soon as we can obtain the connected component $\mathcal{C}$.

- If the genus is zero, there is nothing more to prove since the stratum is connected.
- If the genus is at least two, observe that breaking up a singularity preserves the parity of the spin structure when it is well defined, and $\mathcal{H}$ contains a component of a given parity if and only if $\mathcal{H}_0$ also contains a component of the same parity (see Theorem 2.3).

  We claim that if $p\neq q$, then after breaking up the singularity we are not in a hyperelliptic connected component. Indeed, in the local case, it is easy to see that if $S=S_0\oplus S_1$ is in the hyperelliptic component, then the hyperelliptic involution induces an involution on S_0 and S_1. But $S_1\in\mathcal{H}(-p-q-2,p,q)$, which is not a hyperelliptic component. In the nonlocal case, we see that the length of saddle connection corresponding to the small hole is unique (no other saddle connection has the same length), so the saddle connection is globally preserved by any isometry. Hence if S is in a hyperelliptic connected component, it also induces an involution on the two pieces of surfaces, which is not possible.

  When $p=q$, then $\mathcal{H}_0$ necessarily contains a nonhyperelliptic component (of the same parity as for $\mathcal{C}$), hence starting from a suitable surface and breaking up the singularity of order $p+q$ we obtain the required component $\mathcal{C}$.
- Assume that the genus is one. Recall that in genus one, the components are classified by the rotation number: for a stratum $\mathcal{H}(k_1,\dots,k_r)$, the rotation number can be any positive common divisor of $k_1,\dots,k_r$ except for a stratum of the type $\mathcal{H}(k,-k)$ were the rotation number must be different from k. Let d be the rotation number corresponding to the component $\mathcal{C}$. Then d

divides $p+q, n_1, \dots, m_s$. By hypothesis, $\mathcal{H}(p+q, n_1, \dots, n_r) \neq \mathcal{H}(d, -d)$, hence there exists a surface in $\mathcal{H}(p+q, n_1, \dots, n_r)$ with rotation number d. Breaking up the zero of degree $p+q$ into singularities of degree p, q preserves the rotation number and gives the required surface.

□

Lemma 4.3. *Let S_b^{hor} be a framed surface of genus $g \geq 1$ such that the underlying translation surface is in a nonhyperelliptic connected component $\mathcal{C}$ of a stratum $\mathcal{H} = \mathcal{H}(p, n_1, \dots, n_r, m_1, \dots, m_s)$, with $r, s \geq 0$ and the integers $n_1, \dots, n_r > 0$ and $m_1, \dots, m_s < 0$. Let P be a singularity of degree p. We assume the conditions (S) and (CC) below are both realized:*

- (S) *One of the following condition is fulfilled.*
  - i) $p \geq 2$,
  - ii) $p \leq -2$ *and* $\sum_j m_j \neq -1$ *and* ($g \neq 1$ *or* $r \neq 1$ *or* $s \neq 1$).
- (CC) *If $g = 1$, we are not in the following case: $Rot(S_b) = |p|$ and $n_1, \dots, n_r, m_1, \dots, m_s$ are multiples of p.*

Then, $\sigma_P \in Mon$.

Proof. Denote by $\mathcal{H}_0$ the stratum $\mathcal{H}(p-2, n_1, \dots, n_r, m_1, \dots, m_r)$. As in the previous Lemma, condition (S) imply that we can construct a surface in $\mathcal{H}$ by bubbling a handle $S_1 \in \mathcal{H}(-p, p)$ at a singularity of degree $p-2$ on a surface $S_0 \in \mathcal{H}_0$. We denote by $S = S_0 \oplus S_1 \in \mathcal{H}$ the resulting surface. We must check that we can have $S \in \mathcal{C}$.

Observe that S_0 we can be in any connected component of $\mathcal{H}_0$, S_1 can be in any connected component of $\mathcal{H}(-p, p)$. Observe also that if $S_0 \oplus S_1$ is in a hyperelliptic connected component, then necessarily S_0 and S_1 are in hyperelliptic connected components.

We first assume that $g \geq 2$.

Assume that p is odd then $\mathcal{H}(-p, p)$ does not contain a hyperelliptic component, hence S is not in a hyperelliptic connected component of $\mathcal{H}$. Furthermore, by Theorem 2.3, the stratum $\mathcal{H}$ has only one nonhyperelliptic connected component, therefore $S \in \mathcal{C}$.

Assume that p is even and positive. If $p \geq 6$, the stratum $\mathcal{H}(-p, p)$ has two nonhyperelliptic components (one for each parity of the spin structure). Hence, for any choice of S_0, we can obtain any nonhyperelliptic component with even or odd spin structure. If $p = 4$, the stratum $\mathcal{H}(-4, 4)$ has two components: a nonhyperelliptic one, which has even spin structure (the rotation number is one), and the hyperelliptic one, with odd spin structure (the rotation number is two). If there exists in $\mathcal{H}_0$ a nonhyperelliptic component, we use it and we obtain S in the required components of $\mathcal{H}$. Otherwise, $\mathcal{H}_0 = \mathcal{H}(2,-2)$ or $\mathcal{H}_0 = \mathcal{H}(2,-1,-1)$, so $\mathcal{H} = \mathcal{H}(4,-2)$ or $\mathcal{H} = \mathcal{H}(4,-1,-1)$, which has only one nonhyperelliptic component. If $p = 2$ then $\mathcal{H}_0 = \mathcal{H}(0, n_1, \dots, n_r, m_1, \dots, m_s)$ is a stratum with no hyperelliptic connected component, hence S cannot be in a hyperelliptic component, and the parity of its spin structure is given by that of S_0, which can be odd or even.

The case p even and negative is analogous and left to the reader.

Now we assume that $g = 1$. As in the proof of the previous lemma, let $d > 0$ that divides $p, n_1, \dots, m_s$. We have $d \neq \pm p$, otherwise $n_1, \dots, m_s$ are multiples of p and this case is excluded. Hence, there exists a surface $S_1 \in \mathcal{H}(-p, p)$ with $Rot(S_1) = d$, and therefore $Rot(S) = d$. □

Now we can prove Proposition 3.8 and Proposition 3.9.

Proof of Proposition 3.8. We define the element $\rho = \sum_P \delta_P$, where the sum is taken on all singularities that are not simple poles. Observe that rotating the surface by 2π and keeping track of the horizontal separatrices gives the element ρ which is therefore in Mon.
The case with three singularities is special. Here $\mathcal{H} = \mathcal{H}(p, q, r)$, and denote by P, Q, R the three singularities. Recall that $(r+1)\delta_R = (p+1)\delta_P = (q+1)\delta_Q = 0$ (see Section 3). Since $g = 0$ we also have, $r + 1 = -1 - p - q$, so

$$(r+1)\rho = -(r+1)(\delta_P + \delta_Q) = (p+q+1)(\delta_P + \delta_Q) = \tau_{P,Q} \in Mon.$$

We look at the cases where the hypothesis (S) given in Lemma 4.2 is not satisfied (the genus being zero, the hypothesis (CC) is satisfied).

a) $p + q = -1$. Here, $\tau_{P,Q} = 0$ so there is nothing to prove.
b) $p + q \geq 0$ and $p < 0$ and $\sum_j m_j \geq -1$. This case does not appear since $p + q + \sum_i n_i + \sum_j m_j = -2$.
c) $p + q \leq -2$, and $r = s = 1$. Denote respectively by M, N the two other singularities, and respectively by $m < 0$ and $n > 0$ their degree. Observe that if $m + n \geq 0$, then from Case b) above, the hypothesis (S) is necessarily satisfied. We have $n + m = -2 - (p + q) \geq 0$ so $\tau_{M,N} \in Mon$. As before, the element $\rho = \delta_P + \delta_Q + \delta_M + \delta_N$ is in Mon. Then, the condition $p + q + m + n = -2$ implies $\tau_{M,N} - (n + m + 1)\rho = -\tau_{P,Q}$, so $\tau_{P,Q} \in Mon$.
d) $p+q \leq -2$, and the stratum is of the form $\mathcal{H}(p, q, n_1, \dots, n_r, -1)$ with $r \geq 1$ and $n_1, \dots, n_r > 0$. Denote by M the simple pole, and by $N_1, \dots, N_r$ the singularities of degree respectively

$n_1, \dots, n_r$. As before, we have $\delta_{N_k} = -\tau_{M,N_k} \in Mon$. Also,

$$\tau_{P,Q} = (p+q+1)(\delta_P + \delta_Q) = (p+q+1)(\rho - \sum_k \delta_{N_k}).$$

Hence, $\tau_{P,Q} \in Mon$.

□

Proof of Proposition 3.9. As in the previous proof , we define the element $\rho = \sum_P \delta_P$, where the sum is taken on all singularities that are not simple poles. Recall that $\rho \in Mon$.

We first look at the element $\tau_{P,Q}$. We study the cases where the hypothesis (S) given in Lemma 4.2 is not satisfied.

- $p + q = -1$, there is nothing to prove.
- $p + q \geq 0$ and $p < 0$ and $\sum_j m_j = 0$. Since we suppose that P is not the only pole, this case does not appear.
- $p+q \geq 0$ and $p < 0$ and $\sum_j m_j = -1$. It means that except for the pole P of order p, there is a unique other pole M which is simple. The case $p = -1$ does not appear by hypothesis of the proposition (it corresponds to the case where there are exactly two poles that are simple). If $p < -1$, we denote by $N_1, \dots, N_r$ the singularities of degree $n_1, \dots, n_r$ respectively. We remark that $\delta_{P_i} = -\tau_{N_i,M} \in Mon$ by Lemma 4.2 (since there is a simple pole, the hypothesis (CC) of the lemma is automatically satisfied). Hence we can conclude as in Case d) of the previous proposition.
- $p+q \leq -2$, and the stratum is of the form $\mathcal{H}(p, q, n_1, \dots, n_r, -1)$ with $n_k > 0$. We construct $\tau_{P,Q}$ as in Case d) of the previous proposition.

Hence there remains the case where $g = 1$, the rotation number is d, and $\mathcal{H}(p + q, n_1, \dots, n_r, m_1, \dots, m_s) = \mathcal{H}(d, -d)$.

- $p + q = d$, hence the stratum is $\mathcal{H}(kd, (1-k)d, -d)$ for some $k > 1$ with $p = kd$ and $q = (1-k)d$.
- $p + q = -d$, hence the stratum is $\mathcal{H}(kd, -(1+k)d, d)$ for some $k > 0$ with $p = kd$ and $q = -(1+k)d$.

We postpone these two cases to the end of the proof. In the remaining of the proof we can use that $\tau_{P,Q} \in Mon$ in any cases except these two cases above.

Now we look at the element σ_P. We study the cases where the hypothesis given in Lemma 4.3 are not satisfied.

- $p\leq -2$ and $\sum_j m_j=-1$. There is exactly one other pole M which is simple. Here, $\tau_{P,M}\in Mon$ and we have $\sigma_P=2\tau_{P,M}\in Mon$.
- $p\leq -2$, and $g=r=s=1$. There is exactly one other pole M and one zero N of degree m,n respectively. We have $m+n+p=0$. If $\tau_{M,N}=(m+n+1)(\delta_M+\delta_N)\in Mon$, then

$$(m+n+1)\rho-\tau_{M,N}=(-p+1)\delta_P=(-p-1+2)\delta_P=\sigma_P\in Mon.$$

  Otherwise, from the above study, we have $n=-kp$, $m=(k-1)p$ for some $k\geq 2$ and the rotation number is $|p|$. The case $k=2$ corresponds to the hyperelliptic connected component of $\mathcal{H}(p,-2p,p)$ and therefore does not occur by hypothesis. Hence $k>2$ and therefore $\tau_{P,N}\in Mon$ (we are not in the two exceptional cases of the above study). Also, $\tau_{P,M}\in Mon$ for the same reasons. Hence

$$p\rho-\tau_{P,N}-\tau_{P,M}=(p-n-m)\delta_P=2p\delta_P=-\sigma_P\in Mon.$$

- The hypothesis (CC) fails: the stratum is $\mathcal{H}(\pm d,k_1d,\dots,k_rd)$, and $\deg(P)=p=\pm d$. In this case, as before, we produce σ_P as a combination of "τ" elements. If there is a singularity Q of degree $q=-p$, then observe that $\tau_{P,Q}\in Mon$, hence

$$\begin{aligned}(1+q)\tau_{P,Q} &= (1+q)q\delta_P+(1+q)q\delta_Q=(1-p)(-p)\delta_P\\ &= 2\delta_P=\sigma_P\in Mon.\end{aligned}$$

  If there are least two other singularities P',P'' of degree p, then we have $\tau_{P,P'}+\tau_{P,P''}-\tau_{P',P''}=\sigma_P\in Mon$. So we can assume that there are at most two singularities of degree p and no singularities of degree $-p$.

  If there are two singularities P,P' of degree p. Observe that $\tau_{P,Q}\in Mon$ for each $Q\neq P$. Indeed from the previous study, this may be false only in $\mathcal{H}(d,d,-2d)$ and in $\mathcal{H}(-d,-d,2d)$. But in these cases, the connected component with rotation number d is precisely the hyperelliptic connected component. Also, we have $\sigma_Q\in Mon$. This implies

$$\begin{aligned}\sum_{Q\neq P,P'}(2\tau_{P,Q}-p\sigma_Q) &= 2\left(\sum_{Q\neq P,P'}\deg(Q)\right)\delta_P\\ &= -4p\delta_P=4\delta_P=2\sigma_P\in Mon.\end{aligned}$$

  Hence if p is even, $\sigma_P\in Mon$. If p is odd, we have

$$\begin{aligned} & \rho + \tau_{P,P'} + \sum_{Q \neq P,P'} \left(\tau_{P,Q} - \frac{p+1}{2} \sigma_Q \right) \\ = & \sum_{Q \neq P,P'} \left(\delta_Q + \tau_{P,Q} - (p+1)\delta_Q \right) \\ = & \sum_{Q \neq P,P'} \deg(Q)\delta_P = -2p\delta_P = 2\delta_P = \sigma_P \in Mon. \end{aligned}$$

If P is the only singularity of degree p, we have

$$\sigma_P = \sum_{Q \neq P} (2\tau_{P,Q} - p\sigma_Q) \in Mon.$$

Therefore, we have proven that $\sigma_P \in Mon$ in each case.

Now, we come back to a stratum of the kind $\mathcal{H}(kd, (1-k)d, -d)$ (for some $k > 1$), and we look at the component of rotation number d. We want to produce $\tau_{P,Q}$, where $\deg(P) = kd$ and $\deg(Q) = (1-k)d$. Note that if $k = 2$ we are in the hyperelliptic connected component of $\mathcal{H}(2d, -d, -d)$, hence this case does not occur by hypothesis. Denote by R the other singularity. We have

(1) if d is even, $\sigma_P = 2\delta_P \in Mon$, hence $\delta_P \in Mon$. Similarly, $\delta_Q \in Mon$, so $\tau_{P,Q} \in Mon$.
(2) if d is odd, since $\tau_{P,Q} = (d+1)(\delta_P + \delta_Q)$, we get $\tau_{P,Q} = (d+1)\rho - \frac{d+1}{2}\sigma_R \in Mon$.

The case with a stratum of the kind $\mathcal{H}(dn, -(1+k)d, d)$ is similar. This concludes the proof. □

5. Positive genus

5.1. **A topological invariant.** Here we describe a topological invariant for connected components of $\mathcal{H}^{hor}$ in the following cases:

- there are no simple poles, and there are singularities of odd degree.
- there are exactly two poles that are simple, and some odd singularities of positive degree.

We first assume that there are no simple poles. The invariant is inspired by the well known *parity of spin structure* for translation surfaces with even degree singularities ([8]). See also [2].

For a smooth closed curve γ in S that does not pass through any singularity, define $ind(\gamma)$ to be the index of the Gauss map defined by

γ'. Choose $(\alpha_i,\beta_i)_{i\in\{1,\dots,g\}}$ a collection of smooth simple closed curves representing a sympletic basis for the homology of S, and define

$$\phi(\alpha,\beta)=\sum_{i=1}^{g}(ind(\alpha_i)+1)(ind(\beta_i)+1) \mod 2.$$

When S has no odd degree singularities, $\phi(\alpha,\beta)$ does not depend on the choices of (α,β) and is the parity of the spin structure of S (see [3, 8]).

When there are odd degree singularities, $\phi(\alpha,\beta)$ clearly depends on the choice of (α,β): indeed, if we continuously deform an element α_i or β_i until we "cross an odd singularity", its index changes by a odd value.

Now we choose once for all an ordered pairing of the set of odd degree singularities, *i.e.* we denote by $(P_1^-,P_1^+),\dots,(P_s^-,P_s^+)$ these singularities. For a simple curve γ joining P_j^- to P_j^+, we define $ind(\gamma)$ to be the index (mod 2) of the Gauss map defined by a simple smooth path $\tilde\gamma$, whose image is in a small neighborhood of the image of γ, and such that:

- $\tilde\gamma$ is tangent in its starting point to the fixed horizontal separatrix of P_j^-
- $\tilde\gamma$ is tangent in its ending point to the fixed horizontal separatrix of P_j^+, rotated by π.

Since P_j^+,P_j^- are both of odd degree, their corresponding conical angles are an even multiple of 2π, and hence $ind(\gamma)$ does not depend on the choice of $\tilde\gamma$.

Now, for a fixed choice of $(\alpha_i,\beta_i)_i$, let $\gamma_1,\dots,\gamma_s$ be a collection of simple curves, with no pairwise intersections, with γ_j joining P_j^- to P_j^+, and each γ_j does not intersect the $(\alpha_i,\beta_i)_i$. Then, we define

$$Sp(\alpha,\beta,\gamma)=\phi(\alpha,\beta)+\sum_j ind(\gamma_j) \mod 2.$$

It is obvious that $Sp(\alpha,\beta,\gamma)$ can take two values, for different choices of horizontal separatrices. We will prove that $Sp(\alpha,\beta,\gamma)$ does not depend on the choice of α,β,γ (only on the choice of the oriented pairing of the odd degree singularities). Hence, Sp defines a topological invariant for the connected components of $\mathcal{H}^{hor}$.

Lemma 5.1. *$Sp(\alpha,\beta,\gamma)$ does not depend on the choice of γ.*

Proof. Let γ,γ' be two collections of simple curves as above. Up to making a small perturbation of γ and γ', we can assume that the number of intersection points of any two curves in this collection is finite.

The surface $D = S \backslash \cup_i (\alpha_i \cup \beta_i)$ is a topological disc with $g-1$ holes. By definition γ_1 and γ'_1 have the same end points. If they do not intersect in their interior, then $ind(\gamma_1) = ind(\gamma'_1) + k$, where k is the number of odd singularity of a component of $D \backslash (\gamma_1 \cup \gamma'_1)$. In this case, the number of intersection points (mod 2) between γ'_1 and the $(\gamma_j)_{j \neq 1}$ is k. If γ_1 and γ'_1 have $N > 0$ intersection points we find γ''_1 with the same endpoints as γ_1, such that γ_1 and γ''_1 have no interior intersection point and such that γ''_1 and γ'_1 have $N' < N$ intersection point and we iterate the procedure.

In particular replacing γ_1 by γ'_1 preserves the value:

$$\sum_i ind(\gamma_i) + N(\gamma) \mod 2.$$

where $N(\gamma)$ is the number of self-intersections of the family γ.

Hence, successively replacing γ_i by γ'_i, we obtain:

$$\sum_i ind(\gamma_i) = \sum_i ind(\gamma'_i) \mod 2.$$

□

Lemma 5.2. *$Sp(\alpha, \beta, \gamma) = Sp(\alpha, \beta)$ does not depend on the choice of the symplectic basis α, β.*

Proof. Let (α, β, γ) and $(\alpha', \beta', \gamma')$ be two families of curves as above. We first show that, there exists α'', β'' homotopic to α, β, that do not intersect γ' and such that:

$$Sp(\alpha, \beta, \gamma) = Sp(\alpha'', \beta'', \gamma').$$

By the previous lemma, we can choose γ so that it does not intersect γ'. Let $\gamma'_1 \in \gamma'$, and we assume that it intersects α, β. Consider the last intersection point, *i.e.* $x_0 = \gamma'_1(t_0)$, and α, β do not intersect $(\gamma'_1(t))_{t > t_0}$. We assume for instance that the intersection is with α_1.

Now, we push α_1 until it crosses the endpoint P_1^+. So, $ind(\alpha_1)$ is replaced by $ind(\alpha_1) \pm \deg(P_1^+)$. But now, α_1 intersects γ_1. For ε small enough, the ε-boundary of $\alpha_1 \cup \beta_1$, once removed α_1, β_1 is a topological annulus, hence, we can modify γ_1 in that neighborhood to avoid α_1 (see Figure 7). This replaces $ind(\gamma_1)$ by $ind(\gamma_1) + ind(\beta_1) + 1$. In particular $Sp(\alpha, \beta, \gamma)$ is not changed by this procedure, and the new family (α, β) has one intersection point less with γ'.

Iterating the process, we eventually obtain α'', β'' that do not intersect γ'.

Now, we consider the canonical continuous map $\phi : S \to \overline{S}$, where $\overline{S}$ is the surface obtained by collapsing each curve γ'_i to a single point. The map ϕ induces an homeomorphism from $S \backslash \gamma'$ to its image.

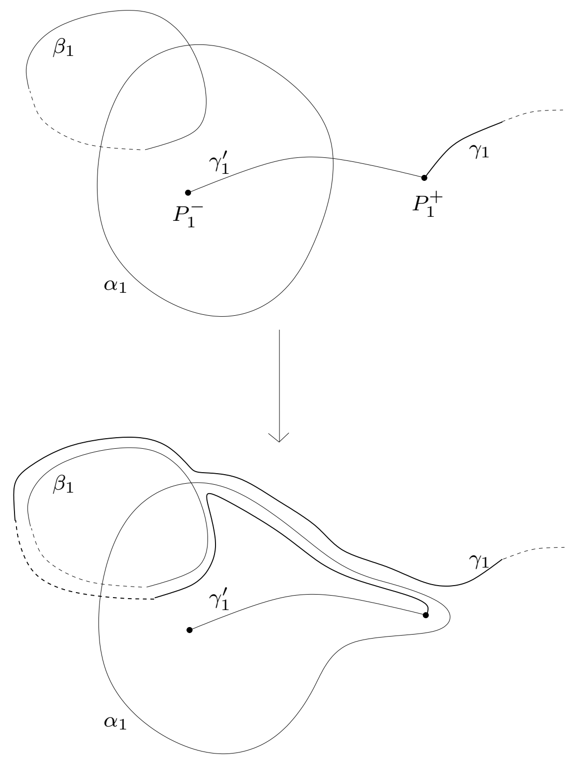

FIGURE 7. Decreasing the number of intersection points.

For a simple closed curve $\overline{c} = \phi(c)$ in $\overline{X}$, that does not pass through the image of a singularity, we define $ind(\overline{c}) = ind(c)$. The map $\theta(\overline{c}) = ind(\overline{c}) + 1 \mod 2$ defines a quadratic form on $H_1(\overline{S}, \mathbb{Z}/2\mathbb{Z})$ (see [7, 8]). Hence its Arf invariant is

$$\sum_i (ind(\alpha_i'') + 1)(ind(\beta_i'') + 1) = \sum_i (ind(\alpha_i') + 1)(ind(\beta_i') + 1) \mod 2.$$

Hence $Sp(\alpha', \beta', \gamma') = Sp(\alpha'', \beta'', \gamma') = Sp(\alpha, \beta, \gamma)$. □

When the stratum is of the form $\mathcal{H}(-1, -1, n_1, \dots, n_r)$, with positive integers $n_1, \dots, n_r$, we define the invariant after first cutting the two infinite cylinder, and gluing them together to make a finite cylinder, on a surface in the stratum $\mathcal{H}(n_1, \dots, n_r)$.

Remark 5.3. The invariant Sp obtained depends on the choice of the pairing $\{(P_1^-, P_1^+), (P_2^-, P_2^+), \dots, (P_s^-, P_s^+)\}$ of the odd degree singularities. We can ask how $Sp(S)$ changes when we replace the pairing by another one. It is enough to study the case when we interchange P_1^- with P_1^+ and when we interchange P_1^- with P_2^-.

(1) For the first case (P_1^- with P_1^+). $Sp(S)$ is clearly replaced by $Sp(S)+1$.
(2) For the second case, we replace again $Sp(S)$ by $Sp(S)+1$. Indeed, consider as before two nonintersecting curves γ_1, γ_2 joining P_1^- to P_1^+ and P_2^- to P_2^+ respectively. Then, deform them until γ_1, γ_2 are tangent on a unique intersection point. Then, we obtain a new pair γ_1', γ_2' joining P_1^- to P_2^+ and P_2^- to P_1^+ such that $Ind(\gamma_1) + Ind(\gamma_2) = Ind(\gamma_1') + Ind(\gamma_2')$. But γ_1', γ_2' intersect (transversally) on one point. From the proof of Lemma 5.2, modifying γ_1', γ_2' so that they don't intersect will change the invariant by adding 1.

In particular, the invariant Sp can be seen as a function from the set of pairings of odd degree singularities to $\mathbb{Z}/2\mathbb{Z}$, satisfying the above conditions.

5.2. **Proof of Theorem 1.1.** We assume first that there are only even degree singularities (or even degree zeroes, and a pair of simple poles.) We also assume that the underlying connected component is not a hyperelliptic one.

Let P be a singularity of the base surface S_p. From Proposition 3.9, the element $\sigma_P = 2\delta_P$ is in Mon. Since the singularity has even degree, $\delta_P \in Mon$. Hence, $Mon = Hor$.

Now we assume that there are odd degree singularities. First observe that if there is a simple pole P (except the case of two simple poles and no other poles), then for any $Q \neq P$, $\tau_{P,Q} = \delta_Q \in Mon$. Hence $Mon = Hor$.

So, we can assume that there are no simple poles. As before, for each singularity P of even degree, we use σ_P to see that $\delta_P \in Mon$. Now, fix a frame, and consider $P_1, \dots, P_{2r}$ the singularities of odd degree. Then, for i from 1 to $2r-1$ successively, we use σ_{P_i} and $\tau_{P_i,P_{i+1}}$ to obtain an element of the form $\delta_{P_i} + k_i\delta_{P_i+1} \in Mon$. For, P_{2r}, we can only get half of possible horizontal separatrices, by using $\sigma_{P_{2r}}$. Hence, we see that Mon is a subgroup of Hor of index at most 2. According to Section 5.1, in this case the invariant Sp can take two values, therefore $Mon \neq Hor$ and Mon is a subgroup of Hor of index 2.

Observe that if there is only one pole Q, for a given singularity P, the element $\tau_{Q,P}$ is not necessarily possible. Hence, we first fix the separatrix of Q by using $\rho = \sum_R \delta_R$, the element in Mon that corresponds to rotating the surface by 2π, and continue as above.

The case with two simple poles is similar and left to the reader.

5.3. **Hyperelliptic connected component.** From [3], a hyperelliptic connected component of the moduli space of meromorphic differentials is necessarily a component of a stratum of the following kind:

- $\mathcal{H}(n, n, p, p)$
- $\mathcal{H}(2n, p, p)$
- $\mathcal{H}(n, n, 2p)$
- $\mathcal{H}(2n, 2p)$

for some, $n > 0$ and $p < 0$.

Let $\mathcal{C}_{hyp}$ be a hyperelliptic connected component of the moduli space of translation surface with poles. Let $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ be the set of framed translation surfaces whose underlying surfaces are in $\mathcal{C}_{hyp}$. We assume that there are two (marked) zeroes N_1, N_2 of the same degree. Denote by i the hyperelliptic involution. Since $i(N_1) = N_2$, the image by i of the marked horizontal separatrix l_1 of N_1 is a horizontal separatrix $i(l_1)$ of N_2. The angle between the marked horizontal separatrix l_2 of N_2 and $i(l_1)$ is an odd multiple of π and is between π and $(2n+1)\pi$ and is invariant by continuous deformations. Hence, it is an invariant Φ_{zeroes} of connected components, which can clearly get $n+1$ values. Similarly, if there are two poles of the same degree, there is an analogous invariant Φ_{poles} for the horizontal separatrices associated to the pair of poles, with $|p+1|$ values.

We have the following lemma:

Lemma 5.4. *Let $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ be a hyperelliptic connected component with framed horizontal separatrices. Let $S^{hor}_b \in \mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$. Let $P \in S^{hor}_b$ be a (marked) singularity.*

- *If there exists another singularity P' of the same degree, then $\tau_{P,P'} \in Mon$.*
- *Otherwise, $\sigma_P \in Mon$.*

Proof. The proof is easy and left to the reader. □

This lemma, associated to the definition of the invariant gives the following theorem.

Theorem 5.5. *Let $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ be a hyperelliptic connected component with marked horizontal separatrices.*

- *If $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}} \subset \mathcal{H}(n, n, p, p)$, for some $n > 0$ and $p < -1$, then $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ has $(n+1)|p+1|$ connected components distinguished by the maps Φ_{zeroes} and Φ_{poles}.*
- *If $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}} \subset \mathcal{H}(n, n, 2p)$, for some $n > 0$ and $p < 0$, or $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}} \subset \mathcal{H}(n, n, -1, -1)$ then $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ has $(n+1)$ connected components distinguished by the map Φ_{zeroes}.*

- *If $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}} \subset \mathcal{H}(2n, p, p)$, for some $n > 0$ and $p < -1$, then $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ has $|p+1|$ connected components distinguished by the maps Φ_{poles}.*
- *If $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}} \subset \mathcal{H}(2n, 2p)$ for some $n > 0$ and $p < -1$ or $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}} \subset \mathcal{H}(2n, -1, -1)$, then $\mathcal{H}^{hor}_{\mathcal{C}_{hyp}}$ is connected.*

Proof. The proof is easy and left to the reader. □

6. Zero genus

Let $\mathcal{H} = \mathcal{H}(n_1, \dots n_r)$ be a stratum of genus zero translation surfaces. In this section, we count the number of connected components of $\mathcal{H}^{hor}(n_1, \dots, n_r)$ and define a topological invariant distinguishing these connected components.

We assume that there are no simple poles. Then, for $i \neq j$, we denote by N_{ij} the (positive) integer:

$$N_{ij} = \gcd\left(\{n_k\}_{k \notin \{i,j\}} \cup \{n_i + 1, n_j + 1\}\right).$$

Let $S \in \mathcal{H}^{hor}(n_1, \dots, n_r)$, and denote $P_1, \dots, P_r$ the (marked) singularities of degree $n_1, \dots, n_r$ respectively. For any $i < j$, let γ_{ij} be a path joining P_i to P_j according to the marked horizontal separatrices (as in Section 5.1). Then $ind(\gamma_{ij})$ is an integer and $ind(\gamma_{ij}) \mod N_{ij}$ does not depend on the choice of γ_{ij} (only on the choice of marked directions).

Now we define $\Phi(S)$ as:

$$\Phi(S) = (ind(\gamma_{ij}))_{i<j} \in \prod_{i<j} \mathbb{Z}/N_{ij}\mathbb{Z}.$$

The map Φ is clearly a locally constant map, and hence, an invariant of connected components of $\mathcal{H}^{hor}(n_1, \dots, n_r)$. Note that Φ depends implicitly on the choice of the ordering of the singularities. Note that the map Sp is also well defined if there are some odd degree singularities.

Theorem 6.1. *Let $\mathcal{H} = \mathcal{H}(n_1, \dots n_r)$ be a stratum of genus zero translation surfaces.*

- *If there exists $i_0 \in \{1, \dots, r\}$ such that $n_{i_0} = -1$, then $\mathcal{H}^{hor}$ is connected.*
- *If all n_i are different from -1 and if there are at most two odd degree singularities, then there are $N = \prod_{i<j} N_{ij}$ connected components of $\mathcal{H}^{hor}$, and two elements S_1 and S_2 of $\mathcal{H}^{hor}$ are in the same connected component if and only if $\Phi(S_1) = \Phi(S_2)$.*

- *Otherwise, there are* $2N$ *connected components of* $\mathcal{H}^{hor}$*, and two elements* S_1 *and* S_2 *of* $\mathcal{H}^{hor}$ *are in the same connected component if and only if* $\Phi(S_1) = \Phi(S_2)$ *and* $Sp(S_1) = Sp(S_2)$*.*

Note that the first part is obvious, since $\tau_{P_{i_0},Q} = \delta_Q$ is in Mon. So, from now, we assume that there are no simple poles.

Lemma 6.2. *The map* Φ *is surjective. Furthermore, if there are at least three odd degree singularities, the map* $\Phi \times Sp$ *is surjective.*

Proof. We first prove that Φ is surjective. Let $i_0 \in \{1, \dots, r\}$. When we replace the marked horizontal separatrix l_{i_0} corresponding to P_{i_0} by the one obtained by rotating l_{i_0} by 2π counterclockwise, it adds to $\Phi(S)$ the element η_{i_0} whose value is

- -1 in the factor $\mathbb{Z}/N_{i_0 j}\mathbb{Z}$ for each $j > i_0$.
- 1 in the factor $\mathbb{Z}/N_{j i_0}\mathbb{Z}$ for each $j < i_0$.
- 0 in the other factors.

Since the integers $\{N_{ij}\}_{i<j}$ are pairwise relatively prime, the element η_{i_0} generates $\prod_{j \neq i_0} \mathbb{Z}/N_{i_0 j}\mathbb{Z}$. In particular, $\eta_1, \dots, \eta_r$ generates the group $\prod_{i<j} \mathbb{Z}/N_{ij}\mathbb{Z}$. So the map Φ is surjective.

When there are at least three odd degree singularities, $N = \prod_{i<j} N_{ij}$ is odd. In particular, choosing i so that n_i is odd, and rotating l_i by $2\pi \prod_{j \neq i} N_{ij}$ does not change $\Phi(S)$, but changes $Sp(S)$ by $1 = \prod_{j \neq i} N_{ij}$ (mod 2), so the map $\Phi \times Sp$ is surjective. □

Lemma 6.3. *Let* $k \in \{1, \dots, r\}$*, for each* $i, j \neq k$*, with* $i \neq j$ *the following elements are in the group* Mon*.*

- $n_i(n_i + 1)\delta_{P_k}$,
- $2n_i n_j \delta_{P_k}$.

Furthermore, the subgroup of Mon *generated by these elements, seen as a subgroup of* $\mathbb{Z}/(n_k + 1)\mathbb{Z}$*, is the subgroup generated by* $\varepsilon_k \prod_{i \neq k} N_{ki}$*, where* $\varepsilon_k = 2$ *if* n_k *is odd and there are at least two other odd singularities, and* $\varepsilon_k = 1$ *otherwise.*

Proof. A direct computation shows that the element $n_i(n_i + 1)\delta_{P_k}$ is given by $(n_i + 1)\tau_{P_k,P_i}$, and the element $2n_i n_j \delta_{P_k}$ is given by $n_j \tau_{P_k,P_i} + n_i \tau_{P_k,P_j} - n_k \tau_{P_i,P_j}$.

The subgroup of $\mathbb{Z}/(n_k + 1)\mathbb{Z}$ generated by these elements is $< d_k >$, where:

$$d_k = \gcd\left(\{2n_i n_j\}_{i \neq j \neq k} \cup \{n_i(n_i + 1)\}_{i \neq k} \cup \{n_k + 1\}\right).$$

Let $p > 2$ be a prime number that divides d_k, $\alpha = \nu_p(d_k)$ its p-adic valuation. By definition of d_k, p^α divides $n_k + 1$, and for each $i \neq k$, p^α divides n_i or $n_i + 1$. It cannot always divides n_i since $(n_k + 1) + \sum_{i \neq k} n_i =$

-1. Also, if there are two indices $i \neq j$, with $i, j \neq k$ such that p^α does not divide n_i and n_j, then p^α does not divide $2n_i n_j$, which contradicts $p^\alpha | d_k$. Hence there is exactly one index $i \neq k$ such that p^α does not divide n_i, hence, divides $n_i + 1$. So, $p^\alpha | N_{ki}$.

Conversely, let $p > 2$ be a prime number and $\alpha = \nu_p(\varepsilon_k \prod_{i \neq k} N_{ki})$. Since the $\{N_{ki}\}_i$ are pairwise relatively prime, there is an index i_0 such that $p^\alpha | N_{ki_0}$. Hence, we easily see that p^α divides d_k.

We have proven that $\nu_p(d_k) = \nu_p(\varepsilon_k \prod_{i \neq k} N_{ki})$ for $p > 2$. Now, we prove the same for the case $p = 2$. If n_k is even, then both d_k and $\varepsilon_k \prod_{i \neq k} N_{ki}$ are odd.

Assume that n_k is odd and there are at least three odd singularities. Denote by $i_0, j_0 \neq k$ the indices of two odd degree singularities different from P_k. We see that the 2-adic valuation of d_k is 1 by considering $2n_{i_0} n_{j_0}$, and the 2-adic valuation of $\varepsilon_k \prod_{i \neq k} N_{ki}$ is also 1 since $\varepsilon_k = 2$ and all the N_{ki} are odd.

Assume that n_k is odd and there are exactly two odd degree singularities P_k, P_{i_0} on the surface. Let $\alpha > 0$ such that $2^\alpha | d_k$. Then 2^α divides $n_k + 1$, and $n_i(n_i + 1)$ for each $i \neq k$. In particular, it divides $(n_{i_0} + 1)$ (since n_{i_0} is odd), and for each $i \notin \{k, i_0\}$, it divides n_i (since $n_i + 1$ is odd). So, $2^\alpha | N_{ki_0}$.

Conversely, let $\alpha > 0$ such that $2^\alpha | \prod_{i \neq k} N_{ki}$. The integer N_{ki} is even if and only if $i = i_0$. Hence, $2^\alpha | N_{ki_0}$ and 2^α divides each n_i, for $i \notin \{k, i_0\}$. So, it divides $n_i n_j$, for each $i, j \neq k$ with $i \neq j$. Finally, $2^\alpha | d_k$.

Hence, we have proven that $d_k = \varepsilon_k \prod_{i \neq k} N_{ki}$. □

Proof of Theorem 6.1. We first assume that there are at most two odd degree singularities, so that for each i, $\varepsilon_i = 1$ in the above lemma. In order to simplify notation, we set $N_{ii} = 1$ for each i.

From Lemma 6.2, Mon has at most

$$N = \frac{\prod_{i \in \{1,\dots,r\}} |n_i + 1|}{\prod_{1 \leq i < j \leq r} N_{ij}}.$$

elements. The theorem will follow if we prove that Mon has exactly this number of elements.

Since for each i, j, $N_{ij} | (n_i + 1)$, there is a canonical map $\mathbb{Z}/(n_i + 1)\mathbb{Z} \to \mathbb{Z}/N_{ij}\mathbb{Z}$, so it induces a canonical map:

$$\Psi : \begin{array}{ccc} \prod_{i=1}^r \mathbb{Z}/(n_i + 1)\mathbb{Z} & \to & \prod_{i,j \in \{1,\dots,r\}} \mathbb{Z}/N_{ij}\mathbb{Z} \\ (x_i)_i & \mapsto & (x_i \mod N_{ij})_{i,j} \end{array}.$$

Since $\{N_{ij}\}_{i,j}$ are relatively prime, by the Chinese Remainder Lemma, the kernel of Ψ is $\prod_i u_i\mathbb{Z}/(n_i+1)\mathbb{Z}$, where $u_i = \prod_{j\neq i} N_{ij}$ and is a subgroup of Mon by the previous lemma.

For a pair (i_0, j_0) of distinct indices, the image by Ψ of the element $-\tau_{P_{i_0},P_{j_0}}$ is the element $E_{i_0j_0}+E_{j_0i_0}$, where E_{ij} is the element which is 1 for the indices (i,j) and 0 everywhere else.

In particular, the image $\Phi(Mon)$ contains at least $\prod_{i<j} N_{ij}$ elements. So Mon contains at least,

$$\frac{\prod_{i=1}^r |n_i+1|}{\prod_{i=1}^r u_i}\prod_{1\leq i<j\leq r} N_{ij} = \frac{\prod_{i=1}^r |n_i+1|}{\prod_{1\leq i<j\leq r} N_{ij}}$$

elements. Therefore it contains exactly that many elements.

Now we assume that there are at least three odd degree singularities. In order to simplify the notation, we define, for $i\neq 0$ $N_{i0}=N_{0i}=\varepsilon_i$, where, $\varepsilon_i=2$ if n_i is odd and $\varepsilon_i=1$ otherwise. From Lemma 6.2, Mon has at most

$$N' = \frac{1}{2}\frac{\prod_{i=1}^r |n_i+1|}{\prod_{1\leq i<j\leq r} N_{ij}}$$

elements. We proceed as before, but replace the map Ψ by the map $\tilde{\Psi}$

$$\tilde{\Psi}: \begin{array}{ccc} \prod_{i=1}^r \mathbb{Z}/(n_i+1)\mathbb{Z} & \to & \prod_{i=1}^r\prod_{j=0}^r \mathbb{Z}/N_{ij}\mathbb{Z} \\ (x_i)_i & \mapsto & (x_i \mod N_{ij})_{(i,j)\in\{1,\dots,r\}\times\{0,\dots,r\}} \end{array}.$$

Since all N_{ij} are odd and pairwise relatively prime, we see as before that the kernel is $\prod_i u_i\mathbb{Z}/(n_i+1)\mathbb{Z}$, where $u_i=\prod_{j=0}^r N_{ij}=\varepsilon_i\prod_{j\neq i}N_{ij}$, and is a subgroup of Mon by the previous lemma.

If n_{i_0} or n_{j_0} is even, then the image by $\tilde{\Psi}$ of the element $-\tau_{P_{i_0},P_{j_0}}$ is the element $E_{i_0j_0}+E_{j_0i_0}$. If both n_{i_0} and n_{j_0} are odd, then we get $E_{i_0j_0}+E_{j_0i_0}+E_{i_00}+E_{j_00}$. Therefore $\tilde{\Psi}((n_{i_0}+n_{j_0})\tau_{P_{i_0},P_{j_0}}) = 2(E_{i_0j_0}+E_{j_0i_0})$. Since $N_{i_0j_0}$ is odd, there is a multiple of $\tau_{P_{i_0},P_{j_0}}$ whose image by $\tilde{\Psi}$ is $E_{i_0j_0}+E_{j_0i_0}$. Finally, we obtain that the image by $\tilde{\Psi}$ of Mon contains at least $2^{n-1}\prod_{i<j}N_{ij}$ elements, where n is the number of odd degree singularities, and therefore, Mon has at least N' elements. Hence it contains exactly that many elements. □

7. Partially marked surfaces

Coming back to the initial motivation of this paper. It is natural to study the moduli space of surfaces where only a subset of the singularities have a marked horizontal separatrix.

Let $g\geq 1$. Let $\mathcal{H}(n_1^{\alpha_1},\dots,n_r^{\alpha_r})$ be a stratum of the moduli space of genus g meromorphic differentials, and let $\mathcal{C}\subset\mathcal{H}(n_1^{\alpha_1},\dots,n_r^{\alpha_r})$

be a nonhyperelliptic connected component. Let $\{n_1^{\beta_1}, \dots, n_r^{\beta_r}\} \subset \{n_1^{\alpha_1}, \dots, n_r^{\alpha_r}\}$, and let $\mathcal{H}_{\mathcal{C}}^{part}$ be the corresponding moduli space of partially framed surfaces.

The following result follows easily from Theorem 1.1.

Corollary 7.1. *Assume that there are non-marked odd degree singularities, then $\mathcal{H}_{\mathcal{C}}^{part}$ is connected. Otherwise, $\mathcal{H}_{\mathcal{C}}^{part}$ has the same number of connected components as $\mathcal{H}_{\mathcal{C}}^{hor}$.*

Assume now that the genus is zero, and denote by $\{P_1, \dots, P_r\}$ the singularities, with $\{P_1, \dots, P_s\}$, $s < r$ the marked ones. Now we define $\Phi_{part}(S)$ as:

$$\Phi_{part}(S) = (ind(\gamma_{ij}))_{1 \le i < j \le s} \in \prod_{1 \le i < j \le s} \mathbb{Z}/N_{ij}\mathbb{Z}.$$

i.e. we restrict the map Φ to the marked singularities.

The following Theorem is an easy corollary of Theorem 6.1.

Corollary 7.2. *Let $\mathcal{H} = \mathcal{H}(n_1, \dots n_r)$ be a stratum of genus zero translation surfaces, such that $\mathcal{H}^{hor}$ is not connected.*

- *If there are some non-marked odd degree singularities, or if there are at most two odd degree singularities, then two elements S_1 and S_2 of $\mathcal{H}^{part}$ are in the same connected component if and only if $\Phi_{part}(S_1) = \Phi_{part}(S_2)$.*
- *Otherwise, two elements S_1 and S_2 of $\mathcal{H}^{hor}$ are in the same connected component if and only if $\Phi_{part}(S_1) = \Phi_{part}(S_2)$ and $Sp(S_1) = Sp(S_2)$.*

Appendix A. More about connected sums

We look back at the construction described in Section 4.1. Let S, S' be translation surfaces, and let $N \in S$, be a singularity of degree $n \geq 0$ and let $N' \in S'$ be a singularity of degree $n' = -2 - n < 0$ with zero residue.

First, we observe that once the scaling of S', the neighborhood U of N and the pointed neighborhood V of N' are fixed, there remains a finite number of possibilities for gluing together $S \backslash U$ and $S' \backslash V$. There are exactly $n + 1$ choices. If S, S' are translation surfaces with marked horizontal separatrices, we can fix this choice by imposing that the separatrix corresponding to N coincides with the separatrix corresponding to N' (rotated by π). Once this combinatorial choice is fixed, the other choices involved in the construction (scaling, U, V) give a connected set of surfaces.

One would like to glue S and S' along several pairs $(N_i, N_i') \in S \times S'$. We assume that for each i the singularity N_i has degree $n_i > 0$, and the singularity N_i' has degree $-2 - n_i < 0$ with zero residue. It is natural to glue successively along (N_1, N_1') then (N_2, N_2') and so on. However, after the first step, the singularities belong to the same surface. self-gluing construction is made analogously, but in general it is not possible any more to shrink one side to make space. However, in this case, shrinking sufficiently S' at first solves this issue (since all singularities N_i' have negative degree). As before, there is a combinatorial choice for each pair which is fixed if the initial surfaces are with marked horizontal separatrices, and two gluings with the same combinatorial choices give surfaces in the same connected component.

Computing the Sp invariant of the new surface is easy. It is enough to consider simple gluings and self-gluings.

(1) For simple gluing, there are two cases: either the two singularities are even, or they are odd. In the first case, the Sp invariant of the new surface is clearly the sum of the Sp invariant of the two surfaces. In the second case, following Remark 5.3 we first choose a suitable pairing of odd degree singularities: we consider pairings of the form $\{(P_1^-, P_1^+)\} \cup \mathcal{P}$ with $P_1^+ = N$ for S and $\{(P_1'^-, P_1'^+)\} \cup \mathcal{P}'$ with $P_1'^- = N'$. We consider the following pairing for the new surface

$$\{(P_1^-, P_1'^+)\} \cup \mathcal{P} \cup \mathcal{P}'.$$

Then, we easily see that, with these pairings, the Sp invariant of the new surface is the sum of the Sp invariant of S and S'.

(2) For self-gluing, the new surface has genus one more than the initial surface. We easily see that the Sp invariant does not change for any pairing when glued singularities are of even order, and for a pairing containing $(N, N') = (P_1^-, P_1^+)$ when the glued singularities have odd order.

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