A short introduction to dispersive PDE
PRELIMINARY VERSION

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Introduction

These notes correspond to (half of) a course taught jointly with Mihai Maris in 2013 on non linear dispersive equations and non linear elliptic equations. The material presented here, from Chapter 1 to Chapter 4, deals with the dispersive equations part and collects what was thaught in class for 16.5 hours. It contains essentially an introduction to the Cauchy problem for the non linear Schrödinger equation (NLS), more precisely the L^2 subcritical and L^2 critical ones. To put things in a wider perspective, the first chapter also presents the non linear wave equation and the $\dot{H}^1$ subcritical NLS, in an informal fashion (not to say naive here and there).

The main goal of these lectures is to describe the functional analytic tools required to state rigorously and solve the Cauchy problem for NLS. The presentation is completely self contained (assuming only some standard background on functional analysis and distribution theory) hoping that the student wishing to understand the proofs from A to Z will find all the material in the present source. We would reach our goal if this could give the minimal autonomy to the student interested in learning more on dispersive PDE. In this spirit, we have decided to include an appendix on the Littlewood-Paley theory; this was not discussed in class but, as is well known, it is a very useful tool for dispersive PDE. We illustrate one application with the proof of homogeneous Sobolev embbedings (with fractional derivatives). We hope to complete the present notes with a section on Strichartz estimates for the wave equation (whose proof uses the full strength of the Littlewood-Paley theory) to provide some additional material to the reader interested in the wave equation.

Chapter 1

A brief overview

In this chapter, we present the Schrödinger and wave equations at a formal level, to record some basic features of these equations and give some flavour of the related analytical problems or tools involved in their resolutions. This will motivate the notions introduced in the next chapters.

We start with some generalities about the linear equations. The Schrödinger equation on $\mathbb{R}^n$ reads

$$i\partial_t u - \Delta u = f, \tag{1.1}$$

where $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ is the Laplacian, f is a given function[1] depending on time, *i.e.* for some interval I,

$$f : I \times \mathbb{R}^n \to \mathbb{C},$$

and $u : I \times \mathbb{R}^n \to \mathbb{C}$ is the unknown function (or distribution). When $f \equiv 0$, one says (1.1) is the homogeneous Schrödinger equation. To solve (1.1), one imposes an initial condition u_0, say at $t = 0$ (assuming implicitly that $0 \in I$),

$$u(0, x) = u_0(x). \tag{1.2}$$

Typically in these lectures, u_0 will belong to some Sobolev space (of nonnegative order). Formally, the equation (1.1) can be seen as an ODE in infinite dimension since, as a function of t, u takes its values in an infinite dimensional space of functions (or distributions) of x. But the important difference with usual ODE of the form

$$\dot{X} + LX = V, \tag{1.3}$$

with $L : \mathcal{B} \to \mathcal{B}$ a linear and *continuous endomorphism* on a Banach space $\mathcal{B}$, is that the linear operator Δ is not a continuous endomorphism on any standard Banach space such as L^2 or H^2, so that the standard Cauchy-Lipschitz Theorem does not apply. One says Δ

[1] or even a temperate distribution

is unbounded (see Exercise 1.1 below). However, under suitable assumptions on u_0 and f which will be detailed further on, one can turn the Cauchy problem (1.1)-(1.2) into an integral equation by using the following so called *Duhamel formula*

$$u(t) = e^{-it\Delta}u_0 + \frac{1}{i}\int_0^t e^{-i(t-s)\Delta}f(s)ds, \tag{1.4}$$

much as the usual method of variation of parameters would give for inhomogeneous ODE (or, even more formally, as if Δ was a matrix).

Exercise 1.1. *Let $\mathcal{B} = L^q(\mathbb{R}^n)$ with $q \in [1, \infty)$. Show that the Laplacian*

$$\Delta : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$$

does not extend to a continuous endomorphism on $\mathcal{B}$. (Hint: Introduce $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $||\varphi||_{L^q} = 1$ and consider the family $\varphi_\epsilon(x) = \epsilon^{-n/q}\varphi(x/\epsilon)$ indexed by $0 < \epsilon < 1$).

The unboundedness of Δ makes the construction of $e^{-it\Delta}$ (which solves formally the homogeneous equation) non obvious since it cannot be defined by the usual exponential series. In general, one solves this problem by using the theory of *semigroups*. However, in this special case where the operator has constant coefficients, the Fourier analysis will allow to define it in a simple way by

$$(e^{-it\Delta}\varphi)(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}e^{it|\xi|^2}\widehat{\varphi}(\xi)d\xi, \tag{1.5}$$

where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$ and $\widehat{\varphi}$ is the Fourier transform of φ. In other words, $e^{-it\Delta}$ is the multiplication by $e^{it|\xi|^2}$ on the Fourier side: this is an important example of *Fourier multiplier* (see Chapter 2).

Similarly, one can consider the wave equation

$$\partial_t^2 u - \Delta u = f, \tag{1.6}$$

which, as a second order equation in time, is subject to two initial conditions

$$u(0, x) = u_0(x), \qquad \partial_t u(0, x) = u_1(x), \tag{1.7}$$

where u_0, u_1 and f are given functions (usually, one assumes that u_0, u_1, f and u are real valued). To rewrite the solution to (1.6)-(1.7) similarly to (1.4), we rewrite first the homogenous equation ($f \equiv 0$) as first order system

$$\partial_t U - \mathcal{A}U = 0,$$

with

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}.$$

Then

$$e^{t\mathcal{A}} = \begin{pmatrix} \cos t\sqrt{-\Delta} & \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \\ -\sqrt{-\Delta}\sin t\sqrt{-\Delta} & \cos t\sqrt{-\Delta} \end{pmatrix}, \tag{1.8}$$

where $\cos(t\sqrt{-\Delta})$ (resp. $\sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$) is the Fourier multiplier defined similarly to (1.5) via the multiplication by $\cos(t|\xi|)$ (resp. $\sin(t|\xi|)/|\xi|$). Then, according to the method of variation of parameters, the solution to

$$\partial_t U - \mathcal{A}U = F$$

is given by

$$U(t) = e^{t\mathcal{A}}U(0) + \int_0^t e^{(t-s)\mathcal{A}}F(s)ds.$$

This expression can be considered as formal (*e.g.* by pretending $\mathcal{A}$ is a finite dimensional matrix) by the non specialist reader, but we point out that this is completely rigorous provided we introduce the semigroup associated to (the relevant closure of) $\mathcal{A}$. Anyway, as said above for the Schrödinger equation, this can be seen as a formal fashion to write down the solution which can be justified afterwards by mean of Fourier multipliers. Using $F = \binom{0}{f}$ and considering only the first component of U, we find

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}f(s)ds, \tag{1.9}$$

for the solution to the Cauchy problem (1.6)-(1.7). Notice that we use here that $-\Delta$ is nonnegative for it corresponds to the multiplication by $|\xi|^2$ on the Fourier side. In the literature, the notation $\sqrt{-\Delta}$ is often replaced by $|\nabla|$ or $|D|$ (in the context of wave equations, D refers sometimes to (D_t, D_x) and one prefers the notation $\nabla = \nabla_x$ for spatial derivatives).

Exercise 1.2. *Work out the details!*

The integral formulations of the Schrödinger and wave equations given by (1.4) and (1.9) respectively will be important to solve the related nonlinear equations which we introduce now. We will consider special types of nonlinearities, namely pure power non linearities, that is

$$\begin{cases} i\partial_t u - \Delta u = -\mu|u|^{\nu-1}u \\ u(0) = u_0 \end{cases} \tag{NLS}$$

and

$$\begin{cases} \partial_t^2 u - \Delta u = -\mu|u|^{\nu-1}u \\ u(0) = u_0 \\ \partial_t u(0) = u_1 \end{cases} \tag{NLW}$$

where $\nu > 1$ and μ are real numbers (more conditions will be specified later on). In the sequel, we will refer to $\mu|u|^{\nu-1}u$ as the *nonlinearity*.

There are basically two different situations: $\mu > 0$, the *defocusing* case, and $\mu < 0$, the *focusing* case. In practice, we consider the cases $\mu = \pm 1$ but, to streamline the discussion, we will keep an abstract real parameter μ.

There are many questions one can ask about such equations. Two basic ones, which are similar to the Cauchy problem for ODE, are

- local well-posedness: for which type of initial data and nonlinearities can one solve the equations *locally* in time, *i.e.* on some open time interval I containing 0 ?
- global well-posedness[2]: for which type of initial data and nonlinearities can one solve the equations *globally* in time, *i.e.* for $t \in \mathbb{R}$?

In the left part of this introduction, we present three basic ideas or notions used to study these questions.

1. Use the Duhamel formula. By this we mean that rather than solving directly (NLS) or (NLW), we will rather recast these equations into integral ones, using the Duhamel formulas (1.4) and (1.9), namely

$$u(t) = e^{-it\Delta}u_0 - \frac{\mu}{i}\int_0^t e^{-i(t-s)\Delta}|u(s)|^{\nu-1}u(s)ds, \tag{1.10}$$

for (NLS), and

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}u_1 - \mu\int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}|u(s)|^{\nu-1}u(s)ds, \tag{1.11}$$

for (NLW). Of course, checking the equivalence of these integral formulations and their PDE counterparts is not completely obvious and requires some analysis. One interest of these formulations is that there are no longer derivatives in (1.10) and (1.11) which suggests one may solve these equations without assuming much smoothness on u.

2. The long time behavior of solutions should depend on the sign of μ. To justify this intuition, we consider first a naive finite dimensional analogue in the following exercise.

Exercise 1.3. *Consider the Hamiltonians (i.e. functions) $H_\pm$ defined on $\mathbb{R}^2$ by*

$$H_\pm(x,\xi) = \xi^2 \pm x^4.$$

[2] the exact definitions of local and global well posedness require more than local or global existence in time. One also asks for some nice dependence on the initial data, but we forget about this aspect in the introduction.

We are interested in the solutions to

$$\begin{cases} \dot{x} = (\partial_\xi H_\pm)(x,\xi) \\ \dot{\xi} = -(\partial_x H_\pm)(x,\xi) \end{cases} \qquad \begin{cases} x(0) = x_0 \\ \xi(0) = \xi_0 \end{cases} \tag{Ham$_\pm$}$$

where $\dot{}$ *means 'time derivative'.*

1. *Assume that the solution to* (Ham$_\pm$) *is defined on an interval* I*. Show that the function* $t \mapsto H_\pm(x(t),\xi(t))$ *is constant on* I*.*

2. *Show that for any* (x_0,ξ_0) *the solution to* (Ham$_+$) *is global in time.*

3. *Show that there are initial data* (x_0,ξ_0) *for which the solution to* (Ham$_-$) *blows up in finite time. (Hint: check that* $(x(t),\xi(t)) = \big(\alpha(a-bt)^{-1}, \frac{\alpha b}{2}(a-bt)^{-2}\big)$ *is a solution provided that* $b^2 = 4\alpha^2$*.)*

The analogy between this exercise and our nonlinear PDE is as follows. One can associate *energy functionals* to (NLS) and (NLW), which take respectively the form

$$E_{\mathrm{Sch}}(u(t)) = \int_{\mathbb{R}^n} \frac{|\nabla u(t,x)|^2}{2} + \mu \frac{|u(t,x)|^{\nu+1}}{\nu+1} dx \tag{1.12}$$

for (NLS), and

$$E_{\mathrm{Wav}}(u(t)) = \int_{\mathbb{R}^n} \frac{(\partial_t u(t,x))^2}{2} + \frac{|\nabla u(t,x)|^2}{2} + \mu \frac{|u(t,x)|^{\nu+1}}{\nu+1} dx, \tag{1.13}$$

for (NLW). The first analogy with Exercise 1.3 is that these energy functionals (or Hamiltonians) are conserved by the flow of the equations in the sense that,

$$u \text{ solves (NLS)} \qquad \Longrightarrow \qquad \frac{d}{dt} E_{\mathrm{Sch}}(u(t)) = 0, \tag{1.14}$$

and

$$u \text{ solves (NLW)} \qquad \Longrightarrow \qquad \frac{d}{dt} E_{\mathrm{Wav}}(u(t)) = 0. \tag{1.15}$$

To justify (1.14) and (1.15), we introduce the L^2 inner product

$$(v,w)_{L^2} = \int_{\mathbb{R}^n} \overline{v} w dx, \tag{1.16}$$

and consider first the wave equation. By using

$$\frac{\partial}{\partial t}|u(t)|^{\nu+1} = \frac{\partial}{\partial t}(|u(t)|^2)^{\frac{\nu+1}{2}} = \frac{\nu+1}{2}\big(\overline{\partial_t u} u + \overline{u}\partial_t u\big)|u|^{\nu-1},$$

and differentiating formally under the integral sign, we find

$$\frac{d}{dt}E_{\mathrm{Wav}}(u(t)) = \frac{1}{2}\int \overline{\nabla \partial_t u} \cdot \nabla u + \overline{\nabla u} \cdot \nabla \partial_t u + \big(\overline{\partial_t u} u + \overline{u}\partial_t u\big)|u|^{\nu-1} dx.$$

By integration by part, we also have

$$\int \overline{\nabla v} \cdot \nabla w dx \;=\; \sum_{j=1}^{n} \int \partial_j \overline{v} \partial_j w dx \;=\; \sum_{j=1}^{n} - \int \overline{v} \partial_j^2 w dx \;=\; (v, -\Delta w)_{L^2}$$

and it is then easy to see that

$$\frac{d}{dt} E_{\text{Wav}}(u(t)) = \text{Re}\left(\partial_t u, \partial_t^2 u - \Delta u + \mu u|u|^{\nu-1}\right)_{L^2},$$

which vanishes if u solves (NLW). Using the same calculation, we find

$$\begin{aligned} \frac{d}{dt} E_{\text{Sch}}(u(t)) &= \text{Re}\ \left(\partial_t u, -\Delta u + \mu u|u|^{\nu-1}\right)_{L^2} \\ &= \text{Im}\ \left(-i\partial_t u, -\Delta u + \mu u|u|^{\nu-1}\right)_{L^2} \\ &= \text{Im}\ || - \Delta u + \mu u|u|^{\nu-1}||_{L^2}^2 = 0. \end{aligned}$$

Interpretation. When $\mu > 0$, these energy functionals are sums of positive terms which are thus controlled individually over the evolution, much as $x_+(t)$ and $\xi_+(t)$ in the second question of Exercise 1.3. For this reason, one expects global well posedness for the defocusing equations (under suitable assumptions on ν and the initial data). In the focusing case where $\mu < 0$, we are in a situation similar to the third question of Exercise 1.3, and we may thus expect possible blow up. This quite rough intuition is correct in the sense that certain focusing (NLS) and (NLW) have indeed solutions blowing up in finite time. We shall see in the next paragraph that this is also related to the power ν.

In these short lectures, we won't have much room (or time!) to study such very interesting issues. However, it is worth mentionning this aspect of the problem since it motivates the study of solutions with low regularity: the control on the energy will give at best a control on the H^1 norms of solutions so if one wants to study (NLS) or (NLW) by using such conservation laws, it is important to use arguments involving only Sobolev norms of order at most 1.

We conclude this part by mentionning that the analogy between the Exercise 1.3 and nonlinear PDE could be pushed much further since (NLS) and (NLW) can be seen as Hamiltonian systems in infinite dimension, but such a point of view is far beyond the scope of these lectures.

3. Scaling properties and critical exponents.
Consider real numbers $\alpha, \beta, \gamma \in \mathbb{R}$ and $\lambda > 0$, and set

$$u_\lambda(t,x) = \lambda^\gamma u(\lambda^\alpha t, \lambda^\beta x).$$

- **Computation 1.** It is not hard to check that,

$$i\partial_t u_\lambda - \Delta u_\lambda + \mu|u_\lambda|^{\nu-1} u_\lambda = \left(\lambda^{\alpha+\gamma} i\partial_t u - \lambda^{2\beta+\gamma}\Delta u + \lambda^{\nu\gamma}\mu|u|^{\nu-1}u\right)_\lambda$$

and

$$\partial_t^2 u_\lambda - \Delta u_\lambda + \mu |u_\lambda|^{\nu-1} u_\lambda = \big(\lambda^{2\alpha+\gamma}\partial_t^2 u - \lambda^{2\beta+\gamma}\Delta u + \lambda^{\nu\gamma}\mu|u|^{\nu-1}u\big)_\lambda.$$

Choosing for the Schrödinger equation,

$$\alpha = 2, \qquad \beta = 1, \qquad \gamma = \frac{2}{\nu-1},$$

we have $\lambda^{\alpha+\gamma} = \lambda^{2\beta+\gamma} = \lambda^{\nu\gamma}$ and therefore, if $T \in (0, +\infty]$,

$$u \text{ solves (NLS) on } (-T, T) \iff u_\lambda \text{ solves (NLS) on } (-\lambda^{-2}T, \lambda^{-2}T). \tag{1.17}$$

For the wave equation, we choose

$$\alpha = 1, \qquad \beta = 1, \qquad \gamma = \frac{2}{\nu-1},$$

and observe similarly that

$$u \text{ solves (NLW) on } (-T, T) \iff u_\lambda \text{ solves (NLW) on } (-\lambda^{-1}T, \lambda^{-1}T).$$

- **Computation 2.** If we keep $\beta = 1$ and $\gamma = \frac{2}{\nu-1}$, it is not hard to check by a simple change of variable that

$$||u_\lambda(0)||_{L^2} = \lambda^{\frac{2}{\nu-1}-\frac{n}{2}} ||u(0)||_{L^2}, \tag{1.18}$$

and similarly, if we set $||\nabla v||_{L^2} = (\sum_j ||\partial_j v||_{L^2}^2)^{1/2}$, that

$$||\nabla(u_\lambda(0))||_{L^2} = \lambda^{\frac{2}{\nu-1}-\frac{n-2}{2}} ||\nabla u(0)||_{L^2}. \tag{1.19}$$

Using a parameter $s \in \{0, 1\}$ to cover both cases of Computation 2, we set

$$||v||_{\dot{H}^0} := ||v||_{L^2}, \qquad ||v||_{\dot{H}^1} := ||\nabla v||_{L^2}.$$

We also observe that

$$\frac{2}{\nu-1} - \frac{n-2s}{2} \geq 0 \qquad \iff \qquad 1 + \frac{4}{n-2s} \geq \nu. \tag{1.20}$$

This is related to the following definition.

Definition 1.4. *If $n > 2s$, define*

$$\nu_{\mathrm{c}} = 1 + \frac{4}{n-2s}.$$

The power $\nu > 1$ is said to be $\dot{H}^s$ subcritical (resp. $\dot{H}^s$ critical, resp. $\dot{H}^s$ supercritical) if $\nu < \nu_{\mathrm{c}}$ (resp. $\nu = \nu_{\mathrm{c}}$, resp. $\nu > \nu_{\mathrm{c}}$).

Here we have given a definition of critical exponent for a given regularity s (as in the lectures by J. Ginibre [2] for instance). One can adopt the symmetric point of view and define the critical regularity $s_{\rm c} := \frac{n}{2} - \frac{2}{\nu-1}$ associated to a given ν (see for instance Tao's book [6]).

To motivate the introduction of Definition 1.4, we now record several remarks illustrating the interest of criticality, in the particular case of (NLS) with $s = 0$ which is the only one we will have time to study.

The first simple remark is that if ν is L^2 subcritical and if we know that (NLS) has global in time solutions for all initial data with *small enough* L^2 *norms*, then we can automatically conclude that (NLS) has global in time solutions for all data $u_0 \in L^2$, regardless their sizes. Let us prove this claim. If u_0 is any L^2 initial datum, consider

$$u_{0,\lambda}(x) = \lambda^{\frac{2}{\nu-1}} u_0(\lambda x).$$

Then by using the subcriticality condition (*i.e.* the strict inequality in (1.20)) and (1.18), we see that

$$||u_{0,\lambda}||_{L^2} \to 0, \qquad \lambda \to 0.$$

Therefore, if λ is small enough, our assumption on small initial data ensures that (NLS) with initial datum $u_{0,\lambda}$ has a solution defined for $t \in \mathbb{R}$. Then, by rescalling, *i.e.* by using (1.17) with $\lambda^{-2}T = +\infty$, we see that (NLS) with initial datum u_0 has also a global in time solution, which proves the claim.

This situation has to be compared to the the one of the critical exponent $\nu = 1 + 4/n$. We will see that, in this case, (NLS) has global in time solutions for small initial data in L^2 but on the other hand, in the focusing case, that there are solutions (with non small initial data) which can blow up in finite time (see Section 4.2). This is an indication that a new non trivial effect shows up for critical exponents. Note that there is of course no contradiction with the previous remark: if ν is L^2 critical, the argument used in the first remark cannot be used to decrease the size of initial data by rescaling since, by criticality, the exponent vanishes in (1.18).

We shall actually see in Chapter 4 that the notion of subcriticality is already relevant for the local well posedness issue (but this aspect is more technical). In particular, we will see that for subcritical exponents, we can solve (NLS) on time intervals depending only on $||u_0||_{L^2}$. This will turn out to be very useful to prove global existence: thanks to a computation similar to the one of energy conservation, one can see that the flow of (NLS) preserves the L^2 norm, since (formally again) if u solves (NLS),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u(t)||_{L^2}^2 &= \mathrm{Re}(u, \partial_t u)_{L^2} \\ &= \mathrm{Re}\ i\big(u, -\Delta u + \mu|u|^{\nu-1}u\big)_{L^2} \\ &= \mathrm{Re}\ i\left(||\nabla u||_{L^2}^2 + \mu||u||_{L^\nu}^\nu\right) = 0. \end{aligned} \tag{1.21}$$

By using this conservation law and the fact that the lifespan of the solution depends on the L^2 norm of the initial condition (for L^2 subcritical exponents) we will be able to prove the global existence. Notice that, similarly to the case of the energy, this conservation law controls only the L^2 norm; this is another reason for considering solutions of limited smoothness.

Chapter 2

Some tools of harmonic analysis

2.1 Lebesgue spaces and real interpolation

In this section, we record several estimates on functions in Lebesgue spaces and operators thereon. In particular, the Marcinkiewicz interpolation Theorem 2.4 will be crucial to prove the so called *Strichartz estimates* which, in turn, allow to solve the fixed point equations (1.10) or (1.11).

We start with an elementary proposition.

Proposition 2.1. *1. Let $q \in [1, \infty]$ and $s > 0$ be such that $\frac{q}{s} \in [1, \infty]$. Then*

$$\big|\big||f|^s\big|\big|_{L^{\frac{q}{s}}} = ||f||_{L^q}^s. \tag{2.1}$$

2. For all real numbers $q_1, q_2 \in [1, \infty]$ such that $\frac{1}{q_1} + \frac{1}{q_2} =: \frac{1}{q} \leq 1$, one has

$$||fg||_{L^q} \leq ||f||_{L^{q_1}} ||g||_{L^{q_2}}. \tag{2.2}$$

3. Let $q_1 \leq q \leq q_2$ all belong to $[1, \infty]$. Then

$$||f||_{L^q} \leq ||f||_{L^{q_1}}^{\theta} ||f||_{L^{q_2}}^{1-\theta}, \tag{2.3}$$

with $\theta \in [0, 1]$ such that

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1 - \theta}{q_2}. \tag{2.4}$$

Proof. The estimate (2.1) is straightforward. To prove the item 2, one observes that

$$||fg||_{L^q}^q = \int |f|^q |g|^q \leq \big|\big||f|^q\big|\big|_{L^{\frac{q_1}{q}}} \big|\big||g|^q\big|\big|_{L^{\frac{q_2}{q}}} = \big|\big|f\big|\big|_{L^{q_1}}^q \big|\big|g\big|\big|_{L^{q_2}}^q$$

where the inequality follows from the standard Hölder inequality and the last equality from (2.1). In the item 3, we may assume that $q < \infty$, otherwise the result is trivial (take $\theta = 0$). Then

$$||f||_{L^q}^q = \int |f|^{\theta q} |f|^{(1-\theta)q} dx,$$

and we observe that
$$|f|^{\theta q} \in L^{\frac{q_1}{\theta q}}, \qquad |f|^{(1-\theta)q} \in L^{\frac{q_2}{(1-\theta)q}}.$$
The condition (2.4) means precisely that $\frac{q_1}{\theta q}$ and $\frac{q_2}{(1-\theta)q}$ are conjugate so, by the Hölder inequality, we get
$$||f||_{L^q}^q \leq ||f^{\theta q}||_{L^{\frac{q_1}{\theta q}}} ||f^{(1-\theta)q}||_{\frac{q_2}{(1-\theta)q}}$$
and we conclude thanks to (2.1). □

To handle properly the equations (1.10) or (1.11), it is convenient to see u as a function of time with values in some L^q space. For this purpose, we introduce the mixed space-time norms L^pL^q, when $p, q \in [1, \infty)$. We shall actually mostly consider the case when $p, q \in (1, \infty)$. We consider functions $F = F(t, x)$ defined for $t \in I$ and $x \in \mathbb{R}^n$, I being a given compact interval. We set
$$||F||_{L_I^pL^q} := \big|\big| ||F(\cdot)||_{L^q(\mathbb{R}^n)} \big|\big|_{L^p(I)},$$
that is,
$$||F||_{L_I^pL^q} = \left(\int_I ||F(t)||_{L^q(\mathbb{R}^n)}^p dt \right)^{1/p}. \tag{2.5}$$
Note that this implicitly uses that $t \mapsto ||F(t)||_{L^q(\mathbb{R}^n)}$ is measurable (see Exercise 2.2). We will define these norms on the space $L^pL_c^q$ which we define as the set of continuous functions $F \in C^0(I \times \mathbb{R}^n)$ such that $||F(t)||_{L^q} < \infty$ for all $t \in I$ and $||F||_{L_I^pL^q} < \infty$.

Exercise 2.2. *Assume that $q \in [1, \infty)$. Check that $t \mapsto ||F(t)||_{L^q}$ is measurable on I.*

In the applications, we shall often consider the subspace
$$C(I, L^q(\mathbb{R}^n)) \cap C^0(I \times \mathbb{R}^n) \subset L^pL_c^q.$$
We note that the introduction of such spaces avoids using (and in particular defining) *Bochner integrals*, as is common in this context (see Cazenave [1] or Ginibre [2]). This elementary approach will be sufficient for our purposes. We will sometimes consider the norm $|| \cdot ||_{L_I^\infty L^q}$ defined in the obvious way by
$$||F||_{L_I^\infty L^q} := \sup_{t \in I} ||F(t)||_{L^q},$$
but only for functions $F \in C(I, L^q)$. In the next proposition, we will always have $p < \infty$.

Proposition 2.3. *1. For all real numbers $p_1, p_2, q_1, q_2 \in [1, \infty)$ such that*
$$\frac{1}{q_1} + \frac{1}{q_2} =: \frac{1}{q} \leq 1 \qquad \text{and} \qquad \frac{1}{p_1} + \frac{1}{p_2} =: \frac{1}{p} \leq 1$$
one has
$$||FG||_{L_I^pL^q} \leq ||F||_{L_I^{p_1}L^{q_1}} ||G||_{L_I^{p_2}L^{q_2}},$$
for all $F \in L^{p_1}L_c^{q_1}$ and all $G \in L^{p_2}L_c^{q_2}$.

2. *Let $p, q \in (1, \infty)$ and let p', q' be the dual exponents, i.e.*

$$\frac{1}{q} + \frac{1}{q'} = 1 \qquad \text{and} \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Then, for all $F \in C(I, L^q(\mathbb{R}^n)) \cap C^0(I \times \mathbb{R}^n)$

$$||F||_{L^p_I L^q} = \sup_{||G||_{L^{p'}_I L^{q'}}=1} \left| \int\int_{I\times\mathbb{R}^n} F G dx dt \right|, \tag{2.6}$$

with $G \in L^{p'} L^{q'}_c$ in the supremum.

Proof. 1. We observe that, for each $t \in I$.

$$||F(t)G(t)||_{L^q} \leq ||F(t)||_{L^{q_1}} ||G(t)||_{L^{q_2}},$$

using (2.2) in space. Using (2.2) in time,

$$\Big| \Big| ||F(\cdot)||_{L^{q_1}} ||G(\cdot)||_{L^{q_2}} \Big| \Big|_{L^p(I)} \leq \Big| \Big| ||F(\cdot)||_{L^{q_1}} \Big| \Big|_{L^{p_1}(I)} \Big| \Big| ||G(\cdot)||_{L^{q_2}} \Big| \Big|_{L^{p_2}(I)},$$

and this yields the result. For the item 2, we observe first that

$$\left| \int\int_{I\times\mathbb{R}^n} F G dx dt \right| \leq ||FG||_{L^1(I\times\mathbb{R}^n)} = ||FG||_{L^1_I L^1} \leq ||F||_{L^p_I L^q} ||G||_{L^{p'}_I L^{q'}}$$

by the first item. This shows that the sup (2.6) is not greater than $||F||_{L^p_I L^q}$. To see that it is not smaller, we proceed as follows. The result is easy if $F(t, x)$ never vanishes (which is the generic situation). In this case, it suffices to consider

$$G(t, x) = \frac{1}{||F||^{p-1}_{L^p_I L^q}} ||F(t)||^{p-q}_{L^q} |F(t, x)|^{q-1} \frac{\overline{F(t, x)}}{|F(t, x)|}, \tag{2.7}$$

for which it is easy to check that

$$||G||_{L^{p'}_I L^{q'}} = 1, \qquad \int\int G F dx dt = ||F||_{L^p_I L^q}.$$

Notice that the non vanishing of F and its continuity ensure that $||F(t)||_{L^q} \neq 0$ for all t. The sup is a max in this case. The case when F may vanish is similar but slightly more technical: one basically constructs a family $G_\epsilon \in L^p L^q_c$ such that

$$||G_\epsilon||_{L^{p'}_I L^{q'}_c} \to 1 \qquad \text{and} \qquad \int\int G_\epsilon F dt dx \to ||F||_{L^p L^q_c}, \qquad \epsilon \to 0,$$

by regularizing the expression (2.7) (*e.g.* by changing $|F(t, x)|^{-1}$ into $(|F(t, x)| + \epsilon)^{-1}$). It is left to the reader as an exercise. □

The last result of this section is a version of the *Marcinkiewicz interpolation theorem*. In the sequel, all L^q spaces are over $\mathbb{R}^n$.

Theorem 2.4. *Assume we are given a map T defined on $L^1 \cup L^2$ such that*

$$T : L^2 \to L^2, \qquad T : L^1 \to L^\infty$$

are both linear and continuous, with continuity estimates

$$||Tf_2||_{L^2} \leq M_2||f_2||_{L^2}, \qquad ||Tf_1||_{L^\infty} \leq M_1||f_1||_{L^1}, \tag{2.8}$$

for all $f_1 \in L^1$, $f_2 \in L^2$. Then, for all $2 < q < \infty$, we have

$$||Tf||_{L^q} \leq C_q M_2^{\frac{2}{q}} M_1^{1-\frac{2}{q}} ||f||_{L^{q'}}, \qquad f \in L^1 \cap L^2, \tag{2.9}$$

with explicit constant $C_q = \left(\frac{q 2^{q+1}}{q'(q-2)}\right)^{\frac{1}{q}}$.

In this result the constant C_q is irrelevant (for our applications), but the explicit dependence on M_1 and M_2 will be very useful. We also point out that there are alternate theorems on interpolation, in particular the classical Riesz-Thorin theorem. Here, we have chosen the Marcinkiewicz Theorem to give an illustration of some techniques of *real* interpolation theory.

The rest of the section is devoted to the proof of this result.

For a given measurable function f, we define its **distribution function** as

$$m_f(t) = \text{meas}\left(\{|f| > t\}\right), \qquad t > 0,$$

where $\{|f| > t\} = \{x \in \mathbb{R}^n \mid |f(x)| > t\}$ and meas$(\cdot)$ is the Lebesgue measure. Notice that it is a non increasing function taking its values in $[0, \infty]$ hence is (Lebesgue) measurable on $\mathbb{R}^+$. If $f \in L^q$ with $1 \leq q < \infty$, we observe that for all $t > 0$,

$$\int |f|^q \geq \int_{|f|>t} |f|^q \geq t^q \int_{|f|>t} = t^q m_f(t),$$

which is the Chebychev-Markov inequality. Therefore, we always have the estimate

$$m_f(t) \leq \frac{||f||_{L^q}^q}{t^q}, \qquad t > 0, \ f \in L^q. \tag{2.10}$$

In particular, m_f takes finite values when $f \in L^q$. A straightforward consequence of (2.8) and (2.10) is that

$$\begin{aligned} m_{Tf_2}(t) &\leq \frac{||Tf_2||_{L^2}^2}{t^2}, \\ &\leq \frac{M_2^2}{t^2}||f_2||_{L^2}^2, \end{aligned} \tag{2.11}$$

for all $t > 0$ and $f_2 \in L^2$. For functions $g \in L^\infty$ we have meas$\{|g| > t\} = 0$ for all $t \geq ||g||_{L^\infty}$. In the context of the theorem, this shows that, for all $t > 0$ and $f_1 \in L^1$,

$$m_{Tf_1}(t) = 0 \qquad \text{if} \ \ t \geq M_1||f_1||_{L^1}, \tag{2.12}$$

since $M_1||f_1||_{L^1} \geq ||Tf_1||_{L^\infty}$.

The next lemma will be crucial in the proof of Theorem 2.4. .

Lemma 2.5. *If* $1 \leq q < \infty$ *and* $f \in L^q$, *then*

$$||f||_{L^q}^q = q \int_0^\infty t^{q-1} m_f(t) dt.$$

Proof. It suffices to consider $q = 1$ for if we know that the result is true in this case, we have for $q > 1$,

$$||f||_{L^q}^q = ||f^q||_{L^1} = \int_0^\infty m_{f^q}(t) dt = \int_0^\infty m_f(t^{1/q}) dt = q \int_0^\infty m_f(s) s^{q-1} ds,$$

by using the change of variable $t = s^q$. If $q = 1$, we write

$$\int_0^\infty m_f(t) dt \quad = \quad \int_0^\infty \left(\int_{|f|>t} dx \right) dt \quad = \quad \iint_{\mathbb{R}^+ \times \mathbb{R}^n} \mathbb{1}_A(t,x) dx dt$$

with

$$A = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \mid |f(x)| > t\},$$

since $\mathbb{1}_A(t,x) = \mathbb{1}_{\{|f|>t\}}(x)$. Using the Fubini Theorem and observing that

$$\mathbb{1}_A(t,x) = \mathbb{1}_{[0,f(x))}(t),$$

we get

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^n} \mathbb{1}_A dt dx = \int_{\mathbb{R}^n} \left(\int_0^{|f(x)|} dt \right) dx = \int_{\mathbb{R}^n} |f(x)| dx,$$

which yields the result. □

In the next lemma, we record a very simple observation which will also be crucial in the proof of the theorem.

Lemma 2.6 (Distribution function of a sum)**.** *If* f, g *are measurable functions then, for all* $t > 0$,

$$m_{f+g}(2t) \leq m_f(t) + m_g(t).$$

Proof. It suffices to observe that if $|f(x) + g(x)| > 2t$ then either $|f(x)| > t$ or $|g(x)| > t$ (otherwise $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq 2t$). Therefore

$$\{|f| + |g| > 2t\} \subset \{|f| > t\} \cup \{|g| > t\},$$

and we get the result by taking the Lebesgue measure. □

Proof of Theorem 2.4. Let $f \in L^1 \cap L^2$. Then $Tf \in L^2 \cap L^\infty$ hence $Tf \in L^q$ by (2.3). By Lemma 2.5, in which we use the change of variable $t \mapsto 2t$, we have

$$||Tf||_{L^q}^q = q 2^q \int_0^\infty m_{Tf}(2t) t^{q-1} dt. \tag{2.13}$$

The idea of the proof is then to find, for each t, a suitable decomposition

$$f = f_{1,t} + f_{2,t}, \qquad f_{1,t} \in L^1,\ f_{2,t} \in L^2,$$

with a nice enough control on $m_{Tf_{1,t}}(t)$ and $m_{Tf_{2,t}}(t)$ to exploit that, by Lemma 2.6[1] and the linearity of T,

$$m_{Tf}(2t) \le m_{Tf_{1,t}}(t) + m_{Tf_{2,t}}(t). \tag{2.14}$$

We will consider

$$f_{2,t} = \left(\mathbb{1}_{|f|\le z} + \frac{z}{|f|}\mathbb{1}_{|f|>z} \right) f$$

for some $z = z(t)$ to be chosen below, and thus

$$f_{1,t} = \left(\mathbb{1}_{|f|>z} - \frac{z}{|f|}\mathbb{1}_{|f|>z} \right) f.$$

We will choose $z(t)$ such that

$$m_{Tf_{1,t}}(t) = 0, \qquad t > 0. \tag{2.15}$$

To determine such a z, we observe that, by (2.12), it is sufficient that $t \ge M_1 ||f_{1,t}||_{L^1}$. We thus need to estimate $||f_{1,t}||_{L^1}$. It is not hard to check that $|f_{1,t}| = (|f| - z)\mathbb{1}_{|f|>z}$ hence

$$||f_{1,t}||_{L^1} \le \int \mathbb{1}_{|f|>z}|f| = \int \mathbb{1}_{|f|>z}\frac{|f|^{q'}}{|f|^{q'-1}} \le \frac{1}{z^{q'-1}}||f||_{L^{q'}}^{q'}. \tag{2.16}$$

Thus, it suffices to choose $z(t)$ so that $tz(t)^{q'-1} \ge M_1 ||f||_{L^{q'}}^{q'}$. A simple calculation shows that

$$z = z(t) = \left(\frac{M_1 ||f||_{L^{q'}}^{q'}}{t} \right)^{\frac{1}{q'-1}} = \left(\frac{M_1 ||f||_{L^{q'}}^{q'}}{t} \right)^{q-1}.$$

will do. With such a z, (2.15) holds true. On the other hand, for each t, (2.11) yields

$$m_{Tf_{2,t}}(t) \le M_2^2 \frac{||f_{2,t}||_{L^2}^2}{t^2}, \tag{2.17}$$

where, by using Lemma 2.5, we write

$$||f_{2,t}||_{L^2}^2 = 2\int_0^\infty m_{f_{2,t}}(s) s ds. \tag{2.18}$$

It is not hard to check that $|f_{2,t}| = \min(|f|, z)$ and then that

$$m_{f_{2,t}}(s) = \begin{cases} m_f(s) & \text{if } \ s < z \\ 0 & \text{otherwise} \end{cases}.$$

[1]this lemma implies that, for a given t, $m_{Tf_{1,t}+Tf_{2,t}}(2s) \le m_{Tf_{1,t}}(s) + m_{Tf_{2,t}}(s)$ for all $s > 0$, which we use in the particular case $s = t$

Therefore

$$||f_{2,t}||_{L^2}^2 = 2\int_0^{z(t)} m_f(s)sds. \tag{2.19}$$

Using (2.13), (2.14), (2.15), (2.17) and (2.19), we obtain

$$||Tf||_{L^q}^q \leq M_2^2 q 2^{q+1} \int_0^\infty \left(\int_0^{z(t)} m_f(s)sds \right) t^{q-3}dt. \tag{2.20}$$

By the Fubini Theorem and the fact that $s < z(t)$ means $t < M_1||f||_{L^{q'}}^{q'}/s^{q'-1}$, we have

$$\begin{aligned}
\int_0^\infty \left(\int_0^{z(t)} m_f(s)sds \right) t^{q-3}dt &= \int_0^\infty \left(\int_0^{M_1||f||_{L^{q'}}^{q'}/s^{q'-1}} t^{q-3}dt \right) m_f(s)sds \\
&= \frac{1}{q-2}\int_0^\infty M_1^{q-2}||f||_{L^{q'}}^{q'(q-2)} s^{(1-q')(q-2)+1} m_f(s)ds \\
&= \frac{M_1^{q-2}||f||_{L^{q'}}^{q'(q-2)}}{q'(q-2)} q' \int_0^\infty s^{q'-1} m_f(s)ds \\
&= \frac{M_1^{q-2}||f||_{L^{q'}}^{q'(q-1)}}{q'(q-2)} \\
&= \frac{M_1^{q-2}}{q'(q-2)}||f||_{L^{q'}}^q,
\end{aligned}$$

using that $q > 2$ and, by elementary computations, that

$$(1-q')(q-2)+1 = q'-1 \qquad q'(q-1) = q.$$

Taking into account the constant in front of the integral in (2.20), the result follows. $\square$

2.2 Fourier analysis

In this section, we record useful results of Fourier analysis, assuming some familiarity of the reader with the Schwartz space and temperate distributions.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is defined as the space of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that, for all multi-indices $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)| < \infty.$$

The Fourier transform is defined on the Schwartz space by

$$(\mathcal{F}\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\varphi(y)dy.$$

We record several of its properties.

Proposition 2.7. *1. $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ into itself.*

2. $\mathcal{F}$ is invertible on $\mathcal{S}(\mathbb{R}^n)$ and we have the inversion formula

$$\varphi(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \widehat{\varphi}(\xi) d\xi.$$

3. The Fourier transform exchanges multiplication and differentiation: for all $\alpha \in \mathbb{N}^n$ and all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}(x^\alpha \varphi) = i^{|\alpha|} \partial_\xi^\alpha (\mathcal{F}\varphi), \qquad \mathcal{F}(\partial_x^\alpha \varphi) = i^{|\alpha|} \xi^\alpha (\mathcal{F}\varphi).$$

4. The L^2 normalized Fourier transform $\mathcal{F}_2 := (2\pi)^{-n/2}\mathcal{F}$ and its formal adjoint

$$\mathcal{F}_2^* \psi(x) := (\mathcal{F}_2 \psi)(-x)$$

satisfy

$$\big(\mathcal{F}_2 \varphi, \psi\big)_{L^2} = \big(\varphi, \mathcal{F}_2^* \psi\big)_{L^2}, \qquad ||\mathcal{F}_2 \varphi||_{L^2} = ||\varphi||_{L^2}, \tag{2.21}$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. (See (1.16) for the L^2 inner product $(.,.)_{L^2}$.)

5. $\mathcal{F}_2$ extends, in a unique fashion, as a unitary operator on $L^2(\mathbb{R}^n)$ which we still denote by $\mathcal{F}_2$.

Given a measurable function a on $\mathbb{R}^n$, we denote by M_a the multiplication by a, *i.e.*

$$(M_a v)(\xi) := a(\xi) v(\xi).$$

The last item of Proposition 2.7 allows to define easily Fourier multipliers on $L^2(\mathbb{R}^n)$.

Definition 2.8. *Given $a = a(\xi) \in L^\infty(\mathbb{R}^n)$, we define the* **Fourier multiplier** *$a(D)$ by*

$$a(D) = \mathcal{F}_2^* M_a \mathcal{F}_2.$$

The function a is called the **symbol** *of the Fourier multiplier.*

Example 1. If $a \equiv 1$, then $a(D) = I$.

Proposition 2.9. *1. For all $a \in L^\infty(\mathbb{R}^n)$ and $u \in L^2(\mathbb{R}^n)$*

$$||a(D)u||_{L^2} \leq ||a||_{L^\infty} ||u||_{L^2}.$$

2. If $a, b \in L^\infty(\mathbb{R}^n)$, then

$$a(D)b(D) = (ab)(D), \qquad a(D)^* = \overline{a}(D).$$

3. *If a belongs to $\mathcal{S}(\mathbb{R}^n)$, then*

$$a(D)u(x) = \int K_a(x,y)u(y)dy,$$

with $K_a(x,y) = (2\pi)^{-n}\widehat{a}(y-x)$.

4. *If a belongs to $\mathcal{S}(\mathbb{R}^n)$ and $q_1 \leq q_2$ both belong to $[1,\infty]$ then*

$$||a(D)u||_{L^{q_2}} \leq C_a||u||_{L^{q_1}}$$

with

$$C_a = (2\pi)^{-n}||\hat{a}||_{L^1}^{\frac{q_1}{q_2}}||\hat{a}||_{L^{q_1'}}^{1-\frac{q_1}{q_2}}.$$

Proof. 1. By unitarity of $\mathcal{F}_2$, we have

$$||a(D)u||_{L^2} = ||a\mathcal{F}_2u||_{L^2} \leq ||a||_{L^\infty}||\mathcal{F}_2u||_{L^2} = ||a||_{L^\infty}||u||_{L^2}.$$

2. It is a straightforward consequence of the unitarity of $\mathcal{F}_2$ together with the fact that $M_a^* = M_{\bar{a}}$ and $M_aM_b = M_{ab}$.
3. If $u \in \mathcal{S}(\mathbb{R}^n)$, the Fubini Theorem allows to write

$$a(D)u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}a(\xi)\widehat{u}(\xi)d\xi = (2\pi)^{-n}\int\int e^{i(x-y)\cdot\xi}a(\xi)u(y)dyd\xi$$

from which the result follows easily, at least when u is a Schwartz function. The result remains true if u is in L^2 by a density argument, since both $a(D)$ and the convolution by $\hat{a}$ (which is integrable) are bounded operators on L^2.
4. By the item 3 of Proposition 2.1, we have

$$||a(D)u||_{L^{q_2}} \leq ||a(D)u||_{L^{q_1}}^{\frac{q_1}{q_2}}||a(D)u||_{L^\infty}^{1-\frac{q_1}{q_2}}.$$

We then use on one hand

$$||a(D)u||_{L^{q_1}} \leq (2\pi)^{-n}||\hat{a}||_{L^1}||u||_{L^{q_1}}$$

since the convolution by a L^1 function preserve L^{q_1}, and on the other hand

$$|a(D)u(x)| = (2\pi)^{-n}\left|\int \widehat{a}(y-x)u(y)dy\right| \leq (2\pi)^{-n}||\widehat{a}||_{L^{q_1'}}||u||_{L^{q_1}}$$

by the Hölder inequality. □

Example 2 (Schrödinger group). For $t \in \mathbb{R}$, we *define* $e^{-it\Delta}$ as the Fourier multiplier by $e^{it|\xi|^2}$

$$e^{-it\Delta} = \mathcal{F}_2^* e^{it|\xi|^2}\mathcal{F}_2.$$

For notational simplicity, we will set

$$U(t) = e^{-it\Delta}.$$

Proposition 2.10. *1. The map $\mathbb{R} \ni t \mapsto U(t)$ is a unitary group on $L^2(\mathbb{R}^n)$, ie*

$$U(t)U(s) = U(t+s), \qquad U(t)^* = U(-t), \qquad U(0) = I.$$

2. *This map is* **strongly continuous***: for all $\varphi \in L^2(\mathbb{R}^n)$, $t \mapsto U(t)\varphi \in L^2(\mathbb{R}^n)$ is continuous.*

3. *Let $I \subset \mathbb{R}$ be an interval containing 0. If $f : I \to L^2(\mathbb{R}^n)$ is a continous $L^2(\mathbb{R}^n)$ valued function, then*

$$I \ni t \mapsto \int_0^t U(t-s)f(s)ds \in L^2(\mathbb{R}^n)$$

is continuous. Here the integral is taken in the Riemann sense[2].

4. *If $t \neq 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$U(t)\varphi(x) = \int_{\mathbb{R}^n} K_t(x,y)\varphi(y)dy$$

with

$$K_t(x,y) = \frac{e^{\pm i\frac{n\pi}{4}}}{|4\pi t|^{\frac{n}{2}}} e^{-\frac{i|x-y|^2}{4t}}, \qquad \pm := \text{sign of } t.$$

Proof. The item 1 follows from the items 1 and 2 of Proposition 2.9 and Example 1. The items 2 and 3 are left to the reader as an exercise. We simply record that, for both items, it suffices to understand why $s \mapsto U(\pm s)f(s)$ is continuous when f is. Let us prove the item 4. We set $\varphi_x(y) = \varphi(x+y)$ so that

$$U(t)\varphi(x) = (2\pi)^{-n/2} \int \overline{e^{-it|\xi|^2}} (\mathcal{F}_2 \varphi_x)(\xi) d\xi$$

where we want to use the Parseval formula (2.21). The function $\xi \mapsto e^{-it|\xi|^2}$ fails to be Schwartz but we can pass to the limit $\epsilon \to 0^+$ in the integral by considering

$$\psi_{z_\epsilon}(\xi) := e^{-z_\epsilon |\xi|^2}, \qquad z_\epsilon = \epsilon + it,$$

and the result follows then from the explicit knowledge of the Fourier transform of Gaussian functions, namely the fact that, if z is complex number with positive imaginary part and $\sqrt{\ }$ the principal determination of the square root,

$$(\mathcal{F}_2^* \psi_z)(y) = \frac{1}{\sqrt{2z}^n} e^{-\frac{|y|^2}{4z}}. \tag{2.22}$$

[2] that is the Riemann integral on a compact interval of a continuous function with values in the Banach space $L^2(\mathbb{R}^n)$

Note that $\sqrt{\epsilon + it} = \left(\frac{\sqrt{\epsilon^2+t^2}+\epsilon}{2}\right)^{1/2} + i\text{sign}(t)\left(\frac{\sqrt{\epsilon^2+t^2}-\epsilon}{2}\right)^{1/2}$. We refer to Exercise 2.12 for the proof of (2.22). $\square$

We record as a Corollary the following straightforward important consequence of the item 4 of Proposition 2.10.

Corollary 2.11. *For all $t \neq 0$, $U(t)$ is a continuous linear map from L^1 to L^∞. Furthermore, there exists $C > 0$ such that*

$$||U(t)||_{L^1 \to L^\infty} \leq C|t|^{-n/2}, \qquad t \neq 0.$$

Exercise 2.12 (Fourier transform of Gaussian functions)**.** *The goal of this exercise is to prove (2.22).*

1. *Check that it suffices to consider $n = 1$ (hint: Fubini).*
2. *Prove that (2.22) is true when $z \in (0, +\infty)$ by checking that*
   (a) *both sides of the equality solve the same ODE of order 1 in y,*
   (b) *both sides coincide at $y = 0$ (hint: use that $\int_{\mathbb{R}} e^{-at^2} dt = (\pi/a)^{1/2}$ if $a > 0$).*
3. *Check that, for a given $y \in \mathbb{R}$, $z \mapsto (\mathcal{F}_2^* \psi_z)(y)$ is holomorphic on the right half plane $\{\text{Re}(z) > 0\}$.*
4. *Conclude (hint: analytic continuation).*

The next exercise gives a first rigorous interpretation of the fact that the Duhamel formula (1.4) solves the Cauchy problem (1.1)-(1.2).

Exercise 2.13. *Let I be an interval containing 0 and $f : I \to L^2(\mathbb{R}^n)$ be continuous. Let $u_0 \in L^2(\mathbb{R}^n)$. Check that*

$$u(t) := U(t)u_0 + \frac{1}{i} \int_0^t U(t-s)f(s)ds$$

solves the Schrödinger equation

$$(i\partial_t - \Delta)u = f,$$

in the distributions sense on $I \times \mathbb{R}^n$.

Solution. Assume first that $f \equiv 0$. We want to show that, for any $\psi \in C_0^\infty(I \times \mathbb{R}^n)$,

$$\langle (i\partial_t - \Delta)u, \psi \rangle = \langle u, -i\partial_t \psi - \Delta \psi \rangle \tag{2.23}$$

vanishes, that is

$$-\iint u(t,x)(i\partial_t + \Delta)\psi(t,x)dxdt = 0.$$

By rewriting $(i\partial_t + \Delta)\psi = \overline{(-i\partial_t + \Delta)\bar{\psi}}$ and using the Fubini Theorem, we can interpret the above integral as

$$\begin{aligned}\int_I \left((i\partial_t - \Delta)\bar{\psi}(t), U(t)u_0\right)_{L^2} dt &= \int_{\mathbb{R}} \left(U(-t)(i\partial_t - \Delta)\bar{\psi}(t), u_0\right)_{L^2} dt \\ &= \left(\int_I U(-t)(i\partial_t - \Delta)\bar{\psi}(t)dt, u_0\right)_{L^2}. \qquad (2.24)\end{aligned}$$

Denoting by $\widehat{\bar{\psi}}$ the Fourier transform with respect to x, we have

$$\begin{aligned}U(-t)(i\partial_t - \Delta)\bar{\psi}(t,x) &= (2\pi)^{-n}\int e^{ix\cdot\xi}e^{-it|\xi|^2}(i\partial_t + |\xi|^2)\widehat{\bar{\psi}}(t,\xi)d\xi \\ &= i(2\pi)^{-n}\frac{d}{dt}\left(\int e^{ix\cdot\xi}e^{-it|\xi|^2}\widehat{\bar{\psi}}(t,\xi)d\xi\right). \qquad (2.25)\end{aligned}$$

Here we have used that $|\xi|^2\widehat{\bar{\psi}}$ is the Fourier transform (in x) of $-\Delta\bar{\psi}$, which follows from the third item of Proposition 2.7. Note also that $\int e^{ix\cdot\xi}e^{-it|\xi|^2}\widehat{\bar{\psi}}(t,\xi)d\xi$ is a Schwartz function of t and x, which is compactly supported in t. Then

$$\int_I U(-t)(i\partial_t - \Delta)\bar{\psi}(t,x)dt = 0$$

as the integral of the derivative of a C_0^∞ function. This implies that (2.24) and thus (2.23) vanish. As a by-product of this computation, more precisely of (2.24), we discover that computing $(i\partial_t - \Delta)U(t)u_0$ in the distribution sense rests on the calculation the action of the adjoint of $U(t)$ on $(i\partial_t - \Delta)\bar{\psi}$.

We consider next the case when $u_0 \equiv 0$ and $f \in C(I, L^2)$. The previous step suggests we have to determine the formal adjoint[3] of the operator $f \mapsto \int_0^t U(t-s)f(s)ds$. To do so, we pick $f \in C(I, L^2)$, $\varphi \in C_0^\infty(I \times \mathbb{R}^n)$ and compute

$$\begin{aligned}\iint \overline{\varphi(t,x)}\left(\int_0^t U(t-s)f(s)ds\right)(t,x)dtdx &= \int_I\int_0^t (\varphi(t), U(t-s)f(s)ds)_{L^2}\, dsdt \\ &= \int_I\int_0^t (U(s-t)\varphi(t), f(s)ds)_{L^2}\, dsdt.\end{aligned}$$

Using that $\int_0^t ds = \int_{[0,t]} ds$ if $t \geq 0$ and $\int_0^t ds = -\int_{[t,0]} ds$, and letting $I = [a,b]$ with $a < 0 < b$, it is not hard to check that

$$\int_I\left(\int_0^t \cdots ds\right)dt = \int_0^b\left(\int_s^b \cdots dt\right)ds - \int_a^0\left(\int_a^s \cdots dt\right)ds,$$

by the Fubini Theorem, from which we get

$$\begin{aligned}\int_I\int_0^t (U(s-t)\varphi(t), f(s)ds)_{L^2}\, dsdt &= \int_0^b\left(\int_s^b U(s-t)\varphi(t)dt, f(s)\right)_{L^2} ds \\ &\quad -\int_a^0\left(\int_a^s U(s-t)\varphi(t)dt, f(s)\right)_{L^2} ds.\end{aligned}$$

[3] *i.e.* tested against $C_0^\infty(I \times \mathbb{R}^n)$ functions

Therefore, (2.23) with $u(t) = -i\int_0^t U(t-s)f(s)ds$ and $\overline{\varphi} := \overline{(i\partial_t - \Delta)\bar{\psi}}$ yields

$$\langle (i\partial_t - \Delta)u, \psi\rangle = -i\int_0^b \left(\int_s^b U(s-t)\varphi(t)dt, f(s)\right)_{L^2} ds + i\int_a^0 \left(\int_a^s U(s-t)\varphi(t)dt, f(s)\right)_{L^2}$$

where, in the right hand side, (2.25) shows that

$$U(s-t)\varphi(t) = i\frac{\partial}{\partial t}U(s-t)\bar{\psi}(t).$$

After integration with respect to t, we get

$$\begin{aligned}
\langle (i\partial_t - \Delta)u, \psi\rangle &= -i\int_0^b \big(-i\bar{\psi}(s), f(s)\big)_{L^2} ds + i\int_a^0 \big(i\bar{\psi}(s), f(s)\big)_{L^2} ds \\
&= \int_a^b (\bar{\psi}(s), f(s))_{L^2} ds \\
&= \iint f(s,x)\psi(s,x)dsdx,
\end{aligned}$$

which is the expected result. □

The definition of Fourier multipliers is not restricted to bounded symbols. As suggested by the item 3 of Proposition 2.7, one can for instance consider the Fourier multiplier by $a_\alpha(\xi) := \xi^\alpha$ which satisfies

$$a_\alpha(D) = (-i)^{|\alpha|}\partial_x^\alpha,$$

at least on the Schwartz space. We also recall that the expression of the wave group (1.8) should involve the Fourier multiplier by $|\xi|\sin(t|\xi|)$ which is not bounded. To handle unbounded Fourier multipliers, which will not be continuous endomorphisms of L^2, we need to use Sobolev spaces.

We first recall the notion of temperate distributions and of their Fourier transform. A **temperate distribution** T is a linear map $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ which is continuous, in the sense that for some $C > 0$ and $N > 0$, we have

$$|\langle T, \varphi\rangle| \leq C \sum_{|\alpha|+|\beta|\leq N} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)|,$$

where $\langle T, \varphi\rangle$ is the usual notation for $T(\varphi)$. The **Fourier transform on** $\mathcal{S}'(\mathbb{R}^n)$ is defined by duality by

$$\langle \mathcal{F}T, \varphi\rangle := \langle T, \mathcal{F}\varphi\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We recall without proof the following result which we consider as part of the background of the reader.

Proposition 2.14. *1. For all $q \in [1,\infty]$, $L^q(\mathbb{R}^n)$ is embedded into $\mathcal{S}'(\mathbb{R}^n)$, via $u \mapsto T_u$ with*

$$\langle T_u, \varphi\rangle = \int_{\mathbb{R}^n} u(x)\varphi(x)dx.$$

2. *The definition of $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^n)$ is compatible with the ones on $\mathcal{S}(\mathbb{R}^n)$ and on $L^2(\mathbb{R}^n)$.*

Definition 2.15. *For $s \in \mathbb{R}$, one defines the space $H^s(\mathbb{R}^n)$ as the space of temperate distributions T such that $\mathcal{F}T$ belongs to $L^2_{\text{loc}}(\mathbb{R}^n)$ and*

$$||T||^2_{H^s} := (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathcal{F}T(\xi)|^2 d\xi < \infty.$$

We record the following proposition for reference.

Proposition 2.16 (Properties of Sobolev spaces)**.** 1. *If $s_1 \leq s_2$, then $H^{s_2} \subset H^{s_1}$.*

2. *If $s \geq 0$, H^s is contained in L^2. Actually, if $s \geq 0$*

$$u \in H^s \qquad \Longleftrightarrow \qquad u \in L^2 \;\; \text{and} \;\; (1+|\xi|^2)^{s/2}(\mathcal{F}u) \in L^2$$

3. *If $s \in \mathbb{N}$, H^s coincides with the subspace of functions $u \in L^2$ such that $\partial^\alpha u \in L^2$ for all $|\alpha| \leq s$, the derivatives being taken in the distributions sense.*

4. *For all real number s, $\mathcal{S}(\mathbb{R}^n)$ is dense in H^s.*

5. *Let $k \in \mathbb{N}$ and $s > k + \frac{n}{2}$. Then $H^s \subset C^k$. In particular, $H^\infty := \cap_{s \geq 0} H^s$ is contained in C^∞.*

We give a short proof for completeness.

Proof. 1 and 2 are trivial (the point of 2 is to emphasize that $H^s \subset L^2$ when $s \geq 0$). The item 3 is left as an exerice (one has basically to see that $\sum_{|\alpha| \leq s} |\xi^\alpha|$ is bounded from above and below by (constants times) $(1+|\xi|^2)^{s/2}$). In 4, it suffices to approximate the L^2 function $(1+|\xi|^2)^{s/2}\mathcal{F}T(\xi)$ by functions φ_j in $\mathcal{S}(\mathbb{R}^n_\xi)$ and observe that $\mathcal{F}^{-1}((1+|\xi|^2)^{-s/2}\varphi_j)$ goes to T in H^s as $j \to \infty$. In 5, when $k = 0$ and $u \in H^s$, we observe that, $\mathcal{F}u(\xi)$ belongs to L^1 by the Hölder inequality since

$$\mathcal{F}u(\xi) = (1+|\xi|^2)^{-s/2} \left((1+|\xi|^2)^{s/2} \mathcal{F}u(\xi) \right)$$

is a product of two L^2 functions. Therefore, the inverse Fourier transform of $\mathcal{F}u(\xi)$ is continuous on $\mathbb{R}^n$ (and goes to zero at infinity by the Riemann-Lebesgue Lemma). The case of $k \geq 1$ is similar. □

Proposition 2.17 (Fourier multipliers on Sobolev spaces)**.** *Let s be a real number.*

1. *If $a \in L^\infty(\mathbb{R}^n)$, then H^s is stable by the Fourier multiplier $a(D)$ and*

$$||a(D)u||_{H^s} \leq ||a||_{L^\infty} ||u||_{H^s}.$$

2. *If p is a measurable function such that, for some $m \in \mathbb{R}$,*

$$|p(\xi)| \leq C(1 + |\xi|)^m,$$

then the Fourier multiplier

$$p(D) := \mathcal{F}_2^* M_p \mathcal{F}_2$$

which maps $\mathcal{S}(\mathbb{R}^n)$ into H^{s-m} extends uniquely to a continous operator from H^s to H^{s-m}.

3. *Two such operators $a(D)$ and $p(D)$ commute.*

Proof. Left as an exercise to the reader. □

Proposition 2.18 (Mollifiers)**.** *Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ such that $\chi(0) = 1$. Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers going to zero and define*

$$\chi_k(\xi) = \chi(\epsilon_k \xi).$$

Then

1. *$\chi(\epsilon_k D)$ maps L^2 into H^∞.*

2. *For all $q \in [1, \infty]$, $(\chi_k(D))_{k \in \mathbb{N}}$ is bounded in $\mathcal{L}(L^q(\mathbb{R}^n))$, i.e. there exists $C > 0$ such that*

$$||\chi_k(D)u||_{L^q} \leq C||u||_{L^q}, \qquad k \geq 0, \ u \in L^q(\mathbb{R}^n).$$

3. *For all $q \in [1, \infty)$, $\chi_k(D)$ converges strongly to the identity on $L^q(\mathbb{R}^n)$, i.e. for all $u \in L^q(\mathbb{R}^n)$,*

$$||\chi_k(D)u - u||_{L^q} \to 0.$$

Proof. The item 1 is obvious for if u belongs to L^2 then $\chi(\epsilon_k D)u$ has a compactly supported Fourier transform. The item 2 is a consequence of the item 4 of Proposition 2.9 by observing that the Fourier transform of $\chi(\epsilon_k \xi)$ is $\epsilon_k^{-n} \widehat{\chi}(x/\epsilon_k)$ which has a L^1 norm independent of ϵ_k. In item 3, due to the a priori uniform boundedness given by the item 2, it suffices to consider a dense subset of L^q such as the Schwartz space. Then, it is easy to check (or even standard to know) that if $u \in \mathcal{S}(\mathbb{R}^n)$ then $\chi(\epsilon_k \xi)\widehat{u}(\xi)$ converges to $\widehat{u}(\xi)$ in $\mathcal{S}(\mathbb{R}^n)$ hence so do their inverse Fourier transforms, which in turn implies the expected convergence in L^q. □

We will need the following exercise in Chapter 4.

Exercise 2.19. *1. Let $u \in H^2(\mathbb{R}^n)$. Show that $t \mapsto e^{-it\Delta}u$ is C^1 on $\mathbb{R}$ with derivative $-ie^{-it\Delta}\Delta u$ (note that $\Delta u \in L^2$ since $u \in H^2$).*

2. Let $\chi \in C_0^\infty(\mathbb{R})$ and $f \in C^1(I, L^2)$, I being an interval. Prove that $t \mapsto e^{-it\Delta}\chi(D)f(t)$ is C^1 with derivative

$$e^{-it\Delta}\chi(D)f'(t) - ie^{it\Delta}\Delta\chi(D)f(t).$$

(Note here that $-\Delta\chi(D)$ is a bounded operator on L^2 since it is the Fourier multiplier by $|\xi|^2\chi(\xi)$.)

Solution. 1. We show first the differentiability namely that, for all t, we have

$$\left|\left|\frac{e^{-i(t+h)\Delta}u - e^{-it\Delta}u}{h} + ie^{-it\Delta}\Delta u\right|\right|_{L^2}^2 \to 0, \qquad h \to 0$$

By the Parseval formula, and up to a multiplicative constant, the above norm reads

$$\int \left|\left(\frac{e^{i(t+h)|\xi|^2} - e^{it|\xi|^2}}{h} - ie^{it|\xi|^2}|\xi|^2\right)\widehat{u}(\xi)\right|^2 d\xi.$$

The bracket goes to zero as $h \to 0$ for all ξ and is uniformly bounded by $C|\xi|^2$ so, using that $|\xi|^2\widehat{u}(\xi)$ belongs to L^2, this integral goes to zero as $h \to 0$ by the dominated convergence theorem. This proves the derivability. The continuity of the derivative in t can be proved as the third item of Proposition 2.10.
2. As in 1, we prove the derivability only. Using the notation $U(t) = e^{-it\Delta}$, we write

$$\begin{aligned} U(t+h)\chi(D)f(t+h) - U(t)\chi(D)f(t) &= \big(U(t+h) - U(t)\big)\chi(D)f(t) \\ &\quad + U(t)\chi(D)\big(f(t+h) - f(t)\big) \\ &\quad + \big(U(t+h) - U(t)\big)\chi(D)\big(f(t+h) - f(t)\big) \\ &= I + II + III. \end{aligned}$$

Since $\chi(D)f(t)$ belongs to H^2 (its Fourier transform is L^2 and compactly supported), I/h has a limit as $h \to 0$ by the previous question. That II/h has a limit as $h \to 0$ follows from the differentiabilty of f in t and the continuity of $U(t)\chi(D)$ on L^2. Finally III/h goes to zero as $h \to 0$ since $(f(t+h) - f(t))/h$ is bounded as $h \to 0$ and

$$\big|\big|\big(U(t+h) - U(t)\big)\chi(D)\big|\big|_{L^2 \to L^2} \leq \sup_{\xi \in \mathbb{R}^n} \left|\int_0^h e^{i(t+s)|\xi|^2}|\xi|^2\chi(\xi)ds\right|$$

is bounded by $C|h|$. □

We conclude this section by recording a few facts and definitions on homogeneous Sobolev spaces and homogeneous Sobolev estimates. We start with the following result. A proof is given in appendix.

Proposition 2.20 (Homogeneous Sobolev inequalities). *If $s \in [0, n/2)$, there exists $C > 0$ such that*

$$||u||_{L^{\frac{2n}{n-2s}}} \leq C\big|\big||D|^s u\big|\big|_{L^2}, \tag{2.26}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. In particular, H^s is contained in $L^{\frac{2n}{n-2s}}$.

Informally, we can rephrase the fact that H^s is contained in $L^{\frac{2n}{n-2s}}$ by saying that a L^2 function u also belongs to $L^{\frac{2n}{n-2s}}$ provided that u has 's derivatives' in L^2.

Definition 2.21. *For $0 \leq s < n/2$, one defines the* **homogeneous Sobolev space** $\dot{H}^s$ *on $\mathbb{R}^n$ as the closure of $\mathcal{S}(\mathbb{R}^n)$ for the norm*

$$||\varphi||_{\dot{H}^s} := \big|\big||\xi|^s \mathcal{F}_2(\varphi)\big|\big|_{L^2(\mathbb{R}^n_\xi)}.$$

It follows from Proposition 2.20 that $\dot{H}^s$ is a subspace of $L^{\frac{2n}{n-2s}}$. In particular, it is a subset of distributions on $\mathbb{R}^n$.

Be careful that $\dot{H}^s$ is not contained in L^2 (unless $s = 0$) due to the low frequencies, namely to the possible non L^2 integrable singularities which are allowed on $\mathcal{F}_2\varphi$ at $\xi = 0$ (they are only square integrable with respect to $|\xi|^{2s}d\xi$). However $\dot{H}^s$ contains H^s.

One interest of the homogeneous Sobolev spaces is to scale precisely as the associated Lebesgue spaces. This means that if u belongs to $\dot{H}^s$ and if we set $u_\lambda(x) = u(\lambda x)$ with $\lambda > 0$ then

$$||u_\lambda||_{\dot{H}^s} = \lambda^{s-\frac{n}{2}}||u||_{\dot{H}^s}, \qquad ||u_\lambda||_{L^{\frac{2n}{n-2s}}} = \lambda^{s-\frac{n}{2}}||u||_{L^{\frac{2n}{n-2s}}}.$$

This follows from a simple calculation, say for $u \in \mathcal{S}(\mathbb{R}^n)$, which is left to the reader. One could prove several results on such spaces, for instance introduce those of negative order and check that there is a natural duality between $\dot{H}^s$ and $\dot{H}^{-s}$. However, we won't need such additional results so we only record the minimal tools for our purposes.

A consequence of Proposition 2.20 is the following result which is a special case of the so called *Hardy-Littlewood-Sobolev inequality.* It will be important in Chapter 3.

Proposition 2.22. *Let $p > 2$ be a real number and let $\delta = \frac{2}{p}$. There exists $C > 0$ such that*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(s)g(t)|}{|t-s|^\delta} ds dt \leq C||f||_{L^{p'}}||g||_{L^{p'}}, \tag{2.27}$$

for all $f, g \in L^{p'}(\mathbb{R})$. Here p' is the conjugate exponent to p, i.e. $\frac{1}{p'} = 1 - \frac{1}{p}$.

In other words, this proposition says that $f * |\cdot|^{-\delta}$ belongs to $L^p(\mathbb{R})$ if $f \in L^{p'}(\mathbb{R})$. Note that $|\cdot|^{-\delta}$ is locally integrable on $\mathbb{R}$ since $0 < \delta < 1$ but it does not belong to any Lebesgue space on $\mathbb{R}$ since $\int_{\mathbb{R}} |t|^{-q\delta}dt$ will diverge either at 0 or at infinity. A proof of Proposition 2.22, splitted into several steps, is suggested in Exercise 2.24 below. It uses the result of the following exercise.

Exercise 2.23. *Let $d \geq 1$ be an integer. The purpose of this exercise is to show that, for all $0 < s < d$,*

$$\mathcal{F}\big(|x|^{-s}\big) = (2\pi)^{\frac{d}{2}} 2^{\frac{d}{2}-s} \frac{\Gamma\big(\frac{d-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)} |\xi|^{s-d},$$

where $x \mapsto |x|^{-s}$ and $\xi \mapsto |\xi|^{d-s}$ are defined on $\mathbb{R}^d$ (as temperate distributions).

1. *Show that, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map*

$$s \mapsto \langle | \cdot |^{-s}, \mathcal{F}\varphi \rangle := \int |x|^{-s} \widehat{\varphi}(x) dx,$$

is holomorphic in the strip $\{0 < \mathrm{Re}(s) < d\}$.

2. *Show that, for $\mathrm{Re}(s) \in (d/2, d)$, $\mathcal{F}(|x|^{-s})$ is locally integrable on $\mathbb{R}^d$. Hint: introduce $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ near 0 and observe that*

$$|x|^{-s}\chi(x) \in L^1, \qquad (1-\chi)(x)|x|^{-s} \in L^2.$$

In the sequel, we denote by $F_s(\xi)$ the locally integrable function $\mathcal{F}(|x|^{-s})$.

3. *For $\mathrm{Re}(s) \in (d/2, d)$, show that*
   - *(a) F_s is continuous on $\mathbb{R}^d \setminus 0$, (hint: $|\xi|^{2d}\mathcal{F}\big((1-\chi)| \cdot |^{-s}\big)$ is continuous on $\mathbb{R}^d$)*
   - *(b) for all $\lambda > 0$ and all $\xi \neq 0$, $F_s(\lambda\xi) = \lambda^{s-d}F_s(\xi)$,*
   - *(c) F_s is radial, i.e. $F_s(R\xi) = F_s(\xi)$ for any orthogonal matrix $R \in O(d)$.*
   - *(d) $F_s(\xi) = c(s)|\xi|^{s-d}$ for some $c(s) \in \mathbb{C}$.*

4. *Show that the result is true for $s \in (d/2, d)$. (Hint: use the identity $\int |x|^{-s}\widehat{\varphi}(x)dx = c(s)\int |\xi|^{s-d}\varphi(\xi)d\xi$ with $\varphi(\xi) = e^{-|\xi|^2/2}$ and introduce polar coordinates.)*

5. *Conclude for all $s \in (0, d)$. (Hint: analytic continuation.)*

Comment. In dimension $d = 3$, we recover that $(2\pi)^{-3}\mathcal{F}(|\xi|^{-2}) = (4\pi)^{-1}|x|^{-1}$, using that $\Gamma(1/2) = \pi^{1/2}$ and $\Gamma(1) = 1$. This yields the fundamental solution to the Laplacian.

Exercise 2.24. *The purpose of this exercise is to prove Proposition 2.22.*

1. *Check that it suffices to show that, for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,*

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\overline{\varphi(s)}\psi(t)}{|t-s|^\delta} dsdt \right| \leq C||\varphi||_{L^{p'}}||\psi||_{L^{p'}}.$$

Hint: approximate $|f|, |g| \in L^{p'}$ by sequences in $\mathcal{S}$ and use Fatou's Lemma.

2. *Check that there exists a positive constant c such that, for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\overline{\varphi(s)}\psi(t)}{|t-s|^{\delta}} ds dt = c \int_{\mathbb{R}} \overline{\widehat{\varphi}(\tau)}\widehat{\psi}(\tau)|\tau|^{\delta-1} d\tau.$$

Hint: use the result of Exercise 2.23 and that $\mathcal{F}$ turns convolutions into products.

3. *Check that there exists $C > 0$ such that, for all $\psi \in \mathcal{S}(\mathbb{R})$,*

$$\big|\big| |\tau|^{\frac{\delta-1}{2}} \widehat{\psi} \big|\big|_{L^2} \leq C ||\psi||_{L^{p'}}.$$

Hint: use (2.26) and the fact that $|(\psi, |D|^{\frac{\delta-1}{2}}\phi)_{L^2}| \leq ||\psi||_{L^{p'}} |||D|^{\frac{\delta-1}{2}}\phi||_{L^p}$ for all $\phi \in \mathcal{S}(\mathbb{R})$ with Fourier transform vanishing in a neighborhood of 0.

4. *Conclude.*

Exercise 2.25 (Wave group). *For $t \in \mathbb{R}$, we define $W(t) : H^1 \times L^2 \to H^1 \times L^2$ by*

$$W(t) = \begin{pmatrix} \cos t|D| & \frac{\sin t|D|}{|D|} \\ -|D|\sin t|D| & \cos t|D| \end{pmatrix},$$

where $\sin t\lambda/\lambda$ is implicitly defined as t for $\lambda = 0$.

1. *Check that each operator in the matrix $W(t)$ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

2. *Check that $W(t)$ has a unique continuous extension (starting from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$) into an operator $\dot{H}^1 \times L^2 \to \dot{H}^1 \times L^2$.*

3. *Check that $(W(t))_{t\in\mathbb{R}}$ is a group on $H^1 \times L^2$ and on $\dot{H}^1 \times L^2$.*

4. *Check that $W(t)$ preserves the norm $(||v||^2_{\dot{H}^1} + ||w||^2_{L^2})^{1/2}$ on $\dot{H}^1 \times L^2$.*

5. *Let I be an open interval containing 0. Prove that if $(u_0, u_1) \in \dot{H}^1 \times L^2$ and $f \in C(I, L^2)$ then*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin t|D|}{|D|}u_1 + \int_0^t \frac{\sin(t-s)|D|}{|D|} f(s) ds$$

solves the wave equation (1.6) in the distributions sense on $I \times \mathbb{R}^n$.

Chapter 3

Strichartz and nonlinear estimates

In this chapter, we introduce suitable functions spaces and prove some related estimates which will be useful to define rigorously and solve the equation (1.10) by mean of the Picard fixed point Theorem. The main result is Theorem 3.9 on Strichartz estimates for the Schrödinger equation.

3.1 Functions spaces

Definition 3.1. *If I is a compact interval and X a Banach space, $C(I, X)$ is the space of functions $f : I \to X$ which are continuous on I. We equip it with the norm*

$$||f||_{L_I^\infty X} := \sup_{t \in I} ||f(t)||_X.$$

We recall that $C(I, X)$ is a Banach space.

Definition 3.2. *Let I be a compact interval and $p > 2$, $q > 2$ two real numbers. We define $\mathcal{M}_I^{p,q}$ as the subspace of functions $f \in C(I, L^2)$ such that*

1. *for almost every $t \in I$, $u(t)$ belongs to L^q,*
2. *the function $t \mapsto ||u(t)||_{L^q}$ is measurable,*
3. *the following norm is finite*

$$||u||_{L_I^p L^q} = \left(\int_I ||u(t)||_{L^q}^p dt \right)^{1/p} < \infty.$$

We equip this space with the norm

$$||u||_{\mathcal{M}_I^{p,q}} := ||u||_{L_I^\infty L^2} + ||u||_{L_I^p L^q}.$$

Definition 3.3. *Let $p > 2$ and $q > 2$ be real numbers, p' and q' be their conjugate and let I be a compact interval. We let*

$$L_I^{p'} L^{q'} := \text{completion of } C(I, L^{q'}) \text{ for the norm } ||\cdot||_{L_I^{p'} L^{q'}}.$$

(See (2.5) for the definition of the norm). We still denote by $||\cdot||_{L_I^{p'} L^{q'}}$ the norm of this completion.

Proposition 3.4. *$\mathcal{M}_I^{p,q}$ is a Banach space.*

Proof. Let $(u_k)_k$ be a Cauchy sequence in $\mathcal{M}_I^{p,q}$. It suffices to see that some subsequence will be convergent in $\mathcal{M}_I^{p,q}$. Since $(u_k)_k$ is a Cauchy sequence in $C(I, L^2)$, we know that $||u_k - u||_{L_I^\infty L^2} \to 0$ for some $u \in C(I, L^2)$. On the other hand, using that $(u_k)_k$ is a Cauchy sequence for the norm $||\cdot||_{L_I^p L^q}$ we can construct a subsequence $(u_{k_j})_j$ such that

$$||u_{k_j} - u_{k_{j+1}}||_{L_I^p L^q} \leq 2^{-j}.$$

For all $t \in I$, we consider

$$f_N(t) := \sum_{j=1}^{N} ||u_{k_j}(t) - u_{k_{j+1}}(t)||_{L^q} \in [0, \infty], \qquad f(t) := \sum_{j=1}^{\infty} ||u_{k_j}(t) - u_{k_{j+1}}(t)||_{L^q} \in [0, \infty].$$

Then we have $||f_N||_{L^p(I)} \leq 1$ and, by the Fatou Lemma[1] applied to f_N^p we also have $||f||_{L^p(I)} \leq 1$. In particular, the sequence f_N converges for almost every t which implies that, for those t, $u_{k_j}(t)$ is a Cauchy sequence in L^q. Since we already know that, as j goes to infinity, $u_{k_j}(t) \to u(t)$ in L^2, we find that

$$\lim_{j \to \infty} u_{k_j}(t) = u(t) \in L^q \text{ for almost every } t \in I.$$

In particular, $u(t)$ belongs to L^q for almost every t. By the Fatou lemma again, we have

$$\int_I ||u_{k_l}(t) - u(t)||_{L^q}^p dt \leq \liminf_{j \to \infty} \int_I ||u_{k_l}(t) - u_{k_j}(t)||_{L^q}^p dt,$$

which shows that u belongs to $L_I^p L^q$ and that $||u - u_{k_l}||_{L_I^p L^q} \to 0$ as l goes to infinity since the right hand side is smaller than $(\sum_{\kappa \geq l} 2^{-\kappa})^p$. This completes the proof. □

To be able to manipulate $L^p L^q$ functions which are unambiguously defined pointwise in time, it will be convenient to use several approximation procedures. This is the purpose of the next two propositions.

Proposition 3.5. *Assume that R_k is a family of continuous linear operators on a Banach space X such that*

$$R_k \to I \quad \text{in the strong sense as} \quad k \to \infty.$$

Then, for all $f \in C(I, X)$, if we set $f_k(t) := R_k(f(t))$, we have $f_k \in C(I, X)$ and

$$||f_k - f||_{L_I^\infty X} \to 0, \qquad k \to \infty.$$

[1] $\int \liminf \leq \liminf \int$

Proof. It relies on the fact that f is *uniformly* continuous on I. Fix first $\epsilon > 0$ and let $C > 1$ such that

$$C \geq \sup_k ||R_k||_{X \to X}.$$

Such a C exists by the uniform boundedness principle. By uniform continuity of f we can find $\delta > 0$ such that $||f(t) - f(s)||_X < \epsilon/3C$ if $|t - s| < \delta$. By compactness of I we can write I as a finite union of intervals $I_j = [t_j, t_{j+1}]$ with $|t_{j+1} - t_j| < \delta/2$. Now for any $t \in I$, we can choose j such that $t \in I_j$ and

$$\begin{aligned} ||f_k(t) - f(t)||_X &\leq ||f_k(t) - f_k(t_j)||_X + ||f_k(t_j) - f(t_j)||_X + ||f(t_j) - f(t)||_X \\ &\leq C||f(t) - f(t_j)|| + ||(R_k - I)f(t_j)||_X + ||f(t_j) - f(t)||_X \\ &\leq \frac{2\epsilon}{3} + ||(R_k - I)f(t_j)||_X, \end{aligned}$$

uniformly with respect to k. Now, since there is a finite number of t_j we can choose k_0 large enough such that $||(R_k - I)f(t_j)||_X < \epsilon/3$ for all j and all $k \geq k_0$. Taking the sup over $t \in I$, we get the result. □

Proposition 3.6. *1. $C(I, L^2 \cap L^q)$ is dense in $\mathcal{M}_I^{p,q}$.*

2. $C(I, L^2 \cap L^{q'})$ is dense in $L_I^{p'} L^{q'}$.

Proof. 1. Consider a sequence of mollifiers R_k as in Proposition 2.18. Let $u \in \mathcal{M}_I^{p,q}$ and define u_k pointwise in time by

$$u_k(t) = R_k(u(t)).$$

We know that u_k belongs to $C(I, L^2)$ and converges to u in this space by Proposition 3.5. By Proposition 2.18 and the item 4 of Proposition 2.9, it is not hard to check that u_k also belongs to $C(I, L^q)$. Furthermore

$$||u_k - u||^p_{L_I^p L^q} = \int_I ||R_k u(t) - u(t)||^p_{L^q} dt \to 0, \qquad k \to \infty, \tag{3.1}$$

by standard dominated convergence since, by Proposition 2.18, $||R_k u(t) - u(t)||_{L^q} \to 0$ for a.e. t and is dominated by $C||u(t)||^p_{L^q}$ independently of k.
2. It suffices to see that $C(I, L^2 \cap L^{q'})$ is dense in $C(I, L^{q'})$ for the $L_I^{p'} L^{q'}$ norm. Indeed, if $f \in C(I, L^{q'})$, then $R_k f$ belongs to $C(I, L^2 \cap L^{q'})$ by the item 4 of Proposition 2.9 and the convergence to f in $L_I^{p'} L^{q'}$ is obtained as in (3.1). □

Although we shall mainly use functions in spaces like $C(I, L^q)$ in this chapter, we will sometimes need to use functions in $L^p L^q_c$ (see prior to Proposition 2.3) and it will be useful to approximate them by functions in $C(I, L^q)$.

Exercise 3.7. *Let I be a compact interval and $G \in L_I^p L_c^q$ with $p, q \in [1, \infty)$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 near 0 and define $G_k(t, x) := \chi(x/k) G(t, x)$.*

1. *Check that $G_k \in C(I, L^{q_1})$ for all $q_1 \in [1, \infty]$.*
2. *Check that $||G - G_k||_{L_I^p L^q} \to 0$ as $k \to \infty$.*

3.2 Strichartz estimates

The Strichartz estimates are linear inequalities which are crucial to solve dispersive PDE at low regularity. We consider here the case of the Schrödinger equation which is simpler than the one of the wave equation (we hope, in a near future, to include a section on the wave equation in appendix).

Definition 3.8. *A pair of real numbers (p, q) is* **Schrödinger admissible** *on $\mathbb{R}^n$ if*

$$p > 2, \qquad q > 2, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{3.2}$$

Here n can be any integer ≥ 1.

We comment that the case $p = 2$ is also allowed (we then require that $q \neq \infty$ *i.e.* that $n \neq 2$) and corresponds to the so called *endpoint* pair, but we won't need to consider this limit case here (furthermore, the proof of the related Strichartz estimates is more difficult in this case, see Keel-Tao [3]). To see the relevance of the notion of Schrödinger admissible pair, we refer to Exercise 3.11 below.

The Strichartz estimates give information on the linear flow, that is on the unitary group $U(t)$ (see Proposition 2.10) and the following related operators. Given a compact interval I and real numbers $t_0, t_1 \in I$, we will consider

$$\begin{aligned} U(\cdot)u_0 &:= [t \mapsto U(t)u_0], \\ D_{t_0} f &:= \left[t \mapsto \int_{t_0}^{t} U(t-s) f(s) ds\right], \\ E_{t_0}^{t_1} &:= \int_{t_0}^{t_1} U(-s) f(s) ds. \end{aligned}$$

We call D_{t_0} the Duhamel operator at initial time t_0. These operators are *a priori* well defined between the following spaces

$$\begin{aligned} U(\cdot) &: L^2 \to C(I, L^2), \\ D_{t_0} &: C(I, L^2) \to C(I, L^2), \\ E_{t_0}^{t_1} &: C(I, L^2) \to L^2. \end{aligned}$$

We recall that the norm $||\cdot||_{\mathcal{M}_I^{p,q}}$ used in the next theorem is given in Definition 3.2.

Theorem 3.9 (Strichartz estimates for the Schrödinger equation)**.** *Let $n \geq 1$ and (p, q) be Schrödinger admissible. Then there exists $C > 0$ such that, for all compact interval I and all $t_0, t_1 \in I$, we have:*

1. *for all $u_0 \in L^2$, $U(\cdot)u_0$ belongs to $\mathcal{M}_I^{p,q}$ and*

$$||U(\cdot)u_0||_{\mathcal{M}_I^{p,q}} \leq C||u_0||_{L^2}. \tag{3.3}$$

2. *For all* $f \in C(I, L^2 \cap L^{q'})$, $D_{t_0} f$ *belongs to* $\mathcal{M}_I^{p,q}$ *and*

$$||D_{t_0} f||_{\mathcal{M}_I^{p,q}} \leq C||f||_{L_I^{p'} L^{q'}}. \tag{3.4}$$

3. *For all* $f \in C(I, L^2 \cap L^{q'})$, $E_{t_0}^{t_1} f$ *belongs to* L^2 *and*

$$||E_{t_0}^{t_1} f||_{L^2} \leq C||f||_{L_I^{p'} L^{q'}}. \tag{3.5}$$

The great interest of Strichartz estimates is the following, say for the homogeneous estimate (3.3): it says that whenever u_0 is in L^2, then $U(t)u_0$ belongs to L^q on an averaged sense (in particular for almost every t), which is quantitative since we have a L^p estimate in time of $||U(t)u_0||_{L^q}$. This is a remarkable fact since we have no derivative in L^2 for u_0 (nor for $U(t)u_0$). This is in strong contrast with Sobolev embeddings (see Proposition 2.20) which show that a function belongs some L^q space with $q > 2$ provided that some (fractional) derivative of the function also belongs to L^2.

The estimates (3.4) and (3.5) mean basically that assuming $f \in L_I^{p'} L^{q'}$ is sufficient; we only assume that f belongs to $C(I, L^2)$ to guarantee that $D_{t_0} f$ and $E_{t_0}^{t_1} f$ are clearly defined. Actually, (3.4), (3.5) and the item 2 of Proposition 3.6 lead in a straightforward fashion to the following corollary.

Corollary 3.10. *For all compact interval* $I \subset \mathbb{R}$ *and all* $t_0, t_1 \in I$, *the operators* D_{t_0} *and* $E_{t_0}^{t_1}$ *extend uniquely from* $C(I, L^2 \cap L^{q'})$ *to continuous operators*

$$\overline{D}_{t_0} : L_I^{p'} L^{q'} \to \mathcal{M}_I^{p,q}, \qquad \overline{E}_{t_0}^{t_1} : L_I^{p'} L^{q'} \to L^2,$$

such that, for some constant C *independent of* I *and* t_0, t_1,

$$\begin{aligned} ||\overline{D}_{t_0} f||_{\mathcal{M}_I^{p,q}} &\leq C||f||_{L_I^{p'} L^{q'}}, & (3.6)\\ ||\overline{E}_{t_0}^{t_1} f||_{L^2} &\leq C||f||_{L_I^{p'} L^{q'}}. & (3.7) \end{aligned}$$

Before proving Theorem 3.9, we justify the interest of the notion of Strichartz admissible pairs. We record first a simple exercise.

Exercise 3.11. *Define*

$$S_\lambda \psi(x) = \psi(\lambda x), \qquad T_\lambda u(t,x) = u(\lambda^2 t, \lambda x).$$

1. *Check that* $S_\lambda U(t) = U(\lambda^{-2} t) S_\lambda$.

2. *Let* $p, q \in [1, \infty)$ *and* I *be an interval. Check that if* ψ *is a Schwartz function*

$$\lambda^{-\left(\frac{2}{p} + \frac{n}{q}\right)} ||U(\cdot)\psi||_{L_I^p L^q} = ||T_\lambda U(\cdot)\psi||_{L^p_{\lambda^{-2} I} L^q} = ||U(\cdot) S_\lambda \psi||_{L^p_{\lambda^{-2} I} L^q}.$$

This exercise shows that (3.2) is precisely the condition which makes the Strichartz estimates scale invariant. Indeed, the second question of Exercise 3.11 shows that if Strichartz estimates as in (3.3) hold true for some pair $(p,q) \in (2,\infty)^2$ then

$$\lambda^{-\left(\frac{2}{p}+\frac{n}{q}\right)}||U(\cdot)\psi||_{L^p_I L^q} = ||U(\cdot)S_\lambda \psi||_{L^p_{\lambda^{-2}I}L^q} \leq C||S_\lambda\psi||_{L^2} = C\lambda^{-\frac{n}{2}}||\psi||_{L^2},$$

and this can hold with a constant independent of the interval I only if (p,q) is admissible.

Theorem 3.9 will be mostly a consequence of the following lemma (where we assume implicitly that (p,q) is Schrödinger admissible).

Lemma 3.12. *There exists $C > 0$ such that for all compact interval I, all $t_0 \in I$ and all $f \in C(I, L^{q'} \cap L^2)$, $D_{t_0} f$ belongs to $\mathcal{M}^{p,q}_I$ and*

$$||D_{t_0} f||_{L^p_I L^q} \leq C||f||_{L^{p'}_I L^{q'}}. \tag{3.8}$$

Proof of Lemma 3.12. That $D_{t_0} f \in C(I, L^2)$ whenever $f \in C(I, L^2)$ follows from the item 3 of Proposition 2.10 (where 0 can be replaced by any t_0). We prove next that, for all $F \in C(I, L^1 \cap L^2)$, $D_{t_0} F$ belongs to $\mathcal{M}^{p,q}_I$ and

$$||D_{t_0} F||_{L^p_I L^q} \leq C||F||_{L^{p'}_I L^{q'}}. \tag{3.9}$$

Let $F_k := R_k F$ with R_k as sequence of mollifiers as in Proposition 2.18. Then F_k belongs $C(I, L^1 \cap L^2)$, by the item 4 of Proposition 2.9, hence also to $C(I, L^{q'})$ by (2.3). We also point out that

$$D_{t_0} F_k(t) = R_k \int_{t_0}^t U(t-s)F(s)ds$$

belongs to $C(I, L^q) \cap C^0(I \times \mathbb{R}^n)$ (on one hand by the item 4 of Proposition 2.9 for L^q and on the other from the items 5 of Proposition 2.16 and 1 of Proposition 2.18 for the space-time continuity). This last property allows to use (2.6) which shows that it suffices to prove that

$$\left| \int\int_{I\times\mathbb{R}^n} G D_{t_0} F_k dxdt \right| \leq C_q ||G||_{L^{p'}_I L^{q'}} ||F_k||_{L^{p'}_I L^{q'}}. \tag{3.10}$$

for all $G \in L^{p'}_I L^{q'}_{\mathrm{c}}$. By Exercise 3.7, we can even assume that $G \in C(I, L^{q'})$. Let us prove (3.10) in this case. We know that $G D_{t_0} F_k \in L^1(I \times \mathbb{R}^n)$ by the item 1 of Proposition 2.3 so the Fubini Theorem allows to write

$$\begin{aligned}
\left| \int\int_{I\times\mathbb{R}^n} G D_{t_0} F_k dxdt \right| &= \left| \int_I \left(\int_{\mathbb{R}^n} G(t)(D_{t_0}F_k)(t)dx \right) dt \right| \\
&= \left| \int_I \int_{t_0}^t \left(\int_{\mathbb{R}^n} G(t)U(t-s)F_k(s)dx \right) dsdt \right| \\
&\leq \int_I \int_I ||G(t)||_{L^{q'}} ||U(t-s)F_k(s)||_{L^q} dsdt.
\end{aligned}$$

Note that we can swap $\int_{t_0}^t ds$ and $\int_{\mathbb{R}^n} dx$ easily since $G \in C(I, L^{q'})$. We then observe that the Marcinkiewicz Theorem 2.4 combined with the unitarity of $U(t-s)$ and Corollary 2.11 yield

$$||U(t-s)F_k(s)||_{L^q} \leq C_q |t-s|^{-\left(\frac{n}{2}-\frac{n}{q}\right)} ||F_k(s)||_{L^{q'}}.$$

Note that here we use that $F_k(s)$ belongs to $L^1 \cap L^2$. The admissibility condition (3.2) shows that $\frac{n}{2} - \frac{n}{q} = \frac{2}{p}$, hence by using the Hardy-Littlewood-Sobolev inequality (2.27) we get (3.10) which in turn yields (3.9) when $F = F_k$. Then, the estimate (3.9) applied to $F_k - F_j$ shows that $D_{t_0}F_k$ is a Cauchy sequence for the $L^p_I L^q$ norm since F_k converges to F in $C(I, L^{q'})$ (by Propositions 2.18 and 3.5) hence also for the $L^{p'}_I L^{q'}$ norm. Since $D_{t_0}F_k$ also converges to $D_{t_0}F$ in $C(I, L^2)$, the completeness of $\mathcal{M}^{p,q}_I$ shows that $D_{t_0}F$ belongs to $\mathcal{M}^{p,q}_I$ and that (3.8) holds. If f belongs to $C(L^2 \cap L^{q'})$ we consider $\tilde{F}_k := \chi(x/k)f$, with $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 near 0. Then $\tilde{F}_k$ belongs to $C(I, L^1 \cap L^2)$ and converges to f in $C(I, L^2 \cap L^{q'})$ by Proposition 3.5. In particular, $D_{t_0}\tilde{F}_k \to D_{t_0}f$ in $C(I, L^2)$. By (3.8) and the completeness of $\mathcal{M}^{p,q}_I$, $D_{t_0}\tilde{F}_k \to D_{t_0}f$ in $\mathcal{M}^{p,q}_I$ and (3.8) holds. □

Proof of Theorem 3.9. We prove first that there exists $C > 0$ such that, for all compact interval I and all $f \in C(I, L^2 \cap L^{q'})$,

$$\left|\left|\int_I U(-t)f(t)dt\right|\right|_{L^2} \leq C||f||_{L^{p'}_I L^{q'}}. \tag{3.11}$$

Denote $I = [a, b]$. Once squared, the norm reads

$$\begin{aligned}\left(\int_I U(-t)f(t)dt, \int_I U(-s)f(s)ds\right)_{L^2} &= \int_I \left(f(t), \int_I U(t-s)f(s)ds\right)_{L^2} dt \\ &= \int_I (f(t), D_a f(t) - D_b f(t))_{L^2}\, dt\end{aligned}$$

by writing $\int_I ds = \int_a^t ds - \int_b^t ds$. Using the Hölder inequality we have

$$|(f(t), D_a f(t) - D_b f(t))_{L^2}| \leq ||f(t)||_{L^{q'}} \left(||D_a f(t)||_{L^q} + ||D_b f(t)||_{L^q}\right)$$

and then, by using (3.8), we get (3.11). Then, for each $t, t_0 \in I$, we consider $I' := [t_0, t]$ (or $[t, t_0]$) and observe that

$$\begin{aligned}||D_{t_0}f(t)||_{L^2} &= \left|\left|U(t)\int_{I'} U(-s)f(s)ds\right|\right|_{L^2} \\ &\leq C||f||_{L^{p'}_{I'} L^{q'}} \\ &\leq C||f||_{L^{p'}_I L^{q'}},\end{aligned}$$

by using (3.11). Together with (3.8), this yields (3.4). The estimate (3.5) is a direct consequence of (3.11). We next consider (3.3). Let u_0 belong to L^2 and R_k be a sequence of mollifiers as in Proposition 2.18. Then $R_k U(\cdot)u_0$ belongs to $C(I, L^2 \cap L^q)$ and for all $G \in C(I, L^2 \cap L^{q'})$ we have

$$\begin{aligned}
\left| \int \int \overline{G(t,x)} R_k U(t) u_0 dx dt \right| &= \left| \int_I (G(t), U(t) R_k u_0)_{L^2} \, dt \right| \\
&= \left| \left(\int_I U(-t) G(t) dt, R_k u_0 \right)_{L^2} \right| \\
&\leq C ||G||_{L_I^{p'} L^{q'}} ||R_k u_0||_{L^2}.
\end{aligned}$$

By Exercise 3.7 the inequality remains valid if $G \in L^{p'} L_c^{q'}$ and, by (2.6), it shows that

$$||R_k U(\cdot) u_0||_{L_I^p L^q} \leq C ||R_k u_0||_{L^2}.$$

We have similar estimates on $(R_k - R_j)U(\cdot)u_0$ which show that $R_k U(\cdot)u_0$ is a Cauchy sequence in $\mathcal{M}_I^{p,q}$ which converges to $U(\cdot)u_0 \in C(I, L^2)$. This shows that $U(\cdot)u_0$ belongs to $\mathcal{M}_I^{p,q}$ and that (3.3) holds. □

3.3 Nonlinear estimates

In this short section, we record fairly elementary estimates to deal with the nonlinearities of (1.10) (or (1.11)) using the same spaces as those involved in Strichartz inequalities.

Proposition 3.13 (Nonlinear estimates). *Let I be a compact interval, $\nu > 1$ be a real number and $q := \nu + 1$.*

1. *The mapping*
$$P_\nu : C(I, L^2 \cap L^q) \ni u \mapsto |u|^{\nu-1} u \in C(I, L^{q'})$$
is well defined.

2. *Let $p > 2$ be a real number such that*
$$\frac{1}{p'} - \frac{\nu}{p} \geq 0. \tag{3.1}$$
Then P_ν has a unique continuous extension
$$\overline{P}_\nu : \mathcal{M}_I^{p,q} \to L_I^{p'} L^{q'},$$
and
$$||\overline{P}_\nu(u) - \overline{P}_\nu(v)||_{L_I^{p'} L^{q'}} \leq \nu 2^{\nu-1} |I|^{\frac{1}{p'} - \frac{\nu}{p}} \left(||u||_{L_I^p L^q}^{\nu-1} + ||v||_{L_I^p L^q}^{\nu-1} \right) ||u - v||_{L_I^p L^q}, \tag{3.2}$$
for all $u, v \in \mathcal{M}_I^{p,q}$.

We point out that if (p, q) is a Schrödinger admissible pair, the strict inequality (resp. equality) in (3.1) means precisely that ν is L^2 subcritical (resp. L^2 critical). We also emphasize that this proposition is designed to handle the L^2 (sub)critical NLS and that the analysis of the $\dot{H}^1$ subcritical equation (where ν can be larger) would require other estimates.

The proof will rely on the following two simple exercises.

Exercise 3.14. *Let $\nu > 1$ be a real number.*

1. *Let $z, \zeta \in \mathbb{C}$ be independent over $\mathbb{R}$. Show that*

$$f(s) := |z + s\zeta|^{\nu-1}(z + s\zeta)$$

is C^1 on $\mathbb{R}$ and that

$$f'(s) = |z + s\zeta|^{\nu-1}\left(\zeta + (\nu - 1)\frac{z + s\zeta}{|z + s\zeta|}\mathrm{Re}\left(\frac{\zeta(\bar{z} + s\bar{\zeta})}{|z + s\zeta|}\right)\right).$$

2. *Show that for all real numbers $a, b \geq 0$ and $\mu > 0$,*

$$(a + b)^\mu \leq 2^\mu(a^\mu + b^\mu).$$

3. *Conclude that for all $z, z' \in \mathbb{C}$ and $\nu > 1$,*

$$\big||z|^{\nu-1}z - |z'|^{\nu-1}z'\big| \leq \nu 2^{\nu-1}\big(|z|^{\nu-1} + |z'|^{\nu-1}\big)|z - z'|.$$

Exercise 3.15. *Let I be a compact interval, $p_2 \geq p_1 \geq 1$ be two real numbers and $f \in L^{p_2}(I)$. Check that*

$$||f||_{L^{p_1}} \leq |I|^{\frac{1}{p_1}-\frac{1}{p_2}}||f||_{L^{p_2}}.$$

Proof of Proposition 3.13. 1. Note first that by $|u|^{\nu-1}u$ we mean the $L^{q'}$ valued function $t \mapsto |u(t)|^{\nu-1}u(t)$. This function is indeed $L^{q'}$ valued since $|u(t)|^\nu \in L^{q/\nu} = L^{q/(q-1)} = L^{q'}$. Furthermore, it is continuous on I since, by the item 3 of Exercise 3.14 and the generalized Hölder inequality (2.2),

$$\begin{aligned}\big|\big||u(t)|^{\nu-1}u(t) - |u(s)|^{\nu-1}u(s)\big|\big|_{L^{q'}} &\leq \nu 2^{\nu-1}\left(||u(t)||_{L^q}^{\nu-1} + ||u(s)||_{L^q}^{\nu-1}\right)||u(t) - u(s)||_{L^q} \\ &\leq C_u||u(t) - u(s)||_{L^q}.\end{aligned}$$

The mapping P_ν is thus well defined.

2. It suffices to prove (3.2) on $C(I, L^2 \cap L^q)$. If we do it, then for any $u \in \mathcal{M}_I^{p,q}$, by using Proposition 3.6, we can define $\overline{P}_\nu(u)$ as the limit of $P_\nu(u_j)$ for any $u_j \in C(I, L^2 \cap L^q)$ converging to u in $\mathcal{M}_I^{p,q}$ and (3.2) will show on one hand that the limit does not depend on the choice of u_j and on the other hand that $\overline{P}_\nu$ is continuous. So let us prove (3.2). Once again, the item 3 of Exercise 3.14 and the generalized Hölder inequality (2.2) yield

$$\big|\big||u(t)|^{\nu-1}u(t) - |v(t)|^{\nu-1}v(t)\big|\big|_{L^{q'}} \leq \nu 2^{\nu-1}\left(||u(t)||_{L^q}^{\nu-1} + ||v(t)||_{L^q}^{\nu-1}\right)||u(t) - v(t)||_{L^q}.$$

By taking the $L^{p'}$ norm in time of both sides and by observing that the right hand side belongs to $L^{\frac{p}{\nu}}(I)$ (using the generalized Hölder inequality (2.2) in time), the result follows from the Exercise 3.15. □

3.4 Global in time estimates

The estimates of Theorem 3.9 and Corollary 3.10 involve constants independent of the interval I. This suggests we could take $I = \mathbb{R}$ in these results. In this section, we describe how to justify this fact. This will turn out to be important to prove the global well posedness of the L^2 critical NLS with small data.

Definition 3.16. *If X is a Banach space, we denote by $C_u(\mathbb{R}, X)$ the space of bounded and uniformly continuous functions $f : \mathbb{R} \to X$. We equip it with the norm*

$$||f||_{L^\infty_{\mathbb{R}} X} := \sup_{t \in \mathbb{R}} ||f(t)||_X.$$

Exercise 3.17. *1. Check that $C_u(\mathbb{R}, X)$ is a Banach space.*

2. Define

$$C_{\mathrm{scat}}(\mathbb{R}, L^2) := \left\{ f \in C(\mathbb{R}, L^2) \mid \lim_{t \to +\infty} U(-t)f(t) \text{ and } \lim_{t \to -\infty} U(-t)f(t) \text{ exist in } L^2 \right\}$$

and check that it is a closed subspace of $C_u(\mathbb{R}, L^2)$.

It follows from this exercise that the space $C_{\mathrm{scat}}(\mathbb{R}, L^2)$ is a Banach space. The index *scat* refers to *scattering*. We will say that a continuous fonction $f : \mathbb{R} \to L^2$ **scatters** as $t \to \pm\infty$ if there are (time independent) $f_\pm \in L^2$ such that

$$||f(t) - U(t)f_\pm||_{L^2} \to 0, \qquad t \to \pm\infty. \tag{3.3}$$

By construction of $C_{\mathrm{scat}}(\mathbb{R}, L^2)$ and the unitarity of $U(t)$, it is clear that all functions of $C_{\mathrm{scat}}(\mathbb{R}, L^2)$ scatter, since if we set $f_\pm = \lim_{t \to \pm\infty} U(-t)f(t)$ then, by unitarity of $U(t)$,

$$||f(t) - U(t)f_\pm||_{L^2} = ||U(-t)f(t) - f_\pm||_{L^2} \to 0, \qquad t \to \pm\infty.$$

In these introductory lectures, will not enter in a detailed description of what scattering theory is but we simply record that (3.3) means that, as time goes to infinity, a function f which scatters behaves like a solution to the 'free' (*i.e.* linear and homogeneous) Schrödinger equation. We shall see that the space $C_{\mathrm{scat}}(\mathbb{R}, L^2)$ is a good one to solve the L^2 critical NLS with small data.

We next introduce the analogue of Definition 3.2 when $I = \mathbb{R}$.

Definition 3.18. *If $p > 2$ and $q > 2$ are real numbers, we define $\mathcal{M}^{p,q}_{\mathbb{R}}$ as the set of functions $u \in C_{\mathrm{scat}}(\mathbb{R}, L^2)$ such that*

1. *for almost every $t \in \mathbb{R}$, $u(t)$ belongs to L^q,*

2. *the function $t \mapsto ||u(t)||_{L^q}$ is measurable,*

3. the following norm is finite

$$||u||_{L^p_{\mathbb{R}} L^q} = \left(\int_{\mathbb{R}} ||u(t)||^p_{L^q} dt \right)^{1/p} < \infty.$$

We equip this space with the norm

$$||u||_{\mathcal{M}^{p,q}_{\mathbb{R}}} := ||u||_{L^\infty_{\mathbb{R}} L^2} + ||u||_{L^p_{\mathbb{R}} L^q}.$$

Exercise 3.19. *Check that $\mathcal{M}^{p,q}_{\mathbb{R}}$ is complete (hint: mimick the proof of Proposition 3.4.)*

Before stating our result on global in time Strichartz estimates, we record another result as an exercise.

Exercise 3.20. *Let $F : \mathbb{R} \to L^2$. Check that $F(t)$ has a limit as $t \to \pm\infty$ if and only if for all $\varepsilon > 0$ there exists $T > 0$ such that $||F(t) - F(s)||_{L^2} < \varepsilon$ for all $t, s > \pm T$.*

Theorem 3.21 (Global in time Strichartz estimates)**.** *Assume that (p, q) is a Schrödinger admissible pair. There exists $C > 0$ such that,*

1. for all $u_0 \in L^2$, $U(\cdot)u_0$ belongs to $\mathcal{M}^{p,q}_{\mathbb{R}}$ and

$$||U(\cdot)u_0||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C||u_0||_{L^2},$$

2. for all $f \in C(\mathbb{R}, L^2 \cap L^{q'})$ such that $||f||_{L^{p'}_{\mathbb{R}} L^{q'}} < \infty$, $D_0 f$ belongs to $\mathcal{M}^{p,q}_{\mathbb{R}}$ and

$$||D_0 f||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C||f||_{L^{p'}_{\mathbb{R}} L^{q'}}.$$

Proof. 1. The estimate of $||U(\cdot)u_0||_{L^p_{\mathbb{R}} L^q}$ follows from the fact that the constant in (3.3) is independent of I. That $U(\cdot)u_0$ belongs to $C_{\text{scat}}(\mathbb{R}, L^2)$ is trivial.
2. By (3.4) where the constant is independent of I, it is clear that $D_0 f$ belongs to $C(\mathbb{R}, L^2)$ and that $||D_0 f||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C||f||_{L^{p'}_{\mathbb{R}} L^{q'}}$. We show that $U(-t)(D_0 f)(t)$ has limits as $t \to \pm\infty$. We consider the case of $+\infty$ (the one of $-\infty$ is similar). For all $t > s$, we have

$$||U(-t)(D_0 f)(t) - U(-s)(D_0 f)(s)||_{L^2} = \left|\left| \int_s^t U(-r) f(r) dr \right|\right|_{L^2} \leq C||f||_{L^{p'}_{[s,t]} L^{q'}}$$

by using (3.5) with $[s, t] = I = [t_0, t_1]$. The conclusion then follows from Exercise 3.20 since $||f||_{L^{p'}_{[s,t]} L^{q'}}$ is arbitrarily small for s, t large enough. $\square$

To extend the Strichartz estimates of Theorem 3.21 to a complete space, we introduce the following definition which is of course the analogue of Definition 3.3.

Definition 3.22. *Let $p > 2$ and $q > 2$ be real numbers, p' and q' be their conjugate. We let*

$$L^{p'}_{\mathbb{R}} L^{q'} := \text{completion of } C(\mathbb{R}, L^{q'}) \text{ for the norm } || \cdot ||_{L^{p'}_{\mathbb{R}} L^{q'}},$$

or, more precisely, the completion of the subset of $C(\mathbb{R}, L^{q'})$ on which the $L^{p'}_{\mathbb{R}} L^{q'}$ norm is finite. We still denote by $|| \cdot ||_{L^{p'}_{\mathbb{R}} L^{q'}}$ the norm of this completion.

Corollary 3.23. *If (p, q) is a Schrödinger admissible pair, the Duhamel operator D_0 initially defined $C(\mathbb{R}, L^2 \cap L^{q'}) \cap L^{p'}_{\mathbb{R}} L^{q'}$ extends in a unique continuous fashion to an operator*

$$\overline{D}_0 : L^{p'}_{\mathbb{R}} L^{q'} \to \mathcal{M}^{p,q}_{\mathbb{R}},$$

which satisfies $||\overline{D}_0 f||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C||f||_{L^{p'}_{\mathbb{R}} L^{q'}}$.

Proof. It suffices to see that $C(\mathbb{R}, L^2 \cap L^{q'}) \cap L^{p'}_{\mathbb{R}} L^{q'}$ is dense in $C(\mathbb{R}, L^{q'}) \cap L^{p'}_{\mathbb{R}} L^{q'}$ for the $L^{p'}_{\mathbb{R}} L^{q'}$ norm. This can be done exactly as for the item 2 of Proposition 3.6 hence we do not repeat the argument. □

We conclude this section with a global in time analogue of Proposition 3.13.

Proposition 3.24 (Nonlinear estimates)**.** *Let $\nu > 1$ be a real number, $q := \nu + 1$ and $p > 2$ be a real number such that*

$$\frac{1}{p'} - \frac{\nu}{p} = 0. \tag{3.4}$$

The mapping

$$P_\nu : \mathcal{M}^{p,q}_{\mathbb{R}} \cap C(\mathbb{R}, L^q) \ni u \mapsto |u|^{\nu-1} u \in C(\mathbb{R}, L^{q'})$$

is well defined and has a unique continuous extension

$$\overline{P}_\nu : \mathcal{M}^{p,q}_{\mathbb{R}} \to L^{p'}_{\mathbb{R}} L^{q'},$$

with

$$||\overline{P}_\nu(u) - \overline{P}_\nu(v)||_{L^{p'}_{\mathbb{R}} L^{q'}} \leq \nu 2^{\nu-1} \left(||u||^{\nu-1}_{L^p_{\mathbb{R}} L^q} + ||v||^{\nu-1}_{L^p_{\mathbb{R}} L^q} \right) ||u - v||_{L^p_{\mathbb{R}} L^q}, \tag{3.5}$$

for all $u, v \in \mathcal{M}^{p,q}_{\mathbb{R}}$.

Proof. That P_ν maps $\mathcal{M}^{p,q}_{\mathbb{R}} \cap C(\mathbb{R}, L^q)$ to $C(\mathbb{R}, L^{q'})$ is a direct consequence of the item 1 of Proposition 3.13. The estimate (3.5) is also valid on $\mathcal{M}^{p,q}_{\mathbb{R}} \cap C(\mathbb{R}, L^q)$ by (3.2). Therefore, the proposition would follow from the density of $\mathcal{M}^{p,q}_{\mathbb{R}} \cap C(\mathbb{R}, L^q)$ in $\mathcal{M}^{p,q}_{\mathbb{R}}$. This is the main point of this proof. Let $R_k = \chi(\epsilon_k D)$ be a sequence of mollifiers as in Proposition 2.18 and $u \in \mathcal{M}^{p,q}_{\mathbb{R}}$. We show that $R_k u \to u$ in $\mathcal{M}^{p,q}_{\mathbb{R}}$ as $k \to \infty$. We check first that $R_k u$

belongs to $C_{\text{scat}}(\mathbb{R}, L^2)$. If $u_\pm := \lim_{t\to\pm\infty} U(-t)u(t)$ then, using that R_k commutes with $U(-t)$ and the item 2 of Proposition 2.18), we have

$$||U(-t)R_k u(t) - R_k u_\pm||_{L^2} = ||R_k(U(-t)u(t) - u_\pm)||_{L^2} \leq C||U(-t)u(t) - u_\pm||_{L^2} \to 0,$$

as $t \to \pm\infty$. This shows that $R_k u$ belongs to $C_{\text{scat}}(\mathbb{R}, L^2)$. We next prove that

$$\sup_{t\in\mathbb{R}} ||R_k u(t) - u(t)||_{L^2} \to 0, \qquad k \to \infty. \tag{3.6}$$

We start by observing that we can write

$$\begin{aligned} ||R_k u(t) - u(t)|| &= ||U(-t)R_k u(t) - U(-t)u(t)||_{L^2} \\ &\leq ||R_k(U(-t)u(t) - u_\pm)||_{L^2} + ||R_k u_\pm - u_\pm||_{L^2} + ||u_\pm - U(-t)u(t)||_{L^2} \end{aligned}$$

where, given $\varepsilon > 0$, we can choose $T > 0$ such that the first and third term are smaller than $\varepsilon/3$ if $\pm t > T$, uniformly with respect to k. Then, using that $R_k \to I$ strongly on L^2, the term in the middle is smaller than $\varepsilon/3$ if k is large enough. We have thus shown that for each $\varepsilon > 0$ we can choose $T > 0$ and $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$

$$\sup_{|t|>T} ||R_k u(t) - u(t)|| \leq \varepsilon.$$

On the other hand, we already know that $\sup_{|t|\leq T} ||R_k u(t) - u(t)||_{L^2} \to 0$ as k goes to infinity, by Proposition 3.5. Therefore, we have proved (3.6). The convergence of $R_k u$ to u for the $L^p_{\mathbb{R}} L^q$ norm is simpler and proved by dominated convergence as in Proposition 3.6. This completes the proof since $R_k u$ belongs to $\mathcal{M}^{p,q}_{\mathbb{R}} \cap C(\mathbb{R}, L^q)$. $\square$

Chapter 4

The Cauchy problem for NLS

In this chapter, we solve the L^2 subcritical non linear Schrödinger equation, for all initial data in L^2, and the L^2 critical one for small initial data in L^2. In both cases, the point is to give a precise meaning to the fixed point equation (1.10) and to solve it.

4.1 The L^2 subcritical NLS

We work in dimension $n \geq 1$ and assume that ν is a L^2 subcritical exponent, namely

$$1 < \nu < 1 + \frac{4}{n}. \tag{4.1}$$

For a given ν as above, we define the real numbers p and q by

$$q := \nu + 1, \qquad p := \frac{4(\nu+1)}{n(\nu-1)}.$$

Exercise 4.1. *1. Check that (p, q) is Schrödinger admissible (in the sense of (3.2)).*

2. Check that

$$\frac{1}{p'} - \frac{\nu}{p} = \frac{n}{4}\left(1 + \frac{4}{n} - \nu\right) > 0.$$

3. Check that $p > q$.

In the previous chapters, we have set up all the tools required to solve rigorously the L^2 subscritical nonlinear Schrödinger equation. We fix $\mu \in \mathbb{R}$. Given $u_0 \in L^2$, we rewrite the equation (1.10) as

$$u = U(\cdot)u_0 - \frac{\mu}{i}\overline{D}_0\overline{P}_\nu(u), \tag{4.2}$$

where we recall that $U(\cdot)u_0$ is the map $t \mapsto U(t)u_0$ and that $\overline{D}_0$ and $\overline{P}_\nu$ are defined respectively in Corollary 3.10 and Proposition 3.13. Note in particular that the right hand side of the above equation is perfectly defined since the subcriticality condition (4.1) implies that $\frac{1}{p'} - \frac{\nu}{p} > 0$ by Exercise 4.1.

We will seek solutions $u \in \cap_I \mathcal{M}_I^{p,q}$, the intersection being taken over all compact intervals containing 0 (see Definition 3.2 for the space $\mathcal{M}_I^{p,q}$). More precisely, this means we shall look for $u \in C(\mathbb{R}, L^2)$ such that, for any compact interval I containing 0, its restriction to I belongs to $\mathcal{M}_I^{p,q}$ and solves (4.2).

This section is entirely devoted to the proof of the following theorem.

Theorem 4.2. *1. For all $u_0 \in L^2$, the equation*

$$u = U(\cdot)u_0 - \frac{\mu}{i}\overline{D}_0\overline{P}_\nu(u), \tag{4.3}$$

has a unique solution $u \in \cap_I \mathcal{M}_I^{p,q}(\mathbb{R})$ (the intersection is taken over all compact intervals containing 0).

2. If we define $\Phi^t(u_0) := u(t)$, we define a (flow) map

$$\Phi^t : L^2 \to L^2$$

which satisfies, for all $u_0 \in L^2$ and all $s, t \in \mathbb{R}$,

$$||\Phi^t(u_0)||_{L^2} = ||u_0||_{L^2}, \qquad \Phi^t \circ \Phi^s = \Phi^{t+s}, \qquad \Phi^0 = I.$$

In addition, for all positive numbers R, T, there exists a constant $C > 0$ such that

$$||\Phi^t(u_0) - \Phi^t(v_0)||_{L^2} \leq C||u_0 - v_0||_{L^2}, \tag{4.4}$$

for all $|t| \leq T$ and all u_0, v_0 such that $||u_0||_{L^2} \leq R$, $||v_0||_{L^2} \leq R$.

3. The solution to (4.3) belongs to $C(\mathbb{R}, L^2) \cap L^\nu_{\text{loc}}(\mathbb{R} \times \mathbb{R}^n)$. It satisfies $u(0) = u_0$ and, in the distributions sense,

$$i\partial_t u - \Delta u = -\mu|u|^{\nu-1}u. \tag{4.5}$$

We will split the proof of this theorem into several propositions or lemmas. For given real numbers $t_0, t_1 \in \mathbb{R}$, it will be convenient to denote

$$K_{t_0} := -\frac{\mu}{i}\overline{D}_{t_0} \circ \overline{P}_\nu : \mathcal{M}_I^{p,q} \to \mathcal{M}_I^{p,q}, \qquad L_{t_0}^{t_1} := \overline{E}_{t_0}^{t_1} \circ \overline{P}_\nu : \mathcal{M}_I^{p,q} \to L^2, \tag{4.6}$$

which are both continuous on $\mathcal{M}_I^{p,q}$ for any compact interval I containing t_0 and t_1 (recall that $\overline{E}_{t_0}^{t_1}$ is defined in Corollary 3.10). The continuity of these maps follows from Corollary 3.10 and Proposition 3.13.

Proposition 4.3 (Local existence)**.** *There exists a constant $C > 1$ such that, for all $t_0 \in \mathbb{R}$ and all $u_0 \in L^2$, the map*

$$u \mapsto U(\cdot - t_0)u_0 + K_{t_0}(u)$$

is a contraction on the closed ball

$$\mathcal{B}_\tau := \left\{ u \in \mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]} \mid ||u - U(\cdot)u_0||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}} \le 2||U(\cdot)u_0||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}} \right\}$$

where τ is given by

$$\tau = C^{-1}||u_0||_{L^2}^{\frac{1}{\frac{n}{4}-\frac{1}{\nu-1}}}. \tag{4.7}$$

Note that $\frac{1}{\frac{n}{4}-\frac{1}{\nu-1}} < 0$ so the existence time grows as $||u_0||_{L^2}$ decreases.

Proof. We note first that, for all $\tau > 0$, $\mathcal{B}_\tau$ is a complete metric space, as a closed ball of the Banach space $\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}$. We then look for conditions on τ such that F, defined by

$$F(u) := U(\cdot - t_0)u_0 + K_{t_0}(u),$$

preserves $\mathcal{B}_\tau$ and is a contraction thereon. By Corollary 3.10 and Proposition 3.13 (with $v = 0$), we know that there exists $C_1 > 0$ such that for all u_0, t_0, τ and all $u \in \mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}$

$$||K_{t_0}(u)||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}} \le C_1|\tau|^{\frac{1}{p'}-\frac{\nu}{p}}||u||^\nu_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}}.$$

In particular, setting $R := 2||U(\cdot)u_0||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}}$ for simplicity, we see that if u belongs to $\mathcal{B}_\tau$, we have

$$||F(u) - U(\cdot)u_0||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}} \le C_1\tau^{\frac{1}{p'}-\frac{\nu}{p}}R^\nu. \tag{4.8}$$

Similarly, for all $u, v \in \mathcal{B}_\tau$, Corollary 3.10 and Proposition 3.13 also yield

$$||F(u) - F(v)||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}} \le C_2\tau^{\frac{1}{p'}-\frac{\nu}{p}}R^{\nu-1}||u - v||_{\mathcal{M}^{p,q}_{[t_0-\tau,t_0+\tau]}}, \tag{4.9}$$

with a constant C_2 independent of t_0, u_0, τ, u, v. Therefore, if we choose τ such that

$$C_1T^{\frac{1}{p'}-\frac{\nu}{p}}R^\nu \le R, \qquad C_2\tau^{\frac{1}{p'}-\frac{\nu}{p}}R^{\nu-1} \le 1/2, \tag{4.10}$$

then $\mathcal{B}_\tau$ is stable by F and F is 1/2 Lipschitz on $\mathcal{B}_\tau$. This is satisfied if $\tau^{\frac{1}{p}-\frac{\nu}{p}}R^{\nu-1}$ is small enough and, using that

$$\tau^{\frac{1}{p}-\frac{\nu}{p}}R^{\nu-1} = \left(\tau^{\frac{1}{\nu-1}-\frac{\nu+1}{\nu-1}\frac{1}{p}}R\right)^{\nu-1}, \qquad \frac{1}{p} = \frac{n}{4} - \frac{n}{2(\nu+1)} = \frac{n}{4}\frac{\nu-1}{\nu+1},$$

it is not hard to check that this holds if τ is of the form (4.7) with C large enough. $\square$

The next proposition will be useful to prove the uniqueness of solutions and also to prove the uniform continuity of the flow on balls.

Proposition 4.4 (Uniquess and stability estimates). *Let $R, T > 0$. There exists $C > 0$ such that*

1. *for all $t_0 \in \mathbb{R}$ and all $I := [t_0 - T, t_0 + T]$,*
2. *for all $\phi, \psi \in \mathcal{M}_I^{p,q}$*
3. *for all $u, v \in \mathcal{M}_I^{p,q}$ such that $||u||_{\mathcal{M}_I^{p,q}} + ||v||_{\mathcal{M}_I^{p,q}} \leq R$ and which solve*

$$u = \phi + K_{t_0}(u), \qquad v = \psi + K_{t_0}(v), \tag{4.11}$$

we have

$$||u - v||_{\mathcal{M}_I^{p,q}} \leq C||\phi - \psi||_{\mathcal{M}_I^{p,q}}.$$

To prove this proposition, and for future purposes, we will need the next lemma.

Lemma 4.5. 1. *Let I be a compact interval and $t_0, t_1 \in I$. If $u \in \mathcal{M}_I^{p,q}$, we have*

$$K_{t_0}(u) = U(\cdot)L_{t_0}^{t_1}(u) + K_{t_1}(u). \tag{4.12}$$

2. *If $t_0 < t_1 < t_2$, if $I = [t_0, t_2]$ and if $u \in \mathcal{M}_I^{p,q}$ solves*

$$\begin{aligned} u &= U(\cdot - t_0)u(t_0) + K_{t_0}(u), && \text{on } [t_0, t_1], && (4.13) \\ u &= U(\cdot - t_1)u(t_1) + K_{t_1}(u), && \text{on } [t_1, t_2] && (4.14) \end{aligned}$$

then it solves both equations on $[t_0, t_2]$.

3. *If $u \in \mathcal{M}_I^{p,q}$, $t_1 \in \mathbb{R}$, $t_0 \in I$ and if we define $u \in \mathcal{M}_{I-t_1}^{p,q}$ by $\widetilde{u}(t) = u(t + t_1)$ then*

$$K_{t_0}(u)(t + t_1) = K_{t_0 - t_1}(\widetilde{u})(t), \qquad t \in I - t_1.$$

The first item gives a rigorous sense to the formula

$$\int_{t_0}^{t} U(t-s)(|u|^{\nu-1}u)(s)ds = U(t)\int_{t_0}^{t_1} U(-s)(|u|^{\nu-1}u)(s)ds + \int_{t_1}^{t} U(t-s)(|u|^{\nu-1}u)(s)ds \tag{4.15}$$

Let us recall that the above expression is formal (at least with the tools we are using) since the map $s \mapsto (|u|^{\nu-1}u)(s)$ is not a continuous L^2 valued function in general.

Proof. 1. If $u \in C(I, L^2 \cap L^\infty)$ then (4.15) makes sense, and coincides with (4.12), since $|u|^{\nu-1}u$ is continuous from I to L^2. If u only belongs to $\mathcal{M}_I^{p,q}$, we let $u_k = R_k u$ with $R_k = \chi(\epsilon_k D)$ as in Proposition 2.18. Using that $u_k \to u$ in $\mathcal{M}_I^{p,q}$ and the continuity of K_{t_0}, K_{t_1} and $L_{t_0}^{t_1}$ on $\mathcal{M}_I^{p,q}$, we get (4.12) by letting $k \to \infty$ in (4.15) applied to u_k.
2. By using (4.12) on $[t_1, t_2]$, we have

$$U(\cdot - t_0)u(t_0) + K_{t_0}(u) = U(\cdot)\big(U(-t_0)u(t_0) + L_{t_0}^{t_1}u\big) + K_{t_1}(u). \tag{4.16}$$

On the other hand, by using (4.13) at $t = t_1$, we also have

$$u(t_1) = U(t_1 - t_0)u(t_0) + K_{t_0}(u)(t_1) = U(t_1)\big(U(-t_0)u(t_0) + L_{t_0}^{t_1}(u)\big)$$

and by plugging this identity in (4.16) we get precisely

$$U(\cdot - t_0)u(t_0) + K_{t_0}(u) = U(\cdot - t_1)u(t_1) + K_{t_1}(u)$$

on $[t_1, t_2]$. Since the right hand side is equal to u on this interval by (4.14) we see that $u = U(\cdot - t_0)u(t_0) + K_{t_0}(u)$ on $[t_1, t_2]$ hence on $[t_0, t_2]$. That (4.14) holds on $[t_0, t_2]$ is proved similarly (using additionally that $L_{t_0}^{t_1}(u) = -L_{t_1}^{t_0}(u)$).
3. If $u \in C(I, L^2 \cap L^q)$, this item follows directly from

$$\int_{t_0}^{t+t_1} U(t + t_1 - s)|u(s)|^{\nu-1}u(s)ds = \int_{t_0-t_1}^{t} U(t - \sigma)|u(\sigma + t_1)|^{\nu-1}u(\sigma + t_1)d\sigma.$$

If $u \in \mathcal{M}_I^{p,q}$, we approximate it as above by a sequence u_k as in the first item and pass to the limit, using that the map $u \mapsto u(\cdot + t_1)$ is continuous from $\mathcal{M}_I^{p,q}$ to $\mathcal{M}_{I-t_1}^{p,q}$. $\square$

Proof of Proposition 4.4. We let A be a constant such that, for all $t_0, t_1 \in \mathbb{R}$, all interval J containing t_0, t_1 and all $w_1, w_2 \in \mathcal{M}_J^{p,q}$,

$$||K_{t_1}(w_1) - K_{t_1}(w_2)||_{\mathcal{M}_J^{p,q}} \leq A|J|^{\frac{1}{p'}-\frac{\nu}{p}} \left(||w_1||_{L_J^pL^q}^{\nu-1} + ||w_2||_{L_J^pL^q}^{\nu-1}\right) ||w_1 - w_2||_{L_J^pL^q}, \quad (4.17)$$

$$||L_{t_0}^{t_1}(w_1) - L_{t_0}^{t_1}(w_2)||_{L^2} \leq A|J|^{\frac{1}{p'}-\frac{\nu}{p}} \left(||w_1||_{L_J^pL^q}^{\nu-1} + ||w_2||_{L_J^pL^q}^{\nu-1}\right) ||w_1 - w_2||_{L_J^pL^q}. \quad (4.18)$$

Such estimates are direct consequences of the Strichartz estimates of Corollary 3.10 and of the nonlinear estimates (3.2). We choose $N \geq 1$ large enough such that

$$2R^{\nu-1}A(T/N)^{\frac{1}{p'}-\frac{\nu}{p}} \leq \frac{1}{2}. \quad (4.19)$$

We let $I_k = [t_0, t_0 + kT/N]$ for $k \leq N$ and show by (finite) induction on k that there exists $C_k > 0$ such that for all $\phi, \psi \in \mathcal{M}_I^{p,q}$ and all $u, v \in \mathcal{M}_I^{p,q}$ solving (4.11), we have

$$||u - v||_{\mathcal{M}_{I_k}^{p,q}} \leq C_k||\phi - \psi||_{\mathcal{M}_{I_k}^{p,q}}. \quad (4.20)$$

The result is trivial if $k = 0$. Assume it holds for k and let us show it holds for $k + 1$. Let us set

$$J_k = [t_0 + kT/N, t_0 + (k + 1)T/N].$$

We start with the inequality

$$||u - v||_{\mathcal{M}_{I_{k+1}}^{p,q}} \leq ||u - v||_{\mathcal{M}_{I_k}^{p,q}} + ||u - v||_{\mathcal{M}_{J_k}^{p,q}} \quad (4.21)$$

and get, using Lemma 4.5 with $t_1 = t_0 + kT/N$,

$$||u - v||_{\mathcal{M}_{J_k}^{p,q}} \leq ||\phi - \psi||_{\mathcal{M}_{J_k}^{p,q}} + ||U(\cdot)\left(L_{t_0}^{t_1}(u) - L_{t_0}^{t_1}(v)\right)||_{\mathcal{M}_{J_k}^{p,q}} + ||K_{t_1}(u) - K_{t_1}(v)||_{\mathcal{M}_{J_k}^{p,q}}.$$

Obviously, we can estimate the first term by

$$||\phi - \psi||_{\mathcal{M}^{p,q}_{J_k}} \le ||\phi - \psi||_{\mathcal{M}^{p,q}_{I_{k+1}}},$$

For the second one, we use (4.18) and the induction assumption to obtain

$$\begin{aligned} ||U(\cdot)\left(L^{t_1}_{t_0}(u) - L^{t_1}_{t_0}(v)\right)||_{\mathcal{M}^{p,q}_{J_k}} &\le C||L^{t_1}_{t_0}(u) - L^{t_1}_{t_0}(v)||_{L^2} \\ &\le 2CR^{\nu-1}A|I_k|^{\frac{1}{p'}-\frac{\nu}{p}}||u-v||_{L^p_{I_k}L^q} \\ &\le 2CR^{\nu-1}A|I_k|^{\frac{1}{p'}-\frac{\nu}{p}}C_k||\phi-\psi||_{\mathcal{M}^{p,q}_{I_k}}. \end{aligned}$$

By (4.17) and (4.19), the last term can be estimated by

$$\begin{aligned} ||K_{t_1}(u) - K_{t_1}(v)||_{\mathcal{M}^{p,q}_{J_k}} &\le 2R^{\nu-1}A|J_k|^{\frac{1}{p'}-\frac{\nu}{p}}||u-v||_{\mathcal{M}^p_{J_k}L^q} \\ &\le \frac{1}{2}||u-v||_{\mathcal{M}^p_{I_{k+1}}L^q}. \end{aligned}$$

Summing up and using (4.21), we obtain

$$\left(1-\frac{1}{2}\right)||u-v||_{\mathcal{M}^{p,q}_{I_{k+1}}} \le \left(C_k + 1 + 2CR^{\nu-1}A|I_k|^{\frac{1}{p'}-\frac{\nu}{p}}C_k\right)||\phi-\psi||_{\mathcal{M}^{p,q}_{I_{k+1}}}$$

which shows that (4.20) holds at step $k+1$ and completes the proof. □

Proposition 4.6 (Conservation of the L^2 norm)**.** *Let I be a compact interval, $t_0 \in I$ and $u \in \mathcal{M}^{p,q}_I$ solve*

$$u = U(\cdot - t_0)u(t_0) + K_{t_0}(u).$$

Then, for all $t \in I$,

$$||u(t)||_{L^2} = ||u(t_0)||_{L^2}.$$

As we will see below, it is worth noticing that the proof of this proposition does not use the subcriticality assumption (4.1). In particular, it would also hold for the L^2 critical case.

Proof. Using the mollifiers of Proposition 2.18, we let $\tilde{u}_k = \chi(\epsilon_k D)u = R_k u$ and define

$$\begin{aligned} v_k(t) &:= R_k U(t-t_0)u(t_0) - \frac{\mu}{i}R_k\int_{t_0}^t U(t-s)|\tilde{u}_k(s)|^{\nu-1}\tilde{u}_k(s)ds \\ &= U(t)R_kU(-t_0)u(t_0) - U(t)R_k\frac{\mu}{i}\int_{t_0}^t U(-s)|\tilde{u}_k(s)|^{\nu-1}\tilde{u}_k(s)ds. \end{aligned}$$

We will use that $t \mapsto U(t)R_kf(t)$ is a L^2 valued $C^1(I)$ function when $f \in C^1(I, L^2)$, with derivative

$$\frac{d}{dt}U(t)R_kf(t) = -i\Delta U(t)R_kf(t) + U(t)R_kf'(t).$$

This follows from Exercise 2.19 and shows on one hand that

$$\frac{d}{dt}v_k(t) = -i\Delta v_k(t) - \frac{\mu}{i}R_k\left(|\tilde{u}_k(t)|^{\nu-1}\tilde{u}_k(t)\right), \tag{4.22}$$

and on the other hand that

$$\frac{d}{dt}||v_k(t)||_{L^2}^2 = 2\mathrm{Re}\,\left(v_k(t), \frac{d}{dt}v_k(t)\right)_{L^2}.$$

Therefore,

$$\begin{aligned}
||v_k(t)||_{L^2}^2 - ||v_k(t_0)||_{L^2}^2 &= -2\int_{t_0}^t \mathrm{Re}\,\left(v_k(s), i\Delta v_k(s) + \frac{\mu}{i}R_k\left(|\tilde{u}_k(s)|^{\nu-1}\tilde{u}_k(s)\right)\right)_{L^2} ds \\
&= -2\mu\int_{t_0}^t \mathrm{Im}\,\left(v_k(s), R_k\left(|\tilde{u}_k(s)|^{\nu-1}\tilde{u}_k(s)\right)\right)_{L^2} ds \\
&= -2\mu\int_{t_0}^t \mathrm{Im}\,\left(\int_{\mathbb{R}^n} \overline{(R_k^* v_k)(s,x)}|\tilde{u}_k(s,x)|^{\nu-1}\tilde{u}_k(s,x)dx\right) ds,
\end{aligned}$$

since $-(v_k(s), \Delta v_k(s))_{L^2} = ||\nabla v_k(s)||_{L^2}^2$ is real. Writing $(R_k^* v_k) = \tilde{u}_k + (R_k^* v_k) - \tilde{u}_k$ and using the mixed Hölder inequality (see Proposition 2.1), we obtain

$$\big|||v_k(t)||_{L^2}^2 - ||v_k(t_0)||_{L^2}^2\big| \leq 2\big|\big|R_k^* v_k - \tilde{u}_k\big|\big|_{L_I^p L^q}\big|\big|P_\nu(\tilde{u}_k)\big|\big|_{L_I^{p'} L^{q'}}. \tag{4.23}$$

Since $\tilde{u}_k \to u$ in $\mathcal{M}_I^{p,q}$ (see Proposition 3.6), $P_\nu(\tilde{u}_k)$ is bounded in $L_I^{p'}L^{q'}$ by Proposition 3.13. On the other hand, using that $K_{t_0}(\tilde{u}_k) \to K_{t_0}(u)$ by Proposition 3.13 and (3.6), we see that $v_k \to u$ in $\mathcal{M}_I^{p,q}$ and

$$\big|\big|R_k^* v_k - \tilde{u}_k\big|\big|_{L_I^p L^q} \leq C\big|\big|v_k - u\big|\big|_{L_I^p L^q} + \big|\big|R_k^* u - \tilde{u}_k\big|\big|_{L_I^p L^q} \to 0.$$

This allows to let k go to infinity in (4.23) whose right hand side goes to zero and the left hand side goes to $|||u(t)||_{L^2}^2 - ||u(t_0)||_{L^2}^2|$. This completes the proof. $\square$

We note that this proof is rigorous justification of the formal computation (1.21).

With the previous results at hand, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. 1. *Uniqueness.* If both u and v solve (4.3) then, for all $T > 0$ we have $u(t) = v(t)$ for all $t \in [-T, T]$ by applying Proposition 4.4 with

$$\phi = \psi = U(\cdot)u_0, \qquad I = [-T, T], \qquad R = ||u||_{\mathcal{M}_I^{p,q}} + ||v||_{\mathcal{M}_I^{p,q}}.$$

Existence. Fix $u_0 \in L^2$ and let τ be as in Proposition 4.3. Then, we can solve (4.3) on $[0, \tau]$; we call $u^{(1)}$ the corresponding solution. By Proposition 4.6 $||u^{(1)}(\tau)||_{L^2} = ||u_0||_{L^2}$, so by Proposition 4.3 with initial time $t_0 = \tau$, we can find $\tilde{u}^{(1)}$ which solve

$$\tilde{u}^{(1)} = U(\cdot - \tau)u^{(1)}(\tau) + K_\tau\big(\tilde{u}^{(1)}\big), \tag{4.24}$$

on $[\tau, 2\tau]$. Then,

$$u^{(2)}(t) := \begin{cases} u^{(1)}(t) & t \in [0,\tau] \\ \tilde{u}^{(1)}(t) & t \in (\tau, 2\tau] \end{cases},$$

is automatically continuous on $[0, 2\tau]$ with values in L^2. Furthermore, $||u||_{L^p_{[0,2\tau]}L^q}$ is also finite, so $u^{(2)}$ belongs to $\mathcal{M}^{p,q}_{[0,2\tau]}$. By construction, $u^{(2)}$ solves (4.3) on $[0,\tau]$ but it also solves (4.24). By the item 2 of Lemma 4.5, it solves (4.3) on $[0, 2\tau]$. We can then repeat this procedure and find a solution $u^{(3)}$ on $[0, 3\tau]$ by gluing to $u^{(2)}$ the solution to

$$\tilde{u} = U(\cdot - 2\tau)u^{(2)}(2\tau) + K_{2\tau}(\tilde{u}) \qquad \text{on } [2\tau, 3\tau],$$

which is well defined by Proposition 4.3 and the fact that $||u^{(2)}(2\tau)||_{L^2} = ||u_0||_{L^2}$ by Proposition 4.6. By induction, we can construct a solution defined on $[0, +\infty)$ (*i.e.* on $[0, k\tau]$ for all $k \in \mathbb{N}$), and then similarly on $(-\infty, 0]$.
2. The flow Φ^t is well defined by the item 1. That it preserves the L^2 norm follows from Proposition 4.6. Let us prove the group relation $\Phi^{t+s} = \Phi^t \circ \Phi^s$. For a given $u_0 \in L^2$, we let

$$v(t) = \Phi^t(\Phi^s(u_0)), \qquad u(t+s) = \widetilde{u}(t) = \Phi^{t+s}(u_0),$$

so that

$$v(t) = U(t)(\Phi^s(u_0)) + K_0(v)(t).$$

and, by the item 3 of Lemma 4.5,

$$\widetilde{u}(t) = U(t+s)u_0 + K_0(u)(t+s) = U(t+s)u_0 + K_{-s}(\widetilde{u})(t).$$

By the item 1 of Lemma 4.5, we have $K_{-s}(\widetilde{u})(t) = U(t)L^0_{-s}(\widetilde{u}) + K_0(\widetilde{u})$, so we get

$$\widetilde{u}(t) = U(t)\left(U(s)u_0 + L^0_{-s}(\widetilde{u})\right) + K_0(\widetilde{u})(t) = U(t)\left(U(s)u_0 + U(s)L^s_0(u)\right) + K_0(\widetilde{u})(t)$$

where the right hand side is precisely $U(t)\Phi^s(u_0) + K_0(\widetilde{u})(t)$. Therefore, v and $\widetilde{u}$ solve the same equation hence they coincide at t. This shows that $\Phi^{t+s}(u_0) = \Phi^t \circ \Phi^s(u_0)$. We next prove the estimate (4.4). It suffices to show that, for any $T > 0$, the $\mathcal{M}^{p,q}_{[-T,T]}$ norm of $\Phi^{\cdot}(u_0) := t \mapsto \Phi^t(u_0)$ is bounded by a constant depending only on $||u_0||_{L^2}$ and T. Then the result will follow from Proposition 4.4 with $\phi = U(\cdot)u_0$ and $\psi = U(\cdot)v_0$ combined with the homogeneous Strichartz inequality (3.3). For a given R, we can fix

$$\tau = C^{-1}R^{\frac{1}{\frac{n}{4} - \frac{1}{\nu - 1}}},$$

so that, for any u_0 such that $||u_0||_{L^2} \leq R$, Proposition 4.3 implies that

$$||\Phi^{\cdot}(u_0)||_{\mathcal{M}^{p,q}_{[-\tau,\tau]}} \leq 3||U(\cdot)u_0||_{\mathcal{M}^{p,q}_{[-\tau,\tau]}} \leq C_S||u_0||_{L^2}$$

with C_S a constant given by Strichartz inequalities. Since $\Phi^\tau(u_0)$ has the same L^2 norm as u_0, we get from the group property

$$||\Phi^{\cdot}(u_0)||_{\mathcal{M}^{p,q}_{[0,2\tau]}} = ||\Phi^{\cdot}(\Phi^\tau(u_0))||_{\mathcal{M}^{p,q}_{[-\tau,\tau]}} \leq C_S||\Phi^\tau(u_0)||_{L^2}.$$

Therefore, we obtain

$$||\Phi^{\cdot}(u_0)||_{\mathcal{M}^{p,q}_{[-\tau,2\tau]}} \leq ||\Phi^{\cdot}(u_0)||_{\mathcal{M}^{p,q}_{[-\tau,\tau]}} + ||\Phi^{\cdot}(u_0)||_{\mathcal{M}^{p,q}_{[0,2\tau]}} \leq 2C_S||u_0||_{L^2},$$

and then, by an induction on k,

$$||\Phi^{\cdot}(u_0)||_{\mathcal{M}^{p,q}_{[-k\tau,k\tau]}} \leq C_k||u_0||_{L^2},$$

which yields the result.
3. We start by observing that, for all compact interval I, the space $\mathcal{M}^{p,q}_I$ is continuously embedded in $L^{\nu+1}(I \times \mathbb{R}^n)$ since $p \geq q$ and since, by the Hölder inequality,

$$||u||_{L^q(I\times\mathbb{R}^n)} \leq |I|^{\frac{1}{q}-\frac{1}{p}}||u||_{L^p_I L^q}.$$

Therefore, $\mathcal{M}^{p,q}_I$ is contained in $L^{\nu}_{\text{loc}}(I\times\mathbb{R}^n)$ and any global solution belongs to $L^{\nu}_{\text{loc}}(\mathbb{R}\times\mathbb{R}^n)$. Then, using the same approximation procedure as in Proposition 4.6, we find that

$$v_k \to u \text{ in } L^2_{\text{loc}}(\mathbb{R}\times\mathbb{R}^n), \qquad R_k(|\tilde{u}_k|^{\nu-1})\tilde{u}_k \to |u|^{\nu-1}u \;\text{ in } L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^n)$$

hence in the distributions sense, and the equation (4.5) follows by letting k go to infinity in (4.22). □

4.2 The L^2 critical case

In this section, we still assume that $n \geq 1$ but now consider the L^2 critical exponent

$$\nu = 1 + \frac{4}{n}.$$

As in the subcritical case, we define the numbers p and q by

$$q := \nu + 1 = \frac{2n+4}{n}, \qquad p := \frac{4(\nu+1)}{n(\nu-1)} = \frac{2n+4}{n},$$

which are Schrödinger admissible in the sense of Definition 3.8.

In the next theorem we show that, regardless the sign of μ, the L^2 critical NLS is globally well posed, but only for small initial data.

Theorem 4.7. *Let $\mu = \pm 1$. There exists a real number $\varepsilon > 0$ such that, if we set $B(\varepsilon) = \{u_0 \in L^2 \mid ||u_0||_{L^2} \leq \varepsilon\}$,*

1. *for all $u_0 \in B(\varepsilon)$, the equation*

$$u = U(\cdot)u_0 - \frac{\mu}{i}\overline{D}_0\overline{P}_\nu(u)$$

has a unique solution in $\mathcal{M}^{p,q}_{\mathbb{R}}$. In particular, this solution scatters as $t \to \pm\infty$, i.e. there are $u_\pm \in L^2$ such that

$$\left|\left|u(t) - e^{-it\Delta}u_\pm\right|\right|_{L^2} \to 0, \qquad t \to \pm\infty.$$

2. *This solution u belongs to $C(\mathbb{R}, L^2) \cap L^{1+\frac{4}{n}}(\mathbb{R} \times \mathbb{R}^{n-1})$ and solves*

$$u(0) = u_0, \qquad i\partial_t u - \Delta u = -\mu |u|^{\frac{4}{n}} u$$

the second equation being taken in the distributions sense.

Proof. 1. The proof is basically the same as the one of Proposition 4.3, up the fact that the contraction will follow from the smallness of the initial data rather than from smallness of the time τ. We give the main lines of the proof for completeness. Let

$$F(u) := U(\cdot)u_0 + K_0(u),$$

with $u_0 \in B(\varepsilon)$ (see (4.6) for K_0). By Corollary 3.23 and Proposition 3.24, we have

$$||K_0(u)||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C||u||^{\nu}_{\mathcal{M}^{p,q}_{\mathbb{R}}},$$

and

$$||F(u) - F(v)||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C\big(||u||^{\nu-1}_{\mathcal{M}^{p,q}_{\mathbb{R}}} + ||v||^{\nu-1}_{\mathcal{M}^{p,q}_{\mathbb{R}}}\big)||u - v||_{\mathcal{M}^{p,q}_{\mathbb{R}}},$$

for all $u, v \in \mathcal{M}^{p,q}_{\mathbb{R}}$. Let $R_\varepsilon := 2C_1\varepsilon$ with C_1 such that $||U(\cdot)u_0||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C_1||u_0||_{L^2}$ for all $u_0 \in L^2$. Such a C_1 exists according to the item 1 of Corollary 3.21. Consider the closed ball $\mathcal{B}_\varepsilon$ of $\mathcal{M}^{p,q}_{\mathbb{R}}$ of radius R_ε centered at 0. Then, if ε is small enough, we have

$$||F(u)||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C_1||u_0||_{L^2} + C\varepsilon^\nu \leq 2C_1\varepsilon$$

and

$$||F(u) - F(v)||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq C\varepsilon^{\nu-1}||u - v||_{\mathcal{M}^{p,q}_{\mathbb{R}}} \leq ||u - v||_{\mathcal{M}^{p,q}_{\mathbb{R}}}/2,$$

for all $u, v \in \mathcal{B}_\varepsilon$. Therefore, if ε is small enough, F is a contraction on the ball $\mathcal{B}_\varepsilon$ and thus has a unique fixed point.
2. This item can be shown as in the subcritical case. We simply note that $p = q$ here and that the embedding of $\mathcal{M}^{p,q}_{\mathbb{R}}$ into $L^{\nu+1}(\mathbb{R} \times \mathbb{R}^n)$ is obvious. □

We end up this section with a few words on finite time blow up for the focusing L^2 critical Schrödinger equation (see for instance [5] where the following calculations can be found). We start with an exercise.

Exercise 4.8. *Let $\psi \in C^2(\mathbb{R}^n)$ and define*

$$u(t, x) := \frac{1}{|t|^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4t} + \frac{i}{t}} \psi\left(\frac{x}{t}\right). \tag{4.25}$$

Show that

$$i\partial_t u = \frac{1}{|t|^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4t} + \frac{i}{t}} \left[\left(\frac{1}{t^2} - \frac{in}{2t} - \frac{|x|^2}{4t^2}\right)\psi\left(\frac{x}{t}\right) - i\frac{x}{t^2} \cdot (\nabla\psi)\left(\frac{x}{t}\right)\right]$$

and that

$$\Delta u = \frac{1}{|t|^{\frac{n}{2}}} e^{-i\frac{|x|^2}{4t} + \frac{i}{t}} \left[\left(-\frac{in}{2t} - \frac{|x|^2}{4t^2}\right)\psi\left(\frac{x}{t}\right) - i\frac{x}{t^2} \cdot (\nabla\psi)\left(\frac{x}{t}\right) + \frac{1}{t^2}(\Delta\psi)\left(\frac{x}{t}\right)\right].$$

It follows from this exercise that if we can find a non zero ψ solving the nonlinear elliptic equation

$$-\Delta\psi + \psi = |\psi|^{\frac{4}{n}}\psi, \tag{4.26}$$

then u given by (4.25) solves the focusing NLS (in the classical sense)

$$i\partial_t u - \Delta u = |u|^{\frac{4}{n}}u, \qquad u(-1,x) = e^{-i\frac{|x|^2}{4}+i}\psi(x), \tag{4.27}$$

for $t \in (-\infty, 0)$ and $x \in \mathbb{R}^n$. Note that the initial time is taken at -1 (by a translation in time we could take it to be 0). If we know additionally that ψ belongs to $L^2(\mathbb{R}^n)$, then

$$|u(t,x)|^2 = \frac{1}{|t|^n}\left|\psi\left(\frac{x}{t}\right)\right|^2 \to ||\psi||_{L^2}^2\delta_0(x), \qquad t \to 0^-. \tag{4.28}$$

If $||\psi||_{L^2}$ is not zero, this means that the solution to the Cauchy problem (4.27) blows up at $t = 0$. Indeed, if the solution was global and continuous in time with values in L^2 then $|u(t,.)|^2$ should converge to $|u(0,.)|^2$ as $t \to 0^-$ in L^1, which is obviously not the case in (4.28). The blow up can also be observed on the Strichartz norm: indeed, the solution to (4.27) satisfies

$$||u(t)||_{L^q} = |t|^{-\frac{n}{2}+\frac{n}{q}}||\psi||_{L^q} = |t|^{-\frac{2}{p}}||\psi||_{L^2},$$

so that, for all $t < 0$,

$$||u||_{L^p_{(-\infty,t]}L^q} = |t|^{-\frac{1}{p}}||\psi||_{L^q} \to +\infty, \qquad t \to 0^-.$$

It turns out that one can indeed find a non trivial solution to (4.26); more precisely, one can find a nonnegative C^2 solution

$$-\Delta Q + Q = Q^{1+\frac{4}{n}}$$

which decays exponentially at infinity, hence is L^2. We refer to the lectures by Mihai Maris for more on this topic. Notice that, in the defocusing case, one cannot find non trivial H^1 and exponentially decreasing solutions to

$$-\Delta\tilde{\psi} + \tilde{\psi} = -|\tilde{\psi}|^{\frac{4}{n}}\tilde{\psi},$$

since this would lead to

$$||\nabla\tilde{\psi}||_{L^2}^2 + ||\tilde{\psi}||_{L^2} = \big(-\Delta\tilde{\psi} + \tilde{\psi}, \tilde{\psi}\big)_{L^2} = -\int_{\mathbb{R}^n}|\psi|^{2+\frac{4}{n}}dx \leq 0$$

hence to $||\tilde{\psi}||_{H^1} = 0$, *i.e.* $\tilde{\psi} = 0$.

Appendix A

Littlewood-Paley decomposition

In this appendix, we provide a self contained proof of the Littlewood-Paley decomposition, which is a very powerful tool to prove various estimates used in dispersive equations (see for instance Tao's book [6] for such applications of this theory). By lack of time, we only give an application to the proof of the homogeneous Sobolev inequalities (2.26), but we hope to complete this appendix by a section on Strichartz estimates for the wave equation.

A.1 The Littlewood-Paley decomposition Theorem

Let us fix a smooth cutoff function $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\phi_0(\xi) \equiv 1 \quad \text{for} \quad |\xi| \leq 1 \qquad \phi_0 \equiv 0 \quad \text{for} \quad |\xi| \geq 2, \tag{A.1}$$

and set

$$\phi(\xi) = \phi_0(\xi) - \phi_0(2\xi).$$

For $u \in L^q(\mathbb{R}^n)$, with $q \in (1, \infty)$, we define the **square function** Su by

$$\begin{aligned} Su(x) &= \left(\sum_{k \in \mathbb{Z}} |\phi(2^{-k}D)u(x)|^2 \right)^{1/2}, \\ &= \lim_{N \to \infty} S_N(x), \end{aligned} \tag{A.2}$$

where $S_N(x) = (\sum_{k=-N}^{N} |\phi(2^{-k}D)u(x)|^2)^2$. This definition makes sense since $\phi(2^{-k}D)u$ belongs to L^q for any k (see the item 4 of Proposition 2.9), hence is a measurable function, so that Su is the pointwise limit of a non decreasing sequence of non negative measurable functions. Therefore Su is a measurable function with values in $[0, \infty]$.

Theorem A.1 (Littlewood-Paley decomposition)**.** *For all $q \in (1, \infty)$, there exists a constant C_q such that*

$$C_q^{-1} ||u||_{L^q} \leq ||Su||_{L^q} \leq C_q ||u||_{L^q},$$

for all $u \in L^q(\mathbb{R}^n)$. Furthermore, S is continuous on L^q since

$$||Su - Sv||_{L^q} \leq C_q||u - v||_{L^q}.$$

Note that the operator S is not linear: it only satisfies $0 \leq S(u+v) \leq Su + Sv$ (the inequalities hold pointwise in x). One says S is sublinear or subadditive. The proof of Theorem A.1 will be given in Section A.3. It will use the following facts which we record as an exercise.

Exercise A.2. *1. Check that the support of ϕ satisfies*

$$\mathrm{supp}(\phi) \subset \left\{ \frac{1}{2} \leq |\xi| \leq 2 \right\}. \tag{A.3}$$

2. Prove that if $u \in L^2$ then

$$\left|\left| u - \sum_{k=-N}^{N} \phi(2^{-k}D)u \right|\right|_{L^2} \to 0, \qquad N \to +\infty. \tag{A.4}$$

Hint: Check first that

$$\sum_{k=1}^{N} \phi(2^{-k}\xi) = \phi_0(2^{-N}\xi) - \phi_0(\xi) \longrightarrow 1 - \phi_0(\xi), \qquad N \to \infty,$$

and that

$$\sum_{k=-N}^{0} \phi(2^{-k}\xi) = \phi_0(\xi) - \phi_0(2^{N+1}\xi) \longrightarrow \phi_0(\xi)\mathbb{1}_{\mathbb{R}^n\backslash 0}(\xi), \qquad N \to \infty.$$

The interest of Theorem A.1 is to allow to localize estimates in frequency as follows.

Corollary A.3. *1. If $q \in [2, \infty)$, then for all $u \in L^q$*

$$||u||_{L^q} \leq C_q \left(\sum_{k \in \mathbb{Z}} ||\phi(2^{-k}D)u||_{L^q}^2 \right)^{1/2}.$$

2. If $q \in (1, 2]$, then for all $u \in L^q$

$$\left(\sum_{k \in \mathbb{Z}} ||\phi(2^{-k}D)u||_{L^q}^2 \right)^{1/2} \leq C_q||u||_{L^q}$$

3. If $q = 2$, then for all $u \in L^2$

$$C_2^{-1}||u||_{L^2} \leq \left(\sum_{k \in \mathbb{Z}} ||\phi(2^{-k}D)u||_{L^q}^2 \right)^{1/2} \leq C_2||u||_{L^2}$$

Proof. 1. Assume that $q \geq 2$. For $u \in L^q$, we write

$$\begin{aligned} ||Su||_{L^q}^q &= \int \left(\sum_{k \in \mathbb{Z}} |\phi(2^{-k}D)u(x)|^2 \right)^{q/2} dx \\ &\leq \left[\sum_{k \in \mathbb{Z}} \left(\int ||\phi(2^{-k}D)u(x)|^2|^{q/2} dx \right)^{2/q} \right]^{q/2} \end{aligned}$$

by interpreting the right hand side of the first line as the $L^{q/2}$ norm, raised to the power $q/2$, of the sum $\sum_k |\phi(2^{-k}D)u(x)|^2$, and then by using the triangle (or Minkowski) inequality in the second line. Note that this argument uses that $q \geq 2$ to guarantee that $(\int |f|^{q/2}dx)^{2/q}$ defines a norm. This show precisely that

$$||Su||_{L^q} \leq \left(\sum_{k \in \mathbb{Z}} ||\phi(2^{-k}D)u||_{L^q}^2 \right)^{1/2},$$

hence yields the result by using the upper bound in Theorem A.1.
2. Assume now that $1 < q \leq 2$ and that $u \in L^q$. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}} ||\phi(2^{-k}D)u||_{L^q}^2 &= \sum_{k \in \mathbb{Z}} \left(\int |\phi(2^{-k}D)u(x)|^q dx \right)^{2/q} \\ &\leq \left[\int \left(\sum_{k \in \mathbb{Z}} |\phi(2^{-k}D)u(x)|^2 \right)^{q/2} dx \right]^{2/q} \end{aligned}$$

by interpreting the right hand side of the first line as the $l^{2/q}$ norm, raised to the power $2/q$, of the sequence $\int |\phi(2^{-k}D)u(x)|^q dx$, and then by using $|| \int f_k(x)dx||_{l_k^{2/q}} \leq \int ||f_k(x)||_{l_k^{q/2}} dx$ in the second line. This means exactly that

$$\left(\sum_{k \in \mathbb{Z}} ||\phi(2^{-k}D)u||_{L^q}^2 \right)^{1/2} \leq ||Su||_{L^q},$$

and the conclusion follows from the lower bound in Theorem A.1. The item 3 is a direct consequence of 1 and 2. □

Corollary A.3 rests on the fact that, if $q \geq 2$ and $1 < q' \leq 2$,

$$|| \cdot ||_{L^q l^2} \leq || \cdot ||_{l^2 L^q}, \qquad || \cdot ||_{l^2 L^{q'}} \leq || \cdot ||_{L^{q'} l^2}.$$

By repeating this argument, we can also consider the mixed space times norms $L_I^p L^q$ (given by $||u||_{L_I^p L^q} = (\int_I ||u(t)||_{L^q}^p dt)^{1/p}$) provided that both p and q are non smaller than 2 or non greater than 2.

Corollary A.4. *Let $p, q \in [2, \infty)$ and $p', q' \in (1, 2]$. Then*

$$||u||_{L^p_I L^q} \leq C_q \left(\sum_{k\in\mathbb{Z}} ||\phi(2^{-k}D)u||^2_{L^p_I L^q} \right)^{1/2}, \tag{A.5}$$

$$\left(\sum_{k\in\mathbb{Z}} ||\phi(2^{-k}D)f||^2_{L^{p'}_I L^{q'}} \right)^{1/2} \leq C_q ||f||_{L^{p'}_I L^{q'}} \tag{A.6}$$

for all compact interval I, all $u \in C(I, L^q)$ and all $f \in C(I, L^{q'})$.

The proof is left to the reader as an exercise. This corollary is very useful to prove Strichartz estimates for the wave equation.

A.2 Homogeneous Sobolev estimates

In this section, we prove Proposition 2.20, namely that for all real number $s \in [0, n/2)$

$$||u||_{L^{\frac{2n}{n-2s}}} \leq C\big|\big||D|^s u\big|\big|_{L^2}$$

for all u in the Schwartz space. Since $2n/(n-2s)$ belongs to $[2, \infty)$, the item 1 of Corollary A.3 shows that

$$||u||_{L^{\frac{2n}{2n-s}}} \leq C \left(\sum_{k\in\mathbb{Z}} ||\phi(2^{-k}D)u||^2_{L^{\frac{2n}{n-2s}}} \right)^{1/2}. \tag{A.7}$$

Choose next $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ near the support of ϕ and $\psi \equiv 0$ near 0. Define

$$\psi_s(\xi) = |\xi|^{-s}\psi(\xi),$$

extended by 0 at $\xi = 0$. This defines a function in $C_0^\infty(\mathbb{R}^n)$. Using that $\psi\phi = \phi$ and the composition properties of Fourier multipliers (see Propositions 2.9 and 2.17), we get

$$\begin{aligned} \phi(2^{-k}D)u &= \psi_s(2^{-k}D)\phi(2^{-k}D)|2^{-k}D|^s u \\ &= 2^{-ks}\psi_s(2^{-k}D)\phi(2^{-k}D)|D|^s u. \end{aligned}$$

Therefore

$$||\phi(2^{-k}D)u||_{L^{\frac{2n}{n-2s}}} \leq 2^{-ks} \left|\left|\psi_s(2^{-k}D)\right|\right|_{L^2 \to L^{\frac{2n}{n-2s}}} \big|\big|\phi(2^{-k}D)|D|^s u\big|\big|_{L^2}.$$

Using the item 4 of Proposition 2.9 with $a(\xi) = \psi_s(2^{-k}\xi)$, we find that

$$\begin{aligned} \left|\left|\psi_s(2^{-k}D)\right|\right|_{L^2 \to L^{\frac{2n}{n-2s}}} &\leq (2\pi)^{-n} \left(\big|\big|\widehat{\psi}_s\big|\big|_{L^1} \right)^{\frac{n-2s}{n}} \left(2^{kn/2} ||\widehat{\psi}_s||_{L^2} \right)^{\frac{2s}{n}} \\ &\leq C2^{ks}, \end{aligned}$$

and thus obtain

$$||\phi(2^{-k}D)u||_{L^{\frac{2n}{n-2s}}} \leq C\big|\big|\phi(2^{-k}D)|D|^s u\big|\big|_{L^2}.$$

Using (A.7) and the item 3 of Corollary A.3, we get the result. □

A.3 Proof of the Littlewood-Paley decomposition

A.3.1 The Calderón-Zygmund Lemma

In the following proposition, m denotes the Lebesgue measure on $\mathbb{R}^n$.

Proposition A.5. *For all $f \in L^1(\mathbb{R}^n)$ and all real number $\lambda > 0$ one can find*

1. *an at most countable family*[1] *of cubes $(Q_j)_{j\in J}$ such that, for all $j \in J$,*

$$\lambda \leq \frac{1}{m(Q_j)} \int_{Q_j} |f(x)|dx < 2^n \lambda, \tag{A.8}$$

and

$$\sum_{j\in J} m(Q_j) \leq \lambda^{-1} ||f||_{L^1}, \tag{A.9}$$

2. *a function $g \in L^1(\mathbb{R}^n)$ such that*

$$|g(x)| \leq 2^n \lambda, \qquad \text{almost everywhere,} \tag{A.10}$$

3. *a family $(b_j)_{j\in J}$ in $L^1(\mathbb{R}^n)$ such that*

$$b_j \equiv 0 \quad \text{outside } Q_j, \qquad \int b_j dx = 0, \tag{A.11}$$

such that

$$f = g + \sum_{j\in J} b_j, \tag{A.12}$$

and

$$||g||_{L^1} + \sum_{j\in J} ||b_j||_{L^1} \leq 4||f||_{L^1}. \tag{A.13}$$

Proof. **Step 1: construction of the cubes.** We start by choosing $k_0 \in \mathbb{N}$ such that

$$2^n \lambda > 2^{-k_0 n} ||u||_{L^1}. \tag{A.14}$$

We then define the cube

$$C_0^0 := [0, 2^{k_0})^n$$

and let $P_0 := (C_N^0)_{N\in\mathbb{N}}$ be the countable collection of all translates of C_0^0 by vectors in $(2^{k_0}\mathbb{Z})^n$. The collection P_0 is obviously a partition of $\mathbb{R}^n$. We then define, for each $k \in \mathbb{N}$, the collection $P_k := (C_N^k)_{N\in\mathbb{N}}$ by

$$C_N^k := 2^{-k} C_N^0. \tag{A.15}$$

[1] possibly empty!

In other words, the partition P_{k+1} is obtained from P_k by dividing each cube into 2^n cubes of half side. In particular, each cube of P_{k+1} has a unique *parent cube* in P_k, *i.e.* for any cube C_N^{k+1} of P_{k+1} there is a unique $N' \in \mathbb{N}$ such that $C_N^{k+1} \subset C_{N'}^k$.

The construction of the cubes $(Q_j)_{j\in\mathbb{N}}$ is as follows. We drop all cubes of P_0 over which the mean value of $|f|$ is $\geq \lambda$ into a set which we call *bad cubes*. In other words, the bad cubes of P_0 are those which satisfy

$$\lambda \leq \frac{1}{m(C_N^0)} \int_{C_N^0} |f(x)| dx < 2^n \lambda,$$

the lower bound following from the definition of bad cubes and the upper bound following from

$$\frac{1}{m(C_N^0)} \int_{C_N^0} |f(x)| dx = 2^{-k_0 n} \int_{C_N^0} |f(x)| dx \leq 2^{-k_0 n} ||f||_{L^1} < 2^n \lambda,$$

by (A.14). For non bad cubes of P_0, we apply the following procedure. We divide each one of them into 2^n cubes of half side (which then all belong to P_1) and among all these new cubes, we drop those over which the mean value of $|f|$ is $\geq \lambda$ into the set of bad cubes. Notice that these new bad cubes are disjoints from the previous bad ones and that they satisfy

$$\lambda \leq \frac{1}{m(C_N^1)} \int_{C_N^1} |f(x)| dx < 2^n \lambda$$

where the lower bound follows again from the very definition of bad cubes and the upper bound from

$$\frac{1}{m(C_N^1)} \int_{C_N^1} |f(x)| dx = \frac{2^n}{m(C_{N'}^0)} \int_{C_N^1} |f(x)| dx \leq \frac{2^n}{m(C_{N'}^0)} \int_{C_{N'}^0} |f(x)| dx < 2^n \lambda,$$

since the parent cube $C_{N'}^0$ of C_N^1 is not a bad cube. By iterating this process, we construct an at most countable family of bad cubes which we denote by $(Q_j)_{j\in J}$ after relabeling. They are disjoints and such that

$$\lambda \leq \frac{1}{m(Q_j)} \int_{Q_j} |f(x)| dx < 2^n \lambda. \tag{A.16}$$

Note that the iterative process either stops at some level k, if the non bad cubes from the previous step (*i.e.* those belonging to P_{k-1}) all give rise to bad cubes after division, or there are arbitrarily small cubes in the complement of $\cup_j Q_j$. If one wants to reformulate this procedure rigorously, one defines iteratively the sets, as long as they are non empty,

$$I_0 = \{N \in \mathbb{N} \mid C_N^0 \text{ is a bad cube}\},$$

and, for $k \geq 1$,

$$I_k = \{N \in \mathbb{N} \mid C_N^k \text{ is a bad cube, disjoint from the } \; C_\nu^\kappa, \; 0 \leq \kappa < k, \; \nu \in I_\kappa\}.$$

We let K be the at most countable set of integers k for which I_k is non empty and define $(Q_j)_{j\in J} := (C^k_N)_{k\in K, N\in I_k}$. If we now define

$$G = \mathbb{R}^n \setminus \cup_{j\in J} Q_j, \tag{A.17}$$

then, for any x in G, there exists a decreasing sequence $(C^k_{N(k,x)})_{k\in\mathbb{N}}$ of cubes of P_k which all contain x and satisfy

$$\frac{1}{m(C^k_{N(k,x)})} \int_{C^k_{N(k,x)}} |f(x)|dx < \lambda. \tag{A.18}$$

Indeed, since each P_k is a partition, there exists a unique $N = N(k,x)$ such that $x \in C^k_N$ and this cube cannot be contained in any bad cube of previous orders $0, \ldots, k-1$ nor be a bad cube of order k (*i.e.* $N \in I_k$), since otherwise x would not belong to G. This implies that (A.18) holds.

Step 2: Construction of the functions. We set

$$b_j(x) = 0 \ \text{ if } \ x \notin Q_j, \qquad b_j(x) = f(x) - \frac{1}{m(Q_j)} \int_{Q_j} f \ \text{ if } \ x \in Q_j, \tag{A.19}$$

and

$$\begin{aligned} g &= \mathbb{1}_G f + \sum_{j\in J} \left(\frac{1}{m(Q_j)} \int_{Q_j} f \right) \mathbb{1}_{Q_j} && \text{(A.20)} \\ &= \mathbb{1}_G f + \sum_{j\in J} \mathbb{1}_{Q_j} f - b_j, && \text{(A.21)} \end{aligned}$$

where in the second line one may recall that $b_j = \mathbb{1}_{Q_j} b_j$. We now check that the cubes Q_j and the functions g and b_j satisfy the expected properties. By (A.16), we know that (A.8) holds. This then implies that

$$m(Q_j) \leq \lambda^{-1} \int_{Q_j} |f(x)|dx,$$

and by summation over j, we get (A.9). We next consider (A.10). It follows from (A.16) and (A.20) that

$$|g(x)| < 2^n \lambda, \qquad \text{for all } x \notin G,$$

so it remains to consider $\mathbb{1}_G g = \mathbb{1}_G f$, that is to show that $|f(x)| \leq 2^n \lambda$ for almost every $x \in G$. By Lemma A.6 below, we know that $A_k f \to f$ in L^1 as $k \to \infty$. In particular, there is a subsequence $A_{k_l} f$ such that

$$\lim_{l\to\infty} A_{k_l} f(x) = f(x), \qquad \text{for almost every } x \in \mathbb{R}^n.$$

On the other hand, if x belongs to G and k is a given integer, we know that the only cube $C_N^k = C_{N(k,x)}^k$ of P_k containing x is not a bad cube, hence that

$$|A_k f(x)| = \frac{1}{m(C_N^k)} \left| \int_{C_N^k} f \right| < \lambda$$

by (A.18). Using this last estimate along the subsequence k_l, we see that $|f(x)| \leq \lambda$ almost everywhere on G, which completes the proof of (A.10). The properties in (A.11) follow directly from (A.19). By (A.17) and the fact that the cubes Q_j are disjoint, (A.12) is a straightforward consequence of (A.21). Using (A.20), we get

$$||g||_{L^1} \leq ||f||_{L^1} + \sum_{j \in J} ||b_j||_{L^1}.$$

On the other hand, it follows easily from the definition of b_j, that is (A.19), that

$$||b_j||_{L^1} \leq 2 \int_{Q_j} |f(x)| dx.$$

The last two inequalities yield clearly (A.13). This completes the proof. □

Lemma A.6. *Let C_N^k be the cubes defined in (A.15). For all $k \in \mathbb{N}$ and $u \in L^1(\mathbb{R}^n)$, define*

$$A_k u = \sum_{N \in \mathbb{N}} \left(\frac{1}{m(C_N^k)} \int_{C_N^k} u(x) dx \right) \mathbb{1}_{C_N^k}.$$

Then $||A_k u - u||_{L^1} \to 0$ as $k \to \infty$.

Proof. Using the easily verified fact that

$$||A_k u||_{L^1} \leq ||u||_{L^1},$$

we may assume that u belongs to $C_0^\infty(\mathbb{R}^n)$ since it is dense in L^1. Fix a large cube Q in $\mathbb{R}^n$ which contains the support of u. Then

$$A_k u = \sum_{C_N^k \cap Q \neq \emptyset} \left(\frac{1}{m(C_N^k)} \int_{C_N^k} u(x) dx \right) \mathbb{1}_{C_N^k}.$$

Since all C_N^k have a bounded side (by $2^{k_0 - k} \leq 2^{k_0}$), there exists $\widetilde{Q}$ independent of k such that

$$\operatorname{supp}(A_k u) \subset \widetilde{Q}. \tag{A.22}$$

Fix then $\varepsilon > 0$. By the uniform continuity of u, we can find $\delta > 0$ such that

$$|x - y| < \delta \qquad \Longrightarrow \qquad |u(x) - u(y)| < \frac{\varepsilon}{m(\widetilde{Q})}.$$

Then, for all $x \in \widetilde{Q}$, if we let $C_N^k(x)$ be the unique cube of the partition P_k containing x, we have

$$A_k u(x) - u(x) = \frac{1}{m(C_N^k(x))} \int_{C_N^k(x)} (u(y) - u(x)) dy$$

so that, if k is large enough to guarantee that $\operatorname{diam}(C_N^k) < \delta$ (this diameter is of order 2^{k_0-k} and is independent of N, see (A.15)), we obtain

$$||A_k u - u||_{L^\infty(\widetilde{Q})} \leq \frac{\varepsilon}{m(\widetilde{Q})}.$$

Using (A.22) and the fact that $\operatorname{supp}(u) \subset Q \subset \widetilde{Q}$, we obtain

$$||A_k u - u||_{L^1} \leq \varepsilon, \qquad \text{for all } k \text{ large enough,}$$

more precisely for all k such that $2^{k_0-k} \ll \delta$. This completes the proof. □

A.3.2 Proof of Theorem A.1

Although the operator S is not linear, one can reduce its study to the one of family of linear operators by a nice randomization technique (we follow here the presentation of Muscalu-Schlag's book [4]). Let us introduce a sequence of random variables

$$(r_k)_{k\in\mathbb{Z}} \;=\; \text{sequence of independent Bernoulli variables with values in } \{-1, 1\}, \tag{A.23}$$

i.e. $\mathbb{P}(r_k = \pm 1) = 1/2$. We denote by $(\Omega, \mathcal{T}, \mathbb{P})$ the probability space on which this sequence is defined. The interest of this randomization technique is the following classical result.

Proposition A.7 (Khinchin's inequality). *For all $q \in [1, \infty)$, there exists $C_q > 0$ such that, for all N and all family $(z_k)_{|k|\leq N} \in \mathbb{C}^{2N+1}$,*

$$C_q^{-1}\big(\sum_{|k|\leq N} |z_k|^2\big)^{q/2} \leq \int_\Omega \big|\sum_{|k|\leq N} r_k(\omega) z_k\big|^q d\mathbb{P} \leq C_q\big(\sum_{|k|\leq N} |z_k|^2\big)^{q/2}.$$

The proof of this proposition rests on the following lemma.

Lemma A.8. *There exists $C > 0$ such that, for all N, all family $(z_k)_{|k|\leq N} \in \mathbb{C}^{2N+1}$ and all $\lambda > 0$,*

$$\mathbb{P}\left(\left\{\big|\sum_{|k|\leq N} r_k z_k\big| > \lambda\big(\sum_{|k|\leq N} |z_k|^2\big)^{1/2}\right\}\right) \leq C e^{-\lambda^2/2}.$$

Proof. Letting $z_k = x_k + i y_k$ be the decomposition into real and imaginary parts, we have

$$\big|\sum_{|k|\leq N} r_k z_k\big|^2 = \big|\sum_{|k|\leq N} r_k x_k\big|^2 + \big|\sum_{|k|\leq N} r_k y_k\big|^2$$

and thus the set we are considering, *i.e.* $\{|\sum_k r_k z_k| > \lambda(\sum_k |z_k|^2)^{1/2}\}$, is contained in

$$\Big\{|\sum_k r_k x_k| > \lambda(\sum_k x_k^2)^{1/2}\Big\} \cup \Big\{|\sum_k r_k y_k| > \lambda(\sum_k y_k^2)^{1/2}\Big\}.$$

This reduces the problem to the case of a real valued family, say (x_k). If all x_k vanish the result is trivial (we compute the probability of an empty set, by the strict inequality). Otherwise, by homogeneity of the condition, we may assume that $\sum x_k^2 = 1$. Then

$$\mathbb{E}\left(\exp(\lambda \sum_{k\le N} r_k x_k)\right) = \mathbb{E}\left(\prod_{|k|\le N} \exp(\lambda r_k x_k)\right) = \prod_{|k|\le N} \mathbb{E}\left(\exp(\lambda r_k x_k)\right) = \prod_{|k|\le N} \cosh(\lambda x_k),$$

using (A.23). Using that[2]

$$\cosh x \le e^{x^2/2}, \qquad x \in \mathbb{R},$$

we get

$$\mathbb{E}\left(\exp(\lambda \sum_{k\le N} r_k x_k)\right) \le e^{\lambda^2/2}. \tag{A.24}$$

Therefore, we obtain

$$\mathbb{P}\left(\Big\{\sum_k r_k x_k > \lambda\Big\}\right) = \mathbb{P}\left(\Big\{\exp(\lambda \sum_k r_k x_k) > e^{\lambda^2}\Big\}\right) \le e^{-\lambda^2} e^{\lambda^2/2} = e^{-\lambda^2/2},$$

using the Tchebychev inequality and (A.24). In a similar fashion, we also have

$$\mathbb{P}\left(\Big\{\sum_k r_k x_k < -\lambda\Big\}\right) = \mathbb{P}\left(\Big\{\exp(-\lambda \sum_k r_k x_k) > e^{\lambda^2}\Big\}\right) \le e^{-\lambda^2/2},$$

by changing x_k into $-x_k$ in (A.24). Using finally that

$$\mathbb{P}\left(\Big\{|\sum_k r_k x_k| > \lambda\Big\}\right) \le \mathbb{P}\left(\Big\{\sum_k r_k x_k > \lambda\Big\}\right) + \mathbb{P}\left(\Big\{\sum_k r_k x_k < -\lambda\Big\}\right),$$

the result follows. $\square$

Proof of Proposition A.7. By homogeneity, we may assume that $\sum_k |z_k|^2 = 1$ (if the sum vanishes, the result is trivial). Let $Z_N(\omega) = \sum_{|k|\le N} r_k(\omega) z_k$. We have

$$\mathbb{E}(|Z_N|^q) = \int_\Omega |Z_N(\omega)|^q d\mathbb{P} = q\int_0^\infty \lambda^{q-1}\mathbb{P}\{|Z_N| > \lambda\} d\lambda \le Cq\int_0^\infty \lambda^{q-1} e^{-\lambda^2/2} d\lambda,$$

[2]check it! (hint: study the function $\frac{x^2}{2} - \ln\cosh(x)$ by differentiating it twice)

using Lemma 2.5 for the second equality and Lemma A.8 for the inequality. This proves the upper bound. We now prove the lower bound. We start by observing that, by independence, the sequence (r_k) is orthornormal[3] , so that

$$\mathbb{E}(|Z_N|^2) = \sum_{|k|\le N} |z_k|^2 = 1. \tag{A.25}$$

On the other hand, by Hölder's inequality on a probability space, we have

$$\mathbb{E}(|Z_N|) \le \mathbb{E}(|Z_N|^q),$$

so it sufficient to consider the case $q = 1$, *i.e.* to bound $\mathbb{E}(|Z_N|)$ from below. We use that

$$\begin{aligned} \mathbb{E}(|Z_N|^2) &= \mathbb{E}(|Z_N|^{4/3}|Z_N|^{2/3}) \\ &\le \mathbb{E}(|Z_N|^4)^{1/3}\mathbb{E}(|Z_N|)^{2/3} \\ &\le C\mathbb{E}(|Z_N|)^{2/3} \end{aligned}$$

where we used the Hölder inequality in the second line and then the already proved upper bound, with $q = 4$, in the third one. Using (A.25), we conclude that $\mathbb{E}(|Z_N|) \ge C^{-3/2}$ which provides the lower bound. $\square$

The proof of Theorem A.1 will mainly follow from a careful study of the family of linear operators

$$L_N^\omega := \sum_{|k|\le N} r_k(\omega)\phi(2^{-k}D),$$

namely by getting suitable bounds uniform with respect to $N \in \mathbb{N}$ and $\omega \in \Omega$. Here is the main technical result of this section.

Proposition A.9. *For all $q \in (1,\infty)$, there exists $C_q > 0$ such that*

$$||L_N^\omega u||_{L^q} \le C_q ||u||_{L^q},$$

for all $N \in \mathbb{N}$, all $\omega \in \Omega$ and all $u \in L^q$.

Let us notice that, for fixed N and ω, the $L^q \to L^q$ boundedness of L_N^ω is a direct consequence of the item 4 of Proposition 2.9. The point of Proposition A.9 is to provide a bound which is uniform with respect to ω and N. Before proving this proposition, we explain how to obtain Theorem A.1.

Proof of Theorem A.1 We prove first the upper bound, *i.e.* that for all $q \in (1,\infty)$, there exists C_q such that

$$||Su||_{L^q} \le C_q ||u||_{L^q}, \qquad \text{for all } u \in L^q. \tag{A.26}$$

[3] if $j \neq k$, $\mathbb{E}(r_j r_k) = \mathbb{P}(r_j = r_k = 1) + \mathbb{P}(r_j = r_k = -1) - \mathbb{P}(r_j = -r_k = 1) - \mathbb{P}(r_j = -r_k = -1)$ which, by independence, reads $\sum_{0\le\nu,\mu\le 1} (-1)^{\mu+\nu}\mathbb{P}(r_j = (-1)^\nu)\mathbb{P}(r_k = (-1)^\mu) = 1/4 + 1/4 - 1/4 - 1/4 = 0$

By monotone convergence, we can write $||Su||_{L^q} = \lim_{N\to\infty} ||S_N u||_{L^q}$ (see (A.2) for S_N). By Proposition A.7 and the Fubini Theorem, we have

$$||S_N u||_{L^q}^q \leq C \int_\Omega \left(\int_{\mathbb{R}^n} \left| \sum_{|k|\leq N} r_k(\omega)\phi(2^k D)u(x) \right|^q dx \right) d\mathbb{P}(\omega)$$

By Proposition A.9, the integral in x is bounded by $C||u||_{L^q}^q$, uniformly with respect to ω and N. Since $\mathbb{P}(\Omega) < \infty$, we get (A.26). We note that, although S is not linear, we have $|Su - Sv| \leq |S(u-v)|$ almost everywhere (by the second triangle inequality) if $u, v \in L^q$. Thus, (A.26) implies that $||Su - Sv||_{L^q} \leq C_q ||u - v||_{L^q}$. The interest of this remark is that if we replace v by a sequence (u_j) which goes to u in L^q, we see that $Su_j \to Su$ in L^q. This is useful to prove the lower bound,

$$||Su||_{L^q} \leq ||u||_{L^q}/C, \qquad u \in L^q, \tag{A.27}$$

since we may assume that u belongs to $\mathcal{S}(\mathbb{R}^n)$. Let us prove (A.27). If u, v belong to the Schwartz space, hence to L^2, we can write

$$(u, v)_{L^2} = \lim_{N\to\infty} \left(\sum_{|k|\leq N} \phi(2^{-k}D)u, \sum_{|j|\leq N} \phi(2^{-j}D)v \right)_{L^2}$$

since $\sum_{|k|\leq N} \phi(2^{-k}D)u \to u$ in L^2 (and the same for v) as N goes to infinity (see Exercise A.2). Then

$$\sum_{|k|\leq N} \sum_{|j|\leq N} \big(\phi(2^{-k}D)u, \phi(2^{-j}D)v\big)_{L^2} \quad = \quad \sum_{\substack{|k|,|j|\leq N \\ |j-k|\leq 2}} \big(\phi(2^{-k}D)u, \phi(2^{-j}D)v\big)_{L^2}$$

since

$$\phi(2^{-k}D)^*\phi(2^{-j}D) = \overline{\phi}(2^{-k}D)\phi(2^{-j}D) = 0 \quad \text{if} \quad |j-k| > 2,$$

for (A.3) implies that if $|j-k| > 2$, $2^{-j}\xi$ and $2^{-k}\xi$ cannot belong simultaneously to the support of ϕ. Therefore,

$$\begin{aligned} |(u,v)_{L^2}| &\leq \sum_{m=-2}^{2} \int \sum_{|k|\leq N} |\phi(2^{-k}D)u(x)\phi(2^{m-k})v(x)|dx \\ &\leq 5 \int |Su(x)||Sv(x)|dx, \end{aligned} \tag{A.28}$$

by estimating the sum over $|k| \leq N$ by the sum over $k \in \mathbb{Z}$ and by the using the Cauchy Schwartz inequality pointwise in x (almost everywhere). Using the Hölder inequality and the upper bound (A.26) for the conjugate exponent q', we obtain

$$\left| \int \overline{u}(x)v(x)dx \right| \leq C||Su||_{L^q}||v||_{L^{q'}},$$

from which (A.27) follows. □

We now turn to the proof of Proposition A.9 to which the rest of the section is devoted.

Proposition A.10. *Proposition A.9 is true for $q = 2$.*

We will see in the proof that one can take $C_2 = 4||\phi||_{L^\infty}$.

Proof. We have

$$\begin{aligned} ||L_N^\omega u||_{L^2}^2 &= \sum_{|k|\leq N} \sum_{j \leq N} \big(r_k(\omega)\phi(2^{-k}D)u, r_j(\omega)\phi(2^{-j}D)u \big)_{L^2} \\ &= \big(u, B_N^\omega u\big)_{L^2}, \end{aligned} \tag{A.29}$$

where

$$B_N^\omega = \sum_{|j|,|k|\leq N} \overline{r_k(\omega)} r_j(\omega) \phi(2^{-k}D)^* \phi(2^{-j}D).$$

It follows from the item 2 of Proposition 2.9 that $B_N^\omega = b_N^\omega(D)$ with

$$b_N^\omega(\xi) = \sum_{|j|,|k|\leq N} \overline{r_k(\omega)} r_j(\omega) \overline{\phi(2^{-k}\xi)} \phi(2^{-j}\xi).$$

We will show that, for all $\xi \in \mathbb{R}^n$,

$$|b_N^\omega(\xi)| \leq 16||\phi||_{L^\infty}^2. \tag{A.30}$$

Indeed, if $\xi = 0$ then $b_N^\omega(\xi) = 0$ since $\phi(0) = 0$. On the other hand, if $\xi \neq 0$ and if we let

$$\mu = \text{integer part of } \ln(|\xi|)/\ln 2, \tag{A.31}$$

we have $2^\mu \leq |\xi| < 2^{\mu+1}$. Using (A.3), the only terms which may not vanish in the sum defining b_N^ω are those for which

$$1/2 \leq |2^{-k}\xi| \leq 2 \qquad \text{and} \qquad 1/2 \leq |2^{-j}\xi| \leq 2.$$

Therefore, j and k must be such that

$$2^{-2} < 2^{\mu-k} \leq 2^2 \qquad \text{and} \qquad 2^{-2} < 2^{\mu-j} \leq 2^2,$$

i.e. $-1 \leq \mu - k \leq 2$ and $-1 \leq \mu - j \leq 2$. Therefore, the sum $b_N(\omega)$ contains at most 16 non vanishing terms, each one being bounded by $||\phi||_{L^\infty}^2$. This proves (A.30). By the item 1 of Proposition 2.9 and (A.29), we get

$$||L_N^\omega u||_{L^2}^2 \leq ||u||_{L^2} ||B_N^\omega u||_{L^2} \leq 16||\phi||_{L^\infty}^2 ||u||_{L^2}^2,$$

so the result follows. □

We next consider the Schwartz kernel of L_N^ω which we denote by K_N^ω. According to the item 3 of Proposition 2.9, it is given by

$$K_N^\omega(x,y) = (2\pi)^{-n} \sum_{|k|\leq N} r_k(\omega) 2^{kn} \widehat{\phi}(2^k(y-x)).$$

It satisfies the following important bounds.

Lemma A.11. *There exists* $B_{\mathrm{CZ}} > 0$ *such that,*

$$\begin{aligned} |K_N^\omega(x,y)| &\leq B_{\mathrm{CZ}} |x-y|^{-n}, && \text{(A.32)} \\ |\nabla_y K_N^\omega(x,y)| &\leq B_{\mathrm{CZ}} |x-y|^{-n-1}, && \text{(A.33)} \end{aligned}$$

for all $N \in \mathbb{N}$, *all* $\omega \in \Omega$ *and all* $x \neq y$.

These estimates are singular at $x = y$. Of course, K_N^ω is a Schwartz function of $x - y$ hence is bounded near $x - y = 0$, as well as its derivatives, but the bounds depend badly on N. The interest of Lemma A.11 is the uniformity of the bounds with respect to the parameters N and ω.

The index CZ in the constantB_{CZ} refers to Calderón-Zygmund since operators whose kernel satisfy bounds as (A.32) and (A.33) are called Calderón-Zygmund operators.

Proof. Using that $\widehat{\phi}$ is a Schwartz function, there exists a constant $C > 0$ such that

$$|\widehat{\phi}(\eta)| + |\nabla \widehat{\phi}(\eta)| \leq C(1+|\eta|)^{-n-2}, \qquad k \in \mathbb{Z},\ \eta \in \mathbb{R}^n, \tag{A.34}$$

Assume now that $x - y \neq 0$. As in (A.31), we may introduce the unique $\mu \in \mathbb{Z}$ such that

$$2^\mu \leq |x-y| < 2^{\mu+1}. \tag{A.35}$$

Splitting the sum according to $k + \mu \leq 0$ and $k + \mu > 0$, and then using (A.34), we get

$$\begin{aligned} |K_N^\omega(x,y)| &\leq \sum_{k+\mu\leq 0} 2^{kn} |\widehat{\phi}(2^k(y-x))| + \sum_{k+\mu>0} 2^{kn} |\widehat{\phi}(2^k(y-x))| \\ &\leq C \sum_{k+\mu\leq 0} 2^{kn} + C \sum_{k+\mu>0} 2^{kn}(1+2^k|y-x|)^{-n-1} \\ &\leq C 2^{-n\mu} \sum_{k'\leq 0} 2^{nk'} + C 2^{-n\mu} \sum_{k'>0} 2^{-k'} \\ &\leq C 2^{-n\mu}. \end{aligned}$$

Using (A.35) again, $2^{-n\mu}$ is bounded by $|x-y|^{-n}$ so we get (A.32). The proof of (A.33) is similar (the change of n into $n+1$ is of course due to the additional factor 2^k obtain after differentiation of $\widehat{\phi}(2^k\cdot)$). □

Lemma A.12. *Let $||\cdot||$ be a norm on $\mathbb{R}^n$. There exists a constant B independent of N and ω such that, for all $t > 0$,*

$$\sup_{||y||<t} \int_{||x||>2t} |K_N^\omega(x,y) - K_N^\omega(x,0)| dx \le B. \tag{A.36}$$

We note that the constant depends on the dimension, the norm $||\cdot||$ and linearly on the constant B_{CZ} in Proposition A.11.

□

Proof. By the Taylor formula and (A.33), the left hand side of (A.36) is bounded by

$$C \int_{||x||>2t} t\big|||x|| - t\big|^{-n-1} dx = C \int_{||z||>2} \big|||z|| - 1\big|^{-n-1} dz$$

where we used the change of variable $z = x/t$. The result follows by using the equivalence of $||z||$ and the euclidean norm $|z|$ for z large. □

In what follows, we will consider cubes on $\mathbb{R}^n$. If Q is a cube, we will denote by Q^* the cube with same center and twice the side.

Lemma A.13. *There exists a constant C independent of N and ω such that, for all cube Q in $\mathbb{R}^n$ and all function $w \in L^1$ supported in Q and such that $\int_Q w(x)dx = 0$, we have*

$$\big|\big|L_N^\omega w\big|\big|_{L^1(\mathbb{R}^n\setminus Q^*)} \le C||w||_{L^1}.$$

Here the constant C can be chosen equal to B in (A.36) for the norm $||x|| := \max(|x_j|)$.

Proof. Let y_0 be the center of Q. Furthermore,

$$L_N^\omega w(x) = \int_Q K_N^\omega(x,y)w(y)dy = \int_Q \big(K_N^\omega(x,y) - K_N^\omega(x,y_0)\big)w(y)dy,$$

since w has mean zero. Observe next that, up to a zero measure set, Q is an open ball for the norm $||x|| = \max_j(|x_j|)$. Denote its radius by $t > 0$. Then

$$\int_Q \big(K_N^\omega(x,y) - K_N^\omega(x,y_0)\big)w(y)dy = \int_{||\tilde{y}||<t} \big(K_N^\omega(x,\tilde{y}+y_0) - K_N^\omega(x,y_0)\big)w(\tilde{y}+y_0)d\tilde{y}$$

and therefore, by taking the modulus, integrating in x (which we write as $y_0 + \tilde{x}$ with $||\tilde{x}|| > 2t$) and using the Fubini Theorem, we get

$$\big|\big|L_N^\omega w\big|\big|_{L^1(\mathbb{R}^n\setminus Q^*)} \le \left(\sup_{||\tilde{y}||<t} \int_{||\tilde{x}||>2t} \big|K_N^\omega(y_0+\tilde{x},\tilde{y}+y_0) - K_N^\omega(y_0+\tilde{x},y_0)\big| d\tilde{x}\right) ||w||_{L^1}.$$

Using

$$K_N^\omega(y_0+\tilde{x},\tilde{y}+y_0) - K_N^\omega(y_0+\tilde{x},y_0) = K_N^\omega(\tilde{x},\tilde{y}) - K_N^\omega(\tilde{x},0)$$

and Lemma A.12, the result follows. □

Proposition A.14. *The family of operators L_N^ω is uniformly of weak type $(1,1)$, i.e. there exists $C > 0$ such that*

$$m\big(\{|L_N^\omega u| > t\}\big) \leq Ct^{-1}||u||_{L^1},$$

for all $t > 0$, all $u \in L^1$, all $N \in \mathbb{N}$ and all $\omega \in \Omega$.

We recall that m stands for the Lebesgue measure on $\mathbb{R}^n$. We note again that one can see in the proof that the constant C depends linearly on the constants $B_{\rm CZ}$, B (see (A.36)) and $C_2 = 4||\phi||_{L^\infty}$.

Proof. Let $t > 0$. Using Proposition A.5 with $\lambda = t$, we write first $u = g + \sum_{j\in J} b_j = g + b$. Then

$$m\big(\{|L_N^\omega u| > t\}\big) \leq m\big(\{|L_N^\omega g| > t/2\}\big) + m\big(\{|L_N^\omega b| > t/2\}\big).$$

Then, by using (A.10), we have $||g||_{L^2}^2 \leq 2^n t||g||_{L^1}$ and therefore

$$m\big(\{|L_N^\omega g| > t/2\}\big) \leq 4t^{-2}||L_N^\omega g||_{L^2}^2 \leq Ct^{-2}||g||_{L^2}^2 \leq Ct^{-1}||u||_{L^1}$$

using Proposition A.10. On the other hand, using that

$$\{|L_N^\omega b| > t/2\} \subset \cup_{j\in J} Q_j^* \cup \{\mathbb{1}_{\mathbb{R}^n\setminus\cup_j Q_j^*}|L_N^\omega b| > t/2\},$$

it follows that

$$\begin{aligned} m\big(\{|L_N^\omega b| > t/2\}\big) &\leq \sum_{j\in J} m(Q_j^*) + 2t^{-1}\sum_{j\in J}\int_{\mathbb{R}^n\setminus\cup_l Q_l^*} |L_N^\omega b_j(x)|dx \\ &\leq 2^n\sum_{j\in J} m(Q_j) + 2t^{-1}\sum_{j\in J}\int_{\mathbb{R}^n\setminus Q_j^*} |L_N^\omega b_j(x)|dx \\ &\leq 2^n t^{-1}||u||_{L^1} + Ct^{-1}\sum_{j\in J}||b_j||_{L^1}, \end{aligned}$$

using (A.9) and Lemma A.13. Using (A.13), we get the result. □

We next give another version of the Marcinkiewicz interpolation Theorem (this one is taken from [7]). Compared to Theorem 2.4, its purpose is to obtain a $L^q \to L^q$ bound rather than a $L^{q'} \to L^q$ one.

Proposition A.15 (Marcinkiewicz interpolation Theorem)**.** *Let T be a map defined on $L^1 \cup L^2$, such that T is linear on L^1 and L^2 and such that*

$$\begin{aligned} \mathrm{meas}\big\{|Tf_1| > t\big\} &\leq M_1||f_1||_{L^1}/t, & \text{(A.37)} \\ \big|\big|Tf_2\big|\big|_{L^2} &\leq M_2||f_2||_{L^2} & \text{(A.38)} \end{aligned}$$

for all $f_1 \in L^1$, all $t > 0$ and all $f_2 \in L^2$. Then, for all $1 < q < 2$, there exists $C_q > 0$ such that

$$||Tf||_{L^q} \leq C||f||_{L^q},$$

for all $f \in L^1 \cap L^2$.

The meaning of this proposition is that if T is both continuous on L^2 (this is (A.38)) and of weak type $(1,1)$ (this is (A.37)) then it is continuous on all intermediate L^q spaces, $1 < q < 2$.

Proof of Proposition A.15. We use the same notation as in the proof of Theorem 2.4. As in (2.13),

$$||Tf||_{L^q}^q = q2^q \int_0^\infty m_{Tf}(2t)t^{q-1}dt.$$

For each $t > 0$, we write $f = f_{1,t} + f_{2,t}$ with

$$f_{1,t} = \mathbb{1}_{\{|f|>t\}}|f|, \qquad f_{2,t} = \mathbb{1}_{\{|f|\le t\}}|f|.$$

Since $f \in L^q$, it is not hard to check that $f_{1,t} \in L^1$ and $f_{2,t} \in L^2$. Then, using Lemma 2.6, (A.37) and (A.38) (which implies (2.11)), we have

$$\begin{aligned} m_{Tf}(2t) &\le m_{Tf_{1,t}}(t) + m_{Tf_{2,t}}(t) \\ &\le M_1 \frac{||f_{1,t}||_{L^1}}{t} + M_2^2 \frac{||f_{2,t}||_{L^2}^2}{t^2} \\ &\le \frac{M_1}{t} \int_{\{|f|>t\}} |f(x)|dx + \frac{M_2^2}{t^2} \int_{\{|f|\le t\}} |f(x)|^2 dx. \end{aligned}$$

Using that

$$\begin{aligned} \int_0^{+\infty} t^{q-2} \left(\int_{\{|f|>t\}} |f(x)|dx \right) dt &= \int_{\mathbb{R}^n} \left(\int_0^{|f(x)|} t^{q-2} dt \right) |f(x)|dx \\ &= \frac{1}{q-1} \int_{\mathbb{R}^n} |f(x)|^{q-1} |f(x)|dx \end{aligned}$$

and similarly

$$\begin{aligned} \int_0^{+\infty} t^{q-3} \left(\int_{\{|f|\le t\}} |f(x)|^2 dx \right) dt &= \int_{\mathbb{R}^n} \left(\int_{|f(x)|}^{+\infty} t^{q-3} dt \right) |f(x)|^2 dx \\ &= \frac{1}{2-q} \int_{\mathbb{R}^n} |f(x)|^{q-2} |f(x)|^2 dx \end{aligned}$$

we obtain

$$||Tf||_{L^q}^q \le q2^q \left(\frac{M_1}{q-1} + \frac{M_2^2}{2-q} \right) ||f||_{L^q}^q$$

which yields the result. □

Proof of Proposition A.9 By Propositions A.10 and A.14 combined with Proposition A.15, we obtain that for any $q \in (1,2]$ there exists C_q such that

$$||L_N^\omega u||_{L^q} \le C_q ||u||_{L^q},$$

for all $\omega \in \Omega$ and $N \geq 0$. This is true for all $u \in L^q$ since it holds on the dense subset $L^1 \cap L^2$, by Proposition A.15, and since we already know that L_N^ω is continuous on L^q (see the comment after the statement of Proposition A.9). This proves the result when $q \in (1,2]$. When $q > 2$, we proceed in a standard fashion by duality: for $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$||L_N^\omega u||_{L^q} = \sup_{\substack{v \in \mathcal{S}(\mathbb{R}^n) \\ v \neq 0}} \frac{\left|\left(L_N^\omega u, v\right)_{L^2}\right|}{||v||_{L^{q'}}}$$

Since adjoint of L_N^ω has a form similar to L_N^ω (we have to replace ϕ by $\bar{\phi}$ - see the item 2 of Proposition 2.9), we have

$$\left|\left(L_N^\omega u, v\right)_{L^2}\right| = \left|\left(u, L_N^{\omega *} v\right)_{L^2}\right| \leq ||u||_{L^q} \left|\left|L_N^{\omega *} v\right|\right|_{L^{q'}} \leq C_{q'} ||u||_{L^q} ||v||_{L^{q'}},$$

since $q' \in (1,2]$. This implies that

$$||L_N^\omega u||_{L^q} \leq C_{q'} ||u||_{L^q},$$

for all u in the Schwartz space, hence for all $u \in L^q$ by density. This completes the proof. $\square$

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