TRACES FORMULAE FOR RELATIVELY HILBERT-SCHMIDT PERTURBATIONS

Jean-Marc Bouclet
Centre for Mathematical Analysis and Its Applications
University of Sussex
Falmer, Brighton BN1 9QH
UK
e-mail : j.m.bouclet@sussex.ac.uk

Abstract : The main goal of this paper is to introduce tools able to replace Krein's spectral shift function when it cannot be defined. We introduce a function η, for a class of perturbations of pseudodifferential operators on $\mathbb{R}^d$, when these perturbation decay as $< x >^{-\rho}$, $\rho > d/2$ at infinity. In Euclidean scattering, we establish a complete asymptotic expansion for η, under the usual non trapping assumption and we deduce a Levinson formula for the Schrödinger operator on $\mathbb{R}^3$. Some results on the general case $\rho > 0$ are also given.

1. Introduction

Consider H_0 a selfadjoint operator on a separable Hilbert space, and a perturbation V, such that $H_1 = H_0 + V$ is also selfadjoint; when this perturbation is relatively trace class, which means that $(H_0 + i)^{-N} - (H_1 + i)^{-N}$ is trace class for some $N > 0$, one can define the spectral shift function (SSF) θ by

$$Tr\,(f(H_1) - f(H_0)) = -\int_{\mathbb{R}} f'(\lambda)\theta(\lambda)d\lambda \qquad \forall\ f \in C_0^\infty(\mathbb{R}).$$

It is a locally integrable function, uniquely defined by the condition $\theta(\lambda) = 0$ when $\lambda << 0$, if H_0 is semi bounded from below, which we shall always assume. The introduction of this function is due to Lifschitz and, from the mathematical point of view, to Krein [15] (see also Yafaev [26] for a self contained presentation). One can view the SSF as a replacement for the counting function of eigenvalues when the spectra of the operators are not discrete, for example in scattering theory where we study absolutely continuous spectrum. In this case the SSF is closely connected to the scattering matrix $S(\lambda)$ of (H_0, H_1) by *Krein's formula*

$$Det_1\ S(\lambda) = e^{2i\pi\theta(\lambda)}, \qquad \text{a.e. on } \sigma_{ac}(H_0),$$

where the Fredholm determinant makes sense since $S(\lambda)-1$ is a trace class operator (see [26] and the beginning of §2 of this article for more details on determinants). The SSF θ is also called the *scattering phase* because of this formula.

Let us introduce the following usefull notation

$$H_\epsilon = H_0 + \epsilon V, \qquad 0 \le \epsilon \le 1.$$

When one studies (relatively) trace class perturbations, some trace formulas can be obtained through the *Birman-Solomyak identity*

$$Tr\,(f(H_1) - f(H_0)) = \int_0^1 Tr\,(f'(H_\epsilon)V)\,d\epsilon; \tag{1}$$

this may be interpreted as a non commutative first order Taylor formula, and is a rather simple consequence of the cyclicity of the trace functional.

All these results hold in particular for the following class of operators acting on $L^2(\mathbb{R}^d)$; $H_0 = -\Delta$, and $H_1 = -\Delta_g + Q$, where $-\Delta_g$ is the Laplace-Beltrami operator associated to a metric $g = (g_{jk}(x))$ on $\mathbb{R}^d$, and $Q = Q(x)$ is a potential satisfying the decay estimate

$$|\partial^\alpha(g_{jk}(x) - \delta_{jk})| + |\partial^\alpha Q(x)| \le C_\alpha(1+|x|)^{-\rho}, \quad \text{for some } \rho > d \tag{2}$$

for all multiindices α. The "relatively trace class" condition is fullfilled since $\rho > d$. The main goal of this paper is to study such operators under weaker conditions on ρ; we will consider $\rho > 0$ and, in more detail, the case $\rho > d/2$. When $\rho > d/2$ the SSF cannot be defined since $(H_0+i)^{-N} - (H_1+i)^{-N}$ is a Hilbert-Schmidt operator (we call $V = H_1 - H_0$ a *relatively Hilbert-Schmidt perturbation*); however, using higher order non commutative Taylor formulae, as inspired by the work of Koplienko [13], we'll show that there exists a function analogous to θ when $\rho > d/2$. Firstly we review the results on the SSF that we're going to generalize.

If we replace $(1+|x|)^{-\rho}$ by $(1+|x|)^{-\rho-|\alpha|}$ in (2) and assume that the metric is *non trapping* (that is if all the geodesics escape to infinity), then it is well known that the SSF is smooth on $(0,\infty)$ and has the following asymptotic expansion

$$\theta'(\lambda) \sim \lambda^{\frac{d}{2}-1} \sum_{k\ge 0} \Gamma_k \lambda^{-k}, \qquad \lambda \nearrow +\infty.$$

This result was proved by Colin-de-Verdière [6] (for $d=3$, Q compactly supported, g euclidean), Guillopé [9] (for all d, g euclidean), and Robert [19] (for non trapping metrics). We also mention the same (difficult) result in obstacle scattering proved by Petkov-Popov [17]. From the asymptotic expansion, one can deduce the so-called *generalized Levinson's formula*, which, in the case that d is odd, can be written as

$$\int_0^{+\infty} \left(\sum_{k=0}^{\nu+\frac{d-1}{2}} \Gamma_k \lambda^{\frac{d}{2}-k-1} - \theta'(\lambda) \right) \lambda^\nu d\lambda = \sum_{\lambda\in\mathcal{E}} \lambda^\nu + \varepsilon_0 \delta_{0\nu};$$

here $\nu \in \mathbb{N}$, $\mathcal{E} \subset (-\infty, 0]$ is the finite family of eigenvalues of H_1 repeated according to their (finite) multiplicities, and $\varepsilon_0 = 1/2$ if 0 is a resonance for H_1 ($\varepsilon_0 = 0$ otherwise). For the proof see [6, 9, 23].

Our final result will be a new Levison's type formula, with θ replaced by a new function. We shall define this new function by following the Taylor formula approch used by Koplienko in [13]. If $V \in \mathbf{S_p}$, the Schatten class of order p (*i.e.* V^p is trace class), he proved that one can define a linear form on a class of rational fractions f by

$$Tr\left(f(H_1) - \sum_{j=0}^{p-1} \frac{1}{j!}\frac{d^j}{d\epsilon^j} f(H_\epsilon)_{|\epsilon=0}\right).$$

Furthermore, when $p = 2$ and $V|H_1 + i|^{-1/2} \in \mathbf{S_2}$ (another kind of relatively Hilbert-Schmidt condition), he showed that there exists a locally integrable function η such that

$$Tr\left(f(H_0+V) - f(H_0) - \frac{d}{d\epsilon} f(H_0 + \epsilon V)_{|\epsilon=0}\right) = \int_{\mathbb{R}} f''(\lambda)\eta(\lambda)d\lambda.$$

In [14], Koplienko applied this fomula to a scattering problem for the Schrödinger operator on the half line. Trace formulae in this framework were also proved more recently by Rybkin [24]. However, in all these results one has to consider (relatively) compact perturbations, which prevents us from studying second order perturbations of the Laplacian; to overcome this difficulty we have to review Koplienko's ideas in the context of pseudodifferential operators.

2. Results

2.1. **Notation.** We will use the Schatten classes $\mathbf{S_p}$, of integer order $p \geq 1$ (see [26]); we recall that these classes of compact operators are defined by

$$A \in \mathbf{S_p} \Leftrightarrow \sigma(|A|) \in l^p(\mathbb{N})$$

where $\sigma(|A|)$ is the spectrum of $|A| = (A^\star A)^{1/2}$ in which the eigenvalues are repeated according to their multiplicities. $\mathbf{S_1}$ is the space of trace class operators and $\mathbf{S_2}$ is the space of Hilbert-Schmidt operators. Recall that $\mathbf{S_1}$ is dense in $\mathbf{S_p}$ for all p and that, for all $A \in \mathbf{S_p}$, one can define the *Fredholm determinant* $Det_p(1 + A)$. When $A \in \mathbf{S_1}$, we have

$$Det_p(1+A) = \prod_{j=0}^{\infty}(1+\lambda_j)\exp\Big(\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\lambda_j^k\Big), \tag{3}$$

if $(\lambda_j)_{j\in\mathbb{N}}$ are the elements of the spectrum of A repeated according to their multiplicities (of course, if $p = 1$ there is no term with exp). For more details on this subject see Gohberg-Krein [8], Simon [25] and Yafaev [26].

In this paper, we use pseudodifferential operators defined by Weyl quantization, which we denote by $Op_h^w(a)$, $a^w(x, hD)$ or a^w; more precisely

$$a^w(x, hD)u(x) = (2\pi h)^{-d} \iint a(\frac{x+y}{2}, \xi) e^{i<x-y,\xi>/h} u(y) dy d\xi$$

when u is in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and the symbol a belongs to $S_\delta(\omega, \mu)$. The symbols in this class satisfy estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta}(\omega(\xi) + 1) < x >^{\mu - \delta|\alpha|}, \tag{4}$$

on $\mathbb{R}^{2d}$, for all α and β. Here $\delta \in [0, 1]$, $\mu \in \mathbb{R}$ and we use the standard notation $< x >= (1 + x^2)^{1/2}$. The function ω is a temperate weight, that is a non negative function satisfying

$$\omega(\xi_1) \leq C(\omega(\xi_2) + 1) < \xi_1 - \xi_2 >^M$$

for all $\xi_1, \xi_2 \in \mathbb{R}^d$, where C and M are positive constants. As usual, the best constants $C_{\alpha\beta}$ in (4) are semi-norms defining our topology on $S_\delta(\omega, \mu)$.

We will need to work with symbols having an expansion with respect to $h \in (0, h_0]$; for $a(h) \in S_\delta(\omega, \mu)$, we write

$$a(h) \sim \sum_{k \geq 0} h^k a_k$$

if there exists a sequence of symbols $a_k \in S_\delta(\omega, \mu - \delta k)$ which is independent of h and, for each $N > 0$, $h^{-N}(a(h) - \sum_{k<N} h^k a_k)$, $h \in (0, h_0]$, is a bounded family in $S_\delta(\omega, \mu - \delta N)$.

Throughout this paper, we shall always assume that $\omega \in S_0(\omega, 0)$, together with the one of the following two conditions,

$$\lim_{\xi \to \infty} \omega(\xi) = +\infty, \tag{5}$$

$$\text{or} \quad 1 + \omega(\xi) \geq c < \xi >^m, \quad \xi \in \mathbb{R}^d \tag{6}$$

for some $c > 0, m > 0$. The operators $H_\epsilon = H_0 + \epsilon V$, $\epsilon \in [0, 1]$, which we will consider are defined as follows.

Assumptions on H_0. $H_0 = p^w(x, hD, h)$ with $p(h) \sim \sum_{k \geq 0} h^k p_k$ in $S_0(\omega, 0)$. We suppose that

- H_0 is symmetric on $\mathcal{S}(\mathbb{R}^d)$ for all $h \in (0, h_0]$,
- there exists $C > 0$ such that $p_0(x, \xi) + C \geq C^{-1}\big(1 + \omega(\xi)\big)$ for all $x, \xi \in \mathbb{R}^d$.

Assumptions on V. $V = v^w(x, hD, h)$ with $v(h) \sim \sum_{k \geq 0} h^k v_k$ in $S_\delta(\omega, -\rho)$ where $\rho > 0$ is a real number and δ will be 0 except for the special case of euclidean scattering where we choose $\delta = 1$. Moreover we suppose that

- $H_0 + V$ is symmetric on $\mathcal{S}(\mathbb{R}^d)$,
- $\big(p_0 + v_0\big)(x, \xi) + C \geq C^{-1}(1 + \omega(\xi))$ for all $x, \xi \in \mathbb{R}^d$.

In the sequel, we shall call such a V a $\rho-$*perturbation* of H_0; in particular it is clear that for any $\epsilon \in [0,1]$, ϵV is still a $\rho-$perturbation of H_0.

Example. Our class of $\rho-$perturbations include all usual long range perturbations of elliptic operators, such as

$$H_0 = L_h(g_0, A_0, V_0), \quad H_0 + V = L_h(g_1, A_1, V_1)$$

where $L_h(g_l, A_l, V_l)$ is the differential operator

$$-G_l^{-1/4} \sum_{1 \le j,k \le d} (h\partial_j + iA_l^j) G_l^{1/2} g_l^{jk} (h\partial_k + iA_l^k) G_l^{-1/4} + V_l$$

defined by the metric $g_l = (g_{l,jk})$ on $\mathbb{R}^d$, the magnetic potential $A_l = (A_l^1, \cdots, A_l^d)$ and the electric potential V_l (as usual $(g_l^{jk}) = (g_{l,jk})^{-1}$ and $G_l = det(g_l)$); we assume that, for $l = 0,\ 1$, g_l, A_l, V_l are smooth, bounded (together with their derivatives), and satisfy

$$|\partial^\alpha (g_1 - g_0)(x)| + |\partial^\alpha (A_1 - A_0)(x)| + |\partial^\alpha (V_1 - V_0)(x)| \le C_\alpha < x >^{-\rho - \delta|\alpha|}$$

for all α.

2.2. **Main results.**

Theorem 2.1. *Assume that V is a ρ-perturbation with $\rho > d/p$ (p integer), and that ω satisfies* (5). *Then*
i) for every bounded interval I, there exists $h_p > 0$ such that, for every $f \in C_0^\infty(I)$ and $h \in (0, h_p]$ the following expression,

$$< u_p, f >= Tr \left(f(H_1) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\epsilon^j} f(H_\epsilon)_{|\epsilon=0} \right)$$

defines a distribution $u_p \in \mathcal{D}'(I)$ (depending on the parameter h).
ii) u_p has a weak asymptotic expansion with respect to h

$$< u_p, f > \sim h^{-d} \sum_{k \ge 0} h^k < u_p^k, f >, \qquad h \searrow 0 \tag{7}$$

where, for all k, u_p^k is a distribution independent of h and explicitely computable in terms of the symbols of H_0 and V.
If ω satisfies (6) *then we can choose h_p independent of I; moreover u_p and the u_p^k are in $\mathcal{S}'(\mathbb{R})$.*

Remark: if we consider elliptic operators, we can choose $h_p = 1$.

Definition 2.2. *u_p is called the* spectral distribution *of order p associated with the pair $(H_0, H_0 + V)$.*

These results are well known in the case $p = 1$, that is for *relatively trace class* perturbations. Moreover in this case $u_1 = \theta'$, with $\theta \in L^1_{loc}$ the SSF. In the general case considered here, we only know that the functional u_p is a distribution; nevertheless, in the special case $p = 2$ we have the following result.

Theorem 2.3. *Assume that $\rho > d/2$. If ω satisfies* (5)*, there exists $\eta \in L^1_{loc}(I)$ (depending on $h \in (0, h_2]$) such that*

$$< u_2, f >= \int_{\mathbb{R}} \eta(\lambda, h) f''(\lambda) d\lambda, \qquad \forall\ f \in C_0^\infty(I).$$

If ω satisfies (6)*, there exists $\kappa \geq 0$, then $\eta \in L^1(\mathbb{R}, (1+|\lambda|)^{-\kappa} d\lambda)$, for all $h \in (0, h_2]$.*

Definition 2.4. *η is called the* Koplienko's function *associated to (H_0, H_0+V), and is uniquely defined if we require that $\eta(\lambda, h) = 0$ when $\lambda < \min(\inf \sigma(H_0), \inf \sigma(H_0 + V))$.*

Notice that we can always assume $\min(\inf \sigma(H_0), \inf \sigma(H_0 + V))$ belongs to the interior of I.

The existence of the function η was first proved by Koplienko in [13], for a general selfadjoint operator H_0 and a perturbation V such that $V|H_0 + i|^{-1/2}$ is Hilbert-Schmidt; in particular this implies that V is a relatively compact perturbation, which prevents us from studying second order perturbations of the Laplacian for example. From this point of view our results cover more cases, but only for pseudodifferential operators.

Now we focus on the special case $\rho > d/2$ in the framework of euclidean scattering, with $d \geq 2$; we assume that

$$H_0 = \omega(hD), \qquad v_k \in S_1(\omega, -k) \cap S_1(\omega, -\rho).$$

As in [21], we consider an open non critical interval $I \subset (0, +\infty)$ for ω (which means that $\nabla\omega$ doesn't vanish on $\omega^{-1}(I)$). Therefore the spectrum of H_0 is absolutely continuous on I, and by the Kato-Kuroda formula, the scattering matrix associated with $(H_0, H_0 + V)$ is almost everywhere a perturbation of the identity on $L^2(\Sigma_\lambda)$ where $\Sigma_\lambda = \omega^1(\{\lambda\})$; moreover we have

$$S(\lambda, h) = 1 - 2i\pi A(\lambda, h) + 2i\pi B(\lambda, h)$$

where

$$A(\lambda, h) = \Xi_\lambda V \Xi_\lambda^\star, \quad B(\lambda, h) = \Xi_\lambda V(H_0 + V - \lambda - i0)^{-1} V \Xi_\lambda^\star$$

and $\Xi_\lambda u = \mathcal{F}_h u_{|\Sigma_\lambda}$, $\mathcal{F}_h u$ being the semiclassical Fourier transform of $u \in \mathcal{S}(\mathbb{R}^d)$. The next theorem shows that we can extend the results concerning the regularity and asymptotic behaviour of the SSF to Koplienko's function η. Part *ii*) of the theorem can be viewed as a generalization of Krein's formula.

Theorem 2.5. *i) For all $h \in (0, h_2]$, η is C^∞ on $I \setminus \sigma_{pp}(H_0 + V)$.*
ii) If $\rho > (d+1)/2$ we have

$$Det_2\ S(\lambda, h) = \exp\big(2i\pi\eta'(\lambda, h) - 2i\pi TrB(\lambda, h)\big). \tag{8}$$

almost everywhere on I.
iii) If I is non trapping and non critical for $\omega + v_0$ then, for every $J \subset\subset I$, there exists $h_J \leq h_2$ such that $\sigma_{pp}(H_0 + V) \cap J = \emptyset$ for all $h \in (0, h_J]$. Moreover, we have the following asymptotic expansion in the $C^\infty(J)$ topology;

$$\eta''(\lambda, h) \sim h^{-d} \sum_{k \geq 0} h^k \gamma_k(\lambda), \qquad h \searrow 0. \tag{9}$$

In *ii*), Det_2 is well defined as the scattering matrix is an Hilbert-Schmidt perturbation of identity.
For *iii*), recall that I is non trapping for $\omega + v_0$ if $\Pi\big(\Phi^t(x, \xi)\big) \to \infty$ as $|t| \nearrow \infty$, uniformly on each compact subset of $\omega^{-1}(I)$. Here Φ^t is the Hamiltonian flow of $\omega + v_0$, and $\Pi : T^\star\mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the base.

As a consequence of the previous theorem, we get the following high energy asymptotic expansion for $h = 1$ and $\lambda \nearrow +\infty$.

Corollary 2.6. *Assume that $H_0 = -\Delta$ and $H_0 + V = L_1(g_1, A_1, V_1)$, with $\rho > d/2$ (and $\delta = 1$). If g is a non trapping metric, then we have the following asymtotic expansion*

$$\eta''(\lambda) \sim \lambda^{\frac{d}{2}-1} \sum_{k \geq 0} \Gamma_k \lambda^{-k}.$$

Furthermore this can be differentiated to any order.

Finally, we consider the special case of the Schrödinger operator $H_0 + V = -\Delta + V$ where $V = V(x)$ is a potential satisfying

$$\partial^\alpha V(x) = \mathcal{O}\big(< x >^{-\rho-|\alpha|} \big).$$

Our result is the following.

Theorem 2.7. *Assume that the dimension $d = 3$ and that $\rho > 5/2$. If 0 is regular in the spectrum of $H_0 + V$, then, for every $\nu \in \mathbb{N}$, we have*

$$\sum_{\lambda \in \sigma_{disc}} \lambda^\nu = \int_0^{+\infty} \left(\sum_{2 \leq k \leq 1+\nu} \Gamma_k \lambda^{\frac{3}{2}-1-k} - \eta''(\lambda) \right) \lambda^\nu d\lambda, \tag{10}$$

where σ_{disc} is the discrete spectrum of $H_0 + V$, and in the sum we repeat the eigenvalues according to their multiplicities.

We say that 0 is *regular* in the spectrum of $H_0 + V$ if it is neither a resonance nor an eigenvalue (see [12] and the last section). (See Reed-Simon [18] for the finiteness of the discrete spectrum). Let's remark that the left hand side of (10) is finite provided that $\rho > 2$. Moreover, if $d = 3$, η is well defined when $\rho > 3/2$, hence when $\rho > 2$, but we have to consider $5/2 < \rho$ (≤ 3) to be able to compute the jump of η' at 0, $\eta'(+0) - \eta'(-0)$, which is done by using estimates proved in [12], where we need a little bit more decay than $< x >^{-2-\varepsilon}$. When there is enough decay with respect to x one can use the expansion of the resolvent of [12], and that's why Colin-de-Verdière, Guillopé and Robert [6, 9, 23] are able to consider the cases when 0 is an eigenvalue or a resonance.

3. Preliminaries on functionnal calculus

3.1. Pseudodifferential functional calculus.

This subsection is entirely devoted to the proof of proposition 3.1 below, which gives results on functions of H_ϵ, and their behaviour with respect to ϵ. This proof consists partially in reviewing Helffer-Robert's article [10], so we might be sketchy on some points of the proof, and insist on the new ones such as estimates in Schatten classes. In the end, we shall also mention some remarks on the continuity of $f(H_0 + \epsilon V)$ with respect to the perturbation V.

In [10], the authors prove that for selfadjoint realizations of semiclassical pseudo-differential operators as H_ϵ we have

$$f(H_\epsilon) \sim \sum_{k\geq 0} h^k a^w_{k,\epsilon v,f}(x, hD)$$

in the sense that $f(H_\epsilon)$ minus the sum truncated at order N is $\mathcal{O}(h^N)$ in operator norm. But we want to estimate (derivatives with respect to ϵ of) the remainder in Schatten classes and control it with respect to ϵ; this is the object of our proposition.

The sequence $a_{k,\epsilon v,f}$ is defined by

$$\begin{aligned} a_{0,\epsilon v,f} &= f(p_0 + \epsilon v_0), \\ a_{k,\epsilon v,f} &= \sum_{1\leq l\leq 2k-1} \frac{(-1)l}{l!} d_{k,l,\epsilon v} f^{(l)}(p_0 + \epsilon v_0), \qquad k \geq 1. \end{aligned}$$

The symbols $d_{k,l,\epsilon v}$ are defined by the expansion of the parametrix for the resolvent $R_\epsilon(z) = (H_\epsilon - z)^{-1}$, that is

$$R_\epsilon(z) \sim \sum_{k\geq 0} h^k Op^w_h \left(\sum_{l=1}^{2k-1} d_{k,l,\epsilon v}(p_0 + \epsilon v_0 - z)^{-1-l} \right), \tag{11}$$

and are universal polynomial functions of the symbols of the expansions of $p(h)$ and $v(h)$. More precisely, it can be easily shown that $d_{k,l,\epsilon v}$ is a linear combination (with coefficients depending only on the dimension d) of

$$\prod_{r=1}^{k} \partial_x^{\alpha_r} \partial_\xi^{\beta_r} (p_{l_r} + \epsilon v_{l_r}), \qquad \text{with} \quad l - \sum_{r=1}^{k} l_r = \sum_{r=1}^{k} |\alpha_r| = \sum_{r=1}^{k} |\beta_r|.$$

Proposition 3.1. *i) There exists $h_0 > 0$ such that, for every $\epsilon \in [0,1]$ and $h \in (0, h_0]$, H_ϵ is essentially selfadjoint on $L^2(\mathbb{R}^d)$. The domain of its selfadjoint extension doesn't depend on ϵ, and there exists $E \in \mathbb{R}$ such that*

$$H_\epsilon \geq E, \qquad \forall\ \epsilon \in [0,1],\ \forall\ h \in (0, h_0].$$

ii) Assume that (5) is verified. Then for every function $f \in C_0^\infty(\mathbb{R})$ and all $N \geq 0$

$$f(H_\epsilon) = \sum_{0 \leq k \leq N} h^k a^w_{k,\epsilon v,f}(x, hD) + h^{N+1} r_{N,\epsilon v,f}(h)$$

with $\epsilon \mapsto r_{N,\epsilon v,f}(h)$ in $C^\infty\big([0,1], \mathcal{L}(L^2)\big)$. Moreover, for every integer $j \geq 1$ and every compact interval I, there exists $h_j > 0$, $C > 0$, $\mathbf{N}$ seminorm of $C_0^\infty(I)$ and $\mathcal{N}$ seminorm of $S_0(\omega, 0)$ such that,

$$(12)\ |||b^w(x, hD) \frac{d^j}{d\epsilon^j} r_{N,\epsilon v,f}(h)|||_{\mathbf{p'_j}} \leq C h^{-d/p'_j} \mathbf{N}(f) \mathcal{N}(b), \qquad p'_j = \max(1, p/j)$$

for all $f \in C_0^\infty(I)$, $b \in S_0(\omega, 0)$, $\epsilon \in [0,1]$ and $h \in (0, h_j]$.
iii) If the assumption (6) is fullfilled, then ii) holds with $C_0^\infty(I)$ replaced by $\mathcal{S}(\mathbb{R})$ and h_j is uniform on $\mathbb{R}$.

The proof of $i)$ is a straightforward extension of an argument given in [10]. For any $E < \min(\inf(p_0), \inf(p_0 + v_0))$, $b_{0,\epsilon,z} := (p_0 + \epsilon v_0 - z)^{-1}$ is a well defined symbol bounded in its class with respect to $\epsilon \in [0,1]$ and $z \notin [E, +\infty)$. This implies that $(H_\epsilon - z) Op_h^w(b_{0,\epsilon,z}) = 1 + \mathcal{O}(h) \in \mathcal{L}(L^2)$ uniformly with respect to ϵ and z, which proves the injectivity of $(H_\epsilon - z)^\star$ when $h \leq h_0$ for some $h_0 = h_0(E)$ small enough, and thus yields the result. The independance of the domain on ϵ is just due to the fact that $H_0 R_\epsilon(z)$ and $H_\epsilon R_0(z)$ are bounded operators, which can be seen by using the resolvent's parametrix.

Up to a translation, we may assume that $E > 0$, and therefore study $f(H_\epsilon)$ for functions f supported in $(0, +\infty)$. This allows us to use the Mellin transform

$$\mathcal{M}[f](\tau) = \int_0^{+\infty} t^{\tau - 1} f(t) dt, \qquad \Re(\tau) < -s$$

when $f \in S^s_+$ (see [10]) which means that f is smooth, supported in $(0, +\infty)$ and such that all seminorms $\sup_{t \geq 0, k \leq j} |(1+t)^{k-s} f^{(k)}(t)|$ are finite, when $j \geq 0$. The use of Mellin transform appears as follows in functionnal calculus

$$(13) \qquad f(H_\epsilon) \ = \ \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{M}[f](\tau) H_\epsilon^{-\tau} d\tau_2, \qquad 0 < \tau_1 < -s$$

and by the Cauchy formula we can write the complex powers of H_ϵ thanks to the resolvent:

$$(14) \qquad H_\epsilon^{-\tau} \ = \ \frac{i}{2\pi} \int_{\mathcal{C}_\Theta} z^{-\tau} R_\epsilon(z) dz, \qquad \tau = \tau_1 + i\tau_2, \qquad \tau_1 > 0,\ \tau_2 \in \mathbb{R}, .$$

We now start the proof of *ii*). This results is based on the two following obvious formulas:

$$\frac{d^j}{d\epsilon^j} R_\epsilon(z) \;=\; j!(-1)^j R_\epsilon(z)\big(V R_\epsilon(z)\big)^j \tag{15}$$

$$\big[R_\epsilon(z), < x >^{\mu_0}\big] \;=\; -R_\epsilon(z)\big[H_\epsilon, < x >^{\mu_0}\big] R_\epsilon(z) \tag{16}$$

with $0 \le \mu_0 \le 1$. As $[H_\epsilon, < x >^{\mu_0}]$ is a pseudodifferential operator with symbol in $S_0(\omega, 0)$, (16) implies that $b^w(x, hD) < x >^{\pm\mu_0} R_\epsilon(z) < x >^{\mp\mu_0}$ is a bounded operator, when $b \in S_0(\omega, 0)$, and if we combine this remark with (15) we get easily that

$$b^w(x, hD) < x >^{\mu_1} \frac{d^j}{d\epsilon^j} R_\epsilon(z) < x >^{\mu_2} \tag{17}$$

is bounded, for all real numbers μ_1, μ_2 such that $\mu_1 + \mu_2 \le j\rho$. Moreover (17) can be easily estimated, in operator norm, polynomially with respect to $\tau_2 = \Im(\tau)$, and uniformly with respect to ϵ, on the same contour as in [10], that is when $z = r\exp(\pm i\Theta(\tau_2))$, with $r \ge 1$ and $\Theta(\tau_2) = \max(2/\pi, 1/|\tau_2|)$. From this remark we deduce that for $h \in (0, h_0]$, in operator norm, we have

$$|||b^w(x, hD) < x >^{\mu_1} \frac{d^j}{d\epsilon^j} r_{N,\epsilon v,f}(h) < x >^{\mu_2} ||| \le C\mathbf{N}(f)\mathcal{N}(b), \tag{18}$$

if $\mu_1 + \mu_2 \le j\rho$. But we want an estimate in a Schatten class; to get it, we shall use the following

Lemma 3.2. *Fix $g \in C_0^\infty(\mathbb{R})$ and let μ, μ' be real numbers such that, for some $q \ge 1$ integer, $\mu + \mu' < -d/q$. Then, there exists $h_1 = h_1(g, q, \mu, \mu')$ such that*

$$h^{d/q} < x >^{\mu} g(H_\epsilon) < x >^{\mu'}$$

ranges over a bounded subset of $\mathbf{S_q}$, when $\epsilon \in [0, 1]$ and $h \in (0, h_1]$.

Proof: Consider $G \in C_0^\infty$ such that $Gg = g$ and write $< x >^{\mu} G(H_\epsilon) < x >^{\mu'} =$

$$< x >^{\mu} a^w_{0,\epsilon v,G}(x, hD) < x >^{\mu'} + h < x >^{\mu} r_{0,\epsilon v,G} < x >^{\mu'} .$$

Multiplying this equality (on the left) by $< x >^{\mu} g(H_\epsilon) < x >^{-\mu}$, we get

$$< x >^{\mu} g(H_\epsilon) < x >^{\mu'} \Big(1 - h < x >^{-\mu'} r_{0,\epsilon v,G} < x >^{\mu'}\Big) =$$

$$< x >^{\mu} g(H_\epsilon) < x >^{-\mu} < x >^{\mu} a^w_{0,\epsilon v,G}(x, hD) < x >^{\mu'} .$$

The term $(\cdots)$ on the left hande side is invertible thanks to (18) for $h \le h_1$ small enough, and $< x >^{\mu} a^w_{0,\epsilon v,G}(x, hD) < x >^{\mu'} = \mathcal{O}(h^{-d/q})$ in $\mathbf{S_q}$ uniformly with respect to ϵ, as its symbol depends continuously on ϵ in $S_0(-\infty, \mu + \mu')$. Moreover, when $\epsilon \in [0, 1]$, $< x >^{\mu} g(H_\epsilon) < x >^{-\mu}$ is a bounded family in $\mathcal{L}(L^2)$, still by using (18), so we get the result. $\square$

Now we are in position to complete the proof of ii), for which we shall proceed by induction. Let $g \in C_0^\infty(\mathbb{R})$ be equal to 1 in a neighbourhoodof I and for $f \in C_0^\infty(I)$ let's define $f(H_\epsilon) = \mathcal{E}_f(\epsilon) + h^N \mathcal{R}_f(\epsilon)$ where $\mathcal{E}_f$ is the pseudodifferential expansion up to order N. Using Leibnitz formula, we get, for every $f \in C_0^\infty(I)$

$$b^w \frac{d}{d\epsilon} f(H_\epsilon) = \big(b^w \mathcal{E}'_f(\epsilon) + h^N b^w \mathcal{R}'_f(\epsilon)\big) g(H_\epsilon) + b^w f(H_\epsilon)\big(\mathcal{E}'_g(\epsilon) + h^N \mathcal{R}'_g(\epsilon)\big).$$

By (18) and the previous lemma, it is clear that $b^w \mathcal{R}'_f(\epsilon) g(H_\epsilon)$ and $b^w f(H_\epsilon)\mathcal{R}'_g(\epsilon)$ satisfie the estimate (12), as we have a gain of $< x >^{-\rho}$ for each derivative.

Finally $\mathcal{E}'_f(\epsilon) g(H_\epsilon) + f(H_\epsilon)\mathcal{E}'_g(\epsilon) = \big(\mathcal{E}_f(\epsilon)\mathcal{E}_g(\epsilon)\big)' + h^N\big(\mathcal{E}'_f(\epsilon)\mathcal{R}_g(\epsilon) + \mathcal{R}_f(\epsilon)\mathcal{E}'_g(\epsilon)\big)$, where the term with the h^N factor satisifies the Schatten estimate (12) by properties of the symbols. The first term, which is purely pseudodifferential can be written $\mathcal{E}'_f(\epsilon) + h^N c^w(x, hD)$ again with the good estimate for $c^w(x, hD, h)$, thus we have proven the result for $j = 1$.
When $j \geq 2$, the Leibnitz formula shows that

$$\frac{d^j}{d\epsilon^j} f(H_\epsilon) = \left(\frac{d^j}{d\epsilon^j} f(H_\epsilon)\right) g(H_\epsilon) + f(H_\epsilon)\frac{d^j}{d\epsilon^j} g(H_\epsilon) + \sum_{l=1}^{j-1} C_j^l \left(\frac{d^l}{d\epsilon^l} f(H_\epsilon)\right) \frac{d^{j-l}}{d\epsilon^{j-l}} g(H_\epsilon)$$

where the sum $\sum_1^{j-1}$ is easy to study by the induction's assumption, and for the first two terms we use the same methods as for $j = 1$. This yields the result with $p'_j = p/j$; to get $\max(1, p/j)$, we just increase the order of expansion, *i.e.*

$$\sum_{k=0}^{N} h^k \frac{d^j}{d\epsilon j} a^w_{k,\epsilon v,f}(x, hD) + \sum_{k=N+1}^{N+M} h^k \frac{d^j}{d\epsilon j} a^w_{k,\epsilon v,f}(x, hD) + h^{N+M+1}\frac{d^j}{d\epsilon j} r_{N+M,\epsilon v,f}(h).$$

If $p/j < 1$, then $j\rho > d$, so the second sum is $\mathcal{O}(h^{N+1-d})$ in trace norm, and the last one is $\mathcal{O}(h^{N+M+1-dj/p})$ in $\mathbf{S_1}$, ie $\mathcal{O}(h^{N+1-d})$ with M big enough. This completes the proof of ii).

The proof of iii) is easier, even if the result is stronger than ii). It is just based on the study of the remainder in the pseudodifferential expansion of the resolvent, therefore we give a rather explicit expression of this remainder:

$$R_\epsilon(z) = \sum_{k=0}^{N} h^k Op_h^w \left(\sum_{l=1}^{2k-1} d_{k,l,\epsilon v}(p_0 + \epsilon v_0 - z)^{-1-l}\right) + h^{N+1} R_\epsilon(z) c^w_{N,\epsilon v,z}(x, hD, h)$$

with $c_{N,\epsilon v,z}(h)$ that satisfies

$$(H_\epsilon - z) \circ \sum_{k=0}^{N} h^k Op_h^w \left(\sum_{l=1}^{2k-1} d_{k,l,\epsilon v}(p_0 + \epsilon v_0 - z)^{-1-l}\right) = 1 - h^{N+1} c^w_{N,\epsilon v,z}(x, hD, h).$$

It is not hard to see that $\partial_\epsilon^j \partial_z^n c_{N,\epsilon v,z}(h) \in S_0((1+\omega)^{-n}, -j\rho)$. If we use these expansions with $N=0$, and differentiate them n times with respect to z we get

$$(H_\epsilon - z)^{-1-n} = Op_h^w\big((p_0+v_0-z)^{-1-n}\big) + h\sum_{j=0}^{n}(H_\epsilon - z)^{-1-j}\partial_z^{n-1}c^w_{0,\epsilon v,z}(h),$$

so if we choose $z=0$, and $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mu_1+\mu_2 < -d/q$, we get easily, with the same arguments as in *ii*), that

$$< x >^{\mu_1} (H_\epsilon - z)^{-1-n} < x >^{\mu_2} (1+\mathcal{O}(h)) =< x >^{\mu_1} Op_h^w\big((p_0+v_0-z)^{-1-n}\big) < x >^{\mu_2}$$

where $\mathcal{O}(h)$ is uniform with respect to ϵ, and must be taken in the operator norm. As the symbol of the right hand side is in $S_0((1+\omega)^{-1-n}, \mu_1+\mu_2)$, the corresponding operator is in $\mathbf{S_q}$ if $m(n+1) > d/q$. Thanks to this, we deduce with standard arguments that, when h is small enough, uniformly with respect to ϵ,

$$b^w(x,hD) < x >^{\mu_1} (H_\epsilon - z)^{-1-n} < x >^{\mu_2} \in \mathbf{S_q}$$

for n big enough, and can be controlled polyniomally in $\mathbf{S_q}$ with respect to $|\sin\Theta|^{-1}$ when z is on the contour Λ_Θ described in the proof of *ii*) (see [10]). Then thanks to this *a priori* estimate, we can control the remainder of the asymptotic expansion of the resolvent and the complex powers, because using integrations by part, we can write the remainder as a linear combination, when $0 \le j \le n$ of

$$\frac{1}{n-\tau}\cdots\frac{1}{1-\tau}\int_{\Lambda_\Theta} z^{n-\tau}(H_\epsilon - z)^{-1-j}\partial_z^{n-j}c^w_{N,\epsilon v,z}(x,hD,h)dz$$

and the result follows easily. $\square$

Remark. It's quite clear by looking at what we wrote, that if $v_k(h) \to v(h)$ in $S_0(\omega,-\rho)$, then $(d/d\epsilon)^p f(H_0+\epsilon V_k) \to (d/d\epsilon)^p f(H_0+\epsilon V)$, in $C^0([0,1],\mathbf{S_1})$, if $\rho > d/p$. Indeed, it is obvious for the pseudodifferential terms, and as far as the remainders are concerned, this is just due to basic properties of convergence of $(H_0+V_j-z)^{-1}$ to $(H_0+V-z)^{-1}$. We shall use this remark to approximate our perturbation by, for example, compactly supported ones.

3.2. **Hilbert-Schmidt estimates.** In this part, we give two Hilbert-Schmidt estimates for functions of selfadjoint operators, which are crucial to prove the existence of Koplienko's function. The second estimate (proposition 3.4 below) is an extension of the one that can be found in [13] and is mainly based, as the first one(proposition 3.3), on the theory of double Stieljes operator integrals (DSOI) of Birman and Solomyak [1].

Proposition 3.3. *Let A and B be two selfadjoint operators, with the same dense domain, on a separable Hilbert space. Assume that $B-A$ is Hilbert-Schmidt and that one of them (and hence both) is (are) lower semi-bounded. Then for every $f \in C_0^\infty(\mathbb{R})$, $f(B)-f(A)$ is Hilbert-Schmidt and*

$$|||f(B)-f(A)|||_{\mathbf{2}} \le |||B-A|||_{\mathbf{2}}||f'||_\infty \tag{19}$$

We briefly recall the (sketch of the) proof of this result for the sake of completeness and to define some notations that we shall use later. First we remark that, given K, L two Hilbert-Schmidt operators, there exists a unique complex Borel measure $\Lambda_{K,L}$ on $\mathbb{R}^2$ such that, if $|\Lambda_{K,L}|(\mathbb{R}^2)$ is its total variation,

$$|\Lambda_{K,L}|(\mathbb{R}^2) \leq |||K|||_{\mathbf{2}}|||L|||_{\mathbf{2}} \tag{20}$$

$$\int f(\lambda_1)g(\lambda_2)d\Lambda_{K,L}(\lambda_1,\lambda_2) = Tr\big(f(A)Kg(B)L^\star\big) \tag{21}$$

for all continuous functions f, g vanishing at infinity. The uniqueness is obvious, and the existence is shown by considering the function Λ that assign to each characteristic function of a rectangle $1_{I\times J}$ the value $Tr\big(E_A(I)KE_B(J)L^\star\big)$, where $E_A(.), E_B(.)$ are the spectral measures of A and B, then by continuing it naturally by linearity to linear combinations of such characteristic functions with non overlapping rectangles and finally to continuous functions vanishing at infinity. Therefore the proof of (19) is a simple consequence of the fact that

$$|||f(A) - f(B)|||_{\mathbf{2}} = \sup_{L\neq 0} \frac{|Tr\big((f(A)-f(B))L^\star\big)|}{|||L|||_{\mathbf{2}}}$$

combined with the identity

$$Tr\big((f(A)-f(B))L^\star\big) = \int \frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2} d\Lambda_{K,L}(\lambda_1,\lambda_2) \tag{22}$$

where $K = B - A$. This last equality is justified because $\big(f(\lambda_1) - f(\lambda_2)\big)/(\lambda_1 - \lambda_2)$ can be naturally continued to a continuous function on $\mathbb{R}^2$ vanishing at infinity, and is proven by application of the Fubini's theorem to the left hande side of (22) that we can write

$$-\int_{-\infty}^{+\infty} \mathcal{M}[f](\tau_1+i\tau_2)\left(\frac{i}{2\pi}\int_{\mathcal{C}_{\Theta(\tau_2)}} z^{-\tau_1-i\tau_2} Tr\big((A-z)^{-1}K(B-z)^{-1}L^\star\big)dz\right)\frac{d\tau_2}{2\pi}$$

and in which we expand the trace using (21). The notations are the ones of the previous subsection, and the use of Mellin transform is legitimate because one can assume that $A > 0$ and $B > 0$. □

Now suppose that A and B are selfadjoint operators with the same domain D, and let $W = B - A$. Assume that

$$A \geq 1 \quad \text{and} \quad B \geq 1 \tag{23}$$

and that, for some integer $n_0 \geq 1$, we have

$$K_1 \stackrel{def}{=} WB^{-n_0} \in \mathbf{S_2}, \quad \text{and} \quad K_2 \stackrel{def}{=} A^{-n_0}W \in \mathbf{S_2} \tag{24}$$

where the last expression means that K_2 can be continued from D to the whole space to a bounded operator which is Hilbert-Schmidt. These conditions ensure that B is a

relatively Hilbert-Schmidt perturbation of A, in the sense that $(A-z)^{-n}-(B-z)^{-n}$ is Hilbert-Schmidt for $n \geq 2n_0+1$ and z in the resolvent sets.
Under assumptions (23) and (24) we have the following

Proposition 3.4. *For every rational function $f(\lambda) = \sum_{j\geq 2n_0+1} a_j\lambda^{-j}$ ($a_j = 0$ for $j >> 0$), $f(A) - f(B)$ is Hilbert-Schmidt and*

$$|||f(A) - f(B)|||_{\mathbf{2}} \leq ||F_1||_\infty |||K_1|||_{\mathbf{2}} + ||F_2||_\infty |||K_2|||_{\mathbf{2}}. \tag{25}$$

F_1 and F_2 are the continuous functions on $[1,+\infty)^2$ defined for $\lambda_1 \neq \lambda_2$ by

$$\begin{aligned} F_1(\lambda_1,\lambda_2) &= -\lambda_2^{2n_0} f(\lambda_2) F_0(\lambda_1,\lambda_2), \qquad \textit{with} \quad F_0(\lambda_1,\lambda_2) = \frac{\lambda_1^{-n_0} - \lambda_2^{-n_0}}{\lambda_1 - \lambda_2} \\ F_2(\lambda_1,\lambda_2) &= -\frac{\lambda_1^{n_0} f(\lambda_1) - \lambda_2^{n_0} f(\lambda_2)}{\lambda_1 - \lambda_2}. \end{aligned}$$

and their L^∞-norms, in (25), are taken over $[1,+\infty)^2$.

Notice that, formally when $n_0 = 0$, $F_1 \equiv 0$ and $||F_2||_\infty = ||f'||_\infty$, so (25) is (19).
Proof: for any $k \geq 2n_0$, by differentiating k times the resolvent identity we get

$$(A-z)^{-1-k} - (B-z)^{-1-k} = \frac{1}{k!}\sum_{l=0}^{k} (A-z)^{-1-l} W (B-z)^{-1-k+l}.$$

So, if $f(\lambda) = \lambda^{-1-k}$ and by evaluating the previous equality at $z = 0$ in which we separate the sum at $l = n_0$, we have, for any Hilbert-Schmidt operator L

$$Tr\big((f(A) - f(B))L^\star\big) = \int F_1(\lambda_1,\lambda_2) d\Lambda_{K_1,L}(\lambda_1,\lambda_2) + \int F_2(\lambda_1,\lambda_2) d\Lambda_{K_2,L}(\lambda_1,\lambda_2)$$

because a simple computation shows that

$$\begin{aligned} \sum_{l=0}^{n_0-1} \lambda_1^{-1-l}\lambda_2^{n_0-1-k+l} &= -f(\lambda_2)\lambda_2^{2n_0} F_0(\lambda_1,\lambda_2) \\ \sum_{l=n_0}^{k} \lambda_1^{-1-l+n_0}\lambda_2^{-1-k+l} &= -\frac{\lambda_1^{n_0-k-1} - \lambda_2^{n_0-k-1}}{\lambda_1 - \lambda_2}. \end{aligned}$$

Then, by linearity, this is true for every rational function of the type that we are considering, and we get the result using (20). □

4. First properties of spectral distributions and applications

In the first part of that section, we do some comments and illustrate the theorem 2.1 by examples. In the second part, we use the results on *heat expansions* to define and give some properties on *regularized zeta functions*, that we shall use later, in the case $\rho > d/2$, to prove a Levinson's formula.

4.1. **Theorem** 2.1 **and applications.** It is clear that the theorem 2.1 is a straightforward consequence of the proposition 3.1, so we don't have anything to prove, and we shall study more precisely the expansions.

If $\rho > d/p$, we have $u_p \sim h^{-d} \sum_{k\geq 0} h^k < u_p^k, f >$ where $< u_p^k, f >=$

$$\sum_{l=1}^{2k-1} \frac{(-1)^l}{l!}(2\pi)^{-d} \int_{\mathbb{R}^{2d}} d_{k,l,v} f^{(l)}(p_0 + v_0) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\epsilon^j} \left(d_{k,l,\epsilon v} f^{(l)}(p_0 + \epsilon v_0)\right)_{|\epsilon=0}$$

when $k \geq 1$, and more simply when $k = 0$

$$< u_p^0, f >= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} f(p_0 + v_0) - \sum_{j=0}^{p-1} \frac{1}{j!} f^{(j)}(p_0) v_0^j.$$

Notice that in all these formulas, $p_0 = p_0(x, \xi)$ which means that it is not necessary to assume that H_0 has constant coefficients. Nevertheless, as the important applications are related to the euclidean Laplacian and as this paper is mainly devoted to the case $\rho > d/2$, here are some explicit calculations, when $p = 2$ for perturbations of the Laplacian.

1. The semiclassical Schrödinger operator. $H_0 = -h^2\Delta/2,\ V = V(x)$.
   If $J \subset (0, +\infty)$ is a non critical interval for V, then $u_p^0(\lambda)$ is smooth on J and is equal to

$$(2\pi)^{-d} mes(S^{d-1}) \int_{\mathbb{R}^d} \left(2\big(\lambda - V(x)\big)_+\right)^{\frac{d}{2}-1} - (2\lambda)^{\frac{d}{2}-1} + (d-2)(2\lambda)^{\frac{d}{2}-2} V(x) dx.$$

2. Perturbation of the metric in the high energy regime. $H_0 = -\Delta,\ H_1 = -\Delta_g$. $H_1 \geq 0$ is the Laplace Beltrami operator (acting on functions) associated to the riemannian metric $g = (g^{jk})$ such that $|\partial^\alpha\big(g^{jk}(x) - \delta_{jk}\big)| = \mathcal{O}(< x >^{-\rho})$. Here we give the leading term of the expansion (7) if we consider H_0 and H_1 with $h = 1$ and $\lambda \nearrow +\infty$; we get

$$u_2^0(\lambda) = \lambda^{d/2-1} (2\pi)^{-d} \frac{mes(S^{d-1})}{2} \int_{\mathbb{R}^d} \sqrt{\det g(x)} - 1 + \frac{1}{2} tr\big(g^{-1}(x) - \mathbf{1}_d\big) dx.$$

3. The Schrödinger operator in the high energy regime. $H_0 = -\Delta,\ H_1 = -\Delta + V(x)$.
   In that case, we get $u_2^0 = u_2^2 = 0$ and

$$u_2^4(\lambda) = \lambda^{d/2-3} \times (2\pi)^{-d} \frac{mes(S^{d-1})}{2} \frac{d(d-2)}{4} \int_{\mathbb{R}^d} V(x)^2 dx.$$

We haven't said anything about the odd terms, because in the high energy limit, for elliptic differential operators, these terms vanish according to the following lemma.

Lemma 4.1. *Assume that $H_0 = h^{2m}P_0$ and $V = h^{2m}V_0$ where P_0 and V_0 are differential operators of order $2m$ such that P_0 and $P_0 + V_0$ are uniformly elliptic on $\mathbb{R}^d$. If V is $\rho-$perturbation of H_0 with $\rho > d/p$ then*

$$u_p^{2k+1} = 0, \qquad \forall\ k \geq 0.$$

Proof: it is a simple consequence of the fact that it is true when $p = 1$ (see [23]) because we integrate odd functions on the sphere, so if $\rho > d$, we have

$$u_p^{2k+1}(.) = u_1^{2k+1}(.,1) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\epsilon^j} u_1^{2k+1}(.,\epsilon)_{|\epsilon=0} = 0$$

if $u_1^j(.,\epsilon)$ is the family of distribution defined by

$$Tr\big(f(H_0 + \epsilon V) - f(H_0)\big) \sim h^{-d} \sum_{j\geq 0} h^j \int_{\mathbb{R}^d} u_1^j(\lambda,\epsilon) f(\lambda) d\lambda.$$

Then, as any $\rho-$perturbation can be approximated by $(d+1)-$perturbations, we get the result in the general case, by a simple density argument. $\square$

4.2. **Elliptic differential operators and Regularized Zeta functions.** Let P_0 and V_0 be scalar symetric differential operators on $\mathbb{R}^d$, of even order $2m$

$$P_0 = \sum_{|\alpha|\leq 2m} p_\alpha(x) D^\alpha, \quad V_0 = \sum_{|\alpha|\leq 2m} v_\alpha(x) D^\alpha$$

such that $\partial^\beta p_\alpha \in L^\infty(\mathbb{R}^d)$ for all α, β and, for $\rho > 0$

$$|\partial^\beta v_\alpha(x)| \leq C_\beta < x >^{-\rho}, \qquad \forall x \in \mathbb{R}^d.$$

We also assume that P_0 and $P_0 + V_0$ are uniformly elliptic on $\mathbb{R}^d$, which means $\sum_{|\alpha|=2m} p_\alpha(x)\xi^\alpha \geq c|\xi|^{2m}$, $\sum_{|\alpha|=2m}(p_\alpha + v_\alpha)(x)\xi^\alpha \geq c|\xi|^{2m}$, for some $c > 0$. Then we have a continous family of selfadjoint operators defined by

$$P_\epsilon = P_0 + \epsilon V_0, \qquad \epsilon \in [0,1].$$

Both are semi bounded from below and thanks to theorem 2.1, we have the following *heat expansion*, if $\rho > d/p$:

$$\mathcal{H}(t) = Tr\left(e^{-tP_1} - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{d\epsilon^j} e^{-tP_\epsilon}_{|\epsilon=0}\right) \sim t^{-d/2m} \sum_{k\geq 0} \alpha_k t^{k/m}, \tag{26}$$

when $t \searrow +0$. Such expansions are already well known when $p = 1$ or for operators on compact manifolds. Notice that $\mathcal{H}(t) =< u_p, f_t >$, with u_p a temperate distribution supported in $[E_0, \infty)$ (with $E_0 = \min(\inf \sigma(P_0), \inf \sigma(P_0 + V_0))$) and $f_t(\lambda) = e^{-t\lambda}$, so $\mathcal{H}(t) = \mathcal{O}(t^N e^{-tE_0})$, as $t \nearrow +\infty$, for some $N \geq 0$. Closely linked to heat kernels are the zeta functions; we define the so called *regularized zeta function* by

$$\zeta(s,E) = Tr\left((P_0+V_0-E)^{-s} - \sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\epsilon^j}(P_0+\epsilon V_0-E)^{-s}_{|\epsilon=0}\right)$$

when $\Re(E) < E_0$. It is clearly well defined and holomorphic with respect to s if $\Re(s) >> 0$, but using the heat expansion we can prove a more precise result, analogous to the ones on compact manifolds, or when $p=1$.

Proposition 4.2. *$s \mapsto \zeta(s,E)$ has a meromorphic continuation to the complex plane. Its poles are simple and localized at the non integer points that can be written*

$$\frac{d}{2m} - \frac{k}{m}, \qquad k \geq 0.$$

Moreover, if d is odd, then $\zeta(-k,E) = 0$ for all $k \in \mathbb{N}$.

Proof: It is quite standard. We write that

$$(\lambda - E)^{-s} = \Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-t(\lambda-E)}dt$$

if $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ is the usual Gamma function. Then we can write

$$\begin{aligned}\zeta(s,E) &= \Gamma(s)^{-1}\int_0^\infty t^{s-1}\mathcal{H}(t)e^{Et}dt \\ &= \Gamma(s)^{-1}\int_0^\infty t^{s-1}\mathcal{H}_n(t)e^{Et}dt + \sum_{k=0}^{n}\alpha_k\Gamma(s-\frac{d-2k}{2m})(-E)^{-s-\frac{d}{2m}+\frac{k}{m}}\end{aligned}$$

with $\mathcal{H}_n(t) = \mathcal{H}(t) - \sum_{k=0}^{n}\alpha_k t^{\frac{-d+2k}{2m}}$. The first term on the right hand side is holomorphic for $\Re(s) >$, as $e^{Et}\mathcal{H}_n(t)$ decays exponentially at infinity, and is $\mathcal{O}(t^{(-d+2n+2)/2m})$ at 0. The localization of the poles and the zeros comes then from the fact that the poles of Γ are on $-\mathbb{N}$. $\square$

5. Koplienko's function

In this section, we shall prove the theorem 2.3 under both assumptions (5) and (6). We omit the dependance on h of the operators, because it is kept fixed. The idea of the proof is very simple and inspired by the original paper [13]. Let's recall its strategy: if the perturbation is relatively trace class, Koplienko's function can be explicitely computed thanks to Birman-Krein's one (lemma 5.1 below). Then we choose a sequence of relatively trace-class perturbations converging to V, and we just have to show that the associated sequence of Koplienko's functions satisfies the Cauchy criterion in some L^1 space, which will be a consequence of the estimates (19) and (25).

5.1. **The assumption** (5)**.**

Lemma 5.1. *Assume that V' is a ρ'-perturbation of H_0 with $\rho' > d$. Then the function*

$$\eta_{V'}(\lambda) = \int_{-\infty}^{\lambda} \theta_{V'}(t)dt + Tr\big(E_0(\lambda)V'\big) \tag{27}$$

is well defined, locally integrable on $\mathbb{R}$ and satisfies, for all $f \in C_0^\infty(\mathbb{R})$

$$Tr\big(f(H_0+V') - f(H_0) - \frac{d}{d\epsilon} f(H_0+\epsilon V')_{|\epsilon=0}\big) = \int_{\mathbb{R}} f''(\lambda)\eta_{V'}(\lambda)d\lambda.$$

Here $\theta_{V'}$ is the Birman-Krein's function of (H_0, H_0+V') and $E_0(\lambda)$ is the spectral projection of H_0 on $(-\infty, \lambda)$.

Proof: $\eta_{V'}$ is well defined and L^1_{loc} because it is the sum of the primitive of $\theta_{V'}$ (recall that $\theta_{V'}(t) = 0$ for $t << 0$) which is absolutely continuous, and $Tr\big(E_0(\lambda)V'\big)$ which is of locally bounded variation and left continuous, thanks to usual properties of spectral projections. By the Birman-Solomyak formula, we have

$$Tr\big(\frac{d}{d\epsilon} f(H_0+\epsilon V')_{|\epsilon=0}\big) = Tr\big(f'(H_0)V'\big) = -Tr\big(\int f''(\lambda)E_0(\lambda)V'd\lambda\big)$$

and as $Tr\big(f(H_0+V') - f(H_0)\big) = \int f''(\lambda)\big(\int_{-\infty}^{\lambda}\theta_{V'}(t)dt\big)d\lambda$ by integration by part, we get the lemma. $\square$

Now we choose a cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi \equiv 1$ on the unit ball. By setting $\chi_j(x) = \chi(x/j)$, we define the sequence V_j as follows

$$V_j = \chi_j V \chi_j, \qquad j \geq 1.$$

It is clear that, for $j \geq j_0$ big enough, V_j is a ρ'-perturbation of H_0 (for any given $\rho' > d$) such that, for some $C > 0$ we have

$$H_0 + \epsilon V \geq -C \qquad \text{and} \qquad H_{\epsilon,j} = H_0 + \epsilon V_j \geq -C, \tag{28}$$

uniformly wtih respect to $j \geq j_0$ and $\epsilon \in [0,1]$.
Let $\eta_j = \eta_{V_j}$ be the sequence of Koplienko's functions defined as in the previous lemma, for the pair (H_0, H_0+V_j), and consider an arbitrary interval $[a,b] \subset \mathbb{R}$. We shall prove that η_j is a Cauchy sequence in $L^1(a,b)$. To do so, we use

$$\int_{(a,b)} |\eta_j(\lambda) - \eta_k(\lambda)|d\lambda = \sup_{f\in C_0^\infty(a,b), |f|\leq 1} \big|\int_{\mathbb{R}} \big(\eta_j(\lambda) - \eta_k(\lambda)\big)f(\lambda)d\lambda\big| \tag{29}$$

combined with the following identity

$$\begin{aligned} \int \big(\eta_j - \eta_k\big)(\lambda)f''(\lambda)d\lambda \;&=\; \int_0^1 Tr\big((f'(H_{\epsilon,j}) - f'(H_{\epsilon,k}))V_j\big)d\epsilon \\ &\quad + \int_0^1 Tr\big((f'(H_{\epsilon,k}) - f'(H_0))(V_j - V_k)\big)d\epsilon \end{aligned} \tag{30}$$

which is an easy consequence of the Birman-Solomyak formula, because for $l = j, k$ we have:

$$\begin{aligned} Tr\big(f(H_0+V_l)-f(H_0)\big) &= \int_0^1 Tr\big(f'(H_0+\epsilon V_l)V_l\big)d\epsilon, \\ \text{and } Tr\left(\frac{d}{d\epsilon}f(H_{\epsilon,k})_{|\epsilon=0}\right) &= \int_0^1 Tr\big(f'(H_0)V_k\big)d\epsilon. \end{aligned}$$

Then, the key estimates are given by:

Lemma 5.2. *For all $\nu > 0$, there exists n such that, for all $\epsilon \in [0,1]$, $f \in C_0^\infty(a,b)$ and $j \geq n$, $k \geq n$ we have*

$$\left|Tr\big((f'(H_{\epsilon,j})-f'(H_{\epsilon,k}))V_j\big)\right| \leq (\nu/2)||f''||_\infty \tag{31}$$

$$\left|Tr\big((f'(H_{\epsilon,k})-f'(H_0))(V_j-V_k)\big)\right| \leq (\nu/2)||f''||_\infty. \tag{32}$$

Before proving this lemma, let us show how we can deduce the proof of theorem 2.3. Using (31), (32) and (30) we get, for $j, k \geq n$

$$\Big|\int \big(\eta_j-\eta_k\big)(\lambda)f''(\lambda)d\lambda\Big| \leq \nu||f''||_{L^\infty(a,b)}, \qquad \forall\ f \in C_0^\infty(a,b). \tag{33}$$

Using the fact that η_l is supported in $(-C,+\infty)$ (thanks to (28)), we can suppose that $-C-1 \leq a < b$. Then, as every function $g \in C_0^\infty(a,b)$ can be written f'' on $(-C-1,+\infty)$ with $f \in C_0^\infty(\mathbb{R})$, we can replace f'' by g in (33), which proves that η_j is a Cauchy sequence using (29). As we already know that $\eta_j'' \to u_2$ in $\mathcal{D}'$ we get the result. □

Proof of lemma 5.2: we just prove (31), the proof of (32) being essentially the same. We want to use the proposition 3.3, so to study bounded operators whose difference is Hilbert-Schmidt we use the following trick: we choose a smooth non decreasing function I such that $I(\lambda) = \lambda$ on $[a-1, b+1]$ and which is constant on each connected component of $\mathbb{R} \setminus [a-2, b+2]$. Then we have $f \circ I = f$ for every $f \in C_0^\infty(a,b)$ so $f'(H_0+\epsilon V_l) = f'\big(I(H_0+\epsilon V_l)\big)$ for all $\epsilon \in [0,1]$ and $l \geq j_0$. Moreover, using the fact that $I(H_0+\epsilon V_l) = I_0(H_0+\epsilon V_l)$ for any function I_0 which coincides with I on $[-C,+\infty[$, we can replace I by a function I_0 verifying $I_0 - I(b+2) \in C_0^\infty(\mathbb{R})$. Then, we choose a function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ near $[a,b]$, so that we can write $\big(f'(H_{\epsilon,j})-f'(H_{\epsilon,k})\big)V_j = (34) + (35)$, where

$$\big(f'\big(I_0(H_{\epsilon,j})\big)-f'\big(I_0(H_{\epsilon,k})\big)\big)\,\phi\big(H_{\epsilon,j}\big)V_j \tag{34}$$

$$f'(H_{\epsilon,k})\big(\phi(H_{\epsilon,j})-\phi(H_{\epsilon,k})\big)V_j. \tag{35}$$

Using the functionnal calculus, it is easy to see that $\big(\phi(H_{\epsilon,j})-\phi(H_{\epsilon,k})\big)V_j$ is arbitrary small in trace norm, when j and k are big enough, uniformly with respect to ϵ, and by the trivial estimate $||f'|| \leq (b-a)||f''||$, we get the existence of $n > 0$ such that

$$|Tr\big((35)\big)| \leq \frac{\nu}{4}||f''||_\infty, \qquad \forall\ \epsilon \in [0,1],\ j \geq n,\ k \geq n.$$

As $\phi(H_{\epsilon,j})V_j$ stays in a bounded subset of $\mathbf{S_2}$, we can estimate $Tr\big((34)\big)$ by the Hilbert-Schmidt norm of the first factor of (34) thanks to propostion 3.3, that is

$$\begin{aligned} |Tr\big((34)\big)| &\leq ||f''||_\infty |||I_0(H_{\epsilon,j}) - I_0(H_{\epsilon,k})|||_{\mathbf{2}} |||\phi(H_{\epsilon,j})V_j|||_{\mathbf{2}} \\ &\leq \frac{\nu}{4}||f''||_\infty, \qquad \forall\, \epsilon \in [0,1],\ j \geq n,\ k \geq n \end{aligned}$$

for n big enough because $I_0(H_{\epsilon,j}) - I_0(H_0)$ is a Cauchy sequence in $\mathbf{S_2}$ uniformly with respect to ϵ (still by the functionnal calculus combined with the fact that $I_0 - \mathrm{const} \in C_0^\infty$). The proof is complete. □

5.2. **The assumption** (6)**.** The proof is based on the same idea as above, but to show the existence of κ, we need polynomial estimates. Precisely, we consider the same sequence of perturbations V_j, but we shall prove that, for some κ big enough, $(\lambda + C + 1)^{-\kappa}\eta_j(\lambda)$ is a Cauchy sequence in $L^1(-C, \infty)$ or, equivalently

$$\eta_j \circ \psi_\kappa^{-1} \text{ is a Cauchy sequence in } L^1(0,1), \quad \text{with } \psi_\kappa(\lambda) = (\lambda + C + 1)^{1-\kappa}.$$

To estimate the $L^1(0,1)$-norms we use that, for any $u \in L^1(0,1)$

$$\int_0^1 |u(t)|dt = \sup_{\hat f} \Big| \int_0^1 \hat f(t)u(t)dt\Big| = \sup_{\hat P} \Big| \int_0^1 \hat P(t)u(t)dt\Big| \tag{36}$$

where $\hat f$ (resp. $\hat P$) describes the set of continuous (resp. polynomial) functions with uniform norm on $[0,1]$ bounded by 1. We also need an analogous formula to (30); it is given by the following lemma in which, for any polynomial P, we denote by $\mathcal{P}$ its primitive such that $\mathcal{P}(0) = 0$. We also choose κ integer, such that

$$(\kappa - 2)m \quad > \quad d.$$

Lemma 5.3. *For any P we can write $\int_0^1 (\eta_j - \eta_k) \circ \psi_\kappa^{-1}(t)P(t)dt =$*

$$\int_0^1 Tr\Big(\big(R(H_{\epsilon,j}) - R(H_{\epsilon,k})\big)V_j\Big)d\epsilon + \int_0^1 Tr\Big(\big(R(H_{\epsilon,k}) - R(H_0)\big)(V_j - V_k)\Big)d\epsilon,$$

where $R(\lambda) = \mathcal{P} \circ \psi_\kappa(\lambda)$ and $j, k \geq j_0$.

Proof: By the assumption on κ and the functionnal calculus, it is clear that $\big(R(H_{\epsilon,j}) - R(H_0)\big)W$ is trace class, when $W \in Op_h^w\big(S_0(\omega, -\rho)\big)$. Then by the change of variable $t = \psi_\kappa(\lambda)$ we get

$$\int_0^1 (\eta_j - \eta_k) \circ \psi_\kappa^{-1}(t)P(t)dt = \int_{\mathbb{R}} (\eta_j - \eta_k)(\lambda)R'(\lambda)d\lambda,$$

and we get the result by the same computations that lead to (30). □

Now, the fact that $\eta_j \circ \psi_\kappa^{-1}$ is a Cauchy sequence (which completes the proof of theorem 2.3) is a consequence of the next lemma.

Lemma 5.4. *For all $\nu > 0$, there exists $n > 0$ such that, for all $j \geq n$, $k \geq n$, $\epsilon \in [0,1]$ and P we have*

$$\left|Tr\big(\big(R(H_{\epsilon,j}) - R(H_{\epsilon,k})\big)V_j\big)\right| \;\leq\; (\nu/2)||P||_{L^\infty(0,1)} \tag{37}$$

$$\left|Tr\big(\big(R(H_{\epsilon,k}) - R(H_0)\big)(V_j - V_k)\big)\right| \;\leq\; (\nu/2)||P||_{L^\infty(0,1)} \tag{38}$$

Proof: As before, we only prove (37). By the assumption on κ for which we may assume that $\kappa/2$ is integer, it is clear that $(H_{\epsilon,j} + C + 1)^{-\kappa/2}V_j$ describes a bounded subset of $\mathbf{S_2}$ when $\epsilon \in [0,1]$ and $j \geq j_0$, so we just have to estimate the Hilbert-Schmidt norm of $\big(R(H_{\epsilon,j}) - R(H_{\epsilon,k})\big)(H_{\epsilon,j} + C + 1)^{\kappa/2} = (39) + (40)$ with

$$f(H_{\epsilon,j}) - f(H_{\epsilon,k}), \qquad f(\lambda) = (\lambda + C + 1)^{\kappa/2}R(\lambda) \tag{39}$$

$$R(H_{\epsilon,k})\left((H_{\epsilon,k} + C + 1)^{\kappa/2} - (H_{\epsilon,j} + C + 1)^{\kappa/2}\right). \tag{40}$$

The Hilbert-Schmidt norm of (40) is clearly lower than the product
$\sup_{\lambda\geq -C}\left|(\lambda + C + 1)^{\kappa-1}\mathcal{P}\big((\lambda + C + 1)^{1-\kappa}\big)\right|\times$

$$|||(H_{\epsilon,k} + C + 1)^{1-\kappa}\big((H_{\epsilon,k} + C + 1)^{\kappa/2} - (H_{\epsilon,j} + C + 1)^{\kappa/2}|||_{\mathbf{2}} \tag{41}$$

with (41) arbitrary small, uniformly with respect to $\epsilon \in [0,1]$, when j and k are big enough, again by the functionnal calculus. Moreover $\sup_{\lambda\geq -C}|\cdots| = \mathcal{O}(||P||_\infty)$ so we can get the estimate (37) for (40) with $\nu/4$ instead of $\nu/2$. Concerning the term (39), the proposition 3.4 proves, with $n_0 = \kappa/2$, that $|||(39)|||_{\mathbf{2}}$ is lower than $(||F_1||_\infty + ||F_2||_\infty)\times$

$$|||\epsilon(V_j - V_k)(H_{\epsilon,j} + C + 1)^{-\kappa/2}|||_{\mathbf{2}} + |||(H_{\epsilon,k} + C + 1)^{-\kappa/2}\epsilon(V_k - V_j)|||_{\mathbf{2}}. \tag{42}$$

This completes the proof because (42) can be made as small as we want (for $j, k \geq j_0 >> 1$), and because $||F_1||_\infty + ||F_2||_\infty = \mathcal{O}(||P||_{L^\infty(0,1)})$. □

6. Euclidean scattering

This section is the most technical part of this paper, as it is the proof of the theorem 2.5. Before starting the proof itself, we recall and define some notations.

$$\begin{aligned} R_\epsilon(z,h) &= (H_\epsilon - z)^{-1}, \\ E'_\epsilon(\lambda,h) &= (2i\pi)^{-1}\big(R_\epsilon(\lambda + i0,h) - R_\epsilon(\lambda - i0,h)\big) \end{aligned}$$

the last formula being the usual Stone's formula for the derivative of the spectral resolution associated to H_ϵ. We note the propagator of H_ϵ by

$$U_\epsilon(t) = U_\epsilon(t,h) = \exp\left(-i\frac{t}{h}H_\epsilon\right).$$

When the subscript ϵ doesn't appear, it always means that $\epsilon = 1$.

We want to prove asymptotic expansions, so we shall have to deal with remainders; if $\mathcal{R}$ is an operator on $L^2(\mathbb{R}^d)$, depending on the parameter $h \in (0,\bar{h}]$, we will write

$$\mathcal{R} = \mathcal{O}_{\mathbf{1},\mathbf{x}}(N) \tag{43}$$

if there exists $C_N > 0$ such that, for all $h \in (0, \bar{h}]$, we have in trace norm

$$|||<x>^N \mathcal{R} <x>^N |||_{\mathbf{1}} \leq C_N h^N, \tag{44}$$

and of course, $\mathcal{R} = \mathcal{O}_{\mathbf{1,x}}(\infty)$ means that $\mathcal{R} = \mathcal{O}_{\mathbf{1,x}}(N)$ for all N.

6.1. **The asymptotic representation formula.** Let's consider $I \subset\subset (0, +\infty)$ an open non critical interval for ω, $J \subset\subset I$ and let $\Omega \in C_0^\infty(\mathbb{R}^d)$ be such that, in a neigbourhood of $\omega^{-1}(J)$ we have

$$\Omega(\xi) = \sqrt{\frac{\omega(\xi)}{|\nabla\omega(\xi)|^2}},$$

and define $\mathcal{D}$ a *conjugate operator* associated with H_0, by

$$\mathcal{D} = \Omega(hD)\big(x.\nabla\omega(hD) + \nabla\omega(hD).x\big)\Omega(hD).$$

By simple computations, we have $[\mathcal{D}, H_0] = 2ihH_0 + 2ihq(hD)$ with $q \equiv 0$ near $\omega^{-1}(J)$, and therefore

$$[\mathcal{D}, H_0]f(H_0) = 2ihH_0 f(H_0)$$

for all $f \in C_0^\infty\big(J\big)$. This remark will be the starting point of the proof of the next proposition which is itself the base of the proof of the theorem 2.5. The aim of the following proposition is to give a formula for $u_2 = \eta''$ (in the weak sense) in which no term with $d/d\epsilon$ appears, as for the spectral shift function.

Proposition 6.1. *Let χ be a smooth function compactly supported on $\mathbb{R}^+$ such that $\chi(\lambda) = \lambda^{-1}$ on J. Then there exists $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that, for all $f \in C_0^\infty(J)$,*

$$\begin{aligned} <\eta'', f> &= Tr\,(\mathcal{A}(h)f(H_0)) + Tr\,\big(\mathcal{B}(h)\big(f(H_1) - f(H_0)\big)\big) \\ &\quad + Tr\,(\mathcal{C}(h)f(H_1)) \; + \; Tr\,(\mathcal{R}(h)f(H_1)), \end{aligned} \tag{45}$$

where, if we set $W = V - (2ih)^{-1}[\mathcal{D}, V]$, we have

$$\mathcal{A}(h) = W\big(\chi(H_1) - \chi(H_0)\big), \qquad \mathcal{B}(h) = W\chi(H_1)$$
$$\mathcal{C}(h) = q(hD)\zeta\big(\chi(H_1) - \chi(H_0)\big), \qquad \mathcal{R}(h) = \mathcal{O}_{\mathbf{1,x}}(\infty).$$

Proof: By continuity of $f(H_0 + V)$ with respect to V, it is sufficient to prove the proposition when V is a ρ'−perturbation, with $\rho' > d$, what we assume from now. Moreover, as in [22] (see also [2] for details) it is not hard to prove the heuristically trivial formula:

$$Tr\big([\mathcal{D}, H_0 + V]f(H_0 + V) - [\mathcal{D}, H_0]f(H_0)\big) = 0,$$

(notice that this makes sense because $\rho' > d$) so that we get

$$\begin{aligned} Tr\big(f(H_0 + V) - f(H_0)\big) &= Tr\big(W\chi(H_0 + V)f(H_0 + V)\big) \\ &\quad + \quad Tr(q(hD)\chi(H_0 + V)f(H_0 + V)). \end{aligned} \tag{46}$$

If we replace V by ϵV we can compute the derivative which gives

$$Tr\big(\frac{d}{d\epsilon}f(H_0+\epsilon V)_{|\epsilon=0}\big) = Tr\big(W\chi(H_0)f(H_0)\big).$$

Then we get the result if we choose $\zeta \in C_0^\infty(\mathbb{R}^d)$ which is $\equiv 1$ on a large enough ball so that $(1-\zeta)q(hD)F_0(H_0+Q) = \mathcal{O}_{\mathbf{1},\mathbf{x}}(\infty)$, which is possible with $F_0 \in C_0^\infty(I)$, $F_0 \equiv 1$ near J, because then $(1-\zeta)q(hD)$ and the symbols of the expansion of $F_0(H_0+Q)$ will have disjoint supports. □

The theorem 2.5 is a simple consequence (as in [21]) of the following one, whose next subsection is devoted to the proof.

Theorem 6.2. *There exist sequences of symbols $a_j^\pm \in S_1(-\infty,-2\rho)$ and $b_j^\pm \in S_1(-\infty,-\rho)$, supported in $\omega^{-1}(I)$ with respect to ξ, such that for all $N>0$, we can write*

$$\begin{aligned}\eta''(\lambda,h) &= \sum_{j=0}^{N} Tr\left(a_j^\pm(x,hD)R_0(\lambda\pm i0,h)\right) + Tr\left(\zeta b_j^\pm(x,hD)E'(\lambda,h)\right)\\ &\quad + Tr\left(\mathcal{R}_{0,N}^\pm(h)R_0(\lambda\pm i0,h)\right) + Tr\left(\mathcal{R}_N(\lambda,h)R(\lambda\pm i0,h)\right)\end{aligned} \tag{47}$$

with $\mathcal{R}_{0,N}^\pm(h) = \mathcal{O}_{\mathbf{1},\mathbf{x}}(N/4)$ and $\mathcal{R}_N(\lambda,h) = \mathcal{O}_{\mathbf{1},\mathbf{x}}(N/4)$ uniformly with respect to $\lambda \in J$. (We use the convention that $Tr(A^\pm) = Tr(A^+) + Tr(A^-)$.)

6.2. **Proof. A. Reduction of the problem.** By the proposition 6.1, we can write

$$\eta''(\lambda,h) = u_A(\lambda,h) + u_B(\lambda,h) + u_C(\lambda,h) + u_R(\lambda,h)$$

with obvious notations.

1. $u_R(\lambda,h) = Tr\big(\mathcal{O}_{\mathbf{1},\mathbf{x}}(\infty)E'(\lambda,h)\big)$ so it is clearly $\mathcal{O}(h^\infty)$ in $C^\infty\big(J\setminus\sigma_{pp}(H)\big)$ and thus we don't have to study it.
2. The first term u_A is quite explicit, and using the asymptotic expansion of $\mathcal{A}(h)$ it is not hard to see that it is smooth on J and has the expected expansion (see [21]).
3. The term u_C is clearly smooth on $J\setminus\sigma_{pp}(H)$, with again the right expansion; such distributions were studied for example in [20] where they were called local spectral densities.

This discussion shows that $u_B(\lambda,h) = Tr\big(\mathcal{B}(h)\big(E_1'(\lambda) - E'(\lambda)\big)\big)$, which must be understood in the weak sense, is the only term that we have to consider. Moreover, if θ is a cut-off function on $\mathbb{R}^d$, we can write $\mathcal{B}(h) = \theta\mathcal{B}(h) + (1-\theta)\mathcal{B}(h)$. The part corresponding to $\theta\mathcal{B}(h)$ is the sum of a local spectral densitie and of a term analogous to u_A, so we just have to focus on $Tr\big((1-\theta)\mathcal{B}(h)\big(E'(\lambda,h) - E_0'(\lambda,h)\big)\big)$, and even , up to remainder terms which are smoother and smoother because of the pseudodifferential expansion of $\mathcal{B}(h)$, we just have to study distributions like

$$Tr\left(c(x,hD)\big(E'(\lambda,h) - E_0'(\lambda,h)\big)\right) \tag{48}$$

where $c \in S_1(-\infty, -\rho)$ is supported in $\{|x| \geq R\} \times \{\omega(\xi) \in \tilde{J}\}$, by choosing $\theta \equiv 1$ on the ball of radius R with R large enough. Here $\tilde{J}$ is an interval such that $J \subset\subset \tilde{J} \subset\subset I$. This will allow us to do a partition of unit in *outgoing/incoming areas* and then use Isozaki-Kitada's constructions; these areas, $\Gamma^{\pm}(R_0, J_0, \sigma_{\pm})$, are defined as the following sets

$$\{(x,\xi) \in \mathbb{R}^{2d} \mid |x| > R_0,\ \omega(\xi) \in J_0,\ \pm\cos(x, \nabla\omega(\xi)) \geq \sigma_{\pm}\}$$

where $\cos(x,y) = < x, y > /|x|.|y|$. $+$ is for outgoing, $-$ for incoming, $\sigma_{\pm}$ are real numbers in $(-1,1)$, and J_0 is an interval $\subset\subset I$.

We do a partition of unit by choosing two functions χ_+ and χ_- in $S_1(-\infty,0)$ supported respectively in $\Gamma^+(R_1, J_1, \sigma_+)$ and $\Gamma^-(R_1, J_1, \sigma_-)$ such that $\chi_+ + \chi_- = 1$ near the support of c, with $\tilde{J} \subset\subset J_1 \subset\subset I$, and $\sigma_{\pm} \in (-1,1)$. As $\big(1 - \chi_+(x,hD) - \chi_-(x,hD)\big)c(x,hD) = \mathcal{O}_{\mathbf{1},\mathbf{x}}(\infty)$, the study of (48) will be deduced from the one of

$$Tr\,\big(c(x,hD)\big(E'(\lambda,h) - E_0'(\lambda,h)\big)\chi_{\pm}(x,hD)\big)$$

or equivalently to the one of its semiclassical Fourier transform, that is

$$Tr\,\big(c(x,hD)\big(U(t,h) - U_0(t,h)\big)\chi_{\pm}(x,hD)\big)\,, \qquad t \in \mathbb{R}. \tag{49}$$

We study only the case of $+$ ($-$ is analogous). Then we remark that we can suppose $t \geq 0$, just because the study when $t \leq 0$ can be reduced to this case thanks to the formula $Tr(T^\star) = \overline{Tr(T)}$ (see [20]). It is important that we may assume $t \geq 0$ because, we can now use Isozaki-Kitada's parametrix, that is

$$U(t)\chi_+(x,hD) = A_N(h)U_0(t)B_N(h)^\star + \mathcal{R}_N(h,t).$$

$A_N(h) = \sum_{j\leq N} h^j J_{\varphi_+}(a_j,h)$ and $B_N(h) = \sum_{j \leq N} h^j J_{\varphi_+}(b_j,h)$ where $J_{\varphi_+}(a,h)$ is operator whose kernel is

$$(2\pi h)^{-d}\int e^{i(\varphi_+(x,\xi) - <y,\xi>)/h} a(x,\xi)d\xi.$$

They are Fourier integral operators associated to the time independant phase function φ_+, where $\varphi_+ \in C^\infty(\mathbb{R}^{2d},\mathbb{R})$ solves the *Hamilton-Jacobi equation* (50)

$$\big(\omega + v_0\big)(x, \partial_x\varphi_+(x,\xi)) = \omega(\xi), \qquad \forall\ (x,\xi) \in \Gamma^+(R_0, J_0, \sigma_0) \tag{50}$$

with $\Gamma^+(R_0, J_0, \sigma_0)$ neighbourhoodof $\Gamma^+(R_1, J_1, \sigma_1)$; it satisfies the following estimates on $\mathbb{R}^{2d}$:

$$|\partial_x^\alpha \partial_\xi^\beta \big(\varphi_+(x,\xi) - < x, \xi >\big)| \leq C_{\alpha\beta} < x >^{1-\rho-|\alpha|}\,. \tag{51}$$

The symbols of A_N and B_N are constructed so that in particular we have

$$A_N(h)B_N(h)^\star = \chi_+(x,hD) + \mathcal{O}_{\mathbf{1},\mathbf{x}}(N/4).$$

and the remainder $\mathcal{R}_N(h,t)$ has the following form

$$h^{N+1}U(t)r_N(x,hD,h) + \int_0^t U(t-s)\mathcal{J}_N(s,h)ds$$

with $r_N(h)$ in a bounded subset of $S_1(-\infty,-N)$ and $(1+s)^{N/4}\mathcal{J}_N(s,h)=\mathcal{O}_{\mathbf{1},\mathbf{x}}(N/4)$, the right hand side being uniform with respect to $s\geq 0$. Thus, in our analysis of (49) we are left to the study, when $t\geq 0$, of

$$Tr\big(c(x,hD)\big(A_N(h)U_0(t)B_N(h)^\star - U_0(t)A_N(h)B_N(h)^\star\big)\big). \tag{52}$$

B. The proof itself. The result will then follow from a carefull study of the symbols of $A_N(h)$ and $B_N(h)$; that's why we recall their constructions in the appendix (see [21] for other details from the semiclassical point of view, and the original paper [11]).

Heuristically, the method is very close to a trick used in [21] and simple; we do as if we could split the trace of the difference into the difference of traces in (52), and then thanks to cyclicity we write it as

$$Tr\left(\big(B_N(h)^\star c(x,hD)A_N(h) - A_N(h)B_N(h)^\star c(x,hD)\big)U_0(t)\right). \tag{53}$$

This actually works because, by the Egorov's theorem (in its semiclassical version)

$$\big(B_N(h)^\star c(x,hD)A_N(h) - A_N(h)B_N(h)^\star c(x,hD)\big) = \tilde c(x,hD,h) \tag{54}$$

is a pseudodifferential which is of trace class as its symbol $\tilde c(h)\in S_1(-\infty,-2\rho)$. Then the conclusion follows as in [21]. So we just have to prove the following lemma.

Lemma 6.3. *Assume that $v_k\in S_1(\omega,-\rho-1)$ when $k\geq 1$, then*

$$(52) = Tr\big(\tilde c(x,hD,h)U_0(t)\big).$$

Proof: Let's define $c_n(x,\xi)=\exp(-x^2/n)c(x,\xi)$; it is clear that in $\mathcal{D}'(0,+\infty)$, (52) is the limit, as $n\nearrow+\infty$, of $(52)_n$ which is the same distribution as (52) with c_n instead of c. But then, $c_n(x,hD)$ is of trace class, so we can use the cyclicity of trace, and thus

$$(52)_n = Tr\big(\tilde c_n(x,hD,h)U_0(t)\big)$$

with $\tilde c_n(h)$ defined by replacing c by c_n in (54). As c_n belongs to the Schwartz class, it is clear that $\tilde c_n(h)$, defined by the semiclassical Egorov's theorem, is also in the Schwartz class, but the non trivial and important point is that $\tilde c_n(x,hD,h)$ converges to $\tilde c(x,hD,h)$ in trace norm. Indeed the assumption on v_k implies that $a_k\in S_1(-\infty,-\rho)$ when $k\geq 1$ so, as c_n converges in $S_1(-\infty,-\rho')$, for all $\rho'<\rho$, this implies that the symbols of

$$B_N(h)^\star c_n(x,hD)J_{\varphi_+}(a_k,h),\quad J_{\varphi_+}(a_k,h)B_N(h)^\star c_n(x,hD)$$

converge in $S_1(-\infty,-\rho-\rho')$ and hence the operators converge in trace norm. Then, if use the remark on a_0-1 of the appendix, we just have to consider the symbols of

$$J_{\varphi_+}(b_j,h)^\star c_n(x,hD)J_{\varphi_+}(1,h) - J_{\varphi_+}(1,h)J_{\varphi_+}(b_j,h)^\star c_n(x,hD) \tag{55}$$

when $j\geq 0$. By the lemma of the appendix, modulo operators that converge in trace norm for the same reasons as above, we can replace b_j by $\tilde\chi^\star_{+,j}$ everywhere. Moreover, using the properties of φ_+ it is easy to see that $J_{\varphi_+}(b_j,h)^\star J_{\varphi_+}(1,h) - b_j(x,hD)^\star$ is a pseudodifferential operator with symbol in $S_1(-\infty,-\rho)$. In the same way, the symbol of $J_{\varphi_+}(b_j,h)^\star c_n(x,hD)J_{\varphi_+}(1,h)$ is the one of $b_j(x,hD)^\star c_n(x,hD)$ modulo a convergent family in $S_1(-\infty,-\rho-\rho')$. This proves that (55) has a symbol which converges in $S_1(-\infty,-\rho-\rho')$, for all $\rho'<\rho$, which entails the result. □

Remark that the assumption $v_k \in S_1(-\infty,-\rho-1)$ for $k\geq 1$ is not very important as we can always assume that this true if we replace the principal symbol $\omega+v_0$ by $\omega+v_0+\cdots+h^jv_j$ with j big enough. Then the non trapping assumption is still verified for h small enough, if it holds for $\omega+v_0$ and the only difference will be that the phase function φ_+ is going to depend smoothly on h, which doesn't change the final result. Anyway, we shall study precisely such a dependance in [5].

6.3. **Connection with the scattering matrix.** In this part, we are going to prove the point *ii*) of theorem 2.5. As several times before, we'll get the result by approximating the perturbation by fast decaying ones (with respect to the space variable); the idea is still simple, but the justification of all formulas need some technical estimates. First let's look at the principle.

Assume that $V_j=\chi(x/j)V\chi(x/j)$ and let $S_j(\lambda,h)$ be the scattering matrix of (H_0,H_0+V_j) . It is a trace class perturbation of the identity on $L^2(\Sigma_\lambda)$, so we can write $Det_2\ S_j(\lambda,h) = Det\ S_j(\lambda,h)\exp\big(-Tr\big(S_j(\lambda,h)-1\big)\big)$ and by the Krein's formula, we get, with θ_j denoting the usual spectral shift function

$$\begin{aligned} Det_2\ S_j(\lambda,h) &= \exp\left(2i\pi\theta_j(\lambda,h)\right)\exp\big(2i\pi\big(Tr(A_j(\lambda,h))+Tr(B_j(\lambda,h))\big)\big) \\ &= \exp\big(2i\pi\eta_j'(\lambda,h)\big)\exp\big(-2i\pi Tr\big(B_j(\lambda,h)\big)\big) \end{aligned} \tag{56}$$

using formula (27) and the fact that, by cyclicity of the trace, we have

$$Tr\big(A_j(\lambda,h)\big)=Tr\big(E_0'(\lambda,h)V_j\big).$$

To let j go to infinity in (56), we need the two following lemmas

Lemma 6.4. *If $\rho>(d+1)/2$, then $A_j(\lambda,h)$ converges to $A(\lambda,h)$ in the Hilbert-Schmidt class, and $B_j(\lambda,h)$ converges to $B(\lambda,h)$ in the trace class. In particular, we have pointwise convergence of $Det_2\ S_j(\lambda,h)$ to $Det_2\ S(\lambda,h)$.*

Lemma 6.5. *$\eta_j''(.,h)$ remains in a bounded subset of $C^0(I,\mathbb{C})$.*

Proof of *ii*)**.** By the last lemma, and the fact that $\eta_j'(.,h)$ converges in the distribution sense, we can extract a subsequence that converges pointwise to $\eta'(.,h)$, using the Ascoli-Arzela's theorem. Then using the other lemma, we get the result just by letting $j\nearrow\infty$ in equality (56). □

Proof of lemma 6.4**.** That A_j and B_j are respectively of Hilbert-Schmidt and trace classes is just du to the fact that when $s>d/2$, $\Xi_\lambda <x>^{-s}$ is an Hilbert-Schmidt operator from $L^2(\mathbb{R}^d)$ to $L^2(\Sigma_\lambda)$ because

$$Tr\big(< x >^{-s} \Xi_\lambda^\star \Xi_\lambda < x >^{-s} \big) = (2\pi h)^{-d} \int < x >^{-2s} dx \int_{\Sigma_\lambda} d\sigma_\lambda < +\infty.$$

So the result will be a consequence of the convergence of $(H_0 + V_j - \lambda - i0)^{-1}$ to $(H_0 + V - \lambda - i0)^{-1}$. Actually it is sufficient to prove that, for any $s > 1/2$, and any $\lambda_0 \in I \setminus \sigma_{pp}(H_0 + V)$, there exists a neighbourhood$U$ of λ_0 such that

$$\sup ||| < x >^{-s} (H_0 + V_j - \lambda \pm i\delta)^{-1} < x >^{-s} ||| < +\infty \tag{57}$$

the sup being taken over $\delta \in (0, 1]$, $\lambda \in U$ and $j \geq j_0$. Using Mourre's technique [16], we get the existence of U_1, neighbourhoodof λ_0, and $c > 0$ such that, for all $f \in C_0^\infty(U_1)$, we have $f(H_0 + V)i[\mathcal{D}, H_0 + V]f(H_0 + V) \geq cf^2(H_0 + V)$, if $\mathcal{D}$ is the conjugate operator defined in the beginning of the section. Using the continuity with respect to V of the functionnal calculus, it is then easy to show that, for $j \geq j_0$ large enough, and $f_0 \equiv 1$ in the neighbourhoodof λ_0, we have the following *Mourre estimate*, uniform with respect to j:

$$f_0(H_0 + V_j)i[\mathcal{D}, H_0 + V_j]f_0(H_0 + V_j) \geq \frac{c}{2} f_0^2(H_0 + V_j)$$

which in particular proves that $\sigma_{pp}(H_0 + V_j) \cap U = \emptyset$, with $U = f_0^{-1}(1)$, using the virial theorem. Then by reviewing the proof of [16], uniformly with respect to the parameter j we get that the sup is finite and this entails the result. □

Proof of lemma 6.5**.** The proof is a consequence of formula (47); as we already know that the estimate (57) holds, we just have to show that the pseudodifferential operators, depending on V_j, have symbols uniformly bounded in their classes, when $j \geq j_0$. To prove that, what we have esssentially to do, is to check that the phases functions $\varphi_{+,j}$ solving the Hamilton-Jacobi equation (for $\omega + v_{j,0}$, $v_{j,0}$ being the principal symbol of V_j) satisfy estimates as (51), uniform with respect to j, in an outgoing area independant on j. This control comes on the same estimates on classical trajectories, proven in [7], but uniform with respect to j. Once we have these estimates we can review the construction of phase functions as in [7] and we get (51) for $\varphi_{+,j}$ with bounds uniform with respect to j; therefore, up to the following lemma, the proof is complete. □

The next lemma says that if v_0, the principal symbol of V ranges over $\mathcal{B}$ a bounded subset of $S_1(\omega, -\rho)$, with $\rho > 0$, and such that $(\omega + v_0)(x, \xi) + C \geq C(1 + \omega(\xi))$ with C uniform on $\mathcal{B}$, then we have a uniform on the hamiltonian flow of $\omega + v_0$ in outgoing (and incoming, analogously) areas independant on v_0. This is the first and main step in studying the dependance of the phase functions with respect to the perturbation, which will be done in details in a later paper [5] (see also [4]). Let's write

$$\Phi^t(x, \xi) = \big(z(t, x, \xi), \zeta(t, x, \xi)\big)$$

the hamiltonian flow of $\omega + v_0$, then we have the following result.

Lemma 6.6. *Let $I \subset (0,\infty)$ be a non critical interval for ω and let $\sigma \in (-1,1)$. Then for all $J \subset\subset I$, there exists $R>0$, $e_0>0$ and $c_1>0$ such that*

$$\begin{aligned} |z(t,x,\xi)| &\geq e_0|x|, \\ |\zeta(t,x,\xi)-\xi| &\leq c_1 <x>^{-\rho}, \\ |z(t,x,\xi)-x-t\nabla\omega(\xi)| &\leq c_1 t <x>^{-\rho} \end{aligned}$$

for all $(x,\xi)\in\Gamma^+(J,R,\sigma)$, $t\geq 0$ and $v_0 \in \mathcal{B}$. Moreover, for all δ small enough we have

$$\big(z(t,x,\xi),\zeta(t,x,\xi)\big) \in \Gamma^+(J+(-\delta,\delta), e_0R, \sigma-\delta).$$

The proof of this lemma is exactly the same as in [7] (see also [2]), and the control with respect to v_0 doesn't cause any trouble.

7. A generalized Levinson's formula

In this section we give the proof of the theorem 2.7. The principle of that proof is exactly the same as the one of Guillopé [9] (see also Colin-de-Verdière [6] and Robert [21]); as the asymptotic expansion of η'' is established (theorem 2.5), the main difficulty consists in computing the jump of η' at 0, $\eta'(+0)-\eta'(-0)$. In our case, we are going to prove that η' is continuous at 0 when 0 is *regular* in the spectrum of $-\Delta+V$ by studying the relations between η and a relative scattering determinant. Before doing it, we recall the strategy of the proof of Levinson's formula, which is based on the study of the *generalized zeta function* introduced in section 4:

$$\zeta(s,E) = \int_{\mathbb{R}} \eta(\lambda)\frac{d^2}{d\lambda^2}(\lambda-E)^{-s}d\lambda.$$

If η is C^1 in a neighbourhoodof 0, then we have, after two integrations by parts

$$\zeta(s,E) = \int_0^{+\infty} \eta''(\lambda)(\lambda-E)^{-s}d\lambda + \sum_{\sigma_{disc}}(\lambda-E)^{-s} \tag{58}$$

(because η' is clearly the counting function of eigenvalues on $(-\infty,0)$). Moreover, using the theorem 2.5, we have an expansion, for all complex number E

$$\eta''(\lambda) \sim \sum_{k\geq 2}\Gamma_k(E)(\lambda-E)^{3/2-k-1}, \qquad \lambda \nearrow +\infty$$

where the coefficients $\Gamma_k(E)$ are holomorphic with respect to E; we use this expansion to continue meromorphically $\zeta(s,E)$ with respect to s by writing (58) as follows

$$\int_0^{+\infty} \big(\eta''(\lambda) - \sum_{k=2}^{l} \Gamma_k(E)(\lambda - E)^{3/2-k-1}\big)(\lambda - E)^{-s} d\lambda \tag{59}$$

$$+ \quad \sum_{k=2}^{l} \Gamma_k(E) \frac{(-E)^{3/2-k-s}}{k+s-3/2} \quad + \quad \sum_{\sigma_{disc}} (\lambda - E)^{-s}$$

so that we can evaluate it when $-z \in \mathbb{N}$ and $l = -z + 1$: in that case (59) = 0 by the proposition 4.2. Finally, as (59) is clearly holomorphic with respect to E when E is in the upper half plane (and not only in $\Re(E) < E_0$, see section 4), we can let $E \to 0$ and thus, the theorem 2.7 is proved.

This discussion shows that the proof of the theorem 2.7 just relies on the following proposition:

Proposition 7.1. *Assume that* 0 *is regular (see* [12] *and below) in the spectrum of* $-\Delta + V$*, where* V *satisfies, for some* $\rho > 5/2$*,*

$$|\partial_x^\alpha V(x)| \leq C_\alpha < x >^{-\rho-|\alpha|}, \qquad x \in \mathbb{R}^d.$$

Then there exists $\lambda_0 > 0$ *such that* η *is* C^1 *on* $(-\lambda_0, \lambda_0)$.

By Jensen-Kato [12], we know that $< x >^{-\nu} R_0(z) < x >^{-\nu'}$, defined for $\Im(z) > 0$, has a continuous extension to $\Im(z) \geq 0$ if $\nu, \nu' > 1/2$ and $\nu + \nu' > 2$. In particular, when z goes to 0, which the reason why we have to suppose $\nu + \nu' > 2$, we get $< x >^{-\nu} G_0 < x >^{-\nu'}$ where G_0 is the operator whose kernel is $\big(4\pi|x-y|\big)^{-1}$. 0 is said to be *regular* if $1+ < x >^{\nu} VG_0 < x >^{-\nu}$ is invertible on $L^2(\mathbb{R}^3)$, when $1/2 < \nu < \rho - 1/2$. If this is verified, it implies that

$$Det_2\big(1 + V(-\Delta - z)^{-1}\big) \neq 0 \ \text{ near } z = 0, \ \ \Im(z) \geq 0. \tag{60}$$

Proof of the proposition: First we observe that if V belongs to C_0^∞ and 0 is regular, then, using the results of [6] and [9], we know that the Birman-Krein function θ (associated to $-\Delta, -\Delta + V$) is continuous in a neighbourhoodof 0, thus

$$\eta'(\lambda) = \theta(\lambda) + \lambda_+^{1/2} \times (2\pi)^{-3} \text{mes}(S^2) \int V(x)dx, \qquad \lambda_+ = \max(0, \lambda)$$

is continuous near 0 and we get the result in this case.

Now consider a general V as in the assumption and let $\chi_j(x) = \chi(x/j)$, $\chi \in C_0^\infty$, be real valued, $\chi \equiv 1$ on the unit ball, and consider the sequence $V_j = \chi_j V$. For j big enough, 0 is then clearly regular for $-\Delta + V_j$, and as V has only finitely many negative eigenvalues, there exists $\lambda_0 > 0$ such that $\sigma(-\Delta + V_j) \cap [-\lambda_0, 0) = \sigma(-\Delta + V) \cap [-\lambda_0, 0) = \emptyset$ for all j big enough. Then the proof of the proposition will follow from the next two lemmas.

Let η_j be the Koplienko's function of the pair $(-\Delta, -\Delta + V_j)$.

Lemma 7.2. *There is a real number φ_j such that*

$$\eta_j'(\lambda) = -\frac{1}{\pi}\arg Det_2\big(1 + V_j R_0(\lambda + i0)\big) + \varphi_j \qquad \forall\ |\lambda| \le \lambda_0.$$

Lemma 7.3. $\big(\arg Det_2\big(1+V_jR_0(\lambda+i0)\big)\big)_j$ *is a Cauchy sequence in $C^0(-\lambda_0,\lambda_0)$, the space of continuous functions on $[-\lambda_0,\lambda_0]$.*

End of proof of the prop.: By lemma 7.3, and the fact that η_j' converges to η' in $\mathcal{D}'$ (because, for example, η_j converges in L^1_{loc} by section 5), we get that φ_j is a convergent sequence, so η_j' is a Cauchy sequence in $C^0(-\lambda_0,\lambda_0)$ and thus η is C^1 near 0. □

In what follows, we set

$$\mathcal{A}_j(z) = V_j R_0(z).$$

Proof of lemma 7.2**:** The branch of $\arg Det_2\big(1+\mathcal{A}_j(z)\big)$ is determined by the condition $\arg Det_2\big(1+\mathcal{A}_j(\lambda+i\delta)\big) \to 0$ when $\delta \to +\infty$. Notice that, by the regularity of 0 and the limiting absorption principle, $Det_2\big(1+A_j(z)\big)$ doesn't vanish for $\Re(z) \in [-\lambda_0,\lambda_0]$, $\Im(z) \ge 0$, and thus $\arg Det_2\big(1+\mathcal{A}_j(\lambda+i0)\big)$ is well defined and continuous on $[-\lambda_0,\lambda_0]$. So, as η_j' is continuous on $[-\lambda_0,\lambda_0]$ too, we just have to show that

$$\eta_j''(\lambda) - \frac{d}{d\lambda}\left(-\frac{1}{\pi}\arg Det_2\big(1+\mathcal{A}_j(\lambda+i0)\big)\right) = 0 \tag{61}$$

on $(-\lambda_0,\lambda_0)$ in the distributions sense. To compute more easily the argument, we approximate V_j by trace class perturbations: we choose a sequence $V_j^k \in \mathcal{S}(\mathbb{R}^3\times\mathbb{R}^3)$ such that

$$V_j^k(x,D)\big(-\Delta+1\big)^{-1} \ \to\ V_j\big(-\Delta+1\big)^{-1}, \qquad \text{in } \mathbf{S_2} \tag{62}$$

$$\eta_{j,k}'' \ \to\ \eta_j'', \qquad \text{in } \mathcal{D}' \tag{63}$$

when $k \nearrow +\infty$, and where $\eta_{j,k}$ is the Koplienko's function of $(-\Delta, -\Delta + V_j^k(x,D))$. According to Yafaev [26], the Birman-Krein function $\theta_{j,k}$ satisfies

$$\theta_{j,k}(\lambda) = \lim_{\delta\searrow 0} -\frac{1}{\pi}\arg Det_1\big(1+\mathcal{A}_j^k(\lambda+i\delta)\big), \qquad a.e. \text{ on } \mathbb{R} \tag{64}$$

if $\mathcal{A}_j^k(z) = V_{j,k}(x,D)R_0(z)$. Moreover, by the Birman-Solomyak's formula, combined with Stone's formula, we have also

$$Tr\big(\frac{d}{d\epsilon}f(-\Delta+\epsilon V_j^k(x,D))_{|\epsilon=0}\big) = \int_{\mathbb{R}} \frac{1}{\pi}\Im\left(Tr\big(\mathcal{A}_j^k(\lambda+i0)\big)\right) f'(\lambda)d\lambda, \tag{65}$$

for all $f \in C_0^\infty(-\lambda_0,\lambda_0)$, and as $\arg Det_2\big(1+\mathcal{A}_j^k(\lambda+i0)\big)$ equals

$$\arg Det_1\big(1+A_{j,k}(\lambda+i0)\big) - \Im\left(Tr\big(A_{j,k}(\lambda+i0)\big)\right),$$

(64) and (65) show that (61) is true if we replace V_j by V_j^k (use the formula (27) and remark that $\pi^{-1}\Im\left(Tr\left(A_{j,k}(\lambda+i0)\right)\right) = Tr\left(V_{j,k}(x,D)E_0'(\lambda)\right)$). Then using (62) and (63) we can let k go to ∞ and we get the lemma. $\square$

Proof of lemma 7.3**:** First we can write $\arg Det_2\left(1+\mathcal{A}_j(\lambda+i0)\right)$ as follows

$$\arg Det_p\left(1+\mathcal{A}_j(\lambda+i0)\right) - \sum_{k=2}^{p-1}\frac{(-1)^k}{k}\Im\left(Tr\left(\mathcal{A}_j(\lambda+i0)^k\right)\right)$$

which is just due to basic properties of Fredholm determinants (see for example the chapter 1 of [26]). Notice that, everything is well defined thanks to cyclicity properties of the trace and determinants because everywhere $\mathcal{A}_j(\lambda+i0)$ can be replaced by $<x>^\nu \mathcal{A}_j(\lambda+i0) <x>^{-\nu}$, in which we may even write $<x>^\nu V_j$ as $<x>^\nu V_j <x>^{\nu-\nu}$. Here ν is a real number >1 such that $\rho-\nu>3/2$ and p an integer such that $\rho-2\nu>3/p$. With such a choice, $<x>^\nu V_j <x>^\nu$ converges in $L^p(\mathbb{R}^3)$, so $<x>^\nu \mathcal{A}_j(.+i0) <x>^{-\nu}$ is a Cauchy sequence in $C^0\left([-\lambda_0,\lambda_0],\mathbf{S_p}\right)$. Thus we are left to study of the sum $\sum_2^{p-1}$ in which each term clearly vanishes when $\lambda<0$. To study them when $\lambda\geq 0$, we remark that

$$\mathcal{A}_j(z)^k - \mathcal{A}_j(\overline{z})^k = \sum_{l=0}^{k-1}\mathcal{A}_j(z)^l V_j\left(R_0(z)-R_0(\overline{z})\right)\mathcal{A}_j(\overline{z})^{k-l-1}$$

which implies that there exists a constant $C>0$, independant on j and λ such that $|\Im\left(Tr(\mathcal{A}_j(\lambda+i0)^k)\right)|\leq$

$$C|||<x>^{-\nu} R_0(\lambda+i0) <x>^{-\nu}|||^{k-1}|||<x>^\nu V_j E_0'(\lambda) V_j <x>^\nu|||_{\mathbf{1}}.$$

The point is that the operator norm is bounded (see [12]) and the trace norm is equal to the trace, as we have a positive operator, that is $\lambda^{1/2}\times$

$$\gamma\int_{\mathbb{R}^3}<x>^{2\nu} V_j^2(x)dx, \qquad (\gamma \text{ universal constant}).$$

So $\Im\left(Tr(\mathcal{A}_j(\lambda+i0)^k)\right) = \mathcal{O}(\lambda^{1/2})$ uniformly with respect to j and thus as small as we want when λ is small enough. This shows that we just have to prove that $\Im\left(Tr(\mathcal{A}_j(.+i0)^k)\right)$ is a Cauchy sequence in $C^0([\lambda_1,\lambda_0])$ for all $\lambda_1>0$, and this is clear because $<x>^{\nu'} \mathcal{A}_j(.+i0) <x>^{-\nu'}$ is convergent in $C^0\left([\lambda_1,\lambda_0],\mathbf{S_2}\right)$, when $\nu'>1/2$ and $\rho-2\nu'>3/2$, just by using standard LAP estimates away from energy 0. The proof is complete. $\square$

Appendix - Symbols of Isozaki-Kitada

Let's define the symbols l_j by $H_1 = \omega + V \sim \sum_{j\geq 0} h^j l_j^w(x, hD)$.
The symbols $a_0, \cdots, a_N$ and $b_0, \cdots, b_N$ are determined successively by asking that

$$H_0 A_N(h) - A_N(h) H_1 = h^N J_{\varphi_+}(r_N(h)) \tag{66}$$

$$A_N(h) B_N(h)^\star = \chi_+(x, hD) + h^N \tilde{r}_N(x, hD, h) \tag{67}$$

with $r_N(h)$ and in a bounded subset of $S_1(-\infty, -N)\big(\Gamma^+(R_0, J_0, \sigma_0)\big)$ (which means that the estimates for the symbols hold in this area) and $\tilde{r}_N(h)$ in a bounded subset of $S_1(-\infty, -N)$.

We have rather explicit formulas for symbols satisfying this:

$$a_0(x, \xi) = \exp\left(\int_0^{+\infty} d(X(s, x, \xi), \xi) ds\right)$$

where $X(s, x, \xi)$ is the solution of the differential equation

$$\begin{aligned} \frac{\partial X}{\partial s} &= \nabla_\xi l_0(X, \partial_x \varphi_+(X, \xi)), \\ X(0, x, \xi) &= x, \end{aligned}$$

and the symbol $d(x, \xi)$ is defined by

$$d(x, \xi) = -\frac{1}{2} tr\big(\big(\partial^2_{\xi\xi} l_0\big)(x, \partial_x \varphi_+(x, \xi)).\partial^2_{x,x}\varphi_+(x, \xi)\big).$$

In particular, we have (by increasing R_1 if necessary) in $\Gamma^+(R_1, J_1, \sigma_1)$

$$a_0(x, \xi) \geq 1/2, \qquad a_0(x, \xi) \to 1 \quad |x| \to \infty$$

which implies the ellipticity of $A_N(h)$ in this outgoing area. The other symbols a_j, $j \geq 1$ are defined inductively by

$$a_j(x, \xi) = \int_0^{+\infty} f_p(X(t, x, \xi), \xi) \exp\left(\int_0^t d(X(s, x, \xi), \xi) ds\right) dt$$

where $f_p(x, \xi)$ is a universal linear combination of

$$\partial_\xi^\beta l_j(x, \partial_x \varphi_+(x, \xi)) \partial_x^{\beta - \alpha} a_k(x, \xi) \partial_x^{\alpha_1} \varphi_+(x, \xi) \cdots \partial_x^{\alpha_n} \varphi_+(x, \xi),$$

with $\alpha \leq \beta$, $\alpha_1 + \cdots + \alpha_n = \alpha$ and $j + k + |\beta| - n = p + 1$. Then we continue a_j on $\mathbb{R}^{2d}$ by multiplying it by a cut-off function that is 1 in a neighbourhoodof $\Gamma^+(R_1, J_1, \sigma_1)$ and that vanishes outside some neighbourhoodof $\mathbb{R}^{2d} \setminus \Gamma^+(R_0, J_0, \sigma_0)$. After that, we determine $b_0, \cdots, b_N$ inductively using (67); in particular, we have

$$\overline{b_0(x, \xi)} = \chi_+(x, \tilde{\eta}(x, x, \xi)) \left(a_0(x, \xi) \Big| \frac{\partial \tilde{\xi}}{\partial \eta}(x, x, \tilde{\eta}(x, x, \xi)) \Big|\right)^{-1}$$

where $\tilde{\eta}(x, y, .)$ is the diffeomorphism of $\mathbb{R}^d$ onto itself defined by

$$\xi \mapsto \int_0^1 \nabla_x \varphi_+(y + t(x - y), \xi) dt \tag{68}$$

and $\tilde{\xi}(x, y, .)$ its inverse. Using (51) we see easily that $\tilde{\eta}(x, x, \xi) - \xi = \mathcal{O}(< x >^{-\rho})$ and $|\partial_\eta \tilde{\xi}(x, x, \eta)| - 1 = \mathcal{O}(< x >^{-\rho})$ together with their derivatives. As we also have

$$a_0 - 1 \in S_1(-\infty, -\rho)(\Gamma^+(R_1, J_1, \sigma_1)) \tag{69}$$

we get

$$b_0 - \overline{\chi_+} \in S_1(-\infty, -\rho). \tag{70}$$

Remark that $\overline{\chi_+}$ is the principal symbol of $\chi_+(x, hD)^\star$. Let's define the sequence of symbols $\tilde{\chi}^*_{+,j}$ by $\chi_+(x, hD)^\star \sim \sum h^j \tilde{\chi}^*_{+,j}(x, hD)$; the next lemma shows that (70) holds for all the b_j's.

Lemma 7.4. *Assume that for all $k \geq 1$ we have*

$$a_k \in S_1(-\infty, -\rho) \tag{71}$$

then we have, for all $j \geq 0$

$$b_j - \tilde{\chi}^*_{+,j} \in S_1(-\infty, -\rho). \tag{72}$$

Proof: we shall write that $a = b\ [\rho]$ to say that $a - b \in S_1(-\infty, -\rho)$, and proceed by induction on j. The result is true at step 0. Let's recall that b_j is defined by

$$\left(a_0 \# b_j\right)_0 = - \sum_{k+l+n=j,\ n<j} \left(a_k \# b_n\right)_l$$

where $(a\#b)_j$ is the sequence of symbols defined by the Egorov's theorem, that is such that $J(a, \varphi_+)J(b, \varphi_+)^\star \sim \sum_{j\geq 0} h^j (a\#b)_j(x, hD)$. Then remark that

$$\begin{aligned} (a_k\#b_n)_l &= 0\ [\rho] \qquad k \geq 1, \\ (a_0\#b_n)_l &= (1\#b_n)_l\ [\rho]\ , \\ (a_0\#b_n)_0 &= \overline{b_n}\ [\rho]. \end{aligned}$$

The first line is a consequence of the assumption (71) and the last two ones are due to (69). Then by using the induction assumption, we get

$$\begin{aligned} \overline{b_j} &= - \sum_{l+n=j,\ n<j} \left(1\#b_n\right)_l \qquad [\rho] \\ &= - \sum_{l+n=j,\ n<j} \left(1\#\tilde{\chi}^\star_{+,j}\right)_l \qquad [\rho] \\ &= - \sum_{l+n=j,\ n<j} \sum_{|\alpha|=l} (-1)^{|\alpha|} \frac{1}{\alpha!} \overline{\partial^\alpha_x D^\alpha_\eta \tilde{\chi}^*_{+,j}(x, \eta)} \qquad [\rho] \end{aligned}$$

Then by adding and substracting the term $n = j$ to the righ hand side of the last line we get the result since $\tilde{\chi}^*_{+,j} = \sum_{|\beta|=j} (\beta!)^{-1} \partial_\eta^\beta D_x^\beta \overline{\chi_+(x,\eta)}$ and

$$\sum_{l+n=j} \sum_{|\alpha|=l,\ |\beta|=k} \frac{\partial_\eta^\beta D_x^\beta}{\beta!} \frac{\partial_\eta^\alpha D_x^\alpha}{\alpha!} (-1)^{|\alpha|} \overline{\chi_+(x,\eta)} = 0$$

which can be easily proved by identifying the symbols expansions of $\chi_+(x, hD)$ and $\big(\chi_+(x, hD)^\star\big)^\star$. The proof is now complete. $\square$

References

[1] M.Sh. Birman, M.Z. Solomyak, *Stieltjes double operator integrals*, Sov. Math. Dokl. 6, 1567-1571 (1965).

[2] J.-M. Bouclet , *Distributions spectrales pour des opérateurs perturbés,* thèse de Doctorat de l'Université de Nantes (2000).

[3] ________, *Fonction spectrale pour des perturbations relativement Hilbert-Schmidt,* C.R. Acad. Sci. Paris, t. 332, Série I, 887-892 (2001).

[4] ________, *Regularized scattering phase for long range perturbations of elliptic operators,* C.R. Acad. Sci. Paris (2001)

[5] ________, *Trace formulas for long range perturbations,* in preparation.

[6] Y. Colin-de-Verdière, *Une formule de trace pour l'opérateur de Schrödinger dans $\mathbb{R}^3$,* Ann. Sci. E.N.S., IV sér. 14, 27-39 (1981).

[7] C. Gerard, A. Martinez , *Prolongement méromorphe de la matrice de scattering pour des problèmes à deux corps à longue portée,* Ann. I.H.P., Phys. Théor. 51, No.1, 81-110, (1989).

[8] I. C. Gohberg, M. G. Krein *Introduction to the theory of linear nonselfadjoint operators,* Translations of Mathematical Monographs, Vol. 18. AMS. Providence, R.I. (1969).

[9] L. Guillopé, *Une formule de trace pour l'opérateur de Schrödinger dans $\mathbb{R}^n$,* thèse de doctorat, Université de Grenoble (1981).

[10] B. Helffer, D. Robert, *Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles,* J. Funct. Anal. 53, 246-268 (1983)

[11] H. Isozaki, H. Kitada *Modified wave operators with time independant modifiers*, J. Fac. Sci., Univ. Tokyo, I A 32, 77-104 (1985).

[12] A. Jensen, T. Kato, *Spectral properties of Schrödinger operators and time decay of the wave functions,* Duke Math. J. 46, 583-611 (1979).

[13] L.S. Koplienko, *Trace formula for non trace-class perturbations,* Sib. Math. J. 25, 735-743 (1984).

[14] ________, *Regularized spectral shift function for one dimensional Schrödinger operator with slowly decreasing potential,* Sib. Math. J. 26, 365-369 (1985).

[15] M. G. Krein, *Perturbation determinants and a formula for the trace of unitary and selfadjoint operators*, Soviet Math. Dokl. 3, 707-710 (1962).

[16] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators,* Commun. Math. Phys. 78, 391-408 (1981).

[17] V. Petkov, G. Popov, *Asymptotic behaviour of the scattering phase for non-trapping obstacles,* Ann. Inst, Fourier 32, No.3, 111-149 (1982).

[18] M. Reed, B. Simon, *Methods of modern mathematical physics. IV : Analysis of operators,* Academic Press (1978).

[19] D. Robert *Asymptotique de la phase de diffusion à haute énergie pour des perturbations du second ordre du Laplacien*, Ann. Sci. Ec. Norm. Super., IV. Ser. 25, No.2, 107-134 (1992).

[20] ________, *On long range scattering for perturbations of Laplace operators*, J. Anal. Math. 59, 189-203 (1992).

[21] ________, *Relative time delay for perturbations of elliptic operators and semiclassical asymptotics,* J. Funct. Anal. 126, No.1, 36-82 (1994).

[22] ________, *On the Weyl formula for obstacles,* Partial differential equations and mathematical physics, L.Hörmander and A. Melin editors, Birkhäuser (1995).

[23] ________, *Semiclassical asymptotics for the spectral shift function* , Buslaev, V. (ed) et al., Differential operators and spectral theory. M. Sh. Birman's 70th anniversary collection. Providence,RI. AMS Transl, Ser.2, 189(41), 187-203 (1999).

[24] A. Rybkin, *On a trace formula of the Buslaev-Faddeev type for a long range potential,* J. Math. Phys 40, No.3, 1334-1343 (1999).

[25] B. Simon *Trace ideals and their applications*, Cambridge University Press, Cambridge (1979).

[26] D. Yafaev, *Mathematical scattering theory. General theory,* vol. 105, A.M.S. Rhode Island (1992).