The Lamplighter Groups

Anthony Saint-Criq

Friday 22nd January, 2021

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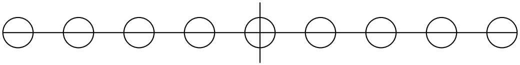
Consider an infinite street with lamp posts every so often :



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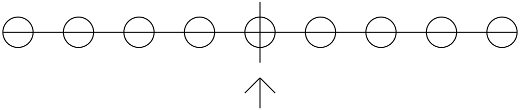

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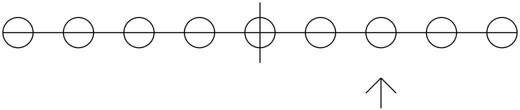

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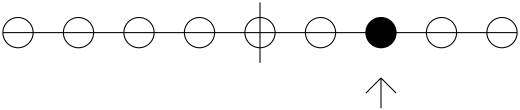

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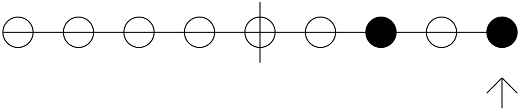

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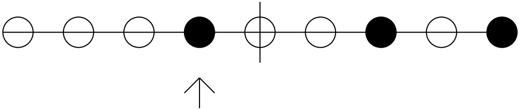

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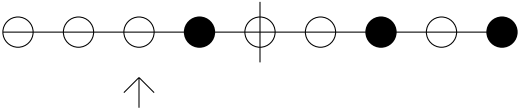

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What is a lamplighter group ?

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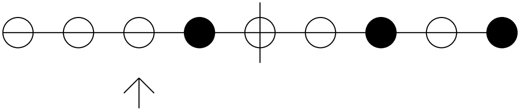

Add a lamplighter to this street. He can move freely, turn on or off some lamps, and end his journey somewhere at the foot of a lamp.

This is an element of L_2.

More formally, we define :

$$L_2 = \mathbb{Z} \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2.$$

The $\mathbb{Z}$ summand corresponds to the ending position of the lamplighter, and the sums of $\mathbb{Z}/2$ are sequences indicating which lamps are turned on or off.

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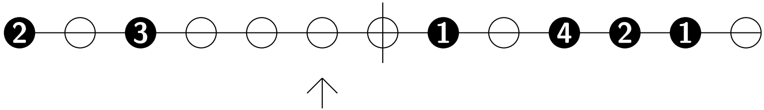
Note that such sequences are **finitely-supported** !

More generally, for $N \geqslant 2$, we define the *rank N lamplighter group* similarly by :

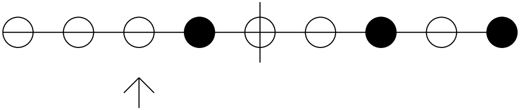
$$L_N = \mathbb{Z} \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/N.$$

This corresponds to having lamps with several states :



How is that a group ? How to compose elements ?

See an element as a set of instructions. The following element :



corresponds to :

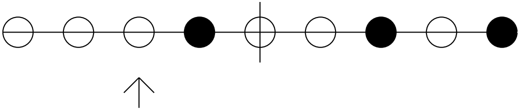
1. Go right twice, switch the state of the lamp.
2. Go right twice, switch the state of the lamp.
3. Go to the left 5 times, switch the state of the lamp.
4. Go to the left once.

We need to make two assumptions :

1. The moving instructions are always written *relative* to the current position (*e.g.* move "to the left x times" or "to the right y times").
2. The turning on or off of the lamps is also relative to the lamp's current state (*e.g.* "add $[z]_{\text{mod } N}$ to the lamp's current state").

Then, composition is simply doing both algorithms one after the other.

This is computed picturally as follows :



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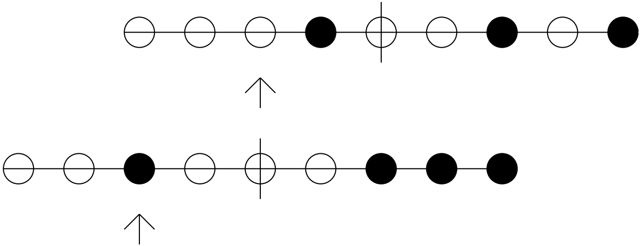

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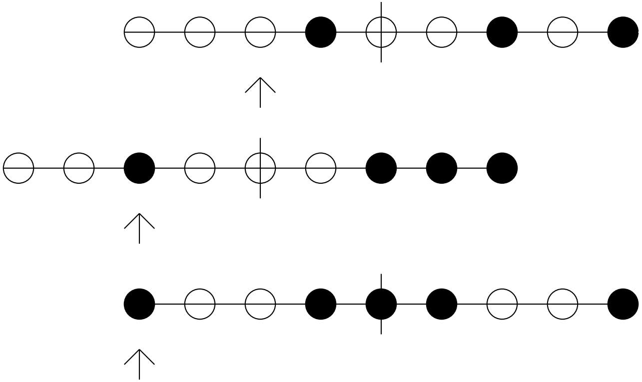
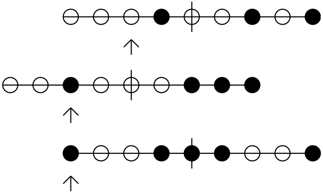


This too can be formalized.



Notice that in terms of ending positions, it corresponds to adding them, and in terms of sequences, it corresponds to shifting the second and computing their product in $\mathbb{Z}/N$ component-wise :

$$(k, (\sigma_n)_{n\in\mathbb{Z}}) \star (\ell, (\varsigma_n)_{n\in\mathbb{Z}}) = (k + \ell, (\sigma_n + \varsigma_{n-k})_{n\in\mathbb{Z}}).$$

$\mathbb{B}$ Properties of L_N

Definition :

Let G and H be two groups. Their (*regular, restricted*) *wreath product* is the group $G \wr H$ defined as follows : take

$$K = \bigoplus_{\omega \in H} G,$$

and define an action of H on K by $h \cdot (g_\omega)_{\omega \in H} = (g_{h^{-1}\omega})_{\omega \in H}$. This gives a morphism $\varphi : H \to \mathrm{Aut}(K)$, which allows to set $G \wr H = K \rtimes_\varphi H$.

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It is a direct verification to check that $L_N = (\mathbb{Z}/N) \wr \mathbb{Z}$ by the very definition.

Proposition :

Let $G = \langle X | R \rangle$ and $H = \langle Y | S \rangle$ be two groups given by presentation, and let $\varphi : H \to \mathrm{Aut}(G)$. We have the following presentation for the semi-direct product :

$$G \rtimes_{\varphi} H \cong \langle X, Y | R, S, yxy^{-1} = \varphi(y)(x),\ (x, y) \in X \times Y \rangle.$$

David L. Johnson. *Presentations of Groups*. Ed. by J. W. Bruce and C. M. Series. illustrated, revised. Vol. 15. London Mathematical Society Student Texts. 1997. Chap. 10, p. 140.

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We can use this to give a presentation of L_N. We already have :

$$\mathbb{Z}/N \cong \langle a | a^N \rangle.$$

Therefore, a presentation for $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/N$ is given by :

$$\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/N \cong \langle a_n,\ n\in\mathbb{Z} | a_n^N, a_n a_m a_n^{-1} a_m^{-1},\ m,n\in\mathbb{Z}\rangle.$$

We also have $\mathbb{Z}\cong\langle t\rangle$. We check that if $\varphi:\mathbb{Z}\to \mathrm{Aut}(\bigoplus \mathbb{Z}/N)$ is given by the definition, then $\varphi(t)(a_n)=a_{n+1}$. This provides :

$$L_N \cong \langle t, a_n,\ n\in\mathbb{Z} | a_n^N, a_n a_m a_n^{-1} a_m^{-1}, t a_n t^{-1} = a_{n+1},\ m,n\in\mathbb{Z}\rangle.$$

We can simplify this presentation by using *Tietze transformations*. Take $a = a_0$. Then : $a_n = t^n a t^{-n}$. We can therefore replace all occurrences of a_n by $t^n a t^{-n}$, to obtain :

$$L_N \cong \langle t, a_n,\ n \in \mathbb{Z} | a_n^N, a_n a_m a_n^{-1} a_m^{-1}, t a_n t^{-1} = a_{n+1},\ m, n \in \mathbb{Z} \rangle$$

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Removing redundancies, we finally obtain :

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- **Can we simplify further ?**

Theorem : (Baumslag, 1961)

Let G and H be finitely presentable groups. Then, their (regular, restricted) wreath product $G \wr H$ is finitely-presentable if and only if G is trivial or H is finite.

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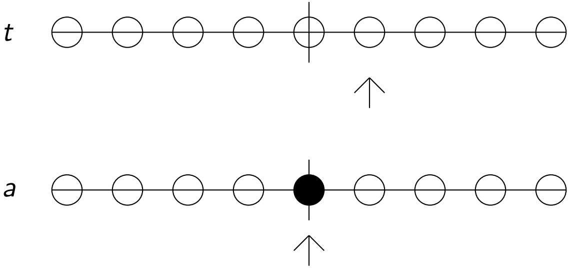
Here, both $\mathbb{Z}$ and $\mathbb{Z}/N$ are finitely presentable. However, $\mathbb{Z}$ is infinite and $\mathbb{Z}/N$ is not trivial, so $L_N = (\mathbb{Z}/N) \wr \mathbb{Z}$ **cannot** be finitely presented. In particular, the presentation we gave **cannot** be simplified further :

$$L_N \cong \langle a, t | a^N, [t^i a t^{-i}, t^j a t^{-j}],\ i, j \in \mathbb{Z} \rangle.$$

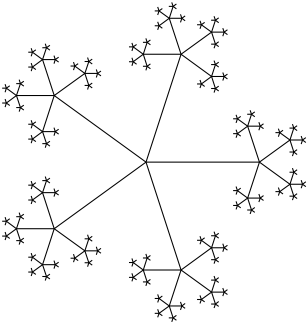
The generators a and t correspond to the following elements :



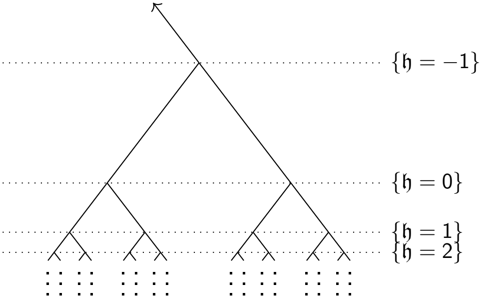
We can indeed check that $L_N = \langle t, a \rangle$ with $t = (1, \mathbf{0})$ and $a = (0, \delta_0)$.

$\mathbb{C}$ The Cayley graph of L_N

We will define a **Diestel-Leader** graph. Choose $p, q \geqslant 2$, and consider the $p+1$ and $(q+1)$-valent trees T_p and T_q. Here is a representation of T_4 :



Fix two height functions $\mathfrak{h}$ and $\mathfrak{h}'$ on T_p and T_q respectively. Each vertex $x \in T_p$ at height $\mathfrak{h}(x) = k \in \mathbb{Z}$ has exactly one neighboor at height $k-1$ and p neighboors at height $k+1$. A **horocycle** is a set $\{\mathfrak{h} = k\} = \{x \in T_p \ / \ \mathfrak{h}(x) = k\}$. In T_2 for instance :



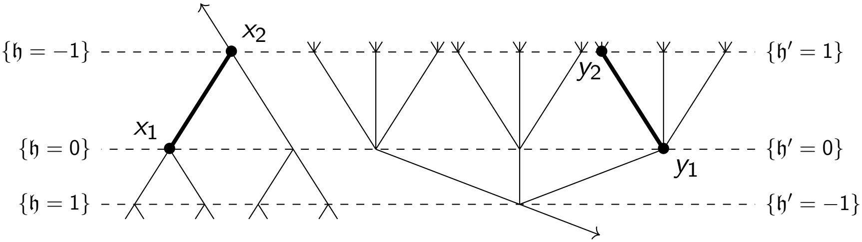
The **Diestel-Leader graph** $\mathrm{DL}(p,q)$ is defined in terms of the previous considerations. Its vertex set is :

$$\mathcal{V}(\mathrm{DL}(p,q)) = \{(x,y) \in T_p \times T_q \ / \ \mathfrak{h}(x) + \mathfrak{h}'(y) = 0\},$$

and there is an edge $(x_1, y_1) \leftrightarrow (x_2, y_2)$ if and only if there are edges $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$ in T_p and T_q respectively.

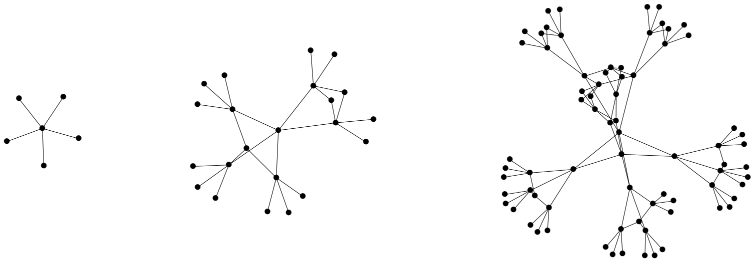
One may show that it does *not* depend on the height functions $\mathfrak{h}$ and $\mathfrak{h}'$, and that $\mathrm{DL}(p,q)$ and $\mathrm{DL}(q,p)$ are graph-isomorphic.

Here is a portion of DL$(2,3)$ as an example, where (x_1, y_1) and (x_2, y_2) are adjacent :

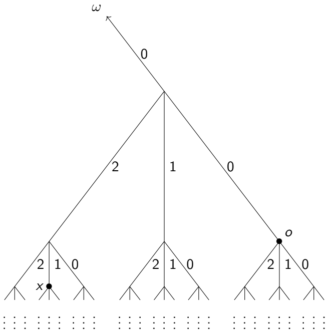


Note that DL(p,q) is *not* a tree, but each vertex has exactly $p+q$ neighboors (DL(p,q) is $(p+q)$-valent).

Here are the balls of radii one, two and three in $\mathrm{DL}(2,3)$:



By defining Σ_p to be the set of sequences $(\sigma_n)_{n\geqslant 0}$ with $\sigma_n \in \mathbb{Z}/p$ and with finite support, one can prove that there is a bijection $\varphi : \Sigma_p \times \mathbb{Z} \to T_p$ compatible with the height function.



Here, x corresponds to $(\ldots\overline{0021}, 1)$, where the sequence $(1, 2, 0, 0, \ldots)$ is represented in number notation. Its parent corresponds to $(\ldots\overline{002}, 0)$.

Taking an element $(k, (\sigma_n)_{n\in\mathbb{Z}})$ in L_N, one can truncate the sequence $(\sigma_n)_{n\in\mathbb{Z}}$ at k to obtain two sequences in Σ_N, and thus two elements of T_N at heights k and $-k$ by the previous labelling :

$$L(\sigma, k) = \varphi((\sigma_{k-n})_{n\geqslant 0}, k),$$

$$R(\sigma, k) = \varphi((\sigma_{k+n+1})_{n\geqslant 0}, -k).$$

Proposition :

The application $\Phi : L_N \to \mathrm{DL}(N)$ defined by $\Phi(\sigma, k) = (L(\sigma, k), R(\sigma, k))$ is one-to-one.

Proof : recall that the labelling $\varphi : \Sigma_N \times \mathbb{Z} \to T_N$ was one-to-one. Assuming $\Phi(k, \sigma) = \Phi(\ell, \varsigma)$, by inspecting the left and right parts, we obtain $k = \ell$ and $\sigma = \varsigma$.

For surjectivity, choose $(x, y) \in \mathrm{DL}(N)$, and write $x = \varphi(\sigma, k)$ and $y = \varphi(\varsigma, -k)$. Define

$$\lambda_n = \begin{cases} \sigma_{k-n} & \text{if } n \leqslant k \\ \varsigma_{n-k-1} & \text{if } n > k \end{cases},$$

which is such that $\Phi(\lambda, k) = (x, y)$.

■

It turns out that $X_N = \{t, ta, ta^2, ..., ta^{N-1}\}$ is also a generating set for L_N. We have :

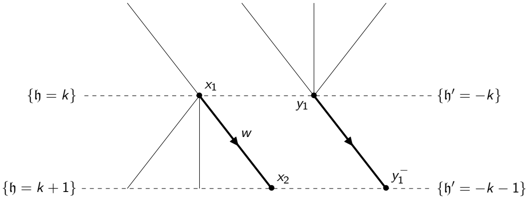
Proposition :

The Cayley graph of L_N with respect to the generating set X_N is $\mathrm{DL}(N)$:

$$\Gamma(L_N, X_N) \cong \mathrm{DL}(N).$$

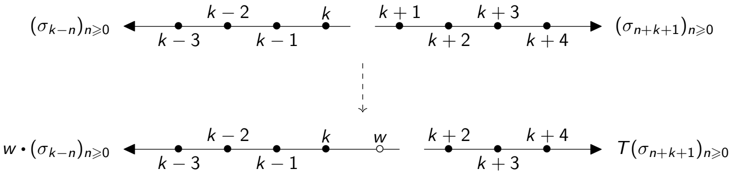
Proof : we will prove that the $2N$ neighboors of a vertex representing $g \in L_N$ are the vertices representing the elements gh for all $h \in X_N$.

Fix an element $(x_1, y_1) \in \mathrm{DL}(N)$, let y_1^- be the parent of y_1, and let x_2 be a child of x_1 downwards an edge labelled w :



Then $(x_1, y_1) = \Phi(\sigma, k)$, and we have $x_1 = L(\sigma, k)$ and $y_1 = R(\sigma, k)$.

Truncating a sequence gives the parent vertex, and appending an element gives a child. This provides :



If we let $(x_2, y_1^-) = \Phi(\tilde{\sigma}, k+1)$, we see that $\tilde{\sigma}$ only differs from σ at position $k+1$, where σ_{k+1} is replaced by w. This corresponds to :

$$(x_2, y_1^-) = \Phi((\sigma, k) \star ta^{\ell})$$

for ℓ such that $\sigma_{k+1} + \ell = w$ in $\mathbb{Z}/N$. Note that $\ell \mapsto \ell + \sigma_{k+1}$ is one-to-one.

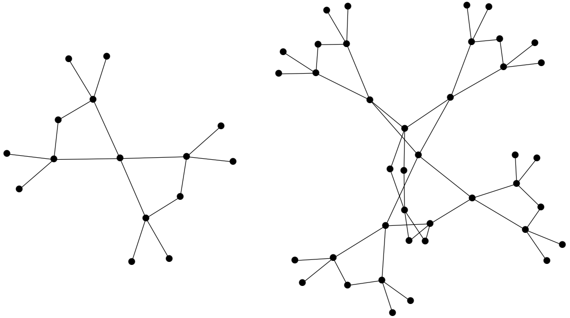
■

$\mathbb{D}$ To go further...

There are lots of problems we didn't cover :

1. What is the word length of an element $g \in L_N$ with respect to the generating set $\{a, t\}$?
2. What are the details of the construction of $\mathrm{DL}(p, q)$ and of the labelling $\varphi : \Sigma_p \times \mathbb{Z} \to T_p$?
3. What is the purpose of the Diestel-Leader graphs ? *Why* are they interesting ?
4. How does the Python script work to generate those images ?

Thank you for your attention !



Balls of radii 2 and 3 in DL(2), that is, in the Cayley graph of L_2.