TRISECTIONS I: EXISTENCE AND EXAMPLES

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Manifolds will be smooth, compact, connected and oriented. Trisections of (closed) 4-manifolds, due to Gay and Kirby [GK16], are a great way to encode the 4-dimensional properties into a simple diagrammatic setting, and in many regards, they are a generalization of these decompositions to the dimension above.

1. Definition and examples

We define an n-dimensional 1-handlebody of genus k to be $Z_k \cong \natural^k(\mathbf{S}^1 \times \mathbf{D}^3)$, and we set $Z_0 = \mathbf{D}^4$. Similarly, a 3-dimensional 1-handlebody of genus g is $Y_g \cong \natural^g(\mathbf{S}^1 \times \mathbf{D}^2)$. Σ_g will denote the closed surface of genus g, with $\partial Y_g \cong \Sigma_g$.

Definition 1. *A **trisection** of a closed 4-manifold X is a decomposition $X = X_1 \cup X_2 \cup X_3$ into three embedded sumbanifolds X_1, X_2, X_3 such that:*

1. $X_i \cong Z_{k_i}$ *for* $i \in \{1, 2, 3\}$.
2. $H_{ij} := X_i \cap X_j \cong Y_g$ *for* $i \neq j \in \{1, 2, 3\}$.
3. $\Sigma := X_1 \cap X_2 \cap X_3 \cong \Sigma_g$.

*The pieces X_1, X_2 and X_3 are called the **sectors** of the trisection, and Σ is the **trisecting surface**. The **genus** of the trisection is that of Σ, and we say that this decomposition is a trisection of **type** $(g; k_1, k_2, k_3)$.*

The schematic way to represent a trisection is shown in Figure 1.

We can make the following observations:

1. $\partial X_1 = H_{12} \cup H_{13}$ is a Heegaard splitting of genus g, and the associated Heegaard surface Σ.

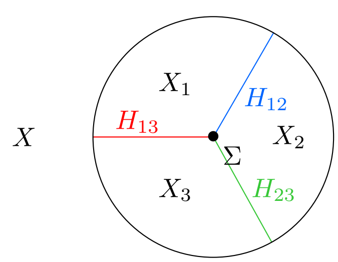


Figure 1. The cartoon picture of a trisection.

(2) $\partial X_1 \cong \#^{k_1}(\mathbf{S}^1 \times \mathbf{S}^2)$. Therefore $k_1 \leqslant g$.
(3) The Waldhausen theorem asserts that $\#^{k_1}(\mathbf{S}^1 \times \mathbf{S}^2)$ admits a *unique* Heegaard splitting of genus $g \geqslant k_1$, up to isotopy.

These observations, which can also be made for ∂X_2 and ∂X_3, already imply some restrictions on the different pieces of the trisection.

One can also make use of the Inclusion-Exclusion principle, and read:

$$\chi(X) = 2 + g - (k_1 + k_2 + k_3). \tag{1}$$

The very first example is that of $\mathbf{S}^4$. Consider that $\mathbf{S}^4 \subset \mathbf{C} \times \mathbf{R}^3$, and cut $\mathbf{C}$ into three sectors almost like in Figure 1:

$$R_i := \{re^{i\theta} \in \mathbf{C} \mid 2i\pi/3 \leqslant \theta \leqslant 2(i+1)\pi/3\},\ i \in \{1,2,3\}.$$

Then, set $X_i = p^{-1}(R_i)$, where $p : \mathbf{S}^4 \to \mathbf{C}$ is the projection onto the first factor. It is an immediate verification to see that $\mathbf{S}^4 = X_1 \cup X_2 \cup X_3$, and that:

(1) $X_i \cong \mathbf{D}^4 \cong Z_0$.
(2) $X_i \cap X_j \cong \mathbf{D}^3 \cong Y_3$.
(3) $X_1 \cap X_2 \cap X_3 \cong \mathbf{S}^2$.

Therefore, this decomposition of $\mathbf{S}^4$ is a genus zero trisection, and it has type $(0; 0, 0, 0)$.

One can also build a trisection of $\mathbf{S}^1 \times \mathbf{S}^3$ in the same fashion: embed $\mathbf{S}^3 \subset \mathbf{C} \times \mathbf{R}^2$, and consider this time the map $p : \mathbf{S}^1 \times \mathbf{S}^3 \to \mathbf{C}$ that projects onto the $\mathbf{C}$-coordinate in the $\mathbf{S}^3$ factor. Set $X_i = p^{-1}(R_i)$ just as before. This time, we see that:

(1) $X_i \cong \mathbf{S}^1 \times \mathbf{D}^3$.
(2) $X_i \cap X_j \cong \mathbf{S}^1 \times \mathbf{D}^2$.
(3) $X_1 \cap X_2 \cap X_3 \cong \mathbf{T}^2$.

The trisection has genus one, and type $(1; 1, 1, 1)$.

Given a trisected closed 4-manifold $X = X_1 \cup X_2 \cup X_3$, define the **spine** of the trisection to be the union

$$Y = H_{12} \cup H_{13} \cup H_{23}.$$

We will use the following theorem from [LP72]:

Theorem 2 (Laudenbach–Poénaru)**.** *Any diffeomorphism of* $\#^k(\mathbf{S}^1 \times \mathbf{S}^2)$ *extends to* $\natural^k(\mathbf{S}^1 \times \mathbf{D}^3)$.

In particular, given a 2-handlebody (a union of 0-, 1- and 2-handles), there is a unique way to cap it off into a closed 4-manifold. Moreover, to go from the spine Y to the whole manifold X, it reduces to attaching such 3- and 4-handles. As such, we have the following:

Corollary 3. *A spine uniquely determines a trisected 4-manifold (up to isotopy).*

Here, the uniqueness is up to equivalence of trisections, which consists in a diffeomorphism of trisected manifolds that maps each piece of one to the corresponding piece of the other.

Now, any handlebody H_{ij} can be uniquely described by a cut system, formed of meridian disks. Denote as $\mathcal{D}_{ij}$ such a choice of a system for H_{ij}, and set $\alpha = \partial\mathcal{D}_{13}$,

$\beta = \partial\mathcal{D}_{12}$ and $\gamma = \partial\mathcal{D}_{23}$. Note that α, β and γ are three sets of g curves that live on the trisecting surface Σ.

The data $(\Sigma; \alpha, \beta, \gamma)$ is sufficient to recover a unique trisected 4-manifold, by the previous observations. Note that any choice of a pair of curves determines a Heegaard diagram for the boundary of the corresponding sector. For instance, $(\Sigma; \alpha, \beta)$ is a Heegaard diagram for ∂X_1.

Any two choices of diagrams for isotopic trisection of the *same* 4-manifold are related by a sequence of handle slides, which translate to sliding the curves on the diagrams. This operation consists of:

(1) Take two distinct curves (in the same system), say α_1 and α_2.
(2) Join them by an arc δ.
(3) Consider a regular neighborhood of $\alpha_1 \cup \delta \cup \alpha_2$ on the surface Σ. It has three components, one of which is not isotopic to either α_1 or α_2.
(4) Keeping α_2, replace α_1 with that third component.

The procedure is depicted in Figure 2.

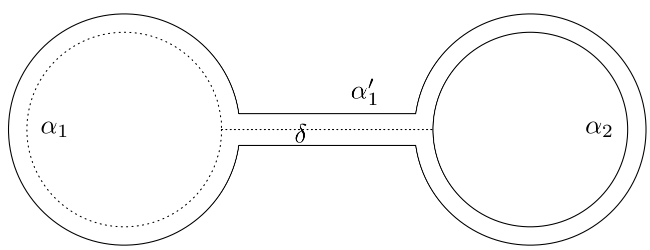


Figure 2. Sliding the α_1 curve over the α_2 curve by means of the arc δ to obtain a new curve α_1'.

We say that two diagrams are **equivalent** if they are related by a sequence of curve sliding and by a diffeomorphism of the surface. Again, we are only allowed to slide α-curves with α-curves, and the same for β and γ curves.

Waldhausen's theorem translates to saying that any Heegaard diagram of genus $g \geqslant k$ for $\#^k(\mathbf{S}^1 \times \mathbf{S}^2)$ is curve-slide equivalent that of Figure 3:

We therefore define:

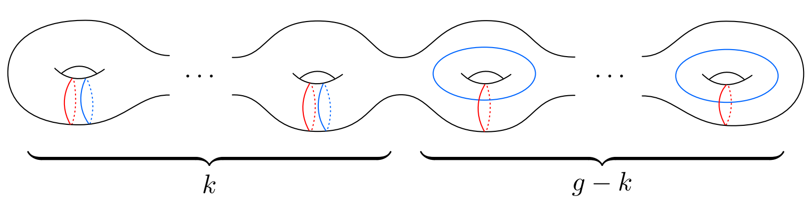


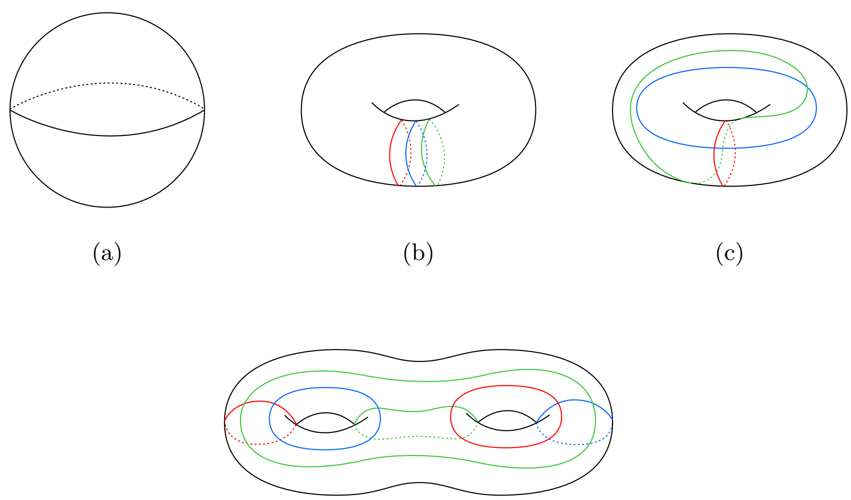
Figure 3. The standard genus g Heegaard diagram for $\#^k(\mathbf{S}^1 \times \mathbf{S}^2)$.

Definition 4. *A* ***trisection diagram*** *is a tuple* $(\Sigma_g; \alpha, \beta, \gamma)$ *where* α, β *and* γ *are sets of* g *curves on* Σ_g, *and such that any pair of curves* $(\Sigma_g; \alpha, \beta)$, $(\Sigma_g; \alpha, \gamma)$ *and* $(\Sigma_g; \beta, \gamma)$ *is curve-slide equivalent to that of Figure 3 for some* k.

By the previous observations, such a diagram uniquely determines a trisected 4-manifold, and conversely, a trisected 4-manifold determines a unique diagram, up to equivalence. That is:

$$^{\text{diagrams}}/_{\text{equivalence}} \sim {}^{\text{trisections}}/_{\text{isotopy}}.$$

The diagrams for the two trisections of $\mathbf{S}^4$ and $\mathbf{S}^1 \times \mathbf{S}^3$ are as in Figure 4.



(a) (b) (c)

(d)

Figure 4. (a) The $(0; 0, 0, 0)$ trisection of $\mathbf{S}^4$. (b) The $(1; 1, 1, 1)$ trisection of $\mathbf{S}^1 \times \mathbf{S}^3$. (c) A $(1; 0, 0, 0)$ trisection diagram for $\mathbf{CP}^2$. (d) A $(2; 0, 0, 0)$ trisection of $\mathbf{S}^2 \times \mathbf{S}^2$.

2. Around a Proof of Existence

We show the following:

Theorem 5. *Any closed 4-manifold admits a trisection.*

Proof. Consider a self-indexing Morse function f on X; that is, $f : X \to \mathbf{R}$ is Morse and the index i critical points are on the level $f^{-1}(i)$. Let k_i denote the number of index i critical points. Without loss of generality, we can assume that $k_0 = k_4 = 1$.

Denote as L the k_2-component link that is the attachment link for the 2-handles. We can assume that L lives inside $f^{-1}(3/2)$. Consider an open regular neighborhood $\nu(L)$ of L inside $f^{-1}(3/2)$, and pick a relative handle decomposition of $f^{-1}(3/2) \smallsetminus \nu(L)$ which consists only of 1-, 2- and 3-handles. Let H_1 be the union of the 2-

and 3-handles, and let H_2 be $\overline{\nu(L)}$ together with the 1-handles. Then H_1 and H_2 are both handlebodies which meet along some central surface Σ of genus g. This means that $f^{-1}(3/2) = H_1 \cup_\Sigma H_2$ is a genus g Heegaard splitting.

L lies completely in H_2, by construction. Therefore, we can flow along a gradient-like vector field for f, and consider the cylinder $H_1 \times [3/2, 5/2]$. Define: $X_1 = f^{-1}([0, 3/2]) \cup H_1 \times [3/2, 2]$, $X_3 = f^{-1}(5/2, 4) \cup H_1 \times [2, 5/2]$. Finally, define X_2 to be the complement $X_2 = X \setminus [\mathring{X}_1 \cup \mathring{X}_3]$. Then, X_1 is a genus k_1 handlebody, and X_3 is a genus k_3 handlebody.

Finally, X_2 is diffeomorphic to $H_2 \times I \cup \{\text{2-handles}\}$, where $H_2 \times I$ is a genus g handlebody obtained from $\nu(L)$ by attaching some 1-handles, and the 2-handles are attached along L so they cancel uniquely k_2 1-handles. Therefore, X_2 is diffeomorhic to a genus $g - k_2$ handlebody.

The intersections are as follows: $X_1 \cap X_2 = H_2$, $X_2 \cap X_3 = H_1$ and $X_1 \cap X_3$ is the result of surgery on H_2 along the link $L \subset \mathring{H}_2$. All three are genus g 3-dimensional handlebodies, and the triple intersection is Σ. ■

Note that we started with a handle decomposition with a unique 0- and a unique 4-handle, and k_i i-handles, and we constructed a $(g; k_1, g - k_2, k_3)$-trisection, where g was a sufficiently large genus of a Heegaard splitting of ∂X_1. The converse is also true:

Proposition 6. *If X admits a $(g; k_1, k_2, k_3)$-trisection, then X admits a handle decomposition with k_1 1-handles, $g - k_2$ 2-handles and k_3 3-handles.*

In particular, we see that if X has a $(g; k_1, k_2, k_3)$-trisection, then:

(1) $k_1, k_2, k_3 \geqslant \mathrm{rk}(\pi_1(X))$. In particular, from $\chi(X) = 2 + g - (k_1 + k_2 + k_3)$, we obtain:
$$g \geqslant \chi(X) - 2 + 3\mathrm{rk}(\pi_1(X)).$$
(2) If one of the $k_i = 0$, then X is simply-connected.
(3) If $k_1 = k_2 = k_3 = g$, then X has a handle decomposition with no 2-handles, and as many 1- and 3-handles. Then, by [LP72] again, we get that $X \cong \#^g(\mathbf{S}^1 \times \mathbf{S}^3)$.

3. Stabilization Moves and Uniqueness

There is a way to "merge" two trisections together: taking their connected sum[1] (that is, it corresponds to taking the connected sum of each corresponding piece of both trisections). A trisection that can be split as the connected sum of two smaller ones is called **reducible**.

On the diagrams, it really corresponds to taking the connected sum of both diagrams. This means that, for instance, $\#^g(\mathbf{S}^1 \times \mathbf{S}^3)$ has a $(g; g, g, g)$-trisection, with a diagram given in Figure 5.

There is a special kind of trisection: a stabilized one. **Stabilization** is the following operation: consider an arc $a_{12} \subset H_{12}$ boundary parallel. Consider an

[1]The choice of the ball on which to take connected sums is important, but I am omitting the details here.

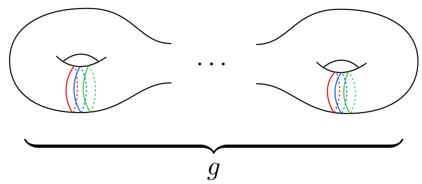


Figure 5. A trisection diagram for $\#^g(\mathbf{S}^1 \times \mathbf{S}^3)$.

open regular neighborhood $\nu(a_{12})$ of a_{12} in X. Then, define:

$$X_1' = X_1 \smallsetminus \nu(a_{12}),\ X_2' = X_2 \smallsetminus \nu(a_{12}) \text{ and } X_3' = X_3 \cup \overline{\nu(a_{12})}.$$

The decomposition $X = X_1' \cup X_2' \cup X_3'$ is a new stabilization of X, not isotopic to the first one, called the **3-stabilization**. We define 1- and 2-stabilizations similarly.

The situation is depicted in Figure 6.

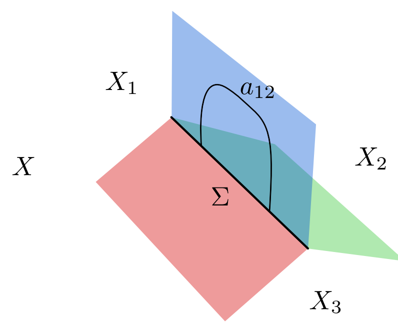


Figure 6. The operation of 3-stabilizing a trisection.

If the starting trisection of X had type $(g; k_1, k_2, k_3)$, then the 3-stabilized one has type $(g+1; k_1, k_2, k_3+1)$. Similar results hold for 1- and 2-stabilizations too.

We claim:

Proposition 7. *Given a trisection, the result of i-stabilization and then j-stabilization is isotopic to that of a j-stabilization and then an i-stabilization.*

Just like the Reidemeister–Singer theorem holds for Heegaard splittings, we have the following for trisections, due to [GK16]:

Theorem 8 (Gay–Kirby)**.** *Any two trisections of the same manifold become isotopic after a certain number of stabilizations of each.*

4. Connections with Kirby Diagrams

We can always place the whole trisection diagram in standard red/blue position. That is: the α and β curves are as in Figure 3, and the green curves get moved along

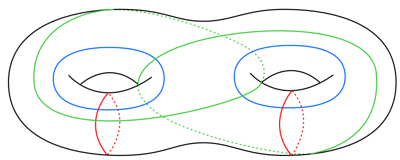


Figure 7. A trisection diagram for $\mathbf{S}^2 \times \mathbf{S}^2$ in standard red/blue position.

in some position. For instance, all the diagrams in Figure 4 but that of $\mathbf{S}^2 \times \mathbf{S}^2$ are in standard position. A standard diagram for it would be that of Figure 7.

Now, in the associated handle decomposition to a trisection, recall that X_1 was the union of the 0- and the 1-handles. This X_1 is obtained by the α and β curves. The 2-handles are attached along the γ curves, and the 3- and 4-handles are uniquely attached to that. This means that we can pass from a trisection diagram to a Kirby diagram by the following procedure:

(1) Put the diagram into standard position.
(2) Draw a 1-handle for each parallel pair of α and β curves. Remove the dual α and β curves.
(3) Each γ curve becomes a framed link for a 2-handle, with framing induced by the surface.

We can compute a Kirby diagram from the previous examples, see Figure 8.

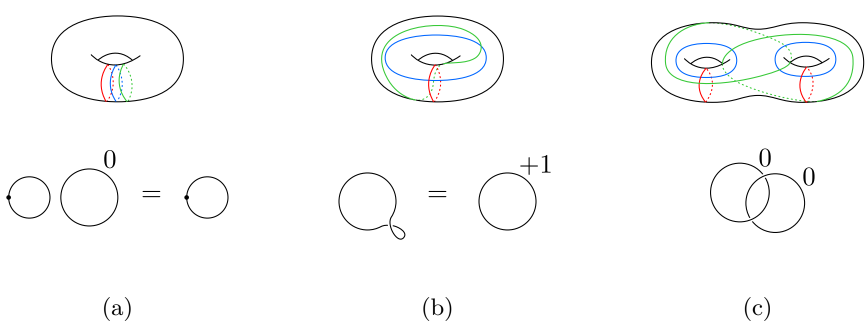


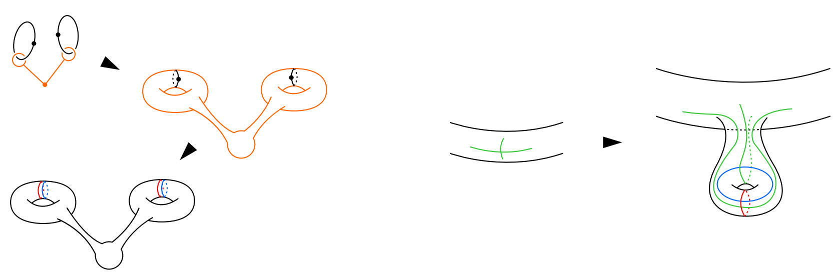
Figure 8. Obtaining a Kirby diagram from a trisection diagram for (a) $\mathbf{S}^1 \times \mathbf{S}^3$, (b) $\mathbf{CP}^2$ and (c) $\mathbf{S}^2 \times \mathbf{S}^2$.

The other way is slightly more subtle: how to go from a Kirby diagram to a trisection diagram? The algorithm, presented (and proved) in [Kep21], goes as follows:

(1) Start with a Kirby diagram. Draw all 1-handles as dotted circles.

(2) Choose a meridian circle to all 1-handles, and connect those to form a wedge of circles.
(3) Thicken this wedge into a surface, so that all 1-handle circles are meridian curves on that surface.
(4) Replace each 1-handle circle with a parallel pair of red/blue curves.
(5) Consider the framed link L of the attachment of the 2-handles. This can be pushed on the surface, but it will have crossings. Stabilize the surface enough so that this does not happen. When doing so, add a dual pair of red/blue curve. The projection of the link on the surface should have framing induced by that surface.
(6) The projection of this link is the set of green curves.

The procedure at steps (2) and (5) are detailed in Figure 9.



(a) (b)

Figure 9. (a) Transforming the circles of the 1-handles into a surface. (b) Resolving crossings on the projection of L on the surface.

For examples of computation, Figure 8 can simply be read in reverse, and each Kirby diagram induces the corresponding trisection diagram. However, for wilder Kirby diagrams, this can become very messy...[2]

References

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[2] Added after the talk: the example I gave (that of $\mathbf{S}^1 \times \mathbf{S}^3$) showed exactly this. In general, it is even wilder to compute, and the choice of the projection onto the surface is important. See [Kep21] for a full algorithm that does not depend on such choices.