Sobolev spaces and capacities theory on path spaces over a compact Riemannian manifold

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August 14, 2002

Abstract

We introduce Sobolev spaces and capacities on the path space $P_{m_0}(M)$ over a compact Riemannian manifold M. We prove the smoothness of the Itô map and the stochastic anti-development map in the sense of stochastic calculus of variation. We establish a Sobolev norm comparison theorem and a capacity comparison theorem between the Wiener space and the path space $P_{m_0}(M)$. Moreover, we prove the tightness of (r,p)-capacities on $P_{m_0}(M)$, $r \in N$, $p > 1$, which generalises a result due to Airault-Malliavin and Sugita on the Wiener space. Finally, we extend our results to the fractional Hölder continuous path space $P^{2m,\alpha}_{m_0}(M)$, $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$.

Mathematics Subject Classification (2002): 58J65, 60H07, 60H30, 31C45
Key words and phrases: Sobolev norm, capacity, Itô map, tightness

1 Introduction

Sobolev spaces and capacities theory on the Wiener space is one of the significant aspects of the Malliavin Calculus. The tightness of (r,p)-capacities $(r \in N, p > 1)$ on the Wiener space plays an important rôle in the study of the finite co-dimensional submanifold (which are usually negligible with respect to the Wiener measure) of the Wiener space through the quasi-sure analysis. The purpose of this paper is to study Sobolev spaces and capacities theory on path spaces over a compact Riemannian manifold.

Let M be a compact connected Riemannian manifold, $d = dimM$, ∇ be a torsion skew-symmetric (TSS) connection on M. For any fixed point $m_0 \in M$, the path space $P_{m_0}(M)$ is defined by

$$P_{m_0}(M) = \{\gamma \in C([0,1], M) : \gamma(0) = m_0\},$$

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[*] Research partially supported by the National Nature Science Foundation of China, the Foundation of Science and Technology of Portugal, and a Postdoctoral Fellowship of Oxford University.

on which we consider the uniform convergence topology. The Itô map I realises a measure theoretic isomorphism between the Wiener space $X = \{x \in C([0,1], R^d) : x(0) = 0\}$ and the path space $P_{m_0}(M)$. The gradient operator [1] ∇ is defined in a natural way on the collection of cylinder functionals on $P_{m_0}(M)$. Using the idea initiated by Bismut [2] and developed by Driver [5], Fang-Malliavin [12] and Cruzeiro-Malliavin [4] among others, one can prove that the stochastic horizontal lift is infinitely differentiable along the Cameron-Martin directions (Theorem 4.5). Moreover, the integration by parts formula on the path space $P_{m_0}(M)$ allows us to prove the closability of ∇^r from $L^p(P_{m_0}(M), E)$ into $L^p(P_{m_0}(M), L^r_{(2)}(H, E))$ (Theorem 4.7), here $r \in N$, $p > 1$, E is a real separable Banach space, $L^r_{(2)}(H, E)$ is the collection of r-linear Hilbert-Schmidt type operators from the Cameron-Martin space $H = \{h = (h^1, \ldots, h^d) \in X : \|h\|_H^2 := \sum_{i=1}^{d} \int_0^1 |\dot{h}^i(s)|^2 ds < \infty\}$ into E (see Section 4.2 below). These two facts lead us to define a family of Sobolev spaces $\{D^{r,p}(P_{m_0}(M), E), r \in N, p > 1\}$ of E-valued functionals on $P_{m_0}(M)$. In particular, we can prove the smoothness (in the sense of stochastic calculus of variation) of the Itô map and the stochastic anti-development map between the Wiener space X and the path space $P_{m_0}(M)$. Following Malliavin[21], we introduce a family of (r,p)-capacities $\{C_{r,p}, r \in N, p > 1\}$ on $P_{m_0}(M)$ and develop the quasi-sure analysis on $P_{m_0}(M)$. Now it is natural to pose the following

Problem: What is the relationship between these Sobolev spaces (respectively, capacities) on the path space and those on the Wiener space? Are these (r,p)-capacities on the path space tight?

Several authors have studied the above problem for $r = 1$ and $p = 2$ or for general $r \in N$ and $p > 1$ in some special cases of M and the TSS connection ∇ on M. In [8], Driver and Röckner proved the tightness of the $(1,2)$-capacity on $P_{m_0}(M)$ which led them to construct the Ornstein-Uhlenbeck process on $P_{m_0}(M)$ by using the general theory of quasi-regular Dirichlet forms [20]. If $M = G$ is a compact Lie group equipped with an $Ad(G)$-invariant Riemannian metric, $m_0 = e$ is the unit element of G, and if ∇ is the left Cartan connection on G, Shigekawa [26] proved that the Itô map $I : X \to P_e(G)$ preserves the (r,p)-Sobolev norms and the (r,p)-capacities between the Wiener space X and the path group $P_e(G)$ and hence is indeed a quasi-homeomorphism. However if one replaces the left Cartan connection by the right Cartan connection or the Levi-Civita connection on G, Driver [6] showed that the quasi-homeomorphism property does not hold even in the case $r = 1$ and $p = 2$.

In this paper we obtain the following results.

Theorem 1.1 *For any $r \in N$, $p \geq 2$, and $\epsilon > 0$, there exists a constant $A = A(r,p,\epsilon)$ such that*

(1) *for any real valued cylinder functional F on $P_{m_0}(M)$,*

$$||F||_{D^{r,p}(P_{m_0}(M))} \leq A||F \circ I||_{W^{2r,p+\epsilon}(X)};$$

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[1] Throughout this paper, we let D denote the gradient operator on the Wiener space and let ∇ denote the gradient operator on the path space. The reader should not confuse the TSS connection ∇ acting on vector fields on M with the gradient operator ∇ acting on functionals on $P_{m_0}(M)$.

(2) *for any* $u \in W^{2r,p+\epsilon}(X)$*, there exists a modification of* $u \circ J$*, denoted by* $\widetilde{u \circ J}$*, such that*

$$||\widetilde{u \circ J}||_{D^{r,p}(P_{m_0}(M))} \leq A||u||_{W^{2r,p+\epsilon}(X)};$$

where $J = I^{-1}$*,* $||\cdot||_{D^{r,p}(P_{m_0}(M))}$ *denotes the* (r,p)*-Sobolev norm on* $P_{m_0}(M)$*, and* $||\cdot||_{W^{2r,p+\epsilon}}$ *denotes the* $(2r, p+\epsilon)$*-Sobolev norm on the Wiener space* X*.*

Theorem 1.2 *Let* $r \in N$*,* $p \geq 2$*, and* $\epsilon > 0$*. Then for any open subset* $O \subset P_{m_0}(M)$*,*

$$C_{r,p}(O) \leq A\widehat{C}_{2r,p+\epsilon}((\widetilde{I})^{-1}(O)),$$

where $A = A(r,p,\epsilon)$ *is the same constant as in Theorem 1.1,* $\widehat{C}_{2r,p+\epsilon}$ *denotes the* $(2r, p+\epsilon)$*-capacity on the Wiener space,* $\widetilde{I} : X \to P_{m_0}(M)$ *is any* ∞*-quasi-continuous modification of the Itô map* I *(see Theorem 3.7 below).*

Theorem 1.3 *For any* $r \in N$*,* $p > 1$*, the* (r,p)*-capacity* $C_{r,p}$ *on* $P_{m_0}(M)$ *is tight. More precisely, there exists a sequence of increasing compact subsets* $\{Z_n\}$ *in* $P_{m_0}(M)$ *such that*

$$\lim_{n\to\infty} C_{r,p}(Z_n^c) = 0.$$

Let $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$. We introduce the fractional Hölder continuous path space $P_{m_0}^{2m,\alpha}(M)$ as

$$P_{m_0}^{2m,\alpha}(M) = \left\{\gamma \in P_{m_0}(M) : \int_0^1 \int_0^1 \frac{d(\gamma(t),\gamma(s))^{2m}}{|t-s|^{1+2m\alpha}} dtds < \infty\right\}$$

(where $d(\cdot,\cdot)$ is the Riemannian distance on M), on which we consider the topology induced by the following distance function $\rho_{2m,\alpha}$: for any $\gamma_1, \gamma_2 \in P_{m_0}^{2m,\alpha}(M)$,

$$\rho_{2m,\alpha}(\gamma_1,\gamma_2) := \left[\int_0^1 \int_0^1 \frac{|d(\gamma_1(t),\gamma_2(t)) - d(\gamma_1(s),\gamma_2(s))|^{2m}}{|t-s|^{1+2m\alpha}} dtds\right]^{\frac{1}{2m}}.$$

By [1] [3], $P_{m_0}^{2m,\alpha}(M)$ is an M-type 2 Banach manifold and the topology on $P_{m_0}^{2m,\alpha}(M)$ induced by $\rho_{2m,\alpha}$ is stronger than the uniform convergence topology on $P_{m_0}^{2m,\alpha}(M)$. Moreover, the Wiener measure μ is supported on $P_{m_0}^{2m,\alpha}(M)$. By a similar method as we used on $P_{m_0}(M)$, stochastic calculus of variation and Sobolev spaces as well as capacities theory can be developed on $P_{m_0}^{2m,\alpha}(M)$. We can prove that Theorem 1.1 and Theorem 1.2 remain true if we replace $P_{m_0}(M)$ by $P_{m_0}^{2m,\alpha}(M)$. As a consequence, we obtain the following

Theorem 1.4 *For any* $m \in N$*,* $m \geq 2$*,* $\alpha \in (\frac{1}{2m}, \frac{1}{2})$*,* $r \in N$*,* $p > 1$*, the* (r,p)*-capacity on* $(P_{m_0}^{2m,\alpha}(M), \rho_{2m,\alpha})$ *is tight.*

A significant aspect of the above results is that they allow us to use quasi-sure analysis to study stochastic analysis and geometry on the loop space $L_{m_0}(M) = \{\gamma \in P_{m_0}(M) : \gamma(1) = m_0\}$ by regarding it as a finite co-dimensional submanifold of $P_{m_0}(M)$. In a forthcoming paper we will use this approach to give an alternative construction of Driver's flow on the loop space $L_{m_0}(M)$ and to prove the quasi-invariance of the pinned Wiener measure. In Section 8.3 of this paper, as an application of Theorem 1.4, we will use the general theory of quasi-regular Dirichlet forms to construct the Ornstein-Uhlenbeck diffusion process on $P_{m_0}^{2m,\alpha}(M)$, $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$.

We would like to end this Introduction by the following two remarks.

Remark 1.5 *Under the notations of Theorem* 1.1 *and* 1.2*, for any* $p \in (1,2)$*, we can also prove that*

$$||F||_{D^{r,p}(P_{m_0}(M))} \leq B||F \circ I||_{W^{2r,2}(X)},$$

$$||\widetilde{u \circ J}||_{D^{r,p}(P_{m_0}(M))} \leq B||u||_{W^{2r,2}(X)},$$

$$C_{r,p}(O) \leq B\widehat{C}_{2r,2}((\widetilde{I})^{-1}(O)),$$

where B *is some constant depending on* r *and* p*. Similar results hold on* $P_{m_0}^{2d,\alpha}(M)$*,* $m \in N$*,* $m \geq 2$*,* $\alpha \in (\frac{1}{2m}, \frac{1}{2})$*.*

Remark 1.6 *All the proofs of Theorem 1.1, 1.2, 1.3 and 1.4 as well as the results in Remark 1.5 are strongly relied on making use of the* L^p*-inequality of Skorohod stochastic integral on the Wiener space (see Proposition 2.1 below), which is a consequence of the Meyer inequality on the Wiener space. By lack of a Meyer inequality on the path space, one can only obtain one-side comparison inequalities in Theorem 1.1, 1.2 and Remark 1.5.*

Acknowledgement The author would like to thank his supervisors: A. B. Cruzeiro, Zhi-Ming Ma and P. Malliavin for suggestions and constant encouragements. He is very grateful to anonymous referees for their careful reading, helpful suggestions and for pointing out some errors in the earlier version of the submitted manuscript.

2 Sobolev spaces and capacities theory on the Wiener space

2.1 Sobolev spaces of E-valued functionals on the Wiener space

Let X be the (classical) Wiener space, i.e., $X = \{x \in C([0,1], R^d) : x(0) = 0\}$, H be the Cameron-Martin space, i.e., $H = \{h \in X : ||h||_H^2 := \int_0^1 |\dot{h}(t)|^2 dt < \infty\}$, μ_0 be the Wiener measure on X (under which $\{x(t), t \in [0,1]\}$ is the standard Brownian motion on R^d). For any real separable Hilbert space K, and for any $r \in N$, $p > 1$, the Sobolev space $W^{r,p}(X,K)$ of K-valued Wiener functionals on X is a standard notion (see e.g. [16] [27] [21] [23]). In [13] [22] [25], the notion of (r,p)-Sobolev space $W^{r,p}(X,E)$ of real separable Banach space E-valued Wiener functionals have been introduced in different ways, $r \in N$, $p > 1$. If E is an M-type 2 Banach space, then the Meyer inequality for E-valued Wiener functionals holds ([22]), which implies that all these definitions in [13] [22] [25] are equivalent. In Section 3 we will define the (r,p)-Sobolev space of E-valued functionals on the path space $P_{m_0}(M)$, $r \in N$, $p > 1$.

Let K be a real separable Hilbert space, $D : L^2(X,K) \to L^2(X, H \otimes K)$ be the gradient operator on K-valued Wiener functionals and D^* or δ be the adjoint of D

with domain being those processes $u \in L^2(X, H \otimes K)$ such that

$$\left| E\left[\int_0^1 < D_s F, \dot{u}(s) >_K ds \right] \right| \leq C ||F||_2,$$

for all $F \in W^{1,2}(X, K)$, where C is some constant depending on u. If $u \in Dom(\delta)$, then $\delta(u)$ is the unique element of $L^2(X, K)$, such that

$$E\left[\int_0^1 < D_s F, \dot{u}(s) >_K ds \right] = E\left[< F, \delta(u) >_K \right].$$

Following [23], we call $\delta(u)$ the Skorohod stochastic integral of the process u and write it as $\delta(u) = \int_0^1 \dot{u}(s) dx(s)$. It is easy to see that $W^{1,2}(X, H \otimes K) \subset Dom(\delta)$.

Proposition 2.1 *Let K be a real separable Hilbert space. For any $p > 1$, there exists a constant $C_p > 0$ such that for any $u \in W^{1,p}(X, H \otimes K)$,*

$$\left\| \int_0^1 \dot{u}(s) dx(s) \right\|_p \leq C_p \left[\left(\int_0^1 ||E(\dot{u}(t))||_K^2 dt \right)^{\frac{1}{2}} + \left\| \left(\int_0^1 \int_0^1 ||D_s \dot{u}_t||_K^2 ds dt \right)^{\frac{1}{2}} \right\|_p \right].$$

Proof. Using the Meyer inequality of K-valued Wiener functionals [27] and the similar argument in [23] (Proposition 3.2.1, p. 158-159) for the special case $K = R$, one can complete the proof of Proposition 2.1. □

We let $\widehat{C}_{r,p}$ denote the (r,p)-capacity on the Wiener space, $r \in N$, $p > 1$. For its definition, we refer the reader to [21] or Section 3 where we will define the (r,p)-capacity on the path space $P_{m_0}(M)$. The following result is very significant in the quasi-sure analysis on the Wiener space and is indeed one of the main motivations of the present paper.

Theorem 2.2 *(Airault-Malliavin [1], Sugita [28]) For any $r \in N$ and $p > 1$, $\widehat{C}_{r,p}$ is tight. More precisely, there exists a sequence of increasing compact subsets $\{K_n\}$ in X such that*

$$\lim_{n \to \infty} \widehat{C}_{r,p}(X \setminus K_n) = 0.$$

A subset S of X is called a *slim set* if $\widehat{C}_{r,p}(S) = 0$ holds for all $r \in N$ and $p > 1$. For any real separable Banach space E, a functional $F : X \to E$ is said to be (r,p)-quasi-continuous, if for all $\epsilon > 0$, there exists an open subset $O \subset X$ such that $\widehat{C}_{r,p}(O) < \epsilon$, and $F : O^c \to E$ is continuous. A functional $F : X \to E$ is said to be ∞-quasi-continuous if it is (r,p)-quasi-continuous for all $r \in N$ and $p > 1$.

Given a measurable functional $F : X \to E$, we call $F^* : X \to E$ an (r,p)-quasi-continuous modification of F, if $F(x) = F^*(x)$ for μ_0-a.s. $x \in X$, and $F^* : X \to E$ is (r,p)-quasi-continuous. Furthermore, we call F an ∞-quasi-continuous modification of F if $F(x) = F^*(x)$ for μ_0-a.s. $x \in X$ and $F^* : X \to E$ is ∞-quasi-continuous. If F_1^*, F_2^* are two ∞-quasi-continuous modifications of F, then F_1^* and F_2^* only differ on a slim set.

Proposition 2.3 *([22]) Assume that the Banach norm $x \to \phi(x) := ||x||_E$ is smooth (in the sense of Fréchet-Gâteaux) in $E \setminus \{0\}$, and its k-th order derivative satisfies*

$$M_k = \sup_{0<||x||_E \le 1} ||\phi^{(k)}||(x) < +\infty, \quad \forall k = 1, \dots, r,$$

where

$$||\phi^{(k)}||(x) = \sup_{||l_i||_E \le 1} < \phi^{(k)}(x), l_1 \otimes \dots l_k >, \quad \forall x \in E \setminus \{0\}.$$

Then any $F \in W^{r,p}(X, E)$ has an (r, p)-quasi-continuous modification, $r \in N \cup \{\infty\}$, $p \in (1, \infty]$.

2.2 The fractional Wiener space $X^{2m,\alpha}$

Note that in Proposition 2.3 we need certain regularity on the Banach norm $x \to \phi(x) := ||x||_E$ in $E \setminus \{0\}$. However the supremum norm $||x||_\infty = \sup\{|x(s)| : s \in [0, 1]\}$ on the classical Wiener space X does not satisfy the required conditions in Proposition 2.3. For the purpose to construct ∞-quasi-continuous modification of the Itô functionals defined as the solution processes of SDEs on R^n by regarding them as some Banach spaces-valued Wiener functionals, the following fractional Wiener space $X^{2m,\alpha}$ have been introduced and widely used in [1] [28] [22].

For any $m \in N$, $m \ge 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, the fractional Wiener space $X^{2m,\alpha}$ is defined as

$$X^{2m,\alpha} := \left\{ x \in X : \int_0^1 \int_0^1 \frac{||x(t) - x(s)||_{R^d}^{2m}}{|t - s|^{1+2m\alpha}} dtds < \infty \right\},$$

on which we consider the fractional Hölder norm $|| \cdot ||_{2m,\alpha}$ given by

$$||x||_{2m,\alpha} = \left(\int_0^1 \int_0^1 \frac{||x(t) - x(s)||_{R^d}^{2m}}{|t - s|^{1+2m\alpha}} dtds \right)^{\frac{1}{2m}}.$$

It is well-known that the Wiener measure μ_0 is supported on $X^{2m,\alpha}$ and $(X^{2m,\alpha}, H, \mu_0)$ is an abstract Wiener space. Furthermore, $x \to ||x||_{2m,\alpha}$ is smooth in $X^{2m,\alpha} \setminus \{0\}$ and satisfies the conditions required in Proposition 2.3. Indeed, $(X^{2m,\alpha}, || \cdot ||_{2m,\alpha})$ is an M-type 2 Banach space. For these, we refer the reader to [1] [28] [22] [3].

3 Itô map and its smoothness

3.1 Orthonormal frame bundle $O(M)$

According to [5] [12] [10], we call ∇ a torsion skew-symmetric (TSS) connection on M if ∇ and its dual connection $\widehat{\nabla}$ are compatible with the Riemannian metric $<, >$, where $\widehat{\nabla}_X Y = \nabla_X Y - T(X, Y)$, for all $X, Y \in \Gamma(TM)$, T is the torsion tensor on M with respect to ∇, i.e., $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, $X, Y \in \Gamma(TM)$. Throughout this paper, we fix a TSS connection ∇ on M.

The orthonormal frame bundle over M, denoted by $O(M)$, is defined by

$$O(M) = \{r = (m, u) : m \in M, u \text{ is an orthonormal frame at } m\},$$

or equivalently,

$$O(M)=\{r=(m,u): m\in M, u \text{ is an isometry from } R^d \text{ onto } T_mM\}.$$

For any fixed connection ∇, we associate it with the $R^d\times so(d)$-valued canonical parallelism (θ,ω) on $O(M)$, the R^d-valued torsion form Θ and the $so(d)$-valued curvature form Ω. See e.g. [17] [12] [21] for the definitions.

An intrinsic Riemannian metric on $O(M)$ is defined as follows: for any given two vector fields X, Y on $O(M)$, let

$$<X,Y>=<\theta(X),\theta(Y)>_{R^d}+<\omega(X),\omega(Y)>_{so(d)},$$

where $<,>_{R^d}$ denotes the standard inner product on R^d, and $<,>_{so(d)}$ denotes the inner product on $so(d)$ given by $<A,B>_{so(d)}=-tr(A^*B),\quad A,B\in so(d)$.

3.2 The Itô map $I: X\to P_{m_0}(M)$

Let $A_1,\ldots,A_d$ be the canonical horizontal vector fields on $O(M)$ with respect to ∇, i.e., the unique vector fields on $O(M)$ such that

$$<\theta,A_i>=e_i,\quad <\omega,A_i>=0,\quad i=1,\ldots,d,$$

where $e_1,\ldots,e_d$ is the canonical basis of R^d. Let $\{r_x(t), t\in[0,1]\}$ be the solution of the following Stratonovich SDE on $O(M)$:

$$\begin{cases} dr_x(t)=\sum\limits_{i=1}^{d}A_i(r_x(t))\circ dx^i(t),\\ r_x(0)=r_0,\end{cases} \tag{3.1}$$

where r_0 is a fixed orthonormal frame at m_0, $x\in X$.

Let $\gamma_x(t)=\pi(r_x(t))$, $t\in[0,1]$. By [16] [5] [12] [14] [11], $\{\gamma_x(t), t\in[0,1]\}$ is a Brownian motion on M, i.e., a diffusion process on M with infinitesimal generator given by Δ_M, i.e., the Laplace-Beltrami operator on M.

Definition 3.1 *The Itô map $I: X\to P_{m_0}(M)$ is defined by*

$$I(x)(t):=\gamma_x(t),\quad t\in[0,1].$$

The Wiener measure μ on $P_{m_0}(M)$ is defined as the law of $\{\gamma_x(t), t\in[0,1]\}$, i.e.,

$$\mu=I_*\mu_0.$$

Remark 3.2 *Indeed, the Itô map $I: X\to P_{m_0}(M)$ depends on the connection ∇. Driver [5] noticed that the Wiener measure μ is independent of the special choice of ∇ in the class of TSS connections on M. Moreover, it is well-known that the Itô map $I: X\to P_{m_0}(M)$ is a measure theoretic isomorphism between the Wiener space X and the path space $P_{m_0}(M)$. We call its inverse, i.e., $J=I^{-1}: P_{m_0}(M)\to X$, the stochastic anti-development map. For its definition, see Section* 4 *below.*

Now for any fixed $t \in [0,1]$, regarding $x \in X \mapsto r_x(t) \in O(M)$ as an $O(M)$-valued Wiener functional on the Wiener space X, by the standard result of Malliavin calculus for SDE on Riemannian manifolds (cf. [16], [23] [21]), $r_x(t) \in W^{\infty,\infty}(X, O(M)) = \cap_{r\in N, p>1} W^{r,p}(X, O(M))$, where $W^{r,p}(X, O(M))$ denotes the (r,p)-Sobolev space of $O(M)$-valued Wiener functionals on X, see e.g. [1] for its definition. More precisely, we have the following

Proposition 3.3 *The Wiener functional $x \to r_x(t)$ is infinitely differentiable along H-directions, i.e., for any $h_1, \ldots, h_n \in H$, the directional derivatives of $r_x(t)$ along $h_1, \ldots, h_n \in H$ can be well defined recursively as follows:*

$$\begin{aligned} D_h r_x(t) &:= \frac{\partial}{\partial \epsilon} r_{x+\epsilon h}(t)\Big|_{\epsilon=0} \in T_{r_x(t)}O(M), \\ &\quad\ldots \\ D_{h_n}\ldots D_{h_1} r_x(t) &:= \frac{\nabla}{\partial \epsilon} D_{h_{n-1}}\ldots D_{h_1} r_{x+\epsilon h_n}(t)\Big|_{\epsilon=0} \in T_{r_x(t)}O(M), \end{aligned}$$

(where $\frac{\nabla}{\partial\epsilon}$ denotes the covariant derivative with respect to the connection ∇ along the smooth curve $\epsilon \longmapsto r_{x+\epsilon h_n}(t)$ on $O(M)$). Moreover, there exist kernel functionals denoted by $D^{\alpha_1\ldots\alpha_n}_{s_1\ldots s_n} r_x(t) \in L^2([0,1]^n, T_{r_x(t)}O(M))$ such that:

(1) *for any $t \in [0, s_1 \vee \ldots \vee s_n]$, $1 \le \alpha_1, \ldots, \alpha_r \le d$,*

$$D^{\alpha_1\ldots\alpha_n}_{s_1\ldots s_n} r_x(t) = 0,$$

(2)

$$D_{h_n}\ldots D_{h_1} r_x(t) = \sum_{\alpha_1,\ldots,\alpha_n=1}^{d} \int_{[0,1]^n} D^{\alpha_1\ldots\alpha_n}_{s_1\ldots s_n} r_x(t) \dot{h}_1^{\alpha_1}(s_1)\ldots \dot{h}_n^{\alpha_n}(s_n) ds_1 \ldots ds_n,$$

(3) *for any $p \ge 1$,*

$$\sup_{s_1,\ldots,s_n\in[0,1]} E_{\mu_0}\Big[\sup_{t\in[s_1\vee\ldots\vee s_n,1]} ||D^{\alpha_1\ldots\alpha_n}_{s_1\ldots s_n} r_x(t)||^p_{T_{r_x(s)}O(M)}\Big] < +\infty.$$

where E_{μ_0} denotes the expectation with respect to the Wiener measure μ_0 on X.

Proof. This can be proved by repeating the argument used in [16] P. 337-340 (cf. also Theorem 2.2.1 and Theorem 2.2.2 of [23] P.102-107), only replacing there the usual derivative on R^n by the covariant derivative on the compact Riemannian manifold $O(M)$. □

3.3 The Itô map $I : X \to P^{2m,\alpha}_{m_0}(M)$

By the same reason as we explained in Section 2.3, for the purpose of constructing ∞-quasi-continuous modification of the solution processes of manifold-valued SDEs (such as the Itô map from the Wiener space to the path space) by regarding them as some Banach manifold -valued functionals, one need also to introduce the fractional Hölder continuous path space $(P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$, $m \in N$, $m \ge 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$. See Section 1 for its definition. It is well known (cf. [1] [22] [3]) that $P^{2m,\alpha}_{m_0}(M)$ is an M-type 2 Banach manifold modelled by the M-type 2 Banach space $(X^{2m,\alpha}, ||\cdot||_{2m,\alpha})$.

Let $\gamma_x(\cdot) = I(x)$ be the Brownian motion defined in Section 3.2. Using Whitney-Nash embedding and the Burkholder-Davis-Gundy inequality, we can prove

$$E\left[\int_0^1 \int_0^1 \frac{d(\gamma_x(t), \gamma_x(s))^{2m}}{|t-s|^{1+2\alpha m}} dtds\right] < +\infty.$$

Hence for μ-a.s. $x \in X$, we have $\gamma_x \in P_{m_0}^{2m,\alpha}(M)$. The Itô map $I : X \to P_{m_0}^{2m,\alpha}(M)$ is now well defined.

3.4 Smoothness of the Itô maps

We now prove the smoothness of the Itô maps in the sense of stochastic calculus of variation. An intrinsic proof can be given by using the differential formula of the Itô maps given in Theorem 4.4. For the convenience of the reader and also for saving the space of the paper, here we would use an extrinsic approach. By Whitney-Nash's embedding, M can be isometrically embedded as a compact submanifold of some Euclidean space R^n. For any $r \in N \cup \{+\infty\}$ and $p \in (1, +\infty]$, we introduce

$$W^{r,p}(X, P_{m_0}(M)) := \{F \in W^{r,p}(X, P_{m_0}(R^n)) : F(X) \subset P_{m_0}(M)\},$$

$$W^{r,p}(X, P_{m_0}^{2m,\alpha}(M)) := \left\{F \in W^{r,p}(X, P_{m_0}^{2m,\alpha}(R^n)) : F(X) \subset P_{m_0}^{2m,\alpha}(M)\right\}.$$

If $F \in W^{\infty,\infty}(X, P_{m_0}(M))$ or $F \in W^{\infty,\infty}(X, P_{m_0}^{2m,\alpha}(M))$, we call it a smooth functional (in the sense of stochastic calculus of variation).

Theorem 3.4 *For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, we have*

$$I \in W^{\infty,\infty}(X, P_{m_0}(M)),$$

$$I \in W^{\infty,\infty}(X, P_{m_0}^{2m,\alpha}(M)).$$

Proof. By Whitney-Nash's embedding theorem, $O(M)$ can be isometrically embedded into some Euclidean space R^l. We can extend the SDE (3.1) on $O(M)$, i.e., $dr_x(t) = \sum\limits_{i=1}^{d} A_i(r_x(t)) \circ dx_t^i$, $r_x(0) = r_0$, to an SDE on R^l:

$$d\bar{r}_x(t) = \sum_{i=1}^{d} \bar{A}_i(\bar{r}_x(t)) \circ dx_t^i, \quad \bar{r}_x(0) = r_0,$$

where $\bar{A}_i$ is a compact supported smooth extension of A_i from $O(M)$ into R^l. Applying Theorem 3.4 of [22] or the result of Example 4.2 in [28], the mapping $x \in X \to \bar{r}_x(\cdot) \in P_{r_0}^{2m,\alpha}(R^l)$ is a smooth functional in the sense of stochastic calculus of variation. Since μ_0-a.s. $x \in X$, $r_x(t) = \bar{r}_x(t)$ for all $t \in [0,1]$, we see that $x \in X \to r_x(\cdot) \in P_{r_0}^{2m,\alpha}(O(M))$ is indeed a smooth functional. Therefore the Itô map $I : X \to P_{m_0}^{2m,\alpha}(M)$ defined by $I(x)(\cdot) = \pi(r_x(\cdot))$ is smooth. Similarly, one can prove $I \in W^{\infty,\infty}(X, P_{m_0}(M))$. □

Remark 3.5 *For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, the Wiener measure μ_0 is supported on $X^{2m,\alpha}$. Since the stochastic calculus of variation on $X^{2m,\alpha}$ is as the same as on X, Theorem 3.4 remains true if we replace X by $X^{2m,\alpha}$, i.e., for any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, we have:*

$$I \in W^{\infty,\infty}(X^{2m,\alpha}, P_{m_0}(M)),$$

$$I \in W^{\infty,\infty}(X^{2m,\alpha}, P_{m_0}^{2m,\alpha}(M)).$$

Proposition 3.6 *For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, the Itô map $I : X \rightarrow P_{m_0}^{2m,\alpha}(M)$ has an ∞-quasi-continuous modification. Namely, there exists a sequence of decreasing open subsets (G_n) of X and a map $\widetilde{I} : X \rightarrow P_{m_0}^{2m,\alpha}(M)$ such that*

$$\lim_{n \to \infty} \widehat{C}_{r,p}(G_n) = 0, \ \forall r \in N, p > 1; \tag{3.2}$$

$$\widetilde{I}|_{G_n^c} : G_n^c \rightarrow (P_{m_0}^{2m,\alpha}(M), \rho_{2m,\alpha}) \text{ is continuous;} \tag{3.3}$$

$$\widetilde{I} = I, \quad \mu_0 - a.s. \tag{3.4}$$

Proof. This is a consequence of Theorem 3.4 and Proposition 2.3. ☐

Now since $P_{m_0}^{2m,\alpha}(M) \subset P_{m_0}(M)$, we can regard $\widetilde{I}$ as a map from X into $P_{m_0}(M)$.

Theorem 3.7 *$\widetilde{I} : X \rightarrow P_{m_0}(M)$ is ∞-quasi-continuous. In other words, $I : X \rightarrow P_{m_0}(M)$ has an ∞-quasi-continuous modification.*

Proof. By Lemma $A.3$ of [1], for any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, there exists a constant $C = C(m, \alpha) > 0$ such that for any $x \in X^{2m,\alpha}$, $||x||_\infty \leq C||x||_{2m,\alpha}$. It follows that $\rho_\infty(\gamma_1, \gamma_2) \leq C\rho_{2m,\alpha}(\gamma_1, \gamma_2)$ holds for any $\gamma_1, \gamma_2 \in P_{m_0}^{2m,\alpha}(M)$, where $\rho_\infty(\gamma_1, \gamma_2) := \sup_{s \in [0,1]} d(\gamma_1(s), \gamma_2(s))$. Therefore the uniform convergence topology $(P_{m_0}^{2m,\alpha}(M), \rho_\infty)$ is weaker than the one on $P_{m_0}^{2m,\alpha}(M)$ induced by the distance function $\rho_{2m,\alpha}$. By this fact and Proposition 3.6, $\widetilde{I}$ is continuous from G_n^c into $(P_{m_0}(M), \rho_\infty)$. ☐

4 Fundamental theorems on the path space $P_{m_0}(M)$

4.1 Stochastic horizontal lift and stochastic anti-development

According to [2] [5] [15] [10], for μ-a.s $\gamma \in P_{m_0}(M)$, we define its stochastic horizontal lift as the unique $O(M)$-valued stochastic process $\{U_t(\gamma), t \in [0, 1]\}$ by solving the parallel transport covariant SDE:

$$\nabla_{\circ d\gamma(t)} U(t) = 0, \ \ U_0 = r_0, \tag{4.1}$$

where $\nabla_{\circ d\gamma_s}$ denotes the stochastic covariant derivative along γ_s with respect to the given TSS connection ∇ on M, $\circ d\gamma_s$ denotes the Stratonovich differential along γ_s, $s \in [0, 1]$. Indeed, for $\gamma = I(x)$, we have $U_s(\gamma) = r_x(s)$, $s \in [0, 1]$.

The stochastic anti-development map, denoted by $J : P_{m_0}(M) \to X$, is defined by:

$$J(\gamma)(t) = \int_0^t U_s^{-1}(\gamma) \circ d\gamma_s, \quad t \in [0,1], \mu - a.s. \gamma \in P_{m_0}(M). \tag{4.2}$$

It is well know that $J = I^{-1}$, i.e., $x(t) = \displaystyle\int_0^t U_s^{-1}(\gamma) \circ d\gamma_s$ is a Brownian motion on R^d (see e.g. [15]). For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, since μ_0-a.s. the sample path of Brownian motion on R^d belongs to $X^{2m,\alpha}$, we see that for μ-a.s. $\gamma \in P_{m_0}(M)$, $x = J(\gamma) \in X^{2m,\alpha}$.

Let $h \in H$. Define a vector field X_h on $P_{m_0}(M)$ by

$$X_h(\gamma)(t) = U_t(\gamma)h(t), \quad \mu - a.s. \gamma \in P_{m_0}(M), \ t \in [0,1]. \tag{4.3}$$

According to [5] [14] [11], the vector field X_h generates a global unique flow denoted by $\{\Phi_s, s \in R\}$ on $P_{m_0}(M)$ such that for μ-a.s. $\gamma \in P_{m_0}(M)$,

$$\begin{array}{rcl} \dfrac{d}{ds}\Phi_s(\gamma) & = & X_h(\Phi_s(\gamma)), \quad s \in R, \\ \Phi_0(\gamma) & = & \gamma. \end{array} \tag{4.4}$$

Moreover, the Wiener measure μ on $P_{m_0}(M)$ is quasi-invariant under the flow Φ_t. Therefore for any measurable functional F on $P_{m_0}(M)$ and for all $t \in R$, $F(\Phi_t(\gamma)) - F(\gamma)$ is well defined for μ-a.s. $\gamma \in P_{m_0}(M)$. We call $\{\Phi_s, s \in R\}$ Driver's flow on $P_{m_0}(M)$ generated by X_h. In the sequel, we also let γ^s denote $\Phi_s(\gamma)$, $s \in R$, μ-a.s. $\gamma \in P_{m_0}(M)$.

4.2 Gradient operator on $P_{m_0}(M)$

Let E be a real separable Banach space. We let E^* denote the topological dual of E, i.e., the space of continuous linear functionals on E. The linear tensor product of H with E, denoted by $H \otimes E$, consists of finite summations of $h \otimes e$, where $h \otimes e$ is the bilinear form on $H \times E$ which maps $(k, l) \in H \times E^*$ into $< h, k > l(e)$. When $E = K$ is a real separable Hilbert space, we also let $H \otimes K$ denote the collection of Hilbert-Schmidt operators from H into K, i.e., $H \otimes K = L_{(2)}(H, K)$, on which we consider the Hilbert-Schmidt norm. For any $r \in N$, let $L^r(H, E)$ be the space of bounded r-linear operators from $H \times \ldots \times H$ into E, on which we consider the operator norm defined by $||A||_{L^r(H,E)} := \sup\limits_{||h_1||_H \leq 1, \ldots, ||h_r||_H \leq 1} ||A(h_1, \ldots, h_r)||_E$. We let $L^r_{(2)}(H, E)$ denote the collection of bounded r-linear operators $A \in L^r(H, E)$ such that: for any $h_1, \ldots, h_r \in H$,

$$A(h_1, \ldots, h_r) = \sum_{\alpha_1, \ldots, \alpha_r = 1}^{d} \int_{[0,1]^r} K^{\alpha_1 \ldots \alpha_r}_{s_1 \ldots s_r} \dot{h}_1^{\alpha_1}(s_1) \ldots \dot{h}_r^{\alpha_r}(s_r) ds_1 \ldots ds_r,$$

and

$$||A||_{L^r_{(2)}(H,E)} := \left[\sum_{\alpha_1, \ldots, \alpha_r = 1}^{d} \int_{[0,1]^r} ||K^{\alpha_1 \ldots \alpha_r}_{s_1 \ldots s_r}||_E^2 ds_1 \ldots ds_r \right]^{1/2} < +\infty.$$

Obviously, $||A||_{L^r(H,E)} \leq ||A||_{L^r_{(2)}(H,E)}$. If $A \in L^r_{(2)}(H,E)$, we call it a Hilbert-Schmidt type operator from $H \times \ldots \times H$ to E. When $r = 1$, we use the notation $L(H,E)$ (respectively, $L_{(2)}(H,E)$) instead of $L^1(H,E)$ (respectively, $L^1_{(2)}(H,E)$).

Definition 4.1 *We call $F : P_{m_0}(M) \to E$ an E-valued cylinder functional on $P_{m_0}(M)$, if there exists a finite partition of $[0,1]$, i.e., $0 \leq s_1 < s_2 < \ldots < s_k \leq 1$, $k \in N$, $l_1, \ldots, l_n \in E$, and $f_1, \ldots, f_n \in C^\infty(M^k, R)$, $n \in N$, such that*

$$F(\gamma) = \sum_{i=1}^n f_i(\gamma_{s_1}, \ldots, \gamma_{s_k}) l_i, \quad \gamma \in P_{m_0}(M). \tag{4.5}$$

Definition 4.2 *For any $F \in \mathcal{FC}^\infty(P_{m_0}(M), E)$, we define its gradient ∇F on $P_{m_0}(M)$ as follows: for μ-a.s. $\gamma \in P_{m_0}(M)$, $\nabla F(\gamma)$ is the unique element in $L_{(2)}(H,E)$ such that: for any $h \in H$, we have*

$$\nabla F(\gamma)(h) = \nabla_h F(\gamma),$$

where $\nabla_h F$ is the derivative of F along the direction $h \in H$:

$$\nabla_h F(\gamma) := \lim_{\epsilon \to 0} \frac{F(\Phi_\epsilon(\gamma)) - F(\gamma)}{\epsilon}, \quad \mu - \text{a.s. } \gamma \in P_{m_0}(M),$$

where $\{\Phi_\epsilon, \epsilon \in R\}$ is Driver's flow on $P_{m_0}(M)$ generated by X_h.

Indeed, for F given by (4.5), we have: for μ-a.s. $\gamma \in P_{m_0}(M)$,

$$\nabla_h F(\gamma) = \sum_{i=1}^n \sum_{j=1}^k f_i^{(j)}(\gamma) \left(U_{s_j}(\gamma) h(s_j)\right) l_i,$$

where $f_i^{(j)}(\gamma) = f_i^{(j)}(\gamma_{s_1}, \ldots, \gamma_{s_k}) \in L(T_{\gamma(s_i)}M, R)$ is the j-th partial differential of the smooth function $f_i \in C^\infty(M^k, R)$ with respect to the j-th variable, $i = 1, \ldots, n$, $j = 1, \ldots, k$. Let $h_i \in H$ be $h_i(\tau) = \tau \wedge s_i$, $\tau \in [0,1]$. Then

$$\nabla F(\gamma) = \sum_{i,j} f_i^{(j)}(\gamma) U_{s_j}(\gamma) h_j \otimes l_i,$$

$$\nabla_\tau^\alpha F(\gamma) = \sum_{i,j} 1_{[\tau < s_j]} f_i^{(j)}(\gamma) \left(U_{s_j}(\gamma) e_\alpha\right) l_i. \qquad (4.6).$$

Obviously, for μ-a.s. $\gamma \in P_{m_0}(M)$, $\nabla_\cdot^\alpha F(\gamma) \in L^2([0,1], E)$. Moreover,

$$\Big|\Big| ||\nabla F||_{L_{(2)}(H,E)} \Big|\Big|_{L^p(P_{m_0}(M),\mu)} = \left|\left| \left[\sum_{\alpha=1}^d \int_0^1 ||\nabla_\tau^\alpha F||_E^2 d\tau \right]^{1/2} \right|\right|_{L^p(P_{m_0}(M),\mu)} < \infty.$$

We call $\nabla_\tau^\alpha F$ the first order Malliavin derivatives of F.

4.3 Tangent process and the differential of the stochastic anti-development

Following [7] [4], an R^d-valued semimartingale ξ is called a tangent process (or an adapted vector field as in [7]) on the Wiener space X, if its Itô differential is given by $d\xi^\alpha(\tau) = \sum\limits_{\beta=1}^{d} a_{\alpha\beta}dx^\beta(\tau) + b^\alpha d\tau, \alpha = 1, \ldots, d$, with $a_{\alpha\beta} = -a_{\beta\alpha}$ and satisfying the regularity assumptions $\xi \in \mathcal{C}^1$, $a_{\alpha\beta} \in \mathcal{C}^1$, where $\mathcal{C}^1$ denotes the class of semimartingales η such that besides its representation in Itô integral, η can be also represented in terms of Stratonovich integrals.

For any $F \in \mathcal{FC}^\infty(X)$ given by $F(x) = f(x_{s_1}, \ldots, x_{s_k})$ and any tangent process ξ, we define the directional derivative of F along ξ in the following way:

$$D_\xi F(x) = \sum_{i=1}^{k} < f^{(i)}(x), \xi(s_i) > .$$

Theorem 4.3 *(Cruzeiro-Malliavin [4]) Let* ξ *be a tangent process on* X, $d\xi^\alpha(\tau) = a_{\alpha,\beta}dx_\beta(\tau) + c_\alpha d\tau$, *such that* $\int_0^1 ||a(\tau)||_{1,p}d\tau < +\infty, \forall p > 1$, *and* $\int_0^1 ||c(\tau)||_2 d\tau < +\infty$. *Then for any* $F \in W^{2,2+}(X) := \cap_{p>2}W^{2,p}(X)$, *we have* $F \in Dom(D_\xi)$, *moreover*

$$D_\xi F = \sum_{\alpha,\beta=1}^{d} \int_0^1 D_{\tau,\alpha}F(x)(a_{\alpha,\beta}(x,\tau)dx_\beta(\tau) + c_\alpha(x,\tau)d\tau),$$

where the stochastic integral is taken in the sense of Skorohod.

We call ξ a regular tangent process if ξ is a tangent process on X and satisfies the conditions of Theorem 4.3.

Theorem 4.4 *Let* ξ *be a regular tangent process, let* $\widetilde{\xi}(\gamma)(\tau) := U_\tau(\gamma)\xi(\tau)$, μ*-a.s.* $\gamma \in P_{m_0}(M), \tau \in [0,1]$. *Then* $J_*(\widetilde{\xi})$ *is a regular tangent process on the Wiener space. Moreover*

$$J_*(\widetilde{\xi})(x,\tau) = \xi(\tau) + \frac{1}{2}\int_0^\tau \widehat{Ric}_{r_x(s)}(\xi(s))ds - \int_0^\tau q_\xi(x,r)dx(r),$$

where J_* *denotes the differential of the stochastic anti-development* $J : P_{m_0}(M) \to X$, *and*

$$\widehat{q}_\xi(x,\tau) = \Theta_{r_x(\tau)}(\cdot, \xi(\tau)) + \int_0^\tau \Omega_{r_x(r)}(\xi(r), \circ dx(r)). \qquad (4.7)$$

where Θ *is the* R^d*-valued torsion form of* ∇, Ω *is the* $so(d)$*-valued curvature form of* ∇, *for any* $a \in R^d$, $r = (m,u) \in O(M)$,

$$\widehat{Ric}_r(a) := \sum_{i=1}^{d} u^{-1} \circ \widehat{R}_m(ue_i, ua)ue_i,$$

where $\widehat{R}$ *denotes the curvature tensor with respect to the dual connection* $\widehat{\nabla}$ *(see Section* 3*),* $(e_1, \ldots, e_d)$ *is the canonical orthonormal basis of* R^d.

Proof. See e.g. [5] [12] [14] [4]. □

4.4 Infinite differentiability of the stochastic horizontal lift

From (4.6), we see that the gradient ∇F of an E-valued cylinder functional F on $P_{m_0}(M)$ is an $L_{(2)}(H, E)$-valued functional on $P_{m_0}(M)$ but is no longer a cylinder functional since the stochastic horizontal lift $U_s(\gamma)$ is involved in ∇F. In order to deal with the high order differentiability of F along H-directions, we need to show the infinite differentiability of the stochastic horizontal lift along Cameron-Martin directions.

Theorem 4.5 *The stochastic horizontal lift $U_s : P_{m_0}(M) \to O(M)$ is infinitely differentiable along H-directions. More precisely, for any $r \in N$, and for any $h, h_1, \ldots, h_{r-1} \in H$, the following covariant derivatives can be defined recursively:*

$$\nabla_h U_s(\gamma) := \frac{\nabla}{\partial \epsilon} U_s(\gamma^\epsilon)\Big|_{\epsilon=0} \in T_{U_s(\gamma)}O(M),$$

$$\ldots$$

$$\nabla^r_{h_1 \ldots h_{r-1} h} U_s(\gamma) := \frac{\nabla}{\partial \epsilon} \nabla_{h_1 \ldots h_{r-1}} U_s(\gamma^\epsilon)\Big|_{\epsilon=0} \in T_{U_s(\gamma)}O(M),$$

where $\gamma^\epsilon = \Phi_\epsilon(\gamma)$ denotes Driver's flow on $P_{m_0}(M)$ generated by the vector field X_h, $\frac{\nabla}{\partial \epsilon}$ denotes the covariant derivative along $\epsilon \to \gamma^\epsilon_s$ with respect to the given TSS connection ∇.

Moreover, for μ-a.s. $\gamma \in P_{m_0}(M)$, there exist $\nabla^{\alpha_1 \ldots \alpha_r}_{\tau_1 \ldots \tau_r} U_s(\gamma) \in L^2([0,1]^r, T_{U_s(\gamma)}O(M))$ such that:

(1) *for any $s \in [0, \tau_1 \vee \ldots \vee \tau_r]$, and any $\alpha_1, \ldots, \alpha_r = 1, \ldots, d$,*

$$\nabla^{\alpha_1 \ldots \alpha_r}_{\tau_1 \ldots \tau_r} U_s(\gamma) = 0,$$

(2)

$$\nabla^r_{h h_1 \ldots h_{r-1}} U_s(\gamma) = \sum_{\alpha_i = 1}^{d} \int_{[0,1]^r} \nabla^{\alpha_1 \ldots \alpha_r}_{\tau_1 \ldots \tau_r} U_s(\gamma) \dot{h}^{\alpha_1}(\tau_1) \ldots \dot{h}^{\alpha_r}_{r-1}(\tau_r) d\tau_1 \ldots d\tau_r,$$

(3) *for any $p > 1$,*

$$\sup_{\tau_1, \ldots, \tau_r \in [0,1]} E\left[\sup_{\tau_1 \vee \ldots \vee \tau_r \le s \le 1} ||\nabla^{\alpha_1 \ldots \alpha_r}_{\tau_1 \ldots \tau_r} U_s(\gamma)||^p_{T_{U_s(\gamma)}O(M)}\right] < +\infty,$$

Proof. First, we have the Bismut formula ([2]):

$$\nabla_h U_s(\gamma) = U_s(\gamma) q_h(\gamma, s) \tag{4.8}$$

where

$$q_h(\gamma, s) = \int_0^s \Omega_{r_{x(u)}}(\circ dx_u, h(u)). \tag{4.9}$$

For a proof, see Eq. (5.10) of [5] (p.321) and Theorem 5.2 of [4] (p.145-147).

By (4.8) and applying the stochastic Fubini theorem ([23]), we have $q_h(x,s) = \int_0^s q_\alpha(\tau,s)\dot{h}_\tau^\alpha d\tau$, where $q_\alpha(\tau,s) = 1_{[\tau\le s]}\int_\tau^s \Omega_{U_u(\gamma)}(\circ dx_u, e_\alpha)d\tau$. Hence

$$\nabla_h U_s(\gamma) = U_s(\gamma)\int_0^s \dot{h}_\tau^\alpha q_\alpha(\tau,s)d\tau.$$

Thus $\nabla_\tau^\alpha U_s(\gamma) = U_s(\gamma)q_\alpha(\tau,s)$. The Burkholder-Davis-Gundy inequality yields the third statement of Theorem 4.5 for $r=1$. See also [4] (Theorem 5.2, p. 145).

By (4.8), in order to define $\nabla_h\nabla_{h_{r-1}}\dots\nabla_{h_1}U_s(\gamma)$, we need only to show that $\nabla_h\nabla_{h_{r-1}}\dots\nabla_{h_2}q_{h_1}(\gamma,s)$ is well defined. Indeed, $q_{h_1}(I(x),s)$ is a Stratonovich stochastic integral. By the standard argument in Malliavin calculus, we can prove that $q_{h_1}(I(x),s)\in W^{\infty,\infty}(X,so(d))$. Let X_h, X_{h_i}, $i=1,\dots,r-1$, be the vector fields defined by (4.3) and set $\xi = J_*(X_h)$, $\xi_i = J_*(X_{h_i})$, $i=2,\dots,r-1$. Then by Theorem 4.4, we have

$$\nabla_h\nabla_{h_{r-1}}\dots\nabla_{h_2}q_{h_1}(\gamma,s) = D_\xi D_{\xi_{r-1}}\dots D_{\xi_2}q_{h_1}(I(x),s).$$

Using Theorem 4.3, one can represent $D_\xi D_{\xi_{r-1}}\dots D_{\xi_2}q_{h_1}(I(x),s)$ as a multiple Skorohod stochastic integral and prove that $D_\xi D_{\xi_{r-1}}\dots D_{\xi_2}q_{h_1}(I(x),s)\in L^p(\mu)$ for any $p>1$. This finishes the proof of Theorem 4.5. □

By the chain rule and Theorem 4.5, we can easily prove the following

Corollary 4.6 *Any $F\in\mathcal{FC}^\infty(P_{m_0}(M),E)$ is infinitely differentiable along H-directions. More precisely, for any $r\in N$ and μ-a.s. $\gamma\in P_{m_0}(M)$, $\nabla^r F(\gamma)\in L^r_{(2)}(H,E)$ and*

$$E\left[\left|\left||\nabla^r F||_{L^r_{(2)}(H,E)}\right|\right|^p\right] < \infty, \quad \forall p>1.$$

4.5 Closability of gradient operators on the path space

Theorem 4.7 *For any $r\in N$ and any $p>1$, ∇^r is closable from $L^p(P_{m_0}(M),E)$ into $L^p(P_{m_0}(M),L^r_{(2)}(H,E))$.*

Proof. Let $F_n\to 0$ in $L^p(P_{m_0}(M),E)$ and $\nabla^r F_n\to u$ in $L^p(P_{m_0}(M),L^r_{(2)}(H,E))$. We need to show that $u=0$. This is equivalent to prove $E[(u,G)]=0$ for any G of the form $G(\gamma)=g(\gamma)h_{j_1}\otimes\dots\otimes h_{j_r}\otimes l^*$, where $l^*\in E^*$, $g\in\mathcal{FC}^\infty(P_{m_0}(M))$, $h_{j_1},\dots,h_{j_r}$ are running over a fixed ONB $\{h_n,n\in N\}$ of H.

We have $E[(\nabla^r F_n,G)]\to E[(u,G)]$ since $\nabla^r F_n\to u$ in $L^p(P_{m_0}(M),L^r_{(2)}(H,E))$ and G is bounded. On the other hand, if we can prove that there exists a functional on $P_{m_0}(M)$, denoted by $(\nabla^*)^r G$, such that for every $F\in\mathcal{FC}^\infty(P_{m_0}(M),E)$,

$$E\left[(\nabla^r F,G)\right] = E\left[(F,(\nabla^*)^r G)\right], \tag{4.10}$$

$$(\nabla^*)^r G\in L^q(P_{m_0}(M),E^*), \quad \forall q>1, r\in N, \tag{4.11}$$

then $E[(F_n,(\nabla^*)^r G)]\to 0$ since $F_n\to 0$ in $L^p(P_{m_0}(M),E)$. Therefore $E[(u,G)]=0$.

Now we verify that (4.10) and (4.11). First, we prove these for $r=1$. By linearity and using the integration by parts formula for real valued cylinder functional on

$P_{m_0}(M)$ (see e.g. [5] [12] [14][15]), we can easily prove (4.10) and by [15] (p.294 formula (2.8) and p.329 Theorem 6.3, 6.4), we can prove

$$\nabla^* G(\gamma) = \left(-\nabla_{h_j} + \delta(h_j)\right) g(\gamma) l^*. \tag{4.12}$$

where for μ-a.s.$\gamma \in P_{m_0}(M)$,

$$\delta(h_j)(\gamma) = \int_0^1 (\dot{h}_j(\tau) + \frac{1}{2}\widehat{Ric}_{r_x(\tau)}(h_j(\tau)), dx(\tau)). \tag{4.13}$$

By [12] [15], $\delta(h) \in L^q(P_{m_0}(M))$, $\forall q > 1$. Hence (4.11) holds for $r = 1$.

Remark 4.8 *Suppose that we have proved* (4.10) *and* (4.11) *for $r \leq k$ with*

$$(\nabla^*)^r (g h_1 \otimes \ldots h_r) = (-\nabla_{h_r} + \delta(h_r)) \ldots (-\nabla_{h_1} + \delta(h_1)) \, g. \tag{4.14}$$

Then ∇^k can be extended to $D^{k,p}(P_{m_0}(M), E)$ which is the completion of $\mathcal{FC}^\infty(P_{m_0}(M), E)$ with respect to the seminorm defined by:

$$||F||_{D^{k,p}(P_{m_0}(M),E)} := ||F||_p + \sum_{i=1}^k \left|\left| ||\nabla^i F||_{L^i_{(2)}(H,E)} \right|\right|_p .$$

Moreover, by the argument of continuity, we can prove that (4.10), (4.11) *and* (4.14) *remain true for $F \in D^{k,p}(P_{m_0}(M), E)$ and for $G(\gamma) = g(\gamma)(h_1 \otimes \ldots \otimes h_r) \otimes l^*$ such that $g \in \cap_{q>1} D^{k,q}(P_{m_0}(M), R)$.*

Now we prove that (4.10), (4.11) and (4.14) hold when $r = k+1$ and $G(\gamma) = g(\gamma)(h_{j_1} \otimes \ldots \otimes h_{j_{k+1}}) \otimes l^*$ with $g \in \mathcal{FC}^\infty(P_{m_0}(M))$. Let $\widetilde{F} = \nabla^k F$ and $\widetilde{G} = \nabla^* G$. By Corollary 4.6, we have $\widetilde{F} \in D^{1,p}(P_{m_0}(M), L^k_{(2)}(H, E))$. Using (4.12), we have $\widetilde{G} = \left[-\nabla_{h_{j_1}} g + \delta(h_{j_1}) g\right] (h_{j_2} \otimes \ldots \otimes h_{j_{k+1}}) \otimes l^*$. By (4.13), we can prove $\delta(h_j) \in \cap_{k \in N, p>1} D^{k,p}(P_{m_0}(M))$ (see Proposition 4.9 below). By this and Corollary 4.6, we have $-\nabla_{h_{j_1}} g + \delta(h_{j_1}) g \in \cap_{q>1} D^{k,q}(P_{m_0}(M), R)$. Therefore by the last sentence of Remark 4.8, we have

$$\begin{aligned} E\left[(\nabla^{k+1} F, G)\right] &= E\left[(\nabla(\widetilde{F}), G)\right] = E\left[(\widetilde{F}, \nabla^* G)\right] \\ &= E\left[(\nabla^k F, \widetilde{G})\right] = E\left[(F, (\nabla^*)^k \widetilde{G})\right]. \end{aligned}$$

Thus we have proved (4.10) for $r = k+1$ by letting $(\nabla^*)^{k+1} G(\gamma) := (\nabla^*)^k (\widetilde{G})(\gamma)$. Moreover, by (4.14) and by the assumption of the induction, also by the last sentence of Remark 4.8, we have

$$\begin{aligned} (\nabla^*)^{k+1} G(\gamma) &= \left[\left(-\nabla_{h_{j_{k+1}}} + \delta(h_{j_{k+1}})\right) \ldots \left(-\nabla_{h_{j_2}} + \delta(h_{j_2})\right)\right] \left(-\nabla_{h_{j_1}} + \delta(h_{j_1}) g\right) l^* \\ &= \left[\left(-\nabla_{h_{j_{k+1}}} + \delta(h_{j_{k+1}})\right) \ldots \left(-\nabla_{h_{j_2}} + \delta(h_{j_2})\right) \left(-\nabla_{h_{j_1}} + \delta(h_{j_1}\right) g\right] l^*. \end{aligned}$$

This proves that (4.14) holds for $r = k+1$.

The rest part of the proof is to check that (4.11) holds for $r = k+1$, i.e.,

$$\left(-\nabla_{h_{j_{k+1}}} + \delta(h_{j_{k+1}})\right) \dots \left(-\nabla_{h_{j_1}} + \delta(h_{j_1})\right) g \in \cap_{q>1} L^q(P_{m_0}(M)),$$

which is an easy consequence of the fact that for any $n \in N$, $h_1, \dots, h_n \in H$,

$$E_\mu \left[|\nabla_{h_1} \dots \nabla_{h_n} g|^p \right] < \infty, \quad \forall p > 1,$$

which is true by Corollary 4.6, and the following

Proposition 4.9 *For any $n \in N$, $h_1, \dots, h_n \in H$, we have*

$$E_\mu \left[|\nabla_{h_1} \dots \nabla_{h_n} \delta(h)|^p \right] < \infty, \quad \forall p > 1. \tag{4.15}$$

Proof. By Eq. (4.13), we can prove that (see Theorem 2 of [19]): for any $r \in N$ and $p > 1$, there is a constant $C = C(r,p)$ such that $||\delta(h) \circ I||_{W^{r,p}(X)} \le C||h||_H$. This implies $\delta(h) \circ I \in W^{\infty,\infty}(X)$. By Theorem 4.4, for $\gamma = I(x)$,

$$\nabla_{h_1} \dots \nabla_{h_n} \delta(h)(\gamma) = D_{\xi_1} \dots D_{\xi_n} (\delta(h) \circ I)(x), \tag{4.16}$$

where ξ_i is the following regular tangent process:

$$\xi_i(t) = h_i(t) + \frac{1}{2} \int_0^t \widehat{Ric}_{r_x(s)}(h_i(s))ds - \int_0^t \widehat{q}_{h_i}(x,s)dx(s), \tag{4.17}$$

with

$$\widehat{q}_{h_i}(x,t) = \Theta_{r_x(\tau)}(\cdot, h_i(\tau)) + \int_0^t \Omega_{r_x(s)}(h_i(s), \circ dx(s)). \tag{4.18}$$

Let $\widehat{\delta(h)} = \delta(h) \circ I$. Then

$$D_{\xi_1} \dots D_{\xi_n} \widehat{\delta(h)} = \sum_{\alpha_1,\dots,\alpha_n=1}^{d} \int_{[0,1]^n} D^{\alpha_1 \dots \alpha_n}_{\tau_1 \dots \tau_n} \widehat{\delta(h)} \circ d\xi_1^{\alpha_1}(\tau_1) \dots \circ d\xi_n^{\alpha_n}(\tau_n), \tag{4.19}$$

where the right-hand-side is a multiple Stratonovich stochastic integral. Applying Theorem 4.3, we can represent the right-hand-side of (4.19) as a multiple Skorohod stochastic integral. Using Proposition 3.3, the L^p-inequality of Skorohod integral (Proposition 2.1), and the fact that all covariant derivatives of the curvature form and the tensor form are bounded on compact Riemannian manifold $O(M)$, one can verify that

$$E_\mu \left[\left| D_{\xi_1} \dots D_{\xi_n} \widehat{\delta(h)} \right|^p \right] < \infty, \quad \forall p > 1.$$

The proof of Proposition 4.9 and Theorem 4.7 are complete. □

Remark 4.10 *Indeed, in view of Theorem 7.3 that we will prove in Section 7, for any $n \in N$ and any $p > 1$, there exists a constant $C = C(n,p)$ such that*

$$||\nabla^n \delta(h)||_{L^p(P_{m_0}(M), H^{\otimes n})} \le C||h||_H. \tag{4.20}$$

This is stronger than (4.15).

5 Sobolev space and capacity on path spaces

5.1 Sobolev space $D^{r,p}(P_{m_0}(M), E)$

The above Theorem 4.7 and Corollary 4.6 lead us to introduce the following

Definition 5.1 *For any $r \in N$ and $p > 1$, we define the Sobolev space $D^{r,p}(P_{m_0}(M), E)$ of E-valued functionals on $P_{m_0}(M)$ as the completion of $\mathcal{FC}^\infty(P_{m_0}(M), E)$ with respect to the norm $|| \cdot ||_{D^{r,p}(P_{m_0}(M),E)}$ defined by*

$$||F||_{D^{r,p}(P_{m_0}(M),E)} := ||F||_{L^p(P_{m_0}(M),\mu)} + \sum_{k=1}^{r} \Big|\Big| ||\nabla^k F||_{L^k_{(2)}(H,E)} \Big|\Big|_{L^p(P_{m_0}(M),\mu)} \tag{5.1}$$

For any $F \in D^{r,p}(P_{m_0}(M), E)$, we can prove that $\nabla^r F$ has an unique kernel $\nabla^{\alpha_1,\dots,\alpha_r}_{\tau_1,\dots,\tau_r} F \in L^p(X, L^2([0,1], E))$, the so-called r-th order Malliavin derivatives of F, such that: for any $h_1, \dots, h_r \in H$,

$$\nabla^r F(\gamma)(h_1, \dots, h_r) = \sum_{1 \le \alpha_i \le d} \int_{[0,1]^r} \nabla^{\alpha_1 \dots \alpha_r}_{\tau_1 \dots \tau_r} F(\gamma) \dot{h}_1^{\alpha_1}(\tau_1) \dots \dot{h}_r^{\alpha_r}(\tau_r) d\tau_1 \dots d\tau_r,$$

and

$$\Big|\Big| ||\nabla^r F||_{L^r_{(2)}(H,E)} \Big|\Big|_p^p = E\left[\sum_{1 \le \alpha_i \le d} \int_{[0,1]^r} ||\nabla^{\alpha_1 \dots \alpha_r}_{\tau_1 \dots \tau_r} F(x)||_E^2 d\tau_1 \dots d\tau_r \right]^{\frac{p}{2}} < \infty.$$

Let

$$D^{\infty,\infty}(P_{m_0}(M), E) = \cap_{r \in N} \cap_{p>1} D^{r,p}(P_{m_0}(M), E).$$

If $F \in D^{\infty,\infty}(P_{m_0}(M), E)$, we call it an E-valued smooth functional (in the sense of stochastic calculus of variation) on $P_{m_0}(M)$. When $E = R$, we let $D^{r,p}(P_{m_0}(M))$ denote $D^{r,p}(P_{m_0}(M), R)$, for all $r \in N \cup \{+\infty\}$ and $p \in (1, +\infty]$.

5.2 (r,p)-capacity on $P_{m_0}(M)$

Definition 5.2 *For any $r \in N$, $p > 1$, the (r,p)-capacity $C_{r,p}$ on $P_{m_0}(M)$ is defined as follows:*

(1) *for any open subset $O \subset P_{m_0}(M)$,*

$$C_{r,p}(O) := inf\{||f||_{D^{r,p}(P_{m_0}(M))} : f \ge 0, \mu - a.s., f \ge 1, \mu - \text{a.s. on } O\}; \tag{5.2}$$

(2) *for any subset $A \subset P_{m_0}(M)$,*

$$C_{r,p}(A) := inf\{C_{r,p}(O) : A \subset O, O \text{ is open in } P_{m_0}(M)\}. \tag{5.3}$$

The notion of *slim set* on $P_{m_0}(M)$, (r,p)-quasi-continuity and (r,p)-quasi-continuous modification as well as ∞-quasi-continuous modification of functionals on $P_{m_0}(M)$ can be similarly defined as in Section 2. If the Banach norm on E satisfies the conditions of Proposition 2.3, then any $F \in D^{r,p}(P_{m_0}(M), E)$ has an (r,p)-quasi-continuous modification, $r \in N \cup \{\infty\}$, $p \in (1, \infty]$.

5.3 Sobolev space and capacity on the path space $P_{m_0}^{2m,\alpha}(M)$

For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, by [1] [3], the Wiener measure μ is supported on $P_{m_0}^{2m,\alpha}(M)$. For μ-a.s. $\gamma \in P_{m_0}^{2m,\alpha}(M)$, the stochastic horizontal lift $\{U_t(\gamma), t \in [0,1]\}$ along $\{\gamma_t, t \in [0,1]\}$ is defined as the unique solution of the Stratonovich covariant SDE (4.1). By analogue of Section 4, for any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, we can define the gradient operator ∇ on $\mathcal{FC}^\infty(P_{m_0}^{2m,\alpha}(M), E)$ (the collection of E-valued cylinder functionals on $P_{m_0}^{2m,\alpha}(M)$) and prove that Theorem 4.4 holds on $P_{m_0}^{2m,\alpha}(M)$. Similarly to the proof of Theorem 4.5, we can show that the stochastic horizontal lift $U : \gamma \in P_{m_0}^{2m,\alpha}(M) \to U_s(\gamma)$ is infinitely differentiable along H-directions. The integration by parts formula on $P_{m_0}^{2m,\alpha}(M)$ can be similarly proved as in the case of $P_{m_0}(M)$ (cf. [12]). For any real separable Banach space E, one can prove that ∇^r is closable from $L^p(P_{m_0}^{2m,\alpha}(M), E)$ into $L^p(P_{m_0}^{2m,\alpha}(M), L^r_{(2)}(H, E))$, $r \in N$, $p > 1$. These lead us to introduce the Sobolev space $D^{r,p}(P_{m_0}^{2m,\alpha}(M), E)$ of E-valued functionals on $P_{m_0}^{2m,\alpha}(M)$ and the associated (r,p)-capacity on $(P_{m_0}^{2m,\alpha}(M), \rho_{2m,\alpha})$, $r \in N$, $p > 1$. We let $||\cdot||_{D^{r,p}(P_{m_0}^{2m,\alpha}(M),E)}$ denote the (r,p)-Sobolev norm of E-valued functionals on $P_{m_0}^{2m,\alpha}(M)$. Without confusion of notation, we let $C_{r,p}$ denote the associated (r,p)-capacity on $(P_{m_0}^{2m,\alpha}(M), \rho_{2m,\alpha})$, $r \in N$, $p > 1$.

Similarly, the notion of slim set, (r,p)-quasi-continuity and (r,p)-quasi-continuous modification can be also defined for any $r \in N \cup \{+\infty\}$ and $p \in (1, +\infty]$ on $P_{m_0}^{2m,\alpha}(M)$, $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$. If the Banach norm $||\cdot||_E$ is smooth (in the sense of Fréchet-Gâteaux) in $E \setminus \{0\}$ and all of its k-th order derivatives ($k \in N$) are bounded on the unit ball of E (see Proposition 2.3), then any smooth functional $F \in D^{\infty,\infty}(P_{m_0}^{2m,\alpha}(M), E)$ has an ∞-quasi-continuous modification, $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$.

6 Sobolev norm comparison theorem: (1)

In this section we prove the first Sobolev norm comparison inequality in Theorem 1.1. For this, we need some preliminary results.

6.1 Change of variables formula of the Itô map

Lemma 6.1 *Let $F \in \mathcal{FC}^\infty(P_{m_0}(M))$ and $\widehat{F} = F \circ I$. Then for μ-a.s. $\gamma \in P_{m_0}(M)$, $\alpha = 1, \ldots, d$, and $\tau \in [0,1]$,*

$$\begin{aligned} \nabla^\alpha_\tau F(\gamma) &= D^\alpha_\tau \widehat{F}(x) + \tfrac{1}{2}\int_\tau^1 (\widehat{Ric}_{r_x(s)}(D_s\widehat{F}(x)), e_\alpha)ds \\ &\quad + \int_0^1 (\widehat{q}_\alpha(\tau, s)D_s\widehat{F}(x), dx_s), \end{aligned} \tag{6.1}$$

where $D^\alpha_\tau \widehat{F}$ denotes the Malliavin derivative of $\widehat{F}$, the stochastic integral is taken in the sense of Skorohod, $x = I^{-1}(\gamma)$, and

$$\widehat{q}_\alpha(\tau, s) := 1_{[\tau,1]}(s)\left[\Theta_{r_x(s)}(\cdot, e_\alpha) + \int_\tau^s \Omega_{r_x(u)}(e_\alpha, \circ dx_u)\right]. \tag{6.2}$$

Proof. Indeed, Theorem 4.4 says that for any $h \in H$,

$$\nabla_h F(\gamma) = D_\xi \widehat{F}(x), \quad \gamma = I(x), \tag{6.3}$$

where ξ is the regular tangent process whose Itô differential is given by

$$d\xi(s) = \left(\dot{h}(s) + \frac{1}{2}\widehat{Ric}_{rx(s)}(h(s))\right) ds - \widehat{q}_h(x,s)dx(s), \tag{6.4}$$

with the initial condition $\xi(0) = 0$, where $\widehat{q}_h(x,s) = \int_0^s \widehat{q}_\alpha(\tau,s)\dot{h}^\alpha(\tau)d\tau$.

Theorem 4.3 and the stochastic Fubini theorem imply that

$$\begin{aligned} D_\xi F(x) &= \sum_{\alpha=1}^d \int_0^1 \int_0^1 (\widehat{q}_\alpha(\tau,s)D_s\widehat{F}(x), dx(s))\dot{h}^\alpha(\tau)d\tau \\ &\quad + \sum_{\alpha=1}^d \int_0^1 \left(D_\tau^\alpha \widehat{F}(x) + \frac{1}{2}\int_\tau^1 (\widehat{Ric}_{rx(s)}(D_s\widehat{F}(x)), e_\alpha)ds\right)\dot{h}^\alpha(\tau)d\tau, \end{aligned}$$

where the stochastic integral is taken in the sense of Skorohod. From the above identity and (6.3), also using $\nabla_h F(\gamma) = \sum_{\alpha=1}^d \int_0^1 \nabla_\tau^\alpha F(\gamma)\dot{h}^\alpha(\tau)d\tau$, we obtain (6.1). $\square$

Corollary 6.2 *The Itô map $I : X \to P_{m_0}(M)$ is not a quasi-homeomorphism between the Wiener space and the path space except that M is flat, i.e., $\widehat{q}_\alpha(\tau,s)(\gamma) = 0$, $\forall \alpha = 1,\ldots,d$, $\forall \tau, s \in [0,1]$, $\mu - a.s. \gamma \in P_{m_0}(M)$.*

Proof. Suppose that $I : X \to P_{m_0}(M)$ is a quasi-homeomorphism between the Wiener space and the path space, then for any $r \in N, p > 1$, $||F|||_{D^{r,p}(P_{m_0}(M))}$ is equivalent to $||F \circ I||_{W^{r,p}(X)}$. Taking $r = 1$ and $p = 2$, using Lemma 6.1 we can conclude that

$$E\left[\int_0^1 \left|\int_0^1 \widehat{q}_\alpha(\tau,s)D_s\widehat{F}(x)dx_s\right|^2 d\tau\right] \le C||F||_{1,2}^2 \tag{6.5}$$

for some constant C.

Let $u(\tau,s) = \widehat{q}_\alpha(\tau,s)D_s\widehat{F}(x)$. Since D_sF is not adapted, $\int_0^1 u(\tau,s)dx_s$ is a Skorohod integral. By the energy identity ([21] P.169), we have

$$\begin{aligned} E\left[\int_0^1 \left|\int_0^1 u(\tau,s)dx_s\right|^2 d\tau\right] &= E\left[\int_{[0,1]^3} D_r u(\tau,s)D_s u(\tau,r)drdsd\tau\right] \\ &\quad + E\left[\int_0^1 \int_0^1 |u(\tau,s)|^2 dsd\tau\right]. \end{aligned} \tag{6.6}$$

The second order derivative D^2F appears in the first term of the right-hand-side of (6.6). From this we see that (6.5) holds if and only if

$$E\left[\int_0^1 \int_0^1 \int_0^1 D_r u(\tau,s)D_s u(\tau,r)drdsd\tau\right] = 0.$$

This can not be true in the general case of Riemannian manifold M with a TSS connection except that $\widehat{q}_\alpha(\tau,s) = 0$ for all μ-a.s. $\gamma \in P_{m_0}(M)$, $\tau, s \in [0,1]$, $\alpha = 1,\ldots,d$, i.e., M is flat. $\square$

6.2 Proof of Theorem 1.1(1): $r = 1$

Since $O(M)$ is compact, we can prove that

$$E\left[\int_0^1 \left| D_\tau^\alpha \widehat{F} + \frac{1}{2}\int_\tau^1 (\widehat{Ric}_{r_x(s)}(D_s\widehat{F}), e_\alpha)ds\right|^2 d\tau\right]^{p/2} \leq C_p ||D\widehat{F}||_p^p. \tag{6.7}$$

By Lemma 6.1, for the proof of Theorem 1.1(1) for $r = 1$, setting $u(x, \cdot) = D\widehat{F}(x)(\cdot)$, we need only to prove: for any $p \geq 2$ and $\epsilon > 0$,

$$E\left[\int_0^1 \left|\int_0^1 \widehat{q}_\alpha(\tau, s)\dot{u}(s)dx_s\right|^2 d\tau\right]^{\frac{p}{2}} \leq C||u||_{1,p+\epsilon}^p. \tag{6.8}$$

Indeed, the left-hand-side is dominated by $\sup\limits_{\tau\in[0,1]} E\left[\left|\int_0^1 \widehat{q}_\alpha(\tau, s)\dot{u}(s)dx_s\right|^p\right]$. By Proposition 2.1, we have

$$\sup_{\tau\in[0,1]} E\left[\left|\int_0^1 \widehat{q}_\alpha(\tau, s)\dot{u}(s)dx_s\right|^p\right] \leq A_1 + A_2,$$

where

$$\begin{aligned} A_1 &= C\sup_{\tau\in[0,1]}\left[\int_0^1 |E[\widehat{q}_\alpha(\tau, s)\dot{u}(s)]|^2 ds\right]^{\frac{p}{2}}, \\ A_2 &= C\sup_{\tau\in[0,1]} E\left[\int_0^1\int_0^1 |D_r(\widehat{q}_\alpha(\tau, s)\dot{u}(s))|^2 dr ds\right]^{\frac{p}{2}}. \end{aligned}$$

Let us first estimate A_1:

$$\begin{aligned} A_1 &\leq C\sup_{\tau\in[0,1]}\left[\int_0^1 (E|\widehat{q}_\alpha(\tau, s)|^2)(E|\dot{u}(s)|^2)ds\right]^{\frac{p}{2}} \\ &\leq C\sup_{\tau\in[0,1]}\left[\sup_{s\in[\tau,1]} E|\widehat{q}_\alpha(\tau, s)|^2\right]^{\frac{p}{2}}\left[\int_0^1 E|\dot{u}(s)|^2 ds\right]^{\frac{p}{2}} \\ &\leq C\sup_{\tau\in[0,1]} E\left[\sup_{s\in[\tau,1]} |\widehat{q}_\alpha(\tau, s)|^p\right]\left[E\left(\int_0^1 |\dot{u}(s)|^2 ds\right)\right]^{\frac{p}{2}} \\ &\leq C\sup_{\tau\in[0,1]} E\left[\sup_{s\in[\tau,1]} |\widehat{q}_\alpha(\tau, s)|^p\right] E\left[\int_0^1 |\dot{u}(s)|^2 ds\right]^{\frac{p}{2}} \\ &\leq C_1||u||_2^p, \end{aligned}$$

where by (6.2) and by the Burkholder-Davis-Gundy inequality,

$$C_1 \leq C\sup_{u\in O(M)} ||\Theta_u||^p + C\sup_{\tau\in[0,1]} E\left[\sup_{s\in[\tau,1]}\left|\int_\tau^s \Omega_{r_x(u)}(e_\alpha, \circ dx(u))\right|^p\right] < +\infty.$$

For the estimate of A_2, using $D_r(\widehat{q}_\alpha(\tau,s)\dot{u}(s)) = (D_r\widehat{q}_\alpha(\tau,s))\dot{u}(s) + \widehat{q}_\alpha(\tau,s)D_r\dot{u}(s)$, we have

$$\begin{aligned}
A_2 &\leq C\sup_{\tau\in[0,1]} E\left[\int_0^1\int_0^1 |D_r\widehat{q}_\alpha(\tau,s)|^2|\dot{u}(s)|^2 dsdr\right]^{\frac{p}{2}} \\
&\quad + C\sup_{\tau\in[0,1]} E\left[\int_0^1\int_0^1 |\widehat{q}_\alpha(\tau,s)|^2|D_r\dot{u}(s)|^2 dsdr\right]^{\frac{p}{2}} \\
&\leq C\sup_{\tau\in[0,1]} E\left[\left(\int_0^1 \sup_{s\in[\tau,1]}|D_r(\widehat{q}_\alpha(\tau,s))|^2 dr\right)\int_0^1|\dot{u}(s)|^2 ds\right]^{\frac{p}{2}} \\
&\quad + C\sup_{\tau\in[0,1]} E\left[\sup_{s\in[\tau,1]}|\widehat{q}_\alpha(\tau,s)|^2\int_0^1\int_0^1|D_r\dot{u}(s)|^2 dsdr\right]^{\frac{p}{2}}.
\end{aligned}$$

Applying the Hölder inequality with exponent $\eta > 1$ and its conjugate $\eta^* = \frac{\eta}{\eta-1}$, we obtain

$$\begin{aligned}
A_2 &\leq C\sup_{\tau\in[0,1]}\left[E\left(\int_0^1\sup_{s\in[\tau,1]}|D_r\widehat{q}_\alpha(\tau,s)|^2 dr\right)^{\frac{p\eta^*}{2}}\right]^{\frac{1}{\eta^*}}\left[E\left(\int_0^1|\dot{u}(s)|^2 ds\right)^{\frac{p\eta}{2}}\right]^{\frac{1}{\eta}} \\
&\quad + C\sup_{\tau\in[0,1]}\left[E\left(\sup_{s\in[\tau,1]}|\widehat{q}_\alpha(\tau,s)|^{p\eta^*}\right)\right]^{\frac{1}{\eta^*}}\left[E\left(\int_0^1\int_0^1|D_r\dot{u}(s)|^2 dsdr\right)^{\frac{p\eta}{2}}\right]^{\frac{1}{\eta}}.
\end{aligned}$$

Using the Burkholder-Davis-Gundy inequality and Proposition 3.3. we can prove

$$C_2 := \sup_{\tau\in[0,1]} E\left[\int_0^1\sup_{s\in[\tau,1]}|D_r\widehat{q}_\alpha(\tau,s)|^2 dr\right]^{\frac{p\eta^*}{2}} < \infty,$$

$$C_3 := \sup_{\tau\in[0,1]} E\left[\sup_{s\in[\tau,1]}|\widehat{q}_\alpha(\tau,s)|^{p\eta^*}\right] < \infty.$$

Therefore

$$A_2 \leq CC_2^{1/\eta^*}||u||_{p\eta}^p + CC_3^{1/\eta^*}||Du||_{1,p\eta}^p \leq C||u||_{1,p\eta}^p.$$

Combining the above estimates for A_1 and A_2, we obtain

$$\sup_{\tau\in[0,1]} E\left[\left|\int_0^1\widehat{q}_\alpha(\tau,s)\dot{u}(s)dx_s\right|\right]^p \leq C\left(||u||_2^p + ||u||_{1,p\eta}^p\right) \leq C||u||_{1,p\eta}^p.$$

Taking $\eta = 1 + \frac{\epsilon}{p}$ in the last inequality, we obtain (6.8). $\square$.

6.3 Proof of Theorem 1.1 (1): $r \geq 2$

By the explicit expression of ∇F in (4.6), we have

$$\nabla_\tau^\alpha F(\gamma) = \sum_{i=1}^n 1_{[\tau\leq s_i]}f^{(i)}(\gamma_x(s_1),\ldots,\gamma_x(s_n))r_x(s_i)r_0^{-1}e_\alpha.$$

By this and the fact that $x \to r_x(s_i)$ is $O(M)$-valued smooth Wiener functional (Proposition 3.3), we see that $\nabla^{\alpha}_{\tau} F \circ I \in W^{\infty,\infty}(X, H)$. Applying Theorem 4.4 to $F_1 = \nabla^{\alpha}_{\tau} F$, using the same argument as in the proof of Lemma 6.1, we can prove

$$\begin{aligned} \nabla^{\alpha_1\alpha_2}_{\tau_1\tau_2} F(\gamma) &= D^{\alpha_1}_{\tau_2}(\nabla^{\alpha_1}_{\tau_1} F \circ I)(x) + \frac{1}{2}\int_{\tau_2}^{\alpha_2} \widehat{Ric}_{r_x(s)}(D_s(\nabla^{\alpha_1}_{\tau_1} F \circ I)(x), e_{\alpha_2}) ds \\ &\quad + \int_0^1 \widehat{q}_{\alpha_2}(\tau_2, s) D_s(\nabla^{\alpha_1}_{\tau_1} F \circ I)(x) dx_s. \end{aligned} \tag{6.9}$$

Similarly, for any $r \in N$, by Corollary 4.6 and Theorem 3.4, we have $\nabla^{\alpha_1\ldots\alpha_r}_{\tau_1\ldots\tau_r} F \circ I \in W^{\infty,\infty}(X, H^{\otimes r})$. Moreover, using the same argument as in the proof of Lemma 6.1, we can prove

$$\begin{aligned} \nabla^{\alpha_1\ldots\alpha_r}_{\tau_1\ldots\tau_r} F(\gamma) &= D^{\alpha_r}_{\tau_r}(\nabla^{\alpha_1\ldots\alpha_{r-1}}_{\tau_1\ldots\tau_{r-1}} F \circ I)(x) + \frac{1}{2}\int_{\tau_r}^{\alpha_r} \widehat{Ric}_{r_x(s)}(D_s(\nabla^{\alpha_1\ldots\alpha_{r-1}}_{\tau_1\ldots\tau_{r-1}} F \circ I)(x), e_{\alpha_r}) ds \\ &\quad + \int_0^1 \widehat{q}_{\alpha_r}(\tau_r, s) D_s(\nabla^{\alpha_1\ldots\alpha_{r-1}}_{\tau_1\ldots\tau_{r-1}} F \circ I)(x) dx_s. \end{aligned} \tag{6.10}$$

Using the same argument as in the proof of Theorem 1.1 (1) for $r = 1$, we can prove

$$\begin{aligned} & E\left[||\nabla^r F||^p_{H^{\otimes r}}\right] := E\left[\left(\sum_{\alpha_1,\ldots,\alpha_r=1}^{d} \int_{[0,1]^r} |\nabla^{\alpha_1\ldots\alpha_r}_{\tau_1\ldots\tau_r} F|^2 d\tau_1 \ldots d\tau_r\right)^{\frac{p}{2}}\right] \\ \leq\ & C_p E\left[\left(\sum_{\alpha_1,\ldots,\alpha_r=1}^{d} \int_{[0,1]^r} \left|D^{\alpha_r}_{\tau_r}(\nabla^{\alpha_1\ldots\alpha_{r-1}}_{\tau_1\ldots\tau_{r-1}} F \circ I)\right|^2 d\tau_1 \ldots d\tau_r\right)^{\frac{p}{2}}\right] \\ & + C_p E\left[\left(\sum_{\alpha_1,\ldots,\alpha_r=1}^{d} \int_{[0,1]^r} \left|\int_{\tau_2}^{1} \widehat{Ric}_{r_x(s)}(D_s(\nabla^{\alpha_1\ldots\alpha_{r-1}}_{\tau_1\ldots\tau_{r-1}} F \circ I), e_{\alpha_r}) ds\right|^2 d\tau_1 \ldots d\tau_r\right)^{\frac{p}{2}}\right] \\ & + C_p E\left[\left(\sum_{\alpha_1,\ldots,\alpha_r=1}^{d} \int_{[0,1]^r} \left|\int_0^1 \widehat{q}_{\alpha_r}(\tau_r, s) D_s(\nabla^{\alpha_1\ldots\alpha_{r-1}}_{\tau_1\ldots\tau_{r-1}} F \circ I)(x) dx_s\right|^2 d\tau_1 \ldots d\tau_r\right)^{\frac{p}{2}}\right] \\ \leq\ & C_p E\left[||D(\nabla^{r-1} F \circ I)||^p_{H^{\otimes r}}\right] + C_p E\left[||D^2(\nabla^{r-1} F \circ I)||^{\frac{p\eta}{2}}_{H^{\otimes(r+1)}}\right]^{\frac{p}{p\eta}}. \end{aligned}$$

That is to say, for any $p \geq 2$ and $\epsilon > 0$,

$$||\nabla^r F||_p \leq C||\nabla^{r-1} F \circ I||_{2,p+\epsilon}. \tag{6.11}$$

We now in a position to state the following inequality.

Proposition 6.3 *Under the above notation, for any* $k = 1, \ldots, r-1$,

$$||\nabla^{r-k} F \circ I||_{2k,p} \leq C||\nabla^{r-k-1} F \circ I||_{2(k+1),p+\epsilon}, \tag{6.12}$$

Proof. We introduce a linear operator $A : \mathcal{FC}^\infty(X, H \otimes K) \to L^2(X, H \otimes K)$. For any $v \in \mathcal{FC}^\infty(X, H \otimes K)$ of the form

$$v(x,s) = \sum_{i=1}^n u_i(x) \otimes h_i(s), \tag{6.13}$$

where $x \in X$, $s \in [0,1]$, $u_i \in \mathcal{FC}^\infty(X, K)$, $\{h_i, i \in N\}$ is an ONB of H, $n \in N$, we define

$$\begin{aligned} Av(x,s) \;&=\; \sum_{i=1}^n u_i(x) \otimes h_i(s) + \frac{1}{2}\sum_{i=1}^n u_i(x) \otimes \int_0^s \int_\tau^1 \widehat{Ric}_{r_x(u)}(\dot{h}_i(u))du d\tau \\ &\quad + \sum_{i=1}^n u_i(x) \otimes \sum_{\alpha=1}^d \left(\int_0^s \int_0^1 (\widehat{q}_\alpha(\tau,u)\dot{h}_i(u), dx(u)) d\tau \right) e_\alpha. \end{aligned} \tag{6.14}$$

Similarly to the proof of Theorem 1.1 (1) for $r = 1$, for any $p \geq 2$ and $\epsilon > 0$, we can prove

$$\|Av\|_p \leq C_p ||v||_{1,p+\epsilon}.$$

Hence we can extend the domain of A from $\mathcal{FC}^\infty(X, H \otimes K)$ to

$$Dom(A) = W^{1,2+}(X, H \otimes K) = \cap_{p>2} W^{1,p}(X, H \otimes K).$$

We now want to show

$$\|Av\|_{r,p} \leq C_{r,p,\epsilon} ||v||_{r+1,p+\epsilon}. \tag{6.15}$$

Indeed, the chain rule implies

$$D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n}(Av) = \sum_{l=1}^n (D^{\alpha_1 \ldots \widehat{\alpha}_{i_1} \ldots \widehat{\alpha}_{i_l} \ldots \alpha_n}_{r_1 \ldots \widehat{r}_{i_1} \ldots \widehat{r}_{i_l} \ldots r_n} A)(D^{\alpha_{i_1} \ldots \alpha_{i_l}}_{r_{i_1} \ldots r_{i_l}} v).$$

where for $v \in \mathcal{FC}^\infty(X, H \otimes K)$ given by (6.13),

$$\begin{aligned} \left[(D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n} A) v \right](x,s) \;&=\; \frac{1}{2} \sum_{i=1}^n u_i(x) \otimes \int_0^s \int_\tau^1 D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n} \widehat{Ric}_{r_x(u)}(\dot{h}_i(u)) du d\tau \\ &\quad + \sum_{i=1}^n u_i(x) \otimes \sum_{\alpha=1}^d \left(\int_0^s \int_0^1 (D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n} \widehat{q}_\alpha(\tau,u) \dot{h}_i(u), dx_u) d\tau \right) e_\alpha \\ &\quad + \sum_{i=1}^n u_i(x) \otimes \sum_{\alpha=1}^d \left(\int_0^s (D^{\alpha_1 \ldots \widehat{\alpha}_i \ldots \alpha_n}_{r_1 \ldots \widehat{r}_i \ldots r_n} \widehat{q}_\alpha(\tau, r_i) \dot{h}_i(r_i), e_{\alpha_i}) d\tau \right) e_\alpha, \end{aligned}$$

or briefly,

$$\begin{aligned} \left[(D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n} A) v \right](x,s) \;&=\; \frac{1}{2} \int_0^s \int_\tau^1 D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n} \widehat{Ric}_{r_x(u)}(\dot{v}(x,u)) du d\tau \\ &\quad + \sum_{\alpha=1}^d \left(\int_0^s \int_0^1 (D^{\alpha_1 \ldots \alpha_n}_{r_1 \ldots r_n} \widehat{q}_\alpha(\tau,u) \dot{v}(x,u), dx_u) d\tau \right) e_\alpha \\ &\quad + \sum_{\alpha=1}^d \left(\int_0^s (D^{\alpha_1 \ldots \widehat{\alpha}_i \ldots \alpha_n}_{r_1 \ldots \widehat{r}_i \ldots r_n} \widehat{q}_\alpha(\tau, r_i) \dot{v}(x, r_i), e_{\alpha_i}) d\tau \right) e_\alpha, \end{aligned} \tag{6.16}$$

In order to prove (6.15), we have the following

Claim: For any $k \in N$, $p \geq 2$ and $\epsilon > 0$, we have

$$||(D^k A)v||_p := \left\{ E\left[\sum_{\alpha_1,\ldots,\alpha_k=1}^{d} \int_{[0,1]^k} ||(D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k} A)v||^2_{H\otimes K} dr_1 \ldots r_k \right]^{p/2} \right\}^{1/p} \leq C||v||_{1,p+\epsilon}. \tag{6.17}$$

Indeed, if this is true, then using $D^k(Av) = \sum_{i=0}^{k} (D^i A)(D^{k-i} v)$, we have

$$\begin{aligned} ||Av||_{r,p} &= \sum_{k=0}^{r} ||D^k(Av)||_p \leq \sum_{k=0}^{r} \sum_{i=1}^{k} ||(D^i A)(D^{k-i} v)||_p \\ &\leq C \sum_{k=0}^{r} \sum_{i=0}^{k} ||D^{k-i} v||_{1,p+\epsilon} \leq C \sum_{k=0}^{r} \sum_{i=0}^{k} ||v||_{k-i+1,p+\epsilon} \\ &\leq C||v||_{r+1,p+\epsilon}. \end{aligned}$$

Proof of Claim (6.17)

By Eq. (6.16), we have $||(D^k A)v||_p < I_1 + I_2 + I_3$, where

$$I_1 := E\left[\int_{[0,1]^k} \left(\int_0^1 \left\| \int_\tau^1 D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k} \widehat{Ric}_{r_x(s)}(\dot{v}(x,s), e_\alpha) ds \right\|_K^2 d\tau \right) dr_1 \ldots dr_k \right]^{p/2},$$

$$I_2 := E\left[\int_{[0,1]^k} \left(\int_0^1 \left\| \sum_{i=1}^{k} (D^{\alpha_1\ldots\widehat{\alpha}_i\ldots\alpha_k}_{r_1\ldots\widehat{r}_i\ldots r_k} \widehat{q}_\alpha(\tau, r_i)\dot{v}(x, r_i), e_{\alpha_i}) \right\|_K^2 d\tau \right) dr_1 \ldots dr_k \right]^{p/2},$$

$$I_3 := E\left[\int_{[0,1]^k} \left(\int_0^1 \left\| \int_0^1 (D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k} \widehat{q}_\alpha(\tau, s)u(x)\dot{v}(x,s), dx(s)) \right\|_K^2 d\tau \right) dr_1 \ldots dr_k \right]^{p/2}.$$

Let us estimate I_i, $i = 1, 2, 3$. For this, notice that by Proposition 3.3 and the chain rule, we have

$$D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k} \widehat{Ric}_{r_x(s)} = 0, \quad \forall s \in [0, r_1 \vee \ldots r_k],$$

$$D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k} \widehat{q}_\alpha(\tau, s) = 0, \quad \forall s \in [0, \tau \vee r_1 \vee \ldots r_k].$$

Thus

$$\begin{aligned}
I_1 &\leq E\left[\int_{[0,1]^{k+1}}\left(\sup_{s\in[\tau\vee r_1\vee\ldots\vee r_k,1]}|D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{Ric}_{r_x(s)}|^2\int_\tau^1||\dot{v}(x,s)||_K^2ds\right)d\tau dr_1\ldots dr_k\right]^{p/2}\\
&\leq E\left[\int_{[0,1]^{k+1}}\sup_{s\in[\tau\vee r_1\vee\ldots\vee r_k,1]}\left|D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{Ric}_{r_x(s)}\right|^2\left(\int_0^1||\dot{v}(x,s)||_K^2ds\right)d\tau dr_1\ldots dr_k\right]^{p/2}\\
&\leq E\left[\int_{[0,1]^{k+1}}\sup_{s\in[\tau\vee r_1\vee\ldots\vee r_k,1]}\left|D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{Ric}_{r_x(s)}\right|^p\left(\int_0^1||\dot{v}(x,s)||_K^2ds\right)^{p/2}d\tau dr_1\ldots dr_k\right]\\
&\leq \left[E\left(\int_{[0,1]^{k+1}}\sup_{s\in[\tau\vee r_1\vee\ldots\vee r_k,1]}\left|D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{Ric}_{r_x(s)}\right|^{p\eta^*}d\tau dr_1\ldots dr_k\right)\right]^{\frac{1}{\eta^*}}\\
&\quad\times\left[E\left(\int_0^1||\dot{v}(x,s)||_K^2ds\right)^{p\eta/2}\right]^{\frac{1}{\eta}}\\
&\leq \sup_{\tau,r_1,\ldots,r_k\in[0,1]}\left[E\left(\sup_{s\in[\tau\vee r_1\vee\ldots\vee r_k,1]}\left|D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{Ric}_{r_x(s)}\right|^{p\eta^*}\right)\right]^{\frac{1}{\eta^*}}\\
&\quad\times\left[E\left(\int_0^1||\dot{v}(x,s)||_K^2ds\right)^{p\eta/2}\right]^{\frac{1}{\eta}}\\
&\leq C_1||v||^p_{p\eta},
\end{aligned}$$

where

$$C_1=\sup_{\tau,r_1,\ldots,r_k\in[0,1]}\left[E\left(\sup_{s\in[\tau\vee r_1\vee\ldots\vee r_k,1]}\left|D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{Ric}_{r_x(s)}\right|^{p\eta^*}\right)\right]^{\frac{1}{\eta^*}}$$

is finite by the chain rule and Proposition 3.3.

Similarly, we can prove

$$I_2\leq C_2||v||^p_{p\eta},$$

with

$$C_2=\sum_{i=1}^k\sup_{\tau,r_1,\ldots,r_k\in[0,1]}\left[E\left(\left|D^{\alpha_1\ldots\widehat{\alpha}_i\ldots\alpha_k}_{r_1\ldots\widehat{r}_i\ldots r_k}\widehat{q}_\alpha(\tau,r_i)\right|^{p\eta^*}\right)\right]^{\frac{1}{\eta^*}},$$

which is finite by using the Burkholder-Davis-Gundy inequality and Proposition 3.3.

Moreover, using the Hölder inequality and Proposition 2.1, we have

$$\begin{aligned}
I_3 &\leq \int_{[0,1]^{k+1}}E\left[\left\|\int_0^1(D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{q}_\alpha(\tau,s)\dot{v}(x,s),dx(s))\right\|^p_K\right]d\tau dr_1\ldots dr_k\\
&\leq \sup_{\tau,r_1,\ldots,r_k\in[0,1]}E\left[\left\|\int_0^1(D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{q}_\alpha(\tau,s)\dot{v}(x,s),dx(s))\right\|^p_K\right]\\
&\leq C(I_{31}+I_{32}),
\end{aligned}$$

where

$$\begin{aligned}
I_{31} &:= \sup_{\tau,r_1,\ldots,r_k\in[0,1]}\left[\int_0^1\left\|E\left(D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{q}_\alpha(\tau,s)\dot{v}(x,s)\right)\right\|^2_K ds\right]^{\frac{p}{2}}\\
I_{32} &:= \sup_{\tau,r_1,\ldots,r_k\in[0,1]}E\left[\int_0^1\int_0^1\left\|D_r\left(D^{\alpha_1\ldots\alpha_k}_{r_1\ldots r_k}\widehat{q}_\alpha(\tau,s)\dot{v}(x,s)\right)\right\|^2_K drds\right]^{\frac{p}{2}}.
\end{aligned}$$

Repeating step by step the argument as we used in the estimate of A_1, A_2 for the proof of (6.8), we can prove

$$I_{31} \leq C_3 ||v||_2^p,$$

$$I_{32} \leq C_4 ||v||_{1,p\eta}^p,$$

with

$$C_3 = \sup_{\tau, r_1, \ldots, r_k \in [0,1]} E\left[\sup_{s \in [\tau \vee r_1 \vee \ldots \vee r_k, 1]} \left| D_{r_1 \ldots r_k}^{\alpha_1 \ldots \alpha_k} \widehat{q}_\alpha(\tau, s) \right|^p \right] < +\infty,$$

$$C_4 = \sup_{\tau, r, r_1, \ldots, r_k \in [0,1]} E\left[\sup_{s \in [\tau \vee r \vee r_1 \vee \ldots \vee r_k, 1]} \left| D_r D_{r_1 \ldots r_k}^{\alpha_1 \ldots \alpha_k} \widehat{q}_\alpha(\tau, s) \right|^p \right] < +\infty.$$

From the above estimates for I_1, I_2 and I_3, letting $\eta = \frac{p+\epsilon}{p}$, we obtain (6.17). The Claim is proved.

6.4 End of Proof of Proposition 6.3 and Theorem 1.1 (1)

Indeed, for any $F \in \mathcal{FC}^\infty(P_{m_0}(M))$, we have $\nabla^{r-k-1} F \circ I \in W^{\infty,\infty}(X, H^{\otimes(r-k-1)})$. By Eq. (6.10) and by the fact that A defined by (6.4) can be extended to $W^{2,\infty}(X, H^{\otimes(r-k)})$, we have $\nabla^{r-k} F \circ I = AD(\nabla^{r-k-1} F \circ I)$. Using (6.15), we obtain

$$\begin{aligned} ||\nabla^{r-k} F \circ I||_{2k,p} &= ||AD(\nabla^{r-k-1} F \circ I)||_{2k,p} \\ &\leq C||D(\nabla^{r-k-1} F \circ I)||_{2k+1,p+\epsilon} \\ &\leq C||\nabla^{r-k-1} F \circ I||_{2(k+1),p+\epsilon}. \end{aligned}$$

This proves Proposition 6.3.

By (6.11), (6.12) and by induction, we have

$$\begin{aligned} ||\nabla^r F \circ I||_p &\leq C||\nabla^{r-1} F \circ I||_{2,p+\epsilon} \leq \ldots \\ &\leq C||\nabla^{r-k} F \circ I||_{2k,p+k\epsilon} \leq \ldots \\ &\leq C||F \circ I||_{2r,p+r\epsilon}. \end{aligned}$$

Replacing ϵ by $\frac{\epsilon}{r}$, we finish the proof of Theorem 1.1(1). □

7 Sobolev norm comparison theorem: (2)

7.1 Smoothness of the stochastic anti-development map

Theorem 7.1 *For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, we have*

$$J \in D^{\infty,\infty}(P_{m_0}(M), X).$$

$$J \in D^{\infty,\infty}(P_{m_0}(M), X^{2m,\alpha}).$$

Proof. The proof of the first statement is easy. We will only prove the second statement. Without loss of generality, we suppose that ∇ is the Levi-Civita connection on M, i.e., Θ is identically zero. For any $h \in H$, by Theorem 4.4 and the stochastic Fubini theorem, we have

$$\begin{aligned}\nabla_h J(\gamma)(t) &= h(t) + \int_0^t q_h(x,r) \circ dx_r \\ &= \sum_{\beta=1}^d \int_0^1 \left(1_{[0,t]}(\tau) + \int_0^t q_\beta(\tau,r) \circ dx_r \right) \dot{h}^\beta(\tau) d\tau,\end{aligned} \tag{7.1}$$

where $q_\beta(\tau,r)$ is given by (6.2). Hence the first order Malliavin derivatives of J are given by

$$\nabla_\tau^\beta J(\gamma)(t) = 1_{[0,t]}(\tau) + \int_0^t q_\beta(\tau,r) \circ dx_r.$$

By the definition of $||\nabla J(\gamma)||_{L_{(2)}(H,X^{2m,\alpha})}$, we have

$$\begin{aligned}||\nabla J(\gamma)||_{L_{(2)}(H,X^{2m,\alpha})} : &= \left[\sum_{\beta=1}^d \int_0^1 \left|\left| \nabla_\tau^\beta J(\gamma)(\cdot) \right|\right|_{2m,\alpha}^2 d\tau \right]^{1/2} \\ &\le \sum_{\beta=1}^d \left[\int_0^1 \left|\left| 1_{[0,\cdot]}(\tau) + \int_0^\cdot q_\beta(\tau,r) \circ dx_r \right|\right|_{2m,\alpha}^2 d\tau \right]^{1/2} \\ &\le d \left[\int_0^1 ||1_{[0,\cdot]}(\tau)||_{2m,\alpha}^2 d\tau \right]^{1/2} + \sum_{\beta=1}^d \left[\int_0^1 \left|\left| \int_0^\cdot q_\beta(\tau,r) \circ dx_r \right|\right|_{2m,\alpha}^2 d\tau \right]^{1/2}\end{aligned}$$

By calculation, we have

$$\int_0^1 ||1_{[0,\cdot]}||_{2m,\alpha}^2 d\tau = \frac{1}{[\alpha m(1-2\alpha m)]^{\frac{1}{m}}} \int_0^1 \left[(1-\tau)^{1-2\alpha m} + \tau^{1-2\alpha m} - 1 \right]^{\frac{1}{m}} d\tau$$

is finite since $\alpha \in (\frac{1}{2m}, \frac{1}{2})$. Therefore

$$||\nabla J(\gamma)||_{L_{(2)}(H,X^{2m,\alpha})} < C + \sum_{\beta=1}^d \left[\int_0^1 \left|\left| \int_0^\cdot q_\beta(\tau,r) \circ dx_r \right|\right|_{2m,\alpha}^2 d\tau \right]^{1/2}. \tag{7.2}$$

For any $p \ge 2$, the Hölder inequality yields

$$\left|\left| ||\nabla J||_{L_{(2)}(H,X^{2m,\alpha})} \right|\right|_p^p \le C + \sum_{\beta=1}^d E \left[\int_0^1 \left|\left| \int_0^\cdot q_\beta(\tau,r) \circ dx_r \right|\right|_{2m,\alpha}^p d\tau \right]. \tag{7.3}$$

Let us estimate $E\left[||\int_0^\cdot q_\beta(\tau,r) \circ dx_r||_{2m,\alpha}^p \right]$. It can be dominated by $B_1 + B_2$, where B_1 corresponds to the Itô integral, i.e., $B_1 := E\left[||\int_0^\cdot q_\beta(\tau,r) \cdot dx_r||_{2m,\alpha}^p \right]$, B_2

corresponds to bounded variation part, i.e.,

$$\begin{aligned} B_2 &= 2^{-p}E\left[\left|\left|\int_0^\cdot d_r q_\beta(\tau,r)\cdot dx_r\right|\right|_{2m,\alpha}^p\right] \\ &= 2^{-p}E\left[\left|\left|\int_0^\cdot 1_{[\tau,1]}(r)Ric_{r_x(r)}(e_\beta)dr\right|\right|_{2m,\alpha}^p\right] \\ &\le 2^{-p}\sup_{u\in O(M)}|Ric_u|\left|\left|\int_0^\cdot 1_{[\tau,1]}(r)dr\right|\right|_{2m,\alpha}^p \end{aligned}$$

is finite since the function $g(s)=s-\tau\in X^{2m,\alpha}$ for $\alpha\in(\frac{1}{2m},\frac{1}{2})$.

Note that $(X^{2m,\alpha},||\cdot||_{2m,\alpha})$ is an M-type 2 Banach space. Applying first the Burkholder-Davis-Gundy inequality for $(X^{2m,\alpha},||\cdot||_{2m,\alpha})$-valued Itô stochastic integral $t\to\int_0^t q_\beta(\tau,r)dx_r$, then the Hölder inequality, we obtain that for any $p\ge 2$,

$$\begin{aligned} B_1 &= E\left[\left|\left|\int_0^\cdot q_\beta(\tau,r)\cdot dx_r\right|\right|_{2m,\alpha}^p\right]\le CE\left[\left|\left|\int_0^\cdot |q_\beta(\tau,r)|^2dr\right|\right|_{2m,\alpha}^{\frac{p}{2}}\right] \\ &= CE\left[\int_0^1\int_0^1\frac{\left|\int_{t_1}^{t_2}|q_\beta(\tau,r)|^2dr\right|^{2m}}{|t_1-t_2|^{1+2\alpha m}}dt_1dt_2\right]^{\frac{p}{4m}} \\ &\le CE\left[\sup_{r\in[\tau,1]}|q_\beta(\tau,r)|^p\right]\left[\int_0^1\int_0^1\frac{|t_1-t_2|^{2m}}{|t_1-t_2|^{1+2\alpha m}}dt_1dt_2\right]^{\frac{p}{4m}} \\ &\le CE\left[\sup_{r\in[\tau,1]}|q_\beta(\tau,r)|^p\right]. \end{aligned}$$

By the Burkholder-Davis-Gundy inequality, we have

$$B_1\le C\sup_{\tau\in[0,1]}E\left[\sup_{r\in[\tau,1]}|q_\beta(\tau,r)|^p\right]=C\sup_{\tau\in[0,1]}E\left[\sup_{r\in[\tau,1]}\left|\int_\tau^r\Omega_{r_x(s)}(e_\beta,\circ dx_s)\right|^p\right]<\infty.$$

Thus

$$\sup_{\tau\in[0,1]}E\left[\left|\left|\int_0^\cdot q_\beta(\tau,r)\circ dx_r\right|\right|_{2m,\alpha}^p\right]<B_1+B_2<+\infty. \tag{7.4}$$

From (7.3) and (7.4), for any $p\ge 2$, we obtain $E\left[||\nabla J||^p_{L_{(2)}(H,X^{2m,alpha})}\right]<+\infty$, which is equivalent to say $J\in D^{1,\infty}(P_{m_0}(M),X^{2m,\alpha})$.

Now let $r=2$. Taking $h\in H$, $k\in H$, by (7.1), we can prove

$$\begin{aligned} \nabla_h\nabla_kJ(\gamma)(t) &= \sum_{\beta,\eta}\int_0^1\int_0^1\left(\int_0^t\nabla_s^\eta q_\beta(\tau,r)\circ dx_r\right)\dot h^\beta(\tau)\dot k^\eta(s)d\tau ds \\ &\quad+\sum_{\beta,\eta}\int_0^t\int_0^s q_{\beta\eta}(\tau,s)\dot h^\beta(\tau)\dot k^\eta(s)d\tau ds \\ &\quad+\sum_{\beta,\eta}\int_0^1\int_0^1\left(\int_0^t\int_s^r q_\beta(\tau,r)q_\eta(s,u)\circ dx_u\circ dx_r\right)\dot h^\beta(\tau)\dot k^\eta(s)d\tau ds, \end{aligned}$$

where $q_{\beta\eta}(\tau, r) = q_\beta(\tau, r)e_\eta$. Thus

$$\begin{aligned}
\nabla^{\beta\eta}_{\tau s} J(\gamma)(t) &= \int_0^t \nabla^\eta_s q_\beta(\tau, r) \circ dx_r + 1_{[0,t]}(s)1_{[0,s]}(\tau)q_{\beta\eta}(\tau, s) \\
&\quad + \int_0^t \int_s^r q_\beta(\tau, r)q_\eta(s, u) \circ dx_u \circ dx_r
\end{aligned}$$

By the definition of $||\nabla^2 J(\gamma)||_{L^2_{(2)}(H,X^{2m,\alpha})}$, we have

$$\begin{aligned}
||\nabla^2 J(\gamma)||_{L^2_{(2)}(H,X^{2m,\alpha})} &= \left[\int_0^1 \int_0^1 \frac{|\nabla^{\beta\eta}_{\tau s} J(\gamma)(t_1) - \nabla^{\beta\eta}_{\tau s} J(\gamma)(t_2)|^{2m}}{|t_1 - t_2|^{1+2\alpha m}} dt_1 dt_2\right]^{\frac{1}{2m}} \\
&\leq T_1(\gamma) + T_2(\gamma) + T_3(\gamma).
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
T_1(\gamma) &= 2\left[\int_0^1 \int_0^1 \frac{\left|\int_{t_1}^{t_2} \nabla^\eta_s q_\beta(\tau, r) \circ dx_r\right|^{2m}}{|t_1 - t_2|^{1+2\alpha m}} dt_1 dt_2\right]^{\frac{1}{2m}}, \\
T_2(\gamma) &= \left[\int_0^1 \int_0^1 \frac{\left|1_{[0,t_1]}(s)1_{[0,s]}(\tau) - 1_{[0,t_2]}1_{[0,s]}(\tau)\right|^{2m} |q_{\beta,\eta}(\tau, s)|^{2m}}{|t_1 - t_2|^{1+2\alpha d}} dt_1 dt_2\right]^{\frac{1}{2m}}, \\
T_3(\gamma) &= 2\left[\int_0^1 \int_0^1 \frac{\left|\int_{t_1}^{t_2} \int_s^r q_\beta(\tau, r)q_\eta(s, u) \circ dx_u \circ dx_r\right|^{2m}}{|t_1 - t_2|^{1+2\alpha m}} dt_1 dt_2\right]^{\frac{1}{2m}}.
\end{aligned}$$

By Itô calculus, we have

$$\int_{t_1}^{t_2} \nabla^\eta_s q_\beta(\tau, r) \circ dx_r = \int_{t_1}^{t_2} \nabla^\eta_s q_\beta(\tau, r)dx_r + +\frac{1}{2}\sum_{\lambda=1}^{d} \int_{t_1}^{t_2} 1_{[\tau,1]}(r)(\nabla^\eta_s \Omega_{r_x(r)})(e_\beta, e_\lambda)e_\lambda dr.$$

Similarly to the proof of (7.4), for any $p \geq 2$, $E\,[T_1^p] \leq C_p E\,[T_{11}^p] + C_p E\,[T_{12}^p]$. Where $E\,[T_{11}^p]$ corresponds to the first term given by the Itô stochastic integral, and $E\,[T_{12}^p]$ corresponds to the bounded variation part in the above decomposition. Using the Burkholder-Davis-Gundy inequality for $(X^{2m,\alpha}, ||\cdot||_{2m,\alpha})$-valued Itô integral $t \to \int_0^t \nabla^\eta_s q_\beta(\tau, r) \cdot dx_r$, we can prove

$$\begin{aligned}
E\,[T_{11}^p] &= C\int_0^1 \int_0^1 E\left[\left|\left|\int_0^\cdot \nabla^\eta_s q_\beta(\tau, r) \cdot dx_r\right|\right|^p_{2m,\alpha}\right] d\tau ds \\
&\leq C\int_0^1 \int_0^1 E\left[\left|\left|\int_0^\cdot |\nabla^\eta_s q_\beta(\tau, r)|^2\, dr\right|\right|^{\frac{p}{2}}_{2m,\alpha}\right] d\tau ds \\
&= C\int_0^1 \int_0^1 E\left[\int_0^1 \int_0^1 \frac{\left|\int_{t_1}^{t_2} |\nabla^\eta_s q_\beta(\tau, r)|^2 dr\right|^{2m}}{|t_1 - t_2|^{1+2\alpha m}} dt_1 dt_2\right]^{\frac{p}{4m}} d\tau ds \\
&\leq C \sup_{s,\tau\in[0,1]} E\left[\sup_{r\in[\tau\vee s,1]} \left|\nabla^\eta_s \left(\int_0^r \Omega_{r_x(s)}(e_\beta, \circ dx_s\right)\right|^p\right]
\end{aligned}$$

is finite by applying again the Burkholder-Davis-Gundy inequality and Proposition 3.3. On the other hand,

$$\begin{aligned}E\left[T_{12}^p\right] &\leq C\sum_{\lambda=1}^d E\left[\int_0^1\int_0^1 \left|\left|\int_0^\cdot 1_{[\tau,1]}(r)(\nabla_s^\eta \Omega_{r_x(r)})(e_\beta,e_\lambda)e_\lambda dr\right|\right|_{2m,\alpha}^p d\tau ds\right]\\ &\leq C\sum_{\lambda=1}^d \sup_{\tau,s\in[0,1]} E\left[\sup_{r\in[\tau,1]}|\nabla_s^\eta\Omega_{r_x(r)}|^p\right]\left|\left|\int_0^\cdot dr\right|\right|_{2m,\alpha}\end{aligned}$$

is finite since the function $g(s)=s\in X^{2m,\alpha}$ for $\alpha\in(\frac{1}{2m},\frac{1}{2})$ and

$$\sup_{s,\tau\in[0,1]} E\left[\sup_{r\in[\tau,1]}|\nabla_s^\eta\Omega_{r_x(r)}|^p\right]<+\infty.$$

Thus

$$E\left[T_1^p\right]<+\infty. \tag{7.6}$$

Using an analogue argument as above, we can also prove

$$E\left[T_3^p\right]<\infty. \tag{7.7}$$

Moreover, it is easy to see that

$$\begin{aligned}T_2(\gamma) &= 2\left[\int_s^1\int_0^s \frac{|q_{\beta,\eta}(\tau,s)|^{2m}}{|t_1-t_2|^{1+2\alpha m}}dt_1dt_2\right]^{\frac{1}{2m}} 1_{[\tau\leq s]}\\ &\leq C\sup_{0\leq\tau\leq s\leq 1}|q_{\beta,\eta}(\tau,s)|,\end{aligned}$$

Hence

$$E\left[T_2^p\right]\leq CE\left[\sup_{s,\tau\in[0,1]}|q_{\beta,\eta}(\tau,s)|^p\right]<\infty. \tag{7.8}$$

From (7.5) to (7.8), we obtain $E\left[||\nabla^2 J||_{2m,\alpha}^p\right]<\infty$, $\forall p\geq 2$. That is to say $J\in D^{2,\infty}(P_{m_0}(M),X^{2m,\alpha})$.

The general case can be proved by analogue, i.e., for any $r\in N$ and $p>1$, we have $\left|\left|\,||\nabla^r J||_{L_{(2)}^r(H,X^{2m,\alpha})}\right|\right|_p<\infty$, where

$$||\nabla^r J(\gamma)||_{L_{(2)}^r(H,X^{2m,\alpha})} := \left[\int_0^1\int_0^1 \frac{\left|\nabla_{\tau_1\ldots\tau_r}^{\alpha_1\ldots\alpha_r}J(\gamma)(t)-\nabla_{\tau_1\ldots\tau_r}^{\alpha_1\ldots\alpha_r}J(\gamma)(s)\right|^{2m,\alpha}}{|t-s|^{1+2m\alpha}}dtds\right]^{\frac{1}{2m}}. \tag{7.9}$$

Therefore $J\in D^{\infty,\infty}(P_{m_0}(M),X^{2m,\alpha})$. □

Remark 7.2 *Since the Wiener measure μ is supported on $P_{m_0}^{2m,\alpha}(M)$, Theorem 4.4 remains true on $P_{m_0}^{2m,\alpha}(M)$. By the same argument as in the proof of Theorem 7.1, we can prove that for any $m\in N$, $m\geq 2$, $\alpha\in(\frac{1}{2m},\frac{1}{2})$,*

$$J\in D^{\infty,\infty}(P_{m_0}^{2m,\alpha}(M),X).$$

$$J\in D^{\infty,\infty}(P_{m_0}^{2m,\alpha}(M),X^{2m,\alpha}).$$

7.2 Proof of Theorem 1.1: (2)

We now prove the second Sobolev norm comparison inequality in Theorem 1.1.

Theorem 7.3 *Let* $r \in N$, $p \geq 2$, $\epsilon > 0$, $u \in W^{2r,p+\epsilon}(X)$. *Then there exists a modification of* $u \circ J$, *denoted by* $\widetilde{u \circ J}$, *such that*

$$||\widetilde{u \circ J}||_{D^{r,p}(P_{m_0}(M))} \leq A||u||_{W^{2r,p+\epsilon}(X)}. \tag{7.10}$$

where $A = A(r,p,\epsilon)$ *is the same constant as in Theorem 1.1. Similarly, the same inequality holds if one replaces* $P_{m_0}(M)$ *by* $P^{2m,\alpha}_{m_0}(M)$, $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$.

Proof. First, we suppose that $u \in \mathcal{FC}^\infty(X)$. By Theorem 7.1 and Theorem 4.4, we have for any $h \in H$:

$$\nabla_h(u \circ J)(\gamma) = \nabla_\xi u(x), \tag{7.11}$$

where $\gamma = I(x)$, $\xi = J_*(X_h)$ is the tangent process given by Eq. (6.4). Similarly to the proof of Lemma 6.1, we have

$$\nabla^\alpha_\tau(u \circ J)(\gamma) = D^\alpha_\tau u(x) + \frac{1}{2}\int_\tau^1 (\widehat{Ric}_{r_x(s)}(D_s u(x)), e_\alpha)ds + \int_0^1 (\widehat{q}_\alpha(\tau,s)D_s u(x), dx_s).$$

Therefore, by a similar argument as in the proof of Theorem 1.1, we can prove

$$||u \circ J||_{D^{r,p}(P_{m_0}(M))} \leq A(r,p,\epsilon)||u||_{W^{2r,p+\epsilon}(X)}. \tag{7.12}$$

where $A(r,p,\epsilon)$ is the same constant as in Theorem 1.1.

Next we suppose that $u \in W^{2r,p+\epsilon}(X)$. Let $u_n \in \mathcal{FC}^\infty(X)$ be a sequence of cylinder functionals such that $||u_n - u||_{W^{2r,p+\epsilon}(X)} \to 0$. For each $n \in N$, $u_n \circ J \in D^{r,p}(P_{m_0}(M))$, and by (7.12), we have

$$||u_n \circ J - u_m \circ J||_{D^{r,p}(P_{m_0}(M))} \leq A(r,p,\epsilon)||u_n - u_m||_{W^{2r,p+\epsilon}(X)}.$$

This implies that $u_n \circ J$ converges to a limit in $D^{r,p}(P_{m_0}(M))$ since $D^{r,p}(P_{m_0}(M))$ is complete. Moreover, we can easily show that this limit does not depend on the choice of u_n. Let $\widetilde{u \circ J} := \lim_{n\to\infty} u_n \circ J$ in $D^{r,p}(P_{m_0}(M))$. Then $\widetilde{u \circ J}$ is a modification of $u \circ J$. Moreover

$$||\widetilde{u \circ J}||_{D^{r,p}(P_{m_0}(M))} \leq \lim\inf_{n\to\infty} ||u_n \circ J||_{D^{r,p}(P_{m_0}(M))} \leq A(r,p,\epsilon)||u||_{W^{2r,p+\epsilon}(X)}.$$

This proves Theorem 7.3 for $J : X \to P_{m_0}(M)$. Similarly, we can prove that (7.10) for any $u \in W^{2r,p+\epsilon}(X)$ and for $J : P^{2m,\alpha}_{m_0}(M) \to X$. □

8 Capacity comparison inequality and the tightness

8.1 The capacity comparison inequality

Theorem 8.1 *Let* $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, $r \in N$, $p \geq 2$, $\epsilon > 0$. *Then for any open subset* $O \subset (P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$,

$$C_{r,p}(O) \leq A\widehat{C}_{2r,p+\epsilon}\left((\widetilde{I})^{-1}(O)\right), \tag{8.1}$$

where $A = A(r,p,\epsilon)$ *is the same constant as in Theorem 1.1,* $\widetilde{I}$ *is any* ∞*-quasi-continuous modification of the Itô map* $I : X \to P^{2m,\alpha}_{m_0}(M)$*. Similar result holds if we replace* $(P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$ *by* $(P_{m_0}(M), \rho_\infty)$*.*

Proof. By Theorem 2.2, there exists a sequence of increasing compact subsets $\{K_n\}$ in X such that

$$\lim_{n\to\infty} \widehat{C}_{r,p}(X \setminus K_n) = 0, \quad \forall r \in N, p > 1. \tag{8.2}$$

Let

$$F_n = K_n \cap G^c_n, \tag{8.3}$$

where G_n is the open subset of X as in Proposition 3.6. By (3.2), (3.3), (3.4), (8.2) and (8.3), it is clear that $\{F_n\}$ is a sequence of increasing compact subsets of X and satisfies the following conditions:

$$\lim_{n\to\infty} \widehat{C}_{r,p}(X \setminus F_n) = 0, \quad \forall r \in N, p > 1; \tag{8.4}$$

$$\widetilde{I}|_{F_n} : F_n \to (P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha}) \text{ is continuous.} \tag{8.5}$$

Note that $(\widetilde{I})^{-1}(O) \cap F_n = (\widetilde{I}|_{F_n})^{-1}(O)$ is a relatively open subset of F_n since $\widetilde{I}|_{F_n} : F_n \to (P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$ is continuous. Hence there exists an open subset O_n of X such that

$$(\widetilde{I})^{-1}(O) \cap F_n = O_n \cap F_n. \tag{8.6}$$

Moreover,

$$\begin{aligned} (\widetilde{I})^{-1}(O) &= \left((\widetilde{I})^{-1}(O) \cap F_n\right) \cup \left((\widetilde{I})^{-1}(O) \setminus F_n\right) \\ &\subset \left((\widetilde{I})^{-1}(O) \cap F_n\right) \cup (X \setminus F_n) \\ &= (O_n \cap F_n) \cup (X \setminus F_n), \end{aligned}$$

whence

$$(\widetilde{I})^{-1}(O) \subset O_n \cup (X \setminus F_n). \tag{8.7}$$

By definition, for any $\eta > 0$, there exists a functional $u \in W^{2r,p+\epsilon}(X, R^+ \setminus \{0\})$ such that

$$u \geq 1, \quad \mu_0 - a.s. \quad \text{on} \quad O_n \cup (X \setminus F_n) \tag{8.8}$$

and

$$\widehat{C}_{2r,p+\epsilon}(O_n \cup (X \setminus F_n)) \leq ||u||_{W^{2r,p+\epsilon}(X)} \leq \eta + \widehat{C}_{2r,p+\epsilon}(O_n \cup (X \setminus F_n)). \tag{8.9}$$

By Theorem 7.3, $u \circ J$ has a modification $\widetilde{u \circ J}$ in $D^{r,p}(P^{2m,\alpha}_{m_0}(M), R^+ \setminus \{0\})$. Moreover, since $u(x) \geq 1$ for μ_0-a.s. $x \in O_n \cup (X \setminus F_n)$, we obtain $u \circ J(\gamma) \geq 1$ for μ-a.s. $\gamma \in$

$I\,(O_n \cup (X \setminus F_n))$, which μ-a.s. equals $(\widetilde{I})\,(O_n \cup (X \setminus F_n))$. Therefore $\widetilde{u \circ J}(\gamma) \geq 1$ for μ-a.s. $\gamma \in \widetilde{I}\,(O_n \cup (X \setminus F_n))$. From this and (8.7), we have

$$\widetilde{u \circ J}(\gamma) \geq 1, \quad \mu - \text{a.s. } \gamma \in O.$$

By the definition of (r,p)-capacity $C_{r,p}$ on $P^{2m,\alpha}_{m_0}(M)$, using Theorem 7.3 and (8.9), we have

$$\begin{aligned} C_{r,p}(O) &\leq ||\widetilde{u \circ J}||_{D^{r,p}(P^{2m,\alpha}_{m_0}(M))} \\ &\leq A||u||_{W^{2r,p+\epsilon}(X)} \\ &\leq A\left(\eta + \widehat{C}_{2r,p+\epsilon}(O_n \cup (X \setminus F_n))\right), \end{aligned} \tag{8.10}$$

where $A = A(r,p,\epsilon)$ is the same constant as in Theorem 1.1.

From the proof of (8.7), we have

$$O_n \cup (X \setminus F_n) = \left((\widetilde{I})^{-1}(O) \cap F_n\right) \cup (X \setminus F_n), \tag{8.11}$$

From (8.10) and (8.11), and letting $\eta \to 0$, we obtain

$$\begin{aligned} C_{r,p}(O) &\leq A\widehat{C}_{2r,p+\epsilon}(O_n \cup (X \setminus F_n)) \\ &\leq A\widehat{C}_{2r,p+\epsilon}((\widetilde{I})^{-1}(O) \cap F_n) + A\widehat{C}_{2r,p+\epsilon}(X \setminus F_n) \\ &\leq A\widehat{C}_{2r,p+\epsilon}((\widetilde{I})^{-1}(O)) + A\widehat{C}_{2r,p+\epsilon}(X \setminus F_n). \end{aligned}$$

Letting n tends to infinity, and using (8.4), we prove (8.1) on $(P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$. Similarly, we can prove (8.1) for any open subset $O \subset (P_{m_0}(M), \rho_\infty)$. □

8.2 Tightness of $C_{r,p}$-capacity

Now we prove the tightness of the (r,p)-capacity on the path spaces, i.e., Theorem 1.3 and 1.4.

Theorem 8.2 *For any $m \in N$, $m \geq 2$, $\alpha \in (\frac{1}{2m}, \frac{1}{2})$, $r \in N$, $p > 1$, the (r,p)-capacity $C_{r,p}$ on $(P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$ is tight, i.e., there exists a sequence of increasing compact subsets $\{Z_n\}$ in $(P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$ such that*

$$\lim_{n \to \infty} C_{r,p}(Z_n^c) = 0.$$

Similar result holds if we replace $(P^{2d,\alpha}_{m_0}(M), \rho_{2d,\alpha})$ by $(P_{m_0}(M), \rho_\infty)$ (see Theorem 1.3).

Proof. Let $\{F_n\}$ be the compact-nest of $\widehat{C}_{2r,p+\epsilon}$ in X constructed in the proof of Theorem 8.1. Set $Z_n = \widetilde{I}(F_n)$. By the compactness of F_n and the continuity of the map $\widetilde{I}|_{F_n} : F_n \to (P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$, $\{Z_n\}$ is a sequence of increasing compact subsets of $(P^{2m,\alpha}_{m_0}(M), \rho_{2m,\alpha})$.

Notice that

$$Z_n^c = \widetilde{I}^{-1}(P^{2m,\alpha}_{m_0}(M) \setminus \widetilde{I}(F_n)) \subset X \setminus F_n. \tag{8.12}$$

By (8.1) and (8.12), we have

$$\begin{aligned} C_{r,p}(Z_n^c) &\leq A\widehat{C}_{2r,p+\epsilon}(\widetilde{I}^{-1}(P_{m_0}^{2m,\alpha}(M)\setminus\widetilde{I}(F_n)) \\ &\leq A\widehat{C}_{2r,p+\epsilon}(X\setminus F_n)\to 0. \end{aligned}$$

Theorem 1.4 is therefore proved. Similarly we can prove Theorem 1.3. □

Corollary 8.3 *(Choquet capacitability) For any $m\in N$, $m\geq 2$, $\alpha\in(\frac{1}{2m},\frac{1}{2})$, $r\in N, p>1$ and any Borel subset $A\subset P_{m_0}^{2m,\alpha}(M)$, there exists a compact subset $K\subset A$ such that*

$$C_{r,p}(A\setminus K)<\epsilon.$$

Similar result holds if we replace $P_{m_0}^{2m,\alpha}(M)$ by $P_{m_0}(M)$.

Proof. By Theorem 1.3, Theorem 1.4 and a standard argument as in [21] (p.108 Theorem 6.1). □

8.3 Hölder continuity of Ornstein-Uhlenbeck process on path space

Let ∇ be the gradient operator on $P_{m_0}^{2m,\alpha}(M)$, $m\in N$, $m\geq 2$, $\alpha\in(\frac{1}{2m},\frac{1}{2})$. Consider the pre-Dirichlet form given by $\mathcal{E}(F,G)=\int_{P_{m_0}^{2m,\alpha}(M)}(\nabla F(\gamma),\nabla G(\gamma))_H d\mu(\gamma)$, where μ is the Wiener measure supported on $P_{m_0}^{2m,\alpha}(M)$, $F,G\in\mathcal{FC}^\infty(P_{m_0}^{2m,\alpha}(M))$. Similarly to the case of $P_{m_0}(M)$, the integration by parts formula on $(P_{m_0}^{2m,\alpha}(M),\mu)$ implies the closability of $\mathcal{E}$ on $L^2(P_{m_0}^{2m,\alpha}(M),\mu)$. By Theorem 1.4, the $\mathcal{E}$-capacity (i.e., $C_{1,2}$) on $P_{m_0}^{2m,\alpha}$ is tight. Hence $\mathcal{E}$ is a quasi-regular Dirichlet form on $L^2(P_{M_0}^{2m,\alpha}(M),\mu)$. By the similar approach used in Driver-Röckner [8] based on the general theory of quasi-regular Dirichlet forms [20], we can construct a diffusion process $(X_t)_{t\geq 0}$ on $P_{m_0}^{2m,\alpha}(M)$ whose infinitesimal generator is given by $L=-\frac{1}{2}\nabla^*\nabla$, i.e., the Ornstein-Uhlenbeck operator on $P_{m_0}^{2m,\alpha}(M)$. In other words, for any fixed $t\geq 0$, μ-a.s. trajectory of the Ornstein-Uhlenbeck process X_t satisfies

$$||X_t||_{2m,\alpha}:=\int_0^1\int_0^1\frac{|X_t(s)-X_t(r)|^{2m}}{|s-r|^{1+2\alpha m}}dtds<\infty.$$

Then, as in [9], we have

$$\sup_{s<r}\frac{d(X_t(s),X_t(r))}{|s-r|^{\alpha_1}}\leq C||X_t||_{2m,\alpha}$$

for some constant $C>0$ and $\alpha_1\in(0,\alpha-\frac{1}{2m})$. Therefore μ-a.s., X_t satisfies the Hölder continuity with exponent $\alpha_1\in(0,\alpha-\frac{1}{2m})$.

8.4 Comparison with the results of Driver-Röckner and Shigekawa

According to [24], one can use the method of [8] to prove the tightness of $C_{1,p}$ on $P_{m_0}(M)$ for any $p>1$. However we cannot follow [8] [24] to prove the tightness of the general (r,p)-capacities on $P_{m_0}(M)$ for $r\geq 2$ and $p>1$. Moreover, it seems that the approach of [8] does not work on $P_{m_0}^{2m,\alpha}(M)$ even for the tightness of $(1,2)$-capacity.

When $M = G$ is a compact connected Lie group equipped with an $Ad(G)$-invariant metric, we define the Itô map $I : X \to P_e(G)$ (where e is the unit element of G) by $I(x) = g_x(\cdot)$, where $g_x(t)$ solves the following SDE on G:

$$\begin{cases} dg_x(t) & = & g_x(t) \circ dx(t), \\ g_x(0) & = & e. \end{cases}$$

By Eq. (4.8) in [26], for any $h \in H$,

$$D_h I(x)(s) = \left(\int_0^s g_x(\tau)\dot{h}(\tau)g_x^{-1}(\tau)d\tau \right) g_x(s), \forall s \in [0,1].$$

By the above identity and the Ad-invariance of the Riemannian metric, Shigekawa [26] proved that $||F||_{D^{r,p}(P_e(G))} = ||F \circ I||_{W^{r,p}(X)}$ and $C_{r,p}(A) = \widehat{C}_{r,p}((\widetilde{I})^{-1}(A))$, $\forall A \subset P_e(G)$. Therefore the Itô map $I : X \to P_e(G)$ is a quasi-homeomorphism. However Corollary 6.2 says that the Itô map $I : X \to P_{m_0}(M)$ cannot be a quasi-homeomorphism except that M is flat. See also Driver [6] for the non-equivalence of the Dirichlet forms defined with respect to the left Cartan connection, the right Cartan connection and the Levi-Civita connection on compact Lie group.

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