On the weak L^p-Hodge decomposition theorem on complete Riemannian manifolds

Xiang-Dong Li

Abstract

Let M be a complete Riemannian manifold, W_k be the Weitzenböck curvature on $\Lambda^k T^*M$. We prove that, if $W_k \geq 0$, then weak L^p-Hodge decomposition theorem holds on $L^p(\Lambda^k T^*M)$ for all $p > 1$. To do so, we prove a probabilistic representation formula for the singular integrals $dd^*\Box_k^{-1}$ and $d^*d\Box_k^{-1}$ and prove that they are bounded in L^p for all $1 < p < \infty$ provided $W_k \geq 0$.

2000 Mathematics Subjet Classification. Primary 58J65; Secondary 60J65.

1 Introduction

During the decade 1930-1940, Sir W. Hodge [13] developed the theory of harmonic integrals on forms over compact Riemannian manifolds. He established the Hodge decomposition theorem on compact Riemannian manifolds. As a consequence, he obtained the De Rham-Hodge isomorphism theorem between the De Rham cohomology and the space of harmonic forms on compact Riemannian manifolds. It might be worth to mention that, the first complete and correct proof of Hodge's theorems is due to H. Weyl [32]. So far, it has been well-known that the Hodge theory has far reaching implications in geometry and topology, in particular, in algebraic and complex geometry.

The Hodge decomposition theory has a very close connection with the study of boundary valued problems and the hydrodynamic systems. In his 1858 paper [12], in order to solve boundary value problems arising from the study of hydrodynamic systems, Helmholtz [12] formulated a result, known as *the fundamental theorem of vector calculus*, states that any sufficiently smooth, rapidly decaying vector field can be decomposed into irrotational (curl-free) and solenoidal (divergence-free) component vector fields. The Helmholtz decomposition, when understood in a suitable form, can be regarded as the Hodge decomposition on $\mathbb{R}^3$. To see another important source of using the Hodge decomposition to study hydrodynamic systems, let us mention that, in his fundamental papers [17, 18, 19], Leray initiated the method of using the L^2-orthogonal projection $P = \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$ (from the space of L^2-vectors on $\mathbb{R}^3$ to its subspace of divergence-free L^2-vectors) to study the Navier-Stokes equations. In [31], Weyl used the method of orthogonal projection to study the boundary problems on domains in $\mathbb{R}^n$. For further developments, see e.g. Lions [22], Schwartz [26] and reference therein.

Since the end of 1940s, many people have tried to generalize the Hodge theory to complete non-compact Riemannian manifolds. Let (M, g) be a complete Riemannian manifold, $n = \mathrm{dim}M$, and $dv(x) = \sqrt{\mathrm{det}g(x)}dx$ be the volume measure on M. Let $C_0^\infty(\Lambda^k T^*M)$ be the space of smooth k-forms with compact support. For any $p > 1$, let $L^p(\Lambda^k T^*M)$ be the space of L^p-integrable k-forms on M. The L^2-inner product on $L^2(\Lambda^k T^*M)$ is defined

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \, dv, \qquad \forall \alpha, \beta \in C_0^\infty(\Lambda^k T^*M).$$

Let $d_k : C_0^\infty(\Lambda^k T^*M) \to C_0^\infty(\Lambda^{k+1} T^*M)$ be the exterior differential operator. The formal L^2-adjoint of d_k with respect to the Riemannian volume measure, $d_k^* : C_0^\infty(\Lambda^{k+1} T^*M) \to C_0^\infty(\Lambda^k T^*M)$, is defined by: for all $\alpha \in C_0^\infty(\Lambda^k T^*M), \beta \in C_0^\infty(\Lambda^{k+1} T^*M)$,

$$\int_M \langle d_k \alpha, \eta \rangle \, dv = \int_M \langle \alpha, d_k^* \beta \rangle \, dv.$$

The Hodge Laplacian on $C_0^\infty(\Lambda^k T^*M)$ is defined by

$$\Box_k = d_{k-1} d_{k-1}^* + d_k^* d_k. \tag{1}$$

It is well-known that, on any complete Riemannian manifold M, $\Box_k$ is closable in $L^p(\Lambda^k T^*M)$ for all $p > 1$, and is essentially self-adjoint on $L^2(\Lambda^k T^*M)$, cf. [29]. In this paper, we use $\Box$ to denote the Hodge Laplacian on $C_0^\infty(\Lambda^{\cdot} T^*M)$.

In 1949, extending the orthogonal method of Weyl [31, 32], Kodaira [16] proved the following weak L^2-Hodge decomposition theorem on complete Riemannian manifolds.

Theorem 1.1 *(Kodaira [16], De Rham [8]) Let M be a complete Riemannian manifold. Then the following orthogonal Hodge decomposition holds in L^2:*

$$L^2(\Lambda^k T^*M) = H_2(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1} T^*M)} \oplus \overline{d^* C_0^\infty(\Lambda^{k+1} T^*M)},$$

where $\overline{\{\cdot\}}$ denotes the L^2 closure of the subspace in $\{\cdot\}$, and

$$H_2(M) = (\mathrm{Ker}\Box) \cap L^2(\Lambda^k T^*M).$$

In 1983, Strichartz [29] introduced the notion of the Riesz transforms on differential forms on complete Riemannian manifolds. Moreover, he pointed out that there exists a deep relationship between the L^p-boundedness of the Riesz transforms on forms and the L^p-Hodge decomposition theory on complete Riemannian manifolds. In their 1993 Acta Math paper [14], using the L^p-boundedness of the Riesz transforms, Iwaniec and Martin proved the following L^p-Hodge decomposition theorem on $\mathbb{R}^n$.

Theorem 1.2 *Let $\omega \in L^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n)$, $1 < p < \infty$, and $k = 1, \ldots, n-1$. Then there is a $(k-1)$-form α and $(k+1)$-form β such that*

$$\omega = d\alpha + d^*\beta$$

and

$$d\alpha, \ d^*\beta \in L^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n),$$

where d^* denotes the L^2-adjoint of d with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover, $d\alpha$ and $d^*\beta$ are unique,

$$\alpha \in \mathrm{Ker} d^* \cap L_1^p(\mathbb{R}^n, \Lambda^{k-1} \mathbb{R}^n), \quad \beta \in \mathrm{Ker} d \cap L_1^p(\mathbb{R}^n, \Lambda^{k+1} \mathbb{R}^n),$$

and there exists a constant $C_p(k,n)>0$ such that

$$\|\alpha\|_{L^p_1(\mathbb{R}^n)}+\|\beta\|_{L^p_1(\mathbb{R}^n)}\leq C_p(k,n)\|\omega\|_p,$$

where $L^p_1(\mathbb{R}^n,\Lambda^k\mathbb{R}^n)$ denotes the homogenuous Sobolev space of k-forms on $\mathbb{R}^n$ whose first order derivatives are L^p-integrable with respect to the Lebesgue measure on $\mathbb{R}^n$, on which $\|\omega\|_{L^p_1(\mathbb{R}^n)}:=\|d\omega\|_p+\|d^*\omega\|_p$. (Indeed, using a later result of Iwaniec-Martin [15], and Bañuelos-Wang [6], the above mentioned constant $C_p(k,n)$ can be independent of n.)

The purpose of this paper is to study the following fundamental problem.

Problem 1.3 *Under which condition on M, can we establish the L^p-Hodge decomposition theorem for some or all $p>1$?*

As in the case $M=\mathbb{R}^n$, we can imagine that a satisfied answer to the above fundamental problem will have important applications in the study of the Navier-Stokes equations on complete Riemannian manifolds. By [29], in order to establish the L^p-Hodge decomposition theorem, we need only to prove that the singular integral operators $dd^*\Box_k^{-1}$ and $d^*d\Box_k^{-1}$ are bounded in $L^p(\Lambda^kT^*M)$ for all $p>1$. Indeed, Strichartz [29] also indicated that $dd^*\Box^{-1}$ and $d^*d\Box^{-1}$ are bounded in L^p for all $p>1$ provided that the Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ are bounded in L^p for all $p>1$. Inspired by Strichartz [29] and based on some earlier works due to Bakry[2], the author [21] proved the Hodge decomposition theorem under the non-negativity conditions on the Weitzenböck curvatures W_i, $i=k-1,\ k,k+1$. More precisely, we have

Theorem 1.4 *(Li [21]) Let M be a complete Riemannian manifold. Suppose that the following Weitzenböck curvatures are non-negative, i.e.,*

$$W_i\geq 0,\qquad i=k-1,\ k,\ k+1.$$

Then the Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ are bounded in $L^p(\Lambda^kT^*M)$ for all $p>1$. Moreover, the Hodge projection

$$H:L^p(\Lambda^kT^*M)\to(\mathrm{Ker}\Box_k)\cap L^p(\Lambda^kT^*M)$$

is bounded in L^p, and the weak L^p-Hodge decomposition holds:

$$L^p(\Lambda^kT^*M)=HL^p(\Lambda^kT^*M)\oplus dd^*\Box_k^{-1}L^p(\Lambda^kT^*M)\oplus d^*d\Box_k^{-1}L^p(\Lambda^kT^*M).$$

More precisely, every $\omega\in L^p(\Lambda^kT^*M)$ has the weak L^p-Hodge decomposition

$$\omega=H\omega+dd^*\Box_k^{-1}\omega+d^*d\Box_k^{-1}\omega. \tag{2}$$

We will establish the so-called Weak L^p-Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvature. Before stating our result, we would like to review some backgrounds in the study of the L^p-Hodge decomposition on complete non-compact Riemannian manifolds.

The main observation of this paper is that, the singular operators $dd^*\Box_k^{-1}$ and $d^*d\Box^{-1}$ are bounded in $L^p(\Lambda^kT^*M)$ for all $p>1$ provided that $W_k\geq 0$.

Now we are in a position to state main result of this paper. Our first result is the following

Theorem 1.5 *Let M be a complete Riemannian manifold. Suppose that the Weitzenböck curvature on $\Lambda^k T^*M$ is uniformly bounded from below by a non-positive constant, i.e.,*

$$W_k \geq -a,$$

where $a \geq 0$ is a constant. Then, for all $p > 1$, the singular integral operators $dd^*(a+\Box_k)^{-1}$ and $d^*d(a+\Box_k)^{-1}$ are bounded in L^p. Moreover, there exists a constant $C > 0$ such that

$$\|dd^*(a+\Box_k)^{-1}\omega\|_p \leq C\|A_1\|(p^*-1)^{3/2}, \tag{3}$$

$$\|d^*d(a+\Box_k)^{-1}\omega\|_p \leq C\|A_2\|(p^*-1)^{3/2}, \tag{4}$$

where

$$p^* = \max\left\{p, \frac{p}{p-1}\right\},$$

and

$$\|A_i\| = \sup_{x\in M} \sup_{\omega\in\Lambda^k T^*M, \omega(x)\neq 0} \frac{|A_i\omega(x)|}{|\omega(x)|}, \quad i = 1, 2.$$

In particular, if the Weitzenböck curvature on $\Lambda^k T^*M$ is non-negative, i.e.,

$$W_k \geq 0,$$

for all $p > 1$, the singular integral operators $dd^*\Box_k^{-1}$ and $d^*d\Box_k^{-1}$ are bounded in L^p. Moreover, there exists a constant $C > 0$ such that

$$\|dd^*\Box_k^{-1}\omega\|_p \leq C\|A_1\|(p^*-1)^{3/2}, \tag{5}$$

$$\|d^*d\Box_k^{-1}\omega\|_p \leq C\|A_2\|(p^*-1)^{3/2}. \tag{6}$$

Based on Theorem 1.5 and using the heat equation approach to the Hodge theory, we prove the following Weak L^p-Hodge decomposition theorem.

Theorem 1.6 *Let M be a complete Riemannian manifold. Suppose that the k-th Weitzenböck curvature is non-negative, i.e.,*

$$W_k \geq 0.$$

Then the singular integral operators $dd^*\Box_k^{-1}$ and $d^*d\Box_k^{-1/2}$ are bounded in $L^p(\Lambda^k T^*M)$ for all $p > 1$. Moreover, the Hodge projection

$$H : L^p(\Lambda^k T^*M) \to (\mathrm{Ker}\Box_k) \cap L^p(\Lambda^k T^*M)$$

is bounded in L^p, and the weak L^p-Hodge decomposition holds:

$$L^p(\Lambda^k T^*M) = HL^p(\Lambda^k T^*M) \oplus dd^*\Box_k^{-1}L^p(\Lambda^k T^*M) \oplus d^*d\Box_k^{-1}L^p(\Lambda^k T^*M).$$

More precisely, every $\omega \in L^p(\Lambda^k T^*M)$ has the weak L^p-Hodge decomposition

$$\omega = H\omega + dd^*\Box_k^{-1}\omega + d^*d\Box_k^{-1}\omega. \tag{7}$$

2 Martingale representation formulas of $dd^*(a+\Box)^{-1}$ and $d^*d(a+\Box)^{-1}$

In this section we prove the martingale representation formula of the singular integral operators $dd^*\Box^{-1}$ and $d^*d\Box^{-1}$ on complete Riemannian manifolds.

Let M be a complete Riemannian manifold, ∇^{TM} be the Levi-Civita connection on TM. We have the following Bochner-Weitzenböck formula

$$\Box = -\Delta^H + W_k, \tag{8}$$

where

$$\Delta^H = \mathrm{Tr}\nabla^2$$

is the covariant Laplace-Beltrami operator on $\Lambda^{\cdot}T^*M$. More precisely, in a local orthonormal basis $e_1, \ldots, e_n$ near $x \in M$, we have

$$\Delta^H = \sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i} - \nabla_{\sum\limits_{i=1}^{n} e_i e_i}. \tag{9}$$

Fix $x \in M$. Let $\{e_1, \ldots, e_n\}$ be a normal orthonomal basis of T_yM for y near x such that $\nabla_{e_i}e_j(x) = 0$, $i, j = 1, \ldots, n$. Let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis of $\{e_1, \ldots, e_n\}$. For $j = 1, \ldots, n$, let $a_j^* = e_j^*\wedge$, and a_j be the interior multiplication induced by e_j, i.e.,

$$\begin{aligned} a_j^*\omega &= e_j^* \wedge \omega, \\ a_j\omega &= i_{e_j}\omega. \end{aligned}$$

Proposition 2.1 *Let $A_1 = \left(a_i a_j^*\right)$, $A_2 = (a_i^* a_j)$, $B = A_1 - A_2$. Then, at the point x, for all $\omega \in C^\infty(\Lambda^{\cdot}T^*M)$, we have*

$$d^*d\omega(x) = \nabla^* A_1 \nabla\omega(x), \tag{10}$$

$$dd^*\omega(x) = \nabla^* A_2 \nabla\omega(x), \tag{11}$$

$$(d^*d - dd^*)\omega(x) = \nabla^* B \nabla\omega(x). \tag{12}$$

Proof. By the standard result in Riemannian geometry, we have

$$\nabla^*\eta = -\sum_{j=1}^{n} \nabla_{e_j}\eta(e_j).$$

Moreover,

$$d = \sum_{j=1}^{n} a_j^* \nabla_{e_j},$$

$$d^* = -\sum_{j=1}^{n} a_j \nabla_{e_j}.$$

Therefore

$$d^*d\omega = -\sum_{i,j} a_i \nabla_{e_i}\left(a_j^* \nabla_{e_j}\omega\right).$$

Since $\nabla_{e_i} e_j(x) = 0$, we deduce that, at the point x, it holds

$$\begin{aligned} d^*d\omega &= -\sum_{i,j} a_i a_j^* \nabla_{e_i}\nabla_{e_j}\omega \\ &= -\sum_{i,j} \nabla_i\left(a_i a_j^* \nabla_{e_j}\omega\right). \end{aligned}$$

This yields

$$d^*d\omega(x) = \nabla^* A_1 \nabla\omega(x) = \nabla^* \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{in} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \nabla_1\omega \\ \dots \\ \nabla_j\omega \\ \dots \\ \nabla_n\omega \end{pmatrix} (x).$$

This proves (10). Similarly, we can prove (11). By (10) and (11), we obtain (12). $\square$

Let X_t be the Brownian motion on M with generator Δ, i.e., the heat semigroup genetated by X_t is $e^{t\Delta}$, where Δ is the Laplace-Beltrami operator on M. Suppose that M is stochastically complete, i.e., the Brownian motion X_t has infinite lifetime, equivalently, the heat kernel of Δ satisfies

$$\int_M p_t(x,y)dy = 1, \quad \forall x \in M, t > 0.$$

Let $M^k \in \mathrm{End}(\Lambda^k T^*_{X_0} M_{X_0}, \Lambda^k T^*_{X_t} M_{X_t})$ be the solution of the following equation

$$\begin{aligned} \frac{\nabla}{\partial t} M_k(t) &= -W_k(X_t) M_k(t) \\ M_k(0) &= 1_{\Lambda^k T^* M}. \end{aligned}$$

For any $a \geq 0$, let

$$\omega_a(x,t) = e^{-t(a+\square)}\omega(x), \quad \forall x \in M, t > 0.$$

Proposition 2.2 *We have the following identity*

$$\omega(X_T) = e^{aT} M_T^{*-1}\omega_a(X_0,T) + \int_0^T e^{a(T-s)} M_T^{*-1} M_s^* \nabla\omega_a(X_s, T-s) dX_s. \tag{13}$$

Proof. By the covariant Itô formula on forms, as $dX_t \cdot dX_t = 2dt$, we have

$$\begin{aligned} & \nabla(e^{-at} M_k^*(t)\omega_a(X_t, T-t)) \\ = \; & e^{-at} M_k^*(t)\nabla\omega_a(X_t, T-t) \circ dX_t + e^{-at} M_k^*(t)\partial_t\omega_a(X_t, T-t)dt \\ & + \frac{\nabla}{\partial t}(e^{-at} M_k^*(t))\omega_a(X_t, T-t)dt \\ = \; & e^{-at} M_k^*(t)\nabla\omega_a(X_t, T-t)dX_t + \frac{1}{2} e^{-at} M_k^*(t)\nabla^2\omega_a(X_t, T-t)(dX_t, dX_t) \\ & + e^{-at} M_k^*(t)(a+\square)\omega_a(X_t, T-t)dt - e^{-at} M_k^*(t)\left(a + W_k(X_t)\right)\omega_a(X_t, T-t)dt \\ = \; & e^{-at} M_k^*(t)\nabla\omega_a(X_t, T-t)dX_t + e^{-at} M_k^*(t)\left(\mathrm{Tr}\nabla^2 + \square - W_k(X_t)\right)\omega_a(X_t, T-t)dt. \end{aligned}$$

By the Bochner-Weitzenböck formula,

$$\Box = -\mathrm{Tr}\nabla^2 + W_k.$$

Hence

$$\nabla(e^{-at}M_k^*(t)\omega_a(X_t, T-t)) = e^{-at}M_k^*(t)\nabla\omega_a(X_t, T-t)dX_t.$$

Integrating from $t = 0$ to $t = T$, we obtain

$$e^{-aT}M_T^*\omega(X_T) = \omega_a(X_0, T) + \int_0^T e^{-as}M_s^*\nabla\omega_a(X_s, T-s)dX_s.$$

This completes the proof of Proposition 2.2. $\Box$

Now we can state the main result of this section.

Theorem 2.3 *Let M be a complete and stochastically complete Riemannian manifold. Then, for all $a \geq 0$, $\omega \in C_0^\infty(\Lambda^k T^*M)$, we have*

$$d(a+\Box)^{-1}d^*\omega = \lim_{T\to\infty} S_{A_1}^T\omega, \tag{14}$$

$$d^*(a+\Box)^{-1}d\omega = \lim_{T\to\infty} S_{A_2}^T\omega, \tag{15}$$

where

$$S_{A_i}^T\omega(x) = E^T\left[\int_0^T e^{a(t-T)}M_TM_t^{-1}A_i\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right], \quad i = 1, 2. \tag{16}$$

In particular, the Beurling-Ahlfors transform

$$S_B\omega := (d^*d - dd^*)(a+\Box)^{-1}\omega$$

has the following martingale transform representation

$$S_B\omega(x) = \lim_{T\to\infty} E^T\left[\int_0^T e^{a(t-T)}M_TM_t^{-1}B\nabla\omega_a(X_t, T-t)dZ_t\middle| Z_T = x\right], \tag{17}$$

Equivalently,

$$S_k\omega = \lim_{T\to\infty}(S_{A_2}^T - S_{A_1}^T)\omega.$$

Proof. To prove (14), we need only to show that, for all $\eta \in C_0^\infty(\Lambda^k T^*M)$, we have

$$\int_M \left\langle \lim_{T\to\infty} S_{A_1}^T\omega(x), \eta(x)\right\rangle dx = \int_M \left\langle dd^*(a+\Box)^{-1}\omega, \eta\right\rangle dx$$

Since the distribution of X_T is $m(dx) = dx$, the property of conditional expectation implies

$$\int_M \langle S_{A_1}^T\omega(x), \eta(x)\rangle dx = E^T\left[(\eta(X_T), \int_0^T e^{a(s-T)}M_TM_s^{-1}A\nabla\omega_a(X_s, T-s)dX_s)\right].$$

Applying (13) to η, we can therefore deduce that

$$\int_M \left\langle S_{A_1}^T \omega(x), \eta(x) \right\rangle dx = I_1^T + I_2^T,$$

where

$$\begin{aligned}
I_1^T &= E^T \left[\left\langle e^{aT} M_T^{*-1} \eta_a(X_0, T), \int_0^T e^{a(s-T)} M_T M_s^{-1} A_1 \nabla \omega_a(X_s, T-s) dX_s \right\rangle \right] \\
I_2^T &= E^T \left[\left\langle \int_0^T e^{a(T-s)} M_T^{*-1} M_s^* \nabla \eta_a(X_s, T-s) dX_s, \int_0^T e^{a(s-T)} M_T M_s^{-1} A_1 \nabla \omega_a(X_s, T-s) dX_s \right\rangle \right].
\end{aligned}$$

Taking the conditional expectation with respect to X_0 and using the martingale property of the Itô integral, we can prove that

$$\begin{aligned}
I_1^T &= E^T \left[\left\langle \eta_a(X_0, T), E\left[\int_0^T e^{as} M_s^{-1} A_1 \nabla \omega_a(X_s, T-s) dX_s \middle| X_0 \right] \right\rangle \right] \\
&= 0.
\end{aligned}$$

Thus we need only to prove

$$\lim_{T\to\infty} I_2^T = \int_M \left\langle dd^*(a+\square)^{-1}\omega(x), \eta(x) \right\rangle dx. \tag{18}$$

By the L^2-isometry of Itô integral, we have

$$\begin{aligned}
I_2^T &= E^T \left[\int_0^T \left\langle e^{a(T-s)} M_T^{*-1} M_s^* \nabla \eta_a(X_s, T-s), e^{a(s-T)} M_T M_s^{-1} A_1 \nabla \omega_a(X_s, T-s) \right\rangle ds \right] \\
&= \int_0^T \int_M \int_M \langle \nabla \eta_a(z, T-s), A_1 \nabla \omega_a(z, T-s) \rangle \, p_t(x,z) dx dz ds \\
&= \int_0^T \int_M \int_M \langle \nabla \eta_a(z, T-s), A_1 \nabla \omega_a(z, T-s) \rangle \, p_t(x,z) dx dz ds.
\end{aligned}$$

By the fact that $\int_M p_t(x,z)dx = 1$, and using integration by parts formula, we have

$$\begin{aligned}
I_2^T &= \int_0^T \int_M \langle \nabla \eta_a(z, T-s), A_1 \nabla \omega_a(z, T-s) \rangle \, dz ds \\
&= \int_0^T \int_M \langle \eta_a(z, T-s), \nabla^* A \nabla \omega_a(z, T-s) \rangle \, dz ds.
\end{aligned}$$

By Proposition 2.1, we have

$$\nabla^* A_1 \nabla = dd^*.$$

Therefore

$$I_2^T = \int_0^T \int_M \langle \eta_a(z, T-s), dd^* \omega_a(z, T-s) \rangle \, dz ds.$$

Changing $T-t$ to t, and using the fact that $dd^*(a+\Box)^{-1}=(a+\Box)^{-1}dd^*$, and by the self-adjointness of the heat semigroup with respect to $dv(x)$, we have

$$\begin{aligned}
I_2^T &= \int_0^T\int_M \left\langle \eta_a(z,t), dd^*(a+\Box)(a+\Box)^{-1}\omega_a(z,t)\right\rangle dzdt\\
&= \int_0^T\int_M \left\langle \eta_a(z,t), (a+\Box)dd^*(a+\Box)^{-1}\omega_a(z,t)\right\rangle dzdt\\
&= \int_0^T\int_M \left\langle e^{-t(a+\Box)}\eta(z), (a+\Box)dd^*(a+\Box)^{-1}e^{(-t(a+\Box)}\omega(z))\right\rangle dzdt\\
&= \int_0^T\int_M \left\langle e^{-t(a+\Box)}\eta(z), (a+\Box)e^{-t(a+\Box)}dd^*(a+\Box)^{-1}\omega(z)\right\rangle dzdt\\
&= \int_0^T\int_M \left\langle \eta(z), (a+\Box)e^{-2t(a+\Box)}dd^*(a+\Box)^{-1}\omega(z)\right\rangle dzdt\\
&= \int_M \left\langle \eta(z), \int_0^T (a+\Box)e^{-2t(a+\Box)}dd^*(a+\Box)^{-1}\omega(z)dt\right\rangle dz.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{T\to\infty} I_2^T &= \int_M \left\langle \eta(z), \lim_{T\to\infty}\int_0^T (a+\Box)e^{-2t(a+\Box)}dd^*(a+\Box)^{-1}\omega(z)dt\right\rangle dz\\
&= \int_M \left\langle \eta(z), \int_0^\infty (a+\Box)e^{-2t(a+\Box)}dd^*(a+\Box)^{-1}\omega(z)ds\right\rangle dz\\
&= \frac{1}{2}\int_M \left\langle \eta(z), (a+\Box)(a+\Box)^{-1}dd^*(a+\Box)^{-1}\omega(z)\right\rangle dz\\
&= \frac{1}{2}\int_M \left\langle \eta, dd^*(a+\Box)^{-1}\omega\right\rangle dv.
\end{aligned}$$

This finishes the proof of (14). Similarly, we can prove (15) and (17). $\Box$

3 The L^p-boundedness of $dd^*(a+\Box)^{-1}$ and $d^*d(a+\Box)^{-1}$

In this section we prove Theorem 1.5. To prove it, we need the following

Lemma 3.1 *For $\omega\in C_0^\infty(\Lambda^kT^*M)$, and $a\geq 0$, let*

$$\omega_a(x,t)=e^{-t(a+\Box)}\omega(x),\quad \forall x\in M, t>0.$$

Then

$$\left(L+\frac{\partial}{\partial t}\right)|\omega_a(x,T-t)|^2=2|\overline{\nabla}\omega_a(x,T-t)|^2+2\left\langle (W+a)\omega_a(x,T-t),\omega_a(x,T-t)\right\rangle, \quad (19)$$

where $\overline{\nabla}=(\nabla_x,\partial_t)$.

Proof. Let $\widetilde{\omega}_a(x,t) = \omega_a(x, T-t)$. By the generalized Bochner-Weitzenböck formula, we have

$$L|\widetilde{\omega}_a|^2 = 2|\nabla\widetilde{\omega}_a|^2 - 2\langle \Box\widetilde{\omega}_a, \omega_a\rangle + 2\langle W\widetilde{\omega}_a, \widetilde{\omega}_a\rangle. \tag{20}$$

Using the fact that

$$\frac{\partial}{\partial t}\omega_a(x, T-t) = (a+\Box)\omega_a(x, T-t),$$

we deduce that

$$L|\widetilde{\omega}_a|^2 = 2|\nabla\widetilde{\omega}_a|^2 - 2\left\langle \frac{\partial}{\partial t}\widetilde{\omega}_a, \widetilde{\omega}_a\right\rangle + 2\langle (W+a)\widetilde{\omega}_a, \widetilde{\omega}_a\rangle. \tag{21}$$

By the elementary identity

$$\frac{\partial}{\partial t}|\omega_a(x, T-t)|^2 = 2\left\langle \frac{\partial}{\partial t}\omega_a(x, T-t), \omega_a(x, T-t)\right\rangle,$$

we can obtain (19) from (21). The proof of Lemma 3.1 is completed. □

Proof of Theorem 1.5. By Theorem 2.3, using the fact that $E[\quad | X_\tau = x]$ is contractive in L^p, and by the Burkholder-Davies-Gundy inequality, we have

$$\|S^T_{A_i}\omega\|_p \leq C_p \sup_{0\leq t\leq T}\|e^{a(t-T)}M_T M_t^{-1}A_i\| \left\| \left\{\int_0^T |\overline{\nabla}\omega_a(X_t, T-t)|^2 dt\right\}^{1/2}\right\|_p,$$

where, cf. [28] (P. 50)

$$C_p = \left\{\frac{1}{2}p(p-1)\left(\frac{p}{p-1}\right)^p\right\}^{p/2}.$$

Under the curvature condition $W_k \geq -a$, and using the Gronwall inequality, we can prove

$$\|M_T M_t^{-1}\| \leq e^{a(T-t)}, \quad \forall t\in[0,T]. \tag{22}$$

Therefore

$$\|S^T_{A_i}\omega\|_p \leq C_p\|A_i\| \left\| \left\{\int_0^T |\overline{\nabla}\omega_a(X_t, T-t)|^2 dt\right\}^{1/2}\right\|_p. \tag{23}$$

Let

$$I = \left\{\int_0^T |\overline{\nabla}\omega_a(X_t, T-t)|^2 dt\right\}^{1/2}.$$

Below we prove that

$$\|I\|_p \leq B_p\|\omega\|_p. \tag{24}$$

By Lemma 3.1, we have

$$\sqrt{2}J \leq \left\{ \int_0^T \left(L + \frac{\partial}{\partial t} \right) |\omega_a(X_t, T-t)|^2 dt \right\}^{1/2} = I.$$

By Itô's formula, we have

$$\begin{aligned} |\omega_a(X_t, T-t)|^2 &= |\omega_a(X_0, T)|^2 + \int_0^t \nabla |\omega_a(X_s, T-s)|^2 dW_s \\ &\quad + \int_0^t \left(L + \frac{\partial}{\partial s} \right) |\omega_a(X_s, T-s)|^2 ds. \end{aligned}$$

Let

$$Z_t = |\omega_a(X_t, T-t)|^2 - |\omega_a(X_0, T)|^2, \quad 0 \leq t \leq T,$$

$$M_t = \int_0^{t \wedge T} \nabla |\omega_a(X_s, T-s)|^2 dW_s,$$

and

$$A_t = \int_0^{t \wedge T} \left(L + \frac{\partial}{\partial s} \right) |\omega_a(X_s, T-s)|^2 ds.$$

Then we have the Doob-Meyer decomposition $Z_t = M_t + A_t$. By Lemma 3.1, A_t is a non-negative increasing process. Thus, Z_t is a submartingle with respect to P_{X_0}.

To estimate $\|I\|_p$, we need to use some submartingale inequalities. By the Lenglart-Lépingle-Pratelli inequality, cf. [28] (Proposition 52 in P. 52), for all $p > 1$, we have

$$E\left[A_T^{\frac{p}{2}} \right] \leq (p)^{p/2} E\left[\sup_{t \in [0,T]} Z_t^{\frac{p}{2}} \right]. \tag{25}$$

On the other hand, using the Doob inequality for submartingale, for all $p > 2$, we have

$$\begin{aligned} E\left[\sup_{t \in [0,T]} Z_t^{\frac{p}{2}} \right] &\leq \left(\frac{p/2}{p/2 - 1} \right)^{p/2} E\left[Z_T^{\frac{p}{2}} \right] \\ &= \left(\frac{p}{p-2} \right)^{p/2} E\left[|\omega(X_T)|^p \right] + E\left[|\omega_a(X_0, T)|^p \right] \\ &= 2 \left(\frac{p}{p-2} \right)^{p/2} \|\omega\|_p^p. \end{aligned}$$

From the above inequalities we get

$$\|I\|_p^p \leq p^{p/2} \left(\frac{p}{p-2} \right)^{p/2} \|\omega\|_p^p.$$

Thus, for all $p > 2$,

$$\|J\|_p \leq \frac{p}{\sqrt{2(p-2)}} \|\omega\|_p.$$

From (22), (23) and (24), we can deduce that, for all $p > 2$, we have

$$\|dd^*\square^{-1}\omega\|_p \leq C\|A_1\|(p-1)^{3/2}. \tag{26}$$

Similarly, we can prove that, for all $p > 2$, we have

$$\|dd^*(a+\square)^{-1}\omega\|_p \leq C\|A_2\|(p-1)^{3/2}. \tag{27}$$

Note that $dd^*(a+\square_k)^{-1}$ and $d^*d(a+\square_k)^{-1}$ are self-adjoint. In fact, we have

$$dd^*(a+\square_k)^{-1} = (a+\square_k)^{-1}dd^* = (dd^*(a+\square_k)^{-1})^*,$$

and

$$d^*d(a+\square_k)^{-1} = (a+\square_k)^{-1}d^*d = (d^*d(a+\square_k)^{-1})^*.$$

In the case $p = 2$, using the Gaffeny integration by parts formula, we have

$$\|\omega - H\omega\|_2^2 = \|dd^*\square^{-1}\omega\|_2^2 + \|d^*d\square^{-1}\|_2^2. \tag{28}$$

This yields that

$$\begin{aligned} \|dd^*\square^{-1}\omega\|_2 &\leq \|\omega\|_2, & (29)\\ \|d^*d\square^{-1}\omega\|_2 &\leq \|\omega\|_2, & (30)\\ \|H\omega\|_2 &\leq \|\omega\|_2. & (31) \end{aligned}$$

By duality argument, we can prove that, for $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|dd^*(a+\square_k)^{-1}\|_{p,p} = \|(dd^*(a+\square_k)^{-1})^*\|_{p,p} = \|dd^*(a+\square_k)^{-1}\|_{q,q},$$

and

$$\|d^*d(a+\square_k)^{-1}\|_{p,p} = \|(d^*d(a+\square_k)^{-1})^*\|_{p,p} = \|d^*d(a+\square_k)^{-1}\|_{q,q}. \tag{32}$$

By (26), (27), (32) and (32), we can therefore obtain that, for all $1 < p < 2$,

$$\|dd^*(a+\square)^{-1}\omega\|_p \leq \frac{C\|A_1\|}{(p-1)^{3/2}}. \tag{33}$$

$$\|dd^*(a+\square)^{-1}\omega\|_p \leq \frac{C\|A_2\|}{(p-1)^{3/2}}. \tag{34}$$

Thus, for all $p > 1$, we have proved (3) and (4). The proof of Theorem 1.5 is completed. □

4 Manifolds with constant Weitzenböck curvature

Theorem 4.1 *Let M be a complete Riemannian manifold. Suppose that the Weitzenböck curvature on $\Lambda^k T^*M$ is a non-positive constant, i.e.,*

$$W_k = -a,$$

where $a \geq 0$ is a constant. Then, for all $p > 1$, the singular integral operators $dd^*(a+\Box_k)^{-1}$ and $d^*d(a+\Box_k)^{-1}$ are bounded in L^p. Moreover, there exists a constant $C > 0$ such that

$$\begin{aligned} \|dd^*(a+\Box_k)^{-1}\omega\|_p &\leq C\|A_1\|(p^*-1), && (35)\\ \|d^*d(a+\Box_k)^{-1}\omega\|_p &\leq C\|A_2\|(p^*-1), && (36) \end{aligned}$$

where

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

In particular, if the Weitzenböck curvature on $\Lambda^k T^*M$ is zero, i.e.,

$$W_k = 0,$$

for all $p > 1$, the singular integral operators $dd^*\Box_k^{-1}$ and $d^*d\Box_k^{-1}$ are bounded in L^p. Moreover, there exists a constant $C > 0$ such that

$$\begin{aligned} \|dd^*\Box_k^{-1}\omega\|_p &\leq C\|A_1\|(p^*-1), && (37)\\ \|d^*d\Box_k^{-1}\omega\|_p &\leq C\|A_2\|(p^*-1). && (38) \end{aligned}$$

Proof. Let $\omega_a(x,t) = e^{-t(a+\Box_k)}\omega(x)$. Since $W_k = -a$, we have

$$M_t = e^{at}U_{t\leftarrow 0}1_{\Lambda^k T_{X_0}},$$

where $U_{t\leftarrow 0} \in \mathrm{End}(T_{X_0}M, T_{X_t}M)$ denotes the stochastic parallel transport along $\{X_s : s \in [0,T]\}$. By Itô's formula, we have

$$\begin{aligned} d(\omega_a(X_t, T-t)) &= \nabla\omega_a(X_t, T-t)\circ dX_t + \partial_t\omega_a(X_t, T-t)dt\\ &= \nabla\omega_a(X_t, T-t)dX_t + \frac{1}{2}\nabla^2\omega_a(X_t, T-t)(dX_t, dX_t)\\ &\qquad +(a+\Box)\omega_a(X_t, T-t)dt\\ &= \nabla\omega_a(X_t, T-t)dX_t + \left(\mathrm{Tr}\nabla^2 + a + \Box\right)\omega_a(X_t, T-t)dt \end{aligned}$$

By the Bochner-Weitzenböck formula,

$$\Box = -\mathrm{Tr}\nabla^2 + W_k = -\mathrm{Tr}\nabla^2 - a.$$

Hence

$$d(\omega_a(X_t, T-t)) = \nabla\omega_a(X_t, T-t)dX_t.$$

Integrating from $t = 0$ to $t = T$, we obtain

$$\omega(X_T) = U_{T\leftarrow 0}\omega_a(X_0, T) + \int_0^T U_{T\leftarrow t}\nabla\omega_a(X_t, T-t)dX_t. \tag{39}$$

By the probabilistic representation formula in Theorem 2.3, we have

$$dd^*(a+\Box)^{-1}\omega = \lim_{T\to\infty} S_{A_1}^T\omega, \tag{40}$$

$$d^*d(a+\Box)^{-1}\omega \quad = \quad \lim_{T\to\infty} S^T_{A_2}\omega, \tag{41}$$

where

$$S^T_{A_i}\omega(x) = E^T\left[\int_0^T U_{T\leftarrow t}A_i\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right], \quad i = 1, 2. \tag{42}$$

Moreover

$$S_B\omega(x) = \lim_{T\to\infty} E^T\left[\int_0^T U_{T\leftarrow t}B\nabla\omega_a(X_t, T-t)dX_t\middle| X_T = x\right]. \tag{43}$$

Using the Burkholder sharp L^p-inequality for martingale transforms, cf. [3], for all $p > 1$, we have

$$\|S^T_{A_i}\omega\|_p \leq (p^*-1)\|U_{T\leftarrow t}A_iU_{t\leftarrow T}\|\left\|\int_0^T U_{T\leftarrow t}A_i\nabla\omega_a(X_t, T-t)dX_t\right\|_p. \tag{44}$$

Substituting (39) into (44), we have

$$\|S^T_{A_i}\omega\|_p \leq (p^*-1)\|A_i\|\,\|\omega(X_T) - U_{T\leftarrow 0}\omega_a(X_0, T)\|_p. \tag{45}$$

Observing that

$$\|\omega(X_T)\|_p = \|\omega\|_p, \tag{46}$$

and

$$\|U_{T\leftarrow 0}\omega_a(X_0, T)\|_p = \|\omega_a(X_0, T)\|_p = \left\|e^{-T(a+\Box)}\omega\right\|_p. \tag{47}$$

By Bismut's formula, we have

$$e^{-t\Box}\omega(x) = E_x\left[M_t^*\omega(X_t)\right] = e^{ta}\left[U_{t\leftarrow 0}\omega(X_t)\right].$$

Thus

$$e^{-t(a+\Box)}\omega(x) = E_x\left[U_{t\leftarrow 0}\omega(X_t)\right].$$

This yields that

$$\left\|e^{-T(a+\Box)}\omega\right\|_p \leq \|\omega\|_p, \quad p = 1, \infty. \tag{48}$$

On the other hand, using the fact that $\Box$ is a non-negative self-adjoint operator on $L^2(\Lambda^k T^*M)$, we have

$$\left\|e^{-T\Box}\omega\right\|_2 \leq \|\omega\|_2, \quad p = 1, \infty.$$

Therefore

$$\left\|e^{-T(a+\Box)}\omega\right\|_2 \leq e^{-aT}\|\omega\|_2. \tag{49}$$

By (48) and (49), and using the Riesz-Thorin convexity theorem, for all $p > 1$, we can prove that

$$\left\| e^{-T(a+\Box)}\omega \right\|_p \quad \leq \quad e^{-2\min\{\frac{1}{p},\ 1-\frac{1}{p}\}aT}\|\omega\|_p. \tag{50}$$

Therefore, if $a > 0$, then, for all $p > 1$, we have

$$\lim_{T\to\infty} \left\| e^{-T(a+\Box)}\omega \right\|_p = 0, \tag{51}$$

and if $a = 0$, we have

$$\lim_{T\to\infty} \left\| e^{-T\Box}\omega \right\|_p \leq \|\omega\|_p. \tag{52}$$

From (40), (41), (44), (46), (47), (51) and (52), we get, if $W_k = 0$, then, for all $p > 1$,

$$\|d^*d\Box^{-1}\omega\|_p \leq \lim_{T\to\infty} \|S^T_{A_1}\omega\|_p \leq 2(p^*-1)\|A_1\|\|\omega\|_p,$$

$$\|dd^*\Box^{-1}\omega\|_p \leq \lim_{T\to\infty} \|S^T_{A_2}\omega\|_p \leq 2(p^*-1)\|A_2\|\|\omega\|_p,$$

and if $W_k = -a$, then, for all $p > 1$,

$$\|d^*d(a+\Box)^{-1}\omega\|_p \leq \lim_{T\to\infty} \|S^T_{A_1}\omega\|_p \leq (p^*-1)\|A_1\|\|\omega\|_p,$$

$$\|dd^*(a+\Box)^{-1}\omega\|_p \leq \lim_{T\to\infty} \|S^T_{A_2}\omega\|_p \leq (p^*-1)\|A_2\|\|\omega\|_p.$$

The proof of Theorem 4.1 is completed. □

5 The weak L^p-Hodge decomposition theorem

In this section we prove the main result of this paper.

Theorem 5.1 *Let M be a complete Riemannian manifold. Suppose that the k-th Weitzenböck curvature is non-negative, i.e.,*

$$W_k \geq 0.$$

Then the singular integral operators $dd^*\Box_k^{-1}$ and $d^*d\Box_k^{-1/2}$ are bounded in $L^p(\Lambda^k T^*M)$ for all $p > 1$. Moreover, the Hodge projection

$$H : L^p(\Lambda^k T^*M) \to (\mathrm{Ker}\Box_k) \cap L^p(\Lambda^k T^*M)$$

is bounded in L^p, and the weak L^p-Hodge decomposition holds:

$$L^p(\Lambda^k T^*M) = HL^p(\Lambda^k T^*M) \oplus dd^*\Box_k^{-1}L^p(\Lambda^k T^*M) \oplus d^*d\Box_k^{-1}L^p(\Lambda^k T^*M).$$

More precisely, every $\omega \in L^p(\Lambda^k T^*M)$ has the weak L^p-Hodge decomposition

$$\omega = H\omega + dd^*\Box^{-1}\omega + d^*d\Box^{-1}\omega. \tag{53}$$

Proof. To prove the theorem, we develop the heat equation approach to the L^p-Hodge theory. This method was initiated by Milgram-Rosenbloom [24] on compact Riemannian manifolds and was developed by Gaffney [11] for the L^2-Hodge theory on complete Riemannian manifolds. By Theorem 1.5, $P_1 = dd^*\Box^{-1}$ and $P_2 = d^*d\Box^{-1}$ are bounded in L^p for all $p > 1$. Therefore

$$\Box\Box^{-1} = dd^*\Box^{-1} + d^*d\Box^{-1} : L^p(\Lambda^k T^*M) \to L^p(\Lambda^k T^*M)$$

is a bounded operator for all $p > 1$.

Let $P_t = e^{-t\Box}$. Since $W_k \geq 0$, by Theorem 2.8 in [21], for all $\omega \in L^p(\Lambda^k T^*M)$, it holds that

$$H\omega = \lim_{t\to\infty} P_t\omega \quad \text{in} \quad L^p(\Lambda^k T^*M).$$

Writing

$$P_t\omega - \omega = \int_0^t \frac{\partial}{\partial s} P_s\omega ds \quad \text{in} \quad L^p(\Lambda^k T^*M),$$

and letting $t \to \infty$, we have

$$H\omega - \omega = \int_0^\infty \frac{\partial}{\partial s} P_s\omega ds \quad \text{in} \quad L^p(\Lambda^k T^*M).$$

Note that

$$\frac{\partial}{\partial s} P_s\omega = -\Box P_s\omega.$$

Hence

$$H\omega - \omega = -\int_0^\infty \Box P_s\omega ds.$$

Therefore, we have the following decomposition in $L^p(\Lambda^k T^*M)$:

$$\omega = H\omega + \int_0^\infty \Box P_s\omega ds.$$

Equivalently, the following decomposition holds in $L^p(\Lambda^k T^*M)$:

$$\omega = H\omega + \int_0^\infty (dd^* + d^*d) P_s\omega ds.$$

By the convention used in the definition of the Riesz transforms, the following decomposition holds in $L^p(\Lambda^k T^*M)$:

$$\omega = H\omega + dd^*\Box^{-1} + d^*d\Box^{-1}\omega. \tag{54}$$

Finally, from (54), we can deduce that

$$\begin{aligned} \|(I - H)\omega\|_p &= \|dd^*\Box^{-1}\omega + d^*d\Box^{-1}\omega\|_p \\ &\leq \|dd^*\Box^{-1}\omega\|_p + \|d^*d\Box^{-1}\omega\|_p \\ &\leq C(\|A_1\| + \|A_2\|)(p^* - 1)^{3/2}\|\omega\|_p. \end{aligned}$$

Thus, the Hodge harmonic projection $H : L^p(\Lambda^k T^*M) \to (\mathrm{Ker}\Box)^\perp \cap L^p(\Lambda^k T^*M)$ is bounded. The proof of Theorem 5.1 is completed. $\Box$

Acknowledgments. The author is very grateful to Professors D. Bakry, T. Coulhon, N. Lohoué and Weiping Zhang for stimulated discussions. Part of this work was supported by a Delegation in CNRS at the University of Paris-Sud during the 2005-2006 academic year. The author would like to thank Professor J.-M. Bismut for his kind invitation and Le Laboratoire de Mathématiques d'Orsay for a very nice hospitality.

References

[1] P. Auscher, T. Coulhon, X. T. Duong, S. Hofmann, Riesz transform on manifolds and heat kernel regularity, Ann. Sci. Ecole Norm. Sup. Paris, (4) **37** (2004), no. 6, 911-957.

[2] D. Bakry, Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, in Séminaire de Probab. XXI, Lect. Notes in Math. **1247**, Springer, Berlin, 1987, 137-172.

[3] D. L. Burkholder, A sharp and strict L^p-inequality for stochastic integrals, Ann. Probab. **15** (1987), 268–273.

[4] R. Bañuelos, P. J. Méndez-Hernández, Space-time Brownian motion and the Beurling-Ahlfors transform, Indiana University Math. J. **52** (2003), 981-990.

[5] R. Bañuelos, A. II. Lindeman, A martingale study of the Beurling-Ahlfors transforms in $\mathbb{R}^n$, J. Funct. Anal. **145** (1997), 224-265.

[6] R. Bañuelos, G. Wang, Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms, Duke Math. J. **80** (1995), 575-600.

[7] T. Coulhon, X. T. Dong, Riesz transforms and related inequalities on complete Riemannian manifolds, Comm. Pure Appl. Math. **56** (2003), no. 12, 1728-1751.

[8] G. De Rham, Variétés Différentielles, Hermain, Paris, 1960.

[9] S. Donaldson, D. Sullivan, Quasiconformal 4-manifolds, Acta Math. **163** (1989), 181-252.

[10] D. Elworthy, Y. LeJan, X.-M. Li, *On the Geometry of Diffusion Operators and Stochastic Flows*, Lect. Notes in Math. **1720**, Springer, Berlin, 1999.

[11] M. Gaffney, Hilbert spaces methodes in the theory of harmonic integrals, Trans. Amer. Math. Soc.**78** (1955), 426-444.

[12] H. Helmholtz, Über Integrale der hydrodynamischen Gle ichungen, welche den Wirbelbewegunggen entsprechen, J. Reine Angew. Math. **55** (1858), 25-55.

[13] W. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, 1941.

[14] T. Iwaniec, G. Martin, Quasiregular mappings in even dimensions, Acta Math. **170** (1993), 29-81.

[15] T. Iwaniec, G. Martin, Riesz transforms and related singular integrals, J. Reine Angew. Math. **473** (1996), 25-57.

[16] K. Kodaira, Harmonic fields in Riemannian manifolds (generalized potential theory), Ann. Math. **50** (1949), 587-665.

[17] J. Leray, Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appl. **12** (1933), 1-82.

[18] J. Leray, Essai sur les mouvements plans d'un liquide visqueux que limitent des parois, J. Math. Pures Appl. **13** (1934), 331-418.

[19] Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. **63** (1934), 193-248.

[20] X.-D. Li, Riesz transforms for symmetric diffusion operators on complete Riemannian manifolds, Revista Mat. Iberoamericana, **22** No. 2 (2006), 591-648.

[21] X.-D. Li, Riesz transforms on forms and L^p-Hodge decomposition on complete Riemannian manifolds, preprint, 2007, submitted to J. Reine Angew. Math.

[22] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*. Vol 1 Incompressible Methods, and Vol 2 Compressible Medels. Oxford Univ. Press, 1996 and 1998.

[23] P. Malliavin, Formule de la moyenne, calcul des perturbations et théorèmes d'annulation pour les formes harmoniques, J. Funct. Anal. **17** (1974), 274-291.

[24] A. Milgram, P. Rosenbloom, Harmonic forms and heat conduction, I: Closed Riemannian manifolds, Proc. Nat. Acad. Sci. **37** (1951), 180-184.

[25] J. W. Robbin, R. C. Rogers, B. Temple, On weak continuity and the Hodge decomposition, Trans. Amer. Math. Soc. **303** (1987), 609-618.

[26] G. Schwartz, Hodge decomposition–a method for solving boundary value problems, Lect. Notes in Math. **1607**, Springer-Verlag, Berlin, 1995.

[27] C. Scott, L^p Theory of differential forms on manifolds, Trans. Amer. Math. Soc. **347** (1995), No. 6, 2075-2096.

[28] I. Shigekawa, *Stochastic Analysis*, Translations of Math. Monographs, Vol. **224**, Amer. Math. Soc., Providence, Rhode Island, 2004.

[29] R. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, J. Funct. Anal. **52** (1983), 48-79.

[30] E.-M. Stein, *Singular Integrals and the Differentiability of Functions*, Princton Univ. Press, 1970.

[31] H. Weyl, The method of orthogonal projection in potential theory, Duke Math. J. **7** (1940), 411-444.

[32] H. Weyl, On Hodge's theory of harmonic integrals, Ann. Math., (2) **44** (1943), 1-6.

Laboratoire Statistique et Probabilités, Université Paul Sabatier, 118, route de Narbonne, 31062, Toulouse Cedex 9, France xiang@math.ups-tlse.fr