

TWO GENERALIZATIONS OF CHEEGER-GROMOLL SPLITTING THEOREM VIA BAKRY-EMERY RICCI CURVATURE

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ABSTRACT. In this paper, we prove two generalized versions of the Cheeger-Gromoll splitting theorem via the non-negativity of the Bakry-Émery Ricci curvature on complete Riemannian manifolds.

1. INTRODUCTION

Cheeger and Gromoll's splitting theorem [CG] played an important role in the study of manifolds with nonnegative or almost nonnegative Ricci curvature. In this paper, we consider the manifolds with nonnegative Bakry-Émery Ricci curvature and prove two generalized versions of the splitting theorem on such manifolds.

Following Bakry-Émery [BE], see also [Q, BQ3, Lo1, LD], given a Riemannian manifold (M, g) and a C^2 -smooth function ϕ , M is said to have nonnegative ∞ -dimensional Bakry-Émery Ricci curvature associated to ϕ if $Ric + Hess(\phi) \geq 0$, where Ric denotes the Ricci curvature of g and $Hess$ denotes the Hessian with respect to g . As pointed out in Lott [Lo1], in general, the splitting theorem does not hold for manifolds with nonnegative ∞ -dimensional Bakry-Émery Ricci curvature. A trivial counterexample is given by the hyperbolic n -space form \mathbb{H}^n , where $Ric + \frac{1}{\delta}Hess(\rho^2) \geq 0$ for some small constant $\delta > 0$ and distance function ρ . Obviously there are many lines in this space but it doesn't split off a line. See [WW].

If the manifold M is compact and its ∞ -dimensional Bakry-Émery Ricci curvature is positive, then $\pi_1(M)$ is finite. This was first proved by X.-M. Li [LM], see also [FG, WW, W, Z]. Also, from Lott's work [Lo1, Theorem 1], a compact manifold with nonnegative ∞ -dimensional Bakry-Émery Ricci curvature has b_1 parallel vector fields where b_1 is the first Betti number of M , which are orthogonal to the gradient field of ϕ . This indicates that the

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universal Riemannian covering space of (M, g) should split off b_1 lines. We confirm this in this paper, as a corollary of the following theorem.

Theorem 1.1. *Let (M, g) be a complete connected Riemannian manifold with $\text{Ric} + \text{Hess}(\phi) \geq 0$ for some $\phi \in C^2(M)$ which is bounded above uniformly on M . Then it splits isometrically as $N \times \mathbb{R}^l$, where N is some complete Riemannian manifold without lines and \mathbb{R}^l is the l -Euclidean space. Furthermore, the function ϕ is constant on each \mathbb{R}^l in this splitting.*

Then the corollary reads as

Corollary 1.2. *Let (M, g) be a closed connected Riemannian manifold with $\text{Ric} + \text{Hess}(\phi) \geq 0$ for some smooth function ϕ on M . Then we have an isometric decomposition for its universal Riemannian covering space: $\widetilde{M} \cong N \times \mathbb{R}^l$, where N is a closed manifold, \mathbb{R}^l is the l -Euclidean space and $l \geq b_1$, the first Betti number of M . Furthermore, the lifting function of ϕ , say $\tilde{\phi}$, is constant on each \mathbb{R}^l -factor.*

If b_1 equals the dimension of M , then (M, g) is the flat torus.

Another generalized version of splitting theorem can be described as follows. According to [Ba94, BQ3], we say that the symmetric diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ satisfies the curvature-dimension condition $CD(0, m)$ if

$$L|\nabla u|^2 \geq \frac{2|Lu|^2}{m} + 2 \langle \nabla u, \nabla Lu \rangle, \quad \forall u \in C_0^\infty(M).$$

Following the notation used in [LD], which is slightly different from [Ba94, BQ3, Lo1], we define the m -dimensional Bakry-Émery Ricci curvature of $L = \Delta - \nabla\phi \cdot \nabla$ on an n -dimensional Riemannian manifold as follows

$$\text{Ric}_{m,n}(L) := \text{Ric} + \text{Hess}(\phi) - \frac{\nabla\phi \otimes \nabla\phi}{m-n},$$

where $m = \dim_{BE}(L) > n$ is called the Bakry-Émery dimension of L , which is a constant and is not necessarily to be an integer. By [Ba94, BQ3, LD], we know that $CD(0, m)$ holds if and only if $\text{Ric}_{m,n}(L) \geq 0$. We now state the following

Theorem 1.3. *Let (M, g) be a complete connected Riemannian n -manifold and $\phi \in C^2(M)$ be a function satisfying that $\text{Ric}_{m,n}(L) \geq 0$ for some constant $m = \dim_{BE}(L) > n$ which is not necessarily to be an integer. Then M splits isometrically as $N \times \mathbb{R}^l$ for some complete Riemannian manifold N without line and the l -Euclidean space \mathbb{R}^l . Furthermore, the function ϕ is constant on each \mathbb{R}^l -factor, and N has non-negative $(m-l)$ -dimensional Bakry-Émery Ricci curvature.*

Our paper provides us with two extensions of the Cheeger-Gromoll splitting theorem on complete Riemannian manifolds via the Bakry-Émery Ricci curvature. We would like to mention that a very relevant and independent paper by Wei and Wylie [WW] has been posted recently in the arxiv. One of their results (see Theorem 1.4 in [WW]) says that if M is an n -dimensional complete Riemannian manifold with $Ric + Hess(f) \geq 0$ for some bounded function f and contains a line, then M splits into $M = N^{n-1} \times \mathbb{R}$ and f is constant along the line.

The paper is organized as follows: In Section 2, we show that the Buseman function associated to the line has parallel gradient field. Then we prove Theorems 1.1, 1.3 and Corollary 1.2 in Section 3. In Section 4, we give some remarks on the Bakry-Émery Ricci curvature and the Cheeger-Gromoll splitting theorem.

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2. ESTIMATION OF THE LAPLACIAN ON BUSEMAN FUNCTION

So far, there have been at least three different proofs of the Cheeger-Gromoll splitting theorem. All these proofs turn to prove that the Busemann function is harmonic. The original proof of Cheeger and Gromoll [CG] uses the Jacobi fields theory and the elliptic regularity. The second one by Eschenburg-Heintze [EH] uses only the Laplacian comparison theorem on distance function and the Hopf-Calabi maximum principle. The third one, given by Schoen-Yau [SY], uses the Laplacian comparison theorem on distance and the sub-mean value inequality rather than the maximum principle. For an elegant description of the proof of [EH], see Besse [Bs]. We will follow the lines given by [Bs, CG, EH, SY] to prove the L -harmonicity of the Buseman function on M .

First we assume (M, g) is a complete Riemannian manifold and $\phi \in C^2(M)$ satisfying that $Ric + Hess(\phi) \geq 0$ over M . Fix $p \in M$ as a base point and denote $\rho(x) = \text{dist}(p, x)$ the distance function. Given any $q \in M$, let $\gamma : [0, \rho] \rightarrow M$ be a minimal normal geodesic from p to q and $\{E_i(t)\}_{i=1}^{n-1}$

be parallel orthonormal vector fields along γ which are orthogonal to $\dot{\gamma}$. Constructing vector fields $\{X_i(t) = \frac{t}{\rho} E_i(t)\}_{i=1}^{n-1}$ along γ and by the second variation formula, we have the estimate

$$\begin{aligned}
\Delta\rho(q) &\leq \int_0^\rho \sum_{i=1}^{n-1} (|\nabla_{\dot{\gamma}} X_i|^2 - \langle X_i, R_{X_i, \dot{\gamma}} \dot{\gamma} \rangle) dt \\
&= \int_0^\rho \left(\frac{n-1}{\rho^2} - \frac{t^2}{\rho^2} Ric(\dot{\gamma}, \dot{\gamma}) \right) dt \\
&\leq \frac{n-1}{\rho} + \int_0^\rho \frac{t^2}{\rho^2} Hess(\phi)(\dot{\gamma}, \dot{\gamma}) dt \\
&= \frac{n-1}{\rho} + \frac{1}{\rho^2} \int_0^\rho t^2 \frac{d^2}{dt^2} (\phi \circ \gamma) dt \\
&= \frac{n-1}{\rho} + \langle \nabla \phi, \dot{\gamma} \rangle (q) - \frac{2}{\rho^2} \int_0^\rho t \frac{d}{dt} (\phi \circ \gamma) dt \\
&= \frac{n-1}{\rho} + \langle \nabla \phi, \dot{\gamma} \rangle (q) - \frac{2}{\rho} \phi(q) + \frac{2}{\rho^2} \int_0^\rho \phi \circ \gamma dt.
\end{aligned}$$

Thus

$$(1) \quad L\rho(q) \leq \frac{n-1}{\rho} - \frac{2}{\rho} \phi(q) + \frac{2}{\rho^2} \int_0^\rho \phi \circ \gamma dt, \quad \forall q \in M \setminus cut(p),$$

where γ is any minimal normal geodesic connecting p and q .

Lemma 2.1. *Let (M, g) be a complete Riemannian manifold and γ be a ray. If $Ric + Hess(\phi) \geq 0$ for some smooth function ϕ which is bounded from above uniformly on M , then the associated Buesman function of γ , say b^γ , satisfies that $Lb^\gamma \geq 0$ in the barrier sense.*

Remark 2.2. We say that a continuous function f on M satisfies $Lf \geq 0$ in the barrier sense, if for any given $q \in M$ and $\epsilon > 0$, there is a C^2 function $f_{q,\epsilon}$ in a neighborhood of q , such that $f_{q,\epsilon} \leq f$ but $f_{q,\epsilon}(q) = f(q)$, and that $Lf_{q,\epsilon} \geq -\epsilon$. Such $f_{q,\epsilon}$ is called a support function of f . We say that $Lf \leq 0$ in the barrier sense if $L(-f) \geq 0$ in the barrier sense.

Proof of Lemma 2.1. We use the same argument as in [Bs, EH], see also Lemma 4.7 of [Z]. Denote $p = \gamma(0)$. The Buseman function along the ray γ is defined by $b^\gamma(q) := \lim_{t \rightarrow \infty} (t - d(q, \gamma(t)))$. By [Bs, EH, Z, SY], b^γ is Lipschitz with 1 as its Lipschitz constant. Following [Bs, EH, Z], for any fixed $q \in M$, we define the support functions around q as follows.

Let δ_{t_k} be a minimal geodesic connecting q and $\gamma(t_k)$. By [Bs, EH, Z], there exists a subsequence of t_k such that the initial vector $\dot{\delta}_{t_k}(0)$ converges to some $X \in T_q M$. Let δ be the ray emanating from q and generated by X . Then q does not belong to the cut-locus of $\delta(r)$ for any $r > 0$. So

$b_r^\gamma(x) = r - d(x, \delta(r)) + b^\gamma(q)$ is C^∞ around q and satisfies that $b_r^\gamma \leq b^\gamma$ with $b_r^\gamma(q) = b^\gamma(q)$. On the other hand, by the estimate (1), we have

$$Lb_r^\gamma(x) = -Ld(\delta(r), x) \geq \frac{n-1}{d(\delta(r), x)} - \frac{2\phi(x)}{d(\delta(r), x)} + \frac{2}{d(\delta(r), x)^2} \int_0^{d(\delta(r), x)} \phi \circ \sigma dt$$

where σ is a minimal geodesic connecting $\delta(r)$ and x . Thus for any given $\epsilon > 0$, when r is large enough, $Lb_r^\gamma \geq -\epsilon$ for x in a small neighborhood of q . This shows that b_r^γ is the desired support function for b^γ . \square

Remark 2.3. If b^γ is smooth at q , then $\nabla b^+(q) = \dot{\delta}(0)$, where δ is the ray emanating from q constructed in the proof of Lemma 2.1. See [CG, Z].

Lemma 2.4 (The Calabi-Hopf maximum principal). *Let (M, g) be a connected complete Riemannian manifold, $\phi \in C^2(M)$, and $L = \Delta - \nabla\phi \cdot \nabla$. Let f be a continuous function on M such that $Lf \geq 0$ in the barrier sense. Then f attains no maximum unless it is a constant.*

Proof. The proof is similar to the one in [C], see also [Bs]. \square

Lemma 2.5. *Let (M, g) and ϕ be as in Lemma 2.1. Suppose M contains a line γ , then the Buseman functions b^\pm associated to rays $\gamma^\pm(t) = \gamma(\pm t), t \geq 0$, are both smooth and satisfy that $Lb^\pm = 0$.*

Proof. By Lemma 2.1, $L(b^+ + b^-) \geq 0$ in the barrier sense. On the other hand, $b^+ + b^- = 0$ on the line γ and the triangle inequality implies that $b^+ + b^- \leq 0$ over M . So $b^+ + b^- \equiv 0$ over M by Lemma 2.4. Now $Lb^+ \geq 0$ and $L(-b^+) = Lb^- \geq 0$ show that $Lb^+ = 0$ in the barrier sense, then from the elliptic regularity theorem b^+ is smooth and $Lb^+ = 0$ in the canonical way, cf. section 6.3-6.4 of [GT]. Similarly b^- is smooth satisfying that $Lb^- = 0$. \square

Next we consider the case where $Ric_{m,n}(L) \geq 0$. We have the following

Lemma 2.6. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $m > n$ such that $Ric_{m,n}(L) \geq 0$. Then the Buseman functions b^\pm associated to rays $\gamma^\pm(t) = \gamma(\pm t), t \geq 0$, are both smooth and satisfy that $Lb^\pm = 0$.*

Proof. Let $b_r^\gamma(x) = t - d(x, \delta(r)) + b^\gamma(q)$ be the support function defined in the proof of Lemma 2.1. By [LD, Remark 3.2](pp. 1317-1318), we have the Laplacian comparison theorem

$$Ld(\cdot, x)|_y \leq \frac{m-1}{d(y, x)}, \quad \forall x \in M, y \in M \setminus \text{cut}(x).$$

This yields that

$$Lb_r^\gamma(x) = -Ld(x, \delta(r)) \geq -\frac{m-1}{d(x, \delta(r))}.$$

Hence, $Lb^+ \geq 0$ holds in the barrier sense. Similarly, $Lb^- \geq 0$ holds in the barrier sense. By the same argument as used in the proof of Lemma 2.5, we can conclude the result. Below we follow [SY] to give an alternative proof of Lemma 2.6. Indeed, for all $\psi \in C_0^\infty(M)$ with $\psi \geq 0$, and for all $t > 0$,

$$\begin{aligned} \int_M Lb_t^\gamma \psi d\mu &= - \int_M Ld(x, \gamma(t)) \psi d\mu \\ &\geq - \int_M \frac{m-1}{d(x, \gamma(t))} \psi d\mu. \end{aligned}$$

Taking $t \rightarrow \infty$, we have $Lb^+ \geq 0$ in the sense of distribution. Similarly, $Lb^- \geq 0$. Hence $L(b^+ + b^-) \geq 0$ holds in the sense of distribution. By the strong maximum principle, since the L -subharmonic function $b^+ + b^-$ has an interior maximum on the geodesic ray γ , it must be identically constant. Thus, $b^+ + b^- = 0$, $Lb^\pm = 0$ and b^\pm are smooth. \square

Lemma 2.7. *Under the conditions as in Lemma 2.5 or Lemma 2.6, ∇b^+ and ∇b^- are unit parallel vector fields.*

Proof. By Remark 2.3, ∇b^\pm are normal vector fields. To show they are parallel, we will use a generalized version of the Bochner-Weitzenböck formula. By Bakry-Emery [BE], for any smooth function ψ , we have

$$(2) \quad L|\nabla\psi|^2 = 2|\nabla^2\psi|^2 + 2\langle \nabla L\psi, \nabla\psi \rangle + 2(\text{Ric} + \text{Hess}(\phi))(\nabla\psi, \nabla\psi).$$

Using Lemma 2.5 or Lemma 2.6 and applying (2) to $\psi = b^\pm$, we see that $0 = L|\nabla b^\pm|^2 \geq 2|\nabla^2 b^\pm|^2$ over M , since $\text{Ric} + \text{Hess}(\phi) \geq 0$ in both cases. Now the result follows. \square

3. PROOF OF THEOREMS 1.1 AND 1.3

Now we are in a position to give a

Proof of Theorems 1.1 and 1.3. By Lemma 2.7, $X = \nabla b^+$ is a parallel unit vector field. Let $\phi(t) = e^{tX}$ be the one-parameter transformation group of isometries generated by X . The level surface $N = \{x | b^+(x) = 0\}$ is a totally geodesic submanifold of M , and the induced metric h_N from g is complete. Define a map $F : N \times \mathbb{R} \rightarrow M$ by

$$F(p, t) = \phi(t)(p).$$

We have $\frac{d}{dt} b^+(\phi(t)p) = |\nabla b^+|^2(\phi(t)p) \equiv 1$. This implies $F(N, t) \subset \{x \in M | b^+(x) = t\}$. We claim that F is bijective. In fact, for any $x \in M$, letting $q \in N$ be the nearest point to x and γ be the shortest normal geodesic from

q to x , then $\dot{\gamma}(q) = X(q)$ and $\gamma(t) = \phi(t)q$ by the uniqueness of the geodesic, as $\phi(t)q$ is obviously a normal geodesic. So $x \in \gamma \subset \text{Im}(F)$. This proves that F is surjective. By the group property $F(\cdot, t) \circ F(\cdot, s) = F(\cdot, t + s)$, F is injective. The claim follows.

Next we prove that F is an isometry. To do so, notice that $F(\cdot, t)$ maps N isometrically onto $\{x \in M | b^+(x) = t\}$ via $\phi(t)$. So it suffices to show that for any vector $v \in TN$, we have

$$\langle dF(\cdot, t)(v), dF\left(\frac{\partial}{\partial t}\right) \rangle = \langle d\phi(t)(v), X \rangle \equiv 0.$$

This is obviously true since $d\phi(t)(TN) \perp X$. So F is an isometry.

Now identifying (M, g) with $(N \times \mathbb{R}, h_N \otimes dt^2)$ and applying (2) to $\psi = b^+$, we get

$$0 = L|\nabla b^+|^2 = 2(\text{Ric} + \text{Hess}(\phi))(\nabla b^+, \nabla b^+) = 2\frac{\partial^2}{\partial t^2}\phi.$$

So ϕ is linear on each line of M . Since ϕ is bounded from above, it must be constant on each line. This proves Theorem 1.1.

Finally, if $\text{Ric}_{m,n}(L) \geq 0$, then (2) yields

$$0 = \frac{\partial^2}{\partial t^2}\phi \geq \frac{1}{m-n} \left| \frac{\partial}{\partial t}\phi \right|^2.$$

So ϕ is constant along each line of M and Theorem 1.3 follows. \square

Proof of Corollary 1.2. By Theorem 1.1 and using the same argument as did by Cheeger and Gromoll in [CG], we conclude the result. \square

Remark 3.1. All the arguments in the proof of Theorem 1.1 depend only on the fact that the limit

$$\overline{\lim}_{\rho \rightarrow \infty} \frac{1}{\rho^2} \int_0^\rho \phi \circ \sigma dt \leq 0$$

on any ray σ , see Estimate (1). If so, then Theorem 1.1 remains hold. In particular, if $\frac{\phi(q)}{d(p,q)} = o(1)$ as $\frac{1}{d(p,q)} \rightarrow 0$, where p is a fixed base point, then Theorem 1.1 and all corollaries considered above still hold.

Finally we state an alternative result about the splitting theorem, where the boundedness of the potential function ϕ is removed.

Corollary 3.2. *Let (M, g) be an open complete connected Riemannian manifold and X be a unit parallel vector field. If $\text{Ric} + \text{Hess}(\phi) \geq cg$ for some $\phi \in C^2(M)$ and a constant $c > 0$, then (M, g) splits off a line. In particular, any open shrinking Ricci soliton with a parallel vector field splits off a line.*

Recall that a Riemannian manifold (M, g) is a shrinking Ricci soliton if there exists a smooth function f such that $\text{Ric} + \text{Hess}(f) = cg$ for some positive constant c .

Proof. By the result of [WW, W], such a manifold M has finite fundamental group $\pi_1(M)$. Denote by $(\widetilde{M}, \widetilde{g})$ the universal Riemannian covering of (M, g) and let \widetilde{X} be the lifting of X . Let $\phi(t) = e^{t\widetilde{X}}$ and N be a maximal integral submanifold of \widetilde{X}^\perp , the distribution orthogonal to \widetilde{X} . Define the map F as in the proof of Theorem 1.1 and Theorem 1.3. Then it can be shown that F is an isometry by the simply connectedness of \widetilde{M} . So we can identify $(\widetilde{M}, \widetilde{g})$ with $(N \times \mathbb{R}, h_N \otimes dt^2)$, where h_N is the restriction of \widetilde{g} on N . Then the vector field \widetilde{X} equals $\frac{\partial}{\partial t}$ and is invariant under the action by $\pi_1(M)$. We claim that $\pi_1(M)$ acts trivially on the \mathbb{R} -factor.

Suppose not, then there is $\alpha \in \pi_1(M)$ and $(z_0, t_0) \in N \times \mathbb{R}$ such that $\alpha(z_0, t_0) = (z_1, t_1)$ with $t_0 \neq t_1$. Then α must maps the line $\{z_0\} \times \mathbb{R}$ isometrically onto the line $\{z_1\} \times \mathbb{R}$ in the same direction and maps the slice $N \times \{t_0\}$ onto the slice $N \times \{t_1\}$, because it preserves $\frac{\partial}{\partial t}$. Denote by $p : N \times \mathbb{R} \rightarrow \mathbb{R}$ the projection to the \mathbb{R} -factor and $i : \mathbb{R} \rightarrow N \times \mathbb{R}$ the injection $i(t) = (z_0, t)$. Then $\bar{\alpha} = p \circ \alpha \circ i$ is a translation on \mathbb{R} with variation $t_1 - t_0 \neq 0$. Now $\{\bar{\alpha}^k = p \circ \alpha^k \circ i\}_{k=1}^\infty$ forms a subgroup of isometry transformation group of \mathbb{R} , which is generated by a translation. This shows that α is a free element of $\pi_1(M)$, which contradicts the finiteness of $\pi_1(M)$. Hence $\pi_1(M)$ acts trivially on \mathbb{R} -factor and consequently the base manifold (M, g) splits off a line. \square

It is natural to pose the following questions.

Question 3.3. Construct a compact Riemannian manifold with negative Ricci curvature somewhere and with positive Bakry-Émery-Ricci curvature everywhere.

Question 3.4. Let (M, g) be an open complete Riemannian manifold with $Ric + Hess(\phi) \geq cg$ for some function $\phi \in C^2(M)$ and some constant $c \in \mathbb{R}$. If (M, g) contains a line, does it really split off a line? In particular, is it true on a shrinking Ricci soliton?

4. SOME REMARKS

In this section, we give some remarks on the Bakry-Émery-Ricci curvature and the Cheeger-Gromoll splitting theorem.

From our personal conversation with Dominique Bakry, we are able to know some interesting history about the introduction of the Bakry-Émery Ricci curvature which was named by Lott in [Lo1]. In the beginning of 1980s, when Bakry studied some problems of the Riesz transforms associated with diffusion operators, he observed that some commutation formulae play an essential role. In [Ba85], he introduced the so-called Γ -operator (*le carré du champs*) and the Γ_2 -operator (*le carré du champs*)

itéré) which are symmetric bilinear derivative operators acting on nice functions. In [BE], Bakry and Émery formulated Γ and Γ_2 in a very general setting and proved that, if $\Gamma_2 \geq \lambda\Gamma$ for some constant $\lambda > 0$, then the logarithmic Sobolev inequality holds. For a symmetric diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ on a Riemannian manifold, Bakry and Émery [BE] obtained the fundamental weighted Bochner-Lichnerowicz-Weitzenböck formula, which says that $Ric + \nabla^2\phi$ is exactly the Ricci curvature term in the Bochner-Lichnerowicz-Weitzenböck decomposition of the Witten Laplacian on one-forms on Riemannian manifolds equipped with weighted volume measures $e^{-\phi}dv$. The notion of the curvature-dimension $CD(K, n)$ -condition for diffusion operators has also been introduced in [BE]. During 1980-2006, the study of the comparison theorems of the Bakry-Émery-Ricci curvature has been extensively developed by Bakry and his collaborators [Ba87, Ba94, BL1, BL2, BQ1, BQ2, BQ3, BL3]. In [BL2], Bakry and Ledoux proved the generalized Myers theorem for Markovian diffusion operators satisfying the $CD(R, n)$ -condition with $R > 0$. In [Q], Qian extended the Myers theorem to complete Riemannian manifolds on which the $(n + \alpha)$ -dimensional Bakry-Émery Ricci curvature $Ric + Hess(h) - \alpha^{-1}\nabla h \otimes \nabla h$ has a uniformly positive lower bound. In [BQ3], Bakry and Qian proved a generalized version of the Bishop-Gomov volume comparison theorem and the generalized Laplacian comparison theorem. In [LD], the second author of this paper gave a new proof of the generalized Laplacian comparison theorem and extended S.-T. Yau's L^∞ -Liouville theorem, P. Li's L^1 -Liouville and L^1 -uniqueness theorem to the elliptic equation or the heat equation associated with symmetric diffusion operators on complete Riemannian manifolds.

In [Lo1], Lott gave a new understanding of the Bakry-Émery Ricci curvature and obtained some interesting results from the point of view of the measured Gromov-Hausdorff convergence. In 2002-2003, Perelman posted three papers on Arxiv. In [P], Perelman used the Bakry-Émery Ricci curvature to give a modified version of R. Hamilton's Ricci flow equation. Under the condition that the weighted measure $e^{-\phi(x)}\sqrt{\det g(x)}dx$ does not change, Perelman [P] proved that the equation of the modified Ricci flow for the Riemannian metric g can be viewed as the gradient flow of the \mathcal{F} -functional introduced in [P] and that the potential function ϕ satisfies a conjugate backward heat equation. He then introduced the so-called \mathcal{W} -entropy functional and proved that the \mathcal{W} -functional is monotonically decreasing along the modified Ricci flow. By the monotonicity of the \mathcal{W} -functional and using an earlier result due to Rothaus on L. Gross' logarithmic Sobolev inequalities, Perelman [P] proved the Little Loop Lemma which was conjectured by R. Hamilton in the 1990s. This result consists of one of the most important steps in Perelman's final resolution of the Poincaré conjecture.

By the Bishop-Gromov volume comparison theorem for the weighted volume measure and using the standard argument as used in Gromov's original proof [G1, G2] for his famous theorem, we can extend the Gromov precompactness theorem to compact Riemannian manifolds with weighted measures via the finite dimensional Bakry-Émery Ricci curvature. More precisely, we have the following

Theorem 4.1. *Let $\mathcal{M}(m, n, d, K)$ be the set of n -dimensional compact Riemannian manifolds (M, g) equipped with C^2 -weighted volume measures $d\mu = e^{-\phi}dv$ such that: $n \leq \dim_{BE}(L) \leq m$, $\text{diam}(M) \leq d$, and $\text{Ric}_{\dim_{BE}(L), n}(L) \geq K$, where $\dim_{BE}(L)$ is the Bakry-Emery dimension of the diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$. Then $\mathcal{M}(m, n, d, K)$ is precompact in the sense of the measured Gromov-Hausdorff convergence.*

To our knowledge, at least in the case $\dim_{BE}(L) = m$ and $\phi \in C^\infty(M)$, Theorem 4.1 has been already pointed out by Lott [Lo1, Remark 3, p. 881]. Indeed, if $L = \Delta - \nabla\phi \cdot \nabla$ is a symmetric diffusion operator with $\text{Ric}_{m, n}(L) \geq K$ for some $m \geq n$ and $K \in \mathbb{R}$, then it is obviously true that $\text{Ric}_{m', n}(L) \geq K$ for all $m' \geq m$. So, if $\dim_{BE}(L) \leq m$ and if $\text{Ric}_{\dim_{BE}(L), n}(L) \geq K$, then obviously we have $\text{Ric}_{m, n}(L) \geq K$. Therefore, Theorem 4.1 can be recaptured from the above mentioned result due to Lott [Lo1], which holds obviously when $\phi \in C^2(M)$.

Theorem 1.3 and Theorem 4.1 were obtained in November 2005. During December 12-18, 2005, the second author of this paper reported Theorem 1.3 and Theorem 4.1 in the First Sino-French Conference of Mathematics organized by Zhongshan University in Zhuhai. During this conference, the second author had a very nice discussion with Prof. J.-P. Bourguignon on the Gromov precompactness theorem on weighted Riemannian manifolds. Prof. Bourguignon also told him that the recent study of M. Kontsevich on the Conformal Fields Theory needs some kind of generalizations of the Gromov precompactness theorem.

To what extent can we say about the geometry on the measured Gromov-Hausdorff limit of the weighted Riemannian manifolds in $\mathcal{M}(m, n, d, K)$? This problem has been a central issue in the study of the Riemannian geometry in the Large. In 2005-2006, Lott-Villali [LV] and Sturm [St1, St2] independently developed a Comparison Geometry of the Ricci curvature on Metric-Measure Spaces. Roughly speaking, they used the optimal transports and the entropy functions to introduce the notions of infinite dimensional Ricci curvature and the finite dimensional Ricci curvature Ric_N as well as the curvature-dimension $\mathbf{CD}(K, N)$ -condition on metric-measure spaces. In [LV], Lott and Villali proved that, if (X, d, μ) is the measured Gromov-Hausdorff limit of a sequence of compact metric-measure spaces (X_i, d_i, μ_i) ,

and if the generalized N -dimensional Ricci curvature Ric_N (resp. the generalized ∞ -dimensional Ricci curvature) on each (X_i, d_i, μ_i) is non-negative, $N \in [1, \infty)$, then (X, d, μ) has non-negative generalized N -dimensional Ricci curvature (resp, generalized ∞ -dimensional Ricci curvature). Similar stability theorem was also proved by Sturm [St2] under the $\mathbf{CD}(K, N)$ -condition. It might be interesting to point out that there are some differences between Lott-Villani's definition of Ric_N and Sturm's definition of the $\mathbf{CD}(K, N)$ -condition on metric-measure spaces. In the case of Riemannian manifolds, their definitions coincide with the standard ones in Riemannian geometry. However, in general they are not the same. Under the condition that $Ric_N \geq 0$ and $Ric_\infty \geq K$ for some $K > 0$ on a compact metric-measure space (X, d, ν) , Lott and Villali proved the so-called Weak Bonnet-Myers theorem, i.e., $\text{diam}(X) \leq C\sqrt{\frac{N}{K}}$ for some universal constant $C > 0$. See Theorem 6.30 in [LV]. On the other hand, under the $\mathbf{CD}(K, N)$ -condition, where $K > 0$ and $N \geq 1$ are two constants, Sturm [St1, St2] proved that $\text{diam}(X) \leq \pi\sqrt{\frac{N-1}{K}}$, which is the same upper bound for the diameter as indicated in the classical Bonnet-Myers theorem on complete N -dimensional Riemannian manifolds with Ricci curvature $Ric \geq K > 0$.

In [Lo2], Lott pointed out that the Cheeger-Gromoll splitting theorem cannot be extended to metric-measure spaces with non-negative finite dimensional Ricci curvature (in the sense of Lott-Villali [LV]). Whether or not the Cheeger-Gromoll splitting theorem can be extended to metric-measure spaces satisfying the curvature-dimension $\mathbf{CD}(0, N)$ condition in the sense of Sturm [St2], or how to introduce a more reasonable definition of the finite dimensional Ricci curvature on metric-measure spaces so that the Cheeger-Gromoll splitting theorem can be extended? This might be an interesting problem for a study in future.

REFERENCES

- [Ba85] D. Bakry, Transformation de Riesz pour les semi-groupes symétriques. Première partie : étude de la dimension 1. Séminaire de Probabilités XIX, 1983/1984, 130-144. Lect. Notes in Math. 1123, Springer, Berlin, 1985.
- [BE] D. Bakry and M. Emery, Diffusion hypercontractives, in Séminaire de Probabilités XIX, 1983/1984, 177-206, Lect. Notes in Math. 1123, Springer, Berlin, 1985.
- [Ba87] D. Bakry, Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorées, in Séminaire de Probabilités XXI, 137-172, Lect. Notes in Math. 1247. Springer, Berlin, 1987.
- [Ba94] D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes, Lect. Notes in Math. 1581, 1-114, 1994.
- [BL1] D. Bakry, M. Ledoux, Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. Invent. Math. 123 (1996), no. 2, 259–281.

- [BL2] D. Bakry, M. Ledoux, Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator. *Duke Math. J.* 85 (1996), no. 1, 253–270.
- [BL3] D. Bakry and M. Ledoux, A logarithmic Sobolev form of the Li-Yau parabolic inequality, *Rev. Mat. Iberoamericana*, 2006.
- [BQ1] D. Bakry and Z.-M. Qian, Harnack inequalities on a manifold with positive or negative Ricci curvature, *Rev. Mat. Iberoamericana* 15 (1999), 143–179.
- [BQ2] D. Bakry and Z.-M. Qian, Some new results on eigenvectors via dimension, diameter and Ricci curvature, *Advances in Math.* 155 (2000), 98–153.
- [BQ3] D. Bakry and Z.-M. Qian, Volume comparison theorems without Jacobi fields. *Current trends in potential theory*, 115–122, *Theta Ser. Adv. Math.*, 4, Theta, Bucharest, 2005.
- [Bs] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.
- [C] E. Calabi, An extension of E. Hopf's maximum principle with an application to Riemannian geometry, *Duke math. J.* 25 (1965) 45–56.
- [CG] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, *J. Diff. Geom.* 6 (1971) 119–128.
- [EH] J. Eschenburg and E. Heintze, An elementary proof of the Cheeger-Gromoll splitting theorem, *Ann. Glob. Analysis and Geometry*, No. 2, 2(1984), 141–151.
- [GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, New York (1983).
- [G1] M. Gromov, *Structures Métriques pour les Variétés Riemanniennes*, *Textes Mathématiques*, 1. CEDIC, Paris, 1981.
- [G2] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, with Appendices by M. Katz, P. Pansu and S. Semmes, Birkhauser, Boston, Basel, Berlin, 1999.
- [FG] M. Fernández-López and E. García-Río, A remark on compact Ricci solitons, preprint.
- [LD] X.-D. Li, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, *J. Math. Pure. Appl.* 84 (2005) 1295–1361.
- [LM] X.-M. Li, On extensions of Myers' theorem. *Bull. London Math. Soc.* 27 (1995), no.4, 392–296.
- [Lo1] J. Lott, Some geometric properties of the Bakry-Émery-Ricci tensor, *Comment. Math. Helv.* 78 (2003), 865–883.
- [LV] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, preprint 2005, to appear in *Ann. Math.*
- [Lo2] J. Lott, Optimal transports and Ricci curvature for metric-measure spaces, preprint, October 4, 2006.
- [P] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, <http://arXiv.org/abs/math/0211159>.
- [Q] Z. Qian, Estimates for weighted volumes and applications, *Quart. J. Math. Oxford Ser. (2)* 48 (1997), 235–242.
- [SY] R. Schoen and S.T. Yau, *Lectures on Differential Geometry*, International Press, 1994.
- [St1] K.-T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.* 196 (2006), 65–131.
- [St2] K.-T. Sturm, On the geometry of metric measure spaces. II, *Acta Math.* 196 (2006), 133–177.

- [WW] G.F. Wei and W. Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, arXiv:math.DG/0706.1120v1.
- [W] W. Wylie, Complete shrinking Ricci solitons have finite fundamental group, to appear in Proc. AMS..
- [Zh] Z.L. Zhang, On the Finiteness of the fundamental group of a compact shrinking Ricci soliton, Colloq. Math., 107 (2007), 297-299.
- [Z] S.H. Zhu, The comparison geometry of Ricci curvature, Comparison Geometry, MSRI Publications, v30 (1997) 221-262.

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