

# Riesz transforms on forms and $L^p$ -Hodge decomposition on complete Riemannian manifolds

Xiang-Dong Li\*

*Institut de Mathématiques, Université Paul Sabatier  
118, route de Narbonne, 31062, Toulouse Cedex 9, France  
E-mail: xiang@math.ups-tlse.fr*

## Abstract

In this paper we study the growth of the upper bound of the  $L^p$ -norm of the Riesz transforms on forms and establish the Weak  $L^p$ -Hodge decomposition theorem on complete Riemannian manifolds with suitable curvature conditions. Let  $M$  be a complete Riemannian manifold,  $d\mu = e^{-\phi} dv$  the weighted volume measure with  $\phi \in C^2(M)$ . Let  $\square_{\phi,k}$  be the weighted Hodge Laplacian on  $k$ -forms,  $W_{\phi,k}$  the Weitzenböck curvature of  $\square_{\phi,k}$  on  $\Lambda^k T^*M$ . In the case  $W_{\phi,k} \geq -a$  and  $W_{\phi,k+1} \geq -a$ , where  $a \geq 0$  is a non-negative constant, we prove that the  $L^p$ -norm of the Riesz transform  $d(a + \square_{\phi,k})^{-1/2}$  satisfies  $\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}$  for all  $p > 1$ , where  $C_k$  is a constant depending only on  $k$ ,  $p^* = \max\{p, \frac{p}{p-1}\}$ . In the case  $W_{\phi,k+1} \geq -a$  and  $W_{\phi,k} = -a$ , we prove that  $\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)$  for all  $p > 1$ . In particular, on  $n$ -dimensional hyperbolic manifolds with constant curvature  $-1$ , we prove that, for all  $k \geq \frac{n-1}{2}$ ,  $p > 1$ , we have  $\|d(k(n-k) + \square_k)^{-1/2}\|_{p,p} \leq C_k(p^* - 1)$ . Finally, we prove the Weak  $L^p$ -Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvature.

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## 1 Introduction

### 1.1 Background

It is well-known that the Riesz transforms  $R_j$  on  $\mathbb{R}^n$ , defined by the principal value of the singular integrals

$$R_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x - y|^{n+1}} dy, \quad j = 1, \dots, n,$$

are weak  $(1, 1)$  and are bounded in  $L^p(\mathbb{R}^n, dx)$  for all  $p > 1$ , see e.g. E. M. Stein [64]. In recent years, there has been considerable interest in finding the exact value or obtaining a good estimate of the  $L^p$ -norm of the Riesz transforms. In 1972, Pichorides [57] proved that the  $L^p$ -norm of the Hilbert transform on the real line

$$Hf(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy$$

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is given by

$$\|H\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall p > 1.$$

Here and throughout of this paper, we denote

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

In 1996, Iwaniec and Martin [34] proved that the  $L^p$ -norm of the Riesz transforms  $R_j$  is given by

$$\|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad j = 1, \dots, n. \quad (1)$$

In [10], Bañuelos and Wang gave an alternative proof of (1) and proved that for all  $p > 1$  the  $L^p$ -norm of the vector Riesz transform  $\nabla(-\Delta)^{-1/2} = (R_1, \dots, R_n)$  has an explicit and dimension-free upper bound

$$\|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1). \quad (2)$$

One of the motivations of the above study can be seen in Donaldson and Sullivan [20] and in Iwaniec and Martin [33, 34], where it has been pointed out that the knowledge of the exact value or a good estimate of the  $L^p$ -norm of the Riesz transform can lead important applications in the study of quasi-conformal mappings and related nonlinear geometric PDEs as well as in the  $L^p$ -Hodge decomposition theory. In their 1993 Acta Math paper [33], using the  $L^p$ -boundedness of the Riesz transforms, Iwaniec and Martin proved the following version of the  $L^p$ -Hodge decomposition theorem on  $\mathbb{R}^n$ .

**Theorem 1.1** *Let  $\omega \in L^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n)$ ,  $1 < p < \infty$ , and  $k = 1, \dots, n-1$ . Then there is a  $(k-1)$ -form  $\alpha$  and  $(k+1)$ -form  $\beta$  such that*

$$\omega = d\alpha + d^*\beta$$

and

$$d\alpha, d^*\beta \in L^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n),$$

where  $d^*$  denotes the  $L^2$ -adjoint of  $d$  with respect to the Lebesgue measure on  $\mathbb{R}^n$ . Moreover,  $d\alpha$  and  $d^*\beta$  are unique,

$$\alpha \in \text{Ker}d^* \cap L_1^p(\mathbb{R}^n, \Lambda^{k-1} \mathbb{R}^n), \quad \beta \in \text{Ker}d \cap L_1^p(\mathbb{R}^n, \Lambda^{k+1} \mathbb{R}^n),$$

and there exists a constant  $C_p(k, n) > 0$  such that

$$\|\alpha\|_{L_1^p(\mathbb{R}^n)} + \|\beta\|_{L_1^p(\mathbb{R}^n)} \leq C_p(k, n)\|\omega\|_p,$$

where  $L_1^p(\mathbb{R}^n, \Lambda^k \mathbb{R}^n)$  denotes the homogeneous Sobolev space of  $k$ -forms on  $\mathbb{R}^n$  whose first order derivatives are  $L^p$ -integrable with respect to the Lebesgue measure on  $\mathbb{R}^n$ , on which  $\|\omega\|_{L_1^p(\mathbb{R}^n)} := \|d\omega\|_p + \|d^*\omega\|_p$ . (Indeed, using (2), the above mentioned constant  $C_p(k, n)$  can be independent of  $n$ .)

It might be interesting to point out that, in his 1858 *Crelle's* paper [32], in order to solve boundary value problems arising from the study of hydrodynamic systems, Helmholtz formulated a result on the splitting of vector fields into vortices and gradients, which can be understood in a suitable form of what is now called the "Hodge decomposition". In his fundamental paper [38], Leray used the  $L^2$ -orthogonal projection  $P = \text{Id} + \nabla(-\Delta)^{-1}\text{div}$  to study the Navier-Stokes equations. So far, it has been well-known that the  $L^p$ -boundedness of the Riesz transforms and the  $L^p$ -Hodge decomposition theorem have important applications in the study of the elliptic and parabolic PDEs [26], the Navier-Stokes equations [47], the boundary valued problems [55], the quasi-conformal mappings and related nonlinear geometric PDEs [20, 33, 34, 58] and the stochastic differential equations [67].

## 1.2 Motivation

To what extent can the classical analysis on Euclidean space be extended to complete Riemannian manifolds? This is an important issue in analysis on complete non-compact Riemannian manifolds. Let  $(M, g)$  be a complete non-compact Riemannian manifold,  $n = \dim M$ ,  $\nu$  the Riemannian volume measure, i.e.,  $d\nu(x) = \sqrt{\det g(x)} dx$ . For any  $k = 1, \dots, n$ , we use  $C_0^\infty(\Lambda^k T^*M)$  to denote  $C_0^\infty(M, \Lambda^k T^*M)$ . Let  $d_k$  be the exterior differential on  $k$ -forms,  $d_k^*$  the formal  $L^2$ -adjoint of  $d_k$  with respect to  $\nu$ . The Hodge Laplacian on  $k$ -forms is defined by

$$\square_k = d_{k-1} d_{k-1}^* + d_k^* d_k.$$

Inspired by the Witten deformation of the Morse theory [69], we consider more general weighted Hodge Laplacians on complete Riemannian manifolds. More precisely, let  $M$  be a complete Riemannian manifold equipped with a weighted volume measure

$$d\mu = e^{-\phi} d\nu,$$

where  $\phi \in C^2(M)$ . Let  $d_{\phi,k}^*$  the formal  $L^2$ -adjoint of  $d_k$  with respect to  $\mu$ , i.e., for all  $\alpha \in C_0^\infty(\Lambda^k T^*M)$  and  $\beta \in C_0^\infty(\Lambda^{k+1} T^*M)$ , we have

$$\int_M \langle d\alpha, \beta \rangle d\mu = \int_M \langle \alpha, d_{\phi,k}^* \beta \rangle d\mu.$$

The weighted Hodge Laplacian on  $k$ -forms with respect to the weighted volume measure  $\mu$ , also called the Witten Laplacian on  $k$ -forms with respect to  $\mu$ , is defined by

$$\square_{\phi,k} = d_{k-1} d_{\phi,k-1}^* + d_{\phi,k}^* d_k.$$

When  $\phi \equiv 0$ , we have  $d_{0,k}^* = d_k^*$  and  $\square_{0,k} = \square_k$ ,  $k = 0, 1, \dots, n$ . For all  $p > 1$ , let  $L^p(\Lambda^k T^*M, \mu)$  be the completion of  $C_0^\infty(\Lambda^k T^*M)$  with respect to the  $L^p$ -norm  $\|\cdot\|_p$  defined by

$$\|\omega\|_p^p := \int_M |\omega(x)|^p d\mu(x).$$

By [15, 65], it is well-known that  $\square_k$  is essentially self-adjoint on  $L^2(\Lambda^k T^*M, \nu)$ . Similarly, we can prove that  $\square_{\phi,k}$  is essentially self-adjoint on  $L^2(\Lambda^k T^*M, \mu)$ .

The purpose of this paper is to study the following fundamental problems.

**Problem 1.2** *Under which conditions on a complete non-compact Riemannian manifold  $M$  and  $\phi \in C^2(M)$ , the Riesz transforms  $d\square_{\phi,k}^{-1/2}$  and  $d_{\phi,k}^* \square_{\phi,k}^{-1/2}$  are bounded in  $L^p$  with respect to the weighted measure  $d\mu = e^{-\phi} d\nu$  for some or all  $p > 1$ ?*

**Problem 1.3** *Under which conditions on a complete non-compact Riemannian manifold  $M$  and  $\phi \in C^2(M)$ , the Weak  $L^p$ -Hodge decomposition theorem holds for some or all  $p > 1$ ?*

When  $\phi \equiv 0$ , Problem 1.2 was originally raised by Strichartz in his 1983 celebrated paper [65] and is called the Strichartz problem by people working in harmonic analysis on complete non-compact manifolds. In [65], Strichartz has already pointed out the deep relationship between the above two problems. When  $p = 2$ , using Gaffney's integration by parts formula and the Kodaira Weak  $L^2$ -Hodge decomposition theorem on complete Riemannian manifolds, it is well-known that the Riesz transforms  $d\square_{\phi,k}^{-1/2}$  and  $d_{\phi,k}^* \square_{\phi,k}^{-1/2}$  are always bounded in  $L^2$ , and the Weak  $L^2$ -Hodge decomposition theorem is always true on all complete Riemannian manifolds with  $C^2$ -weighted volume measures, see e.g. [19, 65, 12]. However, for  $p \neq 2$ , the situation is very complicated. Since 1983, many people have studied the above problems on complete non-compact Riemannian manifolds with various geometric or analytic conditions, see e.g. [65, 48, 5, 62, 71, 72, 16, 17, 2, 14, 49, 42, 43] and reference therein.

### 1.3 Previous results

We now describe some results of Bakry [5]. Using a martingale approach to the Littlewood-Paley-Stein inequalities, Bakry [5] proved that, for any diffusion operator  $L = \Delta - \nabla\phi \cdot \nabla$  on a complete Riemannian manifold  $M$ , where  $\phi \in C^2(M)$ , if the Ricci curvature associated with  $L$  is bounded below by  $-a$ , i.e.,  $Ric(L) = Ric + \nabla^2\phi \geq -a$ , where  $a \geq 0$  is a non-negative constant, then for all  $p > 1$  the Riesz transform  $\nabla(a - L)^{-1/2}$  is bounded in  $L^p$  with respect to the weighted volume measure  $d\mu = e^{-\phi}dv$  and its  $L^p$ -norm is bounded above by a universal constant depending only on  $p$ . In particular, if  $Ric(L) \geq 0$ , then the Riesz transform  $\nabla(-L)^{-1/2}$  is bounded in  $L^p$  for all  $p > 1$  and its  $L^p$ -norm is bounded above by a dimensional free constant. This recaptures an earlier famous result due to P. A. Meyer [53] on the  $L^p$ -boundedness of the Riesz transforms associated with the Ornstein-Uhlenbeck operator on finite or infinite dimensional Gaussian spaces. In [5], Bakry also proved that, if  $M$  is a complete Riemannian manifold with  $W_k \geq -a$  and  $W_{k+1} \geq -a$  for some positive constant  $a > 0$ , where  $W_k$  denotes the  $k$ -th Weitzenböck curvature on  $(M, g)$ , then the Riesz transforms  $d(a + \square_k)^{-1/2}$  associated with the Hodge Laplacian  $\square_k$  is bounded in  $L^p$  for all  $p > 1$ , and there exists a universal constant  $C_{p,k} > 0$  which is independent of  $n = \dim M$  and  $a$ , such that for all  $\omega \in C_0^\infty(\Lambda^k T^*M)$ ,

$$\|d(a + \square_k)^{-1/2}\omega\|_p \leq C_{p,k}\|\omega\|_p. \quad (3)$$

Under the same conditions, the Riesz transform  $d^*(a + \square_{k+1})^{-1/2}$  is bounded in  $L^p$  for all  $p > 1$ . Moreover,

$$\|d^*(a + \square_{k+1})^{-1/2}\omega\|_p \leq C_{p,k}\|\omega\|_p. \quad (4)$$

It is very natural to ask whether the Riesz transforms  $d\square_k^{-1/2}$  and  $d^*\square_{k+1}^{-1/2}$  are bounded in  $L^p$  for all  $1 < p < \infty$  if  $M$  is a complete Riemannian manifold with non-negative Weitzenböck curvatures  $W_i \geq 0$ ,  $i = k, k+1$ . Whether or not this result is true is very important to obtain an affirmative answer to the Strichartz problem and to prove the Weak  $L^p$ -Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvatures. When  $k \geq 1$ , it seems that one cannot find an explicit statement of this result in [5] even though it can be derived from Bakry's  $L^p$ -estimates (3) and (4) with universal constant. Indeed, (3) is equivalent to

$$\|d\omega\|_p \leq C_{p,k}\|(a + \square_k)^{1/2}\omega\|_p, \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M). \quad (5)$$

Using Lemma 5.2 in [5], there exists a constant  $A > 0$  independent of  $a$  and  $p$  such that

$$\|(a + \square_k)^{1/2}\omega\|_p \leq A(\sqrt{a}\|\omega\|_p + \|\square_k^{1/2}\omega\|_p), \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M). \quad (6)$$

Since the universal constants  $C_{p,k}$  and  $A$  are independent of  $a > 0$ , one can take the limit  $a \rightarrow 0$  in (5) and (6), and deduce that

$$\|d\omega\|_p \leq AC_{p,k}\|\square_k^{1/2}\omega\|_p, \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M).$$

Thus, the Riesz transform  $d\square_k^{-1/2}$  is bounded in  $L^p$  for all  $p > 1$  on complete Riemannian manifolds with non-negative Weitzenböck curvatures  $W_i \geq 0$ ,  $i = k, k+1$ . By duality argument, under the same conditions, the Riesz transform  $d^*\square_{k+1}^{-1/2}$  is bounded in  $L^p$  for all  $p > 1$ . Moreover, for all  $p > 1$  and  $k = 1, \dots, n$ , we have

$$\begin{aligned} \|d\square_k^{-1/2}\|_p &\leq AC_{p,k}, \\ \|d^*\square_{k+1}^{-1/2}\|_p &\leq AC_{p,k}. \end{aligned}$$

Inspired by the above mentioned results due to of Pichorides [57], Iwaniec-Martin [34] and Bañuelos-Wang [10] on the  $L^p$ -norm estimates of the Riesz transforms on Euclidean space, it is very natural to ask what is the asymptotic behavior of the constant  $C_{p,k}$  when  $p \rightarrow 1$  and  $p \rightarrow \infty$  for all  $k = 0, 1, \dots, n$ . In our previous paper [43], we developed a new probabilistic approach in the study of the Riesz transforms on complete Riemannian manifolds and proved that, if the Bakry-Emery Ricci curvature associated with  $L = \Delta - \nabla\phi \cdot \nabla$  is non-negative, i.e.,  $Ric(L) = Ric + \nabla^2\phi \geq 0$ , then the  $L^p$ -norm of the Riesz transform  $\nabla(-L)^{-1/2}$  with respect to the weighted volume measure  $\mu$  satisfies

$$\|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1. \quad (7)$$

In particular, on all complete Riemannian manifolds with non-negative Ricci curvature, we proved in [43] that

$$\|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1. \quad (8)$$

These can be considered as a refinement of the above mentioned result of Bakry [5] and a natural generalization of the above mentioned results due to Pichorides [57], Iwaniec-Martin [34] and Bañuelos-Wang [10]. Moreover, using the results of Iwaniec-Martin [34] and Bañuelos-Wang [10], Arcozzi [1] and Larsson-Cohn [37], we pointed out the above estimate (7) and (8) are asymptotic sharp when  $p \rightarrow 1$  and when  $p \rightarrow \infty$ . Now, it is very natural to ask whether one can extend the above estimates (7) and (8) to the Riesz transforms associated with the Witten Laplacians on complete Riemannian manifolds with non-negative Weitzenböck curvature. In this paper, we study this problem and establish the Weak  $L^p$ -Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvature.

#### 1.4 Notation and assumption

Let  $(M, g)$  be a complete non-compact Riemannian manifold,  $v$  the Riemannian volume measure,  $n = \dim M$ . Let  $\nabla$  be the Levi-Civita connection on  $TM$  or the Levi-Civita covariant derivative operator on  $\Lambda T^*M$ . Let  $\phi \in C^2(M)$ , and  $d\mu = e^{-\phi}dv$ . The Hodge Laplacian and the Witten Laplacian on  $k$ -forms are denoted by  $\square_k$  and  $\square_{\phi,k}$  respectively. When acting on  $k$ -forms on  $M$ ,  $k = 0, 1, \dots, n$ , we denote

$$\Delta_\phi = \Delta - \nabla_{\nabla\phi},$$

where  $\Delta = \text{tr}\nabla^2$  is the covariant Laplace-Beltrami operator on  $(M, g)$ . In particular, when  $k = 0$ , we use  $L$  to denote  $\Delta_\phi|_{C_0^\infty(M)}$ . More precisely,

$$L = \Delta - \nabla\phi \cdot \nabla.$$

Note that, for all  $f, g \in C_0^\infty(M)$ , we have the integration by parts formula

$$\int_M \langle \nabla f, \nabla g \rangle d\mu = - \int_M (Lf)gd\mu = - \int_M f(Lg)d\mu.$$

In other words,  $L = -\square_{\phi,0}$ . Similarly to the case  $\phi = 0$  as in [15, 65],  $L$  and  $\square_{\phi,k}$  are essentially self-adjoint on  $L^2(M, \mu)$  and  $L^2(\Lambda^k T^*M, \mu)$ ,  $k = 1, \dots, n$ .

Fix  $x \in M$ . Let  $e_1, \dots, e_n$  be a normal orthonormal basis of  $T_x M$  neat  $x$  such that  $\nabla_{e_i} e_j(x) = 0$  for all  $i, j = 1, \dots, n$ , and let  $e_1^*, \dots, e_n^*$  be its dual basis. By definition, the  $k$ -th Weitzenböck curvature associated with the Witten Laplacian  $\square_\phi$  on  $k$ -forms, is defined as a symmetric endomorphism on  $\Lambda^k T^*M$ , and is given by

$$W_{\phi,k}(x) := \sum_{i,j} e_i^* \wedge i_{e_j} R(e_i, e_j) + d\Lambda^k \nabla^2 \phi,$$

where  $R$  the Riemannian curvature of the Levi-Civita connection  $\nabla$  on  $(M, g)$ ,  $i_{e_j}$  denotes the interior multiplication induced by the contraction of the vector field  $e_j$  on  $\Lambda^k(T^*M)$ ,  $\nabla^2\phi$  denotes the Hessian of  $\phi$  with respect to the Levi-Civita connection  $\nabla$ , and  $d\Lambda^k\nabla^2\phi$  is defined by

$$d\Lambda^k\nabla^2\phi(v_1 \wedge \dots \wedge v_k) = \sum_{i=1}^k v_1 \dots \wedge \nabla^2\phi(v_i) \dots \wedge v_k, \quad v_1, \dots, v_k \in TM.$$

For simplicity, we make use the convention that  $W_{\phi,k} \equiv 0$  for  $k = 0, n+1$ . When  $\phi = 0$ , we denote  $W_k = W_{0,k}$ ,  $k = 0, \dots, n+1$ .

Throughout this paper, we make the following basic assumption:

(A) The heat semigroup  $e^{tL}$  is a Markovian semigroup in the sense that  $e^{tL}1(x) = 1$  for all  $x \in M$ . In other words, the heat semigroup  $e^{tL}$  is stochastically complete.

In the case  $\phi \equiv 0$ , Yau [70] proved that the heat semigroup  $e^{t\Delta}$  is Markovian if  $M$  is a complete Riemannian manifold with Ricci curvature bounded from below. More general criterion for the stochastic completeness of the heat semigroup  $e^{t\Delta}$  were given by Kanp-Li [36], Li-Schoen [39], and Grigor'yan [27, 28]. In the case  $\phi \neq 0$ , Bakry [4] proved that, if the Bakry-Emery Ricci curvature  $Ric(L) = Ric + \nabla^2\phi$  is bounded from below by a negative constant, then  $e^{tL}$  is Markovian. More general criterion for the stochastic completeness of the heat semigroup  $e^{tL}$  are due to Sturm [68] and the author [41] (p. 1306, Theorem 1.4).

## 1.5 Main results

To state the main results of this paper, we first recall the rigorous definitions of the Riesz transforms and the Riesz potentials associated with the Hodge Laplacian on complete Riemannian manifolds.

**Definition 1.4** ([65]) *Let  $M$  be a complete Riemannian manifold,  $\square_{\phi,k}$  be the weighted Hodge Laplacian with respect to the weighted volume measure  $d\mu = e^{-\phi}dv$ . For  $a \geq 0$ , the Riesz transforms  $d(a + \square_{\phi,k})^{-1/2}$  and  $d_{\phi}^*(a + \square_{\phi,k})^{-1/2}$  as well as the Riesz potential  $(a + \square_{\phi,k})^{-1/2}$  are defined as follows:*

(i) *a  $k$ -form  $\omega \in L^p(\Lambda^k T^*M, \mu)$  is in the domain of  $d(a + \square_{\phi,k})^{-1/2}$  if*

$$d(a + \square_{\phi,k})^{-1/2}\omega : = \frac{1}{\Gamma(1/2)} \lim_{\varepsilon \rightarrow 0, A \rightarrow \infty} \int_{\varepsilon}^A de^{-t(a + \square_{\phi,k})}\omega \frac{dt}{\sqrt{t}}$$

*exists in  $L^p(\Lambda^{k+1}T^*M, \mu)$ .*

(ii) *a  $k$ -form  $\omega \in L^p(\Lambda^k T^*M, \mu)$  is in the domain of  $d_{\phi}^*(a + \square_{\phi,k})^{-1/2}$  if*

$$d_{\phi}^*(a + \square_{\phi,k})^{-1/2}\omega : = \frac{1}{\Gamma(1/2)} \lim_{\varepsilon \rightarrow 0, A \rightarrow \infty} \int_{\varepsilon}^A d_{\phi}^*e^{-t(a + \square_{\phi,k})}\omega \frac{dt}{\sqrt{t}},$$

*exist in  $L^p(\Lambda^{k-1}T^*M, \mu)$ .*

(iii) *a  $k$ -form  $\omega \in L^p(\Lambda^k T^*M, \mu)$  is in the domain of  $(a + \square_{\phi,k})^{-1/2}$  if*

$$(a + \square_{\phi,k})_{\phi,k}^{-1/2}\omega := \frac{1}{\Gamma(1/2)} \lim_{\varepsilon \rightarrow 0, A \rightarrow \infty} \int_{\varepsilon}^A de^{-t(a + \square_{\phi,k})}\omega \frac{dt}{\sqrt{t}}$$

*exists in  $L^p(\Lambda^k T^*M, \mu)$ .*

Now we are in a position to state the main results of this paper as follows.

Our first result is a refinement of the above mentioned result due to Bakry [5].

**Theorem 1.5** *Let  $(M, g)$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that the heat semigroup  $e^{tL}$  is conservative, and  $W_{\phi,i} \geq -a$ ,  $i = k, k+1$ , where  $a$  is a non-negative constant. Then, there exists a constant  $C_k > 0$  such that, for all  $1 < p < \infty$ , we have*

$$\|d_k(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}, \quad (9)$$

$$\|d_{\phi,k}^*(a + \square_{\phi,k+1})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}. \quad (10)$$

*In particular, if the  $k$ -th and  $(k+1)$ -th Weitzenböck curvatures are non-negative, i.e.,*

$$W_{\phi,i} \geq 0, \quad i = k, k+1,$$

*then the Riesz transforms  $d\square_{\phi,k}^{-1/2}$  and  $d_{\phi,k+1}^*\square_{\phi,k+1}^{-1/2}$  are bounded in  $L^p$  for all  $p > 1$ . Moreover, for all  $p > 1$ ,*

$$\|d\square_{\phi,k}^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}, \quad (11)$$

$$\|d_{\phi,k+1}^*\square_{\phi,k+1}^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}. \quad (12)$$

Our second result provides us with a reasonable condition on the Weitzenböck curvatures under which the asymptotically sharp  $L^p$ -norm estimates (2), (7) and (8) extend to the Riesz transforms associated with the Witten Laplacians on  $k$ -forms.

**Theorem 1.6** *Let  $(M, g)$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that  $e^{tL}$  is conservative,  $W_{\phi,k} = -a$  and  $W_{\phi,k+1} \geq -a$ , where  $a \geq 0$  is a non-negative constant. Then, there exists a constant  $C_k$ , such that for all  $1 < p < \infty$ , we have*

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq 2C_k(p^* - 1), \quad (13)$$

$$\|d_{\phi,k+1}^*(a + \square_{\phi,k+1})^{-1/2}\|_{p,p} \leq 2C_k(p^* - 1), \quad (14)$$

where

$$C_k := \sup_{\omega \in \Gamma(\Lambda^k T^*M), \nabla\omega \neq 0} \frac{\|d\omega\|_{\infty}}{\|\nabla\omega\|_{\infty}} < +\infty$$

is a positive constant depending only on  $k = 0, 1, \dots, n$ .

Moreover, at least on the Euclidean space, an upper bound of the form  $O(p^* - 1)$  for the  $L^p$ -norm of the Riesz transforms associated with the Hodge Laplacian is asymptotically sharp when  $p \rightarrow 1$  and when  $p \rightarrow \infty$ .

By Gallot-Meyer [25], on the hyperbolic manifold  $M = \Gamma \backslash \mathbb{H}^n(-1)$  of constant sectional curvature  $-1$ , where  $\Gamma$  is any torsion-free discrete subgroup of  $G = SO^+(n, 1)$  (the group of isometries on  $\mathbb{H}^n(-1)$ ), we have

$$W_k = -k(n - k)\text{Id}.$$

From this and Theorem 1.6 we have the following result which is also new in the literature.

**Theorem 1.7** *Let  $M = \Gamma \backslash \mathbb{H}^n(-1)$  be the hyperbolic manifold of constant sectional curvature  $-1$ , where  $\Gamma$  is any torsion-free discrete subgroup of  $G = SO^+(n, 1)$ . Then, for  $k = \frac{n-1}{2}, \dots, n$  if  $n$  is odd,  $k = \frac{n}{2}, \dots, n$  if  $n$  is even, and for all  $p > 1$ , we have*

$$\|d(k(n - k) + \square_k)^{-1/2}\|_{p,p} \leq 2C_k(p^* - 1), \quad (15)$$

$$\|d^*(k(n - k) + \square_{k+1})^{-1/2}\|_{p,p} \leq 2C_k(p^* - 1), \quad (16)$$

where  $C_k$  is a universal constant depending only on  $k$ .

The last main result is the following Weak  $L^p$ -Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvatures, which is a natural extension of Theorem 1.1.

**Theorem 1.8** *Let  $(M, g)$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that  $e^{tL}$  is conservative, and the Weitzenböck curvatures are non-negative:*

$$W_{\phi, i} \geq 0, \quad i = k-1, k, k+1.$$

*Then, for all  $1 < p < \infty$ , every  $\omega \in L^p(\Lambda^k T^*M, \mu)$  has a unique decomposition in  $L^p$ :*

$$\omega = H\omega + d\alpha + d_\phi^* \beta,$$

*where  $H : L^p(\Lambda^k T^*M, \mu) \rightarrow \text{Ker} \square_{\phi, k} \cap L^p(\Lambda^k T^*M, \mu)$  denotes Hodge harmonic projection,*

$$\alpha \in \text{Ker} d_\phi^* \cap L_1^p(\Lambda^{k-1} T^*M, \mu), \quad \beta \in \text{Ker} d \cap L_1^p(\Lambda^{k+1} T^*M, \mu).$$

*Moreover, there exists a constant  $C_k > 0$  (as the same as in Theorem 1.5), such that for all  $1 < p < \infty$ , we have*

$$\begin{aligned} \|\alpha\|_{L_1^p(\Lambda^{k-1} T^*M, \mu)} &\leq C_{k-1}^2 (p^* - 1)^3 \|\omega\|_p, \\ \|\beta\|_{L_1^p(\Lambda^{k+1} T^*M, \mu)} &\leq C_k^2 (p^* - 1)^3 \|\omega\|_p, \end{aligned}$$

*and*

$$\|(I - H)\omega\|_p \leq (C_{k-1}^2 + C_k^2)(p^* - 1)^3 \|\omega\|_p,$$

*where  $L_1^p(\Lambda^k T^*M, \mu)$  denotes the homogenous Sobolev space of  $k$ -forms  $\eta$  on  $M$  such that*

$$\|\eta\|_{L_1^p(\Lambda^k T^*M, \mu)} := \|d\eta\|_p + \|d^* \eta\|_p < +\infty.$$

## 1.6 Remarks

**Remark 1.9** To prove Theorem 1.5 and Theorem 1.6, we will first prove a probabilistic representation formula of the Riesz transforms on differential forms, cf. Theorem 5.3 below. In the proof of Theorem 5.3, we use the Littlewood-Paley identity. However, the proof of Theorem 1.5 and Theorem 1.6 does not need to use the Littlewood-Paley-Stein inequalities.

**Remark 1.10** To prove Theorem 1.5, Theorem 1.6 and Theorem 1.8, we need also to prove the  $L^p$ -strong stability of the heat semigroup and the Poisson semigroup of the Hodge Laplacian or the Witten Laplacians on complete Riemannian manifolds with non-negative Weitzenböck curvature, cf. Theorem 3.4 below. It seems that we cannot find either an explicit statement or a proof of Theorem 3.4 in the literature.

## 1.7 Applications

The method and the main results of this paper will be used in three forthcoming papers [44, 45, 46]. In [44], we use the main results of this paper to prove the Strong  $L^p$ -Hodge decomposition theorem and to prove some vanishing theorems of the  $L^p$ -cohomology on complete Riemannian manifolds with suitable geometric conditions. In [45], we use the main results of this paper to prove some Sobolev inequalities on differential forms and to prove some vanishing theorems of the  $L^{p,q}$ -cohomology on complete Riemannian manifolds with suitable geometric conditions. In [46], we use the method of this paper to prove the  $L^p$ -boundedness of the Riesz transforms associated with the complex Kodaira-Hodge Laplacian on complete Kähler manifolds with non-negative curvature operator, and use this result

to prove the  $L^p$ -estimates and existence theorems of the  $\bar{\partial}$ -operator on complete Kähler manifold with semi-positive curvatures. The main result in [46] can be viewed as a non-trivial extension of the famous Hörmander-Andreotti-Vesentini  $L^2$ -estimate and existence theorem of the  $\bar{\partial}$ -operator on complete Kähler manifolds with semi-positive curvature. In a paper which is in preparation, we will extend the method and the results of this paper to the Riesz transforms associated with the Hodge Laplacian of the Dirac operator and the twisted Dirac operators on complete Riemannian spin manifolds with suitable curvature conditions.

The rest of this paper is organized as follows. In Section 2, we recall the well-known probabilistic representation formulas of the heat semigroup and the Poisson semigroup generated by the Hodge Laplacian or the Witten Laplacians on forms on complete Riemannian manifolds, as well as the well-known semigroup domination inequalities. In Section 3, we prove the strong  $L^p$ -stability of the heat semigroup and the Poisson semigroup on forms on complete Riemannian manifolds with non-negative Weitzenböck curvature. In Section 4, we formulate the Burkholder sharp  $L^p$ -inequality for martingale transforms on complete Riemannian manifolds. In Section 5, we prove the probabilistic representation formulas for the Riesz transforms and the Riesz potentials on forms on complete Riemannian manifolds. In Section 6, we prove Theorem 1.5, Theorem 1.6 and Theorem 1.7. In Section 7, we prove the Weak  $L^p$ -Hodge decomposition theorem (i.e., Theorem 1.8) on complete Riemannian manifolds with non-negative Weitzenböck curvatures.

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## 2 Heat semigroup and Poisson semigroup on forms

### 2.1 The weighted Bochner-Weitzenböck formula

We first recall the weighted Bochner-Weitzenböck formula which has been well-known to experts. To sake the completeness of the paper, we give a proof for it.

**Theorem 2.1** *Let  $\nabla$  be the Levi-Civita connection on  $(M, g)$ ,  $\phi \in C^2(M)$ . Let*

$$\Delta_\phi = \Delta - \nabla_{\nabla\phi}$$

*be the covariant weighted Laplace-Beltrami operator, where  $\Delta = \text{Tr}\nabla^2$  is the covariant Laplace-Beltrami operator on  $k$ -forms. Let*

$$W_{\phi,k} = \sum_{i,j} e_i^* \wedge i_{e_j} R(e_i, e_j) + d\Lambda^k \nabla^2 \phi$$

*be the weighted Weitzenböck curvature, where  $e_1, \dots, e_n$  is a normal orthonormal basis at  $T_y M$  for  $y \in M$  near  $x \in M$  such that  $\nabla_{e_i} e_j(x) = 0$ ,  $\nabla^2 \phi$  denotes the Hessian of  $\phi$  with respect to the Levi-Civita connection  $\nabla$ . Then*

$$\square_{\phi,k} = -\Delta_\phi + W_{k,\phi}. \tag{17}$$

*Proof.* By definition, the  $L^2$ -adjoint of  $d_k$  with respect to the measure  $\mu$ , denoted by  $d_{\phi,k}^* : C_0^\infty(\Lambda^{k+1} T^* M) \rightarrow L^2(\Lambda^k T^* M, \mu)$ , satisfies

$$\int_M \langle d_k \omega, \eta \rangle d\mu = \int_M \langle \omega, d_{\phi,k}^* \eta \rangle d\mu.$$

Standard argument based on integration by parts formula shows that

$$d_{\phi,k}^* = d^* + i_{\nabla\phi},$$

where  $i_{\nabla\phi}$  denotes the interior multiplication induced by the contraction of the vector field  $\nabla\phi$  on  $\Lambda^{k+1}T^*M$ . This yields

$$\begin{aligned}\square_{\phi,k} &= d(d^* + i_{\nabla\phi}) + (d^* + i_{\nabla\phi})d \\ &= (dd^* + d^*d) + (di_{\nabla\phi} + i_{\nabla\phi}d).\end{aligned}$$

Using the Cartan identity

$$di_{\nabla\phi} + i_{\nabla\phi}d = L_{\nabla\phi},$$

we obtain

$$\square_{\phi,k} = \square + L_{\nabla\phi}. \quad (18)$$

By the standard Bochner-Weitzenböck formula, we have

$$\square = -\Delta + \sum_{i<j} e_i^* \wedge i_{e_j} R(e_i, e_j). \quad (19)$$

We now prove the following identity

$$L_{\nabla\phi}\omega = \nabla_{\nabla\phi}\omega + d\Lambda^k\nabla^2\phi\omega. \quad (20)$$

To prove (20), notice that  $\nabla_{e_j}i_{e_i} = 0$  holds at the point  $x$ . Hence

$$\begin{aligned}di_{\nabla\phi}\omega &= \sum_{i,j} e_j^* \wedge \nabla_{e_j} \langle \nabla\phi, e_i \rangle i_{e_i}\omega \\ &= \sum_{i,j} \nabla_{e_j} \langle \nabla\phi, e_i \rangle e_j^* \wedge i_{e_i}\omega + \sum_{i,j} \langle \nabla\phi, e_i \rangle e_j^* \wedge i_{e_i} \nabla_{e_j}\omega \\ &= \sum_{i,j} \nabla^2\phi(e_i, e_j) e_i^* \wedge i_{e_j}\omega + \sum_{i,j} \langle \nabla\phi, e_i \rangle e_j^* \wedge i_{e_i} \nabla_{e_j}\omega,\end{aligned}$$

and

$$\begin{aligned}i_{\nabla\phi}d\omega &= \sum_{i,j} \langle \nabla\phi, e_i \rangle i_{e_i} e_j^* \wedge \nabla_{e_j}\omega \\ &= \sum_{i,j} \langle \nabla\phi, e_i \rangle \nabla_{e_i}\omega - \sum_{i<j} \langle \nabla\phi, e_i \rangle e_j^* \wedge i_{e_i} \nabla_{e_j}\omega \\ &= \nabla_{\nabla\phi}\omega - \sum_{i<j} \langle \nabla\phi, e_i \rangle e_j^* \wedge i_{e_i} \nabla_{e_j}\omega.\end{aligned}$$

Combining the above identities, we obtain (20). From (18)-(20), we obtain (17).  $\square$

## 2.2 Heat semigroup on forms

In this subsection we recall the well-known probabilistic representation formula of the heat semigroup on forms and the semigroup domination inequalities.

Let  $X_t$  be a diffusion process on  $M$  with infinitesimal generator  $L = \Delta - \nabla_{\nabla\phi}$  and with  $X_0 = x$ . By Itô's theory of diffusion process on Riemannian manifolds, there exists a Brownian motion  $W_t$  on  $\mathbb{R}^n$ , such that

$$\begin{aligned}dX_t &= U_t \circ dW_t - \nabla\phi(X_t)dt, \\ X_0 &= x,\end{aligned}$$

where  $U_t \in \text{End}(T_x M, T_{X_t} M)$  denotes the stochastic parallel transport along the path  $\{X_s, s \in [0, t]\}$  with respect to the Levi-Civita connection, and is the unique solution to the following covariant SDE along  $\{X_s, s \in [0, t]\}$ :

$$\nabla_{\circ d X_t} U_t = 0, \quad U_0 = \text{Id}_{T_x M},$$

where  $\circ d$  denotes the Stratonovich stochastic differentiation.

By Itô's formula and the weighted Bochner–Lichnerowicz–Weitzenböck formula, and using the same argument as in the proof of the Feynman–Kac formula for vector valued function on  $\mathbb{R}^n$ , we have the following well-known probabilistic representation formula of the heat semigroup generated by the Witten Laplacian  $\square_{\phi, k}$  on  $k$ -forms:

$$e^{-t \square_{\phi, k}} \omega(x) = E_x [M_{t, k}^* \omega(X_t)], \quad (21)$$

where  $M_{t, k} \in \text{End}(\Lambda^k T_x^* M, \Lambda^k T_{X_t}^* M)$  is the solution of the covariant differential equation

$$\frac{\nabla}{\partial t} M_{t, k} = -W_{\phi, k}(X_t) M_{t, k}, \quad M_{0, k} = \text{Id}_{\Lambda^k T_x^* M}, \quad (22)$$

where  $\frac{\nabla}{\partial t} := U_t \frac{\partial}{\partial t} U_t^{-1}$  denotes the covariant derivative operator with respect to the Levi-Civita connection along the path of  $\{X_s, s \in [0, t]\}$ . For a proof of (21), we refer the reader to Elworthy–Le Jan–X.-M. Li [21].

We have the following well-known semigroup domination inequality, cf. [21].

**Theorem 2.2** *Let  $M$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that  $W_{\phi, k} \geq a$ , where  $a \in \mathbb{R}$ . Then, for all  $\omega \in C_0^\infty(\Lambda^k T^* M)$ ,*

$$\left| e^{-t \square_{\phi, k}} \omega(x) \right| \leq e^{-at} e^{tL} |\omega|(x), \quad \forall x \in M, t > 0. \quad (23)$$

**Remark 2.3** In the case  $k = 0$  and  $\phi \equiv 0$ , the probabilistic representation formula (21) of the heat semigroup  $e^{-t \square}$  and the semigroup domination inequality (23) can be traced back to Malliavin [50].

### 2.3 Poisson semigroup on $k$ -forms

Using the Bochner subordination formula, for all  $\omega \in C_0^\infty(\Lambda^k T^* M)$ , and for all  $(x, t) \in M \times \mathbb{R}^+$ , the Poisson semigroup  $e^{-t \sqrt{a + \square_{\phi, k}}}$  on one-forms can be defined as follows

$$e^{-t \sqrt{a + \square_{\phi, k}}} \omega(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u} (a + \square_{\phi, k})} \omega(x) e^{-u} u^{-1/2} du.$$

In the case  $k = 0$ , for all  $f \in C_0^\infty(M)$ , the Poisson semigroup  $e^{-t \sqrt{a - L}}$  on functions can be given by

$$e^{-t \sqrt{a - L}} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u} (a - L)} f(x) e^{-u} u^{-1/2} du.$$

We can also give the probabilistic representation formulas for the Poisson semigroup  $e^{-t \sqrt{a - L}}$  and the Poisson semigroup  $e^{-t \sqrt{a + \square_{\phi, k}}}$ . To do so, let  $B_t$  be the standard Brownian motion starting from  $B_0 = y > 0$ , and let

$$\tau_y = \inf\{t > 0 : B_t = 0\}.$$

Then it is well known that for all  $\lambda > 0$ , we have

$$E_y [e^{-\lambda\tau_y}] = e^{-y\sqrt{\lambda}}.$$

This leads us to the following probabilistic representation formula for the Poisson semigroup on functions

$$e^{-y\sqrt{a-L}}f(x) = E_y \left[ e^{-(a-L)\tau_y} f(x) \right] = E_y \left[ e^{-a\tau_y} E_x [f(X_{\tau_y})] \right].$$

That is

$$e^{-y\sqrt{a-L}}f(x) = E_{(x,y)} \left[ e^{-a\tau_y} f(X_{\tau_y}) \right].$$

Similarly, for any  $\omega \in C_0^\infty(\Lambda^k T^*M)$ , the Poisson semigroup  $e^{-y\sqrt{a+\square_{\phi,k}}}\omega(x)$  can be represented by

$$e^{-y\sqrt{a+\square_{\phi,k}}}\omega(x) = E_y \left[ e^{-(a+\square_{\phi,k})\tau_y} \omega(x) \right] = E_y \left[ e^{-a\tau_y} E_x [M_{\tau_y,k}^* \omega(X_{\tau_y})] \right].$$

That is

$$e^{-y\sqrt{a+\square_{\phi,k}}}\omega(x) = E_{(x,y)} \left[ e^{-a\tau_y} M_{\tau_y,k}^* \omega(X_{\tau_y}) \right].$$

**Remark 2.4** Let  $B_t$  be the one-dimensional Brownian motion starting at  $B_0 = y > 0$ . Let  $\tau_y = \inf\{t > 0 : B_t = 0\}$ , and define

$$\tilde{B}_t = B_{\tau_y-t}, \quad t \in [0, \tau_y].$$

Let

$$L_y = \sup\{t > 0 : \tilde{B}_t = y\}$$

be the last exiting time of  $\{\tilde{B}_t\}$  from  $y$ . Then, it is well known, see [52, 59], that  $L_y$  has the same law as  $\tau_y$ , i.e.,

$$P(L_y \in dz) = P(\tau_y \in dz) = \frac{y}{\sqrt{2\pi}} z^{-3/2} e^{-y^2/2z} dz. \quad (24)$$

Moreover,  $\{\tilde{B}_t, t \in [0, L_y]\}$  is the conditional 3-dimensional Bessel process starting at 0 and with terminal value  $y$  conditioning at  $L_y$ , i.e.,  $\tilde{B}_{L_y} = y$ . Note that  $\lim_{y \rightarrow \infty} \tau_y = \lim_{y \rightarrow \infty} L_y = \infty$ .

This leads us to identify  $\tilde{B}_t = B_{\tau_y-t}$  with the standard 3-dimensional Bessel process starting from 0 when the initial position of Brownian motion tends infinity, i.e., when  $B_0 = y \rightarrow \infty$ .

### 3 $L^p$ -strong stability of the heat semigroup and the Poisson semigroup

In this section we prove the  $L^p$ -strong stability of the heat semigroup and the Poisson semigroup on complete Riemannian manifolds with non-negative Weitzenböck curvatures. This result will play a crucial role in the proof of Theorem 1.5.

We first prove the following result which is essentially due to Bakry [5].

**Theorem 3.1** Let  $M$  be a complete Riemannian manifold. Suppose that  $W_{\phi,k} \geq -a$ , where  $a > 0$  is a constant. Then, for all  $p \in [1, \infty]$  and  $t > 0$ , we have

$$\left\| e^{-t(a+\square_{\phi,k})} \right\|_{p,p} \leq e^{-2 \min\{\frac{1}{p}, 1-\frac{1}{p}\}at}, \quad (25)$$

$$\left\| e^{-t\sqrt{a+\square_{\phi,k}}} \right\|_{p,p} \leq e^{-2 \min\{\frac{1}{p}, 1-\frac{1}{p}\}\sqrt{at}}. \quad (26)$$

*Proof.* Using the semigroup domination inequality (23) and the fact that  $e^{tL}$  is  $L^p$ -contractive, under the curvature condition  $W_{\phi,k} \geq -a$ , we have

$$\|e^{-t(a+\square_{\phi,k})}\omega\|_p \leq \|e^{tL}|\omega|\|_p \leq \|\omega\|_p, \quad \forall p \in [1, \infty], t > 0.$$

Therefore

$$\|e^{-t(a+\square_{\phi,k})}\|_{p,p} \leq 1, \quad \forall p \in [1, \infty], t > 0.$$

This proves (25) for  $p = 1, \infty$ . On the other hand, for all  $\omega \in L^2(\Lambda^k T^*M, \mu)$ ,

$$\begin{aligned} \frac{\partial}{\partial t} \|e^{-t\square_{\phi,k}}\omega\|_2^2 &= -2 \int_M \langle \square_{\phi,k} e^{-t\square_{\phi,k}}\omega, e^{-t\square_{\phi,k}}\omega \rangle d\mu \\ &\leq 0. \end{aligned}$$

Hence

$$\|e^{-t\square_{\phi,k}}\omega\|_2 \leq \|\omega\|_2.$$

This yields

$$\|e^{-t(a+\square_{\phi,k})}\|_{2,2} \leq e^{-at}, \quad \forall t > 0.$$

By the Riesz convexity interpolation, for all  $p \in (1, 2)$ , letting  $\theta \in (0, 1)$  be such that  $1/p = \theta/1 + 1 - \theta/2$ , i.e.,  $\theta = 2/p - 1$ , we have

$$\begin{aligned} \|e^{-t(a+\square_{\phi,k})}\|_{p,p} &\leq \|e^{-t(a+\square_{\phi,k})}\|_{1,1}^\theta \|e^{-t(a+\square_{\phi,k})}\|_{2,2}^{1-\theta} \\ &\leq e^{-(1-\theta)at} \\ &= e^{-\frac{2(p-1)at}{p}}, \quad \forall t > 0. \end{aligned}$$

Similarly, for all  $p \in (2, \infty)$ , letting  $\theta \in (0, 1)$  be such that  $1/p = 1 - \theta/2 + \theta/\infty$ , i.e.,  $\theta = 1 - 2/p$ , we have

$$\begin{aligned} \|e^{-t(a+\square_{\phi,k})}\|_{p,p} &\leq \|e^{-t(a+\square_{\phi,k})}\|_{\infty,\infty}^\theta \|e^{-t(a+\square_{\phi,k})}\|_{2,2}^{1-\theta} \\ &\leq e^{-(1-\theta)at} \\ &= e^{-\frac{2at}{p}}, \quad \forall t > 0. \end{aligned}$$

These finish the proof of (25) for all  $p \in [1, \infty]$  and  $t > 0$ . Similarly, for all  $p \in [1, \infty]$  and  $t > 0$ , we can prove (26).  $\square$

We would like to point out that Theorem 3.1 plays a crucial role in Bakry's proof of the  $L^p$ -boundedness of the Riesz transforms  $d(a+\square_{\phi,k})^{-1/2}$  on complete Riemannian manifolds with  $W_{\phi,k} \geq -a$  and  $W_{\phi,k+1} \geq -a$  for some positive constant  $a > 0$ . According to [22], Theorem 3.1 says that, under the condition  $W_{\phi,k} \geq -a$  with  $a > 0$ , the heat semigroup  $e^{-t(a+\square_{\phi,k})}$  and the Poisson semigroup  $e^{-t\sqrt{a+\square_{\phi,k}}}$  are exponentially stable when  $t \rightarrow \infty$ . In particular, we obtain the following

**Corollary 3.2** *Under the same conditions as in Theorem 3.1, for all  $p > 1$ , we have*

$$\begin{aligned} \lim_{t \rightarrow \infty} \|e^{-t(a+\square_{\phi,k})}\|_{p,p} &= 0, \\ \lim_{t \rightarrow \infty} \|e^{-t\sqrt{a+\square_{\phi,k}}}\|_{p,p} &= 0. \end{aligned}$$

**Remark 3.3** In the case where  $M$  is a complete non-compact Riemannian manifold with non-negative Weitzenböck curvature  $W_{\phi,k} \geq 0$ , if we do not assume some additional condition on  $M$ , we cannot prove that the heat semigroup  $e^{-t\Box_{\phi,k}}$  and the Poisson semigroup  $e^{-t\sqrt{\Box_{\phi,k}}}$  are exponentially stable. This might be the most important reason why we cannot find an explicit statement in [5] saying that the Riesz transform  $d\Box_{\phi,k}^{-1/2}$  is bounded in  $L^p$  for all  $p > 1$  if  $M$  is a complete Riemannian manifold with  $W_{\phi,k} \geq 0$  and  $W_{\phi,k+1} \geq 0$ .

An important observation of this paper is that, even though the heat semigroup  $e^{-t\Box_{\phi,k}}$  and the Poisson semigroup  $e^{-t\sqrt{\Box_{\phi,k}}}$  are usually not exponentially stable on complete Riemannian manifolds with non-negative Weitzenböck curvatures  $W_{\phi,k} \geq 0$ , they are strong stable in  $L^p$  for all  $p > 1$  in the sense of Theorem 3.4 below. It will play a crucial role in the proof of Theorems 1.5, 1.6 and Theorem 1.8.

**Theorem 3.4** *Let  $M$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that*

$$W_{\phi,k} \geq 0.$$

*Then the heat semigroup  $H_t = e^{-t\Box_{\phi,k}}$  and the Poisson semigroup  $e^{-t\sqrt{\Box_{\phi,k}}}$  are  $L^p$ -strong stable in  $(\text{Ker}\Box_{\phi,k})^\perp \cap L^p(\Lambda^k T^*M, \mu)$ , i.e.,*

$$\lim_{t \rightarrow \infty} \|e^{-t\Box_{\phi,k}}\omega - H_p\omega\|_p = 0, \quad \forall \omega \in L^p(\Lambda^k T^*M, \mu),$$

$$\lim_{t \rightarrow \infty} \|e^{-t\sqrt{\Box_{\phi,k}}}\omega - H_p\omega\|_p = 0, \quad \forall \omega \in L^p(\Lambda^k T^*M, \mu),$$

where

$$H_p : L^p(\Lambda^k T^*M, \mu) \rightarrow (\text{Ker}\Box_{\phi,k}) \cap L^p(\Lambda^k T^*M, \mu)$$

denotes the  $L^p$ -Hodge harmonic projection.

*Proof.* Using the semigroup domination inequality (23), as  $W_{\phi,k} \geq 0$ , we have

$$|e^{-t\Box_{\phi,k}}\omega(x)| \leq e^{tL}|\omega|(x), \quad \forall \omega \in L^p(\Lambda^k T^*M, \mu), x \in M, t > 0.$$

Since  $e^{tL}$  is a Markovian semigroup, it is  $L^p$ -contractive. Therefore

$$\|e^{-t\Box_{\phi,k}}\omega\|_p \leq \|\omega\|_p, \quad \forall t > 0.$$

By Bochner subordination, we have

$$\|e^{-t\sqrt{\Box_{\phi,k}}}\omega\|_p \leq \|\omega\|_p, \quad \forall t > 0.$$

Slightly modifying the argument used in the proof of Theorem 1 (p. 67) in E. M. Stein [63] or Theorem 4.2.12 (p. 293) in Jacob [35], using the fact that  $\Box_{\phi,k}$  is a self-adjoint operator on  $L^2(\Lambda^k T^*M, \mu)$ , and that  $e^{-t\Box_{\phi,k}}$  is  $L^p$ -contractive for all  $p \in [1, \infty]$ , we can prove that  $e^{-t\Box_{\phi,k}}$  has an analytic continuation from the sector  $\Sigma_{\theta(p)}$  to  $L^p(\Lambda^k T^*M, \mu)$ , where

$$\Sigma_{\theta(p)} = \left\{ z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2} \left( 1 - \left| \frac{2}{p} - 1 \right| \right) \right\}, \quad p > 1.$$

By the theory of analytic semigroups, see e.g. Theorem 4.2.7 in Jacob [35], there exists two constants  $\delta \in (0, \frac{\pi}{2})$  and  $M > 0$  such that

$$\Sigma_{\frac{\pi}{2} + \delta} = \left\{ \lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta \right\} \cup \{0\} \subset \rho_p(\Box_{\phi,k}),$$

and

$$\|(\lambda - \square_{\phi,k})^{-1}\|_{p,p} \leq \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma_{\frac{\pi}{2} + \delta} \setminus \{0\},$$

where  $\rho_p(\square_{\phi,k})$  denotes the resolvent set of  $\square_{\phi,k}$  in  $L^p(\Lambda^k T^* M, \mu)$ . Equivalently, the  $L^p$ -spectra of  $\square_{\phi,k}$  satisfies

$$\sigma_p(\square_{\phi,k}) \cap i\mathbb{R} \subset \{0\}.$$

By a theorem of Arendt-Batty-Lyubich-Vũ, see [22] (Ch. VI Theorem 2.21, see also Example 2.23(i) of Ch. VI), we obtain

$$\lim_{t \rightarrow \infty} \|e^{-t\square_{\phi,k}} \omega - H_p \omega\|_p = 0, \quad \forall \omega \in L^p(\Lambda^k T^* M, \mu).$$

This proves that  $e^{-t\square_{\phi,k}}$  is strong stable in  $(\text{Ker} \square_{\phi,k})^\perp \cap L^p(\Lambda^k T^* M, \mu)$ . Similarly, we can prove that the Poisson semigroup  $e^{-t\sqrt{\square_{\phi,k}}}$  is strong stable in  $(\text{Ker} \square_{\phi,k})^\perp \cap L^p(\Lambda^k T^* M, \mu)$ . The proof of Theorem 3.4 is completed.  $\square$

**Remark 3.5** As far as we know, we cannot find an explicit statement of Theorem 3.4 and a proof for it in the literature. The  $L^p$ -contractivity of the heat semigroup  $e^{-t\square_\phi}$  on differential forms plays an important role in the above proof. In [66], Strichartz studied the problem of the  $L^p$ -contractivity of the heat semigroup  $e^{-t\square}$  on differential forms and the  $L^p$ -contractivity of the Hodge-Kodaira projection on complete non-compact Riemannian manifolds. He pointed out that the heat semigroup on  $k$ -forms is always  $L^2$ -contractive but one cannot expect that the  $L^p$ -contractivity of the heat semigroup is always "yes". In p. 353 in [66], he wrote: "If  $e^{t\Delta}$  were  $L^p$  contractive then by taking the limit as  $t \rightarrow \infty$  we would obtain that the Kodaira projection operator  $T$  onto the harmonic  $k$ -forms is  $L^p$ -contractive." Here, according to the notation in [66],  $\Delta$  denotes the negative Hodge-de Rham Laplacian on  $k$ -form.

## 4 Martingale transforms on complete Riemannian manifolds

In this section, we formulate the Burkholder  $L^p$ -sharp inequality for martingale transforms on complete Riemannian manifolds. For this, we borrow some ideas from Balñuelos-Wang [10] and Arcozzi [1].

Let  $\mathcal{F}_t$  be the filtration generated by the  $n$ -dimensional Brownian motion  $W_t$ . Let  $Y_t$  be a real-valued  $L^2$ -martingale on the Brownian filtration  $\mathcal{F}_t$ . Then

$$Y_t = E[Y_t] + \int_0^t H_s \cdot dW_s,$$

where  $H_t$  is a predictable process with value in  $\mathbb{R}^n$ . Let  $A \in M(n, \mathbb{R})$  be an  $n \times n$  real matrix-valued predictable process. Define the martingale transformation

$$(A * Y)_t = \int_0^t A(s) H_s \cdot dW_s.$$

Let  $\|A\| = \sup_{s \geq 0} \|A(s)\|$ , where  $\|A(s)\| = \sup\{\|A(s)v\|_{\mathbb{R}^n} : v \in \mathbb{R}^n, \|v\|_{\mathbb{R}^n} \leq 1\}$ . Then

$$\langle A * Y \rangle_t - \langle A * Y \rangle_s \leq \|A\|^2 \langle Y \rangle_t - \|A\|^2 \langle Y \rangle_s.$$

According to [13, 10, 1], this means that  $(A * Y)_t$  is differentially subordinate to  $\|A\|Y_t$ .

The following theorem is essentially due to Burkholder [13]. It gives the best constant in the  $L^p$ -inequality for the differential subordinate martingales.

**Theorem 4.1** ([13, 10, 1]) *Let  $Y$  and  $A * Y$  be as above. Then, for any  $p > 1$ , we have*

$$\|A * Y\|_p \leq (p^* - 1) \|A\| \|Y\|_p$$

and the constant  $(p^* - 1) \|A\|$  is the best possible here.

Let  $M$  be a complete Riemannian manifold,  $n = \dim M$ , and  $\phi \in C^2(M)$ . Suppose that the  $L$ -diffusion process  $X_t$  is stochastically complete, i.e., the lifetime of the  $L$ -diffusion process is infinite. By Itô's SDE theory, we have

$$dX_t = U_t \circ dW_t - \nabla \phi(X_t) dt,$$

where  $W_t$  is the standard Brownian motion on  $\mathbb{R}^n$ ,  $U_t$  denotes the stochastic parallel transport along the trajectory of  $\{X_s : s \in [0, t]\}$ , i.e.,  $U_t$  satisfies the following covariant SDE on the orthonormal frame bundle  $O(M)$  over  $M$ :

$$\nabla_{\circ dX_t} U_t = 0.$$

Note that the Brownian filtration  $\mathcal{F}_t$  coincides with the filtration generated by  $X_t$ , since we can reconstruct the  $\mathbb{R}^n$ -valued Brownian motion  $W_t$  from the  $M$ -valued  $L$ -diffusion process  $X_t$  in the following way:

$$W_t = \int_0^t U_s^{-1} \circ dX_s + \int_0^t U_s^{-1} \nabla \phi(X_s) ds,$$

where  $\circ dX_s$  denotes the Stratonovich differential along the trajectory of  $\{X_s, s \in [0, t]\}$ .

**Definition 4.2** *Let  $F$  be a vector bundle over  $M$ ,  $\langle \cdot, \cdot \rangle_F$  be an inner product over  $F$ . Let  $E = \Lambda^1 T^* M \otimes F$ . Let  $U_t^F : F_{X_0} \rightarrow F_{X_t}$  be the stochastic parallel transport along the trajectory of  $\{X_s : s \in [0, t]\}$ . An  $\text{End}(E)$ -valued martingale transformer over the  $L$ -diffusion process  $X_t$  on  $M$  is a bounded and continuous process  $A_t$  such that*

$$A_t(\omega) \in \text{End}(\Lambda^1 T_{X_t(\omega)}^* M \otimes F_{X_t(\omega)}).$$

Let  $\Psi_t$  be a continuous, bounded process with values in  $E = \Lambda^1 T^* M \otimes F$  over  $X$ . The martingale transform of the  $F_{X_0}$ -valued Itô stochastic integral

$$(I_\Psi)_t = \int_0^t U_s^{F, -1} \Psi_s U_s dW_s$$

by the martingale transformer  $A \in \text{End}(E)$ , denoted by  $(A * I_\Psi)_t$ , is the  $F_{X_0}$ -valued martingale defined by

$$(A * I_\Psi)_t = \int_0^t U_s^{F, -1} A_s \Psi_s U_s dW_s.$$

The following theorem is a straightforward extension of a result due to Arcozzi [1] where  $F$  is a trivial vector bundle over  $M$ , i.e.,  $F = \mathbb{R}^l$ .

**Theorem 4.3** *Let  $X_t$  be a stochastically complete  $L$ -diffusion process on  $M$ . Let  $A_t$  be a martingale transformer over  $X_t$ . Suppose that*

$$\|A\| = \sup_{s > 0} \sup_{\omega \in \Omega} \|A_s(\omega)\|_{\text{op}} < \infty,$$

where  $\|A_s(\omega)\|_{\text{op}}$  denotes the operator norm of  $A_s(\omega) \in \text{End}(\Lambda^1 T_{X_s}^* M \otimes F_{X_s})$ . Then

$$\|A * I_\Psi\|_p \leq (p^* - 1) \|A\| \|I_\Psi\|_p.$$

*Proof.* By Theorem 4.1, we need only to prove that  $A * I_\Psi$  is differentially subordinate to  $I_\Psi$ . The covariance process of the martingale transformation  $A * I_\Psi$  is

$$\langle A * I_\Psi \rangle_t = \int_0^t \text{Tr}(A_s \Psi_s \otimes A_s \Psi_s) ds$$

Let  $e_1, \dots, e_n$  be an orthonormal basis of  $T_x M$ , and let  $f_1, \dots, f_l$  be an orthonormal basis of  $F_{X_s(\omega)}$ . Then  $\{e_i \otimes f_j, 1 \leq i \leq n, 1 \leq j \leq l\}$  is an ONB of  $E_{X_s(\omega)} = \Lambda^* T_{X_s(\omega)}^* M \otimes F_{X_s(\omega)}$ . By definition, we have

$$\begin{aligned} \text{Tr}(A_s \Psi_s \otimes A_s \Psi_s)(X_s(\omega)) &= \sum_{i=1}^n \sum_{j=1}^l \langle A_s \Psi_s(e_i \otimes f_j), A_s \Psi_s(e_i \otimes f_j) \rangle_{E_{X_s(\omega)}} \\ &= \sum_{i=1}^n \sum_{j=1}^l |\langle A_s \Psi_s, e_i \otimes f_j \rangle_{E_{X_s(\omega)}}|^2 \\ &= \|A_s \Psi_s\|_{E_{X_s(\omega)}}^2 \\ &\leq \|A_s\|_{\text{op}}^2 \|\Psi_s\|_{E_{X_s(\omega)}}^2 \\ &\leq \|A\|^2 \|\Psi_s\|_{E_{X_s(\omega)}}^2. \end{aligned}$$

Hence

$$\langle A * I_\Psi \rangle_t \leq \|A\|^2 \int_0^t \|\Psi_s\|_{E_{X_s}}^2 ds = \|A\|^2 \langle I_\Psi \rangle_t.$$

This yields that  $A * Y$  is a differential subordination to  $Y$ , i.e., we have

$$\langle A * I_\Psi \rangle_t - \langle A * I_\Psi \rangle_s \leq \|A\| \langle I_\Psi \rangle_t - \|A\| \langle I_\Psi \rangle_s, \quad \forall 0 \leq s < t.$$

The proof of Theorem 4.3 is completed.  $\square$

## 5 Martingale representation of Riesz transforms and Riesz potentials

### 5.1 Background radiation process

From this section, let  $\{X_t, t \geq 0\}$  be the diffusion process on  $M$  whose infinitesimal generator is  $L$  and whose initial measure is  $\mu$ , and let  $B_t$  be a one-dimensional Brownian motion starting from  $B_0 = y$  with infinitesimal generator  $\frac{d^2}{dy^2}$ . We have  $dB_t \cdot dB_t = 2dt$ . Following P. A. Meyer [52] and Gundy [30], we introduce the so-called *background radiation process* on  $M \times \mathbb{R}^+$  as follows

$$Z_t := (X_t, B_t).$$

In fact,  $\{Z_t, t \in [0, \tau]\}$  is a diffusion process on  $M \times \mathbb{R}^+$  whose infinitesimal generator is  $L + \frac{d^2}{dy^2}$  and whose initial distribution is  $\mu \otimes \delta_y$  supported on the hypersurface  $M \times \{y\}$  at time  $t = 0$ . The process  $\{Z_t, t \in [0, \tau_y]\}$  terminates at time  $t = \tau$  upon hitting the boundary  $M \times \{0\}$ .

Let  $P_{(x,y)}$  be the probability law of  $Z_t = (X_t, B_t)$  starting at  $(x, y) \in M \times \mathbb{R}^+$ . We define the measures  $\{P_y, y > 0\}$  on the path space  $C([0, \infty), M \times \mathbb{R})$  as

$$P_y(Z_t \in \mathcal{B}) = \int_M P_{(x,y)}(Z_t \in \mathcal{B}) d\mu(x), \quad (27)$$

for all Borel sets  $\mathcal{B} \subset M \times \mathbb{R}^+$ . Let  $E_y$  be the expectation corresponding to  $P_y$ .

## 5.2 Covariant Itô's calculus

The following proposition will be used in the proof of the results of the next sections.

**Proposition 5.1** *For all  $\omega \in C_0^\infty(\Lambda^k T^*M)$  and all  $a \geq 0$ , we have*

$$e^{-at}\omega(X_\tau) = e^{a\tau}M_\tau^{*, -1}\omega_a(Z_0) + \int_0^\tau e^{a(\tau-s)}M_\tau^{*, -1}M_s^* \left( \nabla, \frac{\partial}{\partial y} \right) \omega_a(Z_s) \cdot (U_s dW_s, dB_s). \quad (28)$$

where

$$\omega_a(x, y) := e^{-y\sqrt{a+\square}\phi}\omega(x), \quad \forall (x, y) \in M \times \mathbb{R}^+.$$

*Proof.* By (22), we have

$$\frac{\nabla}{\partial t}(M_t^*) = -M_t^*W_{\phi, k}(X_t). \quad (29)$$

Using the covariant version of the Itô formula acting on differential forms, cf. Elworthy-Le Jan-X.M. Li [21] and Norris [56], we have

$$\begin{aligned} \nabla(e^{-at}M_t^*\omega(X_t)) &= -ae^{-at}M_t^*\omega(X_t)dt + e^{-at}\nabla M_t^*\omega(X_t)dt + e^{-at}M_t^*(\nabla\omega)(X_t) \circ dX_t \\ &= -e^{-at}M_t^*(a + W_{\phi, k}(X_t))\omega(X_t)dt + e^{-at}M_t^*(\nabla\omega)(X_t)dX_t \\ &\quad + e^{-at}M_t^*\nabla^2\omega(X_t)(dX_t, dX_t). \end{aligned}$$

Note that

$$dX_t = U_t \circ dW_t - \nabla\phi(X_t)dt.$$

Hence

$$\begin{aligned} \nabla^2\omega(X_t)(dX_t, dX_t) &= \sum_{i, j} \nabla^2\omega(X_t)(U_t e_i, U_t e_j) dW_t^i dW_t^j \\ &= \sum_{i, j} \nabla^2\omega(X_t)(U e_i, U e_j) \delta_{ij} dt \\ &= \text{Tr} \nabla^2\omega(X_t) dt \\ &= \Delta\omega(X_t) dt. \end{aligned}$$

This yields that

$$\begin{aligned} \nabla(e^{-at}M_t^*\omega(X_t)) &= e^{-at}M_t^*(\nabla\omega)(X_t)dX_t + e^{-at}M_t^*(\Delta - W_{\phi, k}(X_t) - a)\omega(X_t)dt \\ &= e^{-at}M_t^*(\nabla\omega)(X_t)U_t dW_t + e^{-at}M_t^*(\Delta - \nabla_{\nabla\phi} - W_{\phi, k}(X_t) - a)\omega(X_t)dt \\ &= e^{-at}M_t^*(\nabla\omega)(X_t)U_t dW_t - e^{-at}M_t^*(-\Delta\phi + W_{\phi, k}(X_t) + a)\omega(X_t)dt. \end{aligned}$$

By the weighted Bochner-Weitzenböck formula (17), we obtain

$$\nabla(e^{-at}M_t^*\omega(X_t)) = e^{-at}M_t^*(\nabla\omega)(X_t)U_t dW_t - e^{-at}M_t^*(a + \square_{\phi, k})\omega(X_t)dt.$$

Therefore, for all  $\omega \in \text{Ker}(a + \square_\phi)$ , we have

$$\nabla(e^{-at}M_t^*\omega(X_t)) = e^{-at}M_t^*(\nabla\omega)(X_t)U_t dW_t.$$

Integrating from  $s$  to  $t$  along the trajectory of the diffusion process  $X$  we get

$$e^{-at}M_t^*\omega(X_t) = e^{-as}M_s^*\omega(X_s) + \int_s^t e^{-as}M_r^*(\nabla\omega)(X_r)U_r dW_r.$$

Replacing  $X_t$  by the background radiation process  $Z_t = (X_t, B_t)$ , and replacing the  $(a + \square_\phi)$ -harmonic form  $\omega \in \text{Ker}(a + \square_\phi)$  on  $M$  by the Poisson semigroup  $\omega_a(x, y) = e^{-y\sqrt{a + \square_\phi}} \omega(x)$  on  $M \times \mathbb{R}^+$ , we get

$$e^{-at} M_t^* \omega_a(Z_t) = e^{-as} M_s^* \omega_a(Z_s) + \int_s^t e^{-ar} M_r^* \left( \nabla, \frac{\partial}{\partial y} \right) \omega_a(Z_r) \cdot (U_r dW_r, dB_r).$$

In particular, at  $t = \tau$  and  $s = 0$ , we get

$$\omega(X_\tau) = e^{a\tau} M_\tau^{*, -1} \omega(Z_0) + \int_0^\tau e^{a(\tau-s)} M_\tau^{*, -1} M_s^* \left( \nabla, \frac{\partial}{\partial y} \right) \omega_a(Z_s) \cdot (U_s dW_s, dB_s).$$

The proof of Proposition 5.1 is completed.  $\square$

### 5.3 A probabilistic representation formula of $k$ -forms

The formula (28) in Proposition 5.1 gives a probabilistic representation of  $k$ -form in terms of the time derivative and the covariant derivative of its Poisson semigroup composed with the background radiation process on  $M \times \mathbb{R}^+$ . In this subsection we prove a probabilistic representation formula of  $k$ -forms which uses only the time derivative of the Poisson semigroup.

**Theorem 5.2** *Suppose that  $W_{\phi, k} \geq -a$ , where  $a$  is a non-negative constant. Then, for all  $\omega \in C_0^\infty(\Lambda^k T^* M)$ , we have*

$$\frac{1}{2} \omega(x) = \lim_{y \rightarrow \infty} E_y \left[ \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s \mid X_\tau = x \right], \quad (30)$$

where

$$\omega_a(x, y) = e^{-y\sqrt{a + \square_\phi}} \omega(x), \quad \forall (x, y) \in M \times \mathbb{R}^+.$$

*Proof.* Let  $\eta \in C_0^\infty(\Lambda^k T^* M)$ . By Proposition 5.1, we have

$$\eta(X_\tau) = e^{a\tau} M_\tau^{*, -1} \eta_a(X_0, B_0) + \int_0^\tau e^{a(\tau-s)} M_\tau^{*, -1} M_s^* \left( \nabla, \frac{\partial}{\partial y} \right) \eta(X_s, B_s) \cdot (U_s dW_s, dB_s).$$

Hence

$$\begin{aligned} & \int_M \left\langle E_y \left[ \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s \mid X_\tau = x \right], \eta(x) \right\rangle d\mu(x) \\ &= E_y \left[ \left\langle \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s, \eta(X_\tau) \right\rangle \right] \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= E_y \left[ \left\langle \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s, e^{a\tau} M_\tau^{*, -1} \eta_a(X_0, B_0) \right\rangle \right], \\ I_2 &= E_y \left[ \left\langle \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s) dB_s, \int_0^\tau e^{a(\tau-s)} M_\tau^{*, -1} M_s^* \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(Z_s) \cdot (U_s dW_s, dB_s) \right\rangle \right]. \end{aligned}$$

Using the martingale property of the Itô stochastic integral, we have

$$\begin{aligned} I_1 &= E_y \left[ \left\langle \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s, \eta_a(X_0, B_0) \right\rangle \right] \\ &= E_y \left[ \left\langle E \left[ \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s \mid (X_0, B_0) \right], \eta_a(X_0, B_0) \right\rangle \right] \\ &= 0. \end{aligned}$$

By the  $L^2$ -isometry of the Itô stochastic integral, we have

$$\begin{aligned} I_2 &= E_y \left[ \int_0^\tau \left\langle M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(Z_s), M_\tau^{*, -1} M_s^* \frac{\partial}{\partial y} \eta_a(Z_s) \right\rangle ds \right] \\ &= E_y \left[ \int_0^\tau \left\langle \frac{\partial}{\partial y} \omega_a(Z_s), M_s^{-1, *} M_\tau^* M_\tau^{*, -1} M_s^* \frac{\partial}{\partial y} \eta_a(Z_s) \right\rangle ds \right] \\ &= E_y \left[ \int_0^\tau \left\langle \frac{\partial}{\partial y} \omega_a(Z_s), \frac{\partial}{\partial y} \eta_a(Z_s) \right\rangle ds \right]. \end{aligned}$$

The Green function of the background radiation process is given by  $2(y \wedge z)$ . Thus

$$E_y \left[ \int_0^\tau \left\langle \frac{\partial}{\partial y} \omega_a(X_s, B_s), \frac{\partial}{\partial y} \eta_a(X_s, B_s) \right\rangle ds \right] = 2 \int_M \int_0^\infty (y \wedge z) \left\langle \frac{\partial}{\partial z} \omega_a(x, z), \frac{\partial}{\partial z} \eta_a(x, z) \right\rangle dz d\mu(x).$$

Using the spectral decomposition, we have the Littlewood-Paley identity

$$\lim_{y \rightarrow \infty} \int_M \int_0^\infty (y \wedge z) \left\langle \frac{\partial}{\partial z} \omega_a(x, z), \frac{\partial}{\partial z} \eta_a(x, z) \right\rangle dz d\mu(x) = \int_M \langle \omega(x), \eta(x) \rangle d\mu(x).$$

Therefore

$$\langle \omega, \eta \rangle_{L^2(\mu)} = 2 \lim_{y \rightarrow \infty} \int_M \left\langle E_y \left[ \int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) \cdot dB_s \mid X_\tau = x \right], \eta(x) \right\rangle d\mu(x).$$

Since the above identity holds for all  $\eta \in C_c^\infty(M, \Lambda^k T^* M)$ , we get

$$\omega(x) = 2 \lim_{y \rightarrow \infty} E_y \left[ \int_0^\tau M_\tau M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) \cdot dB_s \mid X_\tau = x \right].$$

The proof of Theorem 5.2 is completed.  $\square$

#### 5.4 Representation of Riesz transforms on $k$ -forms

Following [65, 5], we consider the Riesz transforms  $d(a + \square_{\phi, k})^{-1/2}$  and  $d_\phi^*(a + \square_{\phi, k})^{-1/2}$  associated with the Witten Laplacian on  $k$ -forms. To simplify the notations, let

$$\begin{aligned} R_a^1(\square_{\phi, k}) &:= d(a + \square_{\phi, k})^{-1/2} \in \text{End}(\Lambda^k T^* M, \Lambda^{k+1} T^* M), \\ R_a^2(\square_{\phi, k}) &:= d_\phi^*(a + \square_{\phi, k})^{-1/2} \in \text{End}(\Lambda^k T^* M, \Lambda^{k-1} T^* M). \end{aligned}$$

We have the following probabilistic representation formulas of the Riesz transforms on  $k$ -forms.

**Theorem 5.3** *For all  $\omega \in C_0^\infty(\Lambda^k T^* M)$ ,  $a \geq 0$ , let*

$$Q_{k, a} \omega(x, y) = e^{-y\sqrt{a + \square_{\phi, k}}} \omega(x), \quad \forall x \in M, y \geq 0.$$

*Then, for all  $x \in M$ , we have*

$$-\frac{1}{2} R_a^1(\square_{\phi, k}) \omega(x) = \lim_{y \rightarrow \infty} E_y \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau, k+1} M_{s, k+1}^{-1} d_k Q_{k, a}(\omega)(Z_s) dB_s \mid X_\tau = x \right], \quad (31)$$

$$-\frac{1}{2} R_a^2(\square_{\phi, k}) \omega(x) = \lim_{y \rightarrow \infty} E_y \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau, k-1} M_{s, k-1}^{-1} d_{\phi, k}^* Q_{k, a}(\omega)(Z_s) dB_s \mid X_\tau = x \right]. \quad (32)$$

*Proof.* By Theorem 5.2, for all  $\omega \in C_0^\infty(\Lambda^k T^*M)$ , we have

$$\frac{1}{2}\omega(x) = \lim_{y \rightarrow +\infty} E_y \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau,k} M_{s,k}^{-1} \frac{\partial}{\partial y} Q_{k,a}\omega(Z_s) dB_s \mid X_\tau = x \right].$$

Replacing  $\omega$  by  $d_k(a + \square_{\phi,k})^{-1/2}\omega$ , we obtain

$$\begin{aligned} & -\frac{1}{2}R_a^1(\square_{\phi,k})\omega(x) \\ = & \lim_{y \rightarrow \infty} E_y \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} \sqrt{a + \square_{\phi,k+1}} Q_{k+1,a} d_k(a + \square_{\phi,k})^{-1/2}\omega(Z_s) dB_s \mid X_\tau = x \right]. \end{aligned}$$

Using the commutative formula

$$d_k \sqrt{a + \square_{\phi,k}} \omega = \sqrt{a + \square_{\phi,k+1}} d_k \omega,$$

we obtain

$$\frac{1}{2}R_a^1(\square_{\phi,k})\omega(x) = - \lim_{y \rightarrow \infty} E_y \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} Q_{k+1,a} d_k \omega(Z_s) dB_s \mid X_\tau = x \right].$$

Using again  $d_k Q_{k,a} = Q_{k+1,a} d_k$ , we prove (31). Similarly, we can prove (32).  $\square$

**Remark 5.4** When  $k = 0$ , the formula (31) was proved in [43]. It is a natural extension of the well-known Gundy-Varopoulos probabilistic representation formula of the Riesz transforms on  $\mathbb{R}^n$  [31] and Gundy's representation formula of the P. A. Meyer Riesz transforms on the Wiener space [30].

## 5.5 Representation of Riesz potential on $k$ -forms

In this subsection we give a probabilistic representation of the Riesz potential  $\square_{\phi,k}^{-1/2}$  and the Bessel potential  $(a + \square_{\phi,k})^{-1/2}$  on  $k$ -forms, even though we do not need it in the study of the Riesz potentials on Riemannian manifolds.

**Theorem 5.5** *Under the same notations as in Theorem 5.3, for all  $a \geq 0$ , we have*

$$\frac{1}{2}(a + \square_{\phi,k})^{-1/2}\omega(x) = -E_\infty \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau,k} M_{s,k}^{-1} e^{-B_s \sqrt{a + \square_{\phi,k}}} \omega(X_s) dB_s \mid X_\tau = x \right].$$

*In particular, for  $a = 0$ ,*

$$\frac{1}{2}\square_{\phi,k}^{-1/2}\omega(x) = -E_\infty \left[ \int_0^\tau M_{\tau,k} M_{s,k}^{-1} e^{-B_s \sqrt{\square_{\phi,k}}} \omega(X_s) dB_s \mid X_\tau = x \right].$$

*Proof.* Applying the general representation formula (30) to  $(a + \square_{\phi,k})^{-1/2}\omega$ , the formula (33) follows. Taking  $a = 0$ , we get (33).  $\square$

## 6 Proof of Theorem 1.5 and Theorem 1.6

In this section we prove Theorem 1.5 and Theorem 1.6. It would be interesting to ask whether one can give an analytic proof of Theorem 1.5 without using "the magic world of Brownian motion".

## 6.1 Proof of Theorem 1.5

In this subsection, we prove Theorem 1.5. More precisely, we prove the following

**Theorem 6.1** *Let  $M$  be a complete Riemannian manifold,  $\phi \in C^2(M)$ . Suppose that  $e^{tL}$  is conservative,  $W_{\phi,k} \geq -a$  and  $W_{\phi,k+1} \geq -a$ , where  $a$  is a non-negative constant. Then, there exists a constant  $C_k > 0$  such that, for all  $p > 1$ ,*

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq CA_k(p^* - 1)^{3/2}, \quad (33)$$

$$\|d_\phi^*(a + \square_{\phi,k+1})^{-1/2}\|_{p,p} \leq CA_k(p^* - 1)^{3/2}, \quad (34)$$

where  $C > 0$  is a constant independent of  $p$  and  $k$ ,  $A_k$  denotes the uniform norm in the following inequality

$$\|d\omega\|_\infty \leq A_k \|\nabla\omega\|_\infty, \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M).$$

*Proof.* In the case  $p = 2$ , it is well-known that, using the Gaffney integration by parts formula [23], the Riesz transform  $d(a + \square_{\phi,k})^{-1/2}$  is bounded in  $L^2$  on all complete Riemannian manifolds and its  $L^2$ -norm is less than 1. Below, we consider the case  $p \neq 2$ .

For all  $p > 1$ , since conditional expectation  $E[\cdot | X_\tau = x]$  is contractive in  $L^p$ , we have

$$\begin{aligned} \|R_a^1(\square_\phi)\omega\|_p^p &= 2^p \int_M \left| E_y \left[ \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s \mid X_\tau = x \right] \right|^p d\mu(x) \\ &\leq 2^p \int_M E_y \left[ \left| \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s \right|^p \mid X_\tau = x \right] d\mu(x) \\ &= 2^p E_y \left[ \left| \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s \right|^p \right]. \end{aligned}$$

Let

$$I_y = \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_{a,k}\omega(X_s, B_s) dB_s.$$

We have

$$\|d(a + \square_{\phi,k})^{-1/2}\omega\|_p \leq 2 \lim_{y \rightarrow \infty} \|I_y\|_p. \quad (35)$$

We now estimate  $\|I_y\|_p$ . Notice that, at any fixed point  $x \in M$ , for all  $\omega \in \Gamma(\Lambda^k T^*M \otimes \mathbb{R})$ ,

$$d\omega(x, \cdot) = \sum_{i=1}^n e_i^* \wedge \nabla_{e_i}\omega(x, \cdot),$$

where  $e_1, \dots, e_n$  is a normal orthonormal basis at  $T_x M$  such that  $\nabla_{e_i} e_j(x) = 0$  for all  $i, j = 1, \dots, n$ , and  $e_1^*, \dots, e_n^*$  is its dual basis. Let  $\bar{\nabla} = \nabla^{T^*M \otimes \mathbb{R}}$ . Then, for all  $\omega \in \Gamma(\Lambda^k T^*M \otimes \mathbb{R})$ , we have

$$d\omega(x, y) = \sum_{j=1}^n e_j^* \wedge \nabla_{e_j}\omega(x, y) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e_1^* \wedge & \dots & e_j^* \wedge & \dots & e_n^* \wedge & 0 \end{pmatrix} \begin{pmatrix} \nabla_{e_1}\omega \\ \dots \\ \nabla_{e_j}\omega \\ \dots \\ \nabla_{e_n}\omega \\ \partial_y\omega \end{pmatrix} (x).$$

Let  $A$  denote the  $(n+1) \times (n+1)$  operator-valued matrix before  $(\nabla\omega, \partial_y\omega)$ . Then

$$d\omega(x, y) = A \bar{\nabla}\omega(x, y).$$

Hence, the Itô integral  $I_y$  can be reformulated as a martingale transform given by

$$I_y = \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A \bar{\nabla} Q_{a,k} \omega(X_s, B_s) \cdot (U_s dW_s, dB_s). \quad (36)$$

By the Burkholder-Davies-Gundy inequality, we have

$$\|I_y\|_p \leq C_p \left\| \left\{ \int_0^\tau |e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A|^2 |\bar{\nabla} Q_{a,k} \omega(X_s, B_s)|^2 ds \right\}^{1/2} \right\|_p,$$

where (cf. Shigekawa [61] p. 50)

$$C_p = \left\{ \frac{1}{2} p(p-1) \left( \frac{p}{p-1} \right)^p \right\}^{1/2}.$$

Note that

$$A_k = \|A\| := \sup_{\omega \in \Gamma(\Lambda^k T^* M), \nabla \omega \neq 0} \frac{\|d\omega\|_\infty}{\|\nabla \omega\|_\infty} < +\infty$$

is a positive constant depending only on  $k = 0, 1, \dots, n$ . Moreover, under the curvature assumptions  $W_{\phi,k+1} \geq -a$ , we have

$$\sup_{s \in [0, \tau]} \|e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1}\| \leq 1.$$

Therefore

$$\|I_y\|_p \leq A_k C_p \|J_y\|_p \quad (37)$$

where

$$J_y = \left\{ \int_0^\tau |\bar{\nabla} Q_{a,k} \omega(X_s, B_s)|^2 ds \right\}^{1/2}.$$

**Proposition 6.2** *Let  $p > 1$ . Then, for all  $a \geq 0$ ,  $\omega \in L^p(\Lambda^k T^* M, \mu)$  (with additional assumption  $H_p \omega = 0$  if  $a = 0$ ), we have*

$$\lim_{y \rightarrow \infty} \left\| \left\{ \int_0^\tau |\bar{\nabla} Q_{a,k} \omega(X_s, B_s)|^2 ds \right\}^{1/2} \right\|_p \leq B_p \|\omega\|_p, \quad (38)$$

where

$$B_p = \begin{cases} \frac{(2p)^{1/2}}{(p-1)^{3/2}}, & p \in (1, 2), \\ 1, & p = 2, \\ \frac{p}{\sqrt{2(p-2)}}, & p > 2. \end{cases}$$

*Proof.* Let  $\omega_a(x, y) = Q_{a,k} \omega(x, y)$ . By the Bochner-Weitzenböck formula, as  $W_{\phi,k} \geq -a$ , we have

$$\begin{aligned} \left( \frac{\partial^2}{\partial y^2} + L \right) |\omega_a(x, y)|^2 &= 2 |\bar{\nabla} \omega_a(x, y)|^2 + 2 \left\langle \left( \frac{\partial^2}{\partial y^2} - \square_{\phi,k} + W_{\phi,k} \right) \omega_a(x, y), \omega_a(x, y) \right\rangle \\ &= 2 |\bar{\nabla} \omega_a(x, y)|^2 + 2 \langle (a + W_{\phi,k}) \omega_a(x, y), \omega_a(x, y) \rangle \\ &\geq 2 |\bar{\nabla} \omega_a(x, y)|^2. \end{aligned}$$

Therefore

$$\sqrt{2}J_y \leq \left\| \left\{ \int_0^\tau \left( \frac{\partial^2}{\partial y^2} + L \right) |\omega_a(X_s, B_s)|^2 ds \right\}^{1/2} \right\|_p,$$

Let  $N_t = |\omega_a(X_{\tau \wedge t}, B_{\tau \wedge t})|^2 - |\omega_a(X_0, B_0)|^2$ . Then  $N_t$  is a continuous submartingale with the Doob-Meyer decomposition  $N_t = M_t + A_t$ , where  $M_t$  is the continuous martingale

$$M_t = N_t - \int_0^{\tau \wedge t} \left( \frac{\partial^2}{\partial y^2} + L \right) |\omega_a(X_s, B_s)|^2 ds,$$

and  $A_t$  is the bounded variation part of  $N_t$  given by

$$A_t = \int_0^{\tau \wedge t} \left( \frac{\partial^2}{\partial y^2} + L \right) |\omega_a(X_s, B_s)|^2 ds.$$

(i) Let  $p = 2$ . By Doob's stopping time theorem, we have

$$E[A_\infty] = E[N_\infty] = \|\omega_a(X_\tau, B_\tau)\|_2^2 - \|\omega_a(X_0, B_0)\|_2^2 \leq \|\omega\|_2.$$

This proves (38) with  $B_2 = 1$ .

(ii) Let  $p > 2$ . By the Lenglart-Lépingle-Pratelli inequality, we have

$$E \left[ A_\infty^{\frac{p}{2}} \right] \leq p^{\frac{p}{2}} E \left[ \sup_{t \geq 0} |N_t|^{\frac{p}{2}} \right]$$

Moreover, using Doob's martingale inequality, we have

$$E \left[ \sup_{t \geq 0} |N_t|^{\frac{p}{2}} \right] \leq \left( \frac{p/2}{p/2 - 1} \right)^{\frac{p}{2}} E \left[ |N_\infty|^{\frac{p}{2}} \right].$$

Hence

$$\begin{aligned} \|A_\infty^{1/2}\|_p^2 &\leq \frac{p^2}{p-2} \|N_\infty\|_{\frac{p}{2}} \\ &= \frac{p^2}{p-2} \left\| |\omega(X_\tau)|^2 - |\omega_a(X_0, B_0)|^2 \right\|_{\frac{p}{2}} \\ &\leq \frac{p^2}{p-2} \left( \|\omega(X_\tau)\|_{\frac{p}{2}} + \|\omega_a(X_0, B_0)\|_{\frac{p}{2}} \right) \\ &= \frac{p^2}{p-2} \left( \|\omega(X_\tau)\|_p^2 + \|\omega_a(X_0, B_0)\|_p^2 \right) \\ &= \frac{p^2}{p-2} \left( \|\omega\|_p^2 + \left\| e^{-y\sqrt{a+\square_\phi}\omega} \right\|_p^2 \right). \end{aligned}$$

By Theorem 3.1 and Theorem 3.4, we can therefore deduce that

$$\lim_{y \rightarrow \infty} J_y \leq \frac{1}{\sqrt{2}} \lim_{y \rightarrow \infty} \left\| \left\{ \int_0^\tau \left( \frac{\partial^2}{\partial y^2} + L \right) |\omega_a(X_s, B_s)|^2 ds \right\}^{1/2} \right\|_p \leq \frac{p}{\sqrt{2(p-2)}} \|\omega\|_p. \quad (39)$$

This proves (38) with  $B_p = \frac{p}{\sqrt{2(p-2)}}$  for all  $p > 2$ .

(iii) Let  $1 < p < 2$ . By [71] (p. 641 Lemma 6.1) or [42] (p. 622 Lemma 4.5), we have

$$|\nabla \omega_a(x, y)|^2 \leq \frac{1}{p(p-1)} |\omega_a(x, y)|^{2-p} \liminf_{\varepsilon \rightarrow 0} \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}}.$$

This implies

$$\begin{aligned} J_y &\leq \frac{1}{\sqrt{p(p-1)}} \left\| \left\{ \int_0^\tau |\omega_a(X_s, B_s)|^{2-p} \liminf_{\varepsilon \rightarrow 0} \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}} ds \right\}^{1/2} \right\|_p \\ &\leq \frac{1}{\sqrt{p(p-1)}} \left\| \sup_{s \geq 0} |\omega_a(X_s, B_s)|^{\frac{2-p}{2}} \left\{ \int_0^\tau \liminf_{\varepsilon \rightarrow 0} \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}} ds \right\}^{1/2} \right\|_p. \end{aligned}$$

Using the Hölder inequality to  $2/(2-p)$  and  $2/p$ , we get

$$J_y \leq \frac{1}{\sqrt{p(p-1)}} \left\| \sup_{s \geq 0} |\omega_a(X_s, B_s)| \right\|_p^{\frac{2-p}{2}} \left\| \int_0^\tau \liminf_{\varepsilon \rightarrow 0} \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}} ds \right\|_1^{1/2}. \quad (40)$$

We need to prove two preliminary results.

**Proposition 6.3** *For all  $1 < p < 2$ , we have*

$$\lim_{y \rightarrow \infty} \left\| \int_0^\tau \liminf_{\varepsilon \rightarrow 0} \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{\frac{p}{2}} ds \right\|_1 \leq 2 \left( \frac{p}{p-1} \right)^p \|\omega\|_p^p. \quad (41)$$

*Proof.* Similarly to Yoshida [71] (p. 644), we set  $f(x, y) = (|\omega_a|^2(x, y) + \varepsilon^2)^{p/2}$ , and  $N_t = f(X_t, B_t)$ . Then  $N_t = M_t + A_t$  is a  $P_{(x,y)}$ -submartingale, where  $M_t$  is the martingale part given by

$$M_t = N_t - N_0 - \int_0^t \left( \frac{\partial^2}{\partial y^2} + L \right) f(X_s, B_s) ds,$$

and  $A_t$  is the bounded variation part given by

$$A_t = \int_0^{t \wedge \tau} \left( \frac{\partial^2}{\partial y^2} + L \right) f(X_s, B_s) ds.$$

By the Lenglart-Lépingle-Pratelli inequality, we have

$$\begin{aligned} E[A_\tau] &\leq 2E \left[ \sup_{t > 0} |N_t - N_0| \right] \\ &= 2E \left[ \sup_{t > 0} (|\omega_a|^2(X_t, B_t) + \varepsilon^2)^{p/2} \right] + 2E \left[ (|\omega_a|^2(X_0, B_0) + \varepsilon^2)^{p/2} \right]. \end{aligned}$$

Using the elementary inequality  $(a+b)^{p/2} \leq a^{p/2} + b^{p/2}$  for  $p \in (1, 2)$ , and  $a, b \geq 0$ , we have

$$E[A_\tau] \leq 2E \left[ \sup_{t > 0} |\omega_a|^p(X_t, B_t) \right] + 2E[|\omega_a|^p(X_0, B_0)] + 4\varepsilon^p.$$

By Doob's martingale inequality, we have

$$E \left[ \sup_{t > 0} |\omega_a|^p(X_t, B_t) \right] \leq \left( \frac{p}{p-1} \right)^p E[|\omega_a|^p(X_\tau, B_\tau)].$$

Therefore

$$E[A_\tau] \leq 2 \left( \frac{p}{p-1} \right)^p E[|\omega_a|^p(X_\tau, B_\tau)] + 2E[|\omega_a|^p(X_0, B_0)] + 4\varepsilon^p.$$

Taking  $\varepsilon \rightarrow 0$  and using Fato's lemma, we have

$$\begin{aligned}
& E \left[ \liminf_{\varepsilon \rightarrow 0} \int_0^\tau \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{p/2} ds \right] \\
& \leq \lim_{\varepsilon \rightarrow 0} E \left[ \int_0^\tau \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{p/2} ds \right] \\
& \leq 2 \left( \frac{p}{p-1} \right)^p E [|\omega_a|^p(X_\tau, B_\tau)] + 2E [|\omega_a|^p(X_0, B_0)] \\
& = 2 \left( \frac{p}{p-1} \right)^p \|\omega(X_\tau)\|_p^p + 2E [|\omega_a|^p(X_0, B_0)] \\
& = 2 \left( \frac{p}{p-1} \right)^p \|\omega\|_p^p + 2E [|\omega_a|^p(X_0, B_0)].
\end{aligned}$$

By Theorem 3.1 and Theorem 3.4, for all  $\omega \in L^p(\Lambda^k T^*M, \mu)$  (with  $H_p \omega = 0$  in the case  $a = 0$ ), we have

$$\lim_{y \rightarrow \infty} E [|\omega_a|^p(X_0, B_0)] = \lim_{y \rightarrow \infty} \left\| e^{-y\sqrt{a+\square_{\phi,k}}}\omega \right\|_p^p = 0.$$

Hence

$$\lim_{y \rightarrow \infty} E \left[ \liminf_{\varepsilon \rightarrow 0} \int_0^\tau \left( \frac{\partial^2}{\partial y^2} + L \right) (|\omega_a(X_s, B_s)|^2 + \varepsilon^2)^{p/2} ds \right] \leq 2 \left( \frac{p}{p-1} \right)^p \|\omega\|_p^p.$$

This completes the proof of (41).  $\square$

**Proposition 6.4** *For all  $1 < p < 2$ , we have*

$$\left\| \sup_{s \geq 0} |\omega_a(X_s, B_s)| \right\|_p \leq \frac{p}{p-1} \|\omega\|_p. \quad (42)$$

*Proof.* Using the semigroup domination inequality (23), as  $W_{\phi,k} \geq -a$ , we have

$$|e^{-t\square_{\phi,k}}\omega(x)| \leq e^{at}e^{tL}|\omega|(x), \quad \forall x \in M, t > 0.$$

From this and using the Bochner subordination, we get

$$|\omega_a(x, y)| = \left| e^{-y\sqrt{a+\square_{\phi,k}}}\omega(x) \right| \leq e^{-y\sqrt{-L}}|\omega|(x) = |\omega|(x, y).$$

This yields that

$$\left\| \sup_{s \geq 0} |\omega_a(X_s, B_s)| \right\|_p \leq \left\| \sup_{s \geq 0} |\omega|(X_s, B_s) \right\|_p.$$

Since  $|\omega|(x, y) = e^{-y\sqrt{-L}}|\omega|(x)$  is a  $\left( \frac{\partial^2}{\partial y^2} + L \right)$ -harmonic function,  $|\omega|(X_{\tau \wedge t}, B_{\tau \wedge t})$  is a martingale. By Doob's martingale inequality, we have

$$\left\| \sup_{s \geq 0} |\omega|(X_{\tau \wedge t}, B_{\tau \wedge t}) \right\|_p \leq \frac{p}{p-1} \|\omega|(X_\tau, B_\tau)\|_p = \frac{p}{p-1} \|\omega\|_p.$$

The maximal inequality (42) is proved.  $\square$

*End of Proof of Proposition 6.2.* It remains to prove (38) for  $1 < p < 2$ . Combining (40), (41) and (42), for all  $1 < p < 2$ , we have

$$J \leq \frac{1}{\sqrt{p(p-1)}} \sqrt{2 \left( \frac{p}{p-1} \right)^p \left( \frac{p}{p-1} \right)^{\frac{2-p}{2}}} \|\omega\|_p. \quad (43)$$

This completes the proof of (38) for all  $1 < p < 2$  with

$$B_p = \frac{(2p)^{1/2}}{(p-1)^{3/2}}.$$

□

*End of Proof of Theorem 6.1.* By (35), (37) and Proposition 6.2, we have

$$\begin{aligned} \|d(a + \square_{\phi,k})^{-1/2}\omega\|_p &\leq \frac{2pC_p A_k}{\sqrt{2(p-2)}} \|\omega\|_p, \quad \forall p > 2, \\ \|d(a + \square_{\phi,k})^{-1/2}\omega\|_p &\leq \frac{2C_p(2p)^{1/2} A_k}{(p-1)^{3/2}} \|\omega\|_p, \quad \forall 1 < p < 2. \end{aligned}$$

Note that

$$C_p = \left\{ \frac{1}{2} p(p-1) \left( \frac{p}{p-1} \right)^p \right\}^{1/2}.$$

Therefore, for all  $p > 2$ ,

$$\frac{2pC_p}{\sqrt{2(p-2)}} = \sqrt{\left( \frac{p}{p-1} \right)^p \frac{p^3(p-1)}{p-2}} \leq \sqrt{e}(p-1)^{3/2}(1+o(1)),$$

and for all  $1 < p < 2$ ,

$$\frac{2C_p(2p)^{1/2}}{(p-1)^{3/2}} = 2 \left( \frac{p}{p-1} \right)^{1+p/2} \leq 2\sqrt{e} \left( \frac{1}{p-1} \right)^{3/2} (1+o(1)).$$

This finishes the proof of (33) in Theorem 3.4.

It remains to prove (34). Let  $(\ )_{\phi}^*$  denote the  $L^2$ -adjoint of the operator in  $(\ )$  with respect to  $d\mu = e^{-\phi}dv$ . Then

$$d_{\phi,k-1}^*(a + \square_{\phi,k})^{-1/2} = ((a + \square_{\phi,k})^{-1/2}d_{k-1})_{\phi}^* = (d_{k-1}(a + \square_{\phi,k-1})^{-1/2})_{\phi}^*.$$

Using the duality argument, for all  $p > 1$ , we have

$$\|d_{\phi,k-1}^*(a + \square_{\phi,k})^{-1/2}\|_{p,p} = \|(a + \square_{\phi,k})^{-1/2}d_{k-1}\|_{q,q} = \|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q} \quad (44)$$

Indeed, for all  $\omega \in L^p(\Lambda^k T^*M, \mu)$  and all  $\eta \in L^q(\Lambda^{k-1} T^*M, \mu)$ , we have

$$\begin{aligned} \|d_{\phi,k-1}^*(a + \square_{\phi,k})^{-1/2}\omega\|_p &= \sup_{\|\eta\|_q=1} \int_M \langle d_{\phi,k-1}^*(a + \square_{\phi,k})^{-1/2}\omega, \eta \rangle d\mu \\ &= \sup_{\|\eta\|_q=1} \int_M \langle \omega, (a + \square_{\phi,k})^{-1/2}d_{k-1}\eta \rangle d\mu \\ &\leq \sup_{\|\eta\|_q=1} \|\omega\|_p \|(a + \square_{\phi,k})^{-1/2}d_{k-1}\eta\|_q \\ &\leq \|(a + \square_{\phi,k})^{-1/2}d_{k-1}\|_{q,q} \|\omega\|_p \\ &= \|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q} \|\omega\|_p. \end{aligned}$$

Thus

$$\|d_{\phi,k-1}^*(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq \|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q}.$$

Similarly, we can prove that

$$\|d_{k-1}(a + \square_{\phi,k-1})^{-1/2}\|_{q,q} \leq \|d_{\phi,k-1}^*(a + \square_{\phi,k})^{-1/2}\|_{p,p}.$$

This proves (44). By (33) and (44), we finish the proof of (34) in Theorem 3.4. □

## 6.2 Proof of Theorem 1.6 and Theorem 1.7

By (36), the Itô integral  $I_y$  can be reformulated as a martingale transform on  $M \times \mathbb{R}$ :

$$I_y = \int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A U_s U_s^{-1} \bar{\nabla} \omega_a(X_s, B_s) \cdot (U_s dW_s, dB_s).$$

By the Burkholder sharp  $L^p$ -inequality for martingale transforms, cf. Theorem 4.3, we get

$$\begin{aligned} \|I_y\|_p &\leq (p^* - 1) \sup_{s \in [0, \tau]} \|e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A U_s\|_{\text{op}} \\ &\quad \times \left\| \int_0^\tau U_s^{-1} \bar{\nabla} \omega_a(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p. \end{aligned}$$

Under the curvature assumptions  $W_{\phi,k+1} \geq -a$ , we have

$$\sup_{s \in [0, \tau]} \|e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} A U_s\|_{\text{op}} \leq A_k.$$

Therefore

$$\|I_y\|_p \leq (p^* - 1) A_k \left\| \int_0^\tau U_s^{-1} \bar{\nabla} \omega_a(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p. \quad (45)$$

By Itô's formula, we can prove that

$$\begin{aligned} U_\tau^{-1} \omega_a(X_\tau, B_\tau) - \omega_a(X_0, B_0) &= \int_0^\tau U_s^{-1} \bar{\nabla} \omega_a(X_t, B_t) (U_t dW_t, dB_t) \\ &\quad - \int_0^\tau U_s^{-1} \left( \frac{\partial^2}{\partial y^2} + \Delta - \nabla \phi \right) \omega_a(X_s, B_s) ds. \end{aligned}$$

By the Bochner-Weitzenböck formula  $\square_{\phi,k} = -\Delta_\phi + W_{\phi,k}$ , we have

$$\begin{aligned} \left( \frac{\partial^2}{\partial y^2} + \Delta - \nabla \phi \right) \omega_a(x, y) &= (\square_{\phi,k} + a + \Delta - \nabla \phi) \omega_a(x, y) \\ &= (-\Delta_\phi + W_{\phi,k} + a + \Delta - \nabla \phi) \omega_a(x, y) \\ &= (a + W_{\phi,k}) \omega_a(x, y). \end{aligned}$$

Therefore, as  $W_{\phi,k} = -a$ , it holds that

$$\int_0^\tau U_s^{-1} \bar{\nabla} \omega_a(X_t, B_t) (U_t dW_t, dB_t) = U_\tau^{-1} \omega_a(X_\tau, B_\tau) - \omega_a(X_0, B_0). \quad (46)$$

Combining (45) with (46), we obtain

$$\begin{aligned} \|R_a^1(\square_{\phi,k})\omega\|_p &\leq 2(p^* - 1) A_k \|U_\tau^{-1} \omega_a(X_\tau, B_\tau) - \omega_a(X_0, B_0)\|_p \\ &= 2(p^* - 1) A_k \|U_\tau^{-1} \omega(X_\tau) - \omega(X_0, B_0)\|_p \\ &\leq 2(p^* - 1) A_k [\|\omega(X_\tau)\|_p + \|\omega_a(X_0, B_0)\|_p]. \end{aligned}$$

Since  $X_\tau$  has the law  $\mu$ , we have

$$\|\omega(X_\tau)\|_p = \|\omega\|_p.$$

On the other hand, by Theorem 3.4 and Theorem 3.1, for all  $p > 1$ , and  $\omega \in L^p(\Lambda^k T^* M, \mu)$  (with the additional condition  $H_p \omega = 0$  when  $a = 0$ ), we have

$$\lim_{y \rightarrow \infty} \|\omega_a(X_0, B_0)\|_p = \lim_{y \rightarrow \infty} \left\| e^{-y\sqrt{a+\square_{\phi,k}}} \omega \right\|_p = 0. \quad (47)$$

Thus

$$\|R_a^1(\square_{\phi,k})\omega\|_p \leq 2(p^* - 1)A_k\|\omega\|_p.$$

This implies that

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq 2A_k(p^* - 1).$$

From this and using (44), we have

$$\|d_\phi^*(a + \square_{\phi,k+1})^{-1/2}\|_{p,p} \leq 2A_k(p^* - 1).$$

The proof of Theorem 1.6 is completed. Note that when  $k = 0$ , we have  $A_0 = 1$ .

In the particular case where  $M = \Gamma \backslash \mathbb{H}^n(-1)$ , we have  $W_k = -k(n - k)\text{Id}$ . From this and using Theorem 1.6, we can easily prove Theorem 1.7.  $\square$

### 6.3 Examples

**Example 6.5** Let  $M = \mathbb{R}^n$ ,  $\phi = 0$ , and  $d\mu(x) = dx$ . Then  $\|d\square_k^{-1/2}\|_{p,p} \leq C_k(p^* - 1)$  for all  $p > 1$  and  $k = 0, 1, \dots, n$ . Let  $\omega = \omega_I dx^I$ , where  $I = (i_1, \dots, i_k)$ ,  $1 \leq i_1 < \dots < i_k \leq n$ . Then  $\square_k \omega = -\Delta \omega_I dx^I$ , and

$$d\square_k^{-1/2}\omega = d(-\Delta)^{-1/2}\omega_I dx^I = \sum_{j \notin I} \frac{\partial}{\partial x_j} (-\Delta)^{-1/2}\omega_I dx^j \wedge dx^I.$$

By the estimate (1) of Pichorides [57], Iwaniec and Martin [34] and Bañuelos-Wang [10],

$$\|d\square_k^{-1/2}\|_{p,p} = \sup_{\omega \neq 0} \frac{\|d\square_k^{-1/2}\omega\|_p}{\|\omega\|_p} \geq \sup_{\omega_I \neq 0} \frac{\|R_j \omega_I\|_p}{\|\omega\|_p} = \|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right).$$

Thus, when  $p \rightarrow 1$ , we have

$$\|d\square_k^{-1/2}\|_{p,p} \geq \frac{2}{\pi} \frac{1}{p-1} (1 + o(1)).$$

Combining this with  $\|d\square_k\|_{p,p} \leq C_k(p^* - 1)$ , when  $p \rightarrow 1$ , we have

$$\frac{2}{\pi} \frac{1}{p-1} (1 + o(1)) \leq \|d\square_k^{-1/2}\|_{p,p} \leq \frac{C_k}{p-1}.$$

Similarly, when  $p \rightarrow \infty$ , we have  $\|d\square_k^{-1/2}\|_{p,p} \geq \cot\left(\frac{\pi}{2p^*}\right) \geq \frac{2}{\pi}(p-1)(1 + o(1))$ . Hence

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d\square_k^{-1/2}\|_{p,p} \leq C_k(p-1).$$

By (44), we have  $\|d^*\square_k^{-1/2}\|_{p,p} = \|d\square_{k-1}^{-1/2}\|_{q,q}$ . The above estimates for  $d\square_k^{-1/2}$  lead to similar estimates for the  $L^p$ -norm of  $d^*\square_k^{-1/2}$ . More precisely, when  $p \rightarrow 1$ ,

$$\frac{2}{\pi}(p-1)^{-1}(1 + o(1)) \leq \|d^*\square_k^{-1/2}\|_{p,p} \leq C_{k-1}(p-1)^{-1},$$

and when  $p \rightarrow \infty$ ,

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d^*\square_k^{-1/2}\|_{p,p} \leq C_{k-1}(p-1).$$

Thus, in the Euclidean case, an upper bound of the form  $O(p^* - 1)$  of the  $L^p$ -norm of the Riesz transforms  $d\square_k^{-1/2}$  and  $d^*\square_k^{-1/2}$  are asymptotically sharp when  $p \rightarrow 1$  and when  $p \rightarrow \infty$ .

**Example 6.6** Let  $M = \mathbb{R}^n$ ,  $\phi(x) = \frac{\|x\|^2}{2}$ , and  $\mu$  be the standard Gaussian measure on  $\mathbb{R}^n$ . Then  $L = \Delta - x \cdot \nabla$  is the Ornstein-Uhlenbeck operator. The  $k$ -th Witten Laplacian on the Gaussian space is given by

$$\square_{\phi,k} = k - L, \quad k = 0, \dots, n,$$

and the  $k$ -th Weitzenböck curvature is

$$W_{\phi,k} = k, \quad k = 0, \dots, n.$$

By Theorem 1.5, for all  $p > 1$  and all  $a \geq 0$ , we have

$$\|d(a + \square_{\phi,k})^{-1/2}\|_{p,p} \leq C_k(p^* - 1)^{3/2}.$$

On the other hand, taking again  $\omega = \omega_I dx^I$  as in the above example, we get

$$d\square_{\phi,k}^{-1/2}\omega = d(k - L)^{-1/2}\omega_I dx^I = \sum_{j \notin I} \frac{\partial}{\partial x_j} (k - L)^{-1/2}\omega_I dx^j \wedge dx^I,$$

from which we get

$$\|d\square_{\phi,k}^{-1/2}\|_{p,p} = \sup_{\omega \neq 0} \frac{\|d\square_{\phi,k}^{-1/2}\omega\|_p}{\|\omega\|_p} \geq \sup_{\omega_I \neq 0} \frac{\|\frac{\partial}{\partial x_j} (k - L)^{-1/2}\omega_I\|_p}{\|\omega_I\|_p} = \left\| \frac{\partial}{\partial x_j} (k - L)^{-1/2} \right\|_{p,p}.$$

For  $n = 1$ , let  $D = d/dx$ . Then  $L = D^2 - xD$ . By Larsson-Cohn [37], for any  $a > 0$ , taking  $f_a(x) = x/a$  for  $|x| \leq a$  and  $f_a(x) = \text{sign}(x)$  for  $|x| > a$ , where  $x \in \mathbb{R}$ , we have

$$\limsup_{a \rightarrow 0} \frac{\|D^2(-L)^{-1/2}f_a\|_p}{\|f'_a\|_p} \geq \frac{2}{\pi} \frac{1}{p-1} (1 + o(1)).$$

Notice that  $D^2(-L)^{-1/2}f_a = D(1-L)^{-1/2}f'_a$ . Taking  $\omega_I(x_1, \dots, x_n) = f_a(x_j)$ , we can prove that when  $p \rightarrow 1$ ,

$$\|d\square_{\phi,1}^{-1/2}\|_{p,p} \geq \frac{2}{\pi} \frac{1}{p-1} (1 + o(1)).$$

Therefore, when  $p \rightarrow 1$ , we have

$$\frac{2}{\pi} \frac{1}{p-1} (1 + o(1)) \leq \|d\square_{\phi,1}^{-1/2}\|_{p,p} \leq \frac{C_1}{(p-1)^{3/2}}.$$

Similarly, we can prove that, when  $p \rightarrow \infty$ , we have  $\|d\square_{\phi,1}^{-1/2}\|_{p,p} \geq \frac{2}{\pi}(p-1)(1 + o(1))$ . Hence

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d\square_{\phi,1}^{-1/2}\|_{p,p} \leq C_1(p-1)^{3/2}.$$

By (44), we have  $\|d_\phi^*\square_{\phi,2}^{-1/2}\|_{p,p} = \|d\square_{\phi,1}^{-1/2}\|_{q,q}$ . The above estimates for  $d\square_{\phi,1}^{-1/2}$  lead to similar estimates for the  $L^p$ -norm of  $d_\phi^*\square_{\phi,2}^{-1/2}$ . More precisely, when  $p \rightarrow 1$ ,

$$\frac{2}{\pi}(p-1)^{-1}(1 + o(1)) \leq \|d_\phi^*\square_{\phi,2}^{-1/2}\|_{p,p} \leq C_1(p-1)^{-3/2},$$

and when  $p \rightarrow \infty$ ,

$$\frac{2}{\pi}(p-1)(1 + o(1)) \leq \|d_\phi^*\square_{\phi,2}^{-1/2}\|_{p,p} \leq C_1(p-1)^{3/2}.$$

It would be very interesting to know whether we can replace  $(p-1)^{3/2}$ ,  $(p-1)^{-3/2}$  in the above estimates by  $(p-1)$ ,  $(p-1)^{-1}$  respectively.

## 7 $L^p$ -Hodge decomposition on complete Riemannian manifolds

In this section we prove the Weak  $L^p$ -Hodge decomposition theorem on complete Riemannian manifolds with non-negative Weitzenböck curvatures.

First, we recall the Kodaira Weak  $L^2$ -Hodge decomposition theorem on complete Riemannian manifolds. Let  $M$  be a complete Riemannian manifold. Then, for all  $k = 0, 1, \dots, n$ , the Kodaira Weak  $L^2$ -Hodge decomposition theorem holds (cf. De Rham [19]):

$$L^2(\Lambda^k T^* M, \nu) = \text{Ker} \square_k \oplus \overline{d_{k-1} \Omega^{k-1}(M)} \oplus \overline{d_k^* \Omega^{k+1}(M)},$$

where

$$\Omega^i(M) = C_0^\infty(\Lambda^i T^* M), \quad i = k-1, k+1.$$

In the case if either 0 is a isolated point of the  $L^2$ -spectrum  $\sigma_2(\square_k)$  or  $0 \notin \sigma_2(\square_k)$ , the Green operator  $\square_k^{-1}$  exists on  $(\text{Ker} \square_k)^\perp$ . In this case, the Strong  $L^2(\nu)$ -Hodge orthogonal decomposition holds: every  $\omega \in L^2(\Lambda^k T^* M, \nu)$  has the unique  $L^2$ -orthogonal decomposition:

$$\omega = H\omega + dd^* \square^{-1} \omega + d^* d \square^{-1} \omega,$$

where  $H\omega$  denotes the harmonic component part of  $\omega$  in  $L^2(\Lambda^k T^* M, \nu)$ .

The above results remain valid for the Witten Laplacians on complete Riemannian manifolds equipped with weighted volume measures. More precisely, the following Weak  $L^2(\mu)$ -Hodge decomposition holds

$$L^2(\Lambda^k T^* M, \mu) = \text{Ker} \square_{\phi, k} \oplus \overline{d_{k-1} \Omega^{k-1}(M)} \oplus \overline{d_{\phi, k}^* \Omega^{k+1}(M)}.$$

Moreover, in the case if either 0 is a isolated point of the  $L^2$ -spectrum  $\sigma_2(\square_{\phi, i})$  or  $0 \notin \sigma_2(\square_{\phi, i})$ ,  $i = k-1, k, k+1$ , the Green operators  $\square_{\phi, i}^{-1}$  exist on  $(\text{Ker} \square_{\phi, i})^\perp$ ,  $i = k-1, k, k+1$ . In this case, for every  $\omega \in L^2(\Lambda^k T^* M, \mu)$ , we have the Strong  $L^2(\mu)$ -Hodge orthogonal decomposition:

$$\omega = H\omega + dd_{\phi}^* \square_{\phi}^{-1} \omega + d_{\phi}^* d \square_{\phi}^{-1} \omega, \quad (48)$$

where  $H\omega$  denotes the harmonic projection of  $\omega$  from  $L^2(\Lambda^k T^* M, \mu)$  to  $\text{Ker} \square_{\phi, k} \cap L^2(\Lambda^k T^* M, \mu)$ .

Using the commutation formulas  $d \square_{\phi} = \square_{\phi} d$  and  $d_{\phi}^* \square_{\phi} = \square_{\phi} d_{\phi}^*$ , we have

$$\begin{aligned} dd_{\phi}^* \square_{\phi, k}^{-1} &= d \square_{\phi, k-1}^{-1/2} \square_{\phi, k-1}^{-1/2} d_{\phi, k-1}^* = (d \square_{\phi, k-1}^{-1/2}) (d \square_{\phi, k-1}^{-1/2})^*, \\ d_{\phi}^* d \square_{\phi, k}^{-1} &= d_{\phi}^* \square_{\phi, k+1}^{-1/2} \square_{\phi, k+1}^{-1/2} d = (d_{\phi}^* \square_{\phi, k+1}^{-1/2}) (d_{\phi}^* \square_{\phi, k+1}^{-1/2})^*. \end{aligned}$$

Therefore, the Riesz transforms  $d \square_{\phi, k-1}^{-1/2}$  and  $d_{\phi}^* \square_{\phi, k+1}^{-1/2}$  are naturally involved in the  $L^2$ -Hodge orthogonal decomposition formula on  $L^2(\Lambda^k T^* M, \mu)$  whenever they make sense.

Suppose that the Riesz transform  $d \square_{\phi, k}^{-1/2}$  is a bounded operator from  $L^p(\Lambda^k T^* M, \mu)$  into  $L^p(\Lambda^{k+1} T^* M, \mu)$  for a fixed  $p > 1$ . By the duality argument, we can easily prove that  $\square_{\phi, k}^{-1/2} d_{\phi}^* = (d \square_{\phi, k}^{-1/2})^*$  is bounded from  $L^q(\Lambda^{k+1} T^* M, \mu)$  into  $L^q(\Lambda^k T^* M, \mu)$ , where  $q = \frac{p}{p-1}$ . Moreover,

$$\|(d \square_{\phi, k}^{-1/2})^*\|_{p, p} = \|d \square_{\phi, k}^{-1/2}\|_{q, q}.$$

Since  $d_{\phi}^* \square_{\phi, k+1}^{-1/2} = \square_{\phi, k}^{-1/2} d_{\phi}^*$ , we obtain

$$\|d_{\phi}^* \square_{\phi, k+1}^{-1/2}\|_{p, p} = \|d \square_{\phi, k}^{-1/2}\|_{q, q}.$$

Hence, if the Riesz transforms  $d\Box_{\phi,k-1}^{-1/2}$  and  $d\Box_{\phi,k}^{-1/2}$  are bounded in  $L^p(\mu)$  and in  $L^q(\mu)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p, q < \infty$ , then, for all  $\omega \in L^p(\Lambda^k T^* M, \mu)$ , we have

$$\|\omega - H\omega\|_p \leq \left[ \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q} + \|d\Box_{\phi,k}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{q,q} \right] \|\omega\|_p. \quad (49)$$

Now we are able to give the proof of Theorem 1.8.

*Proof of Theorem 1.8.* By Theorem 1.5 and using the fact that  $\|(d\Box_{\phi,k-1}^{-1/2})^*\|_{p,p} = \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q}$ , where  $q = \frac{p}{p-1}$ , we get

$$\begin{aligned} \|dd_\phi^* \Box_{\phi,k}^{-1} \omega\|_p &= \|(d\Box_{\phi,k-1}^{-1/2})(d\Box_{\phi,k-1}^{-1/2})^* \omega\|_p \\ &\leq \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \|(d\Box_{\phi,k-1}^{-1/2})^*\|_{p,p} \|\omega\|_p \\ &= \|d\Box_{\phi,k-1}^{-1/2}\|_{p,p} \|d\Box_{\phi,k-1}^{-1/2}\|_{q,q} \|\omega\|_p \\ &\leq C_{k-1}^2 (p^* - 1)^{3/2} (q^* - 1)^{3/2} \|\omega\|_p \\ &= C_{k-1}^2 (p^* - 1)^3 \|\omega\|_p. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|d_\phi^* d\Box_{\phi,k}^{-1} \omega\|_p &\leq \|d\Box_{\phi,k}^{-1/2}\|_{p,p} \|d\Box_{\phi,k}^{-1/2}\|_{q,q} \|\omega\|_p \\ &\leq C_k^2 (p^* - 1)^3 \|\omega\|_p. \end{aligned}$$

Therefore, the operators  $dd_\phi^* \Box_{\phi,k}^{-1}$  and  $d_\phi^* d\Box_{\phi,k}^{-1}$  are bounded in  $L^p$  for all  $p > 1$ . Consequently,

$$\Box_{\phi,k} \Box_{\phi,k}^{-1} = dd_\phi^* \Box_{\phi,k}^{-1} + d_\phi^* d\Box_{\phi,k}^{-1} : L^p(\Lambda^k T^* M) \rightarrow L^p(\Lambda^k T^* M)$$

is a bounded operator for all  $p > 1$ .

Let  $P_t = e^{-t\Box_{\phi,k}}$ . Since  $W_k \geq 0$ , by Theorem 3.4, for all  $\omega \in L^p(\Lambda^k T^* M, \mu)$ , it holds that

$$H\omega = \lim_{t \rightarrow \infty} P_t \omega \quad \text{in } L^p(\Lambda^k T^* M, \mu).$$

Writing

$$P_t \omega - \omega = \int_0^t \frac{\partial}{\partial s} P_s \omega ds \quad \text{in } L^p(\Lambda^k T^* M, \mu),$$

and letting  $t \rightarrow \infty$ , we have

$$H\omega - \omega = \int_0^\infty \frac{\partial}{\partial s} P_s \omega ds \quad \text{in } L^p(\Lambda^k T^* M, \mu).$$

Note that

$$\frac{\partial}{\partial s} P_s \omega = -\Box P_s \omega.$$

Hence

$$H\omega - \omega = - \int_0^\infty \Box P_s \omega ds.$$

Therefore, we have the following decomposition in  $L^p(\Lambda^k T^* M, \mu)$ :

$$\omega = H\omega + \int_0^\infty \Box P_s \omega ds.$$

Equivalently, for all  $\omega \in L^p(\Lambda^k T^* M, \mu)$ , we have

$$\omega = H\omega + \int_0^\infty (dd_\phi^* + d_\phi^*d)P_s\omega ds.$$

By the definition of the Riesz transforms, we obtain the following decomposition in  $L^p(\Lambda^k T^*M, \mu)$ :

$$\omega = H\omega + dd_\phi^*\square_{\phi,k}^{-1} + d_\phi^*d\square_{\phi,k}^{-1}\omega. \quad (50)$$

Moreover, from (48), (49) and (50), we can deduce that

$$\begin{aligned} \|(I - H)\omega\|_p &= \|dd_\phi^*\square_{\phi,k}^{-1}\omega + d_\phi^*d\square_{\phi,k}^{-1}\omega\|_p \\ &\leq \|dd_\phi^*\square_{\phi,k}^{-1}\omega\|_p + \|d_\phi^*d\square_{\phi,k}^{-1}\omega\|_p \\ &\leq (C_{k-1}^2 + C_k^2)(p^* - 1)^3\|\omega\|_p. \end{aligned}$$

Thus, the Hodge harmonic projection  $H : L^p(\Lambda^k T^*M, \mu) \rightarrow (\text{Ker}\square_\phi)^\perp \cap L^p(\Lambda^k T^*M, \mu)$  is bounded. The proof of Theorem 1.8 is completed.  $\square$

The argument used in the proof of Theorem 1.8 goes back to [65] and has been indicated in our earlier papers [40, 42]. It can be considered as a natural extension of the heat equation approach initiated by Milgram-Rosenbloom [54] for the Hodge decomposition theorem on compact Riemannian manifolds and developed by Gaffney [24] for the  $L^2$ -Hodge decomposition theory on complete Riemannian manifolds.

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