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Valuations and local uniformization

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Abstract.

We give the principal notions in valuation theory, the value group and the residue field of a valuation, its rank, the compositions of valuations, and we give some classical examples. Then we introduce the Riemann-Zariski variety of a field, with the topology defined by Zariski. In the last part we recall the result of Zariski on local uniformization and give a sketch of the proof in the case of an algebraic surface.

§ Introduction

In these notes, we are going to give an idea of the proof of the resolution of singularities of an algebraic surface by O. Zariski. This proof is based on the theory of the valuations of algebraic function fields and could be seen as one of the most important applications of this theory in algebraic geometry.

In the first part of the paper we give the principal definitions and properties of valuations that we need for resolution. We don't speak about the problems of extension of valuations in a field extension, neither the problems of ramification.

In the second part we define the Riemann-Zariski variety of a field, what is called "abstract Riemann surface" or "Riemann manifold" by Zariski, and we give the principal property of this space.

In the last part we give a sketch of the proof of local uniformization in the case of an algebraic surface over an algebraically closed field of characteristic zero, and how we can deduce the resolution.

All the results on valuations of this paper are classical, we give a proof of some of them, otherwise we send back the reader to the books of Bourbaki ([Bo]), Endler ([En]), Ribenboim ([Ri]) or Zariski and Samuel ([Za-Sa]), or to the articles of Zariski and of the author ([Va]).

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§1. Valuations

1.1. Valuation rings and valuations

A commutative ring A is called a local ring if the non-units form an ideal, this ideal is the unique maximal ideal $\mathfrak{m}$ of A and we note $(A, \mathfrak{m})$ the local ring. The quotient field $k = A/\mathfrak{m}$ is called the residue field of $(A, \mathfrak{m})$. We don't assume that the ring A is noetherian, in Zariski's terminology such a ring is called a "quasi-local ring".
Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be two local rings, we say that B *dominates* A, and we note $A \preceq B$, if $A \subset B$ and $\mathfrak{m} = A \cap \mathfrak{n}$. If we assume $A \subset B$, then B dominates A if and only if $\mathfrak{m} \subset \mathfrak{n}$.
We deduce from the definition that if A is dominated by B, we have an inclusion between the residue fields: $A/\mathfrak{m} \subset B/\mathfrak{n}$.

Example 1. If $(A, \mathfrak{m})$ is a noetherian local ring, the completion $\hat{A}$ of A for the $\mathfrak{m}$-topology dominates A.
Let A and B be two integral domains with $A \subset B$, then for any prime ideal $\mathfrak{q}$ in B, the local ring $B_{\mathfrak{q}}$ dominates the local ring $A_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime ideal of A defined by $\mathfrak{p} = A \cap \mathfrak{q}$.
Let $f\colon X \longrightarrow Y$ be a morphism between algebraic varieties, or schemes, for any point x in X, the local ring $\mathcal{O}_{Y,y}$ dominates the local ring $\mathcal{O}_{X,x}$, where $y = f(x)$.

Let K be a field, the relation B *dominates* A, or A *is dominated by* B, defines a partial ordering on the set of the local rings contained in K. Then we can give the following definition.

Definition. Let V be an integral domain; then V is a *valuation ring* of K if K is the fraction field of V and if V is a maximal element of the set of local rings contained in K ordered by the relation of domination. If V is an integral domain, we say that V is a valuation ring if V is a valuation ring of its fraction field.

With this definition, it is easy to prove the existence of valuation rings, more precisely, we have the following result.

Proposition 1.1. ([Bo], Chap. 6, §1, n°2, Théorème 2, page 87.) *Let A be a subring of a field K and $h\colon A \longrightarrow L$ be a morphism of A in L an algebraically closed field, then there exists a valuation ring V of K with $A \subset V$ and a morphism $f\colon V \longrightarrow L$ which extends h and such that $max(V) = f^{-1}(0)$.*

Proof. We consider the set $\mathcal{H} = \big\{(B, f) \,/\, B \subset K \,\text{and}\, f\colon B \to L\big\}$; we order $\mathcal{H}$ by $(B, f) \preceq (C, g)$ if $B \subset C$ and g extends f. Any totally ordered subset $\big((B_\alpha, f_\alpha)\big)$ of $\mathcal{H}$ has an upper bound (B, f) in $\mathcal{H}$, with

$B = \cup_\alpha B_\alpha$, then by Zorn's lemme the set $\mathcal{H}$ contains a maximal element (W, g) and if we note $\mathfrak{p}$ the kernel of $g\colon W \longrightarrow L$, the local ring $W_\mathfrak{p}$ is the valuation ring satisfying the required condition.

Corollary. *Any local subring A of a field K is dominated by at least one valuation ring V of K.*

Remark 1.1. In the cases we shall consider, we have a ground field k and we have the following result. Let A be a k-subalgebra of K and $h\colon A \longrightarrow L$ a k-morphism from A in an algebraically closed field L, then there exists a valuation ring V of K wich is a k-algebra, with $A \subset V$ and a k-morphism $f\colon V \longrightarrow L$ which extends h and such that $max(V) = f^{-1}(0)$. In particular we get that the ground field k is included in $V \setminus max(V)$.

We are going to give now the principal characteristic properties of valuation rings.

Theorem 1.2. ([Bo], Chap. 6, §2, n°2, Théorème 1, page 85.) *Let V be an integral domain, contained in a field K, then the following conditions are equivalent:*

a) V is a valuation ring of K;

b) let $x \in K$, then $x \notin V \Longrightarrow x^{-1} \in V$;

c) K is the fraction field of V and the set of ideals of V is totally ordered by inclusion;

c') K is the fraction field of V and the set of principal ideals of V is totally ordered by inclusion.

Remark 1.2. From the condition b), we deduce that any valuation ring is integrally closed. In fact, we have the following result:

let A be an integral domain and K a field containing A, then the intersection of all the valuation rings V of K with $A \subset V$ is the integral closure of A in K.

From the condition c), we deduce that any finitely generated ideal of a valuation ring is principal.

Let Γ be an additive abelian totally ordered group. We add to Γ an element $+\infty$ such that $\alpha < +\infty$ for every α in Γ, and we extend the law on $\Gamma_\infty = \Gamma \cup \{+\infty\}$ by $(+\infty) + \alpha = (+\infty) + (+\infty) = +\infty$.

Definition. Let A be a ring, a *valuation* of A with values in Γ is a mapping ν of A in Γ_∞ such that the following conditions are satisfied:

1) $\nu(x.y) = \nu(x) + \nu(y)$ for every $x, y \in A$,

2) $\nu(x+y) \geq \min\big(\nu(x), \nu(y)\big)$ for every $x, y \in A$,

3) $\nu(x) = +\infty \Longleftrightarrow x = 0$.

Remark 1.3. The condition 1) means that the valuation ν is a homomorphism of $A \setminus \{0\}$ with the multiplicative law in the group Γ, hence we have $\nu(1) = 0$ and more generally, for any root of unity z, i. e. $z^n = 1$ for some $n > 0$, we have also $\nu(z) = 0$ because Γ has no torsion.

From the conditions 1) and 3) it follows that if there is a valuation ν on A, then A is an integral domain. More generally, if we have a mapping $\nu\colon A \longrightarrow \Gamma_\infty$ with the conditions 1), 2) and with $\nu(0) = +\infty$, but if we don't assume that ν takes the value $+\infty$ only for 0, the set $\mathcal{P} = \nu^{-1}\{+\infty\}$ is a prime ideal of A and ν induces a valuation on the integral domain $A/\mathcal{P}$.
If A is an integral domain, any valuation ν on A with values in Γ extends in a unique way in a valuation of the fraction field K of A with values in Γ.
The set of elements of Γ vhich are values of elements of $A \setminus \{0\}$ generates a subgroup Γ' of Γ and we have $\Gamma' = \nu(K^*)$.

The valuation ν defined by $\nu(x) = 0$ for any x in $A \setminus \{0\}$ is called *the trivial valuation.*

Proposition 1.3. ([Bo], Chap. 6, §3, n°1, Proposition 1, page 97.) *Let ν be a valuation of A, then for any family $\big\{x_1, \ldots, x_n\big)$ in A we have the inequality:*

$$\nu\Big(\sum_{i=1}^{n} x_i\Big) \geq \min\big\{\nu(x_1)\ldots, \nu(x_n)\big\}\,.$$

More over, if the minimum is reached by only one of the $\nu(x_i)$ we have the equality:

$$\nu\Big(\sum_{i=1}^{n} x_i\Big) = \min\big\{\nu(x_1)\ldots, \nu(x_n)\big\}\,.$$

Proposition 1.4. *Let ν be a valuation of a field K with values in a group Γ, then the set A of elements x of K with $\nu(x) \geq 0$ is a valuation ring of K and the maximal ideal $max(A)$ is the set of elements x of K with $\nu(x) > 0$.*
Conversely, we can associate to any valuation ring V of K a valuation ν of K with values in a group Γ such that V is the inverse image $\nu^{-1}\big(\{\alpha \in \Gamma \mid \alpha \geq 0\}\big)$.

Proof. We deduce from the conditions 1) and 2) of the definition of a valuation that the set $A = \{x \in K \mid \nu(x) \geq 0\}$ is a subring of K, and by property b) of theorem 1.2 we get that A is a valuation ring.

To get the converse, we are going to construct the group Γ and the mapping ν from the ring V. More generally, if C is an integral domain with fraction field K, the set $U(C)$ of invertible elements of C is a subgroup of the multiplicative group K^* and we call Γ_C the quotient group. The divisibility relation on C defines a partial order on Γ_C, compatible with the group structure, and we deduce from the remark following the theorem 1.2 that Γ_C is totally ordered if and only if C is a valuation ring. Then the canonical mapping $K^* \longrightarrow \Gamma_C = K^*/U(C)$ induces a valuation ν on K with $C = \{x \mid \nu(x) \geq 0\}$ and with values in the group Γ_C.

Definition. The valuation ring V associated to the valuation ν of K is called the valuation ring of ν and we note it $V = R_\nu$, and the field $\kappa(V) = V/max(V)$ is called *the residue field* of ν and we denote it κ_ν. The subgroup $\Gamma' = \nu(K^*)$ of Γ is called *the value group* of ν and we note it $\Gamma' = \Gamma_\nu$. We deduce from the proof of the proposition that the value group Γ_ν is isomorphic to $\Gamma_V = K^*/U(V)$. In general we shall assume that Γ is the value group, i. e. that ν is surjective from K^* into Γ.

We say that two valuations ν and ν' of a field K are *equivalent* if they have the same valuation rings, i. e. $R_\nu = R_{\nu'}$.

Proposition 1.5. ([Bo], Chap. 6, §3, n°2, Proposition 3, page 99.) *Two valuations ν and ν' of K are equivalent if and only if there exists an order preserving isomorphism φ of Γ_ν onto $\Gamma_{\nu'}$ such that $\nu' = \varphi \circ \nu$.*

We make no distinction between equivalent valuations and we identify them.

We often consider a fixed field k; all the fields K are extensions of k and we say that a valuation ν of K is a valuation of K/k if the restriction of ν to k is trivial, i.e. if for all elements x in k^* we have $\nu(x) = 0$.
If V is the valuation ring of K associated to the valuation ν, this is equivalent to demand to V to be a k-algebra. The natural map $k \longrightarrow V$ has its image included in $V \setminus max(V)$, then we get an inclusion $k \subset \kappa_\nu$, i.e. the residue field of the valuation is also an extension of k.

1.2. Rank of a valuation and composite valuation

Definition. A subset Δ of a totally ordered group Γ is called a *segment* if Δ is non-empty and if for any element α of Γ which belongs to Δ, all the elements β of Γ which lie between α and $-\alpha$, i. e. such that $-\alpha \leq \beta \leq \alpha$ or $\alpha \leq \beta \leq -\alpha$, also belong to Δ.

A subgroup Δ of Γ is called an *isolated subgroup* if Δ is a segment of Γ.

Proposition 1.6. ([Bo], Chap. 6, §4, n°2, Proposition 3, page 108.) *The kernel of an order preserving homorphism of totally ordered groups of Γ in Γ' is an isolated subgroup of Γ.*

Conversely, if Δ is an isolated subgroup of a totally ordered group Γ, the quotient group Γ/Δ has a structure of totally ordered group and the canonical morphism $\Gamma \longrightarrow \Gamma/\Delta$ is ordered preserving.

The set of all the segments Δ of Γ is totally ordered by the relation of inclusion, then we can give the following definition.

Definition. The ordinal type of the totally ordered set of proper isolated subgroups Δ of Γ is called *the rank* of the group Γ.

Let ν be a valuation of a field K, with value group Γ, and let V be the valuation ring associated to ν. For any part A of V containing 0 we denote by Δ_A the set of all the elements Γ_∞ in the complementary of $\big(\nu(A)\big) \cup \big(-\nu(A)\big)$.

Theorem 1.7. ([Za-Sa], Chap.VI, §10, Theorem 14, page 40.) *If $\mathcal{I}$ is an ideal of V, $\mathcal{I} \neq V$, then $\Delta_{\mathcal{I}}$ is a segment in Γ. The mapping $\mathcal{I} \longrightarrow \Delta_{\mathcal{I}}$ is a bijection from the set of all proper ideals of V onto the set of all segments of Γ, which is order-reversing for the relation of inclusion.*
Moreover, the segment $\Delta_{\mathcal{I}}$ is an isolated subgroup of Γ if and only if $\mathcal{I}$ is a prime ideal of V.

The maximal ideal $max(V)$ is the prime ideal corresponding to the isolated subgroup $\Delta = \{0\}$, and the ideal (0) is the prime ideal corresponding to the isolated subgroup $\Delta = \Gamma$.

Definition. We define *the rank* of a valuation ν as the rank of its value group.

Remark 1.4. From the theorem, we see that the set of the prime ideals of the valuation ring V associated to the valuation ν is totally ordered by inclusion and the rank of the valuation ν is by definition the ordinal type of the set of prime ideals of V.

When the ordinal type of the set of prime ideals of the valuation ring V is finite, we say that the valuation ν is of finite rank and we denote $\text{rank}(\nu) = n$ with $n \in \mathbb{N}$. Otherwise we say that the valuation is of infinite rank.

Corollary. *The rank of the valuation ν is equal to the Krull dimension of the valuation ring V associated to ν.*

For any element α in the group Γ we can define the ideals $\mathcal{P}_\alpha(V)$ and $\mathcal{P}_{\alpha+}(V)$ of the valuation ring V by:

$$\mathcal{P}_\alpha(V) = \big\{x \in V \,/\, \nu(x) \geq \alpha\big\} \quad \text{and} \quad \mathcal{P}_{\alpha+}(V) = \big\{x \in V \,/\, \nu(x) > \alpha\big\}\,.$$

We can also define the Rees-like algebras introduced in [Te], 2.1, associated to this family of ideals:

$$\mathcal{A}_\nu(R_\nu) = \bigoplus_{\alpha\in\Gamma} \mathcal{P}_\alpha(R_\nu) v^{-\alpha} \subset R_\nu[v^\Gamma] \quad \text{and} \quad \mathbf{gr}_\nu(V) = \bigoplus_{\alpha\in\Gamma} \mathcal{P}_\alpha(V)/\mathcal{P}_{\alpha+}(V)\,.$$

Remark 1.5. If the value group Γ is not isomorphic to the group of integers $\mathbb{Z}$, then there may exist ideals $\mathcal{I}$ of V which are different from ideals $\mathcal{P}_\alpha$ or $\mathcal{P}_{\alpha+}$ for all α in Γ.
If the value group Γ of ν is equal to $\mathbb{Q}$, for any real number $\beta > 0$ in $\mathbb{R}\smallsetminus\mathbb{Q}$, the set $\mathcal{I} = \{x \in V \,/\, \nu(x) \geq \beta\}$, which is also equal to $\{x \in V \,/\, \nu(x) > \beta\}$, is an ideal of V, but there is no α in Γ such that $\mathcal{I}$ is equal to $\mathcal{P}_\alpha$ or $\mathcal{P}_{\alpha+}$.
If the value group Γ of the valuation ν is of rank bigger than one, and if $\mathcal{P}$ is a prime ideal of the valuation ring V different from (0) and from the maximal ideal $max(V)$, there doesn't exist α in Γ such that $\mathcal{P}$ is equal to $\mathcal{P}_\alpha$ or to $\mathcal{P}_{\alpha+}$.

Proposition 1.8. ([Bo], Chap.6, §4, n°1, Proposition 1, page 110; [Va], Proposition 3.3, page 547.) *Let V be a valuation ring of a field K.*
a) Any local ring R with $V \subset R \subset K$ is a valuation ring of K, and the maximal ideal $max(R)$ of R is contained in V and is a prime ideal of V.
b) The mapping $\mathcal{P} \mapsto V_\mathcal{P}$ is a bijection from the set of prime ideals of V onto the set of local rings R with $V \subset R \subset K$, which is order-reversing for the relation of inclusion. The inverse map is defined by $R \mapsto max(R)$.

Proof. From the condition b) of theorem 1.2 we see that the ring R is a valuation ring and that $max(R)$ is an ideal of V. Since $max(R)$ is a prime ideal of R, it is also a prime ideal of V.
For any prime ideal $\mathcal{P}$ of V the local ring $V_\mathcal{P}$ is such that $V \subset V_\mathcal{P} \subset K$, and if $\mathcal{P} \subset \mathcal{Q}$ we have $V_\mathcal{Q} \subset V_\mathcal{P}$. We can verify that the maximal ideal $\mathcal{P}V_\mathcal{P}$ of the local ring $V_\mathcal{P}$ is equal to $\mathcal{P}$.

Let V be a valuation ring of a field K associated to a valuation ν of value group Γ, and we assume that ν is of finite rank r. We denote respectively $\mathcal{P}_i$, Δ_i and V_i, $0 \leq i \leq r$, the prime ideals of V, the isolated subgroups of Γ and the local subrings of K containing V, with the relations $\mathcal{P}_i = max(V_i)$, $V_i = V_{\mathcal{P}_i}$ and $\Delta_i = \Delta_{\mathcal{P}_i}$. We have the inclusions:

$$\begin{aligned}
(0) &= \mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_{r-1} \subset \mathcal{P}_r = \max(V)\\
V &= V_r \subset V_{r-1} \subset \ldots \subset V_1 \subset V_0 = K\\
(0) &= \Delta_r \subset \Delta_{r-1} \subset \ldots \subset \Delta_1 \subset \Delta_0 = \Gamma.
\end{aligned}$$

We shall study later the relations between the valuations ν_i associated to the valuation rings V_i and the valuation ν, more precisely the relations between their value groups Γ_i and the isolated subgroups Δ_i of Γ (cf. the proposition 1.11).

Example 2. The trivial valuation of K, i.e. the valuation ν defined by $\nu(x) = 0$ for all the non-zero elements x of K, is the unique valuation of rank 0.

Example 3. The valuation ν of K is of rank one if and only if the value group Γ of ν is isomorphic to a subgroup of $(\mathbb{R}, +)$. It is equivalent to say that the group Γ is *archimedean*, i.e. Γ satisfies the following condition: if α and β are any two elements of Γ with $\alpha > 0$, then there exists an integer n such that $n\alpha > \beta$. The valuation ring V associated to ν is of dimension 1 and we deduce from the proposition that V is a maximal subring of K for the relation of inclusion.

Definition. We say that a totally ordered group Γ is a *discrete* group if it is of finite rank r and if all the quotient groups Δ_{i+1}/Δ_i, where the Δ_i are the isolated subgroups of Γ, are isomorphic to $\mathbb{Z}$. It is equivalent to say that the ordered group Γ is isomorphic to a subgroup of $(\mathbb{Z}^n, +)$ with the lexicographic order. We say that a valuation ν is *discrete* if its value group Γ is a discrete group.

If the value group of ν is discrete of rank one, i.e. if ν is discrete valuation of rank one, we can assume that the value group is $\mathbb{Z}$.

Proposition 1.9. ([Bo], Chap. 6, §3, n°6, Proposition 9, page 105.) *Let A be a local integral domain, then the following conditions are equivalent:*

a) A is a discrete valuation ring of dimension 1*;*
b) A is principal;
c) the maximal ideal $max(A)$ is principal and A is noetherian;
d) A is a noetherian valuation ring.

We see that in that case, if we assume that the value group Γ of the valuation ν associated to the ring A is the ring $\mathbb{Z}$, the maximal ideal $\mathfrak{m} = max(A)$ is generated by any element x in A such that $\nu(x) = 1$. Then any element y of the fraction field K of A can be written $y = ux^n$, with $u \in A \setminus \mathfrak{m}$ and with $n \in \mathbb{Z}$, and we have $\nu(y) = n$. We say that the valuation ν is the $\mathfrak{m}$-*adic* valuation, i.e. the valuation defined by the relation: $\nu(y) \geq n$ if and only if $y \in \mathfrak{m}^n$.
The only ideals of A are the ideals $\mathcal{P}_n(A) = \{x \in A \,/\, \nu(x) \geq n\}$,(cf. remark 1.5), and $\mathcal{P}_n(A)$ is the principal ideal generated by x^n.

Definition. Let Γ be a commutative group, then the maximum number of rationally independent elements of Γ is called *the rational rank* of the group Γ. We define *the rational rank* of a valuation ν as the rational rank of its value group Γ.

The rational rank is an element of $\mathbb{N}\cup\{+\infty\}$, we denote it rat.rank$(\Gamma)$, and we have rat.rank$(\Gamma) = \dim_{\mathbb{Q}}\big(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}\big)$.

The rational rank of a group is zero if and only if Γ is a torsion group. If Γ is a value group of a valuation, Γ is totally ordered, then its rational rank is zero if and only if $\Gamma = \{0\}$, i.e. if and only if the valuation is the trivial valuation.

Proposition 1.10. ([Bo], Chap. 6, §10, n°2, Proposition 3, page 159.) *Let Γ be a commutative group and Γ' a subgroup of Γ. Then we have the equality:*

$$\textit{rat.rank}(\Gamma) = \textit{rat.rank}(\Gamma') + \textit{rat.rank}\big(\Gamma/\Gamma'\big) .$$

If Γ is a totally ordered group we have the inequality:

$$\textit{rank}(\Gamma) \leq \textit{rank}(\Gamma') + \textit{rat.rank}\big(\Gamma/\Gamma'\big) .$$

Corollary. *The rank of a valuation ν is never greater than its rational rank:*

$$\textit{rank}(\nu) \leq \textit{rat.rank}(\nu) .$$

Let ν be a valuation of a field K, with value group Γ and valuation ring V, and we denote $\mathfrak{m}$ the maximal ideal of V. If the rank of ν is bigger than one, there exists a proper isolated subgroup Δ of Γ, $\Delta \neq (0)$, and let $\mathfrak{m}'$ the prime ideal of V associated to Δ by the theorem 1.7. We know by proposition 1.8 that $\mathfrak{m}'$ is a prime ideal of V and that the local ring $V' = V_{\mathfrak{m}'}$ is a valuation ring of K with $\mathfrak{m}'$ as maximal ideal and such that $V \subset V'$. We denote ν' the valuation of K associated to V' and Γ' the value group of ν'.

Proposition 1.11. ([Za-Sa], Chap.VI, §10, Theorem 17, page 43.)
a) The value group Γ' is isomorphic to the quotient group Γ/Δ, and the valuation $\nu' \colon K^* \longrightarrow \Gamma'$ is the composition of $\nu \colon K^* \longrightarrow \Gamma$ and $\lambda \colon \Gamma \longrightarrow \Gamma/\Delta$.
b) The quotient ring $\bar{V} = V/\mathfrak{m}'$ is a valuation ring of the residue field $\kappa_{\nu'} = V'/\mathfrak{m}'$ of the valuation ν', and the value group of the valuation $\bar{\nu}$ associated to $\bar{V}$ is isomorphic to Δ.

Proof. a) The valuations ν and ν' are defined as the natural applications $\nu \colon K^* \longrightarrow \Gamma = K^*/U(V)$ and $\nu' \colon K^* \longrightarrow \Gamma' = K^*/U(V')$. Then

we deduce from $V \subset V'$ and $\mathfrak{m} \subset \mathfrak{m}'$ that $U(V)$ is included in $U(V')$ and that ν' is equal to $\lambda \circ \nu$. We have to show that the kernel of λ is isomorphic to the isolated subgroup Δ, which is a consequence of the relation between the prime ideal $\mathfrak{m}'$ and $\Delta = \Delta_{\mathfrak{m}'}$.
b) Since V is a valuation ring of K, the quotient ring $\bar{V}$ is also a valuation ring of the fraction field $\bar{K}$ of $\bar{V}$ and we have $\bar{K} = V'/\mathfrak{m}'$. To show that the value group of the valuation $\bar{\nu}$ is equal to the group Δ it is enough to remark that we have the exact sequence: $0 \longrightarrow \bar{K}^*/U(\bar{V}) \longrightarrow K^*/U(V) \longrightarrow K^*/U(V') \longrightarrow 0$.

Definition. ([Za-Sa], Chap.VI, §10.) The valuation ν is called *the composite valuation with the valuations ν' and $\bar{\nu}$* and we write $\nu = \nu' \circ \bar{\nu}$.

Corollary. *If ν is the composite valuation $\nu' \circ \bar{\nu}$ we have the equalities:*

$$rank(\nu) = rank(\nu') \ + \ rank(\bar{\nu})$$

$$rat.rank(\nu) = rat.rank(\nu') \ + \ rat.rank(\bar{\nu})\,.$$

Conversely, if we have a valuation ν' of a field K and a valuation $\bar{\nu}$ of the residue field $\bar{K} = \kappa_{\nu'}$, we can define the composite valuation $\nu = \nu' \circ \bar{\nu}$.

Proposition 1.12. ([Va], Proposition 4.2, page 552.) *Let ν' be a valuation of K with valuation ring V' and residue field $\kappa_{\nu'} = \bar{K}$ and $\bar{\nu}$ be a valuation of $\bar{K}$, then the composite valuation $\nu = \nu' \circ \bar{\nu}$ is the valuation of the field K associated to the valuation ring V defined by $V = \big\{x \in V' \,/\, \bar{\nu}(\bar{x}) \geq 0\big\}$.*

We notice that the residue field of the composite valuation ν is equal to the residue field $\kappa_{\bar{\nu}}$ of the valuation $\bar{\nu}$.

Remark 1.6. If we have the valuations ν' of K and $\bar{\nu}$ of $\kappa_{\nu'}$, the composite valuation $\nu = \nu' \circ \bar{\nu}$ defines an extension of the value group Γ' of ν' by the value group $\bar{\Gamma}$, i.e. an exact sequence of totally ordered groups: $0 \longrightarrow \bar{\Gamma} \longrightarrow \Gamma \longrightarrow \Gamma' \longrightarrow 0$.
If this exact sequence splits, the value group Γ is isomorphic to the group $(\Gamma' \times \bar{\Gamma})$ with the lexicographic order. If the valuation ν' is a discrete valuation of rank one, i.e. for $\Gamma' \simeq \mathbb{Z}$, the exact sequence always splits and we can describe the composite valuation $\nu = \nu' \circ \bar{\nu}$ in the following way. The maximal ideal of the valuation ring V' associated to ν' is generated by an element u and we can associate to any non zero element x in K the non zero element $\bar{y}$ in the residue field $\kappa_{\nu'}$ which is the class of $y = x.u^{-\nu'(x)}$. The composite valuation ν is then defined by $\nu(x) = \big(\nu'(x), \bar{\nu}(\bar{y})\big)$.

Remark 1.7. If ν_1 is a valuation of a field K and if ν_2 is a valuation of the residue field κ_1 of ν_1, we have defined the composite valuation ν of K, $\nu = \nu_1 \circ \nu_2$. By induction we may define in the same way the composite valuation $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$, where each valuation ν_i is a valuation of the residue field κ_{i-1} of the valuation ν_{i-1}, $1 \leq i \leq r$, with $\nu_0 = \nu$. For any $1 \leq t \leq r$, we decompose the valuation ν as $\nu = \nu'_{(t)} \circ \bar{\nu}_{(t)}$, where $\nu'_{(t)} = \nu_1 \circ \ldots \circ \nu_t$ is a valuation of K and $\bar{\nu}_{(t)} = \nu_{t+1} \circ \ldots \circ \nu_r$ is a valuation of the residue field $\kappa_{\nu'_{(t)}}$ of $\nu'_{(t)}$, with $\kappa_{\nu'_{(t)}} = \kappa_t$. If we denote $V_{(t)}$ the valuation ring of K associated to $\nu'_{(t)}$, the family of valuations $\big(\nu'_{(1)}, \ldots, \nu'_{(r)} = \nu\big)$ corresponds to the sequence $V = V_{(r)} \subset \ldots \subset V_{(1)} \subset K$. We call the valuation $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$, *the composite valuation with the family* $\big(\nu_1, \nu_2, \ldots, \nu_r\big)$.

Let ν_1 and ν_2 be two valuations of a field K and let $(V_1, \mathfrak{m}_1)$ and $(V_2, \mathfrak{m}_2)$ be the valuation rings respectively associated to ν_1 and ν_2. We assume that there exists a valuation ring V of K, $V \neq K$, which contains the rings V_1 and V_2, then there exists a non trivial valuation ν of K such that the valuations ν_1 and ν_2 are composite with ν. More precisely, there exist two valuations $\bar{\nu}_1$ and $\bar{\nu}_2$ of the residue field κ_ν with $\nu_1 = \nu \circ \bar{\nu}_1$ and $\nu_2 = \nu \circ \bar{\nu}_2$. This is also equivalent to say that there exists an non zero subset $\mathfrak{m}$ of $V_1 \cap V_2$ which is a prime ideal of the two rings V_1 and V_2.

Definition. ([Za-Sa], Chap.VI, §10, page 47.) Two valuations ν_1 and ν_2 of a field K are said *independent* if they are not composite with a same non trivial valuation ν.
A family $\big\{\nu_1, \nu_2, \ldots, \nu_k\big\}$ of valuations of a field K is called a *family of independent valuations* if any two of them are independent.

In fact we can define a partial order on the set of all the valuations of a field K by $\nu_1 \preceq \nu_2$ if and only if $V_2 \subset V_1$, where V_i is the valuation ring associated to ν_i, $i = 1, 2$. This equivalent to say that ν_2 is composite with ν_1, i.e. that there exists a valuation $\bar{\nu}$ of the residue field κ_{ν_1} such that $\nu_2 = \nu_1 \circ \bar{\nu}$. If ν_1 and ν_2 are two valuations of K, we can define the valuation $\nu = \nu_1 \wedge \nu_2$ as the "biggest" valuation ν such that $\nu \preceq \nu_1$ and $\nu \preceq \nu_2$. This valuation $\nu = \nu_1 \wedge \nu_2$ is the valuation associated to smallest valuation ring V of K which contains the valuation rings V_1 and V_2 associated to ν_1 and ν_2. Then two valuations ν_1 and ν_2 of K are independent if and only if $\nu_1 \wedge \nu_2$ is the trivial valuation.
If ν is a valuation of rank one, the valuation ring V associated to ν is maximal among the valuation rings of K, i.e. the valuation ν is minimal among the non trivial valuations of K. Then for any valuation ν' of K, ν and ν' are not independent if and only if $\nu \preceq \nu'$, i.e. if and only if ν'

is composite with ν. If ν and ν' are two distinct valuations of K of rank one, then ν and ν' are independent.

The notion of independence of valuations is important because of the following result which is called *the approximation theorem.*

Theorem 1.13. ([Za-Sa], Chap.VI, §10, Theorem 18, page 47.) *Let* $\{\nu_1, \nu_2, \ldots, \nu_k\}$ *be a family of independent valuations of a field* K*; given* k *arbitrary elements* $x_1, \ldots, x_k$ *of* K *and* k *arbitrary elements* $\alpha_1, \ldots, \alpha_k$ *of the value groups* $\Gamma_1, \ldots, \Gamma_k$ *of the valuations* $\nu_1, \ldots, \nu_k$ *respectively, then there exists an element* x *of* K *such that*

$$\nu_i(x - x_i) = \alpha_i\,, \quad i = 1, 2, \ldots, k\,.$$

1.3. Extension of a valuation

Let K be a field and let L be an overfield of K. If μ is a valuation of L, the restriction of μ to K is a valuation of K, the value group Γ_ν of ν is a subgroup of the value group Γ_μ and the valuation ring R_ν associated to ν is equal to $R_\mu \cap K$ where R_μ is the valuation ring associated to μ.

Definition. We say that the valuation μ is *an extension of the valuation* ν *to* L.

Remark 1.8. In fact the valuation ring R_ν is dominated by the valuation ring R_μ. More generally if V and W are valuation rings of K and L respectively, we have W dominates V if and only if $V = W \cap K$. Since the valuation ring R_μ dominates the valuation ring R_ν, we have an inclusion of the residue fields $\kappa_\nu \subset \kappa_\mu$.

Proposition 1.14. *For any valuation* ν *of a field* K *and for any overfield* L *of* K*, there exists at least one valuation* μ *of* L *wich is an extension of* ν.

Proof. By the corollary at the proposition 1.1 there exists at least one valuation ring W of the field L which dominates the valuation ring V associated to the valuation ν. Then the valuation μ associated to W is an extension of ν.

Let ν be a valuation of a field K and let μ be any extension of ν to an overfield L of K. We want to study the extensions Γ_μ and κ_μ of respectively the value group Γ_ν and the residue field κ_ν of ν.

Definition. The ramification index of μ *relative to* ν is the index of the subgroup Γ_ν in Γ_μ:

$$e(\mu/\nu) \;=\; \left[\Gamma_\mu : \Gamma_\nu\right].$$

The residue degree of μ *relative to* ν is the degree of the extension of the residue fields:

$$f(\mu/\nu) \;=\; \big[\kappa_\mu : \kappa_\nu\big]\,.$$

The ramification index and the residue degree are elements of $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$.

Remark 1.9. If μ' is an extension of μ to an overfield L' of L, then μ' is an extension of ν and we have the equalities:

$$e(\mu'/\nu) = e(\mu'/\mu)e(\mu/\nu) \quad \text{and} \quad f(\mu'/\nu) = f(\mu'/\mu)f(\mu/\nu)\,.$$

Proposition 1.15. *Let* ν *be a valuation of a field* K *and let* μ *be an extension of* ν *to an overfield* L*; if the field extension* $L|K$ *is finite of degree* n*, then we have the inequality*

$$e(\mu/\nu)f(\mu/\nu) \leq n\,.$$

We deduce that the ramification index $e(\mu/\nu) \;=\; \big[\Gamma_\mu \;:\; \Gamma_\nu\big]$ *and the residue degree* $f(\mu/\nu) = \big[\kappa_\mu : \kappa_\nu\big]$ *are finite.*

Proof. Let r and s be two integers with $r \leq e(\mu/\nu)$ and $s \leq f(\mu/\nu)$, and we want to show $rs \leq n$. There exist r elements $x_1, x_2, \ldots, x_r$ of L such that for any (i,j) with $i \neq j$, $\mu(x_i) \not\equiv \mu(x_j) \bmod \Gamma_\nu$, and there exist s elements $y_1, y_2, \ldots, y_s$ in the valuation ring R_μ such that their images $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s$ in the residue field κ_μ are linearly independent over κ_ν. It is enough to show that the rs elements $x_i y_j$, $1 \leq i \leq r$ and $1 \leq j \leq s$, of L are linearly independent over K.
We assume that there exists a non trivial relation

$$(*) \qquad \sum a_{i,k} x_i y_k \;=\; 0\;, \quad \text{with } a_{i,k} \in K\;.$$

We choose an (j,m) such that for all (i,k) we have $\mu(a_{j,m}x_j y_m) \leq \mu(a_{i,k}x_i y_k)$. Since $\mu(y_k) = 0$ for all the y_k and since $\mu(x_j) - \mu(x_i) \notin \Gamma_\nu$ for all $i \neq j$, we have $\mu(a_{j,m}x_j y_m) \neq \mu(a_{i,k}x_i y_k)$ for $i \neq j$. If we multiply the relation $(*)$ by $(a_{j,m}x_j)^{-1}$ we get a relation $\sum b_k y_k + z = 0$, with $b_k = a_{j,k}/a_{j,m} \in R_\mu \cap K$ and $z \in max(R_\mu)$, then we get in the residue field κ_μ the relation $\sum \bar{b}_k \bar{y}_k = 0$ with $\bar{b}_m = 1$. This is a non trivial relation of linear dependence of the $\bar{y}_k$ over κ_ν, which is impossible by hypothesis.

Proposition 1.16. *If* L *is an algebraic extension of* K *the quotient group* Γ_μ/Γ_ν *is a torsion group, i.e. every element has finite order, and the residue field* κ_μ *is an algebraic extension of* κ_ν*.*

Proof. We can write $L = \lim_{\to} L_\alpha$, where the fields L_α are finite extensions of K. Then the value group is equal to $\bigcup_\alpha \Gamma_\alpha$ where $\big(\Gamma_\alpha = \mu(L_\alpha^*)\big)$ is a filtered family of groups with $\big[\Gamma_\alpha;\Gamma_\nu\big] < +\infty$ for all the α. If we denote κ_α the residue field of the valuation $\mu_{|L_\alpha}$, then the residue field κ_μ is equal to $\lim_{\to} \kappa_\alpha$ where the κ_α are finite extensions of κ_ν.

Remark 1.10. If K is an algebraic extension of k, the unique extension to K of the trivial valuation of k is also the trivial valuation of K. Then if K is algebraic over k the unique valuation of K/k, i.e. the unique valuation of K which is trivial on k, is the trivial valuation. More generally, if K is any extension of k, a valuation of K/k is also a valuation of $K/\bar{k}$, where $\bar{k}$ is the algebraic closure of k in K.

Let ν be a valuation of a field K with value group Γ and residue field κ, then for any algebraic extension L of K and for any extension μ of ν to L we can consider that the value group Γ_μ is contained in the divisible closure Γ^* of Γ and that the residue field κ_μ is contained in the algebraic closue $\bar{\kappa}$ of κ. The divisible closure Γ^* of Γ is the quotient of the group $\Gamma \times (\mathbb{N} \setminus \{0\})$ by the equivalence relation $\sim$ defined by $(\alpha, p) \sim (\beta, q) \iff p\beta = q\alpha$, endowed with the natural addition and ordering.

Corollary. *If L is an algebraic extension of K then the rank and the rational rank of the valuation μ are equal respectively to the rank and the rational rank of the valuation μ. Moreover if L is a finite extension of K, μ is a discrete valuation if and only if ν is a discrete valuation.*

Remark 1.11. If the extension $L|K$ is algebraic but not finite, we can find an extension μ of a discrete valuation ν of K which is not a discrete valuation.

Let ν be a valuation on a field K and let $L|K$ be an extension, then we want to study the set of all the extensions μ of ν to L. We know that the valuation ring V associated to ν is an integrally closed ring and we consider the integral closure $\bar{V}$ of V in the extension L. In general the ring $\bar{V}$ is not a local ring but we always have $\bar{V} = \bigcap W$ where the rings W are the valuation rings associated to all the extensions of ν to L. In the case of an algebraic extension we have the following result.

Theorem 1.17. ([Bo], Chap. 06, §8, n°6, Proposition 6, page 147.) *Let $L|K$ be an algebraic extension and let ν be a valuation of K with valuation ring V, then there is a bijection from the set of the maximal ideals of the integral closure $\bar{V}$ of V in L onto the set of the extensions of ν to L, which is defined in the following way:*

for any maximal ideal $\mathfrak{p}$ of $\bar{V}$ the local ring $\bar{V}_{\mathfrak{p}}$ is a valuation ring of L which dominates V, and we associate to $\mathfrak{p}$ the valuation μ of L associated to $\bar{V}_{\mathfrak{p}}$.

We can deduce from the theorem that if W and W' are valuation rings of L which dominate V, they are not comparable with respect to the inclusion, then if μ and μ' are two extensions of ν to L, they are not comparable with respect to relation $\preceq$. If the valuation ν is of finite order, it's a consequence of the corollary of the proposition 1.16 and of the corollary of the proposition 1.11.

If L is a finite extension of K, for any valuation ν of K the set $\mathcal{V} = \mathcal{V}_L(\nu)$ of valuations μ of L which are extensions of ν to L is finite. In fact we have the inequality $\text{card}(\mathcal{V}) \leq [L : K]_{sep}$, where $[L : K]_{sep}$ is the separable degree of the extension $L|K$. If the extension $L|K$ is purely inseparable, there exists only one extension μ of ν: for any element x in L there exists an integer $n \geq 0$ such that x^{p^n} belongs to K, where p is the characteristic of K, then the valuation μ is defined by $\mu(x) = p^{-n}\nu\big(x^{p^n}\big)$.

For a finite extension $L|K$ we have the following important result:

Theorem 1.18. ([Za-Sa], Chap.VI, §11, Theorem 19, page 55 and Theorem 20, page 60.) *Let L be a finite extension of K of degree n, let ν be a valuation of K and let $\mu_1, \mu_2, \ldots, \mu_g$ be the extensions of ν to L. If e_i and f_i are respectively the ramification index and the residue degree of μ_i relative to ν, then:*

$$e_1 f_1 + e_2 f_2 + \ldots + e_g f_g \leq n .$$

If we assume that the integral closure $\bar{V}$ in L of the valuation ring V associated to ν is a finite V-module, then we have equality:

$$e_1 f_1 + e_2 f_2 + \ldots + e_g f_g = n .$$

Remark 1.12. If the extension $L|K$ is separable and if ν is discrete valuation of rank one, which is equivalent to say that the valuation ring V is noetherian, the integral closure $\bar{V}$ is a finite V-module and we always have equality. But it is possible to find inseparable extension $L|K$ and a discrete rank one valuation ν of K, or a separable extension $L|K$ and a non-discrete valuation ν of K, such that the equality fails.

Let ν be a valuation of a field K and let $L|K$ be a normal algebraic extension. Then the Galois group $G = Gal(L/K)$ acts transitively on the set $\mathcal{V}$ of the extensions of ν to L. To any extension μ of ν to L, we can associate subgroups of G called the *decomposition group* $G^d(\mu) = G_Z$, the *inertia group* $G^i(\mu) = G_T$ and the *ramification group* $G^r(\mu) = G_V$. The ramification theory of valuations, which is the study of the properties of these groups, is very important part of the theory of valuations. Since we don't need it in the following, we are not going to develop this theory here.

Let ν be a valuation of a field K and let ν' be an extension of ν to an extension K' of K of positive transcendence degree. We called V and V', Γ and Γ' and κ and κ' respectively the valuation rings, the residue fileds and the value groups of the valuations ν and ν'. We want to study the relations between the transcendence degree of the extensions K'/K and κ'/κ and the rational rank and the rank of the quotient group Γ'/Γ. We first consider the case $K' = K(x)$, whith x transcendental over K.

Proposition 1.19. ([Bo], Chap. 6, §10, n°1, Proposition 1 and Proposition 2, page 157.) *Let ν be a valuation of a field K with value group Γ and residue field κ.*
a) If Γ" is a totally ordered group which contains Γ and if ξ is an element of Γ" satisfying the condition $n.\xi \in \Gamma \Longrightarrow n = 0$, there exists a valuation ν' and only one which is an extension of ν to $K' = K(x)$, with values in the group Γ" and such that $\nu'(x) = \xi$. Then the value group of ν' is equal to $\Gamma' = \Gamma + \mathbb{Z}.\xi$ and the residue field $\kappa_{\nu'}$ of ν' is equal to κ.
b) There exists a valuation ν' and only one which is an extension of ν to the field $K' = K(x)$ such that $\nu'(x) = 0$ and such that the image t of x in the residue field $\kappa_{\nu'}$ is transcendental over κ. Then the value group Γ' of ν' is equal to Γ and the residue field $\kappa_{\nu'}$ is equal to $\kappa(t)$.

Remark 1.13. There may exist valuations ν' which are extension of ν to the extension $K' = K(x)$ with value group Γ' such that the quotient Γ'/Γ is a nontrivial torsion group, or with residue field κ' a non trivial algebraic extension of κ.

Proof. For any element ξ of a totally ordered group Γ" containing Γ, the map ν' from the polynomial ring $K[x]$ to Γ" by $\nu'\big(\sum a_j x^j\big) = \min\big(\nu(a_j) + j.\xi\big)$, is a valuation of $K[x]$ which extends to a valuation ν' of the field $K' = K(x)$ and ν' is an extension of ν.

In the first case, we see that if μ is any extension of ν with $\nu(x) = \xi$ we have $\mu(a_i x^i) = \nu(a_j) + i.\xi$ and as for $i \neq j$, $\nu(a_i) + i.\xi \neq \nu(a_j) + j.\xi$, we must have $\mu\big(\sum a_j x^j\big) = \min\big(\nu(a_j) + j.\xi\big)$ (cf. proposition 1.3). Then there exists only one extension of ν which is the valuation ν' that we have defined, and the value group is obviously the group $\Gamma + \mathbb{Z}.\xi$. Any element y of $K' = K(x)$ may be written $y = x^n b(1+u)$, with $n \in \mathbb{Z}$, $b \in K^*$ and $u \in K'$, $\nu'(u) > 0$, then if $\nu'(y) = 0$, i.e. if y is in $V' \setminus max(V')$ where

V' is the valuation ring associated to ν', $n = 0$ and the residue class $\bar{y}$ of y in $\kappa_{\nu'}$ is equal to the residue class of b in κ, and then $\kappa_{\nu'} = \kappa$.

In the second case we want to show that there is only one extension μ of ν, and that this valuation μ is again defined by $\mu\big(\sum a_j x^j\big) = \min\big(\nu(a_j) + j.\xi\big) = \min\big(\nu(a_j)\big)$. Let y be in K', and we may assume that $y = \sum a_j x^j$ with $a_j \in V$ and with $\nu(a_l) = 0$ for one index l. Then by the proposition 1.3 we have $\mu(y) \geq \min\big(\nu(a_j)\big) = 0$. The image $\bar{y}$ of y in the residue field κ_μ is equal to $\sum \bar{a}_j t^j$, and since t is transcendental over κ, we have $\bar{y} \neq 0$ which is equivalent to $\mu(y) = 0$.

We consider now the general case, K' is an extension of K with transcendental degree tr.deg.K'/K and let ν' be an extension of a valuation ν of K to K'. We denote V, κ, Γ and V', κ', Γ' respectively the valuation ring, the residue field, the value group of ν and ν'.

Theorem 1.20. ([Bo], Chap. 6, §10, n°3, Théorème 1, page 161.) *Let $x_1, \ldots, x_s$ be elements of the valuation ring V' such that their images $\bar{x}_1, \ldots, \bar{x}_s$ in the κ' are algebraically independent over κ, and let $y_1, \ldots, y_r$ be elements of K' such that the images of $\nu'(y_1), \ldots, \nu'(y_r)$ in the quotient group Γ'/Γ are linearly independent over $\mathbb{Z}$. Then the $r + s$ elements $x_1, \ldots, x_s, y_1, \ldots, y_r$ of K' are algebraically independent over K. If we denote ν" the restriction of the valuation ν' to the field K" $= K(x_1, \ldots, x_s, y_1, \ldots, y_r)$, the value group Γ" of ν" is equal to $\Gamma + \mathbb{Z}.\nu'(y_1) + \ldots + \mathbb{Z}.\nu'(y_r)$ and the residue field κ" is equal to $\kappa(\bar{x}_1, \ldots, \bar{x}_s)$.*

For a polynomial $f = \sum a_{(\underline{\beta},\underline{\gamma})} x^{\underline{\beta}} y^{\underline{\gamma}}$ in $K[x_1, \ldots, x_s, y_1, \ldots, y_r]$, the valuation of f is defined by:

$$\nu''(f) = \min_{(\underline{\beta},\underline{\gamma})} \nu''\Big(a_{(\underline{\beta},\underline{\gamma})} x^{\underline{\beta}} y^{\underline{\gamma}}\Big) = \min_{(\underline{\beta},\underline{\gamma})} \Big(\nu\big(a_{(\underline{\beta},\underline{\gamma})}\big) + \sum_{1 \leq j \leq r} \gamma_j \nu'\big(y_j\big)\Big) .$$

Proof. We make a proof by induction on $r + s$, and it is enough to consider the two cases $r = 1$ and $s = 0$ or $r = 0$ and $s = 1$. Then the result is a consequence of the proposition 1.19.

Corollary. *a) We have the inequality:*

$$\textit{rat.rank}\big(\Gamma'/\Gamma\big) \; + \; \textit{tr.deg.}\kappa'/\kappa \; \leq \; \textit{tr.deg.}K'/K .$$

Moreover if we have equality and if we assume that K' is a finitely generated extension of K, the group Γ'/Γ is a finitely generated $\mathbb{Z}$-module and the residue field κ' is a finitely generated extension of κ.
b) We have the inequality:

$$\textit{rank}(\nu') \; + \; \textit{tr.deg.}\kappa'/\kappa \; \leq \; \textit{rank}(\nu) \; + \; \textit{tr.deg.}K'/K .$$

Moreover if we have equality, and if we assume that K' is a finitely generated extension of K and that Γ is discrete, i.e. $\Gamma \simeq (\mathbb{Z}^h, +)_{\text{lex}}$, then the residue field κ' is a finitely generated extension of κ and Γ' is discrete.

Example 4. Let K be a field and let ν be a valuation of K, then from the theorem 1.20 there exists a unique extension ν' of ν to the purely transcendental extension $K' = K(x_1, \ldots, x_s)$ of K such that $\nu'(x_i) = 0$ for all i and such that the images $\bar{x}_1, \ldots, \bar{x}_s$ in the residue field $\kappa_{\nu'}$ are algebraically independent over κ_ν. The valuation ν' is defined by

$$\nu'\Big(\sum a_{\underline{\beta}} x^{\underline{\beta}}\Big) = \min_{\underline{\beta}}\Big(\nu(a_{\underline{\beta}})\Big)\,.$$

The valuation ν' is called *the Gauss valuation.*
Let L/K be an extension of transcendence degree s, let ν be a valuation of K and μ be an extension μ of ν to L such that $\text{tr.deg.}\kappa_\mu/\kappa_\nu = \text{tr.deg.}L/K = s$. Then there exist s elements $x_1, \ldots, x_s$ of L, algebraically independent over K such that L is an algebraic extension of $K' = K(x_1, \ldots, x_s)$ and such that μ is the extension of a Gauss valuation ν' of K'.

Let k be a field, K be an extension of k and we consider a valuation ν of K/k, i.e. that ν is a valuation of K which induces the trivial valuation on k, and let κ be the residue field of ν. We define the dimension of the valuation ν by the following.

Definition. The *dimension* of the valuation ν is the transcendence degree of the residue field κ of ν over the field k: $\dim(\nu) = \text{tr.deg.}\kappa/k$.

Remark 1.14. Let k be a field, K be an extension of k and let ν be a valuation of K/k, with residue filed κ. We can apply the corollary in the case of a valuation ν of K/k, where K is an extension of k and we find the inequalities:

$$\text{rank}(\nu) \;+\; \dim(\nu) \;\leq\; \text{rat.rank}(\nu) + \;\dim(\nu) \;\;\leq\; \text{tr.deg.}K/k\,.$$

If we assume that K is *a function field over* k, i.e. that K is a finitely generated extension of k, and if we have the equality $\text{rat.rank.}(\nu) + \dim(\nu) = \text{tr.deg.}K/k$ then the value group Γ is a finitely generated $\mathbb{Z}$-module and the residue field κ is finitely generated over k, moreover if we have the equality $\text{rank}(\nu) + \dim(\nu) = \text{tr.deg.}K/k$, the valuation ν is discrete.

Remark 1.15. Let k be a field, K be an extension of k and let ν be a valuation of K/k which is composite $\nu = \nu' \circ \bar{\nu}$, where ν' is a valuation of K/k with residue field κ' and $\bar{\nu}$ is a valuation of κ'/k. We deduce from the proposition 1.11 that the valuations ν and $\bar{\nu}$ have the same residue field κ, and that we have the equalities rank(ν) = rank(ν')+ rank$(\bar{\nu})$. Hence, if we are in the case of equality for the rank for the valuation ν:

$$\text{rank}(\nu) \ + \ \dim(\nu) \ = \ \text{tr.deg}.K/k\,,$$

we are also in the case of equality for the rank for the valuations ν' and $\bar{\nu}$:

$$\begin{aligned} \text{rank}(\nu') \ + \ \dim(\nu') \ &= \ \text{tr.deg}.K/k\ , \\ \text{rank}(\bar{\nu}) \ + \ \dim(\bar{\nu}) \ &= \ \text{tr.deg}.\kappa'/k\,. \end{aligned}$$

We have the same result for the rational rank of the valuations ν, ν' and $\bar{\nu}$.

Definition. Let K be a field and let ν a valuation on K with value group Γ and residue field κ. Let K^* be an extension of K and let ν^* be an extension of ν to K^*, with value group Γ^* and residue field κ^*, then we say that the valued field (K^*,ν^*) is *an immediate extension* of the valued field (K,ν) if the group Γ is canonically isomorph to Γ^* and the field κ is canonically isomorph to κ^*.

This equivalent to the following condition:

$$\forall x^* \in K^* \quad \exists x \in K \quad \text{such that} \quad \nu^*(x^* - x) > \nu^*(x^*)\ .$$

A valued field (K,ν) is called *a maximal valued field* if there doesn't exist any immediate extension. It is possible to prove that for any valued field (K,ν) there exists an immediate extension (K^*,ν^*) which is a maximal valued field, but in general this extension is not unique ([Ku]).

1.4. Examples

Example 5. Prime divisor ([Za-Sa], Chap.VI, §14, page 88.)
Let K be a function field over a field k, of transcendence degree d, then *a prime divisor* of K over k is a valuation ν of K/k which have dimension $d-1$, i.e. such that tr.deg.$\kappa/k = d-1$ where κ is the residue field of ν. Since the valuation ν is non trivial, we have rank$(\nu) \geq 1$, and we deduce from the corollary of the theorem 1.20 that we have rank$(\nu) = 1$, hence we are in the case of equality for the rank for the valuation ν: rank(ν)+ dim(ν) = tr.deg.K/k, the valuation ν is discrete of rank one, i.e. its value group is isomorphic to $\mathbb{Z}$, and its residue field κ is finitely generated over k. We deduce from the proposition 1.9 that the valuation ring V associated to ν is a noetherian ring.
Furthermore, we can always find a normal integral domain R, finitely generated over k, having K as fraction field, and a prime ideal $\mathfrak{p}$ of height one of R, such that the valuation ring V is equal to the local ring

$R_{\mathfrak{p}}$. The valuation ν is the "$\mathfrak{p}$-adic valuation", i.e. the valuation defined by $\nu(g) = \max\{n \in \mathbb{N} \ / \ g \in \mathfrak{p}^n\}$, for any g in R. If we consider the affine algebraic variety X associated to R, $X = \mathrm{Spec}R$, the prime ideal $\mathfrak{p}$ defines a *prime Weil divisor* D on X, i.e. a reduced irreducible closed subscheme of codimension one, and the valuation ν is the valuation defined by the order of vanishing along the divisor D. Moreover the valuation ring associated to ν is equal to the local ring $\mathcal{O}_{X,D}$ of the generic point of D in X. Conversely, any prime Weil divisor D on a normal algebraic variety X defines a prime divisor ν of the function field $K = F(X)$ of the variety X, and the valuation ring associated to ν is the local ring $\mathcal{O}_{X,D}$.

Example 6. Composition of prime divisors
Let K be a function field over a field k, of transcendence degree d, and let ν be a valuation of K/k of rank r and of dimension $d-r$. Then we are in the case of equality for the rank in the corollary of the theorem 1.20, and we know that the value group Γ of ν is isomorphic to $(\mathbb{Z}^r, +)_{\text{lex}}$ and that the residue field κ of ν is also finitely generated over k. We deduce from the remark 1.15 that if we write ν as a composite valuation $\nu = \nu' \circ \bar{\nu}$, with $\mathrm{rank}(\nu') = 1$, then ν' is a prime divisor of K and $\bar{\nu}$ is a valuation of the residue field κ' of ν' which satisfies also the equality for the rank. By induction we can write the valuation ν as the composite of a family of valuations $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$ (cf remark 1.7). All the valuations ν_i, $1 \leq i \leq r$, are discrete valuations of rank one, the residue field κ_i of the valuation ν_i is a function field over k and the valuation ν_{i+1} is a prime divisor of κ_i.

Example 7. Field of generalized power series ([Za-Sa], Chap. VI, §15, Example 2, page 101.)
We want to construct a valuation with a preassigned value group. More precisely, let Γ be a totally ordered group and let k be a ground field, and we want to find a field K, extension of k, and a valuation ν of K/k whose value group is isomorphic to the group Γ.
We define the ring R of *generalized power series* of a variable x with coefficients in the field k and with exponents in the group Γ by the following: R is the set of the expressions ξ of the form $\xi = \sum_{\gamma \in \Gamma} c_\gamma x^\gamma$ whose support $\mathrm{supp}(\xi) = \{\gamma \in \Gamma \ / \ c_\gamma \neq 0\}$ is a well ordered subset of Γ. We recall that an ordered set A is well ordered if any non empty subset B of A has a minimal element. Then we can define an addition and a multiplication on R in the usual way: for $\xi = \sum_\gamma c_\gamma x^\gamma$ and $\zeta = \sum_\gamma d_\gamma x^\gamma$

in R, we put:

$$\xi + \zeta = \sum_\gamma s_\gamma x^\gamma \qquad \text{with} \quad s_\gamma = c_\gamma + d_\gamma \ ,$$

$$\xi.\zeta = \sum_\gamma m_\gamma x^\gamma \qquad \text{with} \quad m_\gamma = \sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \ ,$$

this last sum is well defined because since the supports of ξ and ζ are well ordered sets, for any $\gamma \in \Gamma$, there exists only a finite number of couples (α, β) in $\mathrm{supp}(\xi) \times \mathrm{supp}(\zeta)$ with $\alpha + \beta = \gamma$.
Hence we have a ring, in fact a k-algebra, and we denote it by $R = k[[x^\Gamma]]$. The ring R is integral, we call its fraction field $K = Fr(R)$ *the field of generalized power series* and we denote it by $K = k((x^\Gamma))$.
We can define a valuation ν on $K = k((x^\Gamma))$. Let ξ be an element of the ring $k[[x^\Gamma]]$, $\xi = \sum_{\gamma\in\Gamma} c_\gamma x^\gamma$, $\xi \neq 0$, then we put $\nu(\xi) = \min(\mathrm{supp}(\xi))$, this is well defined because since the support of ξ is a non empty well ordered subset of Γ, it has a minimal element; and for $\xi = 0$ we put $\nu(0) = +\infty$. It is easy to prove that the valuation ν is a valuation of $K = k((x^\Gamma))$, which is trivial on k and such that its residue field κ is equal to k and its value group is Γ.
Moreover, the valued field $(k((x^\Gamma)), \nu)$ is maximal, i.e. there exist no immediate extension. ([Ri], Chap. D, Corollaire au Théorème 2, page 103.)

Example 8. Valuations of *k(x,y)/k*
We are going to give two examples of valuations of K/k, where K is the pure transcendental extension $K = k(x, y)$ of k of degree 2. We construct these valuations by their restrictions to the polynomial ring $R = k[x, y]$.
i) The first one is a valuation ν on $R = k[x, y]$ whose value group is the group of rational numbers $\mathbb{Q}$.
We put:

$$\nu(y) = 1 \quad \text{and} \quad \nu(x) = 1 + \frac{1}{2} = \frac{3}{2} \ .$$

The "first" element z_2 of R, i.e. a polynomial in x, y of minimal degree, such that the value $\nu(z)$ is not uniquely determined by the values in $z_0 = y$ and in $z_1 = x$ is the element $z_2 = x^2 + y^3$. We must have $\nu(z_2) \geq 3$ and we put:

$$\nu(z_2) = 3 + \frac{1}{3} = \frac{10}{3} \ .$$

We can define a sequence (z_n) of elements of R such that for any integer $n \geq 2$, the value $\nu(z_n)$ is not determined by the values on ν in the z_r for $r < n$. For any n the value $\nu(z_n)$ is a rational number $\gamma_n = \dfrac{p_n}{n+1}$, with p_n a positive integer and $(p_n, n+1) = 1$. The sequence (z_n) is constructed by induction in the following fashion, we assume we have found the elements z_r, $0 \leq r \leq n$ and the values $\nu(z_r) = \gamma_r$, then we put:

$$z_{n+1} = z_n + y^{p_n} \ ,$$

with the value:

$$\nu(z_{n+1}) = \gamma_{n+1} = p_n + \frac{1}{n+2} = \frac{p_{n+1}}{n+2} \quad \text{with } p_{n+1} = p_n(n+2) + 1 \ .$$

We have constructed a valuation ν of R, hence a valuation of the field $K = k(x, y)$, whose value group is $\mathbb{Q}$ and it is easy to see that the residue filed κ of ν is the ground field k.

There is another construction of such a valuation in [Za 1], I §6, page 648.

More generally, for any preassigned value group Γ of $\mathbb{R}$, we can construct a valuation with value group Γ. ([Za-Sa], Chap.VI, §15, Example 3, page 102 or [ML-Sc].)

ii) The second example is what we call *an analytic arc* on the surface $\mathbf{A}^2_k = \mathrm{Spec}R$. ([Za 1], I §5, page 647.) // Let $\hat{R}$ be the completion of the ring R, $\hat{R}$ is the ring of power series $k[[x, y]]$, and let $\hat{K}$ its fraction field $\hat{K} = k((x, y))$. We consider an element t of $\hat{R}$:

$$t = x + \sum_{i=1}^{+\infty} c_i y^i \ , \quad c_i \in k^* \quad \text{for all} \quad i \geq 1 \ ,$$

which is not algebraic over the field K.
We define the valuation $\hat{\nu}$ of $\hat{K}/k$ with values in the group $\hat{\Gamma} = (\mathbb{Z}^2, +)_{\mathrm{lex}}$, by:

$$\hat{\nu}(y) = (0, 1) \quad \text{and} \quad \hat{\nu}(t) = (1, 0) \ .$$

For any element ξ of $\hat{R}$, with $\xi = \sum_{i=1}^{+\infty} d_i y^i$ and $d_1 \neq 0$, we have $\hat{\nu}(\xi) = (0, 1)$, hence since the coefficients c_i are non zero, for any $N \geq 1$ we have:

$$\hat{\nu}\Big(\sum_{i+n}^{+\infty} c_i y^i\Big) \ = \ \hat{\nu}\big(y^N\big) \ = \ (0, N) \ ,$$

then for any $N \geq 1$ we get:

$$\hat{\nu}\big(x + \sum_{i=1}^{N-1} c_i y^i\big) \ = \ (0, N) \ .$$

We define the valuation ν on $K = k(x, y)$ as the restriction to K of the valuation $\hat{\nu}$ of $\hat{K}$, then the value group Γ is equal to $\mathbb{Z}$, i.e. the valuation ν is a discrete valuation of rank one. But this valuation is not a prime divisor, its residue field is equal to k.

We may describe the valuation ν on R in a different way. We have an injective k-morphism φ of $R = k[x, y]$ to the power series ring $k[[y]]$ defined by

$$\varphi(x) = x - t = -\sum_{i=1}^{+\infty} c_i y^i \quad \text{and} \quad \varphi(y) = y \ .$$

Then the valuation ν is the restriction to R of the y-adic valuation on $k[[y]]$.

Example 9. A non finitely generated residue field ([Za 2], Chap.3 II , footnote 12, page 864.)
Let K be a function field over a field k and let ν be a valuation of K/k, then if we don't assume that we have the equality rat.rank$(\nu)+$ $\dim(\nu) =$ tr.deg.K/k, it may happen that the residue field κ of ν is not finitely generated over k.

Let k be a field and $K = k(x, y, z)$ be an extension of k with x, y and z algebraically independent elements over k and we consider the valuation ν *defined by the formal power serie in* y*:*

$$z = x^{1/2}.y + x^{1/4}.y^2 + x^{1/8}y^3 \ldots = \sum_{n \geq 1} x^{1/2^n}.y^n \ .$$

We can give the following description of the valuation ν.
Let $A = k[x, y, z]$ be the polynomial ring and let $R = \cup_{n\geq 1} k\big[\big[x^{1/2^n}, y, z\big]\big]$ be the ring of formal power series in y, z and the $x^{1/2^n}$, $n \geq 1$. Let $f = z - \sum_{n\geq 1} x^{1/2^n}.y^n$ be in R and $\bar{R}$ be the quotient ring $R/(f)$. Then the map $A \longrightarrow \bar{R}$ induced by $A \subset R$ is an injection. We consider on $\bar{R}$ the y-adic valuation μ, i.e. the valuation defined by the order in y. If we denote $\bar{k}$ the field $\bar{k} = \cup_{n\geq 1} k((x^{1/2^n}))$ and L the fraction field of $\bar{R}$, L is an extension of $\bar{k}$ and μ is a discrete rank one valuation of $L/\bar{k}$, then the valuation ring W associated to μ is a noetherian $\bar{k}$-algebra and the residue field κ_μ is an extension of $\bar{k}$. The valuation ν is the restriction of μ to A, we have for instance $\nu(x) = 0$, $\nu(y) = \nu(z) = 1$.
We shall see that the residue field κ of ν is not finitely generated over k, in fact we have $\bar{k} \subset \kappa$, i.e. all the elements $x^{1/2^n}$, for $n \geq 1$, belong to κ. If we denote by $[u]$ the image in $\bar{R}$ of any element $u \in R$, we have $\mu([z/y - x^{1/2}]) = 1$ and the residue class of $[z/y]$ in κ_μ is equal to $x^{1/2}$. Then we have also that $x^{1/2}$ is the residue class of z/y in κ. In

the same way we can write $[z^2/y^2 - x] = [(z/y - x^{1/2})(z/y + x^{1/2})] = [2x^{1/2}x^{1/4}y + \ldots]$, and we find that the residue class of $(z^2 - xy^2)/y^3$ in κ is equal to $2x^{3/4}$, and we can continue in this fashion to show that all the elements $x^{1/2^n}$, for $n \geq 1$, belong to κ.
Since the valuation ν is the restriction of discrete valuation μ, ν is also a discrete valuation of rank one, i.e. its value group Γ is equal to $\mathbb{Z}$, hence rank.(ν) = rat.rank.$(\nu) = 1$, and we have seen that the transcendence degree of the residue field κ over k is equal to one, i.e. $\dim(\nu) =$ tr.deg.$\kappa/k = 1$. Then for this valuation we have rank$(\nu)+\dim(\nu) = 2 <$ tr.deg.$K/k = 3$.

§2. Riemann variety

2.1. Center of a valuation

Let K be a field and ν be a valuation of K, we denote V the valuation ring associated to ν and $\mathfrak{m}$ its maximal ideal.

Definition. Let A be a subring of K with $A \subset V$, i.e. such that $\nu(x)$ is non negative for all the elements x of A, then the *center* of the valuation ν on A is the ideal $\mathfrak{p}$ of A defined by $\mathfrak{p} = A \cap \mathfrak{m}$.

Remark 2.1. The center $\mathfrak{p}$ of the valuation ν on A is the unique prime ideal $\mathfrak{q}$ of A such that the valuation ring V dominates the local ring $A_{\mathfrak{q}}$. If A is a local ring, the center of ν on A is the maximal ideal of A if and only if V dominates A.

Let X be *an algebraic variety over a field* k, i.e. an irreducible reduced scheme of finite type over k, and let $K = F(X)$ be the function field of X, then K is a finitely generated extension of k and the dimension of the variety X is equal to the transcendence degree of K over k. We want to define the center of a valuation ν of K/k, or more generally of a valuation ν of L/k where L is an extension of K, on the variety X.

We consider first that X is an affine variety, $X = \mathrm{Spec}A$, where A is an integral k-algebra of finite type, with $A \subset L$. If A is contained in the valuation ring V associated to ν, i.e. if the valuation ν is non negative for all the elements $x \in A$, the center of ν on X is the point ξ of X corresponding to the prime ideal $\mathfrak{p}$ where $\mathfrak{p}$ is the center of the valuation ν on A, i.e. $\mathfrak{p} = A \cap \mathfrak{m}$. Since the center $\mathfrak{p}$ is a prime ideal of A, the closed subscheme Z of X defined by $\mathfrak{p}$ is an integral subscheme, i.e. an irreducible reduced subscheme of X, and ξ is the generic point of Z. We say also that the closed subscheme Z is the center of the valuation ν on the affine variety X. If A is not contained in the valuation ring V, then we say that the valuation ν has no center on X or that the center Z of the valuation ν on X is the empty set.

We want to generalise this definition for any algebraic variety X over k, with function field K, and say that the center of a valuation ν of a field L, with $K \subset L$, on the variety X is a point ξ of X such that the local ring $\mathcal{O}_{X,\xi}$ is dominated by the valuation ring V associated to ν. This is equivalent to say that we have a morphism of $T = \mathrm{Spec}V$ to X such that the image of the closed point t of T, corresponding to the maximal ideal of V, is the point ξ, and that this morphism induces the inclusion $K \subset L$, i.e. that the image of the generic point of T is the generic point of X.

Before defining the center of a valuation on any algebraic variety, we shall recall the valuation criterions of separatedness and of properness. ([EGA], Proposition 7.23 and Théorème 7.3.8, or [Ha], Chap.II, Theorem 4.3 and Theorem 4.7.)

Valuative criterion of separatedness. *let X and Y be noetherian schemes, let $f\colon X \longrightarrow Y$ be a morphism of finite type, then f is separated if and only if for every field L, for every valuation ring V of L and for every morphism $g\colon U = \mathrm{Spec}L \longrightarrow X$ and $h\colon T = \mathrm{Spec}V \longrightarrow Y$ forming a commutative diagram*

$$\begin{array}{ccc} U = \mathrm{Spec}\, L & \xrightarrow{g} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ T = \mathrm{Spec}\, V & \xrightarrow{h} & Y \end{array}$$

there exists at most one morphism $\bar{h}\colon T \longrightarrow X$ making the whole diagram commutative.

Valuative criterion of properness. *let X and Y be noetherian schemes, let $f\colon X \longrightarrow Y$ be a morphism of finite type, then f is proper if and only if for every field L, for every valuation ring V of L and for every morphism $g\colon U = \mathrm{Spec}L \longrightarrow X$ and $h\colon T = \mathrm{Spec}V \longrightarrow Y$ forming a commutative diagram*

$$\begin{array}{ccc} U = \mathrm{Spec}\, L & \xrightarrow{g} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ T = \mathrm{Spec}\, V & \xrightarrow{h} & Y \end{array}$$

there exists a unique morphism $\bar{h}\colon T \longrightarrow X$ making the whole diagram commutative.

The valuative criterion of separatedness will give the unicity of the center of a valuation on an algebraic variety and the valuative criterion of properness will give a condition on a vatiety for any valuation to have a center on X.

Proposition 2.1. *Let X be an algebraic variety over k and let ν be a valuation of a field L, extension of the function field $K = F(X)$ of X, then there exists at most one point ξ of X such that the local ring $\mathcal{O}_{X,\xi}$ is dominated by the valuation ring V associated to ν.*
Moreover the irreducible closed subvariety Z of X defined by $Z = \overline{\{\xi\}}$ is the subset of the points $x \in X$ whose the local ring $\mathcal{O}_{X,x}$ is contained in the valuation ring V associated to ν.

Proof. Since X is an algebraic variety, the morphism $f\colon X \longrightarrow \mathrm{Spec}k$ is separated, and the unicity of the point ξ is a consequence of the valuative criterion of separatedness, where the morphism $g\colon U = \mathrm{Spec}L \longrightarrow X$ is defined by the inclusion $F(X) \subset L$ and where the morphism $h\colon T = \mathrm{Spec}V \longrightarrow \mathrm{Spec}k$ is defined because the valuation ν is trivial on k. Then there exists at most one morphism $\bar{h}\colon T = \mathrm{Spec}V \longrightarrow X$, i.e. at most one point ξ on the variety X such that its local ring $\mathcal{O}_{X,\xi}$ is dominated by the valuation ring V.
To show that the set $Z = \overline{\{\xi\}}$ is equal to $\left\{x \in X \,/\, \mathcal{O}_{X,x} \subset V\right\}$, we can assume that X is an affine variety $X = \mathrm{Spec}A$ and that Z is exactly the closed subscheme of X defined by the center $\mathfrak{p}$ of the valuation ν on A. Then it is enough to see that for any prime ideal $\mathfrak{q}$ of A we have $\mathfrak{p} \subset \mathfrak{q}$ if and only if $A_{\mathfrak{q}} \subset V$.

Definition. The *center of the valuation ν on the variety X* is the point ξ, when it exists, defined in the proposition. We say also that the center of the valuation ν on the variety X is the subvariety $Z = \overline{\{\xi\}}$. If there doesn't exist ξ we say that the valuation ν has no center on the variety X or that the center Z is empty.

The valuation ν may have no center on the variety X, for instance if X is an affine variety $X = \mathrm{Spec}A$, with A non contained in V. But, if X is a projective variety, any valuation ν has a center on X. In fact we have the following result. We recall that an algebraic variety X over a field k is complete if the morphism $X \longrightarrow \mathrm{Spec}k$ is proper.

Theorem 2.2. *If X is a complete variety over a field k, any valuation ν of L/k, L an extension of the function field $K = F(X)$ of X, has a center on X.*
Conversely, the variety X is complete over k if all the valuations ν of K/k have a center on X.

Proof. If X is a complete variety, the morphism $f\colon X \longrightarrow \mathrm{Spec}k$ is proper, and we can apply the criterion of properness where the morphism $g\colon U = \mathrm{Spec}L \longrightarrow X$ is defined by the inclusion $F(X) \subset L$ and where the morphism $h\colon T = \mathrm{Spec}V \longrightarrow \mathrm{Spec}k$ is defined because the valuation

ν is trivial on k. Then we obtain a morphism $\bar{h}\colon T = \mathrm{Spec} V \longrightarrow X$ and the image of the closed point of T is the center of the valuation ν on X.

If X is an algebraic variety over a field k, then the transcendence degree of the function field $K = F(X)$ of X over k is equal to the dimension of X. Let ν be a valuation of the function field K, then we are going to show that the dimension of ν, i.e. the transcendence degree of the residue field κ of ν over k, is always bigger or equal to the dimension of its center Z on X.

Proposition 2.3. *Let X be an algebraic variety over a field k with function field $K = F(X)$, and let ν be a valuation of K/k with residue field κ. Then if the center Z of ν on X is non empty whe have dim$Z \leq$ dim(ν). Moreover if we have a strict inequality, there exists a proper birational morphism $Y \longrightarrow X$ such that the dimension of the center of ν on Y is equal to dim(ν).*

Proof. Let Z be the center of the valuation ν on X and let ξ be the generic point of Z. We denote by A the local ring $\mathcal{O}_{X,\xi}$ of X in ξ and by $\mathfrak{p}$ its maximal ideal, then the valuation ring V associated to ν dominates A and we have $A/\mathfrak{p} \subset V/\mathfrak{m}$, i.e. an inclusion of the function field $F(Z)$ of Z in the residue field κ. Since Z is an algebraic variety over k we have dim$Z =$tr.deg.$F(Z)/k$, then dim$Z \leq$tr.deg.κ/k.

If the inequality is strict, let $x_1, \ldots, x_r$ be elements of V such that their images $\bar{x}_1, \ldots, \bar{x}_r$ in κ is a transcendental basis of κ over $F(Z)$, and we can write $x_i = p_i/q$ with p_i and q in A, $i = 1, \ldots, r$. We consider an ideal $\mathcal{I}$ of $\mathcal{O}_X$ which is locally generated by $q, p_1, \ldots, p_r$ and Y the blowing up of $\mathcal{I}$ in X. Then the center Z' of ν on Y satisfies $F(Z)(\bar{x}_1, \ldots, \bar{x}_r) \subset F(Z')$ and we obtain dim$Z' =$ tr.deg.κ/k.

Remark 2.2. Let ν be a prime divisor of the function field K of an algebraic variety X over k (cf exemple 5), let Z be the center of ν on X and we assume that Z is non empty. Then we have codim$Z \geq 1$ and we deduce from the proposition 2.3 that there exists a proper birational morphism $\pi\colon Y \longrightarrow X$ such that the center D of ν on Y is a prime Weil divisor, moreover if we choose Y normal, the valuation ring V associated to ν is equal to the local ring $\mathcal{O}_{Y,D}$ of D in Y.
Conversely, if we consider a prime Weil divisor Z on an algebraic variety X over k, then we deduce from the proposition 2.3 that any non trivial valuation ν of the function field K of X with center Z is a prime divisor. It is possible to show that the set of prime divisors ν of the function field K of X which have center Z on X is finite and non empty. Moreover if the variety X is normal there exists only one prime divisor ν with

center Z, this the valuation ν associated to the local ring $\mathcal{O}_{X,Z}$, which is a noetherian valuation ring ([Za-Sa]).

Let X be an algebraic variety over a field k with function field $K = F(X)$ and let ν be a valuation of K/k with residue field κ. We assume that the transcendence degree of κ over k is positive, then there exists non trivial valuations $\bar{\nu}$ of κ/k and we can define the composite valuation $\nu' = \nu \circ \bar{\nu}$ which is also a valuation of K/k. If the center Z of ν on X is non empty the function field $\bar{K} = F(Z)$ of Z is contained in the residue field κ and we can consider the center on Z of a valuation $\bar{\nu}$ of κ/k.

Proposition 2.4. *The center on Z of the valuation $\bar{\nu}$ is equal to the center on X of the composite valuation $\nu' = \nu \circ \bar{\nu}$.*

Proof. We may assume that X is an affine variety $X = \mathrm{Spec}A$. We denote respectively V, V', $\bar{V}$ and $\mathfrak{m}$, $\mathfrak{m}'$, $\bar{\mathfrak{m}}$ the valuation rings associated to ν, ν', $\bar{\nu}$ and their maximal ideals, then we have $(0) \subset \mathfrak{m} \subset \mathfrak{m}' \subset V' \subset V \subset K$ and $\bar{\mathfrak{m}} \subset \bar{V} = V'/\mathfrak{m} \subset \bar{K} = V/\mathfrak{m}$. The centers of the valuations ν and ν' on X are defined by the prime ideals $\mathfrak{p} = A \cap \mathfrak{m}$ and $\mathfrak{p}' = A \cap \mathfrak{m}'$ of A, and the center of the valuation $\bar{\nu}$ on $Z = \mathrm{Spec}\bar{A}$, with $\bar{A} = A/\mathfrak{p}$ is defined by the prime ideal $\bar{\mathfrak{p}} = \bar{A} \cap \bar{\mathfrak{m}}$ of $\bar{A}$. Then the proposition is a consequence of the equality $\bar{\mathfrak{p}} = \mathfrak{p}'/\mathfrak{p}$.

We have seen that if ν' is a composite valuation $\nu' = \nu \circ \bar{\nu}$ of K/k, where K is the function field of an algebraic variety X over k, the center Z' of ν' is contained in the center Z of ν. This a consequence of the proposition 2.4 if the center Z of ν is non empty. If the center Z is empty, no local ring $\mathcal{O}_{X,x}$ for $x \in X$ is contained in the valuation ring V associated to ν, then none is contained in the valuation ring V' associated to ν' since we have $V' \subset V$. More generally if the valuation ν is composite with the family $\big(\nu_1, \nu_2, \ldots, \nu_r\big)$, and if we denote $\nu'_{(t)}$ the valuation of K defined by $\nu'_{(t)} = \nu_1 \circ \ldots \circ \nu_t$, $0 \leq t \leq r$, and ξ_t the center of $\nu'_{(t)}$ on the variety X, we obtain a family $\big(\xi_1, \xi_2, \ldots, \xi_r\big)$ of points of X such that ξ_t is a specialization of ξ_{t+1}, i.e. $\xi_t \in \overline{\{\xi_{t+1}\}}$, for $1 \leq t \leq r-1$.

Conversely, we have the following result.

Theorem 2.5. ([Za-Sa], Chap.VI, §16, Theorem 37, page 106.) *Let X be an algebraic variety over a field k of dimension d, let r be an integer such that $r \leq d$ and let $\big(\xi_1, \xi_2, \ldots, \xi_r\big)$ be a family of points of X such that ξ_t is a specialization of ξ_{t+1}, $1 \leq t \leq r-1$. Then there exists a valuation ν composite with a family $\big(\nu_1, \nu_2, \ldots, \nu_r\big)$, such that the center of the composite valuation $\nu'_{(t)} = \nu_1 \circ \ldots \circ \nu_t$ is the point ξ_t, for $t = 1, \ldots, r$.*

Remark 2.3. Let ν be a valuation of K/k of rank r, where K is the function field of an algebraic variety X over k of dimension d. We assume that the center Z of ν on X is non empty and that we have $\dim Z = d - r$. Then we are in the case of equality of the corollary of the theorem 1.20 for the rank, and we know that we can write the valuation ν as $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$ (cf example 6), where each valuation ν_i is a prime divisor. In that case the center ξ_t of the valuation $\nu'_{(t)} = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_t$ defines a divisor in $\overline{\{\xi_{t+1}\}}$ and the valuation ν is *the composition of the orders of vanishing along these divisors* (cf remark 1.6).

2.2. Riemann variety

Let k be a field, we want to study the set of all the valuations ν of K/k where K is an extension of k, i.e. the set of all the valuations ν of K which are trivial on k.

Definition. ([Za-Sa], Chap.VI, §17, page 110.) *The Riemann variety* or *the Rieman manifold* or *the abstract Riemann surface* of K relative to k is the set of all the valuations ν of K which are trivial on k. We denote this set by $S = S(K/k)$.

More generally we can define the Riemann variety of K/k when k is a subring of K, not necessarily a field. In that case the Riemann variety is the set of all the valuations of K which are not negative on k, i.e. the valuations of K such that the valuation ring V associated to ν contains k.

Remark 2.4. We deduce from the remark 1.10 that the Riemann variety $S(K/k)$ and $S(K/\bar{k})$ are isomorphic, where $\bar{k}$ is the integral closure of k in K, and if K is an algebraic extension of k, the Riemann variety contains one unique element, the trivial valuation.

We give sometimes another definition of the Riemann variety, we consider only the non trivial valuations ν of K which are trivial on k, and we denote this set $S^*(K/k)$, i.e. $S(K/k) = S^*(K/k) \cup \{\nu_0\}$ where ν_0 is the trivial valuation of K. With this definition, if K is an algebraic extension of k, the Riemann variety $S^*(K/k)$ is empty.

We introduce a topology in the Riemann variety $S = S(K/k)$, by defining a basis of open sets.

Definition. Let A be a subring of K containing k, then we denote $E(A)$ the set of all the valuations ν of K/k which are non negative on A, i.e. the set defined by $E(A) = \big\{\nu \in S(K/k) \,/\, A \subseteq V_\nu\big\}$, where V_ν is the valuation ring associated to ν. We define the topology in S by taking as basis of open sets the family of all the sets $E(A)$ where A range over the family of all k-subalgebras of K which are finitely generated over k. We call this topology *the Zariski topology.*

If A and A' are two finitely generated k-subalgebras, we denote $[A, A']$ the subalgebra of K generated by A and A'. This algebra is finitely generated over k and we notice that the intersection $E(A) \cap E(A')$ is equal to $E([A, A'])$. Therefore the intersection of two basic open subsets is again a basic open subset, and hence we have indeed defined a topology in S. Any finitely generated k-subalgebra A of K is of the form $A = k[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ are elements of K. Then we can write the basic open set $E(A) = E(k[x_1]) \cap \dots \cap E(k[x_n])$, and hence the topology in S is generated by the open sets $E(k[x]) = \{\nu \in S(K/k) \,/\, \nu(x) \geq 0\}$, where x range K^*. We may also notice that if A and A' are two k-subalgebras of K with $A \subset A'$, then we have $E(A') \subseteq E(A)$.

Theorem 2.6. *Let ν be a valuation of K/k, then the closure of the set $\{\nu\}$ consisting of the single element ν in S is the set of all the valuations ν' of K/k which are composite with ν:*

$$\overline{\{\nu\}} = \left\{\nu' \in S \;/\; \nu' \textit{ is composite with } \nu\right\}.$$

More precisely the closure $\overline{\{\nu\}}$ is isomorphic to the Riemann variety $S(\kappa/k)$ of the residue field κ of the valuation ν.

Proof. Let ν and ν' be two valuations of K, then ν' is composite with ν if and only if the valuation ring V' associated to ν' is contained in the valuation ring V associated to ν. If ν' is in the closure of $\{\nu\}$, for any finitely generated k-subalgebra A of K we have $\nu' \in E(A) \Longrightarrow \nu \in E(A)$, i.e. $A \subseteq V' \Longrightarrow A \subseteq V$, hence V' is contained in V. Conversely, il ν' is not in the closure of $\{\nu\}$, there exists a finitely generated k-algebra A with $A \subseteq V'$ and $A \nsubseteq V$, hence V' is not contained in V.

We deduce from the proposition 1.12 that the map ϕ of $\overline{\{\nu\}}$ to the Riemann variety $S(\kappa/k)$ of the residue field κ of ν, which sends a composite valuation $\nu' = \nu \circ \bar{\nu}$ to the valuation $\bar{\nu}$ of κ, is a bijection. By definition of the valuation ring $\bar{V}$ associated to $\bar{\nu}$, we see that for all the elements x in the valuation ring V, we have $\nu'(x) \geq 0$ if and only if $\bar{\nu}(\bar{x}) \geq 0$, where $\bar{x}$ is the image of x in κ, hence the map ϕ is an homeomorphism.

Remark 2.5. Let ν_0 be the trivial valuation of K, all the valuations of K are composite with ν_0. The valuation ring associated to ν_0 is the field K, hence the valuation ν_0 belongs to all the non empty open sets $E(A)$, and ν_0 is a generic point of the Riemann variety $S(K/k)$.

Even if we consider the variety $S^*(K/k) = S(K/k) \smallsetminus \{\nu_0\}$, we see that this space is never a Hausdorff space, in the case where k is a field.

Theorem 2.7. ([Za-Sa], Chap.VI, §17, Theorem 40, page 113.) *The Riemann variety $S = S(K/k)$ is quasi-compact, i.e. every open covering of S contains a finite subcovering.*

Proof. We give a sketch of the Chevalley's proof which is exposed with more details in [Za-Sa] or in [Va].
Any valuation ν of K is uniquely determined by its valuation ring, hence to know a valuation it is enough to know the sets of the elements x of K where ν is positive, equal to zero or negative and we can consider the Riemann variety $S = S(K/k)$ as a subset of the set Z^K of the applications of K to $Z = \{+, 0, -\}$.
We define a topology in Z by taking as open sets $\emptyset$, $\{0, +\}$ and Z and we introduce the product topology on Z^K. Then the induced topology in S has for basis of open sets the sets E defined as follows: $E = \{\nu \in S \,/\, \nu(x_i) \geq 0, i = 1, 2, \ldots, r\}$ where $\{x_1, x_2, \ldots, x_r\}$ is a finite subset of K. This definition agree with the preceding definition, hence we can consider the Riemann variety as a subset of the topological space Z^K.
We shall modify temporarily the topology on Z^K, we introduce the discrete topology on Z, then Z is compact and by Tychonoff's theorem the product space Z^K is also compact. With this new topology S becomes closed in Z^K, hence is compact. Since this topology is stronger that the preceding one, we deduce that the Riemann variety is quasi-compact with the Zariski topology.

We shall show that the Riemann variety $S(K/k)$ may be regarded as the projective limit of an inverse system of integral schemes: $S = \varprojlim X_\alpha$. More precisely, if k is a field anf K a function field over k, i.e. a finitely generated extension of k, we define a *model* M of K (over k) as an algebraic variety M over k such that K is the function field of M. We say that M is a complete, resp. projective, model of K if M is a complete, resp. projective, algebraic variety over k.

We call L the set of local k-subalgebras P of K, and for any P we denote $m(P)$ its maximal ideal: $L = \big\{P \,\text{local}\; k\text{-algebra} \,/\, k \subset P \subset K\big\}$. For any k-subalgebra A of K, non necessarily local, we call $L(A)$ the subset of L of the local k-algebras P containing A: $L(A) = \big\{P \in L \,/\, A \subset P\big\}$. Then we define a topology in L such that the set of the $L(A)$, for A ranging the finitely generated k-algebras, is a basis of open sets.
Let A be a finitely generated k-subalgebra of K and let Spec$A$ be the affine scheme associated to A, then we can define a map $f_A \colon L(A) \to$

Spec A by $f_A(P) = m(P) \cap A = \mathfrak{p}$. We have a topology in Spec$A$, the Zariski topology, such that the closed subsets are the sets $V(\mathcal{I}) = \{\mathfrak{p} \in \operatorname{Spec} A \,/\, \mathcal{I} \subset \mathfrak{p}\}$, where $\mathcal{I}$ range the ideals of A. Moreover the closed subset $V(\mathcal{I})$ is isomorphic to the affine scheme Spec$A/\mathcal{I}$.

Proposition 2.8. *The map f_A is continuous from $L(A)$ to* Spec$A$*, and induces an homeomorphism of $V(A)$ into* Spec$A$*, where $V(A)$ is the subset of $L(A)$ defined by $V(A) = \{A_{\mathfrak{p}} \,/\, \mathfrak{p} \in \operatorname{Spec} A\}$.*

Proof. To show that the map f_A is continuous, we have to show that the inverse image of any open subset O of Spec$A$ is open in $L(A)$, and we may consider only the open sets $O = D(x) = \{\mathfrak{p} \in \operatorname{Spec} A \,/\, x \notin \mathfrak{p}\}$, and we recall that $D(x)$ is isomorphic to the affine scheme SpecA_x. We shall see that the inverse image $f_A^{-1}\big(D(x)\big)$ is equal to the open set $L(A_x)$. A local ring P of $L(A)$ belongs to $f_A^{-1}\big(D(x)\big)$ if and only if the prime ideal $\mathfrak{p} = m(P) \cap A$ doesn't contain x, i.e. x doesn't belong to the maximal ideal $m(P)$, and as x belongs to $A \subset P$ and P is local this equivalent to demand to x^{-1} to belong to P, hence to demand to A_x to be contained in P.

By definition the map f_A induces a bijection of the subset $V(A)$ into Spec$A$ and we have to show that f_A identify the topology in $V(A)$ induced by the topology of L to the Zariski topology in Spec$A$. Any open set in $V(A)$ is a finite intersection of sets $O(x)$ of the following type $O(x) = \{P \in V(A) \,/\, x \in P\}$, for x a non zero element of the fraction field of A, and it is enough that the set $f_A\big(O(x)\big)$ is open in Spec$A$. In fact we see that this set is the complementary in Spec$A$ of the closed subset $V(\mathcal{I})$ where $\mathcal{I}$ is the ideal $\mathcal{I} = (A : x) = \{c \in A \,/\, cx \in A\}$. An element x of the fraction field of A belongs to the local ring $A_{\mathfrak{p}}$ if and only if we have $x = a/b$ with $a \in A$ and $b \in A \setminus \mathfrak{p}$, i.e. if and only if there exists b with $b \in \mathcal{I}$ and $b \notin \mathfrak{p}$.

Remark 2.6. If M is a model of the field K, i.e. if M is an algebraic variety over k with function field K, then we can associate to any point x of M the local ring $\mathcal{O}_{M,x}$ in L. By the preceding proposition, the map f defined by $f(x) = \mathcal{O}_{M,x}$ is a homeomorphism of M, with the Zariski topology, into a subset of L.

In the same way, if we associate to any valuation ν of the Riemann variety $S = S(K/k)$ the valuation ring $V = V_\nu$, we see that S is a subset of L. And by definition we see that the topology in L induces the Zariski topology in S.

Let A be an integral k-algebra, finitely generated over k and with fraction field K, then the set of valuations ν of the Riemann variety $S(K/k)$ which have a non empty center on the affine scheme $X = \operatorname{Spec} A$

is equal to the open set $E(A)$. We can define a map g_A of $E(A)$ to X by $g_A(\nu) = x$ where $x = \mathfrak{p}$ is the center of ν on X, and as the center $\mathfrak{p}$ is by definition equal to $A \cap m(V)$, where $m(V)$ is the maximal ideal of the valuation ring V, this map is the restriction of the map f_A, hence is continuous. More generaly, we have the following result.

Proposition 2.9. *Let X be an algebraic variety over k with function field K. The set of valuations ν of K/k which have a non empty center on X is an open set $U(X)$ of the Riemann variety $S(K/k)$ and the map g_X which associate to any valuation ν of $U(X)$ its center x_ν on X is continuous. Moreover, the variety X is complete if and only if the open set $U(X)$ is equal to the whole Riemann variety, and we get a continuous map $g_X : S(K/k) \longrightarrow X$.*

Proof. We deduce from the proposition 2.1 that the center of a valuation ν of K/k on X is well defined, and from the theorem 2.2 that all the valuations ν of K/k have a center on X if and only if the variety X is complete over k.
We can write the algebraic variety X as the union of a finite number of affine open sets $X = \bigcup_{i=1}^n X_i$, where $X_i = \mathrm{Spec} A_i$ and A_i is a finitely generated k-algebra with fraction field K. Then the subset $U(X)$ of the valuations ν in $S(K/k)$ which have a center on the variety X is the union of the open subsets $E\big(A_i\big)$, hence $U(X)$ is open in $S(K/k)$. The restriction of the map g_X on each subset $E\big(A_i\big)$ is the continuous map f_{A_i}, hence the map g_X is also continuous of $U(X)$ to X.

Let X and X' be two algebraic varieties over k, with the same function field K, and let $h \colon X' \longrightarrow X$ be a birational morphism of X' to X. Let ν be a valuation of K/k belonging to the open set $U(X')$, then there exits a birational morphism f' of $T = \mathrm{Spec} V$ to X', where V is the valuation ring associated to ν, and the image ξ' of the closed point t of T is the center of ν on X'. Then the composite morphism $f = h \circ f'$ is a birational morphism of $T = \mathrm{Spec} V$ to X, hence the valuation ν has also a center on X, i.e. the valuation ν belongs to the open set $U(X)$, and this center $\xi = f(t)$ is equal to the image $h(\xi')$ of the center of ν on X'.
We can also notice that for any point x' of X', its image $x = h(x')$ is the point of X such that the local ring $\mathcal{O}_{X,x}$ is dominated by the local ring $\mathcal{O}_{X',x'}$. Then if ξ' is the center of the valuation ν on X', we have the local ring $\mathcal{O}_{X'.\xi'}$ which is dominated by V, and since the relation of domination is transitive, the local ring $\mathcal{O}_{X,\xi}$ is dominated by V, where $\xi = h(\xi')$, i.e. ξ is the center of ν on X.

We have shown that the open set $U(X')$ is contained in the open set $U(X)$ and that the restriction of the map g_X to $U(X')$ is equal to $h \circ g_{X'}$.

Moreover, if the morphism $h\colon X' \longrightarrow X$ is proper, for any valuation ν of K/k having a center ξ on X, we can apply the valuative criterion of properness to the following commutative diagram:

$$\begin{array}{ccc} U = \operatorname{Spec} K & \xrightarrow{j} & X' \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle h} \\ T = \operatorname{Spec} V & \xrightarrow{f} & X \end{array}$$

where the morphism $f: T \longrightarrow X$ is defined by the existence of the center of the valuation on X. Then we deduce the existence of a morphism $f': V \longrightarrow X'$ and the image ξ' of the closed point t of T is the center of the valuation ν on X' and $h(\xi')$ is equal to ξ. Hence, if the birational morphism $h\colon X' \longrightarrow X$ is proper, any valuation ν having a center on X has also a center on X', i.e. the open sets $U(X)$ and $U(X')$ are equal.

We have proven the following result.

Proposition 2.10. *Let X and X' be two algebraic varieties over a field k, with the same function field K. If there exists a birational morphism $h\colon X' \longrightarrow X$, then we have the inclusion $U(X') \subset U(X)$ in the Riemann variety $S(K/k)$.*
Moreover, the morphism $h\colon X' \longrightarrow X$ is proper if and only if we have equality $U(X') = U(X)$.

Let X be an algebraic variety over k, with function field K, and let $U = U(X)$ the open subset of the Riemann variety $S = S(K/k)$ of the valuations ν having a center on X. For any algebraic variety Y such that there exists a proper birational morphism $h_Y: Y \longrightarrow X$, the open subset $U(Y)$ of S is equal to U and the continuous map $g_Y: U \longrightarrow Y$ satisfies $g_X = h_Y \circ g_Y$. Let Y and Y' be two algebraic varieties with proper birational morphisms $h_Y: Y \longrightarrow X$ and $h_{Y'}: Y' \longrightarrow X$, we denote $Y \prec Y'$ if there exists a morphism $h_{Y',Y}$ of Y' to Y such that $h_{Y'} = h_Y \circ h_{Y',Y}$, in that case the morphism $h_{Y',Y}$ is also proper birational. We call $\mathcal{D}$ the inverse system of the (Y, h_Y) with the relation $\prec$ and we may define the projective limit

$$\mathcal{X} = \varprojlim_{\mathcal{D}} Y\,.$$

This projective limit is the subset of the product space $\prod_{\mathcal{D}} Y$ of the elements $\bar{x} = \big(x_Y\big)$ such that $h_{Y',Y}\big(x_{Y'}\big) = x_Y$ for any couple (Y, Y') with $Y \prec Y'$. We introduce in $\mathcal{X}$ the topology induced by the product topology on $\prod_{\mathcal{D}} Y$, and the natural maps $t_Y: \mathcal{X} \longrightarrow Y$ are continuous.

Theorem 2.11. ([Za-Sa], Chap.VI, §17, Theorem 41, page 122.)
There exists a natural homeomorphism $g\colon U \longrightarrow \mathcal{X}$ of the open subset

$U = U(X)$ of the Riemann variety $S(K/k)$ to the projective limit $\mathcal{X} = \varprojlim_{\mathcal{D}} Y$. Hence the Riemann variety $S(K/k)$ may be identified with the projective limit of the inverse system $\mathcal{C}$ of the complete algebraic varieties Z over k, with function field K: $S = \varprojlim_{\mathcal{C}} Z$.

Remark 2.7. The algebraic varieties Y of the inverse system $\mathcal{D}$ are models of the function field K, and if we have two algebraic varieties Y and Y' in $\mathcal{D}$ with $Y \prec Y'$, we say that Y' dominates Y. To calculate the projective limit $\mathcal{X} = \varprojlim_{\mathcal{D}} Y$ of the inverse system $\mathcal{D}$, we can consider a cofinal subset $\mathcal{D}'$ of $\mathcal{D}$, i.e. a subfamily $\mathcal{D}'$ of $\mathcal{D}$ such that for any Y in $\mathcal{D}$ there exists an element Z in $\mathcal{D}'$ with $Y \prec Z$.
The algebraic varieties of the system $\mathcal{C}$ are the complete models of K, and the Chow lemma says that the inverse system of projective models $\mathcal{P}$ is cofinal in the inverse system of complete models, hence we have also the equality $S(K/k) = \varprojlim_{\mathcal{P}} P$.

Remark 2.8. The complete algebraic varieties are quasi-compact topological spaces, hence we could deduce the quasi-compacity of the Riemann variety as projective limit of quasi-compact spaces, but to prove the theorem 2.11 we use the quasi-compacity of $S(K/k)$.

Proof. For every couple (Y, Y') of the inverse system $\mathcal{D}$ with $Y \prec Y'$, the maps $g_Y \colon U \to Y$ and $g_{Y'} \colon U \to Y'$ are continuous and satisfy $g_Y = h_{Y',Y} \circ g_{Y'}$. Hence we obtain a continuous map g of the open subset U of S in the projective limit $\mathcal{X}$, such that $t_Y \circ g = g_Y$ on U.
We shall show that this map $g \colon U \longrightarrow \mathcal{X}$ is onto. Let $\bar{x} = (x_Y)$ be a point in $\mathcal{X}$ and let R_Y the local ring of the point $x_Y = t_Y(\bar{x})$, $R_Y = \mathcal{O}_{Y,x_Y}$. Since for $Y \prec Y'$ the local ring R_Y is dominated by $R_{Y'}$, the ring $R = \bigcup_{Y \in \mathcal{D}} R_Y$ is a local ring, contained in K, with maximal ideal $max(R) = \bigcup_{Y \in \mathcal{D}} max(R_Y)$. There exists a valuation ring V, associated to a valuation ν of K/k, which dominates the local ring R. For all the Y in $\mathcal{D}$ the valuation ring dominates also the local rings $R_Y = \mathcal{O}_{Y,x_Y}$, then the center of the valuation ν on Y is x_Y, i.e. ν belongs to the open subset U and its image by g_y is x_Y, hence the image of ν by the map g is the point $\bar{x}$.
To show that the map $g \colon U \longrightarrow \mathcal{X}$ is injective, we shall show that for any point $\bar{x} = (x_Y)$ of $\mathcal{X}$, the local ring $R = \bigcup_{Y \in \mathcal{D}} R_Y$ defined by $R_Y = \mathcal{O}_{Y,x_Y}$, is a valuation ring of K. Let w be an element of K, and we have to show that either w, either w^{-1} belongs to the ring R. We can write $w = u/v$ with u and v in R, and there exist Y' and Y'' in $\mathcal{D}$ such that $u \in R_{Y'}$ and $v \in R_{Y''}$, and since there exists Y in $\mathcal{D}$ with $Y' \prec Y$ and $Y'' \prec Y$, we may assume that u and v belong to the same local ring R_Y. Let $\mathcal{I}$ be a sheaf of ideals on the variety Y such that $\mathcal{I}_{Y,x_Y}$ is equal

to the ideal (u, v) of the local ring $R_Y = \mathcal{O}_{Y,x_Y}$ and let $r\colon Z \longrightarrow Y$ be the blowing up of center $\mathcal{I}$ in Y. Then Z belongs to the inverse system $\mathcal{D}$ and let x_Z be the point in Z with $x_Z = t_Z(\bar{x})$. By definition the ideal $\mathcal{I}R_Z$ is principal, i.e. is generated by one of the elements u or v. If $\mathcal{I}R_Z$ is generated by u, then $w^{-1} = v/u$ belongs to R_Z anf if $\mathcal{I}R_Z$ is generated by v, $w = u/v$ belongs to R_Z, hence we deduces that w or w^{-1} belongs to R.
We have to prove that the map $g\colon U \longrightarrow \mathcal{X}$ is closed. We deduce from the proposition 2.8 that the maps $g_Y : U \longrightarrow Y$ are closed, then the map $g\colon U \longrightarrow \mathcal{X}$ is also closed because for any closed subset F of U we have $g(F) = \mathcal{X} \cap \left(\prod_{Y\in\mathcal{D}} g_Y(F)\right)$.

Proposition 2.12. *A valuation ν of K/k is a closed point of the Riemann variety $S(K/k)$ if and only if the residue field κ of ν is an algebraic extension of k, i.e. if and only if the valuation ν is zero-dimensional.*

Proof. The valuation ν is a closed point of $S(K/k)$ if and only if $\overline{\{\nu\}}$ is reduced to one point, hence from the theorem 2.6, if and only if the Riemann variety $S(\kappa/k)$ of the valuations of the residue field κ of ν which are trivial on k, contains one element, and we deduce from the remark 2.4 that this is equivalent to demand to κ to be an algebraic extension of the field k. By definition of the dimension of a valuation ν of K/k, this also equivalent to say that the dimension of ν is zero.

§3. Uniformization and resolution of singularities

3.1. The general problem

Let X be a scheme, a point x of X is said *non-singular*, or *simple*, if the local ring $\mathcal{O}_{X,x}$ is a regular ring. If we assume that X is an excellent scheme, for instance if X is a scheme of finite type over a field K, and if X is reduced, the set of all the non-singular points of X is a dense open subset X_{reg} of X. We say that the scheme X is *non-singular* if all the points x of X are non-singular, i.e. if $X = X_{reg}$. Hence all the connected components of X are irreducible. By definition *a resolution of singularities* of a reduced scheme X is a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ of a non-singular scheme $\tilde{X}$ onto X, which induces an isomorphism over the non-singular open subset X_{reg} of X. We may also demand more conditions on the morphism π, for instance that the exceptional locus, i.e. the closed subset E in $\tilde{X}$ where π is not an isomorphism, $E = \tilde{X} \setminus \pi^{-1}(X_{reg})$, is a normal crossings divisor in $\tilde{X}$, or that π is a composition of blowups in regular centers.
If we assume that X is an algebraic variety over a field k, any resolution

of singularities $\pi\colon \tilde{X} \longrightarrow X$ of X will give a non-singular model $\tilde{X}$ of the function field $K = F(X)$ of X, i.e. a non-singular algebraic variety $\tilde{X}$ over k with function field $F(\tilde{X})$ equal to K. Moreover, if the variety X is complete, the variety $\tilde{X}$ is also complete because the morphism π is proper, then we get a complete non-singular model $\tilde{X}$ of the function field K. Then we may also define a problem, which is weaker than the resolution of singularities, by the following:

let K be a function field over a field k, then does there exist a complete non-singular model Y of K?

The stategy of Zariski to solve the problem of the resolution of singularities of an algebraic variety X over a field k, with function field $F(X) = K$, is to study all the valuations of K/k, which belong to the open subset $U(X)$ of the Riemann variety $S(K/k)$, and to try to find for each valuation ν of $U(X)$ a model Y of K such that the center ξ of ν on Y is a non-singular point, i.e. such that the local ring $\mathcal{O}_{Y,\xi}$ is a regular ring. This is this problem we call *the local uniformization of a valuation ν.*
We may notice that there are also two ways to define the problem of the uniformization, one we call the abstract form, or the invariantive form in Zariski's terminology, and one we call the strong form, or the projective form in Zariski's terminology ([Za 2]).

Uniformization problem in the abstract form. *Let K be a function field over a field k and let ν be a valuation of K/k, then does there exist a complete model V of K over k on wich the center ξ of the valuation ν is a non-singular point?*

Uniformization problem in the strong form. *Let X be an algebraic variety over a field k, with function field K, and let ν be a valuation of K/k which belongs to the open subset $U(X)$ of the Riemann variety $S(K/k)$, then does there exist a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ of an algebraic variety $\tilde{X}$ onto X, such that the center $\tilde{\xi}$ of the valuation ν on $\tilde{X}$ is a non-singular point?*

Remark 3.1. Zariski gives a different definition of the uniformization problem in the strong form. He considers a model X of the field K and a valuation ν of K/k with center ξ on X and he wants to find a new model $\tilde{X}$ of K such that the center $\tilde{\xi}$ of ν on $\tilde{X}$ is a non-singular point and such that the local ring $\mathcal{O}_{X,\xi}$ is contained in the local ring $\mathcal{O}_{\tilde{X},\tilde{\xi}}$. Since the two local rings $\mathcal{O}_{X,\xi}$ and $\mathcal{O}_{\tilde{X},\tilde{\xi}}$ are dominated by the valuation ring V associated to the valuation ν, we have $\mathcal{O}_{X,\xi}$ contained in $\mathcal{O}_{\tilde{X},\tilde{\xi}}$ if and only if $\mathcal{O}_{X,\xi}$ is dominated by $\mathcal{O}_{\tilde{X},\tilde{\xi}}$. Hence if there exists a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ the local ring $\mathcal{O}_{X,\xi}$ is contained in the local ring $\mathcal{O}_{\tilde{X},\tilde{\xi}}$, and conversely if the local ring $\mathcal{O}_{X,\xi}$ is contained in the local ring $\mathcal{O}_{\tilde{X},\tilde{\xi}}$, there exists locally in a neighbourhood of $\tilde{\xi}$ a birational morphism π of $\tilde{X}$ to X with $\pi(\tilde{\xi}) = \xi$.

To see how the uniformization problem is a step to get the resolution of singularities, we need the following result.

Proposition 3.1. (cf. [Za 2], Chap.II, §5, , page 855.) *Let X be an algebraic variety over a field k, with function field K, then the set of valuations ν of K/k wich have a center ξ on X which is a non-singular point of X is an open subset of the Riemann variety $S(K/k)$.*

Proof. The set V of valuations ν of K/k wich have a non-singular center ξ on X is a subset of the open subset $U = U(X)$ of valuation which have a center on X, and to prove that V is open we have to show that V is stable under generalization, i.e. that for any valuation ν in V and for any valuation μ in U with $\nu \in \overline{\{\mu\}}$, we have μ which belongs to V. If ξ and ζ are the centers on X respectively of the valuations ν and μ, then ζ is again a generalization of ξ, because the map $g_X : U \longrightarrow X$ is continuous. Since the subset of non-singular points of an algebraic variety is an open subset, we see that if ν has a non-singular center ξ on X, then μ has also a non-singular center ζ on X.

Corollary. *The uniformization theorem for zero-dimensional valuations implies the uniformization theorem for all the valuations of K/k.*

Proof. By the proposition it is enough to show that for any valuation μ of K/k, there exists a zero-dimensional valuation ν such that $\nu \in \overline{\{\mu\}}$. If the valuation μ is of dimension $d > 0$, then by definition deg.tr.κ/k is positive and there exists a zero-dimensional valuation $\bar{\nu}$ of κ/k. Hence the composite valuation $\nu = \mu \circ \bar{\nu}$ is a zero-dimensional valuation of K/k which belongs to $\overline{\{\mu\}}$.

However, to prove the uniformization theorem we do not prove the result for zero-dimensional valuations and then use the corollary to get the result for all the valuations of K/k, i.e. we don't uniformize a zero-dimensional valuation $\nu = \mu \circ \bar{\nu}$ to get the uniformization of the valuation μ. We do the converse, we first uniformize the valuation μ and we then use this result to get the uniformization theorem for the valuations ν which are composite with μ. The reason for this is that we get the proof by induction on the rank of the valuations and we have seen that for $\nu = \mu \circ \bar{\nu}$ we have rank(ν) = rank(μ)+ rank$(\bar{\nu})$ by proposition 1.10 ([Za 2], Chap.III, §7, , page 857).

We may enonce the local version of the uniformization theorem in the strong form in the following way. Let R be an integral finitely generated k-algebra, $R = k[x_1, x_2, \ldots, x_n]$ and let ν be a valuation of K, where K is the fraction field of R, $K = Fr(R)$, with $\nu(x) \geq 0$ for all the elements x in R, i.e. we assume that the valuation ν has a center on R. We denote V the valuation ring associated to ν, $\mathfrak{m}$ its maximal ideal and $\mathfrak{p} = R \cap \mathfrak{m}$ the center of ν on R. Then there exists a finitely generated k-algebra S, with fraction field $Fr(S) = K$, i.e. we have $S = R[u_1, u_2, \ldots, u_t]$ with $u_i \in K$ for $i = 1, 2, \ldots t$, such that the center $\mathfrak{q} = S \cap \mathfrak{m}$ of ν on S is regular, i.e, such that the local ring $S_\mathfrak{q}$ is regular, and such that the local ring $R_\mathfrak{p}$ is contained in the local ring $S_\mathfrak{q}$. Since the local rings $R_\mathfrak{p}$ and $S_\mathfrak{q}$ are dominated by the valuation ring V, we have also $R_\mathfrak{p}$ dominated by $S_\mathfrak{q}$ and the inclusion $R_\mathfrak{p} \subset S_\mathfrak{q}$ induces a birational correspondence $\pi\colon \mathrm{SpecS} \dashrightarrow \mathrm{Spec}R$ which is defined in a neighbourhood of $\mathfrak{q}$ and with $\pi(\mathfrak{q}) = \mathfrak{p}$. We may replace the ring S by $S^* = S[v_1, v_2, \ldots, v_s]$ in such a way that the ring S^* is a regular ring and R is contained in S^*. This ring S^* corresponds to a non-singular affine open subvariety $U = \mathrm{Spec}S^*$ of the affine variety $\mathrm{Spec}S$, which contains the non-singular point $\mathfrak{q}$, a such subvariety U exists because the set of non-singular points of $\mathrm{Spec}S$ is open. Then we get a birational morphism $\pi^*\colon \mathrm{Spec}S^* \longrightarrow \mathrm{Spec}R$ with $\mathrm{Spec}S^*$ a non-singular affine algebraic variety and such that the valuation ν has a center on $\mathrm{Spec}S^*$.
Let d be the dimension of the ring R, then d is equal to the transcendence degree of the fraction field K over k. We can find d elements $\xi_1, \xi_2, \ldots, \xi_d$ of K algebraically independent over k, and the field K is an algebraic extension of $k(\xi_1, \xi_2, \ldots, \xi_d)$. Let X be the affine algebraic variety associated to the k-algebra R. If we write $R = k[X_1, X_2, \ldots, X_n]/I$, where $X_1, X_2, \ldots, X_n$ are algebraically independent over k, then X is the closed subvariety of the n-dimensional affine space $\mathbf{A}_k^n$ defined by the ideal I of the polynomial ring $k[X_1, X_2, \ldots, X_n]$. We say that the k-algebra R is *an hypersurface ring* if we may write $R = k[x_1, x_2, \ldots, x_{d+1}]$ with $d = \dim R$, i.e. if the affine variety X associated to R is an hypersurface in the affine space $\mathbf{A}_k^{d+1}$. In that case the ideal I is generated by only one element, $I = (f)$.

Proposition 3.2. (cf. [Za 2], Chap.IV, §9, , page 858.) *Let k be a field of characteristic zero and let K be a function field over k. If the uniformization problem is resolved for all the k-algebras R with fraction field K which are hypersurface rings, then it is resolved for any k-algebra with fraction field K.*

Proof. Let $R = k[x_1, x_2, \ldots, x_n]$ be an integral k-algebra with fraction field K and let ν be a valuation of K/k with valuation ring V, we assume that ν is non negative on R, i.e. $R \subset V$ and let $\mathfrak{p} = R \cap \mathfrak{m}$ be the center of ν on R. By the Emmy Noether normalization theorem there exists d elements $y_1, y_2, \dots y_d$, with $d = \dim R = \text{tr.deg.} K/k$, such that R is integral over the k-algebra $k[y_1, y_2, \dots, y_d]$. Then K is a finite extension of $L = k(y_1, y_2, \dots, y_d)$, and since the characteristic of k is zero, there exists z in K with $K = L(z)$ and we may assume $z \in R$. Let R^* be the k-algebra $R^* = k[y_1, y_2, \dots, y_d, z]$, then R^* satisfies $R^* \subset R$, $Fr(R^*) = Fr(R) = K$ and R is integral over R^*. Moreover, by construction R^* is an hypersurface ring. Let $\mathfrak{p}^*$ be the center of the valuation ν on R^*, $\mathfrak{p}^* = R^* \cap \mathfrak{m}$, and by hypothesis there exists an uniformization of ν over R^*, i.e. a k-algebra S with $R^* \subset S$, $Fr(S) = Fr(R^*) = K$, and such that the center $\mathfrak{q} = S \cap \mathfrak{m}$ of ν on S is non-singular. Since the ring $S_\mathfrak{q}$ is regular, $S_\mathfrak{q}$ is integrally closed in its fraction field K, then we get also $R_\mathfrak{p} \subset S_\mathfrak{q}$ and S is a uniformization of ν over R.

The most important result on the uniformization problem is the theorem of Zariski for algebraic varieties over a field of characteristic zero. For varieties over a field of positive characteristic, we have the theorem for the dimensions $d \leq 3$, and there are also results for some special valuations ([Kn-Ku]).

Uniformization theorem. ([Za 2]) *Let X be an algebraic variety over an arbitrary ground field k of characteristic zero, with function field K, and let ν be a valuation of K/k which belongs to the open subset $U(X)$ of the Riemann variety $S(K/k)$, then there exists a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ of an algebraic variety $\tilde{X}$ onto X, such that the center $\tilde{\xi}$ of the valuation ν on $\tilde{X}$ is a non-singular point.*

Now, if we assume that we have the uniformization theorem we shall see that the resolution is a consequence of *a gluing problem of a finite number of local uniformizations.* More precisely, let X be a complete algebraic variety over a field k, with function field K, and we assume that for any valuation ν of K/k, there exists a proper birational morphism $\pi(\nu)\colon \tilde{X}(\nu) \longrightarrow X$ such that the center $\xi(\nu)$ of ν on $\tilde{X}(\nu)$ is a non-singular point. By the proposition 3.1, there exists an open subset $V(\nu)$ of the Riemann variety $S(K/k)$, such that the center of any valuation μ in $V(\nu)$ is also a non-singular point of $\tilde{X}(\nu)$. By the quasi-compacity of the Riemann variety (theorem 2.7), there exists a finite number of valuations $\nu_1, \nu_2, \dots, \nu_t$ of K/k such that the family $V(\nu_1), V(\nu_2), \dots, V(\nu_t)$ is a covering of the Riemann variety $S(K/k)$. We have obtained a finite family of proper birational morphisms $\pi_i\colon \tilde{X}_i \longrightarrow X$, $i = 1, 2, \dots, t$, such that for any valuation ν of K/k there exists i such that ν has a non-singular center on $\tilde{X}_i$. Hence, the problem of the resolution of singularities is reduced to the following gluing problem. Let X be an algebraic variety, and let $\pi_1 \colon \tilde{X}_i \longrightarrow X$ and $\pi_2 \colon \tilde{X}_2 \longrightarrow X$ be two proper birational morphisms. Then, does there exist a variety Y and two proper birational morphisms $\rho_1 \colon Y \longrightarrow \tilde{X}_1$ and $\rho_2 \colon Y \longrightarrow \tilde{X}_2$, with $\pi_1 \circ \rho_1 = \pi_2 \circ \rho_2$, and such that the open subset Y_{reg} of non-singular points of Y satisfies ${\rho_1}^{-1}\big(\tilde{X}_{1reg}\big) \cup {\rho_2}^{-1}\big(\tilde{X}_{2reg}\big) \subset Y_{reg}$.

The Zariski proof of resolution of singularities of surfaces over an algebraically closed field k of characteristic zero is the first algebraic or arithmetic proof. He used the uniformization theorem and the theory of integrally closed ideals to show that we can obtain the resolution of singularities by a finite sequence of normal blowing ups ([Za 1]. Zariski give a new proof of the resolution of singularities for surfaces and later a proof of resolution of singularities for three dimensional varieties over an arbritary ground field of characteristic zero by using the method of the uniformization of valuations and by showing that it is possible to solve the gluing problem ([Za 3], [Za 4]).

3.2. The case of algebraic surfaces

In this section, we give an idea of the Zariski proof of the uniformization theorem for surfaces over an algebraically closed field of characteristic zero ([Za 1]).

Let k be an algebraically closed field of characteristic zero, let K be a function field over k with transcendence degree $d = 2$. First of all we shall study all the valuations ν of K which are trivial on the ground field k. Let ν be a non trivial valuation of K/k, we recall that we have the inequalities (remark 1.14):

$$\mathrm{rank}(\nu) \ + \ \dim(\nu) \ \leq \ \mathrm{rat.rank}(\nu) + \ \dim(\nu) \ \ \leq \ \mathrm{tr.deg.}K/k \ = 2\,,$$

where the dimension of the valuation is the transcendence degree of the residue field κ of ν over the ground field k. Since the valuation ν is non trivial its rank is positive, then we have the four following possibilities:

$$\begin{array}{rllllll} i) & \mathrm{rank}(\nu) = 1 & = & \mathrm{rat.rank}(\nu) = 1 & \text{and} & \dim(\nu) = 1\,. \\ ii) & \mathrm{rank}(\nu) = 1 & = & \mathrm{rat.rank}(\nu) = 1 & \text{and} & \dim(\nu) = 0\,; \\ iii) & \mathrm{rank}(\nu) = 1 & < & \mathrm{rat.rank}(\nu) = 2 & \text{and} & \dim(\nu) = 0\,; \\ iv) & \mathrm{rank}(\nu) = 2 & = & \mathrm{rat.rank}(\nu) = 2 & \text{and} & \dim(\nu) = 0\,; \end{array}$$

We are going to give a description of the valuation ν in the four cases.

Remark 3.2. Since K is finitely generated over the ground field k, we deduce from the corollary of the theorem 1.20 that in the cases i),

iii) and *iv*) the value group Γ of the valuation ν is finitely generated over $\mathbb{Z}$ and the residue field κ is finitely generated over k, and moreover in the cases *i*) and *iv*) the value group Γ is discrete, i.e. is isomorphic to $(\mathbb{Z}, +)$ or to $(\mathbb{Z}^2, +)_{\text{lex}}$.
Moreover since we have assumed that the field k is algebraically closed, in the cases *ii*), *iii*) and *iv*) the residue field κ, which is algebraic over k, is equal to k.

i) $\mathbf{rank}(\nu) = \mathbf{rat.rank}(\nu) = 1,\ \mathbf{dim}(\nu) = 1$

The valuation ν is a prime divisor of the function field K (cf example 5). If X is a model of the field K such that the valuation ν has a center ξ on X, the dimension of the center $Z = \overline{\{\xi\}}$ is either zero, i.e. ξ is a closed point of X, either a curve, which we may call an *algebraic arc* on X.
Let $U = \mathrm{Spec} R$ be an affine open neighbourhood of ξ in X, then R is contained in the valuation ring V associated to ν, and let $\mathfrak{p}$ the center of ν in R, i.e. the prime ideal of R corresponding to ξ. There exists an affine normal model $Y = \mathrm{Spec} S$ of K, i.e. S is a finitely generated k-algebra, integrally closed in its fraction field K, and there exists a prime ideal $\mathfrak{q}$ of S with height one, such that $R \subset S$ and $S_{\mathfrak{q}} = V$. Hence the valuation ν is the $\mathfrak{q}$-adic valuation, and we may write $\nu(f) = \mathrm{order}_{\mathfrak{q}}(f)$ for any element f in S.
The residue field κ of the valuation ν is a function field of transcendence degree one over the ground field k. Let C be the center of the valuation ν on $\mathrm{Spec} S$, i.e. let C be the affine algebraic curve defined by the prime ideal $\mathfrak{q}$, then the residue field of the valuation is the function field of C: $\kappa = F(C)$.

ii) $\mathbf{rank}(\nu) = \mathbf{rat.rank}(\nu) = 1,\ \mathbf{dim}(\nu) = 0$

We consider first the case where the valuation ν is discrete, i.e. that its value group Γ is isomorphic to $\mathbb{Z}$, and we may assume $\Gamma = \mathbb{Z}$. Let u be an element of the field K such that $\nu(u) = 1$. Then for any element x in K, $x \neq 0$, we have $\nu(x) = n_0$ with $n_0 \in \mathbb{Z}$, hence $\nu(x/u^{n_0}) = 0$. Since the residue field κ of the valuation ν is equal to k, there exists a uniquely determined element c_0 of k such that x/u^{n_0} and c_0 have the same image in κ, i.e. such that $\nu(x/u^{n_0} - c_0) > 0$. Hence we may define x_1 in K such that:

$$x = c_0 u^{n_0} + x_1, \quad \text{with } \nu(x_1) = n_1 > n_0 \ .$$

By induction we may construct uniquely determined sequences (c_i) in k, (n_i) in $\mathbb{Z}$ and x_i in K by:

$$x = c_0 u^{n_0} + c_1 u^{n_1} + c_2 u^{n_2} + \ldots + c_{i-1} u^{n_{i-1}} + x_i \text{ with } \nu(x_i) = n_i > n_{i-1} \ .$$

Let $\xi(x) = \sum_{i \geq 0} c_i u^{n_i}$ be the power series expension for x, hence the map $x \mapsto \xi(x)$ defines an injective morphism of K to the field $k((u))$ of integral power series of u with coefficients in k. The restriction to K of the u-adic valuation of $k((u))$, i.e. the valuation by the order in u, is the valuation ν.
If we choose any model X of K such that the valuation ν has a center on X, for instance if we choose a complete model of K, then the center of the valuation is a closed point p of X because the dimension of the center is always non greater than the dimension of the valuation. In a neighbourhood of the center p, we may choose coordinates $(x_1, x_2, \dots, x_n)$, i.e. we choose an affine neighbourhood $U = \text{Spec } k[x_1, x_2, \dots, x_n]$, with $\nu(x_i) \geq 0$ for $i = 1, 2, \dots, n$, and we may write the power series expensions for these coordinates:

$$x_i = c_{0,i} u^{n_{0,i}} + c_{1,i} u^{n_{1,i}} + c_{2,i} u^{n_{2,i}} + \dots \quad 1 \leq i \leq n \ .$$

These expensions represent an *analytic arc* on U which is not algebraic, i.e. which is not supported by an algebraic curve in U.

We consider now the case where the value group Γ is not discrete, by hypothesis we may assume $\mathbb{Z} \subset \Gamma \subset \mathbb{Q}$. There exists a family of prime numbers $P = \{p_i\}$, which may be finite or infinite, and for any prime number p_i in P a number n_i, $1 \leq n_i \leq \infty$ such that the value group Γ consists of all the rational numbers whose denominators are of the form $p_1^{a_1} p_2^{a_2} p_3^{n_3} \dots$ with $0 \leq a_i \leq n_i$ for any i.
Since Γ is non discrete, the denominators of the elements of Γ are not bounded, hence if all the numbers n_i are finite, the family P must be infinite.

iii) $\mathbf{rank}(\nu) = 1$, $\mathbf{rat.rank}(\nu) = 2$, $\mathbf{dim}(\nu) = 0$

Since the field K is finitely generated over k, we deduce from the corollary of the theorem 1.20 that the value group Γ is finitely generated subgroup of $\mathbb{R}$ with $\dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) = 2$. Then the group Γ is generated by two elements linearly independent over $\mathbb{Q}$ and we may assume $\Gamma = \mathbb{Z} \oplus \beta\mathbb{Z}$, with $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \geq 0$.
Let x and y be elements of K with $\nu(x) = 1$ and $\nu(y) = \beta$, then x, y are algebraically independent over k and K is a finite extension of the field $K^* = k(x, y)$. We denote R^* the polynomial ring $R^* = k[x, y]$ and ν^* the restriction of the valuation ν to K^*, since $\beta \notin \mathbb{Q}$ the value $\nu^*(x^i y^j) = i + j\beta$ is equal to $\nu^*(x^{i'} y^{j'}) = i' + j'\beta$ if and only if $(i, j) = (i', j')$, then the valuation ν^* is defined on the polynomial ring R^* by the following: for any polynomial $f = \sum_{i,j} a_{i,j} x^i y^j$ in R^*, we have $\nu^*(f) = \min\{i + j\beta \ / \ a_{i,j} \neq 0\}$.

The valuation ν^* on K^* is obtained by putting formally $y = x^\beta$, i.e. we may define an injective morphism of K^* in the field $k((x^\Gamma))$ of power series of x with exponents in the group Γ (cf example), and the valuation ν^* is the restriction of the natural valuation μ defined on $k((x^\Gamma))$ to the field K^*. The equation $y = x^\beta$ represents formally an arc on the affine plane $\mathbf{A}^2_k = \mathrm{Spec}R^*$, which we call a *transcendental branch* on $\mathbf{A}^2_k$.

iv) $\mathbf{rank}(\nu) = \mathbf{rat.rank}(\nu) = 2,\ \mathbf{dim}(\nu) = 0$

The valuation ν is composite of prime divisors: $\nu = \nu' \circ \bar{\nu}$, the valuation ν' is a prime divisor of K and the valuation $\bar{\nu}$ is a prime divisor of the residue field κ' of ν', which a function field of transcendence degree one over k (cf example 6).
For the valuation ν' we are in the case i), and we can consider an affine normal model $Y = \mathrm{Spec}S$, with S integrally closed in its fraction field K and such that the center $\mathfrak{q}$ of ν' in S is a height one prime ideal. Then the residue field κ' is equal to the function field of the algebraic curve C and the valuation $\bar{\nu}$ is a valuation of $F(C)$ whose center ξ is also the center of the valuation ν. The local ring $S_\mathfrak{q}$ is the valuation ring associated to the discrete valuation ν' of rank one, and let u be a generator of its maximal ideal $\mathfrak{q}S_\mathfrak{q}$. Then we can write the composite valuation $\nu = \nu' \circ \bar{\nu}$ in the following form:

$$\nu(f) \ = \ \big(\nu'(f), \bar{\nu}(\overline{f})\big) \ = \ \big(\mathrm{order}_\mathfrak{q}(f), \bar{\nu}(\overline{f})\big) \ ,$$

where we denote $\overline{f}$ the image of $fu^{-\nu'(f)}$ in the residue field κ'.
Moreover, if the center ξ of the valuation $\bar{\nu}$ is a regular point of the curve C, the local ring $\mathcal{O}_{C,\xi}$ is the discrete rank one valuation ring associated to $\bar{\nu}$ and we can write

$$\bar{\nu}(\overline{f}) = \mathrm{order}_\mathfrak{m}(\overline{f}) \ ,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{C,\xi}$.

Uniformization

We shall give an idea of Zariski's proof of the uniformization theorem in the abstract form for a valuation of rank one and of rational rank two, i.e. in the case *iii)*, the proof in the other cases are simpler, cases *i)* and *iv)*, ore are quite similar, case *ii)* ([Za 1]).

Let Γ be the value group of the valuation ν, we may write $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}$, with $\tau \in \mathbb{R} \smallsetminus \mathbb{Q}$ and $\tau > 0$, and we choose two elements x and y in the function field K such that $\nu(x) = 1$ and $\nu(y) = \tau$. Then the elements x and y are algebraically independent over k and since the characteristic of k is zero, there exists a primitive element z for the algebraic extension $K/k(x,y)$, i.e. $K = k(x,y,z)$, and we choose z with $\nu(z)$ positive.

Hence we consider the affine model $M = \mathrm{Spec}R$ of the function field K where R is an hypersurface ring $R = k[x, y, z]$, i.e. M is a surface defined in the affine space $\mathbb{A}_k^3 = \mathrm{Spec}k[X, Y, Z]$ by a polynomial $f(X, Y, Z) = \sum a_{r,s,t} X^r Y^s Z^t$ which satisfies the equality:

$$f(x, y, z) \ = \ \sum a_{r,s,t} x^r y^s z^t \ = \ 0 \ . \tag{1}$$

Since we have $\nu(x)$, $\nu(y)$ and $\nu(z)$ positive, the center ξ of the valuation ν is the point $\xi = (0, 0, 0)$, then $a_{0,0,0} = 0$. Moreover, since $f(x, y, z) = 0$ there must exist at least two monomials $a_{r,s,t} x^r y^s z^t$ in $f(x, y, z)$ for which the minimal value $\nu\big(x^r y^s z^t\big)$ is reached. If $x^{r_1} y^{s_1} z^{t_1}$ and $x^{r_2} y^{s_2} z^{t_2}$ are two distinct terms of f which have the same value, then we have $t_1 \neq t_2$. In fact if $t_1 = t_2$, we get $\nu(x^{r_1} y^{s_1}) = \nu(x^{r_2} y^{s_2})$, then $r_1 + s_1\tau = r_2 + s_2\tau$ and since $\tau \notin \mathbb{Q}$ this implies $r_1 = r_2$ and $s_1 = s_2$.
We may write the polynomial f in the form:

$$f(x, y, z) \ = \ \sum_{i=1}^{d} a_{r_i,s_i,t_i} x^{r_i} y^{s_i} z^{t_i} \ + \ \sum_{j=d+1}^{D} a_{r_j,s_j,t_j} x^{r_j} y^{s_j} z^{t_j} \ , \tag{2}$$

such that the monomials $x^{r_i} y^{s_i} z^{t_i}$, $1 \leq i \leq d$, have the same value $\nu(x^{r_i} y^{s_i} z^{t_i}) = \gamma$ and such that the monomials $x^{r_j} y^{s_j} z^{t_j}$, $d+1 \leq j \leq D$, have a value $\nu(x^{r_j} y^{s_j} z^{t_j}) > \gamma$, and we may assume $t_1 < t_2, \ldots < t_d$. If $\nu(z) = u + v\tau$, with u and v in $\mathbb{Z}$, and if $\gamma = M + N\tau$, with M and N in $\mathbb{Z}$, we get the equalities:

$$M = r_1 + ut_1 = r_2 + ut_2 = \ldots = r_d + ut_d \ , \tag{3}$$

$$N = s_1 + vt_1 = s_2 + vt_2 = \ldots = s_d + vt_d \ , \tag{4}$$

and for any $j > d$ we have:

$$(r_j + ut_j) + (s_j + vt_j)\tau \ > \ M + N\tau \ . \tag{5}$$

We may assume that $f(0, 0, z)$ is not identically zero, i.e. that the z-axis Z does not lie on the model $M \subset \mathbf{A}_k^3$; it is enough if necessary to replace z by $z + c'x + c''y$ for sufficiently general c' and c'' in k. We may write $f(0, 0, z) = \sum_{t=m}^{T} a_{0,0,t} z^t$, with $a_{0,0,m} \neq 0$, i.e. $\xi = (0, 0, 0)$ is an m-fold point of the zero-dimensional subvariety $Z \cap M$. If ξ is a regular point of $Z \cap M$, then ξ is also a regular point of the variety M, i.e. the center of the valuation ν on the model M is a non-singular point. Hence we may assume that ξ is a singular point of $Z \cap M$, which is equivalent to $m > 1$, and the polynomial $f(x, y, z)$ contains the term az^m with $a = a_{0,0,m} \in k^*$. Moreover, since we have $\nu(z^m) \geq \gamma = \nu(x^{r_d} y^{s_d} z^{t_d})$, we must have

$$t_d \leq m \; . \tag{6}$$

We consider the expansion of τ in continued fraction:

$$\tau = h_1 + \cfrac{1}{h_2 + \cfrac{1}{h_3 + \dots}} \; .$$

We denote (f_i/g_i) the sequence of convergent fractions of τ:

$$\frac{f_i}{g_i} = h_1 + \cfrac{1}{h_2 + \cfrac{1}{\dots + \cfrac{1}{h_i}}} \quad , \quad \text{with} \quad \big(f_i, g_i\big) = 1 \; ,$$

and moreover for all $i \geq 2$ we have:

$$f_{i-1}g_i - f_i g_{i-1} = (-1)^{i-1} \quad \text{and} \quad (-1)^{i-1}\big(\tau - f_i/g_i\big) > 0 \; .$$

Since we have $\lim_{q\to\infty} f_q/g_q = \tau$ and since we have a finite number of terms $x^{r_j}y^{s_j}z^{t_j}$ with $\nu(x^{r_j}y^{s_j}z^{t_j}) > \gamma$, we can find an integer p, sufficiently high, such that for any $q \geq p-1$, we have also the inequality

$$(r_j + ut_j) + (s_j + vt_j)\frac{f_q}{g_q} > M + N\frac{f_q}{g_q} \; , \tag{7}$$

for all these terms $x^{r_j}y^{s_j}z^{t_j}$, $d+1 \leq j \leq D$.

We want to construct a new ring $R_1 = k[x_1, y_1, z_1]$, with $R \subset R_1 \subset K$, or in other words we want to construct a birational morphism $M_1 \longrightarrow M$, such that the situation is better in some sense for the singularity of the center ξ_1 of ν on M_1. We pass from the elements x, y, z to the new elements x_1, y_1, z_1 of the function field K, by doing the following *Cremona transformation*:

$$x = x_1^{g_p}y_1^{g_{p-1}}, \quad y = x_1^{f_p}y_1^{f_{p-1}}, \quad z = x^u y^v(z_1 + c) \; , \tag{8}$$

where c is the element of k^* determined as follows: since $\nu(z) = u + v\tau = \nu(x^u y^v)$, and since the residue field of the valuation ν is equal to the ground field k, there exists a unique element c in k^* such that $\nu(z - cx^u y^v) > \nu(z)$. Hence we have

$$\nu(z_1) = \nu\Big(\frac{z}{x^u y^v} - c\Big) > 0 \; . \tag{9}$$

If we denote $\varepsilon = (-1)^{p-1}$, we deduce from the equality $f_{p-1}g_p - f_p g_{p-1} = \varepsilon$ and from the inequalities $\varepsilon(-\tau + f_{p-1}/g_{p-1}) > 0$ and $\varepsilon(\tau - f_p/g_p) > 0$, that we have:

$$(10) \qquad x_1 \ = \ \left(\frac{x^{f_{p-1}}}{y^{g_{p-1}}}\right)^{\varepsilon} \quad \text{and} \quad y_1 \ = \ \left(\frac{y^{g_p}}{x^{f_p}}\right)^{\varepsilon} ,$$

whence:
(11)

$$\nu(x_1) \ = \ \varepsilon g_{p-1}\left(\frac{f_{p-1}}{g_{p-1}} - \tau\right) \ > \ 0 \quad \text{and} \quad \nu(y_1) \ = \ \varepsilon g_p\left(\tau - \frac{f_p}{g_p}\right) \ > \ 0 \ .$$

Then the center of the valuation ν on R_1 is the maximal ideal (x_1, y_1, z_1), i.e. the center of ν on the model $M_1 = \mathrm{Spec} R_1$ is the point $\xi_1 = (0,0,0)$.

The ring R_1 is again an hypersurface ring, i.e. R_1 is isomorphic to the quotient ring $k[X_1, Y_1, Z_1]/(f_1)$ where f_1 is the polynomial defined as follows. Every monomial $x^r y^s z^t$ is transformed by the Cremona transformation into the polynomial $x_1^{r'} y_1^{s'} (z_1 + c)^t$ with $r' = (r+tu)g_p + (s+tv)f_p$ and $s' = (r+tu)g_{p-1} + (s+tv)f_{p-1}$. Hence, we deduce from the equalities (3) and (4) and from the inequality (7) that all the terms of f are divisible by the monomial $x_1^{Mg_p + Nf_p} y_1^{Mg_{p-1} + Nf_{p-1}}$, and that this monomial is the biggest factor of all the terms. We may write:

$$(12) \qquad f(x,y,z) \ = \ x_1^{Mg_p + Nf_p} y_1^{Mg_{p-1} + Nf_{p-1}} . f_1(x_1, y_1, z_1) \ ,$$

with f_1 irreducible.
More precisely we notice that all the terms $x^r y^s z^t$ of f with minimal value have exactly $x_1^{Mg_p + Nf_p} y_1^{Mg_{p-1} + Nf_{p-1}}$ as factor, and that the other terms acquire a factor $x_1^{r'} y_1^{s'}$ with $r' > Mg_p + Nf_p$ and $s' > Mg_{p-1} + Nf_{p-1}$. Then we deduce from (2) that the polynomial $f_1(x_1, y_1, z_1)$ has the form:
(13)

$$f_1(x_1, y_1, z_1) \ = \ (z_1 + c)^{t_1} \left(\sum_{i=1}^{d} a_{r_i, s_i, t_i} (z_1 + c)^{t_i - t_1}\right) \ + \ x_1 y_1 g(x_1, y_1, z_1) \ .$$

We put $a_{r_i, s_i, t_i} = a_i$ for $i = 1, 2, \ldots, d$ and we notice $h(u)$ the polynomial defined by $h(u) = a_1 + a_2 u^{t_2 - t_1} + \ldots + a_d u^{t_d - t_1} = \sum_{j=0}^{t_d - t_1} h_j u^j$, hence we have

$$f_1(x_1, y_1, z_1) = (z_1 + c)^{t_1} h(z_1 + c) + x_1 y_1 g(x_1, y_1, z_1) \ .$$

Since the center ξ_1 of the valuation on M_1 is the point $(0,0,0)$ of $\mathbf{A}_k^3$, we have $(0,0,0) \in M_1$, i.e. we must have $f_1(0,0,0) = 0$, and since $c \neq 0$,

$u = c$ is a root of the polynomial $h(u)$. If c is an m_1-fold root of $h(u)$, i.e. if we have $h(u) = (u - c)^{m_1} h'(u)$ with $h'(c) \neq 0$, then ξ_1 is also a m_1-root of $Z_1 \cap M_1$, where Z_1 is the z_1-axis. Since $\deg h(u) = t_d - t_1$ and from the inequality (6) we deduce:

$$(14) \qquad m_1 \ \leq \ t_d - t_1 \ \leq \ m \ .$$

To prove the theorem of uniformization, we have to show that with this process, we can get a point ξ_1 which is *better* than the point ξ. If we have $m_1 < m_0 = m$, then we have succeeded, we can make an induction on m_i and for $m_N = 1$ we have a non-singular point.
We assume that we have $m_1 = m$, then from (14) we deduce that we have:

$$t_1 \ = \ 0 \ , \quad t_d \ = \ m \quad \text{and} \quad h(u) \ = \ a_d(u - c)^m \ .$$

Then the sum of the terms of minimal value of $f(x, y, z)$ is equal to:

$$\begin{aligned} \sum_{i=1}^{d} a_i x^{r_i} y^{s_i} z^{t_i} &= x_1^{Mg_p + Nf_p} y_1^{Mg_{p-1} + Nf_{p-1}} z_1^m \\ &= h_0 x^{b_0} y^{c_0} + h_1 x^{b_1} y^{c_1} z^1 + \ldots + h_m x^{b_m} y^{c_m} z^m \ , \end{aligned}$$

with $h_j \neq 0$ and $\nu(x^{b_j} y^{c_j} z^j) = \gamma$ for all j, and we may deduce $\nu(z) = b_{m-1} + c_{m-1}\tau$.
Then we have the following result: if by the transformation (8) the multiplicity m does not decrease, the value $\nu(z)$ is of the form $\nu(z) = u + v\tau$ with $u, v \geq 0$.

We assume that we have choosen the element z with $\nu(z) = u + v\tau$, with $u, v \geq 0$. Since the residue field of the valuation ν is equal to the ground field k, there exists a unique $c \in k^*$ such that $\nu(z - cx^u y^v) > \nu(z)$. Let

$$\begin{aligned} z^{[1]} &= z - cx^u y^v \ , \qquad \nu(z^{[1]}) > \nu(z) \\ f^{[1]}(x, y, z^{[1]}) &= f(x, y, cx^u y^v + z^{[1]}) \ = \ f(x, y, z) \ . \end{aligned}$$

We have found a new presentation of the model M as closed subvariety of $\mathbf{A}_k^3$, now defined by the polynomial $f^{[1]}$. The center ξ of the valuation ν on M belongs to the $z^{[1]}$-axis $Z^{[1]}$, and the multiplicity of ξ on $Z^{[1]} \cap M$ is also equal to m, i.e. $z^{[1]} = 0$ is also a m-fold root of $f^{[1]}(0, 0, z^{[1]}) = 0$. We can apply a Cremona transformation such as (8), and we get a new model M_1 with a center ξ_1 of "multiplicity" m_1. If we have $m_1 < m$, then we have got a better variety.
We assume that we have again the equality $m_1 = m$, then we deduce

from the previous result that the value of $z^{[1]}$ is also of the form

$$\nu(z^{[1]}) \;=\; u_1 + v_1\tau \;, \quad \text{with } u_1, v_1 \geq 0 \;.$$

Then we put again

$$z^{[2]} \;=\; z^{[1]} - c_1 x^{u_1} y^{v_1} \;,$$

where the constant $c_1 \in k^*$ is choosen such that $\nu(z^{[2]}) > \nu(z^{[1]})$, and we make the same construction as before.

Hence we may assume that we have found by induction a sequence $(z^{[i]})$, $o \leq i \leq p$, of elements of K, with $z^{[0]} = z$, such that

$$(15) \quad z^{[i+1]} \;=\; z^{[i]} - c_i x^{u_i} y^{v_i} \quad \text{and} \quad \nu(z^{[i+1]}) \;>\; \nu(z^{[i]}) \;=\; u_i + v_i\tau \;.$$

Each $z^{[i]}$ defines a polynomial $f^{[i]}(x, y, z^{[i]})$ and an embedding of the model M in $\mathbf{A}^3_k$, or equivalently, if we have fixed the first embedding defined by $f(x, y, z)$, each element $z^{[i]}$ defines a curve $C^{[i]}$ on $\mathbf{A}^3_k$, $C^{[i]}$ corresponds to the $z^{[i]}$-axis on $\mathbf{A}^3_k$. The center ξ of the valuation ν on M belongs to all the curves $C^{[i]}$ and the multiplicity $m = \mathrm{mult}_\xi(C^{[i]} \cap M)$ does not depend from i.
Such as (8), we can associate to each $z^{[i]}$, $0 \leq j \leq p-1$ a Cremona transform $M_1^{[i]} \longrightarrow M$, and the center $\xi_1^{[i]}$ of the valuation ν on this model $M_1^{[i]}$ satisfies

$$m_1^{[i]} \;=\; \mathrm{mult}_{\xi_1^{[i]}}(C_1^{[i]} \cap M_1^{[i]}) \;=\; m \;.$$

If for $j = p$ we have an inequality, i.e. $m_1^{[p]} < m$, then we consider the new model $M_1 = M_1^{[p]}$ of K for which the situation is better.

If for $j = p$ we have again an equality $m_1^{[p]} = m$, then we have shown that we can find a new element $z^{[p+1]}$ of K with $z^{[p+1]} = z^{[p]} - c_p x^{u_p} y^{v_p}$ and $\nu(z^{[p+1]}) > \nu(z^{[p]})$.
Hence it is enough to show that it is impossible to find an infinite sequence $(z^{[i]})$ of elements of K which satisfies the property (15).
By definition, for any i, $i \geq 0$, we have

$$f^{[i+1]}(x, y, z^{[i+1]}) \;=\; f^{[i]}(x, y, c_i x^{u_i} y^{v_i} + z^{[i+1]}) \;,$$

hence we deduce

$$\frac{\partial f^{[i+1]}}{\partial z^{[i+1]}} \;=\; \frac{\partial f^{[i]}}{\partial z^{[i]}} \;.$$

Since in each polynomial $f^{[i]}$ the term in $(z^{[i]})^m$ must be among the minimum value terms, it follows that

$$\nu\left(\frac{\partial f^{[i]}}{\partial z^{[i]}}\right) \;\geq\; (m-1)\nu(z^{[i]}) \;\geq \nu(z^{[i]}) \;,$$

hence we have for all i the inequality

$$\nu(z^{[i]}) = u_i + v_i\tau \;\leq\; \nu\left(\frac{\partial f}{\partial z}\right) . \tag{16}$$

If we have an infinite sequence $\big(z^{[i]}\big)$, we find an infinite sequence $\big((u_i, v_i)\big)$ in $\mathbb{N}^2$ such that the sequence of real numbers $\alpha_i = u_i + v_i\tau$ is increasing, but in that case the sequence (α_i) is non bounded, which contradicts the inequality (16).

In fact we have proven that the sequence is finite, i.e. there exists a curve $C = C^{[p]}$ which is *better* than the other ones. In particular we have shown that the Cremona transformation associated to this curve C will give a new situation which is better than the initial one because the multiplicity m_1 is srictly smaller than the multiplicity m. This is the curve which corresponds to the *maximal contact*.

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