Severi varieties and self rational maps of $K3$ surfaces

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Introduction

0.1 Notations. We deal in this paper with complex projective $K3$ surfaces, *i.e.* smooth K-trivial complex projective surfaces without irregularity. Let $\varphi : S \dashrightarrow S$ be a dominant self rational map. Suppose $\mathrm{Pic}(S) = \mathbf{Z}$. Then there exists a positive integer l such that $\varphi^*\mathcal{O}_S(1) \cong \mathcal{O}_S(l)$. It is the algebraic degree of φ, that is the degree of the polynomials defining φ. There always exists an elimination of indeterminacies

$$\begin{tikzcd} \widetilde{S} \arrow[d, "\tau"'] \arrow[dr, "\widetilde{\varphi}"] & \\ S \arrow[r, dashed, "\varphi"'] & S \end{tikzcd}$$

i.e. a commutative diagram, where $\widetilde{\varphi}$ is a morphism and τ is a finite sequence of blow-ups. One defines the topological degree of φ as $\deg \varphi := \deg \widetilde{\varphi}$. It is the number of points in the inverse image of a generic point $x \in S$ under the action of φ. We write R for the ramification divisor of $\widetilde{\varphi}$ (it is the zero divisor of the Jacobian $\bigwedge^2 d\widetilde{\varphi}$).

0.2 Self-rational maps. The main goal of this article is to study the geometric and numerical properties of self-rational maps $S \dashrightarrow S$, where S is a $K3$ surface with Picard group $\mathbf{Z}$, in order to attack the following conjecture.

Conjecture 1 *For a generic projective $K3$ surface S, there does not exist any dominant rational map*

$$\varphi : S \dashrightarrow S$$

satisfying $\varphi^*\mathcal{O}_S(1) \cong \mathcal{O}_S(l)$, $l > 1$.

It is another problem to find complex projective manifolds X equipped with a self morphism $f : X \to X$. It has already been studied by Beauville ([Bea01]), Fujimoto and Nakayama ([Fuj02], [Nak02], [FN05]), or Amerik, Rovinsky and Van de Ven ([ARV99]). Toric and abelian varieties are obvious examples of such manifolds, and one does not know any other example than those deduced from these two obvious ones. This leads us to conjecture that there does not exist any other example at all, or in other words that in case $\mathrm{Pic}(X) = \mathbf{Z}$ and $\kappa(X) \geqslant 0$, there does not exist any f with $\deg f > 1$. Beauville ([Bea01]) proves in this direction that a complete intersection of p hypersurfaces of respective degrees $d_1, \ldots, d_p$ in $\mathbf{P}^{n+p}$ ($n \geqslant 2$) does not admit any endomorphism of degree strictly greater than 1, as soon as at least one of the d_i's is greater than 3. One easily sees that a $K3$ surface cannot possess any dominant endomorphism of degree strictly larger than 1. Indeed, if S is a $K3$ surface and $f : S \to S$ a dominant morphism, then the relation

$$K_S = f^* K_S + R$$

(where R is the ramification divisor of f), combined with the fact that $K_S = 0$, proves that f is necessarily smooth. It is thus an étale cover of S by itself, and therefore an automorphism of S, because all $K3$ surfaces are simply connected.

Amerik and Campana ([AmCa05]), or Cantat ([Can05]) are more generally interested in the search of manifolds equipped with dominant self rational maps. Again, we have easy examples deduced from the toric and abelian ones. If S is a $K3$ surface equipped with an elliptic pencil $|F|$ and a relatively ample line bundle L of relative degree d, then we can construct $\mu_{d+1} : S \dashrightarrow S$ of degree $(d+1)^2$, *via* multiplication in the fibers of the pencil : the image of a point x on a generic fiber F is defined as the unique point $y \in F$ satisfying

$$\mathcal{O}_F\left((d+1)x - y\right) = L|_F \,.$$

Kummer surfaces are smooth models of quotients of complex tori under the action of an involution. One can therefore construct self rational maps of degree strictly greater than 1 on them, by descending the homotheties on the tori. Note that these two examples only concern special $K3$ surfaces.

In greater dimensions, one has examples that cannot be deduced from the two obvious ones. Voisin constructs in [Voi04] a self rational map of degree 16 of the variety X of lines in a cubic hypersurface $V \subset \mathbf{P}^5$. X is 4-dimensional and hyperkähler, has $\mathrm{Pic}(X) \cong \mathbf{Z}$, and is deformation equivalent to the punctual Hilbert scheme $S^{[2]}$ of a $K3$ surface S of degree 14 ([BD85]). One gets this self rational map by mapping the generic line $l \subset V$ to the residual line l' to l in $P \cap V$, where P is the unique 2-plane tangent to V along l. This map does not respect any fibration in virtue of the following theorem of Amerik and Campana ([AmCa05]) : if X is a projective manifold satisfying $K_X = 0$ and $\mathrm{NS}(X) = \mathbf{Z}$, then any rational fibration $g : X \dashrightarrow B$ $(0 < \dim B < \dim X)$ has fibers of general type. Voisin's example shows in particular that conjecture 1 is specific to the case of surfaces.

The holomorphic dynamical point of view gives a new insight into this problem. Given a complex manifold X equipped with a transformation $f : X \to X$, one gets a discrete dynamical system by iterating f (see [Can05]). Amerik and Campana associate a meromorphic fibration $g : X \dashrightarrow T$ to any dominant self rational map $f : X \dashrightarrow X$. Its general fiber X_t is the Zariski closure of the orbit of a general point in X_t. This allows them to prove that if X is a complex projective manifold satisfying both $K_X = 0$ and $\mathrm{NS}(X) \cong \mathbf{Z}$, and if $f : X \dashrightarrow X$ is a dominant self rational map of degree greater than 2, then the orbit of a general point of X under the action of f is Zariski-dense.

Such dynamical facts induce results concerning potential density in the K-trivial case (a variety X over some field k is said to be potentially dense if there exists some finite extension $k \to k'$, such that the set of k'-rational points is Zariski-dense in X, see *e.g.* [HT06]). As a simple corollary of their results exposed above, Amerik and Campana get : if X is a smooth projective variety defined over $\bar{k}$ (k a non countable field), such that $K_X = 0$ and $\mathrm{Pic}(X) = \mathbf{Z}$, and if there exists $f : X \dashrightarrow X$ with $\deg f > 1$, then X has potential density. Cantat gets on his side, and using dynamical methods, a large part of the proof of the following theorem, due to Bogomolov and Tschinkel ([BT00]) : a projective $K3$ surface defined over some number field k is potentially dense, as soon as it can be realized as an elliptic fibration.

Eventually, this dynamical study allows Cantat to show that if X is a projective Calabi-Yau manifold of dimension n, and if there exists $f : X \dashrightarrow X$ satisfying some dilating property, then there exists a dominant rational map $\mathbf{C}^n \dashrightarrow X$ (in particular, X satisfies the Kobayashi conjecture, see *e.g.* [Voi03]). This leads him to ask the following questions concerning a generic algebraic $K3$ surfaces S (in addition to the question of the existence of a dominant self-rational map $\varphi : S \dashrightarrow S$ with $\deg \varphi > 1$, to which this article gives a conjectural answer) : does S have potential density ? does S admit a dominant rational map $\mathbf{C}^2 \dashrightarrow S$?

0.3 Severi varieties. We present in this article a result relating conjecture 1 and the irreducibility of Severi varieties for $K3$ surfaces.

Nodal plane curves (*i.e.* plane curves with only non-degenerate singularities) are a classical topic. A historical reason for this is the fact that every smooth curve is birationally equivalent to a nodal plane curve, *via* a series of projections. Let $V_{d,g}$ be the variety parametrizing plane irreducible curves of degree d and geometric genus g. It is called a Severi variety. It is the closure in the projective space parametrizing all plane curves of degree d of the locus of irreducible nodal curves of genus g. Severi gave an uncomplete proof of the fact that all varieties $V_{d,g}$ are irreducible, and it is only in 1986 that Harris actually proved this (see [Har86]).

A natural generalization of this is the study of nodal curves on a projective surface S equipped with a fixed ample effective line bundle L. $V_{k,h}$ then denotes the closure in $|kL|$ of the locus of irreducible nodal curves of geometric genus h, and is again called a Severi variety. The following questions arise naturally : when are the $V_{k,h}$ non empty ? What are their dimensions ? Are they irreducible ? smooth ? They are studied by Chiantini and Ciliberto in [CC99]. Also Greuel, Lossen and Shustin ([GLS00]), Keilen ([Kei03]) give some numerical criteria for generalized Severi varieties to be irreducible.

We focus here on universal Severi varieties for $K3$ surfaces. Fix an integer $g \geqslant 2$, and write $\mathcal{M}_{K3,g}$ for the moduli space of $K3$ surfaces equipped with an indivisible, ample line bundle L of self-intersection $2g-2$ (we call these $K3$ surfaces of genus g). There exists a universal family $\mathcal{S}_g \to \mathcal{M}^\circ_{K3,g}$ over an open subset of $\mathcal{M}_{K3,g}$. To a generic point $m \in \mathcal{M}^\circ_{K3,g}$ corresponds a $K3$ surface S_m with Picard group

$$\mathrm{Pic}(S_m) = \mathbf{Z} \cdot L_m,$$

where L_m is an ample and indivisible divisor class, satisfying $L_m^2 = 2g-2$ (see *e.g.* [Pal85]). A generic member of the complete linear system $|L_m|$ is a smooth curve of geometric genus g. For integers $k, h \geqslant 1$, we define the universal Severi variety

$$\mathcal{V}_{k,h} \longrightarrow \mathcal{M}^\circ_{K3,g}$$

to be the variety whose fiber over a generic $m \in \mathcal{M}^\circ_{K3,g}$ is the closure in $|kL_m|$ of the locus

$$\{C \in |kL_m| \text{ s.t. } C \text{ is irreducible, nodal, and of geometric genus } h\}.$$

By the genus formula, all curves in the complete linear system $|kL_m|$ have arithmetic genus $p_a(k) = 1 + (kL_m)^2/2 = 1 + k^2(g-1)$.

The deformation theory of nodal curves on $K3$ surfaces works very well. In particular, we know that for a $K3$ surface S, the Severi variety $V_{k,h}$ is smooth and of the expected dimension ; if S is generic, then $V_{k,h}$ is non empty (see section 1). The only question that remains open is whether the $V_{k,h}$ are irreducible or not. For $h = 0$, it is clear that the answer is no : there are finitely many rational curves in the linear system $|kL|$. The corresponding Severi variety is then a disjoint union of points, which of course is not irreducible.

It is perfectly possible that the universal Severi variety $\mathcal{V}_{k,h}$ is irreducible even if the fibers $V_{k,h}$ are reducible. The question of the irreducibility of the $\mathcal{V}_{k,h}$ is the closest to the initial problem of Severi, where the projective plane plays the role of a universal space for complete non singular curves, since they all are birationally equivalent to plane curves with at most nodes as singularities. We conjecture that all $\mathcal{V}_{k,h}$ are irreducible. The following less optimistic version is however sufficient for our purpose.

Conjecture 2 *Let $\varepsilon > 0$ be given. If k is great enough with regard to ε, then for all integer h satisfying*

$$\varepsilon p_a(k) \leqslant h \leqslant p_a(k),$$

the universal Severi variety $\mathcal{V}_{k,h}$ is irreducible.

0.4 Results. In section 2, we prove the following.

Theorem 3 *Let $g, l \geqslant 2$ be given. If for $m \in \mathcal{M}^{\circ}_{K3,g}$ generic there exists a dominant rational map $\varphi_m : S_m \dashrightarrow S_m$ satisfying $\varphi_m^* \mathcal{O}_S(1) \cong \mathcal{O}_S(l)$, then for k great enough the universal Severi variety $\mathcal{V}_{kl,p_a(k)}$ possesses at least two irreducible components.*

To prove this, we look at the images under the action of φ of the curves in $|kL|$. We show that they are elements of $|klL|$, and are generically nodal and of geometric genus $p_a(k)$. This gives a way to construct two distinct irreducible components of $\mathcal{V}_{kl,p_a(k)}$, the first one parametrizing curves whose respective singularity 0-cycles are all rationally equivalent to a constant, and the second one parametrizing curves with non constant singularity 0-cycle modulo rational equivalence. By singularity 0-cycle of a curve $C \subset S$, we mean the sum of all singular points of C, seen as a 0-cycle on S.

Now asymptotically, we have $p_a(k)/p_a(lk) \sim_{k\to\infty} 1/l^2$. We thus get the following result, which gives a way to tackle conjecture 1.

Corollary 4 *Conjecture 2 on the Severi varieties implies conjecture 1 on self-rational maps.*

In section 3, we gather numerical constraints between the topological and numerical degrees (*i.e.* $\deg \varphi$ and l with the notations of 0.1) of a dominant self-rational map $S \dashrightarrow S$, where S is a given generic $K3$ surface. This restricts the possibilities for the existence of such self-rational maps, and may lead to special cases of conjecture 1. The most significant results we get in this direction are the following.

Theorem 5 *Let S be a $K3$ surface of genus g, with $\mathrm{Pic}(S) = \mathbf{Z}$. We assume there exists a dominant self rational map $\varphi : S \dashrightarrow S$ with $\deg \varphi > 1$. Then we have the following.*
(i) There exists an integer λ, such that $\deg \varphi = \lambda^2$. In addition, $2g-2$ necessarily divides $l - \lambda$ (note that we do not know the sign of λ).
(ii) There exist positive integers $\beta_1, \ldots, \beta_p$, such that

$$l^2 = \deg \varphi + (2g-2) \textstyle\sum_i \beta_i^2 .$$

$\sum_i \beta_i$ is always divisible by 2. If we can eliminate the indeterminacies without any chain of successive blow-ups of length strictly larger than 2, then

$$\deg \varphi \leqslant 1 + \frac{1}{24} \left[p + 4(g-1) \textstyle\sum_i \beta_i \right].$$

We prove this by studying the geometry of an elimination of indeterminacies of φ. To do this, we use the intersection tree of the irreducible exceptional curves that appear in the elimination of indeterminacies. We get on our way the following result, which allows us to control the complexity of such an elimination of indeterminacies.

Proposition 6 *(i) The depth of the tree is less than $\deg \varphi - 2$. If this maximal depth is achieved, then all exceptional curves project on the same point of S.*
(ii) If the tree has two connected components of depths l_1 and l_2, then $l_1 + l_2 \leqslant \deg \varphi - 2$.
(iii) If all irreducible exceptional curves are disjoint, or equivalently, if the indeterminacy can be solved with one blow-up, then there are at most $8(\deg \varphi - 1)$ such irreducible exceptional curves.

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1 Families of nodal curves on $K3$ surfaces

We recall here classical results on families of nodal curves in general, and on families of nodal curves on $K3$ surfaces in particular. We use a description with infinitesimal deformations, and refer to [Voi02] (chapter 14) for more details (see [Tan82] for a slightly different point of view).

Let X be a smooth, n-dimensional variety, equipped with an ample line bundle L. We write $X_{(\delta)}$ for the open subset of the symmetric product $\mathrm{Sym}^{\delta} X$ corresponding to sums $x_1 + \cdots + x_\delta$ of δ pairwise distinct points. We look at hypersurfaces in the complete linear system $|L|$ with at least δ singular points, and therefore introduce the incidence variety

$$\mathcal{I} = \{(D, x_1 + \cdots + x_\delta) \in |L| \times X_{(\delta)} \text{ s.t. } D \text{ is singular at } x_1, \ldots, x_\delta\}.$$

Let $\pi : \mathcal{I} \to |L|$ be the natural projection. In the neighbourhood of a point $(f, x) \in |L| \times X_{(\delta)}$ (where f is an equation for a hypersurface D and $x = x_1 + \cdots + x_\delta$), $\mathcal{I}$ is defined by the $\delta(n+1)$ equations

$$f(x_i) = \frac{\partial f}{\partial z_1}(x_i) = \cdots = \frac{\partial f}{\partial z_n}(x_i) = 0$$

$(1 \leqslant i \leqslant \delta)$, where $z_1, \ldots, z_n$ denote with a little abuse of notations local holomorphic coordinates at the neighbourhood of every x_i. We thus have

$$\dim(\mathcal{I}) \geqslant \dim |L| - \delta.$$

$\dim |L| - \delta$ is called the expected dimension of $\mathcal{I}$.

After differentiation, we get the equations of the tangent space $T_{\mathcal{I},(f,x)}$ in $T_{|L|,f} \times T_{X_{(\delta)},x}$: $(g, h_1 + \cdots + h_\delta) \in T_{|L|,f} \times T_{X_{(\delta)},x}$ lies in $T_{\mathcal{I},(f,x)}$ if and only if one has for every x_i

$$\left\{ \begin{array}{l} df_{x_i}(h_i) + g(x_i) = 0 \\ d\left(\frac{\partial f}{\partial z_j}\right)_{x_i}(h_i) + \frac{\partial g}{\partial z_j}(x_i) = 0 \quad (1 \leqslant j \leqslant n), \end{array} \right.$$

that is

$$\left\{ \begin{array}{l} g(x_i) = 0 \\ \mathrm{Hess}_{x_i}(f)(h_i) = -\left(\frac{\partial g}{\partial z_1}(x_i), \ldots, \frac{\partial g}{\partial z_n}(x_i)\right), \end{array} \right.$$

since all differentials df_{x_i} vanish. The kernel of the differential π_* at the point (f, x) is given by $g = 0$, so π is an imbedding at the point $(x_1 + \cdots + x_\delta, D)$ if and only if $\mathrm{Hess}_{x_1}(f), \ldots, \mathrm{Hess}_{x_\delta}(f)$ are non degenerate, *i.e.* if and only if the δ points $x_1, \ldots, x_\delta$ are non degenerate singular points of D. In this case, the image of π_* at the point (f, x) is simply

$$\{g \in T_{|L|,f} \text{ s.t. } g(x_1) = \ldots = g(x_\delta) = 0\},$$

and the tangent space to the projection of $\mathcal{I}$ on $|L|$ at D identifies with

$$\mathrm{H}^0(X, \mathcal{O}_X(D) \otimes I_x)/\mathrm{H}^0(X, \mathcal{O}_X),$$

where $I_x \subset \mathcal{O}_X$ is the ideal sheaf defining x. $\mathcal{I}$ is of the expected dimension $\dim |L| - \delta$ if and only if the δ non degenerate singular points impose independent conditions on the linear system $|L|$. In this case, the non degenerate singular points of D can be independently smoothed by deformation.

Theorem 1.1 *Let S be a generic $K3$ surface, and L an ample and indivisible line bundle on it. For any positive integer k, and any $h \leqslant p_a(k)$, the quasi-projective variety*

$$V^{\circ}_{k,h} = \{C \in |kL| \text{ s.t. } C \text{ is irreducible, nodal, and of geometric genus } h\}$$

is non empty, smooth, and of the expected dimension h.

Proof. Riemann-Roch formula for surfaces gives $\dim |kL| = p_a(k)$, so h is the expected dimension of $V_{k,h}$. Suppose we have some irreducible nodal curve $C \in |kL|$ with precisely $\delta = p_a(k) - h$ nodes. Let Z denote the singularity 0-cycle of C. The former infinitesimal calculations give the Zariski-tangent space

$$T_{V_{k,h},C} \cong \mathrm{H}^0(S, \mathcal{O}_S(C) \otimes I_Z)/\mathrm{H}^0(S, \mathcal{O}_S) \cong \mathrm{H}^0(C, \mathcal{O}_C(C) \otimes I_Z).$$

The canonical bundle K_S being trivial, we get by adjunction formula

$$\mathcal{O}_C(C) \cong \mathcal{O}_C(K_S + C) \cong \omega_C.$$

Since C is nodal, we have $K_{\widetilde{C}} = \nu^* K_C(-2Z)$, where $\nu : \widetilde{C} \to C$ is a normalization of C. Therefore

$$\mathrm{H}^0(C, \mathcal{O}_C(C) \otimes I_Z) \cong \mathrm{H}^0(C, \nu_* K_{\widetilde{C}}),$$

which gives $\dim T_{V_{k,h},C} = g(C) = h$. Since $\dim V_{k,h} \geqslant \dim |kL| - \delta = h$, this proves that $V^{\circ}_{k,h}$ is smooth and of the expected dimension.

It therefore only remains to show that there actually exists an irreducible nodal curve $C \in |kL|$ with precisely $\delta = p_a(k) - h$ nodes. It is enough to find an irreducible rational nodal curve in $|kL|$, since it gives a genus h curve by smoothing exactly h of its nodes. This is given by Chen's theorem below.

$\square$

Theorem 1.2 (Chen, [Che99]) *Consider $n \geqslant 3$ and $k > 0$. For S a generic $K3$ surface in $\mathbf{P}^n$, the complete linear system $|\mathcal{O}_S(k)|$ contains an irreducible rational curve with only nodes as singularities.*

2 Link between conjectures 1 and 2

In this section, we prove that conjecture 2 implies conjecture 1. We start with the following result, which for a generic $K3$ surface S, describes the geometric action of a dominant self-rational map $\varphi : S \dashrightarrow S$ on a generic curve $C \in |\mathcal{O}_S(k)|$.

Proposition 2.1 *Let S be a $K3$ surface of genus $g \geqslant 2$, with $\mathrm{Pic}(S) = \mathbf{Z}$, and assume there exists a dominant rational map $\varphi : S \dashrightarrow S$ satisfying $\varphi^* \mathcal{O}_S(1) \cong \mathcal{O}_S(l)$. We consider $C \in |\mathcal{O}_S(k)|$ generic.*
(i) Its image $\varphi(C)$ lies in $|\mathcal{O}_S(kl)|$.
(ii) For k big enough, $\varphi(C)$ is irreducible and nodal, and C and $\varphi(C)$ have the same geometric genus $p_a(k)$.

Proof. *(i)* Consider

$$\begin{tikzcd} \widetilde{S} \arrow[d, "\tau"'] \arrow[dr, "\widetilde{\varphi}"] & \\ S \arrow[r, dashed, "\varphi"'] & S \end{tikzcd}$$

an elimination of indeterminacies of φ, and write R for the ramification divisor of $\widetilde{\varphi}$. Since $K_{\widetilde{S}} = \widetilde{\varphi}^* K_S + R$ and K_S is trivial, R is entirely exceptional. In other words, φ is smooth away from the indeterminacy locus.

Since $C \in |\mathcal{O}_S(k)|$ is generic, we can assume that it avoids the indeterminacy locus. Then $\varphi_{|C}$ is locally an imbedding, and in particular we have the equality of homology classes

$$[\varphi(C)] = \varphi_*[C]$$

(φ_* and φ^* are defined as $\widetilde{\varphi}_*\tau^*$ and $\tau_*\widetilde{\varphi}^*$ respectively). We then compute the intersection product

$$\langle [\varphi(C)]\,, L\rangle = \langle \varphi_*[C], L\rangle = \langle [C], \varphi^* L\rangle\,,$$

where L is the divisor class corresponding to $\mathcal{O}_S(1)$. Finally, we have $\langle [\varphi(C)]\,, L\rangle = kl(2g-2)$, and therefore $\varphi(C) \in |klL|$.

(ii) We define a scheme $S \times_S S$, which is pointwise the set of pairs of points in S having the same image under φ, by considering a morphism $\varphi_U : U \to S$ (U Zariski-open subset of S) representing the rational map φ, and taking $S \times_S S$ to be the Zariski-closure of $U \times_S U$ in $S \times_{\mathrm{Spec}\,\mathbf{C}} S$.

We claim that for k big enough, $\varphi_{|C}$ is everywhere injective but at a finite number of points of C, or equivalently that

$$C \times_S C \subset S \times_S S$$

only possesses a finite number of points outside from the diagonal. To prove this, we define the incidence variety

$$J = \{(C, x_1 + x_2) \in |\mathcal{O}_S(k)| \times S_{(2)} \text{ s.t. } (x_1, x_2) \in C \times_S C\},$$

which parametrizes the pairs of distinct points of S having the same image by φ. It is given by the equations

$$\begin{cases} x_1, x_2 \in C \\ x_1 \neq x_2 \\ \varphi(x_1) = \varphi(x_2). \end{cases}$$

Now the projection of J on $S_{(2)}$ is $S \times_S S$, which is pointwise the set of sums $x_1 + x_2$ with $x_1 \neq x_2$ and $\varphi(x_1) = \varphi(x_2)$, and is of dimension 2. When k is large enough, the fibers of J over its projection on $S_{(2)}$ are of dimension $\dim |\mathcal{O}_S(k)| - 2$ (see 2.2), so

$$\dim J = \dim |\mathcal{O}_S(k)|.$$

The fiber of J over generic $C \in |\mathcal{O}_S(k)|$ is thus necessarily zero-dimensional, and our claim is proved. It follows that for generic $C \in |kL|$, $\varphi_{|C}$ is of degree 1 onto its image $\varphi(C)$. We can assume C to be smooth. Then it is the normalization of $\varphi(C)$, and these two curves have the same geometric genus.

A similar argument shows that for $C \in |\mathcal{O}_S(k)|$ generic, there cannot exist three pairwise distinct points on C having the same image under the action of φ. So, since C is smooth and $\varphi_{|C}$ is a local imbedding, all singular points of $\varphi(C)$ occur as the identification of two distinct points in C by φ. We shall now prove that for generic $C \in |\mathcal{O}_S(k)|$, these singular points are all nodes. Write $p : \mathbf{P}(T_S) \to S$ for the canonical projection of the projectivized holomorphic tangent bundle, and consider the incidence variety

$$J' \subset |\mathcal{O}_S(k)| \times \mathbf{P}(T_S) \times \mathbf{P}(T_S),$$

defined by the equations

$$(C, u_1, u_2) \in J' \iff \begin{cases} u_1, u_2 \in \mathbf{P}(T_C) \\ p(u_1) \neq p(u_2) \\ \varphi \circ p(u_1) = \varphi \circ p(u_2) \\ \varphi_* u_1 = \varphi_* u_2. \end{cases}$$

It parametrizes the couples of tangent directions of S at two different points, that are sent by the differential φ_* on a couple of colinear tangent directions at the same point of S (*i.e.* exactly the situations that yield degenerated singularities on $\varphi(C)$). The image of the projection of J' on $\mathbf{P}(T_S) \times \mathbf{P}(T_S)$ is given by the conditions $p(u_1) \neq p(u_2)$, $\varphi \circ p(u_1) = \varphi \circ p(u_2)$ and $\varphi_* u_1 = \varphi_* u_2$, which yield three independent equations, so it is of dimension 3. When k is large enough, the generic fiber of J' over its projection on $\mathbf{P}(T_S) \times \mathbf{P}(T_S)$ is of codimension 4 in $|\mathcal{O}_S(k)|$ (see 2.2). Then

$$\dim J' = \dim |\mathcal{O}_S(k)| - 1,$$

and the fiber of J' over generic $C \in |\mathcal{O}_S(k)|$ is necessarily empty, which concludes the proof.

□

Remark 2.2 *Proposition 2.1 works as soon as $k \geqslant 4$ when $g \geqslant 3$, and as soon as $k \geqslant 6$ when $g = 2$.*

Indeed, if $g \geqslant 3$ (resp. $g = 2$), then the line bundle $\mathcal{O}_S(k)$ is very ample for $k \geqslant 2$ (resp. $k \geqslant 3$). This is sufficient to ensure that two distinct points of S impose independent conditions on $|\mathcal{O}_S(k)|$ and thus that the argument concerning J is correct.

Now let x_1 and x_2 be two distinct points in some projective space $\mathbf{P}^N$, and $u_1 \in \mathbf{P}(T_{\mathbf{P}^N, x_1})$, $u_2 \in \mathbf{P}(T_{\mathbf{P}^N, x_2})$ be two tangent directions. As soon as the line defined by (x_1, u_1) (resp. (x_2, u_2)) does not pass through x_2 (resp. x_1), we are sure that (x_1, u_1) and (x_2, u_2) impose independent conditions on the linear system of quadrics in $\mathbf{P}^N$. So when $g \geqslant 3$ and $k \geqslant 4$ (resp. $g = 2$ and $k \geqslant 6$), the claim about the dimension of the generic fiber of J' is true.

Theorem 2.3 *Let $g, l \geqslant 2$ be given. If for $m \in \mathcal{M}^{\circ}_{K3,g}$ generic there exists a dominant rational map $\varphi_m : S_m \dashrightarrow S_m$ satisfying $\varphi_m^* \mathcal{O}_S(1) \cong \mathcal{O}_S(l)$, then for k great enough the universal Severi variety $\mathcal{V}_{kl, p_a(k)}$ possesses at least two irreducible components.*

Theorem 2.3 is one of the main results of this article. The key of the proof is the construction of two irreducible components of $V_{kl, p_a(k)}$ for S generic, such that the rational equivalence class (in $\mathrm{CH}_0(S)$) of the singularity 0-cycle is constant for the curves parametrized by the first component, and non constant for those parametrized by the second component. We give these two constructions in lemmas 2.4 and 2.6.

Lemma 2.4 *Under the hypotheses of proposition 2.1, and for k large enough, there exists an irreducible component of $V_{kl, p_a(k)}$ on which the application*

$$C \in V_{kl, p_a(k)} \mapsto \mathrm{cl}\,(Z_C) \in \mathrm{CH}_0(S)$$

is constant.

Here $V_{kl, p_a(k)}$ is a Severi variety related to the single surface S, and for $C \in V_{kl, p_a(k)}$, $\mathrm{cl}\,(Z_C)$ is the rational equivalence class of the singularity 0-cycle Z_C of the curve C.

Proof. By proposition 2.1, for $C \in |kL|$ generic and k large enough, $\varphi(C)$ is an irreducible nodal curve in $|klL|$, with geometric genus $p_a(k)$, and therefore $\varphi(C) \in V_{kl, p_a(k)}$. $V_{kl, p_a(k)}$ is of the expected dimension $p_a(k)$ by theorem 1.1, while $|kL|$ is a projective space of dimension

$p_a(k)$. So the subset of $V_{kl,p_a(k)}$ parametrizing the images of curves in $|kL|$ under the action of φ is an irreducible component V' of $V_{kl,p_a(k)}$.

Let C be a generic curve in $|kL|$, and write $Z_{\varphi(C)}$ for the 0-cycle of the singular points of its image $\varphi(C)$, seen as a 0-cycle in S. From the proof of proposition 2.1, we know that $\varphi_{|C} : C \to \varphi(C)$ is a normalization of $\varphi(C)$. The latter being an irreducible nodal curve, we have

$$2Z_{\varphi(C)} = K_{\varphi(C)} - (\varphi_{|C})_* K_C,$$

as 0-cycles in $\varphi_* C$. This proves that for another generic curve $C' \in |kL|$, the singularity 0-cycle $Z_{\varphi(C')}$ of the image $\varphi(C')$ is rationally equivalent to $Z_{\varphi(C)}$, as 0-cycles in S. Indeed, since C and C' are rationally equivalent, the adjunction formula tells us that $K_C = (K_S + C)|_C$ and $K_{C'} = (K_S + C')|_{C'}$ are rationally equivalent, as 0-cycles on S. $\varphi(C)$ and $\varphi(C')$ are rationally equivalent as well, since they both are in $|klL|$, and the adjunction formula tells us that $K_{\varphi(C)} = (K_S + \varphi(C))|_{\varphi(C)}$ and $K_{\varphi(C')} = (K_S + \varphi(C'))|_{\varphi(C')}$ are rationally equivalent, as 0-cycles on S.

$\square$

Remark 2.5 *In fact, one gets*

$$\mathrm{cl}\left(Z_{\varphi(C)}\right) = \frac{1}{2}k^2(l^2-1)L^2 = \frac{p_a(kl)-p_a(k)}{2g-2}L^2 \in \mathrm{CH}_0(S).$$

$\delta = p_a(kl) - p_a(k)$ is the number of nodes of $\varphi(C)$ for C generic. Using [BV04], we get

$$\mathrm{cl}\left(Z_{\varphi(C)}\right) = \delta c_X \in \mathrm{CH}_0(S),$$

where c_X is the rational equivalence class of any point of S that lie on a rational curve.

We now construct an irreducible component of $V_{kl,p_a(k)}$, which parametrizes curves with non constant rational equivalence class for their singularity 0-cycles.

Lemma 2.6 *Under the hypotheses of proposition 2.1, and for k large enough, there exists an irreducible component of $V_{kl,p_a(k)}$ on which the application*

$$C \in V_{kl,p_a(k)} \mapsto \mathrm{cl}\left(Z_C\right) \in \mathrm{CH}_0(S)$$

is non constant.

Proof. By theorem 1.1, there exists an irreducible family of dimension $p_a(k)-1$ of irreducible curves $C \in |kL|$, with only one node as singularity. We write Z_C for the 0-cycle on S defined by the singular point of C. $\varphi(C)$ is generically an irreducible curve in $|klL|$, with exactly $\delta+1$ nodes as singularities, $\delta = p_a(kl) - p_a(k)$ (this is proposition 2.1). In fact, one has

$$Z_{\varphi(C)} = \varphi_* Z_C + Z'_{\varphi(C)}$$

as 0-cycles on S, where $Z'_{\varphi(C)}$ is the sum of the singular points that appear when applying φ. As C moves, $Z'_{\varphi(C)}$ has constant rational equivalence class in $\mathrm{CH}_0(S)$, exactly as in the proof of lemma 2.4.

We thus have an irreducible $(p_a(k)-1)$-dimensional family of curves $\varphi(C) \in |klL|$. For each $\varphi(C)$, we smooth one of the nodes that are in $Z'_{\varphi(C)}$. This eventually gives an irreducible, $p_a(k)$-dimensional family of irreducible, nodal curves in $|klL|$, with exactly δ nodes. Such a family is an irreducible component V'' of $V_{kl,p_a(k)}$.

We claim that the rational equivalence class of the singularity 0-cycles $Z_{C'}$ of curves C' parametrized by V'' is non constant. This can be seen by the following simple consideration. For any points $x, y \in S$, and if k is large enough, we can find a curve $C \in |kL|$ with a node at x as its only singular point, and such that $\varphi(C)$ is nodal, with nodes at $\varphi(x)$ and y. Smoothing y, we get curves C' in V'', with singularity 0-cycle

$$Z_{C'} = \varphi(x) + Z'_{\varphi(C)} - y.$$

Since $Z'_{\varphi(C)}$ has constant rational equivalence class, fixing x and letting y move, we see that that the rational equivalence class $\mathrm{cl}\,(Z_{C'}) \in \mathrm{CH}_0(S)$ cannot be constant.

$\square$

Proof of theorem 2.3 We write the Stein factorization

$$\begin{tikzcd} & \mathcal{V}_{k,p} \arrow[dl] \arrow[d] \\ \widetilde{\mathcal{M}^\circ_{K3,g}} \arrow[r] & \mathcal{M}^\circ_{K3,g} \end{tikzcd}$$

of the projective morphism $\mathcal{V}_{k,p} \to \mathcal{M}^\circ_{K3,g}$. $\mathcal{V}_{k,p} \to \widetilde{\mathcal{M}^\circ_{K3,g}}$ is a projective morphism with connected fibers, while $\widetilde{\mathcal{M}^\circ_{K3,g}} \to \mathcal{M}^\circ_{K3,g}$ is finite. A point of $\widetilde{\mathcal{M}^\circ_{K3,g}}$ over $m \in \mathcal{M}^\circ_{K3,g}$ represents a connected component of $(\mathcal{V}_{k,l})_m$. The monodromy of this morphism thus acts as a subgroup of the permutation group of the connected components of fibers of $\mathcal{V}_{k,p} \to \mathcal{M}^\circ_{K3,g}$. Irreducibility of $\mathcal{V}_{k,p}$ is equivalent to the fact that the monodromy acts transitively on the components of the fibers $V_{k,l}$ (see *e.g.* [Har86]).

If there exists a dominant rational map $\varphi_m : S_m \dashrightarrow S_m$, satisfying $\varphi_m^* \mathcal{O}_{S_m}(1) \cong \mathcal{O}_{S_m}(l)$ for generic $m \in \mathcal{M}^\circ_{K3,g}$, then we have by lemmas 2.4 and 2.6 two irreducible components V'_m and V''_m of each generic fiber $(\mathcal{V}_{kl,p_a(k)})_m$ that are algebraically distinguished, since for curves parametrized by the first one, all singularity 0-cycles are rationally equivalent, and for curves parametrized by the other one, they are not. It follows that there exists an open subset

$$\begin{tikzcd} \widetilde{\mathcal{M}^\circ_{K3,g}} \arrow[d] \\ U \subset \mathcal{M}^\circ_{K3,g}, \end{tikzcd}$$

such that all fibers over U contain at least two points that are algebraically distinguished. The monodromy cannot exchange these two points. In particular it does not act transitively, and $\mathcal{V}_{kl,p_a(k)}$ is not irreducible.

$\square$

3 Properties of a self-rational map on a $K3$ surface

This section is devoted to the study of a dominant self-rational map on a given $K3$ surface. The observation of the geometry of an elimination of indeterminacies gives properties that this map must satisfy, and which of course restrain the possibilities for such a self-rational map to exist. We first get numerical relations between the algebraic and topological degree that are always valid. We then make further remarks depending on the complexity of the elimination of indeterminacies, and give a way to control this complexity.

The notations are as follows. S is a generic algebraic $K3$ surface. We assume in particular that $\mathrm{Pic}(S) = \mathbf{Z} \cdot L$, where L is effective and satisfies $L^2 = 2g-2$ ($g \in \mathbf{N}^*$). $\varphi : S \dashrightarrow S$ is a dominant rational map, and l the positive integer such that $\varphi^*\mathcal{O}_S(1) \cong \mathcal{O}_S(l)$. We assume $l > 1$. We consider an elimination of indeterminacies of φ, *i.e.* a commutative diagram

$$\begin{tikzcd} \widetilde{S} \arrow[d, "\tau"'] \arrow[dr, "\widetilde{\varphi}"] & \\ S \arrow[r, dashed, "\varphi"'] & S, \end{tikzcd}$$

where τ is a finite sequence of blow-ups

$$\widetilde{S} = S_p \xrightarrow{\varepsilon_p} S_{p-1} \xrightarrow{\varepsilon_{p-1}} \cdots \xrightarrow{\varepsilon_2} S_1 \xrightarrow{\varepsilon_1} S_0 = S.$$

We write F_i for the exceptional divisor which appears with ε_i, and E_i for $\varepsilon_p^* \circ \cdots \circ \varepsilon_{i+1}^* F_i$ ($1 \leqslant i \leqslant p$). $(\tau^*L, E_1, \ldots, E_p)$ is an orthogonal basis of $\mathrm{Pic}(\widetilde{S})$, and $E_i^2 = -1$ ($1 \leqslant i \leqslant p$).

3.1 Numerical properties

We start with a numerical observation coming from Hodge theory.

Proposition 3.1 *There exists an integer λ, such that*

$$\deg \varphi = \lambda^2.$$

Proof. Let ω be a global, nowhere vanishing, holomorphic 2-form on S. Since K_S is trivial, and $K_{\widetilde{S}} = \tau^* K_S + E_1 + \cdots + E_p$, where the E_i's are exceptional divisors, any global holomorphic 2-form on $\widetilde{S}$ is a multiple of $\tau^*\omega$. In particular there exists $\lambda \in \mathbf{C}$ such that $\widetilde{\varphi}^*\omega = \lambda \tau^* \omega$.

We write $\mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$ for the transcendental rational cohomology of S, that is the orthogonal in $\mathrm{H}^2(S, \mathbf{Q})$ of the Neron-Severi group $\mathrm{NS}(S)$, with respect to the intersection form $\langle\ ,\ \rangle$. We shall show that $\widetilde{\varphi}^*\eta = \lambda\tau^*\eta$ for all $\eta \in \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$

We clearly have $\mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}} \cong \mathrm{H}^2(\widetilde{S}, \mathbf{Q})_{\mathrm{tr.}}$ *via* τ^*, and since $\widetilde{\varphi}^*$ sends the transcendental cohomology classes of S to transcendental cohomology classes in $\widetilde{S}$, there exists a Hodge structure morphism

$$\psi : \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr}} \longrightarrow \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr}},$$

such that for all $\eta \in \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$, one has $\widetilde{\varphi}^*\eta = \tau^*(\psi(\eta))$. Now $\omega \in \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$, and $\widetilde{\varphi}^*\omega = \lambda\tau^*\omega$, so the eigenspace E_λ relative to λ for ψ is non nempty.

Suppose $E_\lambda \subset \mathrm{H}^2(S, \mathbf{C})_{\mathrm{tr}}$ is a proper subspace for S generic. Since $\lambda \in \mathbf{C}$ is algebraic over $\mathbf{Q}$, the equations defining E_λ are contained in the countable set of equations with coefficients in $\overline{\mathbf{Q}}$. This says that, when S moves, $\omega \in E_\lambda$ is contained in a countable union of proper linear subspaces of $\mathrm{H}^2(S, \mathbf{C})$. This contradicts the surjectivity of the period map for $K3$ surfaces : its image is an open set of a projective quadric in $\mathbf{P}\left(\mathrm{H}^2(S, \mathbf{C})\right)$ (see *e.g.* [Pal85]). We thus have $E_\lambda = \mathrm{H}^2(S, \mathbf{C})_{\mathrm{tr}}$ for S generic.

ψ acts on $\mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$ as multiplication by λ, so λ is necessarily a rational number. From the two equalities

$$\int_{\widetilde{S}} \widetilde{\varphi}^*\omega \wedge \widetilde{\varphi}^*\overline{\omega} = \deg(\varphi) \int_S \omega \wedge \overline{\omega},$$

and

$$\int_{\widetilde{S}} \widetilde{\varphi}^*\omega \wedge \widetilde{\varphi}^*\overline{\omega} = \lambda^2 \int_{\widetilde{S}} \tau^*\omega \wedge \tau^*\overline{\omega} = \lambda^2 \int_S \omega \wedge \overline{\omega},$$

we get $\deg(\varphi) = \lambda^2$. Since $\deg \varphi$ is an integer, and λ is a rational number, λ is necessarily an integer.

$\square$

We shall now prove a divisibility property involving λ. To do so, we need some further notations. Let us write

$$\widetilde{\varphi}^* L = l\tau^* L - \sum_{1 \leqslant i \leqslant p} \alpha_i E_i$$

for some integers $\alpha_1, \ldots, \alpha_p$. Since $\mathrm{Pic}(S) = \mathbf{Z} \cdot L$, and the E_i's are effective, there also exist non negative integers $\beta_1, \ldots, \beta_p$, such that for all i

$$\widetilde{\varphi}_* E_i = \beta_i L.$$

Note that by projection formula,

$$\alpha_i = \widetilde{\varphi}_*(\widetilde{\varphi}^* L \cdot E_i) = L \cdot \widetilde{\varphi}_* E_i = \beta_i L^2 = (2g-2)\beta_i.$$

Lemma 3.2 $2g-2$ *necessarily divides* $l - \lambda$.

Proof. Let λ be as in the proof of proposition 3.1. We have $\widetilde{\varphi}^* \eta' = \lambda \tau^* \eta'$ for all class $\eta' \in \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$. We have on the other hand $\widetilde{\varphi}^* c_1(L) = l\tau^* c_1(L) - (2g-2) \sum_i \beta_i [E_i]$.

Any $\eta \in \mathrm{H}^2(S, \mathbf{Q})$ decomposes over $\mathbf{Q}$ into $\eta = \eta' + \eta''$, where $\eta' \in \mathrm{H}^2(S, \mathbf{Q})_{\mathrm{tr.}}$, and $\eta'' = (\langle \eta, c_1(L) \rangle / (2g-2)) c_1(L)$. Then

$$\begin{aligned}
\widetilde{\varphi}^* \eta &= \lambda \tau^* \eta' + l \tau^* \eta'' - \langle \eta, c_1(L) \rangle \sum_i \beta_i [E_i] \\
&= \lambda \tau^* \eta + (l - \lambda) \tau^* \eta'' - \langle \eta, c_1(L) \rangle \sum_i \beta_i [E_i].
\end{aligned}$$

The intersection product is unimodular, and $c_1(L)$ is indivisible. So there exists a class $\eta_1 \in \mathrm{H}^2(S, \mathbf{Z})$, such that

$$\langle \eta_1, c_1(L) \rangle = 1.$$

It decomposes over $\mathbf{Q}$ into $\eta_1 = \eta_1' + \eta_1''$, and the equality

$$(l - \lambda) \tau^* \eta_1'' = \widetilde{\varphi}^* \eta_1 - \lambda \tau^* \eta_1 + \langle \eta_1, c_1(L) \rangle \sum_i \beta_i [E_i]$$

shows that

$$(l - \lambda) \tau^* \eta_1'' = \frac{l - \lambda}{2g-2} \tau^* c_1(L)$$

is an integral cohomology class. Since $c_1(L)$ is indivisible, this shows that $2g-2$ necessarily divides $l - \lambda$.

$\square$

We now look more specifically at the geometry of the elimination of indeterminacies. To have a more accurate description of the situation, we consider the proper transforms $\widehat{F}_i \subset \widetilde{S}$ of the F_i's, and introduce their intersection tree. Later on, we will call it the exceptional tree, or the ramification tree (recall that since K_S is trivial, the ramification divisor of $\widetilde{\varphi}$ and the total exceptional divisor of τ are equal, *cf.* proof of proposition 2.1). Its vertices are the $\widehat{F}_i$'s, and two vertices are connected if and only if the two corresponding divisors meet in $\widetilde{S}$. The descendants of a vertex $\widehat{F}_i$ are the vertices situated below $\widehat{F}_i$ in the tree, *i.e.* those corresponding to divisors whose projection by $\varepsilon_{i+1} \circ \cdots \circ \varepsilon_p$ is contained in F_i. The depth m_i of a vertex $\widehat{F}_i$ is the number of ancestors of $\widehat{F}_i$ in the tree, *i.e.* the number of points situated above $\widehat{F}_i$. The depth of the tree is the maximal depth of its vertices.

Example 3.3 *The following exceptional tree*

$$
\begin{tikzcd}
 & \widehat{F_1} \arrow[d, dash] & & \widehat{F_2} \arrow[d, dash] \\
 & \widehat{F_3} \arrow[dl, dash] \arrow[dr, dash] & & \widehat{F_4} \\
\widehat{F_5} & & \widehat{F_6} &
\end{tikzcd}
$$

is obtained by first blowing up S along two points ; $\widehat{F_1}$ and $\widehat{F_2}$ are the exceptional divisors above these two points. One then blows up the resulting surface along one point on $\widehat{F_1}$, and one point on $\widehat{F_2}$. Write $\widehat{F_3}$ (resp. $\widehat{F_4}$) for the exceptional divisor appearing above the blown up point on $\widehat{F_1}$ (resp. $\widehat{F_2}$). One finally blows up along two points of $\widehat{F_3}$. The descendants of $\widehat{F_3}$ are $\widehat{F_3}$, $\widehat{F_5}$ and $\widehat{F_6}$. Its ancestors are $\widehat{F_1}$ and $\widehat{F_3}$. Its depth is 2. The depth of the tree is 3.

This being set, $E_i = \varepsilon_p^* \circ \cdots \circ \varepsilon_{i+1}^* F_i$ is clearly the sum of all descendants of $\widehat{F_i}$ in the tree. In the above example we have $E_3 = \widehat{F_3} + \widehat{F_5} + \widehat{F_6}$. The canonical divisor of $\widetilde{S}$ is $K_{\widetilde{S}} = \varepsilon_p^* \circ \cdots \circ \varepsilon_1^* K_S + E_1 + \cdots + E_p$. Since K_S is trivial we have

$$K_{\widetilde{S}} = \sum_{1 \leqslant i \leqslant p} E_i = \sum_{1 \leqslant i \leqslant p} m_i \widehat{F_i}.$$

It is also the ramification divisor of the map $\widetilde{\varphi}$.

Let F be an exceptional divisor, such that τ does not contain any blow up along a point of F (*i.e.* an exceptional divisor which appears at the bottom of the exceptional tree). For a suitable choice of notations, this divisor can be supposed to be F_p. If F collapses under the action of $\widetilde{\varphi}$, then there necessarily exists a morphism $\widetilde{\varphi}_{p-1} : S_{p-1} \to S$, and a commutative diagram

$$
\begin{tikzcd}
S_p \arrow[d, "\varepsilon_p"'] \arrow[dddr, "\widetilde{\varphi}"] & \\
S_{p-1} \arrow[d, "\varepsilon_{p-1}"'] \arrow[ddr, "\widetilde{\varphi}_{p-1}"] & \\
\vdots \arrow[d, "\varepsilon_1"'] & \\
S \arrow[r, dashed, "\varphi"'] & S,
\end{tikzcd}
$$

that is another elimination of indeterminacies of φ involving one less exceptional divisor. We may thus assume τ to be minimal, in the sense that $\widetilde{\varphi}$ does not contract to a point any exceptional divisor which appears at the end of the exceptional tree.

The following equality is obtained simply by computing the self-intersection $(\widetilde{\varphi}L)^2$. It is the most important relation between $\deg \varphi$ and l. We use the minimality of τ to show the positivity of the β_i's.

Proposition 3.4 *The β_i's are all positive. In addition, the algebraic degree l and the topological degree of φ satisfy*

$$l^2 = \deg \varphi + (2g-2) \sum_{1 \leqslant i \leqslant p} \beta_i^2.$$

Proof. We have $E_i = \widehat{F}_i + \widehat{F}_{i_1} + \cdots + \widehat{F}_{i_q}$, where $\widehat{F}_i, \widehat{F}_{i_1}, \ldots, \widehat{F}_{i_q}$ are all the descendants of $\widehat{F}_i$ in the exceptional tree. Therefore

$$\beta_i = \gamma_i + \gamma_{i_1} + \cdots \gamma_{i_q},$$

where $\widetilde{\varphi}_* \widehat{F}_{i_s} = \gamma_{i_s} L$, $1 \leqslant s \leqslant q$. The γ_{i_s}'s are *a priori* non negative integers. $\widehat{F}_i$ has at least one descendant $\widehat{F}_{i_j}$ at the end of the exceptional tree. By minimality of τ, $\widetilde{\varphi}$ cannot contract $\widehat{F}_{i_j}$ to a point, and we have $\gamma_{i_j} \geqslant 1$. Finally

$$\beta_i \geqslant \gamma_{i_j} > 0.$$

We get the relation between l and $\deg \varphi$ simply by computing in two different ways the self-intersection $(\widetilde{\varphi}^* L)^2$. We have on the one hand

$$(\widetilde{\varphi}^* L)^2 = (\deg \widetilde{\varphi}) L^2 = (\deg \varphi)(2g-2),$$

and on the other hand

$$(\widetilde{\varphi}^* L)^2 = l^2 (\tau^* L)^2 + \sum_{1 \leqslant i \leqslant p} \alpha_i^2 E_i^2 = (2g-2) l^2 - (2g-2)^2 \sum_{1 \leqslant i \leqslant p} \beta_i^2,$$

which yields the announced formula.

□

We now get the following arithmetic property on the β_i's by some Riemann-Roch computations.

Lemma 3.5 *$\sum_{1 \leqslant i \leqslant p} \beta_i$ is even.*

Proof. We first show that $\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}}$ is a locally free sheaf of rank $r := \deg \varphi$. Since it is clearly torsion free, it is enough to show that any section defined on a punctured open set $U \setminus \{x_0\}$ extends in a unique way to a section defined over U (see [Bar77], lemma 1). So let $U \subset S$ be an open set, $x_0 \in U$, and $f \in \widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}}(U \setminus \{x_0\})$. f can be seen as a holomorphic function on $\widetilde{S}$, defined over $\widetilde{\varphi}^{-1}(U \setminus \{x_0\})$. If the fiber of $\widetilde{\varphi}$ above x_0 is a finite set of points, then the result is clear. Otherwise the fiber contains an irreducible exceptional curve F. f cannot be singular along F, since this would give by restriction a global section of $\mathcal{O}_{\widetilde{S}}(mF)_{|F}$ for some positive m, which is impossible, since $F^2 < 0$. So f has only isolated singularities along F, and therefore extends to a function over $\widetilde{\varphi}^{-1}(U)$.

Now it is an easy consequence of Grauert's theorem that $R^i \widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}} = 0$ for $i > 0$. This gives $\widetilde{\varphi}_! \mathcal{O}_{\widetilde{S}} = \widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}}$, and we thus have

$$\begin{aligned} ch(\widetilde{\varphi}_! \mathcal{O}_{\widetilde{S}}).td(T_S) &= \left(r[S] + c_1(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}}) + \frac{c_1(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}})^2 - 2c_2(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}})}{2} \right).([S]+2) \\ &= r[S] + c_1(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}}) + \left(\frac{c_1(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}})^2 - 2c_2(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}})}{2} + 2r \right). \end{aligned}$$

On the other hand, we have

$$\widetilde{\varphi}_* \left(ch(\mathcal{O}_{\widetilde{S}}).td(T_{\widetilde{S}}) \right) = \widetilde{\varphi}_* \left([\widetilde{S}] - \frac{1}{2}(E_1 + \cdots + E_p) + 2 \right) = r[S] - \frac{1}{2} \left(\textstyle\sum_i \beta_i \right) L + 2,$$

so the Grothendieck-Riemann-Roch formula gives

$$c_1(\widetilde{\varphi}_* \mathcal{O}_{\widetilde{S}}) = -\frac{1}{2} \left(\textstyle\sum_i \beta_i \right) L.$$

Since L is indivisible, the lemma follows.

□

3.2 Complexity of an elimination of indeterminacies

To motivate the study of the complexity of the elimination of indeterminacies, we first show that we have further numerical constraints on φ when the elimination of indeterminacies is not too complicated. The following numerical property is true under the hypothesis that the exceptional tree has depth smaller than 2.

Proposition 3.6 *If the differential $d\widetilde{\varphi}$ does not vanish identically along any curve of $\widetilde{S}$, then the topological degree of φ satisfies the inequality*

$$\deg \varphi \leqslant 1 + \frac{1}{24}\left[p + 4(g-1)\textstyle\sum_i \beta_i\right].$$

The condition on the differential is satisfied as soon as the total depth of the exceptional tree is non greater than 2.

Proof. We follow an idea of Amerik, Rovinsky and Van de Ven ([ARV99], see [Bea01] as well). The fiber bundle $\Omega^1_S(2)$ is generated by its global sections, so by lemma 1.1 of [ARV99] a generic section $\sigma \in \mathrm{H}^0(S, \Omega^1_S(2))$ has isolated zeroes. With the assumption made on $d\widetilde{\varphi}$ this is also true for the pull-back section $\widetilde{\varphi}^*\sigma \in \mathrm{H}^0(\widetilde{S}, \Omega^1_{\widetilde{S}}(2\widetilde{\varphi}^*L))$. Counting these zeroes yields the inequality on Chern classes

$$c_2\left(\Omega^1_{\widetilde{S}}(2\widetilde{\varphi}^*L)\right) \geqslant (\deg\varphi) c_2\left(\Omega^1_S(2)\right).$$

The left-hand side of this inequality is

$$c_2(\Omega^1_{\widetilde{S}}) + 2\widetilde{\varphi}^* c_1(L)\cdot c_1(\Omega^1_{\widetilde{S}}) + 4\widetilde{\varphi}^* c_1(L)^2,$$

and its right-hand side is

$$\deg\varphi\left[c_2(\Omega^1_S) + 2c_1(L)\cdot c_1(\Omega^1_S) + 4c_1(L)^2\right].$$

Now $\widetilde{\varphi}^* c_1(L)^2 = (\deg\varphi) c_1(L)^2$, so we get

$$c_2(\Omega^1_{\widetilde{S}}) + 2\widetilde{\varphi}^* c_1(L)\cdot c_1(\Omega^1_{\widetilde{S}}) \geqslant \deg\varphi\left[c_2(\Omega^1_S) + 2c_1(L)\cdot c_1(\Omega^1_S)\right],$$

that is

$$\chi_{\mathrm{top}}(\widetilde{S}) + 2\widetilde{\varphi}^* L\cdot K_{\widetilde{S}} \geqslant \deg\varphi\left[\chi_{\mathrm{top}}(S) + 2L\cdot K_S\right],$$

where χ_{top} denotes the topological Euler-Poincaré characteristic, that is the alternated sum of the Betti numbers. It is 24 for all $K3$ surfaces. $\widetilde{S}$ is obtained from S by successively blowing up along p points so $\chi_{\mathrm{top}}(\widetilde{S}) = 24 + p$. We also have $K_S = 0$, and

$$\widetilde{\varphi}^* L\cdot K_{\widetilde{S}} = (l\tau^*L - \textstyle\sum_i \alpha_i E_i)\cdot(E_1 + \cdots + E_p) = \textstyle\sum_i \alpha_i.$$

We eventually get

$$24 + p + 2\textstyle\sum_i \alpha_i \geqslant 24\deg\varphi,$$

which yields the desired inequality with the relations $\alpha_i = (2g-2)\beta_i$.

Now suppose $d\widetilde{\varphi}$ vanishes identically along a curve C in $\widetilde{S}$. Then C necessarily collapses under the action of $\widetilde{\varphi}$, and it appears with multiplicity at least 2 in its ramification divisor. Indeed, let f be some local equation for C. If $d\widetilde{\varphi}$ vanishes with order μ along C, then it writes

$$d\widetilde{\varphi} = \begin{pmatrix} f^\mu g_{11} & f^\mu g_{12} \\ f^\mu g_{21} & f^\mu g_{22} \end{pmatrix}$$

in some local holomorphic coordinate system, with the g_{ij} holomorphic, and $\bigwedge^2 d\widetilde{\varphi}$ vanishes with order 2μ along C. If the total depth of the exceptional tree is less than 2, the only curves which appear with multiplicity greater than 2 in the ramification divisor are at the end of the tree, and cannot be contracted to a point by $\widetilde{\varphi}$ by minimality of the elimination of indeterminacies. So in this case, $d\widetilde{\varphi}$ does not vanish identically along any curve of $\widetilde{S}$.

□

The first step towards a control of the complexity of the elimination of indeterminacies is made with the following basic remark. It shows that the depth of the exceptional tree is controlled by the topological degree.

Proposition 3.7 *(i) The depth m of the exceptional tree always satisfy*

$$m \leqslant \deg \varphi - 2.$$

(ii) If the tree has two connected components of depths m' and m'', then

$$m' + m'' \leqslant \deg \varphi - 2.$$

In particular, if one has equality in (i), then the tree only has one connected component.

Proof. *(i)* Since the ramification divisor of $\widetilde{\varphi}$ is $\sum_i m_i \widehat{F}_i$, it is clear that

$$m = \max m_i \leqslant \deg \varphi - 1.$$

Now suppose there exists an irreducible exceptional curve F that has depth $\deg \varphi - 1$ in the exceptional tree. Then it is at the end of the tree, and therefore is not contracted. F appears in the ramification divisor with multiplicity $\deg \varphi - 1$, and thus

$$\widetilde{\varphi}^{-1}\left(\widetilde{\varphi}(F)\right) = (\deg \varphi) F + E,$$

where E is contracted by $\widetilde{\varphi}$. In particular, E is exceptional for τ as F is. It follows that $\widetilde{\varphi}^{-1}\left(\widetilde{\varphi}(F)\right)$ is supported on the exceptional divisor of τ, which implies that it has negative self-intersection. This contradicts the fact that

$$\widetilde{\varphi}^{-1}\left(\widetilde{\varphi}(F)\right)^2 = (\deg \varphi) \widetilde{\varphi}(F)^2 > 0.$$

(ii) If the tree has two connected components of depths m' and m'', then we have two irreducible exceptional curves F' and F'' of depths m' and m'', that are not contracted, and that do not meet in $\widetilde{S}$. The image curves $\widetilde{\varphi}(F')$ and $\widetilde{\varphi}(F'')$ intersect in S, because their images have their class proportional to $c_1(L)$. Let x be an intersection point. There are at least two distinct points $x' \in F'$ and $x'' \in F''$ in $\widetilde{\varphi}^{-1}(x)$. Since F' and F'' appear with multiplicities m' and m'' in the ramification divisor of $\widetilde{\varphi}$, x' and x'' appear with multiplicities $m'+1$ and $m''+1$ in $\widetilde{\varphi}^{-1}(x)$. This implies

$$m' + m'' + 2 \leqslant \deg \widetilde{\varphi}.$$

□

Remark 3.8 *In fact, (ii) can be extended as follows : if there exist two distinct curves F' and F'' at the end of the exceptional tree, which have depths m' and m'', then*

$$m' + m'' + 2 \leqslant \deg \widetilde{\varphi}.$$

Now the following result gives control on another aspect of the elimination of indeterminacies, namely the number of blown-up points on S. It says that in case all irreducible exceptional curves are disjoint, then their number is bounded from above. The hypothesis is equivalent to the fact that we can eliminate the indeterminacies of φ by the single blow up of finitely many distinct points on S.

Proposition 3.9 *If the exceptional tree has depth 1, then*

$$p \leqslant 8(\deg\varphi - 1).$$

Proof. In this case, the ramification divisor of $\widetilde{\varphi}$ is $E_1 + \cdots + E_p$, where the E_i's are disjoint $\mathbf{P}^1$'s, and by minimality of the elimination of indeterminacies, $\widetilde{\varphi}$ does not contract any of them. So the differential $d\widetilde{\varphi}$ is surjective, and we have an exact sequence

$$0 \to \widetilde{\varphi}^*\Omega_S \to \Omega_{\widetilde{S}} \to \bigoplus_i L_i \to 0,$$

where each L_i is a line bundle on the exceptional curve E_i. This gives

$$\begin{array}{rcl} c_2\left(\Omega_{\widetilde{S}}\right) & = & c_2\left(\widetilde{\varphi}^*\Omega_S\right) + c_1\left(\widetilde{\varphi}^*\Omega_S\right) \cdot c_1\left(\bigoplus_i L_i\right) + c_2\left(\bigoplus_i L_i\right) \\ & = & (\deg\varphi) c_2\left(\Omega_S\right) + \sum_i c_2(L_i). \end{array}$$

By restriction, we get on each E_i an exact sequence

$$0 \to K_i \to \Omega_{\widetilde{S}|E_i} \to L_i \to 0,$$

where K_i is a line bundle on E_i. We have a map $K_i \to \Omega_{E_i}$, given by the composition

$$\begin{tikzcd} & \Omega_{\widetilde{S}|E_i} \arrow[d] \\ K_i \arrow[ur, "(d\widetilde{\varphi})^t"] \arrow[r] & \Omega_{E_i}. \end{tikzcd}$$

Since $\widetilde{\varphi}$ is ramified along E_i, the map $K_i \to I_i/I_i^2$ induced by $(d\widetilde{\varphi})^t$ is zero (here $I_i \subset \mathcal{O}_{\widetilde{S}}$ is the ideal sheaf of E_i). This shows that the above map $K_i \to \Omega_{E_i}$ is an injection, and thus that

$$\deg K_i \leqslant \deg \Omega_{E_i} = -2$$

(as line bundles on $E_i \cong \mathbf{P}^1$). On the other hand, $\deg(\Omega_{\widetilde{S}|E_i}) = -1$ by the conormal exact sequence, so one has

$$\deg(L_i) = \deg(\Omega_{\widetilde{S}|E_i}) - \deg(K_i) \geqslant 1.$$

We write d_i for $\deg(L_i)$ (*i.e.* $L_i = \mathcal{O}_{E_i}(d_i)$). The restriction exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}}(-(d_i+1)E_i) \to \mathcal{O}_{\widetilde{S}}(-d_iE_i) \to \mathcal{O}_{E_i}(d_i) \to 0$$

gives the two relations

$$\left\{ \begin{array}{rcl} c_1(L_i) - (d_i+1)E_i & = & -d_iE_i \\ c_2(L_i) - (d_i+1)E_i \cdot c_1(L_i) & = & 0, \end{array} \right.$$

and therefore

$$c_2(L_i) = -d_i - 1 \leqslant -2.$$

So eventually

$$\begin{aligned} c_2\left(\Omega_{\widetilde{S}}\right) &= (\deg\varphi)c_2\left(\Omega_S\right) + \textstyle\sum_i c_2(L_i) \\ &\leqslant 24\deg\varphi - 2p, \end{aligned}$$

and since one knows that $c_2\left(\Omega_{\widetilde{S}}\right) = 24 + p$, one gets the announced inequality.

$\square$

In case $\deg\varphi = 4$, propositions 3.7 and 3.9 work very well. The following proposition sums up what they learn us in this case.

Example 3.10 *If* $\deg\varphi = 4$*, then either the exceptional tree has depth* 1 *and there are less than* 24 *blown-up points, or it has depth* 2 *and there is only one blown up point.*

Note that example 3.10 shows that if $\deg\varphi = 4$, then the hypotheses of proposition 3.6 are always satisfied. To conclude, we compute the first possible couples $(\deg\varphi, l)$ for which there could actually be a self-rational map φ, according to all numerical properties gathered above. Recall that $K3$ surfaces of genera 2, 3, 4 and 5 are respectively double covers of $\mathbf{P}^2$, quartics in $\mathbf{P}^3$, complete intersections of a cubic and a quadric in $\mathbf{P}^4$, and complete intersections of three quadrics in $\mathbf{P}^5$.

Example 3.11 *For* $\deg\varphi = 4$*, the first possible values of* l *possible are given by*

g	2	3	4	5
l	$6, 8, 10, \dots$	$6, 10, 14, \dots$	$8, 10, 14, \dots$	$6, 10, 14, \dots$

For $\deg\varphi = 9$*, we get*

g	2	3	4	5
l	$5, 7, 9, \dots$	$5, 7, 9, \dots$	$9, 15, 21, \dots$	$5, 11, 13, 19, \dots$

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