

Standing waves in nonlinear Schrödinger equations

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Abstract. In the theory of nonlinear Schrödinger equations, it is expected that the solutions will either spread out because of the dispersive effect of the linear part of the equation or concentrate at one or several points because of nonlinear effects. In some remarkable cases, these effects counterbalance, and special solutions that neither disperse nor focus appear, the so-called *standing waves*. For the physical applications as well as for the mathematical properties of the equation, a fundamental issue is the stability of waves with respect to perturbations. Our purpose in these notes is to present various methods developed to study the existence and stability of standing waves. We prove the existence of standing waves by using a variational approach. When stability holds, it is obtained by proving a coercivity property for a linearized operator. Another approach based on variational and compactness arguments is also presented. When instability holds, we show by a method combining a virial identity and variational arguments that the standing waves are unstable by blow-up.

Keywords. Nonlinear Schrödinger equation, standing waves, orbital stability, instability, blow-up, variational methods.

AMS classification. 35Q55, 35Q51, 35B35, 35A15.

1 Introduction

In these lecture notes, we consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0. \quad (1.1)$$

Equation (1.1) arises in various physical and biological contexts, for example in nonlinear optics, for Bose–Einstein condensates, in the modelling of the DNA structure, etc. We refer the reader to [14, 73] for more details on the physical background and references.

Here, the unknown u is a complex valued function of $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ for $N \geq 1$,

$$u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}.$$

In most of the applications, t stands for the time and x for the space variable, but sometimes it can be the converse, for example in nonlinear optics. The number i is the imaginary unit ($i^2 = -1$), ∂_t is the first derivative with respect to the time t , Δ denotes the Laplacian with respect to the space variable x ($\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$). Finally, $p \in \mathbb{R}$ is such that $p > 1$, which means that the equation is superlinear.

To simplify the exposition, we have restricted ourselves to the study of nonlinear Schrödinger equations with a power-type nonlinearity, but one can also consider more general versions of these equations, see [14, 73] and the references cited therein.

The purpose of these notes is to study the properties of particular solutions of (1.1) of the form $e^{i\omega t}\varphi(x)$ with $\omega \in \mathbb{R}$ and φ satisfying

$$-\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0. \quad (1.2)$$

Such solutions are called *standing waves*. They are part of a more general class of solutions arising for various nonlinear equations like Korteweg–de Vries, Klein–Gordon or Kadomtsev–Petviashvili equations. These equations enjoy special solutions whose *profile* remains unchanged under the evolution in time. These special solutions are called *solitary waves* or *solitons* (see [14, 18, 20, 73] for an overview of physical and mathematical questions around solitary waves).

This kind of phenomena was discovered in 1834 by John Scott Russel. The Scottish engineer was supervising work in a canal near Edinburgh when he observed that the brutal stop of a boat in the canal was creating a wave that does not seem to vanish. Indeed, he was able to follow this wave horseback on a distance of several miles. His life long, he studied this surprising phenomenon but without being able to give it a theoretical justification. In fact, most of the scientists of that time did believe that such a wave, which does not disperse, could not exist. The first theoretical justification of the existence of solitary waves was given by Korteweg and de Vries in 1895. They derived an equation for the motion of water admitting solitary wave solutions. But one had to wait until the 1950's to see the beginning of an intensive research from mathematicians and physicists about solitary waves.

Heuristically, such solutions appear because of the balance of two contradictory effects: the dispersive effect of the linear part, which tends to flatten the solution as time goes on (see Figure 1), and the focusing effect of the nonlinearity, which tends to concentrate the solution, provided the initial datum is large enough (see Figure 2). For

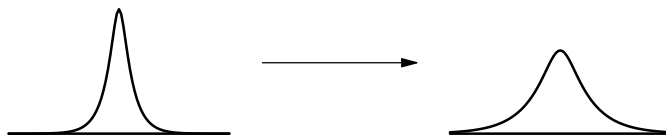


Figure 1. Dispersive effect.

some special initial data, each effect compensates the other and the general profile of the initial data remains unchanged (see Figure 3).

In the study of standing waves, two types of questions arise naturally.

- (i) Do standing waves exist, i.e., does (1.2) admit nontrivial solutions? If yes, what kind of properties for the solutions of (1.2) can be shown: Are they regular, what is their decay at infinity, can we find variational characterizations for at least some of these solutions?

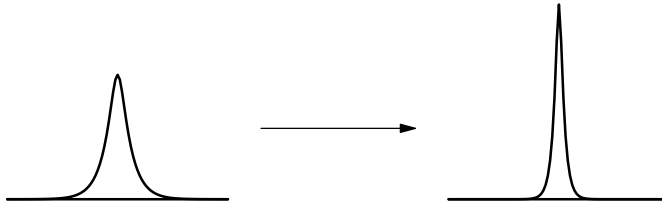


Figure 2. Focusing effect.

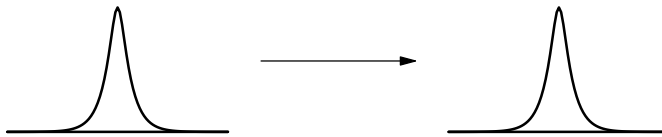


Figure 3. Balance between dispersion and focusing.

- (ii) If standing waves exist, are they stable (in a sense to be made precise) as solutions of (1.1)? If they are unstable, what is the nature of instability?

The first type of questions is related to the study of existence of solutions for semilinear elliptic problems. The methods used in this context are often of variational nature and solutions are obtained by minimization under constraint or with min-max arguments. The second type of questions concerns the dynamics of the evolution equation. Nevertheless, both problems are intimately related. Indeed, informations of variational nature on the solutions of the stationary equation are essential to derive stability or instability results.

The rest of these notes is divided into four sections and an appendix. We first give basic results on the Cauchy problem for (1.1) in Section 2. In Section 3, we study the existence of standing waves. Section 4 is devoted to stability whereas Section 5 deals with instability. The proof of a stability criterion is given in the appendix.

Notation. The space of complex measurable functions whose r -th power is integrable will be denoted by $L^r(\mathbb{R}^N)$ and its standard norm by $\|\cdot\|_{L^r(\mathbb{R}^N)}$. When $r = 2$, the space $L^2(\mathbb{R}^N)$ will be endowed with the real inner product

$$(u, v)_2 = \operatorname{Re} \int_{\mathbb{R}^N} u \bar{v} dx \text{ for } u, v \in L^2(\mathbb{R}^N).$$

The space of functions from $L^r(\mathbb{R}^N)$ whose distributional derivatives of order less than or equal to m are elements of $L^r(\mathbb{R}^N)$ will be denoted by $W^{m,r}(\mathbb{R}^N)$. If $m = 1$ and

$r = 2$, we shall denote $W^{1,2}(\mathbb{R}^N)$ by $H^1(\mathbb{R}^N)$, its usual norm by $\|\cdot\|_{H^1(\mathbb{R}^N)}$ and the duality product between the dual space $H^{-1}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ by $\langle \cdot, \cdot \rangle$. If $m = r = 2$, we denote $W^{2,2}(\mathbb{R}^N)$ by $H^2(\mathbb{R}^N)$.

For notational convenience, we shall sometimes identify a function and its value at some point. For example, we shall write $e^{i\omega t}\varphi(x)$ for the function $(t, x) \mapsto e^{i\omega t}\varphi(x)$. Similarly, we shall say that $|x|v \in L^2(\mathbb{R}^N)$ if the function $x \mapsto |x|v(x)$ belongs to $L^2(\mathbb{R}^N)$. For a solution u of (1.1), we shall denote by $u(t)$ the function $x \mapsto u(t, x)$. Therefore, depending on the context, u may denote either a mapping from \mathbb{R} to some function space or a mapping from $\mathbb{R} \times \mathbb{R}^N$ to \mathbb{C} .

We make the convention that when we take a subsequence of a sequence (v_n) we denote it again by (v_n) .

The letter C will denote various positive constants whose exact values may change from line to line but are not essential in the course of the analysis.

2 The Cauchy problem

For the physical properties of the model as well as for the mathematical study of the equation, it is interesting to look for quantities conserved along the time. At least formally, equation (1.1) admits three conserved quantities. The first one is the *mass* or *charge*: If u satisfies (1.1) with initial datum $u(0) = u_0$ then

$$Q(u(t)) := \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 \equiv Q(u_0). \quad (2.1)$$

The conservation of mass is obtained from multiplying (1.1) with \bar{u} , integrating over \mathbb{R}^N and taking the imaginary part. The second conserved quantity is the *energy* (multiply (1.1) by $\partial_t \bar{u}$, integrate over \mathbb{R}^N and take the real part),

$$E(u(t)) := \frac{1}{2} \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{p+1} \|u(t)\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \equiv E(u_0). \quad (2.2)$$

Finally, multiplying (1.1) by $\nabla \bar{u}$, integrating over \mathbb{R}^N and taking the real part, we obtain the conservation of *momentum*,

$$M(u(t)) := \operatorname{Im} \int_{\mathbb{R}^N} u(t) \nabla \bar{u}(t) dx \equiv M(u_0).$$

Among these three conserved quantities, the mass and the energy are real-valued, but the momentum may be complex-valued, which makes it less convenient to use.

To benefit from the conserved quantities, it is natural to search solutions of (1.1) in function spaces where these quantities are well-defined. Consequently, we look for solutions of (1.1) in $H^1(\mathbb{R}^N)$ and restrict the range of p to be subcritical for the Sobolev embedding of $H^1(\mathbb{R}^N)$ into $L^{p+1}(\mathbb{R}^N)$, i.e., $1 < p < 1 + \frac{4}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ if $N = 1, 2$. Then we have the following result concerning the well-posedness of the Cauchy problem for (1.1) (see [14] and the references cited therein).

Proposition 2.1. *Let $1 < p < 1 + \frac{4}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ if $N = 1, 2$. For every $u_0 \in H^1(\mathbb{R}^N)$ there exists a unique maximal solution u of (1.1), $T_{\min} \in [-\infty, 0)$, $T^{\max} \in (0, +\infty]$ such that $u(0) = u_0$ and*

$$u \in \mathcal{C}((T_{\min}, T^{\max}); H^1(\mathbb{R}^N)) \cap \mathcal{C}^1((T_{\min}, T^{\max}); H^{-1}(\mathbb{R}^N)).$$

Furthermore, we have the conservation of charge, energy and momentum: For all $t \in (T_{\min}, T^{\max})$,

$$Q(u(t)) = Q(u_0), \quad E(u(t)) = E(u_0), \quad M(u(t)) = M(u_0). \tag{2.3}$$

Finally, we have the blow-up alternative:

- If $T_{\min} > -\infty$ then $\lim_{t \downarrow T_{\min}} \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 = +\infty$.
- If $T^{\max} < +\infty$ then $\lim_{t \uparrow T^{\max}} \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 = +\infty$.

From now on, it will be understood that $1 < p < 1 + \frac{4}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ if $N = 1, 2$.

In view of Proposition 2.1, it is natural to ask under which circumstances global existence holds or blow-up occurs. The following proposition gives an answer to the question of global existence.

Proposition 2.2. *If $1 < p < 1 + \frac{4}{N}$ then (1.1) is globally well-posed, i.e., for any solution u given by Proposition 2.1, $T_{\min} = -\infty$ and $T^{\max} = +\infty$.*

The proof of Proposition 2.2 relies on the Gagliardo–Nirenberg inequality (see [1]): There exists a constant $C > 0$ such that for all $v \in H^1(\mathbb{R}^N)$

$$\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \leq C \|\nabla v\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2}} \|v\|_{L^2(\mathbb{R}^N)}^{p+1 - \frac{N(p-1)}{2}}. \tag{2.4}$$

Proof of Proposition 2.2. Let $1 < p < 1 + \frac{4}{N}$ and u be a solution of (1.1) as in Proposition 2.1. We prove the assertion by contradiction. Assume that $T^{\max} < +\infty$ and therefore $\lim_{t \uparrow T^{\max}} \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 = +\infty$. By (2.4), we have

$$E(u(t)) \geq \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 \left(\frac{1}{2} - C \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2} - 2} \|u(t)\|_{L^2(\mathbb{R}^N)}^{p+1 - \frac{N(p-1)}{2}} \right).$$

In view of the conservation of charge and energy (see (2.3)), this implies

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 \left(1 - \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2} - 2} \right) < C \text{ for all } t \in (T_{\min}, T^{\max}). \tag{2.5}$$

Now, since $p < 1 + \frac{4}{N}$, we have $\frac{N(p-1)}{2} - 2 < 0$ and thus letting $\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2$ go to $+\infty$ when t goes to T^{\max} leads to a contradiction in (2.5). Arguing in the same way if $T_{\min} > -\infty$ leads to the same contradiction and completes the proof. \square

Concerning blow-up, we can give the following sufficient condition.

Proposition 2.3. Assume that $p \geq 1 + \frac{4}{N}$ and let $u_0 \in H^1(\mathbb{R}^N)$ be such that

$$|x|u_0 \in L^2(\mathbb{R}^N) \text{ and } E(u_0) < 0.$$

Then the solution u of (1.1) corresponding to u_0 blows up in finite time for positive and negative time, that is

$$T_{\min} > -\infty \text{ and } T^{\max} < +\infty.$$

The proof of Proposition 2.3 relies on the *virial theorem* (the term “virial theorem” comes from the analogy to the virial theorem in classical mechanics).

Proposition 2.4 (Virial theorem). Let $u_0 \in H^1(\mathbb{R}^N)$ be such that $|x|u_0 \in L^2(\mathbb{R}^N)$ and let u be the solution of (1.1) corresponding to u_0 . Then $|x|u(t) \in L^2(\mathbb{R}^N)$ for all $t \in (T_{\min}, T^{\max})$ and the function $f : t \mapsto \|xu(t)\|_{L^2(\mathbb{R}^N)}^2$ is of class \mathcal{C}^2 and satisfies

$$\begin{aligned} f'(t) &= 4\operatorname{Im} \int_{\mathbb{R}^N} \bar{u}(t)x \cdot \nabla u(t) dx, \\ f''(t) &= 8P(u(t)), \end{aligned}$$

where P is given for $v \in H^1(\mathbb{R}^N)$ by

$$P(v) := \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 - \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}. \quad (2.6)$$

The virial theorem comes from the work of Glassey [31] in which the identities for f' and f'' were formally derived. For a rigorous proof, see [14].

Proof of Proposition 2.3. First, we remark that

$$P(u(t)) = 2E(u(t)) + \frac{4 - N(p-1)}{2(p+1)} \|u(t)\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.$$

Since $p \geq 1 + \frac{4}{N}$, we have in view of the conservation of energy (see (2.3))

$$P(u(t)) \leq 2E(u(t)) = 2E(u_0) < 0 \text{ for all } t \in (T_{\min}, T^{\max}).$$

Therefore, by Proposition 2.4, we have

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2(\mathbb{R}^N)}^2 \leq 16E(u_0) \text{ for all } t \in (T_{\min}, T^{\max}).$$

Integrating twice in time gives

$$\|xu(t)\|_{L^2(\mathbb{R}^N)}^2 \leq 8E(u_0)t^2 + \left(4\operatorname{Im} \int_{\mathbb{R}^N} \bar{u}_0 x \cdot \nabla u_0 dx\right) t + \|xu_0\|_{L^2(\mathbb{R}^N)}^2. \quad (2.7)$$

The right member of (2.7) is a polynomial of order two with the main coefficient negative. Hence for $|t|$ large the right-hand side of (2.7) becomes negative, which is in contradiction with $\|xu(t)\|_{L^2(\mathbb{R}^N)}^2 \geq 0$. This implies $T_{\min} > -\infty$ and $T^{\max} < +\infty$ and finishes the proof. \square

3 Existence, uniqueness and properties of solitons

We will use the following definition of standing waves for (1.1).

Definition 3.1. A *standing wave* or *soliton* of (1.1) is a solution of the form $e^{i\omega t}\varphi(x)$ with $\omega \in \mathbb{R}$ and φ satisfying

$$\begin{cases} -\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \\ \varphi \in H^1(\mathbb{R}^N) \setminus \{0\}. \end{cases} \tag{3.1}$$

Many techniques have been developed to study the existence of solutions to problems of type (3.1) (see, e.g., [3]). In this section, we look for solutions of (3.1) by using variational methods (see, e.g., [71] for a general overview).

For the study of solutions of (3.1), we define a functional $S : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ by setting for $v \in H^1(\mathbb{R}^N)$

$$S(v) := \frac{1}{2}\|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \frac{\omega}{2}\|v\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.$$

The functional S is often called *action*. It is standard that S is of class \mathcal{C}^2 (see, for example, [78]) and for $v \in H^1(\mathbb{R}^N)$ the Fréchet derivative of S at v is given by

$$S'(v) = -\Delta v + \omega v - |v|^{p-1}v.$$

Therefore, φ is a solution of (3.1) if and only if $\varphi \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $S'(\varphi) = 0$. In other words, the nontrivial critical points of S are the solutions of (3.1). Therefore, to prove existence of solutions of (3.1) it is enough to find a nontrivial critical point of S .

This section is divided as follows. First, we prove that if solutions to (3.1) exist then they are regular, exponentially decaying at infinity and satisfy some functional identities. Next, we prove the existence of a nontrivial critical point of S . Finally, we derive various variational characterizations for some special solutions of (3.1).

3.1 Preliminaries

Before studying existence of solutions to (3.1), it is convenient to prove that, if such solutions exist, they necessarily enjoy the following properties.

Proposition 3.2. *Let $\omega > 0$. If $\varphi \in H^1(\mathbb{R}^N)$ satisfies (3.1) then φ is regular and exponentially decaying. More precisely,*

- (i) $\varphi \in W^{3,r}(\mathbb{R}^N)$ for all $r \in [2, +\infty)$, in particular $\varphi \in \mathcal{C}^2(\mathbb{R}^N)$;
- (ii) there exists $\varepsilon > 0$ such that $e^{\varepsilon|x|}(|\varphi| + |\nabla\varphi|) \in L^\infty(\mathbb{R}^N)$.

Sketch of proof. Point (i) follows from the usual elliptic regularity theory by a bootstrap argument (see [30]). We just indicate how to initiate the bootstrap and refer to [14] for a detailed proof. Let φ be a solution of (3.1). Suppose that $\varphi \in L^q(\mathbb{R}^N)$ for some $q > p$. Since $|\varphi|^{p-1}\varphi \in L^{\frac{q}{p}}(\mathbb{R}^N)$ and φ satisfies

$$-\Delta\varphi + \omega\varphi = |\varphi|^{p-1}\varphi, \tag{3.2}$$

it follows that $\varphi \in W^{2, \frac{q}{p}}(\mathbb{R}^N)$. Choosing any $q \in (p, +\infty)$ if $N = 1, 2$ and $q = \frac{2N}{N-2}$ if $N \geq 3$ and repeating the previous argument recursively, we obtain after a finite number of steps that $\varphi \in W^{2, r}(\mathbb{R}^N)$ for any $r \in [2, +\infty)$. This implies that $|\varphi|^{p-1}\varphi \in W^{1, r}(\mathbb{R}^N)$ for any $r \in [2, +\infty)$ and it follows from (3.2) that $\varphi \in W^{3, r}(\mathbb{R}^N)$ for all $r \in [2, +\infty)$, hence (i).

If φ is radial, i.e., $\varphi(x) = \varphi(|x|)$ then φ satisfies

$$-\varphi'' - \frac{N-1}{r}\varphi' + \omega\varphi - |\varphi|^{p-1}\varphi = 0.$$

Here, we have employed that $\Delta\varphi = \varphi'' + \frac{N-1}{r}\varphi'$ if φ is radial. In this case, the exponential decay of φ at infinity follows from the classical theory of second-order ordinary differential equations (see, for example, [37]). See [14] for a proof of (ii) for non-radial solutions of (3.1). □

Lemma 3.3. *If $\varphi \in H^1(\mathbb{R}^N)$ satisfies (3.1) then the following identities hold:*

$$\|\nabla\varphi\|_{L^2(\mathbb{R}^N)}^2 + \omega\|\varphi\|_{L^2(\mathbb{R}^N)}^2 - \|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = 0, \tag{3.3}$$

$$\|\nabla\varphi\|_{L^2(\mathbb{R}^N)}^2 - \frac{N(p-1)}{2(p+1)}\|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = 0. \tag{3.4}$$

In the literature, (3.4) is called the *Pohozaev identity*, since it was first derived by Pohozaev in 1965, see [64]. Actually, one can obtain this type of identity for a large class of nonlinearities, see [9]. The set

$$\{v \in H^1(\mathbb{R}^N); v \neq 0, \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \omega\|v\|_{L^2(\mathbb{R}^N)}^2 - \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = 0\} \tag{3.5}$$

is called the *Nehari manifold*.

Proof of Lemma 3.3. Let $\varphi \in H^1(\mathbb{R}^N)$ be a solution of (3.1).

To obtain (3.3), we simply multiply (3.1) by $\bar{\varphi}$ and integrate over \mathbb{R}^N .

For the proof of (3.4), recall first that we have defined in the beginning of this section a C^2 -functional $S : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ by setting for $v \in H^1(\mathbb{R}^N)$

$$S(v) := \frac{1}{2}\|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \frac{\omega}{2}\|v\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}$$

and that if φ is a solution of (3.1) then $S'(\varphi) = 0$.

For $\lambda > 0$, let $\varphi_\lambda(\cdot) := \lambda^{N/2}\varphi(\lambda\cdot)$. It follows from straightforward calculations that

$$\|\nabla\varphi_\lambda\|_{L^2(\mathbb{R}^N)}^2 = \lambda^2\|\nabla\varphi\|_{L^2(\mathbb{R}^N)}^2, \|\varphi_\lambda\|_{L^2(\mathbb{R}^N)}^2 = \|\varphi\|_{L^2(\mathbb{R}^N)}^2,$$

$$\text{and } \|\varphi_\lambda\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = \lambda^{\frac{N(p-1)}{2}}\|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.$$

On the one hand, we have

$$S(\varphi_\lambda) = \frac{\lambda^2}{2}\|\nabla\varphi\|_{L^2(\mathbb{R}^N)}^2 + \frac{\omega}{2}\|\varphi\|_{L^2(\mathbb{R}^N)}^2 - \frac{\lambda^{\frac{N(p-1)}{2}}}{p+1}\|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}$$

and

$$\frac{\partial}{\partial \lambda} S(\varphi_\lambda)|_{\lambda=1} = \|\nabla \varphi\|_{L^2(\mathbb{R}^N)}^2 - \frac{N(p-1)}{2(p+1)} \|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}. \tag{3.6}$$

On the other hand, we have

$$\frac{\partial}{\partial \lambda} S(\varphi_\lambda)|_{\lambda=1} = \left\langle S'(\varphi), \frac{\partial \varphi_\lambda}{\partial \lambda} \Big|_{\lambda=1} \right\rangle \tag{3.7}$$

and, since φ is solution of (3.1), $S'(\varphi) = 0$, which implies

$$\frac{\partial}{\partial \lambda} S(\varphi_\lambda)|_{\lambda=1} = 0. \tag{3.8}$$

Combining (3.6) and (3.8) proves (3.4).

Note that the right-hand side of (3.7) is well-defined for $\omega > 0$ since $\frac{\partial \varphi_\lambda}{\partial \lambda} \Big|_{\lambda=1} = \frac{1}{2}\varphi + x \cdot \nabla \varphi$ is in $H^1(\mathbb{R}^N)$ because of the regularity and exponential decay of φ stated in Proposition 3.2. If $\omega \leq 0$, the previous calculations are only formal. However, it is possible to give a rigorous proof of (3.4) also for $\omega \leq 0$, see [9]. \square

From Lemma 3.3 it is easy to derive a necessary condition for the existence of solutions to (3.1).

Corollary 3.4. *If $\omega \leq 0$ then (3.1) has no solution.*

Proof. We prove the assertion by contradiction. Let $\omega \leq 0$ and suppose that (3.1) has a solution $\varphi \in H^1(\mathbb{R}^N)$. By (3.3) and (3.4), φ satisfies

$$\frac{(N-2)p - (N+2)}{2(p+1)} \|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = -\omega \|\varphi\|_{L^2(\mathbb{R}^N)}^2. \tag{3.9}$$

Since $p < 1 + \frac{4}{N-2}$ if $N \geq 3$ and $p < +\infty$ if $N = 1, 2$, the left-hand side of (3.9) is negative for any N , whereas the right-hand side is non negative by assumption, which is a contradiction. \square

In the rest of these notes, it will be understood that $\omega > 0$.

3.2 Existence

Our main result in this section is the following.

Theorem 3.5. *There exists a nontrivial critical point φ_ω of S , that is a solution of (3.1).*

Several techniques are available to prove the existence of a nontrivial critical point of S . The functional S is clearly unbounded from above and below, and this prevents to find a critical point simply by global minimization or maximization. To overcome this difficulty, other techniques based on minimization under constraint were developed (see, e.g., [8, 9, 17, 64, 70] and the references cited therein). In what follows, we

present a different approach based on the mountain pass theorem of Ambrosetti and Rabinowitz [4] and somehow inspired from [38]. This approach was suggested to us by Louis Jeanjean. We first prove that the functional S has a mountain pass geometry. Thus there exists a Palais–Smale sequence at this level. It clearly converges to a critical point of S weakly in $H^1(\mathbb{R}^N)$. Proving that this sequence is non-vanishing and taking advantage of the translation invariance of (3.1), we get the existence of a nontrivial critical point.

We define the *mountain pass level* c by setting

$$c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} S(\gamma(s)), \quad (3.10)$$

where Γ is the set of admissible paths:

$$\Gamma := \{\gamma \in \mathcal{C}([0, 1]; H^1(\mathbb{R}^N)); \gamma(0) = 0, S(\gamma(1)) < 0\}. \quad (3.11)$$

Lemma 3.6. *The functional S has a mountain pass geometry, i.e., $\Gamma \neq \emptyset$ and $c > 0$.*

Proof. Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$. Then, for any $s > 0$,

$$S(sv) = \frac{s^2}{2} (\|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \omega \|v\|_{L^2(\mathbb{R}^N)}^2) - \frac{s^{p+1}}{p+1} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1},$$

and it is clear that if s is large enough then $S(sv) < 0$. Let $C > 0$ be such that $S(Cv) < 0$ and $\gamma : [0, 1] \rightarrow H^1(\mathbb{R}^N)$ be defined by $\gamma(s) := Csv$. Then $\gamma \in \mathcal{C}([0, 1]; H^1(\mathbb{R}^N))$, $\gamma(0) = 0$ and $S(\gamma(1)) < 0$; thus $\gamma \in \Gamma$ and Γ is nonempty. Now, we clearly have

$$S(v) \geq \frac{\min\{1, \omega\}}{2} \|v\|_{H^1(\mathbb{R}^N)}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1},$$

and by the continuous embedding of $H^1(\mathbb{R}^N)$ into $L^{p+1}(\mathbb{R}^N)$, we find

$$S(v) \geq \frac{\min\{1, \omega\}}{2} \|v\|_{H^1(\mathbb{R}^N)}^2 - \frac{C}{p+1} \|v\|_{H^1(\mathbb{R}^N)}^{p+1}.$$

Let $\varepsilon > 0$ be small enough to have

$$\delta := \frac{\min\{1, \omega\}\varepsilon^2}{2} - \frac{C\varepsilon^{p+1}}{p+1} > 0.$$

Then for any $v \in H^1(\mathbb{R}^N)$ with $\|v\|_{H^1(\mathbb{R}^N)} < \varepsilon$, we have $S(v) > 0$. This implies that for any $\gamma \in \Gamma$, we have $\|\gamma(1)\|_{H^1(\mathbb{R}^N)} > \varepsilon$, and by continuity of γ there exists $s_\gamma \in [0, 1]$ such that $\|\gamma(s_\gamma)\|_{H^1(\mathbb{R}^N)} = \varepsilon$. Therefore,

$$\max_{s \in [0,1]} S(\gamma(s)) \geq S(\gamma(s_\gamma)) \geq \delta > 0.$$

This implies for the mountain pass level c defined in (3.10) that

$$c \geq \delta > 0,$$

and thus S has a mountain pass geometry. \square

Combined with Ekeland’s variational principle (see, e.g., [78]), Lemma 3.6 immediately implies the existence of a Palais–Smale sequence at the mountain pass level. More precisely:

Corollary 3.7. *There exists a Palais–Smale sequence $(u_n) \subset H^1(\mathbb{R}^N)$ for S at the mountain pass level c , i.e., (u_n) satisfies, as $n \rightarrow +\infty$,*

$$S(u_n) \rightarrow c \text{ and } S'(u_n) \rightarrow 0. \tag{3.12}$$

To find a critical point for S , we proceed now in two steps: First, we prove that the sequence (u_n) is, following the terminology of the concentration-compactness theory of P.-L. Lions, *non-vanishing*; second, we prove that, up to translations, (u_n) converges weakly in $H^1(\mathbb{R}^N)$ to a nontrivial critical point of S .

Lemma 3.8. *The Palais–Smale sequence (u_n) is non-vanishing: there exist $\varepsilon > 0$, $R > 0$ and a sequence $(y_n) \in \mathbb{R}^N$ such that for all $n \in \mathbb{N}$, we have*

$$\int_{B_R(y_n)} |u_n|^2 dx > \varepsilon, \tag{3.13}$$

where $B_R(y) := \{z \in \mathbb{R}^N; |y - z| < R\}$.

To prove Lemma 3.8, we will use the following lemma, which is a kind of Sobolev embedding result (see [50]).

Lemma 3.9. *Let $R > 0$. Then there exists $\alpha > 0$ and $C > 0$ such that for any $v \in H^1(\mathbb{R}^N)$, we have*

$$\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \leq C \left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v|^2 dx \right)^\alpha \|v\|_{H^1(\mathbb{R}^N)}^2.$$

Proof. Let $(Q_k)_{k \in \mathbb{N}}$ be a sequence of open cubes of \mathbb{R}^N of same volume such that $Q_k \cap Q_l = \emptyset$ if $k \neq l$, $\cup_{k \in \mathbb{N}} Q_k = \mathbb{R}^N$ and for each k there exists y_k with $Q_k \subset B_R(y_k)$. By Hölder’s inequality and the embedding of $H^1(Q_k)$ into $L^{p+1}(Q_k)$, there exists $C > 0$ independent of k such that for any $v \in H^1(\mathbb{R}^N)$, we have

$$\int_{Q_k} |v|^{p+1} dx \leq C \left(\int_{Q_k} |v|^2 dx \right)^\alpha \int_{Q_k} (|\nabla v|^2 + |v|^2) dx.$$

with $\alpha = \frac{N}{p+1} - \frac{N-2}{2}$ if $N \geq 3$ and $\alpha = 1$ if $N = 1, 2$. Summing over $k \in \mathbb{N}$, this implies that

$$\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \leq C \left(\sup_{k \in \mathbb{N}} \int_{Q_k} |v|^2 dx \right)^\alpha \|v\|_{H^1(\mathbb{R}^N)}^2.$$

Now, since for any k there exists y_k with $Q_k \subset B_R(y_k)$ the conclusion follows. □

Proof of Lemma 3.8. We prove the result by contradiction. Assume that the Palais–Smale sequence (u_n) is vanishing, that is, for all $R > 0$, we have

$$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0. \quad (3.14)$$

Let $\varepsilon > 0$. For n large enough, we have $\left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx\right) < \varepsilon$ thanks to (3.14). Then Lemma 3.9 implies that, still for n large enough,

$$S(u_n) \geq \frac{1}{2} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 + \frac{\omega}{2} \|u_n\|_{L^2(\mathbb{R}^N)}^2 - \varepsilon \|u_n\|_{H^1(\mathbb{R}^N)}^2,$$

and therefore

$$S(u_n) \geq \left(\frac{\min\{1, \omega\}}{2} - \varepsilon\right) \|u_n\|_{H^1(\mathbb{R}^N)}^2.$$

Consequently, taking $\varepsilon < \frac{\min\{1, \omega\}}{2}$, we infer that

$$(u_n) \text{ is bounded in } H^1(\mathbb{R}^N), \quad (3.15)$$

which allows to get

$$\langle S'(u_n), u_n \rangle \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (3.16)$$

Combining (3.15) with (3.14) and Lemma 3.9 implies

$$\lim_{n \rightarrow +\infty} \|u_n\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} = 0.$$

Consequently, we have

$$S(u_n) - \frac{1}{2} \langle S'(u_n), u_n \rangle = -\frac{p-1}{2(p+1)} \|u_n\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (3.17)$$

On the other hand, we infer from (3.12) and (3.16) that

$$S(u_n) - \frac{1}{2} \langle S'(u_n), u_n \rangle \rightarrow c \text{ as } n \rightarrow +\infty,$$

which is a contradiction with (3.17) since $c > 0$. Hence, (u_n) is non-vanishing. \square

Proof of Theorem 3.5. Let $v_n := u_n(\cdot + y_n)$, where (u_n) is given by Corollary 3.7 and (y_n) by Lemma 3.8. Since the functional S is invariant under translations in space, (v_n) is still a Palais–Smale sequence for S :

$$S(v_n) \rightarrow c \text{ and } S'(v_n) \rightarrow 0. \quad (3.18)$$

From (3.15) we infer that (v_n) is bounded in $H^1(\mathbb{R}^N)$. Thus there exists $v \in H^1(\mathbb{R}^N)$ such that, possibly for a subsequence only, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. From (3.18), we infer that $S'(v) = 0$, i.e., v is a critical point for S . To show that v is nontrivial, we remark that, since the embedding $H^1(B_R(0)) \hookrightarrow L^2(B_R(0))$ is compact, (3.13) implies $v \neq 0$. Setting $\varphi_\omega := v$ finishes the proof. \square

The following corollary will be useful in the next subsection.

Corollary 3.10. *The critical point φ_ω is below the level c , i.e., $S(\varphi_\omega) \leq c$.*

Proof. First, since $S'(\varphi_\omega) = 0$, we have

$$\begin{aligned} S(\varphi_\omega) &= S(\varphi_\omega) - \frac{1}{p+1} \langle S'(\varphi_\omega), \varphi_\omega \rangle \\ &= \frac{p-1}{2(p+1)} \left(\|\nabla \varphi_\omega\|_{L^2(\mathbb{R}^N)}^2 + \omega \|\varphi_\omega\|_{L^2(\mathbb{R}^N)}^2 \right). \end{aligned}$$

By virtue of weak convergence of (v_n) towards φ_ω in $H^1(\mathbb{R}^N)$, this gives

$$S(\varphi_\omega) \leq \frac{p-1}{2(p+1)} \liminf_{n \rightarrow +\infty} \left(\|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 + \omega \|v_n\|_{L^2(\mathbb{R}^N)}^2 \right). \tag{3.19}$$

As in (3.16), we have

$$\langle S'(v_n), v_n \rangle \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

and combining with (3.18) and (3.19), we obtain

$$S(\varphi_\omega) \leq \liminf_{n \rightarrow +\infty} \left(S(v_n) - \frac{1}{p+1} \langle S'(v_n), v_n \rangle \right) = c,$$

which completes the proof. □

3.3 Variational characterizations

Among the solutions of (3.1), some are of particular interest.

Definition 3.11. A solution φ of (3.1) is said to be a *ground state* or *least energy solution* if $S(\varphi) \leq S(v)$ for any solution v of (3.1). We define the *least energy level* m by

$$m := \inf\{S(v); v \text{ is a solution of (3.1)}\}. \tag{3.20}$$

The set of all least energy solutions is denoted by \mathcal{G} ,

$$\mathcal{G} := \{v \in H^1(\mathbb{R}^N); v \text{ is a solution of (3.1) and } S(v) = m\}. \tag{3.21}$$

Ground states play an important role in the theory of nonlinear Schrödinger equations. In particular, in the critical case $p = 1 + \frac{4}{N}$, they appear in an essential way in the derivation of global existence results (see [75]) and in the description of the blow-up phenomenon (see [57, 58, 59, 60] and the references cited therein).

Proposition 3.12. *The solution φ_ω of (3.1) found in Theorem 3.5 is a ground state and it is at the mountain pass level c defined in (3.10):*

$$S(\varphi_\omega) = m = c.$$

In addition, φ_ω is a minimizer of S on the Nehari manifold (see (3.5)), i.e., it solves the following minimization problem

$$d := \min\{S(v); v \in H^1(\mathbb{R}^N) \setminus \{0\}, I(v) = 0\} \quad (3.22)$$

where $I(v) := \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \omega\|v\|_{L^2(\mathbb{R}^N)}^2 - \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}$.

The various variational characterizations of the critical point φ_ω of S as a ground state or as a minimizer of S on the Nehari manifold will be useful when we will deal with the stability or instability issues.

Before proving Proposition 3.12, we need some preparation.

Lemma 3.13. *The following inequality holds:*

$$c \leq d, \quad (3.23)$$

where c is the mountain pass level defined in (3.10) and d is given by (3.22).

Proof. Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ be such that $I(v) = 0$. Following [41, 42, 65], the idea is to construct a path γ in Γ (recall that Γ was defined in (3.11)) such that S reaches its maximum on γ at v . From the proof of Lemma 3.6, we know that for C large enough the path $\gamma : [0, 1] \rightarrow H^1(\mathbb{R}^N)$ defined by $\gamma(s) := Csv$ belongs to Γ . It is easy to see that

$$\frac{\partial}{\partial s} S(sv) = s(\|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \omega\|v\|_{L^2(\mathbb{R}^N)}^2 - s^{p-1}\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}).$$

Therefore, we have at $s = 1$

$$\frac{\partial}{\partial s} S(sv) \Big|_{s=1} = I(v) = 0,$$

and consequently,

$$\begin{aligned} \frac{\partial}{\partial s} S(sv) &> 0 & \text{if } s \in (0, 1), \\ \frac{\partial}{\partial s} S(sv) &< 0 & \text{if } s \in (1, +\infty). \end{aligned}$$

Thus S reaches its maximum on γ at v . This implies

$$c \leq S(v) \text{ for all } v \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ with } I(v) = 0,$$

which finishes the proof. \square

Proof of Proposition 3.12. We first recall that, by Corollary 3.10, we have

$$S(\varphi_\omega) \leq c \quad (3.24)$$

and by Lemma 3.13,

$$c \leq d. \quad (3.25)$$

Now, we remark that, since by (3.3) any solution v of (3.1) satisfies $I(v) = 0$, we have

$$d \leq m. \tag{3.26}$$

Since φ_ω is a solution of (3.1), it follows from the definition of the least energy level m (see (3.20)) that

$$m \leq S(\varphi_\omega). \tag{3.27}$$

Combining (3.24)–(3.27) gives

$$S(\varphi_\omega) = c = d = m$$

and completes the proof. □

Remark 3.14. For more general nonlinearities, the equality $m = c$ between the mountain pass level and the least energy level still holds (see [41, 42]).

3.4 Uniqueness

It turns out that we are able to describe explicitly the set \mathcal{G} of ground states (see (3.21)).

Theorem 3.15. *There exists a real-valued, positive, spherically symmetric and decreasing function $\Psi \in H^1(\mathbb{R}^N)$ such that*

$$\mathcal{G} = \{e^{i\theta}\Psi(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}.$$

Therefore, the ground state is unique up to translations and phase shifts. It would exceed the scope of these notes to give a proof of Theorem 3.15 and we just indicate some references. First, a simple and general proof that any complex-valued ground state φ is of the form $e^{i\theta}\tilde{\varphi}$ with $\theta \in \mathbb{R}$ and $\tilde{\varphi}$ a real positive ground state was recently given in [16] (see also [36]). The fact that all real positive ground states of (3.1) are radial up to translations was first proved by Gidas, Ni and Nirenberg in 1979 (see [29]) by using the so-called moving planes method. Alternatively, the same result can be deduced by the method of Lopes [54], which relies on symmetrizations with respect to suitably chosen hyperplanes combined with a unique continuation theorem. Recently, Maris [55] developed for the symmetry of minimizers for a large class of problems a new method that furnishes a simpler proof of radial symmetry of real ground states, see [12], without even assuming a priori that they are positive. Uniqueness follows from a result of Kwong [47] in 1989.

Remark 3.16. When $N = 1$, it turns out that the set of all solutions of (3.1) (not only those of least energy) is precisely the set of ground states

$$\mathcal{G} = \{e^{i\theta}\Psi(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\} = \{v; v \text{ is a solution of (3.1)}\}.$$

Moreover, we are able to give an explicit formula for Ψ ,

$$\Psi(x) = \left[\frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} x \right) \right]^{\frac{1}{p-1}}.$$

For $p = 3$ and $\omega = 1$, the shape of Ψ is given in Figure 4.

In higher dimensions, it is known that there exist solutions of (3.1) that are not ground states (see [8, 10, 52]), and no explicit solution is available.

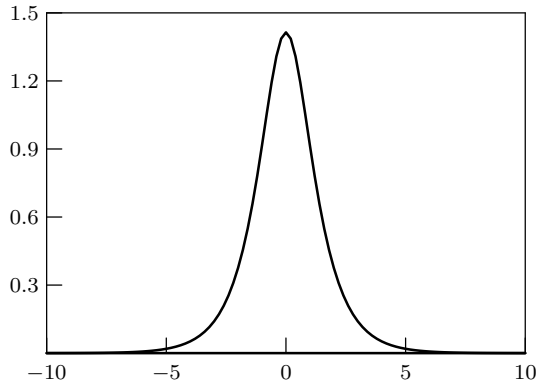


Figure 4. The function Ψ for $N = 1$, $p = 3$ and $\omega = 1$.

4 Stability

In this section, we will prove that for $\varphi \in \mathcal{G}$ the standing wave $e^{i\omega t}\varphi(x)$ is stable (in a sense to be made precise) if $1 < p < 1 + \frac{4}{N}$.

In his report [67] of 1844, Russel was already mentioning the observed remarkable stability properties of solitary waves. From the mathematical point of view, the concept of stability that comes naturally in mind for the standing waves $e^{i\omega t}\varphi(x)$ of (1.1) is *stability in the sense of Lyapunov*, which means uniformly continuous dependence on the initial data: For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in H^1(\mathbb{R}^N)$,

$$\|u_0 - \varphi\|_{H^1(\mathbb{R}^N)} < \delta \implies \|u(t) - e^{i\omega t}\varphi\|_{H^1(\mathbb{R}^N)} < \varepsilon \text{ for all } t,$$

where u is the maximal solution of (1.1) with $u(0) = u_0$. However, with this definition, all standing waves would be unstable (see Remark 4.2), which is contradictory with what is observed for such phenomena. Therefore, we have to look for a different notion of stability.

The main reason for standing waves being unstable in the sense of Lyapunov is that (1.1) admits translation and phase shift invariance: If $u = u(t, x)$ is a solution of (1.1) then for all $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^N$, $e^{i\theta}u(\cdot, \cdot - y)$ is still a solution of (1.1). In some sense, the equation does not prescribe the behavior of the solutions in the “direction” of translations and phase shifts. To take this fact into account, we define the concept of *orbital stability*, which is Lyapunov stability up to translations and phase shifts.

Definition 4.1. Let φ be a solution of (3.1). The standing wave $e^{i\omega t}\varphi(x)$ is said to be *orbitally stable in $H^1(\mathbb{R}^N)$* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^N)} < \delta$ then the maximal solution $u = u(t)$ of (1.1) with $u(0) = u_0$ exists for all $t \in \mathbb{R}$ and

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \inf_{y \in \mathbb{R}^N} \|u(t) - e^{i\theta}\varphi(\cdot - y)\|_{H^1(\mathbb{R}^N)} < \varepsilon.$$

Otherwise, the standing wave is said to be *unstable*.

Remark 4.2. The concept of orbital stability is optimal in the following sense (see [14, 15]).

- Space translations are necessary: For a solution φ of (3.1), $\varepsilon > 0$ and $y \in \mathbb{R}^N$ with $|y| = 1$, let $\varphi_\varepsilon(x) := e^{i\varepsilon x \cdot y} \varphi(x)$. Then it is easy to check that the solution of (1.1) with initial datum φ_ε is

$$u_\varepsilon(t, x) = e^{i\varepsilon(x \cdot y - \varepsilon t)} e^{i\omega t} \varphi(x - 2\varepsilon t y).$$

We clearly have $\varphi_\varepsilon \rightarrow \varphi$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$, but for any $\varepsilon > 0$

$$\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \|u_\varepsilon(t) - e^{i\theta} \varphi\|_{H^1(\mathbb{R}^N)} = 2\|\varphi\|_{H^1(\mathbb{R}^N)}.$$

Conversely, note that if we consider (1.1) in the subspace of $H^1(\mathbb{R}^N)$ consisting of radial functions then no space translation can occur and we can omit the translations in the definition of orbital stability.

- Phase shifts are necessary: For a solution φ of (3.1) and $\varepsilon > 0$, let $\varphi_\varepsilon(x) := (1 + \varepsilon)^{\frac{1}{p-1}} \varphi((1 + \varepsilon)^{\frac{1}{2}} x)$. Then it is easy to check that the solution of (1.1) with initial datum φ_ε is

$$u_\varepsilon(t, x) = e^{i\omega(1+\varepsilon)t} (1 + \varepsilon)^{\frac{1}{p-1}} \varphi((1 + \varepsilon)^{\frac{1}{2}} x).$$

We clearly have $\varphi_\varepsilon \rightarrow \varphi$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$ but for any $\varepsilon > 0$

$$\sup_{t \in \mathbb{R}} \inf_{y \in \mathbb{R}^N} \|u_\varepsilon(t) - \varphi(\cdot - y)\|_{H^1(\mathbb{R}^N)} \geq \|\varphi\|_{H^1(\mathbb{R}^N)}.$$

Remark that φ_ε is a solution of (3.1) with ω being replaced by $\omega(1 + \varepsilon)$.

The main result of this section is the following.

Theorem 4.3. *Let φ be a ground state of (3.1). If $1 < p < 1 + \frac{4}{N}$ then the standing wave $e^{i\omega t} \varphi(x)$ is orbitally stable.*

Remark 4.4. The range of p is optimal: We will see in the next section that instability occurs if $p \geq 1 + \frac{4}{N}$.

Remark 4.5. Except for $N = 1$, where all solutions of (3.1) are ground states, Theorem 4.3 asserts only the stability of standing waves corresponding to ground states. It is an open problem to describe the behavior of other standing waves of (1.1) under perturbations (see nevertheless [33, 74] and the references cited therein).

The first rigorous stability study is due to Benjamin [5] for solitary waves of the Korteweg–de Vries equation. Roughly speaking, two different approaches are possible in the study of stability of standing waves. The first one was introduced by Cazenave [13] and then developed by Cazenave and Lions [15]. It relies on variational and compactness arguments. The main step in this approach consists in characterizing the ground states as minimizers of the energy on a sphere of $L^2(\mathbb{R}^N)$. To use

this approach, it is essential to have a uniqueness result for ground states as Theorem 3.15, otherwise a weaker result is obtained: stability of the set of ground states (see Remark 4.9). The second approach was introduced by Weinstein [76, 77] (see also [66]) and then considerably generalized by Grillakis, Shatah and Strauss [34, 35]. A criterion based on a form of coercivity for $S''(\varphi)$ is derived in this work and allows to prove stability of standing waves. Sufficient conditions for instability are also given in [34, 35].

The rest of this section is devoted to the proofs of Theorem 4.3. In Subsection 4.1, we present the proof using the Cazenave–Lions method and in Subsection 4.2 we present the proof using Grillakis–Shatah–Strauss method.

4.1 Cazenave–Lions method

The proof of Theorem 4.3 by the Cazenave–Lions method relies on the following compactness result.

Proposition 4.6. *Let $1 < p < 1 + \frac{4}{N}$. For any $\tau > 0$, define*

$$\Sigma_\tau := \{v \in H^1(\mathbb{R}^N); \|v\|_{L^2(\mathbb{R}^N)}^2 = \tau\}.$$

Consider the minimization problem

$$-\nu_\tau := \inf\{E(v); v \in \Sigma_\tau\},$$

where E is the functional defined in (2.2): For $v \in H^1(\mathbb{R}^N)$,

$$E(v) = \frac{1}{2} \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.$$

Then $\nu_\tau < +\infty$, and if $(v_n) \subset H^1(\mathbb{R}^N)$ is such that

$$\|v_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \tau \text{ and } E(v_n) \rightarrow -\nu_\tau \text{ as } n \rightarrow +\infty,$$

then there exist a sequence $(y_n) \subset \mathbb{R}^N$ and a function $v \in H^1(\mathbb{R}^N)$ such that, possibly for a subsequence only,

$$v_n(\cdot - y_n) \rightarrow v \text{ strongly in } H^1(\mathbb{R}^N).$$

In particular, $v \in \Sigma_\tau$ and $E(v) = -\nu_\tau$.

The proof of Proposition 4.6 is based on the concentration-compactness principle of Lions [50, 51]. We refer to [14, 15] for a detailed proof.

Remark 4.7. For $p > 1 + \frac{4}{N}$, it is easy to see that $\nu_\tau = +\infty$. Indeed, let $v \in \Sigma_\tau$. Using the scaled functions $v_\lambda(\cdot) := \lambda^{\frac{N}{2}} v(\lambda \cdot)$, we obtain $\|v_\lambda\|_{L^2(\mathbb{R}^N)}^2 = \|v\|_{L^2(\mathbb{R}^N)}^2 = \tau$, but $\lim_{\lambda \rightarrow +\infty} E(v_\lambda) = -\infty$. Therefore, Proposition 4.6 cannot hold in this case.

Next we characterize the ground states of (3.1) as minimizers of the energy on a sphere of $L^2(\mathbb{R}^N)$.

Proposition 4.8. *Let $1 < p < 1 + \frac{4}{N}$. Then*

- (i) *there exists $\tau_{\mathcal{G}} > 0$ such that $\|\varphi\|_{L^2(\mathbb{R}^N)}^2 = \tau_{\mathcal{G}}$ for all $\varphi \in \mathcal{G}$,*
- (ii) *$\varphi \in \mathcal{G}$ if and only if $\varphi \in \Sigma_{\tau_{\mathcal{G}}}$ and $E(\varphi) = -\nu_{\tau_{\mathcal{G}}}$.*

Sketch of proof. Point (i) follows immediately from Theorem 3.15. We refer to [14] for the proof of (ii). □

Proof of Theorem 4.3 by the Cazenave–Lions method. The result is proved by contradiction. Assume that there exist $\varepsilon > 0$ and two sequences $(u_{n,0}) \subset H^1(\mathbb{R}^N)$, $(t_n) \subset \mathbb{R}$ such that

$$\|u_{n,0} - \varphi\|_{H^1(\mathbb{R}^N)} \rightarrow 0 \text{ as } n \rightarrow +\infty, \tag{4.1}$$

$$\inf_{\theta \in \mathbb{R}} \inf_{y \in \mathbb{R}^N} \|u_n(t_n) - e^{i\theta} \varphi(\cdot - y)\|_{H^1(\mathbb{R}^N)} > \varepsilon \text{ for all } n \in \mathbb{N}, \tag{4.2}$$

where u_n is the maximal solution of (1.1) with initial datum $u_{n,0}$. Set $v_n(x) := u_n(t_n, x)$. By (4.1) and the conservation of charge and energy (see (2.3)), we have, as $n \rightarrow +\infty$,

$$\|v_n\|_{L^2(\mathbb{R}^N)}^2 = \|u_n(t_n)\|_{L^2(\mathbb{R}^N)}^2 = \|u_{n,0}\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \|\varphi\|_{L^2(\mathbb{R}^N)}^2 = \tau_{\mathcal{G}}, \tag{4.3}$$

$$E(v_n) = E(u_n(t_n)) = E(u_{n,0}) \rightarrow E(\varphi) = -\nu_{\tau_{\mathcal{G}}}. \tag{4.4}$$

By (4.3), (4.4) and Proposition 4.6, there exist $(y_n) \subset \mathbb{R}^N$ and $v \in H^1(\mathbb{R}^N)$ such that

$$\|v_n(\cdot - y_n) - v\|_{H^1(\mathbb{R}^N)} \rightarrow 0 \text{ as } n \rightarrow +\infty, \tag{4.5}$$

$$v \in \Sigma_{\tau_{\mathcal{G}}} \text{ and } E(v) = -\nu_{\tau_{\mathcal{G}}}. \tag{4.6}$$

By (4.6) and Proposition 4.8, we have $v \in \mathcal{G}$. Therefore, by Theorem 3.15, there exist $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^N$ such that $v = e^{i\theta} \varphi(\cdot - y)$. Remembering that $v_n = u_n(t_n)$ and substituting this in (4.5), we get

$$\|u_n(t_n) - e^{i\theta} \varphi(\cdot - (y - y_n))\|_{H^1(\mathbb{R}^N)} \rightarrow 0 \text{ when } n \rightarrow +\infty,$$

which is a contradiction with (4.2) and finishes the proof. □

Remark 4.9. It is essential for the proof of Theorem 4.3 by the Cazenave–Lions method that the set of ground states \mathcal{G} can be explicitly described by $\{e^{i\theta} \Psi(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}$ as in Theorem 3.15. This uniqueness result is far from being obvious even with our simple pure power nonlinearity. For general nonlinearities, such uniqueness results are usually not available. Without this uniqueness result, one obtains a result corresponding to a weaker notion of stability: stability of the set \mathcal{G} . More precisely, the concept of stability would read as follows: The set of ground states \mathcal{G} is said to

be *stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^N)} < \delta$ for some $\varphi \in \mathcal{G}$ then

$$\sup_{t \in \mathbb{R}} \inf_{\psi \in \mathcal{G}} \|u(t) - \psi\|_{H^1(\mathbb{R}^N)} < \varepsilon,$$

where u is the maximal solution of (1.1) with $u(0) = u_0$.

4.2 Grillakis–Shatah–Strauss method

The method of Grillakis–Shatah–Strauss is a powerful tool to derive stability or instability results. The results in [34, 35] state, roughly speaking, the following: If we consider a family (φ_ω) of solutions of the stationary equation, the standing wave $e^{i\omega t} \varphi_\omega(x)$ is stable if

$$\frac{\partial}{\partial \omega} \|\varphi_\omega\|_{L^2(\mathbb{R}^N)}^2 > 0 \tag{4.7}$$

and unstable if

$$\frac{\partial}{\partial \omega} \|\varphi_\omega\|_{L^2(\mathbb{R}^N)}^2 < 0, \tag{4.8}$$

provided some spectral assumptions on the linearized operator $S''(\varphi_\omega)$ are satisfied. Slope conditions of the type (4.7)–(4.8) can easily be checked when the stationary equation admits some scaling invariance, typically when the nonlinearity is of power type. For example, in the case of (3.1), it is easy to see that if φ_1 is a solution of (3.1) with $\omega = 1$ then the scaling $\varphi_\omega(\cdot) := \omega^{\frac{1}{p-1}} \varphi_1(\omega^{\frac{1}{2}} \cdot)$ provides a family of solutions of (3.1) for $\omega \in (0, +\infty)$ such that

$$\begin{aligned} \frac{\partial}{\partial \omega} \|\varphi_\omega\|_{L^2(\mathbb{R}^N)}^2 > 0 & \quad \text{if } p < 1 + \frac{4}{N}, \\ \frac{\partial}{\partial \omega} \|\varphi_\omega\|_{L^2(\mathbb{R}^N)}^2 < 0 & \quad \text{if } p > 1 + \frac{4}{N}. \end{aligned}$$

In such situations, the main difficulty is to handle the spectral conditions (see, e.g., [25, 26, 49]). If (3.1) has no scaling invariance, it becomes very difficult to obtain the slope conditions (4.7)–(4.8) (see nevertheless [27, 28, 56]). An alternative is to use a stability criterion derived from the work [34]. This stability criterion fits better the case of general nonlinearities, as, e.g., in [19, 23, 24, 40, 44, 46]. See also [72] for a review of the different ways to obtain stability for general nonlinearities thanks to Grillakis–Shatah–Strauss’ results.

Although these notes are restricted to nonlinear Schrödinger equations with power-type nonlinearities, our goal is to provide the reader with methods applicable in rather general situations and this is why we choose to present the proof of Theorem 4.3 using the following stability criterion.

Proposition 4.10 (Stability criterion). *Let φ be a solution of (3.1). Suppose that there exists $\delta > 0$ such that we have*

$$\langle S''(\varphi)v, v \rangle \geq \delta \|v\|_{H^1(\mathbb{R}^N)}^2 \tag{4.9}$$

for all $v \in H^1(\mathbb{R}^N)$ satisfying the orthogonality conditions

$$(v, \varphi)_2 = (v, i\varphi)_2 = \left(v, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N. \tag{4.10}$$

Then the standing wave $e^{i\omega t}\varphi(x)$ is orbitally stable in $H^1(\mathbb{R}^N)$.

Let us heuristically explain why the assumptions of Proposition 4.10 lead to stability. The idea comes from the theory of Lyapunov stability for an equilibrium in dynamical systems. A good candidate for a Lyapunov functional would be the functional S . Indeed, let us suppose for a moment that the coercivity condition (4.9) holds for any $v \in H^1(\mathbb{R}^N)$ (this is not the case, as we will see in the sequel). Let u be a solution of (1.1) with initial datum u_0 close to φ in $H^1(\mathbb{R}^N)$. A Taylor expansion gives

$$\begin{aligned} S(u(t)) - S(\varphi) &= \langle S'(\varphi), u(t) - \varphi \rangle + \frac{1}{2} \langle S''(\varphi)(u(t) - \varphi), u(t) - \varphi \rangle \\ &\quad + o(\|u(t) - \varphi\|_{H^1(\mathbb{R}^N)}^2). \end{aligned}$$

Since φ is a solution of (3.1), $S'(\varphi) = 0$. Combined with (4.9), this would give, for some constant $C > 0$ independent of t ,

$$S(u(t)) - S(\varphi) \geq C\|u(t) - \varphi\|_{H^1(\mathbb{R}^N)}^2. \tag{4.11}$$

Since S is a conserved quantity this would give an upper bound on $\|u(t) - \varphi\|_{H^1(\mathbb{R}^N)}$, hence stability. Of course, as we already know (see Remark 4.2), this cannot be true since stability is possible only up to translations and phase shifts. In fact, translation and phase shift invariance generates, as we will see in the sequel, a kernel for $S''(\varphi)$ of the form

$$\ker\{S''(\varphi)\} = \text{span} \left\{ i\varphi, \frac{\partial \varphi}{\partial x_j}; j = 1, \dots, N \right\}.$$

To avoid this kernel, we require the coercivity condition (4.9) only for $v \in H^1(\mathbb{R}^N)$ satisfying

$$(v, i\varphi)_2 = \left(v, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N,$$

which allows phase shifts and translations in the right-hand side of (4.11). The other orthogonality condition $(v, \varphi)_2 = 0$ is related to the conservation of mass. Indeed, since the mass is conserved, the evolution takes place, in some sense, in the tangent space of the sphere of $L^2(\mathbb{R}^N)$ at φ and therefore it is enough to ask for S to satisfy the coercivity condition (4.9) on this tangent space to get stability. The rigorous proof of Proposition 4.10 is involved and we have postponed it to the appendix.

In view of Proposition 4.10, it is clear that Theorem 4.3 follows immediately from the following proposition.

Proposition 4.11. *Let $1 < p < 1 + \frac{4}{N}$ and φ be a ground state of (3.1). Then there exists $\delta > 0$ such that for all $w \in H^1(\mathbb{R}^N)$ satisfying*

$$(w, \varphi)_2 = (w, i\varphi)_2 = \left(w, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N, \tag{4.12}$$

we have

$$\langle S''(\varphi)w, w \rangle \geq \delta \|w\|_{H^1(\mathbb{R}^N)}^2.$$

Before giving the proof of Proposition 4.11, some preparation is necessary. First, from Theorem 3.15, we can assume without loss of generality that the ground state φ is real, positive, and radial. Note that uniqueness is not involved here. It is convenient to split w in the real and imaginary part. We set $w = u + iv$ for real-valued $u, v \in H^1(\mathbb{R}^N)$. Then the operator $S''(\varphi)$ can be separated into a real and an imaginary part L_1 and L_2 such that

$$\langle S''(\varphi)w, w \rangle = \langle L_1u, u \rangle + \langle L_2v, v \rangle.$$

Here, L_1 and L_2 are two bounded operators defined on $H^1(\mathbb{R}^N)$ restricted to real-valued functions, with values in $H^{-1}(\mathbb{R}^N)$, and given by

$$\begin{aligned} L_1u &= -\Delta u + \omega u - p\varphi^{p-1}u, \\ L_2v &= -\Delta v + \omega v - \varphi^{p-1}v. \end{aligned}$$

Until the end of this subsection, the functions considered will be real-valued. In particular, $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ will be restricted to real-valued functions.

Proposition 4.11 follows immediately from the two following lemmas.

Lemma 4.12. *There exists $\delta_1 > 0$ such that for all $u \in H^1(\mathbb{R}^N)$ satisfying*

$$(u, \varphi)_2 = \left(u, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N,$$

we have

$$\langle L_1u, u \rangle \geq \delta_1 \|u\|_{H^1(\mathbb{R}^N)}^2.$$

Lemma 4.13. *There exists $\delta_2 > 0$ such that for all $v \in H^1(\mathbb{R}^N)$ satisfying*

$$(v, \varphi)_2 = 0,$$

we have

$$\langle L_2v, v \rangle \geq \delta_2 \|v\|_{H^1(\mathbb{R}^N)}^2.$$

It is not hard to see that the operators L_1 and L_2 admit restrictions \tilde{L}_1 and \tilde{L}_2 with domain $H^2(\mathbb{R}^N)$ that are unbounded self-adjoint operators in $L^2(\mathbb{R}^N)$ (the so-called Friedrichs extensions, see, e.g., [43]).

The proofs of Lemmas 4.12 and 4.13 rely on the analysis of the spectra of \tilde{L}_1 and \tilde{L}_2 . The following lemma gives the general structure of these spectra.

Lemma 4.14. *The spectra of \tilde{L}_1 and \tilde{L}_2 consist of essential spectrum in $[\omega, +\infty)$ and of a finite number of eigenvalues of finite multiplicity in $(-\infty, \omega']$ for all $\omega' < \omega$.*

Proof. We first remark that since \tilde{L}_1 and \tilde{L}_2 are self-adjoint operators, their spectra lie on the real line.

The spectra of \tilde{L}_1 and \tilde{L}_2 are bounded from below. Indeed, for all $u \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} \langle L_2 u, u \rangle &= \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \omega \|u\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} \varphi^{p-1} u^2 dx \\ &\geq \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \omega \|u\|_{L^2(\mathbb{R}^N)}^2 - p \int_{\mathbb{R}^N} \varphi^{p-1} u^2 dx = \langle L_1 u, u \rangle. \end{aligned}$$

Since $\varphi \in L^\infty(\mathbb{R}^N)$, there exists $C > 0$, independent of u , such that

$$p \int_{\mathbb{R}^N} \varphi^{p-1} u^2 dx \leq C \|u\|_{L^2(\mathbb{R}^N)}^2$$

and therefore

$$\langle L_2 u, u \rangle \geq \langle L_1 u, u \rangle \geq (\omega - C) \|u\|_{L^2(\mathbb{R}^N)}^2.$$

Now, since φ is exponentially decaying, \tilde{L}_1 and \tilde{L}_2 are compactly perturbed versions of

$$-\Delta + \omega : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N).$$

It is well known that the essential spectrum of $-\Delta + \omega$ is $\sigma_{\text{ess}}(-\Delta + \omega) = [\omega, +\infty)$, thus, by Weyl's Theorem (see, e.g., [43]),

$$\sigma_{\text{ess}}(\tilde{L}_1) = \sigma_{\text{ess}}(\tilde{L}_2) = [\omega, +\infty).$$

Since, for $j = 1, 2$, $\sigma(\tilde{L}_j) \setminus \sigma_{\text{ess}}(\tilde{L}_j)$ consists of isolated eigenvalues of finite multiplicity and $\sigma(\tilde{L}_j)$ is bounded from below, this completes the proof of the lemma. \square

From now on, we consider L_1 and L_2 separately. We begin with L_2 .

Lemma 4.15. *There exists $\tilde{\delta}_2 > 0$ such that for all $v \in H^1(\mathbb{R}^N)$ with $(v, \varphi)_2 = 0$, we have*

$$\langle L_2 v, v \rangle \geq \tilde{\delta}_2 \|v\|_{L^2(\mathbb{R}^N)}^2.$$

Proof. We remark that $L_2 \varphi = S'(\varphi) = 0$ since φ is a solution of (3.1). This means that 0 is an eigenvalue of L_2 with φ being an eigenfunction. But $\varphi > 0$, and it is well known (see, e.g., [11]) that this implies that 0 is the first simple eigenvalue of \tilde{L}_2 . Let $v \in H^1(\mathbb{R}^N)$ be such that $(v, \varphi)_2 = 0$. Then, by the min-max characterization of eigenvalues (see, e.g., [11]), there exists $\tilde{\delta}_2 > 0$ independent of v (in fact, $\tilde{\delta}_2$ is the second eigenvalue of \tilde{L}_2) such that

$$\langle L_2 v, v \rangle \geq \tilde{\delta}_2 \|v\|_{L^2(\mathbb{R}^N)}^2.$$

\square

Proof of Lemma 4.13. The proof is carried out by contradiction. Assume the existence of a sequence $(v_n) \subset H^1(\mathbb{R}^N)$ such that

$$\|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 + \omega \|v_n\|_{L^2(\mathbb{R}^N)}^2 = 1, \quad (v_n, \varphi)_2 = 0 \quad \text{and} \quad \langle L_2 v_n, v_n \rangle \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since $\|v_n\|_{H^1(\mathbb{R}^N)}^2 \leq \max\{\omega, \omega^{-1}\}(\|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 + \omega \|v_n\|_{L^2(\mathbb{R}^N)}^2)$, (v_n) is bounded in $H^1(\mathbb{R}^N)$ and there exists $v \in H^1(\mathbb{R}^N)$ such that, possibly for a subsequence only,

$$v_n \rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}^N).$$

In particular, we have $(v, \varphi)_2 = 0$ and by Lemma 4.15

$$\langle L_2 v, v \rangle \geq 0. \tag{4.13}$$

By the embedding of $H^1(\mathbb{R}^N)$ into $L^{2q}(\mathbb{R}^N)$ for $q \in (1, (N-2)/N)$, we have $v_n \rightharpoonup v$ in $L^{2q}(\mathbb{R}^N)$. In view of the compact embedding of $H^1(\mathbb{R}^N)$ into $L_{\text{loc}}^{2q}(\mathbb{R}^N)$, we also get $v_n^2 \rightharpoonup v^2$ in $L^q(\mathbb{R}^N)$. Since φ is exponentially decaying, we have $\varphi^{p-1} \in L^{q'}(\mathbb{R}^N)$, where q' is the conjugate exponent of q , $\frac{1}{q} + \frac{1}{q'} = 1$. Therefore,

$$\int_{\mathbb{R}^N} \varphi^{p-1} v_n^2 dx \rightarrow \int_{\mathbb{R}^N} \varphi^{p-1} v^2 dx \quad \text{as } n \rightarrow +\infty. \tag{4.14}$$

From (4.14) and by the weak lower semi-continuity of the $H^1(\mathbb{R}^N)$ -norm, we infer that

$$\langle L_2 v, v \rangle \leq \liminf_{n \rightarrow +\infty} \langle L_2 v_n, v_n \rangle = 0. \tag{4.15}$$

Combined with (4.13), (4.15) implies $\langle L_2 v, v \rangle = 0$. Since $(\varphi, v)_2 = 0$, we obtain by Lemma 4.15

$$v \equiv 0. \tag{4.16}$$

On the other hand,

$$0 = \liminf_{n \rightarrow +\infty} \langle L_2 v_n, v_n \rangle = 1 - \int_{\mathbb{R}^N} \varphi^{p-1} v^2 dx.$$

Hence $\int_{\mathbb{R}^N} \varphi^{p-1} v^2 dx = 1$, which is in contradiction with (4.16). \square

We now turn our attention to L_1 . The proof of Lemma 4.12 is more delicate, essentially since the spectrum of \tilde{L}_1 contains nonpositive eigenvalues. Furthermore, φ is no longer an eigenfunction. We first deal with the negative eigenvalues of \tilde{L}_1 .

Lemma 4.16. *The operator \tilde{L}_1 has only one negative eigenvalue $-\lambda_1$ with a corresponding eigenfunction $e_1 \in H^2(\mathbb{R}^N)$ such that $\|e_1\|_{L^2(\mathbb{R}^N)} = 1$.*

Proof. Let $\tilde{S} : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ be the restriction of S to real-valued functions. It is not hard to see that φ is also a mountain pass critical point of \tilde{S} . Then it is well known (see, for example, [3]) that the Morse index of \tilde{S} at φ is at most 1 (recall that the Morse

index is the number of negative eigenvalues of $\tilde{S}''(\varphi)$. We remark that $L_1 = \tilde{S}''(\varphi)$. Therefore, \tilde{L}_1 has at most one negative eigenvalue. On the other hand,

$$\begin{aligned} \langle L_1\varphi, \varphi \rangle &= \|\nabla\varphi\|_{L^2(\mathbb{R}^N)}^2 + \omega\|\varphi\|_{L^2(\mathbb{R}^N)}^2 - p\|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &= -(p-1)\|\varphi\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} < 0, \end{aligned}$$

where the second equality follows from the Nehari identity (3.3). Therefore, \tilde{L}_1 has exactly one negative eigenvalue $-\lambda_1$, and we can pick up an eigenfunction $e_1 \in H^2(\mathbb{R}^N)$ such that $\|e_1\|_{L^2(\mathbb{R}^N)} = 1$. □

The following result originates from the work of Weinstein [76].

Lemma 4.17. *The second eigenvalue of \tilde{L}_1 is 0 and*

$$Z := \ker \tilde{L}_1 = \text{span} \left\{ \frac{\partial\varphi}{\partial x_j}; j = 1, \dots, N \right\}.$$

It is out of reach in these notes to give a complete proof of Lemma 4.17. Therefore, we only give a partial proof and refer to [2] for a complete one.

Partial proof of Lemma 4.17. We remember that φ satisfies

$$-\Delta\varphi + \omega\varphi - \varphi^p = 0. \tag{4.17}$$

Differentiating (4.17) with respect to x_j gives

$$-\Delta \frac{\partial\varphi}{\partial x_j} + \omega \frac{\partial\varphi}{\partial x_j} - p\varphi^{p-1} \frac{\partial\varphi}{\partial x_j} = 0.$$

This is allowed since $\varphi \in W^{3,2}(\mathbb{R}^N)$ implies $\frac{\partial\varphi}{\partial x_j} \in H^2(\mathbb{R}^N)$. Hence 0 is an eigenvalue of \tilde{L}_1 and

$$\text{span} \left\{ \frac{\partial\varphi}{\partial x_j}; j = 1, \dots, N \right\} \subset \ker \tilde{L}_1.$$

We admit that the reverse inclusion also holds. □

Lemma 4.18. *The space $H^1(\mathbb{R}^N)$ can be decomposed as $H^1(\mathbb{R}^N) = E_1 \oplus Z \oplus E_+$, where $E_1 = \text{span}\{e_1\}$ and E_+ is the image of the spectral projection corresponding to the positive part of the spectrum of \tilde{L}_1 .*

Note that in the direct sum $E_1 \oplus Z \oplus E_+$ the spaces are mutually orthogonal with respect to the inner product of $L^2(\mathbb{R}^N)$.

Proof of Lemma 4.18. The assertion follows immediately from Lemmas 4.14, 4.16, 4.17 and the spectral decomposition theorem (see, e.g., [43, p. 177]). □

Lemma 4.19. For all $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ satisfying

$$(u, \varphi)_2 = \left(u, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N, \tag{4.18}$$

we have

$$\langle L_1 u, u \rangle > 0. \tag{4.19}$$

Proof. Let $u \in H^1(\mathbb{R}^N)$ satisfying (4.18). We first look for a function ψ such that

$$L_1 \psi = -\omega \varphi.$$

To do this, we use the scaled functions φ_λ defined by $\varphi_\lambda(\cdot) := \lambda^{\frac{1}{p-1}} \varphi(\lambda^{\frac{1}{2}} \cdot)$ for $\lambda > 0$. The functions φ_λ satisfy

$$-\Delta \varphi_\lambda + \omega \lambda \varphi_\lambda - \varphi_\lambda^p = 0. \tag{4.20}$$

Differentiating (4.20) with respect to λ gives at $\lambda = 1$

$$-\Delta \psi + \omega \psi - p \varphi^{p-1} \psi = -\omega \varphi$$

for $\psi := \frac{\partial \varphi_\lambda}{\partial \lambda} \Big|_{\lambda=1} = \frac{1}{p-1} \varphi + \frac{1}{2} x \cdot \nabla \varphi$. Note that $\psi \in H^1(\mathbb{R}^N)$ since $x \cdot \nabla \varphi \in H^1(\mathbb{R}^N)$ by Proposition 3.2. Furthermore, since φ is radial, φ is even in each x_j . Therefore, ψ is also even in each x_j whereas $\frac{\partial \varphi}{\partial x_j}$ is odd in x_j . This implies

$$\left(\psi, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N.$$

We decompose u and ψ with respect to $H^1(\mathbb{R}^N) = E_1 \oplus Z \oplus E_+$: There exist $\alpha, \beta \in \mathbb{R}$ and $\xi, \eta \in E_+$ such that

$$\begin{aligned} u &= \alpha e_1 + \xi, \\ \psi &= \beta e_1 + \eta. \end{aligned}$$

If $\alpha = 0$ then $\xi \neq 0$ since $u \neq 0$, and we have

$$\langle L_1 u, u \rangle = \langle L_1 \xi, \xi \rangle > 0.$$

Therefore, u satisfies (4.19). Now, we suppose $\alpha \neq 0$. We easily find:

$$\begin{aligned} \langle L_1 \psi, \psi \rangle &= -\omega \left(\varphi, \frac{1}{p-1} \varphi + \frac{1}{2} x \cdot \nabla \varphi \right)_2 \\ &= -\omega \left(\frac{1}{p-1} \|\varphi\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \int_{\mathbb{R}^N} \varphi x \cdot \nabla \varphi dx \right). \end{aligned}$$

With integration by parts, it is easy to see that

$$\int_{\mathbb{R}^N} \varphi x \cdot \nabla \varphi dx = -N \|\varphi\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} \varphi x \cdot \nabla \varphi dx.$$

Therefore, $\int_{\mathbb{R}^N} \varphi x \cdot \nabla \varphi dx = -\frac{N}{2} \|\varphi\|_{L^2(\mathbb{R}^N)}^2$ and

$$\langle L_1 \psi, \psi \rangle = -\omega \left(\frac{1}{p-1} - \frac{N}{4} \right) \|\varphi\|_{L^2(\mathbb{R}^N)}^2.$$

Since $\omega > 0$ and $p < 1 + \frac{4}{N}$ this implies

$$\langle L_1 \psi, \psi \rangle < 0$$

and thus $\beta \neq 0$.

On E_+ , L_1 defines a positive definite quadratic form, and we have the Cauchy-Schwarz inequality

$$\langle L_1 \xi, \eta \rangle^2 \leq \langle L_1 \xi, \xi \rangle \langle L_1 \eta, \eta \rangle, \quad \xi, \eta \in E_+.$$

Therefore, we find

$$\langle L_1 u, u \rangle = -\alpha^2 \lambda_1 + \langle L_1 \xi, \xi \rangle \geq -\alpha^2 \lambda_1 + \frac{\langle L_1 \xi, \eta \rangle^2}{\langle L_1 \eta, \eta \rangle}. \tag{4.21}$$

On the other hand, we get

$$0 = -\omega (\varphi, u)_2 = \langle L_1 \psi, u \rangle = -\alpha \beta \lambda_1 + \langle L_1 \xi, \eta \rangle$$

and thus obtain

$$\langle L_1 \xi, \eta \rangle = \alpha \beta \lambda_1.$$

This gives

$$\begin{aligned} -\alpha^2 \lambda_1 + \frac{\langle L_1 \xi, \eta \rangle^2}{\langle L_1 \eta, \eta \rangle} &= -\alpha^2 \lambda_1 + \frac{\alpha^2 \beta^2 \lambda_1^2}{\langle L_1 \eta, \eta \rangle} \\ &= -\alpha^2 \lambda_1 + \frac{\alpha^2 \beta^2 \lambda_1^2}{\beta^2 \lambda_1 + \langle L_1 \psi, \psi \rangle} \\ &= \frac{-\alpha^2 \lambda_1 \langle L_1 \psi, \psi \rangle}{\langle L_1 \eta, \eta \rangle} > 0. \end{aligned}$$

Combined with (4.21), this finishes the proof. □

Proof of Lemma 4.12. The proof of Lemma 4.12 follows the same lines as the proof of Lemma 4.13. We omit the details. □

5 Instability

In this section, we will prove that the standing waves are unstable by blow-up in finite time if $1 + \frac{4}{N} \leq p < 1 + \frac{4}{N-2}$. More precisely, for any ground state $\varphi \in \mathcal{G}$, we will find initial data, as close to φ as we want, such that the solutions of (1.1) corresponding

to these initial data will all blow up in finite time. As in Proposition 2.3, our basic tool will be the virial theorem (Proposition 2.4). However, since the energy of the ground state is nonnegative, it is not possible to argue directly as in Proposition 2.3. To overcome this difficulty, we follow the approach of [48], which is a recent improvement of the method introduced by Berestycki and Cazenave in [6, 7]. At the heart of this method is a new variational characterization of the ground states as minimizers of the action S on constraints related to the Pohozaev identity (3.4). Compared to [6, 7], the main feature of the approach of [48] is that we do not need to solve directly a new minimization problem; instead, we take advantage of the variational characterizations of the ground states obtained in Section 3.

Before stating our results, we give a precise definition of instability by blow-up.

Definition 5.1. Let φ be a solution of (3.1). The standing wave $e^{i\omega t}\varphi(x)$ is said to be *unstable by blow-up in finite time* if for all $\varepsilon > 0$ there exists $u_{\varepsilon,0} \in H^1(\mathbb{R}^N)$ such that

$$\|u_{\varepsilon,0} - \varphi\|_{H^1(\mathbb{R}^N)} < \varepsilon$$

but the corresponding maximal solution u_ε of (1.1) in the interval $(T_{\min}^\varepsilon, T_\varepsilon^{\max})$ satisfies $T_{\min}^\varepsilon > -\infty, T_\varepsilon^{\max} < +\infty$ and thus

$$\lim_{t \downarrow T_{\min}^\varepsilon} \|u_\varepsilon(t)\|_{H^1(\mathbb{R}^N)} = +\infty \text{ and } \lim_{t \uparrow T_\varepsilon^{\max}} \|u_\varepsilon(t)\|_{H^1(\mathbb{R}^N)} = +\infty.$$

We start with the simplest case $p = 1 + \frac{4}{N}$. The following result is due to Weinstein [75].

Theorem 5.2. *Let $p = 1 + \frac{4}{N}$. Then for every solution φ of (3.1) the standing wave $e^{i\omega t}\varphi(x)$ is unstable by blow-up in finite time.*

Proof. First, we remark that $E(v) = P(v)$ (recall that P was defined in (2.6)) for all $v \in H^1(\mathbb{R}^N)$ since $p = 1 + \frac{4}{N}$. From the Pohozaev identity (3.4), we have

$$E(\varphi) = P(\varphi) = 0.$$

Let $u_{\varepsilon,0}$ be defined by $u_{\varepsilon,0} := (1 + \varepsilon)\varphi$. Then it is easy to see that $E(u_{\varepsilon,0}) < 0$. In view of the exponential decay of φ (see Proposition 3.2), we have $|x|u_{\varepsilon,0} \in L^2(\mathbb{R}^N)$. The conclusion follows from Proposition 2.3. \square

We consider now the general case.

Theorem 5.3. *Let $p > 1 + \frac{4}{N}$. For all $\varphi \in \mathcal{G}$, the standing wave $e^{i\omega t}\varphi(x)$ is unstable by blow-up in finite time.*

Remark 5.4. Theorem 5.2 asserts the instability of standing waves corresponding to any solution of (3.1) whereas Theorem 5.3 concerns only standing waves corresponding to ground states. It is still an open problem to prove instability by blow-up for any solution of (3.1) if $p > 1 + \frac{4}{N}$ (see [74] for a review of related results and open problems).

For the proof of Theorem 5.3 it is not possible to mimic the proof of Theorem 5.2. Indeed, for $p > 1 + \frac{4}{N}$, the identity $E(v) = P(v)$ does not hold any more. Moreover, $E(\varphi) > 0$ for $\varphi \in \mathcal{G}$, which prevents to use Proposition 2.3.

The scaling $v_\lambda(\cdot) := \lambda^{N/2}v(\lambda\cdot)$ will play an important role in the proof. In the following lemma, we investigate the behavior of different functionals under the scaling.

Lemma 5.5. *Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ be such that $P(v) \leq 0$. Then there exists $\lambda_0 \in (0, 1]$ such that*

- (i) $P(v_{\lambda_0}) = 0$,
- (ii) $\lambda_0 = 1$ if and only if $P(v) = 0$,
- (iii) $\frac{\partial}{\partial \lambda} S(v_\lambda) = \frac{1}{\lambda} P(v_\lambda)$,
- (iv) $\frac{\partial}{\partial \lambda} S(v_\lambda) > 0$ for $\lambda \in (0, \lambda_0)$ and $\frac{\partial}{\partial \lambda} S(v_\lambda) < 0$ for $\lambda \in (\lambda_0, +\infty)$,
- (v) $\lambda \mapsto S(v_\lambda)$ is concave on $(\lambda_0, +\infty)$.

Proof. A simple calculation leads to

$$P(v_\lambda) = \lambda^2 \|\nabla v\|_{H^1(\mathbb{R}^N)}^2 - \lambda^{\frac{N(p-1)}{2}} \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.$$

Recalling that $\frac{N(p-1)}{2} > 2$ (because of $p > 1 + \frac{4}{N}$), we infer that $P(v_\lambda) > 0$ for λ small enough. Thus, by continuity of P , there must exist $\lambda_0 \in (0, 1]$ such that $P(v_{\lambda_0}) = 0$. Hence (i) is proved. If $\lambda_0 = 1$, it is clear that $P(v) = 0$. Conversely, suppose that $P(v) = 0$. Then

$$\begin{aligned} P(v_\lambda) &= \lambda^2 P(v) + \left(\lambda^2 - \lambda^{\frac{N(p-1)}{2}}\right) \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &= \left(\lambda^2 - \lambda^{\frac{N(p-1)}{2}}\right) \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \end{aligned}$$

and, since $\frac{N(p-1)}{2} > 2$, this implies that $P(v_\lambda) > 0$ for all $\lambda \in (0, 1)$. Hence (ii) follows. From a simple calculation, we obtain

$$\begin{aligned} \frac{\partial}{\partial \lambda} S(v_\lambda) &= \lambda \|\nabla v\|_{H^1(\mathbb{R}^N)}^2 - \lambda^{\frac{N(p-1)}{2}-1} \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &= \lambda^{-1} P(v_\lambda). \end{aligned}$$

Hence (iii) is shown. To see (iv), we remark that

$$\begin{aligned} P(v_\lambda) &= \lambda^2 \lambda_0^{-2} P(v_{\lambda_0}) + \left(\lambda^2 \lambda_0^{\frac{N(p-1)}{2}-2} - \lambda^{\frac{N(p-1)}{2}}\right) \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &= \lambda^2 \left(\lambda_0^{\frac{N(p-1)}{2}-2} - \lambda^{\frac{N(p-1)}{2}-2}\right) \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}. \end{aligned}$$

Therefore, since $\frac{N(p-1)}{2} - 2 > 0$, $P(v_\lambda) > 0$ for $\lambda < \lambda_0$ and $P(v_\lambda) < 0$ for $\lambda > \lambda_0$. Combined with (iii), this gives (iv). Finally,

$$\begin{aligned} \frac{\partial^2}{\partial \lambda^2} S(v_\lambda) &= \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 - \lambda^{\frac{N(p-1)}{2}-2} \left(\frac{N(p-1)}{2} - 1 \right) \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\ &= \lambda^{-2} P(v_\lambda) - \lambda^{\frac{N(p-1)}{2}-2} \left(\frac{N(p-1)}{2} - 2 \right) \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \end{aligned}$$

which implies $\frac{\partial^2}{\partial \lambda^2} S(v_\lambda) < 0$ for $\lambda > \lambda_0$ since $\frac{N(p-1)}{2} > 2$. Hence (v) follows. \square

The proof of Theorem 5.3 runs in three steps. First, we derive a new variational characterization for the ground states of (3.1). Then we use this characterization to define a set \mathcal{I} , invariant under the flow of (1.1), such that any initial datum in \mathcal{I} gives rise to a blowing up solution of (1.1). Finally, we prove that the ground states can be approximated by sequences of elements of \mathcal{I} .

We consider the minimization problem

$$d_{\mathcal{M}} := \inf_{v \in \mathcal{M}} S(v),$$

where the constraint \mathcal{M} is given by

$$\mathcal{M} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\}; P(v) = 0, I(v) \leq 0\}.$$

Recall that I was defined in Proposition 3.12.

Lemma 5.6. *The following equality holds:*

$$m = d_{\mathcal{M}},$$

where m is the least energy level defined in (3.20).

Proof. Let $\varphi \in \mathcal{G}$. Because of Lemma 3.3, we have $I(\varphi) = P(\varphi) = 0$. Thus it is clear that $\varphi \in \mathcal{M}$. Therefore, $S(\varphi) \geq d_{\mathcal{M}}$ and thus

$$m \geq d_{\mathcal{M}}. \tag{5.1}$$

Now, let $v \in \mathcal{M}$. If $I(v) = 0$ then $S(v) \geq m$ by Proposition 3.12. Suppose that $I(v) < 0$. We use the scaling $v_\lambda(\cdot) := \lambda^{N/2} v(\lambda \cdot)$. Then

$$I(v_\lambda) = \lambda^2 \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \omega \|v\|_{L^2(\mathbb{R}^N)}^2 - \lambda^{\frac{N(p-1)}{2}} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}$$

implies $\lim_{\lambda \rightarrow 0} I(v_\lambda) = \omega \|v\|_{L^2(\mathbb{R}^N)}^2 > 0$. By continuity of I , there exists $\lambda_1 < 1$ such that $I(v_{\lambda_1}) = 0$. Therefore, by Proposition 3.12,

$$m \leq S(v_{\lambda_1}). \tag{5.2}$$

From $P(v) = 0$ and Lemma 5.5, we deduce that

$$S(v_{\lambda_1}) < S(v). \tag{5.3}$$

Combining (5.2) and (5.3) gives $m \leq S(v)$, hence

$$m \leq d_{\mathcal{M}}. \tag{5.4}$$

The conclusion follows from (5.1) and (5.4) □

Now, we define the set \mathcal{I} by

$$\mathcal{I} := \{v \in H^1(\mathbb{R}^N); I(v) < 0, P(v) < 0, S(v) < m\}$$

and use Lemma 5.6 to prove that \mathcal{I} is invariant under the flow of (1.1).

Lemma 5.7. *If $u_0 \in \mathcal{I}$ then the corresponding maximal solution u in (T_{\min}, T^{\max}) of (1.1) satisfies $u(t) \in \mathcal{I}$ for all $t \in (T_{\min}, T^{\max})$.*

Proof. Let $u_0 \in \mathcal{I}$ and u be the corresponding maximal solution of (1.1) in (T_{\min}, T^{\max}) . Since S is a conserved quantity for (1.1), we find

$$S(u(t)) = S(u_0) < m \text{ for all } t \in (T_{\min}, T^{\max}). \tag{5.5}$$

The assertion is proved by contradiction. Suppose that there exists t such that

$$I(u(t)) \geq 0.$$

Then, by the continuity of I and u , there exists t_0 such that

$$I(u(t_0)) = 0.$$

By Proposition 3.12, this implies

$$S(u(t_0)) \geq m,$$

which is in contradiction with (5.5). Therefore,

$$I(u(t)) < 0 \text{ for all } t \in (T_{\min}, T^{\max}). \tag{5.6}$$

Finally, suppose that for some $t \in (T_{\min}, T^{\max})$

$$P(u(t)) \geq 0.$$

Still by continuity, there exists t_1 such that

$$P(u(t_1)) = 0.$$

From (5.6), we also have $I(u(t_1)) < 0$, and thus by Lemma 5.6

$$S(u(t_1)) \geq m,$$

which is another contradiction. Therefore,

$$P(u(t)) < 0 \text{ for all } t \in (T_{\min}, T^{\max}),$$

and this completes the proof. □

Lemma 5.8. *Let $u_0 \in \mathcal{I}$ and u be the corresponding maximal solution of (1.1) in (T_{\min}, T^{\max}) . Then there exists $\delta > 0$ independent of t such that $P(u(t)) < -\delta$ for all $t \in (T_{\min}, T^{\max})$.*

Proof. Let $t \in (T_{\min}, T^{\max})$ and set $v := u(t)$ and $v_\lambda(\cdot) := \lambda^{N/2}v(\lambda \cdot)$. By Lemma 5.5, there exists $\lambda_0 < 1$ such that $P(v_{\lambda_0}) = 0$. If $I(v_{\lambda_0}) \leq 0$, we keep λ_0 , otherwise if $I(v_{\lambda_0}) > 0$ there exists $\tilde{\lambda}_0 \in (\lambda_0, 1)$ such that $I(v_{\tilde{\lambda}_0}) = 0$ and we replace λ_0 by $\tilde{\lambda}_0$. In any case, by Proposition 3.12 or Lemma 5.6,

$$S(v_{\lambda_0}) \geq m. \tag{5.7}$$

Now, by (v) in Lemma 5.5, we get

$$S(v) - S(v_{\lambda_0}) \geq (1 - \lambda_0) \frac{\partial}{\partial \lambda} S(v_\lambda) \Big|_{\lambda=1}. \tag{5.8}$$

From (iii) in Lemma 5.5, we obtain that

$$\frac{\partial}{\partial \lambda} S(v_\lambda) \Big|_{\lambda=1} = P(v). \tag{5.9}$$

Furthermore, $P(v) < 0$ and $\lambda_0 \in (0, 1)$ implies

$$(1 - \lambda_0)P(v) > P(v). \tag{5.10}$$

Combining (5.7)–(5.10) gives

$$S(v) - m > P(v).$$

Let $-\delta := S(v) - m$. Then $\delta > 0$ since $v \in \mathcal{I}$, and δ is independent of t since S is a conserved quantity. In conclusion, for any $t \in (T_{\min}, T^{\max})$,

$$P(u(t)) < -\delta$$

and this ends the proof. □

Lemma 5.9. *Let $u_0 \in \mathcal{I}$ be such that $|x|u_0 \in L^2(\mathbb{R}^N)$. Then the corresponding maximal solution u of (1.1) in (T_{\min}, T^{\max}) blows up in finite time, i.e., $T_{\min} > -\infty$, $T^{\max} < +\infty$,*

$$\lim_{t \downarrow T_{\min}} \|u(t)\|_{H^1(\mathbb{R}^N)} = +\infty \text{ and } \lim_{t \uparrow T^{\max}} \|u(t)\|_{H^1(\mathbb{R}^N)} = +\infty.$$

Proof. By Lemma 5.8, there exists $\delta > 0$ such that

$$P(u(t)) < -\delta \text{ for all } t \in (T_{\min}, T^{\max}).$$

Remembering from Proposition 2.4 that $\frac{\partial^2}{\partial t^2} \|xu(t)\|_{L^2(\mathbb{R}^N)}^2 = 8P(u(t))$, we get by integrating twice in time

$$\|xu(t)\|_{L^2(\mathbb{R}^N)}^2 \leq -4\delta t^2 + C(t + 1).$$

As in the proof of Proposition 2.3, this leads to a contradiction for large $|t|$. Therefore, $T_{\min} > -\infty, T^{\max} < +\infty$ and, by Proposition 2.1,

$$\lim_{t \downarrow T_{\min}} \|u(t)\|_{H^1(\mathbb{R}^N)} = +\infty \text{ and } \lim_{t \uparrow T^{\max}} \|u(t)\|_{H^1(\mathbb{R}^N)} = +\infty.$$

□

Proof of Theorem 5.3. In view of Lemma 5.9, it remains to find a sequence in \mathcal{I} converging to φ in $H^1(\mathbb{R}^N)$. We define $\varphi_\lambda(\cdot) := \lambda^{N/2}\varphi(\lambda \cdot)$. Then, by Lemma 5.5,

$$I(\varphi_\lambda) < 0, P(\varphi_\lambda) < 0, S(\varphi_\lambda) < m,$$

thus $\varphi_\lambda \in \mathcal{I}$ for all $0 < \lambda < 1$. Furthermore, by Proposition 3.2, φ is exponentially decaying and so is φ_λ . Therefore, $|x|\varphi_\lambda \in L^2(\mathbb{R}^N)$ for all $0 < \lambda < 1$. It is clear that $\varphi_\lambda \rightarrow \varphi$ when $\lambda \rightarrow 1$. Because of Lemma 5.9, the solution of (1.1) corresponding to the initial datum φ_λ blows up in finite time for any $0 < \lambda < 1$. This completes the proof. □

Remark 5.10. When the nonlinearity is more general, it may be impossible to use the virial theorem to obtain a result of instability by blow-up. In such cases, a good alternative to prove instability would be to follow the method introduced by Shatah and Strauss in [69] and then developed further in [34, 35]. Other methods, based on modifications of the original idea of Shatah and Strauss, are also available, see for example [21, 22, 32, 45, 61]. Note that proving instability by these methods does not necessarily provide information on the long time behavior (blow-up or global existence) of solutions starting near a standing wave.

Remark 5.11. The type of method employed here to prove instability by blow-up of standing waves is not restricted to nonlinear Schrödinger equations. See, for example, [6, 39, 53, 62, 63, 68] and the references cited therein for results on the instability by blow-up for standing waves of nonlinear Klein–Gordon equations.

6 Appendix

The appendix is devoted to the proof of Proposition 4.10.

For φ being a solution of (3.1), we define a *tubular neighborhood* of φ of size $\varepsilon > 0$ in $H^1(\mathbb{R}^N)$ by

$$U_\varepsilon(\varphi) := \{v \in H^1(\mathbb{R}^N); \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|e^{i\theta}v(\cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)} < \varepsilon\}.$$

Before proving Proposition 4.10, some preliminaries are needed.

Lemma 6.1. *Let φ be a solution of (3.1). For all $\delta > 0$ there exists $\varepsilon > 0$ such that*

$$\|e^{i\theta}\varphi(\cdot - y) - \varphi\|_{L^2(\mathbb{R}^N)} < \varepsilon, \quad \theta \in \mathbb{R}, y \in \mathbb{R}^N,$$

implies $|(\theta, y)| < \delta$. Here, $|\cdot|$ denotes the standard norm in $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^N$.

Proof. The proof is carried out by contradiction. Let $\delta > 0$ and assume that for all $n \in \mathbb{N}$ there exists (θ_n, y_n) such that

$$\|e^{i\theta_n}\varphi(\cdot - y_n) - \varphi\|_{L^2(\mathbb{R}^N)} < \frac{1}{n} \text{ and } |(\theta_n, y_n)| > \delta.$$

Possibly for a subsequence only, we have $e^{i\theta_n}\varphi(\cdot - y_n) \rightarrow \varphi$ almost everywhere as $n \rightarrow +\infty$, and in fact everywhere by regularity of φ (see Proposition 3.2). Suppose that (y_n) is unbounded. Then, possibly for a subsequence only, $|y_n| \rightarrow +\infty$, but by exponential decay of φ (see Proposition 3.2), this implies that for all $x \in \mathbb{R}^N$,

$$e^{i\theta_n}\varphi(x + y_n) \rightarrow 0,$$

which is a contradiction with $\varphi \not\equiv 0$. Therefore, (y_n) is bounded and converges, possibly passing to a subsequence, to some $y \in \mathbb{R}^N$. Since each θ_n can be chosen in $[0, 2\pi)$, we also have $\theta_n \rightarrow \theta$ for some $\theta \in [0, 2\pi)$. By hypothesis, we have

$$|(\theta, y)| \geq \delta, \tag{6.1}$$

but for all $x \in \mathbb{R}^N$

$$e^{i\theta}\varphi(x + y) = \varphi(x). \tag{6.2}$$

We claim that (6.2) can hold true only if $y = \theta = 0$. Indeed, let x_0 be such that $\varphi(x_0) \neq 0$. If $y \neq 0$ then $\lim_{n \rightarrow +\infty} e^{in\theta}\varphi(x_0 + ny) = \varphi(x_0) \neq 0$, what is in contradiction with the decay of φ at infinity. Therefore, $y = 0$ and this immediately implies $\theta = 0$. This is in contradiction with (6.1) and finishes the proof. \square

Lemma 6.2. *Let φ be a solution of (3.1). There exist $\varepsilon > 0$ and two functions $\sigma : U_\varepsilon(\varphi) \rightarrow \mathbb{R}$ and $Y : U_\varepsilon(\varphi) \rightarrow \mathbb{R}^N$ such that for all $v \in U_\varepsilon(\varphi)$*

$$\|e^{i\sigma(v)}v(\cdot - Y(v)) - \varphi\|_{L^2(\mathbb{R}^N)} = \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|e^{i\theta}v(\cdot - y) - \varphi\|_{L^2(\mathbb{R}^N)}.$$

Furthermore, the function $w := e^{i\sigma(v)}v(\cdot - Y(v))$ satisfies

$$(w, i\varphi)_2 = \left(w, \frac{\partial\varphi}{\partial x_j}\right)_2 = 0 \text{ for all } j = 1, \dots, N. \tag{6.3}$$

Proof. For the sake of simplicity, we assume that φ is real-valued and radial. Let $\Phi : \mathbb{R} \times \mathbb{R}^N \times H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ be defined by

$$\Phi(\theta, y, v) := \frac{1}{2} \|e^{i\theta}v(\cdot - y) - \varphi\|_{L^2(\mathbb{R}^N)}^2 = \frac{1}{2} \|v - e^{-i\theta}\varphi(\cdot + y)\|_{L^2(\mathbb{R}^N)}^2$$

and let $F : \mathbb{R} \times \mathbb{R}^N \times H^1(\mathbb{R}^N) \rightarrow \mathbb{R}^{N+1}$ be the derivative of Φ with respect to (θ, y) :

$$F(\theta, y, v) := D_{\theta, y}\Phi(\theta, y, v) = \begin{pmatrix} (e^{i\theta}v(\cdot - y), i\varphi)_2 \\ -\left(e^{i\theta}v(\cdot - y), \frac{\partial\varphi}{\partial x_1}\right)_2 \\ \vdots \\ -\left(e^{i\theta}v(\cdot - y), \frac{\partial\varphi}{\partial x_N}\right)_2 \end{pmatrix}.$$

We have $F(0, 0, \varphi) = 0$, and the derivative of F is given by the diagonal matrix

$$D_{\theta,y}F(0, 0, \varphi) = \begin{pmatrix} \|\varphi\|_{L^2(\mathbb{R}^N)}^2 & & & 0 \\ & \|\frac{\partial\varphi}{\partial x_1}\|_{L^2(\mathbb{R}^N)}^2 & & \\ & & \ddots & \\ 0 & & & \|\frac{\partial\varphi}{\partial x_N}\|_{L^2(\mathbb{R}^N)}^2 \end{pmatrix}.$$

Since φ is a solution of (3.1), φ is not zero and, therefore, we have $\|\varphi\|_{L^2(\mathbb{R}^N)}^2 > 0$ and $\|\frac{\partial\varphi}{\partial x_j}\|_{L^2(\mathbb{R}^N)}^2 > 0$ for all $j = 1, \dots, N$. Consequently, the matrix $D_{\theta,y}F(0, 0, \varphi)$ is positive definite and by the implicit function theorem there exists $\varepsilon > 0, \delta > 0$ and $(\sigma, Y) : V_\varepsilon \rightarrow \Omega$, where

$$V_\varepsilon := \{v \in H^1(\mathbb{R}^N); \|v - \varphi\|_{H^1(\mathbb{R}^N)} < \varepsilon\} \text{ and } \Omega := \{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N; |(\theta, y)| < \delta\},$$

such that for all $v \in V_\varepsilon$

$$F(\sigma(v), Y(v), v) = 0 \tag{6.4}$$

and

$$\|e^{i\sigma(v)}v(\cdot - Y(v)) - \varphi\|_{L^2(\mathbb{R}^N)} = \inf_{(\theta,y) \in \Omega} \|e^{i\theta}v(\cdot - y) - \varphi\|_{L^2(\mathbb{R}^N)}. \tag{6.5}$$

Now we extend (6.5) to all $(\theta, y) \in \mathbb{R} \times \mathbb{R}^N$. Assume that there exists $(\tilde{\theta}, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^N$ such that $|(\tilde{\theta}, \tilde{y})| > \delta$ and

$$\|e^{i\tilde{\theta}}v(\cdot - \tilde{y}) - \varphi\|_{L^2(\mathbb{R}^N)} < \|e^{i\sigma(v)}v(\cdot - Y(v)) - \varphi\|_{L^2(\mathbb{R}^N)}.$$

For $v \in V_\varepsilon$, we have

$$\|e^{i\tilde{\theta}}\varphi(\cdot - \tilde{y}) - \varphi\|_{L^2(\mathbb{R}^N)} < \|e^{i\tilde{\theta}}(\varphi(\cdot - \tilde{y}) - v(\cdot - \tilde{y}))\|_{L^2(\mathbb{R}^N)} + \|e^{i\tilde{\theta}}v(\cdot - \tilde{y}) - \varphi\|_{L^2(\mathbb{R}^N)} < 2\varepsilon.$$

If ε is small enough, this implies by Lemma 6.1 that $|(\tilde{\theta}, \tilde{y})| < \delta$, which is a contradiction. Therefore, we have, for all $v \in V_\varepsilon$,

$$\|e^{i\sigma(v)}v(\cdot - Y(v)) - \varphi\|_{L^2(\mathbb{R}^N)} = \inf_{(\theta,y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta}v(\cdot - y) - \varphi\|_{L^2(\mathbb{R}^N)}.$$

It remains to extend (σ, Y) to $U_\varepsilon(\varphi)$. Let $v \in U_\varepsilon(\varphi)$. Then there exists (θ^*, y^*) such that

$$\|e^{i\theta^*}v(\cdot - y^*) - \varphi\|_{L^2(\mathbb{R}^N)} < \varepsilon.$$

Define

$$\sigma(v) := \sigma(e^{i\theta^*}v(\cdot - y^*)) + \theta^*$$

and

$$Y(v) := Y(e^{i\theta^*}v(\cdot - y^*)) + y^*.$$

Then it is not hard to see that this definition is independent of the choice of (θ^*, y^*) and allows us to extend (σ, Y) to $U_\varepsilon(\varphi)$. The orthogonality relations in (6.3) follow from the relation (6.4). □

Recall that Q was defined in (2.1) and E in (2.2).

Lemma 6.3. *Under the hypothesis of Proposition 4.10, there exist $\varepsilon > 0$ and $C > 0$ such that for all $v \in U_\varepsilon(\varphi)$ satisfying $Q(v) = Q(\varphi)$, we have*

$$E(v) - E(\varphi) \geq C \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} v(\cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)}^2.$$

Proof. For the sake of simplicity, we assume that φ is radial. Let $\varepsilon > 0$ and $v \in U_\varepsilon(\varphi)$. For ε small enough, let (σ, Y) be as in Lemma 6.2 and define

$$w := e^{i\sigma(v)} v(\cdot - Y(v)). \quad (6.6)$$

Then by Lemma 6.2 the function w satisfies

$$(w, i\varphi)_2 = \left(w, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for all } j = 1, \dots, N. \quad (6.7)$$

Let $\lambda \in \mathbb{R}$ and $z \in H^1(\mathbb{R}^N)$ be such that

$$(z, \varphi)_2 = 0 \quad (6.8)$$

and

$$w - \varphi = \lambda \varphi + z. \quad (6.9)$$

Since φ is radial up to translations, we have

$$\left(\varphi, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for } j = 1, \dots, N. \quad (6.10)$$

Combining (6.7), (6.9) and (6.10) we get

$$\left(z, \frac{\partial \varphi}{\partial x_j} \right)_2 = 0 \text{ for } j = 1, \dots, N. \quad (6.11)$$

Moreover, we have

$$(\varphi, i\varphi)_2 = \operatorname{Re}(-i\|\varphi\|_{L^2(\mathbb{R}^N)}^2) = 0,$$

and therefore

$$(z, i\varphi)_2 = 0. \quad (6.12)$$

From (6.8), (6.11) and (6.12), we see that z satisfies the orthogonality conditions in (4.10) and therefore

$$\langle S''(\varphi)z, z \rangle \geq \delta \|z\|_{H^1(\mathbb{R}^N)}^2. \quad (6.13)$$

By a Taylor expansion, we obtain

$$Q(\varphi) = Q(v) = Q(w) = Q(\varphi) + \langle Q'(\varphi), w - \varphi \rangle + O(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2).$$

But $Q'(\varphi) = \varphi$ and therefore

$$\langle Q'(\varphi), w - \varphi \rangle = (\varphi, w - \varphi)_2 = (\varphi, \lambda \varphi + z)_2 = \lambda \|\varphi\|_{L^2(\mathbb{R}^N)}^2,$$

where the last equality follows from (6.8). Consequently,

$$\lambda = O(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2). \tag{6.14}$$

Now, another Taylor expansion gives

$$\begin{aligned} S(v) - S(\varphi) &= S(w) - S(\varphi) \\ &= \langle S'(\varphi), w - \varphi \rangle + \frac{1}{2} \langle S''(\varphi)(w - \varphi), w - \varphi \rangle + o(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2). \end{aligned} \tag{6.15}$$

Since φ is a solution of (3.1), we have $S'(\varphi) = 0$ and therefore

$$\langle S'(\varphi), w - \varphi \rangle = 0. \tag{6.16}$$

Furthermore, from (6.9) we get

$$\langle S''(\varphi)(w - \varphi), w - \varphi \rangle = \lambda^2 \langle S''(\varphi)\varphi, \varphi \rangle + 2\lambda \operatorname{Re} \langle S''(\varphi)\varphi, z \rangle + \langle S''(\varphi)z, z \rangle. \tag{6.17}$$

From (6.14) we have

$$\lambda^2 \langle S''(\varphi)\varphi, \varphi \rangle = o(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2). \tag{6.18}$$

Since

$$\|z\|_{H^1(\mathbb{R}^N)}^2 \leq 2\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2 + 2\lambda^2 \|\varphi\|_{H^1(\mathbb{R}^N)}^2,$$

we have by (6.14) that

$$\|z\|_{H^1(\mathbb{R}^N)}^2 = O(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2), \tag{6.19}$$

and therefore

$$2\lambda \operatorname{Re} \langle S''(\varphi)\varphi, z \rangle = o(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2). \tag{6.20}$$

Combining (6.17), (6.18) and (6.20), we get

$$\langle S''(\varphi)(w - \varphi), w - \varphi \rangle = \langle S''(\varphi)z, z \rangle + o(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2).$$

Together with (6.15)–(6.16), this gives

$$S(v) - S(\varphi) \geq \frac{1}{2} \langle S''(\varphi)z, z \rangle + o(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2). \tag{6.21}$$

But since $Q(v) = Q(\varphi)$, we have

$$E(v) - E(\varphi) = S(v) - S(\varphi)$$

and with (6.13), (6.19) and (6.21), we obtain

$$E(v) - E(\varphi) \geq \frac{\delta}{2} \|w - \varphi\|_{H^1(\mathbb{R}^N)}^2 + o(\|w - \varphi\|_{H^1(\mathbb{R}^N)}^2).$$

Therefore, we have for ε small enough

$$E(v) - E(\varphi) \geq \frac{\delta}{4} \|w - \varphi\|_{H^1(\mathbb{R}^N)}^2.$$

Setting $C := \delta/4$ and remembering how w was defined in (6.6) gives

$$E(v) - E(\varphi) \geq C \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} v(\cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)}^2,$$

which finishes the proof. \square

Proof of Proposition 4.10. The assertion is proved by contradiction. Assume that there exists $u_{n,0}$ and $\varepsilon > 0$ such that

$$\|u_{n,0} - \varphi\|_{H^1(\mathbb{R}^N)} \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (6.22)$$

but for all $n \in \mathbb{N}$

$$\sup_{t \in (T_{\min}^n, T_{\max}^n)} \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} u_n(t, \cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)} > \varepsilon$$

where u_n is the maximal solution of (1.1) in (T_{\min}^n, T_{\max}^n) corresponding to $u_{0,n}$. By continuity, we can pick up the first time t_n such that

$$\inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} u_n(t_n, \cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)} = \varepsilon$$

In view of (6.22) and the conservation of charge and energy (see (2.3)), it is clear that

$$\begin{aligned} E(u_n(t_n)) &= E(u_{n,0}) \rightarrow E(\varphi) \\ Q(u_n(t_n)) &= Q(u_{n,0}) \rightarrow Q(\varphi) \end{aligned} \text{ as } n \rightarrow +\infty.$$

Let $v_n := \frac{u_n(t_n)}{\|u_n(t_n)\|_{L^2(\mathbb{R}^N)}} \|\varphi\|_{L^2(\mathbb{R}^N)}$. Then

$$Q(v_n) = Q(\varphi), E(v_n) \rightarrow E(\varphi) \text{ and } \|v_n - u_n(t_n)\|_{H^1(\mathbb{R}^N)} \rightarrow 0.$$

Choosing ε sufficiently small, we can apply Lemma 6.3 to get

$$\inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} v_n(\cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)}^2 \leq C(E(v_n) - E(\varphi)) \rightarrow 0.$$

On the other hand, we have

$$\begin{aligned} \varepsilon &= \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} u_n(t_n, \cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)} \\ &\leq \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^N} \|e^{i\theta} v_n(\cdot - y) - \varphi\|_{H^1(\mathbb{R}^N)} + \|u_n - v_n\|_{H^1(\mathbb{R}^N)}, \end{aligned}$$

which yields a contradiction for n large. \square

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