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ORBITAL STABILITY OF STANDING WAVES OF A SEMICLASSICAL NONLINEAR SCHRÖDINGER-POISSON EQUATION

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Abstract. We study the orbital stability of single-spike semiclassical standing waves of a nonhomogeneous in space nonlinear Schrödinger-Poisson equation. When the nonlinearity is subcritical or supercritical we prove that the nonlocal Poisson-term does not influence the stability of standing waves, whereas in the critical case it may create instability if its value at the concentration point of the spike is too large. The proofs are based on the study of the spectral properties of a linearized operator and on the analysis of a slope condition. Our main tools are perturbation methods and asymptotic expansion formulas.

1. Introduction

In this paper, we are concerned with the following nonlinear Schrödinger-Poisson equation:

$$-i\epsilon\Psi_t-\epsilon^2\Delta_x\Psi+W(x)\Psi+K(x)\left(|x|^{-1}*K(x)|\Psi|^2\right)\Psi-|\Psi|^{p-1}\Psi=0, \quad (1.1)$$

where $\Psi=\Psi(x,t):\mathbb{R}^3\times\mathbb{R}\to\mathbb{C}$, $\epsilon>0$ is a small parameter meant to tend to 0, $W,K:\mathbb{R}^3\to\mathbb{R}$ and $1<p<5$. These types of equations, sometimes also referred to as Schrödinger-Maxwell equations, arise in various physical and mathematical contexts. In the theory of Bose-Einstein condensates, Ψ is the wave function of the condensate and W stands for an external potential. The constant ϵ represents the Planck constant (often denoted by $\hbar$). The fact that ϵ tends to 0 is modeling the transition between quantum and classical mechanics, hence the terminology of *semiclassical analysis.* The nonlocal term in (1.1) corresponds to the interaction of a charged wave with its own electrostatic field (as was introduced by Benci and Fortunato [7]). We refer to the books of Cazenave [10] and Sulem and Sulem [42] for more on the physical and mathematical background as well as to the

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papers [7, 12, 13, 16, 25, 26, 39] for a particular emphasis on Schrödinger-Poisson/Maxwell equations.

Among solutions of (1.1), some are of particular interest: the *standing waves*. They are solutions appearing because of the balance between the dispersion generated by the linear part of (1.1) and nonlinear effects. Precisely, a standing wave is a solution of the form

$$\Psi(x,t) = \exp\left(\frac{i\omega}{\epsilon}t\right)v(x), \quad \text{where } \omega > 0 \text{ and } v : \mathbb{R}^3 \to \mathbb{R}.$$

For a function of this type (1.1) is satisfied if and only if v is a solution of the stationary Schrödinger-Poisson equation

$$-\epsilon^2 \Delta v + [W(x) + \omega]\, v + K(x)\left(|x|^{-1} * K(x)v^2\right) v - |v|^{p-1}v = 0. \tag{1.2}$$

In the study of standing waves, two main questions arise naturally: existence and stability (see e.g. [33] for an introduction to the theory for standing waves).

When $K \equiv W \equiv 0$, sufficient and necessary conditions for the existence of solutions to (1.2) for all $\epsilon > 0$ are known since the fundamental work of Berestycki and Lions [9]. When $W \not\equiv 0$ and $K \equiv 0$, the study of existence for solutions to (1.2) when $\epsilon \to 0$ (the so-called *semiclassical limit*) was initiated by Floer and Weinstein [19] and followed by a large amount of work (see e.g. [1, 18, 35, 43] for the existence of spike solutions, [24, 29, 38, 44] for multi-bump solutions, and the more recent works [3, 6] for solutions concentrating around a sphere). The case $K \equiv W \equiv 1$ has recently attracted the attention of many authors, see e.g. [4, 12, 14, 15, 17, 31, 32, 40] and the references therein. In particular, [13, 16, 39] are concerned with the semiclassical limit. We also refer to [5, 45] when $K \equiv 1$ and the potential W is nontrivial.

When not only W, but also K, is nontrivial, the difficulty of having nonhomogeneity in space is combined within the nonlocal term. To our knowledge, the only existence results for the semiclassical states with nontrivial potentials are due to Ianni and Vaira in [26] for the existence of single spikes (namely solutions concentrating at non-degenerate critical points of the potential W) and in [25, 27] for the existence of solutions concentrating on spheres.

In this paper, we are interested in the stability properties of the single spike semiclassical standing waves found in [26] (see Proposition 2.1 for a precise statement of the existence result of [26]). For standing waves, it is well known that the relevant concept of stability is *orbital stability*, namely Lyapunov stability up to phase shifts. Precisely, the concept of orbital stability is the following.

Definition 1.1. *A standing wave* $\exp(\frac{i\omega}{\epsilon}t)v(x)$ *of* (1.1) *is said to be* orbitally stable *in* $H^1(\mathbb{R}^3,\mathbb{C})$ *if for any* $\delta > 0$ *there exists* $\gamma > 0$ *such that if* $w_0 \in H^1(\mathbb{R}^3,\mathbb{C})$ *satisfies* $\|w_0 - v\|_{H^1(\mathbb{R}^3,\mathbb{C})} < \gamma$ *then the maximal solution* $\Psi(\cdot,t)$ *of* (1.1) *with* $\Psi(\cdot,0) = w_0$ *exists for all* $t \geq 0$ *and*

$$\sup_{t\geq 0}\inf_{\theta\in\mathbb{R}} \|\Psi(\cdot,t) - \exp(i\theta)v\|_{H^1(\mathbb{R}^3,\mathbb{C})} < \delta.$$

Otherwise the standing wave is said to be unstable. *By extension, we shall say that a solution of* (1.2) *is* stable/unstable *if the corresponding standing wave is stable/unstable.*

The study of the orbital stability of standing waves for nonlinear Schrödinger equations has seen the contributions of many authors since the pioneering works of Berestycki and Cazenave [8], Cazenave and Lions [11], and Weinstein [46, 47] (see e.g. [20, 21, 28, 32, 34]).

In the case $K \equiv W \equiv 0$, least energy solutions of (1.2) are stable if $p < 1 + \frac{4}{3}$ and unstable if $p \geqslant 1 + \frac{4}{3}$. For this reason, when talking about stability, the exponent $p = 1 + \frac{4}{3}$ is called the *critical exponent*. Accordingly, we shall say that we are in the *subcritical, critical or supercritical case* if, respectively, $p < 1 + \frac{4}{3}$, $p = 1 + \frac{4}{3}$ or $p > 1 + \frac{4}{3}$.

Very few works are concerned with the stability of standing waves at the semiclassical limit. When $K \equiv 0$ and W is nontrivial, stability of spikes was studied in [22, 36, 37]. As in the case $K \equiv W \equiv 0$, the single-spike standing waves concentrating at a local non-degenerate minimum of the potential W are stable if $p < 1 + \frac{4}{3}$ and unstable if $p > 1 + \frac{4}{3}$ (see [22, 37]). Moreover, in dimension 1 and for $p < 5$, it was proved in [37] that standing waves concentrating at a local non-degenerate maximum of the potential W are unstable. The critical case $p = 1 + \frac{4}{3}$ has been treated by Lin and Wei [36]. In this case, conversely to what happens for $K \equiv W \equiv 0$, the single-spike standing waves concentrating at a local non-degenerate minimum of W are stable. On the other hand, the single-spike standing waves concentrating at more general non-degenerate critical points of W (for example local non-degenerate maxima) are unstable under some extra assumptions.

Our goal in this paper is to investigate further the stability of semiclassical standing waves for (1.1), when not only W, but also K, is nontrivial, treating at the same time the nonhomogeneity in space generated by the potentials K and W and the presence of a nonlocal term.

Here, as in the rest of the paper, the potentials K and W satisfy the assumptions (K1)-(K2), (V1)-(V3) of [26] (see Proposition 2.1). We denote by v_ϵ the single-spike solutions for (1.2) at a non-degenerate critical point of

W found in [26] and by $\Psi_\epsilon(x,t) := \exp\left(\frac{i\omega}{\epsilon}t\right)v_\epsilon(x)$ the corresponding standing waves. We assume that the family v_ϵ is C^1 in ω uniformly in ϵ with value in $H^1(\mathbb{R}^3)$.

Our main results are the following.

Theorem 1. *Let $p < 1 + \frac{4}{3}$. Let x_0 be a non-degenerate critical point for the potential W and let m denote the number of negative eigenvalues of the matrix* $\mathrm{Hess}W(x_0)$. *If the parameter ϵ is small enough, then Ψ_ϵ is orbitally stable if x_0 is a local minimum and orbitally unstable if m is odd.*

Theorem 2. *Let $p > 1 + \frac{4}{3}$. Let x_0 be a non-degenerate critical point for the potential W and let m denote the number of negative eigenvalues of the matrix* $\mathrm{Hess}W(x_0)$. *If the parameter ϵ is small enough, then Ψ_ϵ is orbitally unstable if x_0 is a local minimum or if m is even.*

Theorem 3. *Let $p = 1 + \frac{4}{3}$. Let x_0 be a non-degenerate critical point for the potential W such that $\Delta W(x_0) - K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}} C \neq 0$, where the constant C is explicitly known and positive. Let m denote the number of negative eigenvalues of the matrix* $\mathrm{Hess}W(x_0)$. *If the parameter ϵ is small enough, then Ψ_ϵ is orbitally stable if x_0 is a local minimum and*

$$\Delta W(x_0) > K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}} C,$$

while it is orbitally unstable if x_0 is a local minimum and

$$\Delta W(x_0) < K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}} C,$$

or if the quantity

$$m - \frac{1}{2}\Big(1 + \frac{\Delta W(x_0) - K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}} C}{|\Delta W(x_0) - K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}} C|}\Big)$$

is even.

Remark 1.2. When p is subcritical or supercritical (i.e., $p \neq 1 + \frac{4}{3}$), the stability results given in Theorem 1 and Theorem 2 are independent of the value of K and of its derivatives in the concentration point x_0. In particular the results are identical to those obtained for the nonlinear Schrödinger equation without the non-local term $K(x)\left(|x|^{-1} * K(x)\Psi^2\right)\Psi$ (see [22, 37]). Conversely, when p is critical (i.e., $p = 1 + \frac{4}{3}$), the potential K has an influence on stability through its value at x_0: For example, if x_0 is a local minimum of W, then there is stability when $K(x_0)^2$ is small and instability when $K(x_0)^2$ is large.

If $K(x_0) = 0$, in the critical case, we get the same stability result obtained in the case $K \equiv 0$ by Lin and Wei [36]: Ψ_ϵ is orbitally stable if x_0 is a minimum for W, unstable if $m - \frac{1}{2}\big(1 + \frac{\Delta W(x_0)}{|\Delta W(x_0)|}\big)$ is even.

To prove Theorem 1, Theorem 2 and Theorem 3 we work within the framework introduced by Grillakis, Shatah and Strauss [22, 23] to study orbital stability for a large class of Hamiltonian systems. In our case, the results of [22, 23] allow us to determine whether there is stability or instability provided two pieces of information are available:

(i) The *spectral information*: the number of eigenvalues of L_ϵ, the linearized operator corresponding to (1.2) (see (2.11) for a precise definition).

(ii) The *slope information*: the sign of $D(\omega) := \frac{\partial}{\partial \omega}\|u_\epsilon\|_{L^2(\mathbb{R}^3)}$ (where u_ϵ is a re-scaled version of v_ϵ, see Section 2 for details).

We denote by $n(L_\epsilon)$ the number of negative eigenvalues of L_ϵ and set $p(D(\omega)) = 0$ if $D(\omega) < 0$, $p(D(\omega)) = 1$ if $D(\omega) > 0$. Then, according to the theory developed in [22, 23], the standing wave Ψ_ϵ is orbitally stable if $n(L_\epsilon) = p(D(\omega))$ and orbitally unstable if $n(L_\epsilon) - p(D(\omega))$ is odd.

To obtain the spectral information, our approach is the following (see [34, 36] for related arguments). We analyze the spectrum of the linearized operator L_ϵ by a perturbation method. When $\epsilon \to 0$, L_ϵ converges, at least formally, toward an operator L_0 whose spectrum is well known. Thanks to the perturbation theory for linear operators, we show that the spectrum of L_ϵ is close to the one of L_0 when ϵ is small. Then we study the splitting of the 0 eigenvalue of L_0 into negative or positive eigenvalues for L_ϵ. For this purpose, we perform an ϵ-expansion of the eigenvalues close to 0 of L_ϵ and find that their signs are related to the eigenvalues of the matrix $\mathrm{Hess}W(x_0)$.

To deal with the slope information, we use an asymptotic expansion of v_ϵ (see Proposition 2.5) in the subcritical and supercritical case. The critical case is more difficult to handle, since when $\epsilon = 0$ the function $D(\omega)$ has some degeneracy, in the sense that $D(\omega) = 0$, and we need to develop a method inspired from the one introduced by Lin and Wei [36]. It relies on the analysis of a function R^ϵ_ω satisfying $L_\epsilon R^\epsilon_\omega = -u_\epsilon$. The main point of the analysis is to decompose R^ϵ_ω in terms of the eigenfunctions in the kernel of L_ϵ, a limit function R_0, and some small remainder. This decomposition, along with some remarkable identities, allows us to perform an ϵ-expansion for $D(\omega)$ and to find its sign for ϵ small.

The paper is organized as follows: in Section 2, after collecting some notation and useful definitions, we recall the existence result proved in [26] for bound states v_ϵ of (1.2) concentrating at a non-degenerate critical point of the potential W and infer some useful properties of these solutions. Next, in Section 3, we study the spectrum of the linearized operator L_ϵ as ϵ goes to zero while in Section 4 we determine the sign of $D(\omega)$. Finally, in Section 5, we conclude the proofs of Theorem 1, Theorem 2 and Theorem 3.

2. Preliminaries

Let us fix some notation. For $f : \mathbb{R}^3 \mapsto \mathbb{R}$ smooth, we denote its partial derivatives by $f_i := \frac{\partial}{\partial x_i} f(x)$, and $f_{ij} := \frac{\partial}{\partial x_i \partial x_j} f(x)$. We indicate the gradient by $\nabla f(x) := (f_i)_{i=1,2,3}$ and the Hessian matrix by $\mathrm{Hess} f(x) := (f_{ij})_{i,j=1,2,3}$. We write δ_{ij} to denote the Kronecker symbol; i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} .$$

The symbol $\perp_{L^2}$ means the orthogonality relation in the Hilbert space $L^2(\mathbb{R}^3)$. For x_0 given, we use the notation $x_\epsilon := \epsilon x + x_0$. For any $\lambda > 0$, let U_λ be the unique positive radial solution (see e.g. [2]) of

$$-\Delta u + \lambda^2 u - u^p = 0, \quad x \in \mathbb{R}^3. \tag{2.1}$$

A simple computation gives $U_\lambda(x) = \lambda^{\frac{2}{p-1}} U_1(\lambda x)$. Moreover, it is known that it satisfies the following decay properties: $U_\lambda(s), U_\lambda'(s) \leq Ce^{-\lambda s}$, $|s| > 1$.

We define also

$$L_0 v := -\Delta v + \lambda^2 v - pU_\lambda^{p-1} v,$$

and

$$R_0 := \frac{1}{p-1} U_\lambda + \frac{1}{2} x \cdot \nabla U_\lambda. \tag{2.2}$$

It is easy to see that

$$L_0 (U_\lambda)_{jh} = p(p-1) U_\lambda^{p-2} (U_\lambda)_j (U_\lambda)_h, \tag{2.3}$$

$$L_0 R_0 = -\lambda^2 U_\lambda. \tag{2.4}$$

We shall also need to consider the translated function $U_{\lambda,\epsilon} := U_\lambda(\cdot - \xi_\epsilon)$, where $\xi_\epsilon \in \mathbb{R}^3$ is given by Proposition 2.1 below. Obviously $U_{\lambda,\epsilon}$ satisfies also (2.1) and, setting $R_{0,\epsilon} := R_0(\cdot - \xi_\epsilon)$ and $L_{0,\epsilon} v := -\Delta v + \lambda^2 v - pU_{\lambda,\epsilon}^{p-1} v$, we have identities analogous to (2.3) and (2.4).

We now recall the existence result for positive bound states of (1.2) proved in [26].

Proposition 2.1. *Let $p \in (1,5)$ and make the following assumptions on W and K:*

(V1) *$W \in C^\infty(\mathbb{R}^3)$, W and its derivatives are uniformly bounded.*
(V2) $\inf_{\mathbb{R}^3} \{W + \omega\} > 0$.
(V3) *There exists $x_0 \in \mathbb{R}^3$ such that $\nabla W(x_0) = 0$.*
(K1) *$K \in C^\infty(\mathbb{R}^3)$, K and its derivatives are uniformly bounded.*
(K1) $K \geq 0$.

Let x_0 be a non-degenerate critical point for W. Then, for ϵ small enough, there exists $v_\epsilon \in H^1(\mathbb{R}^3), v_\epsilon > 0$, such that v_ϵ is a solution of (1.2) and

$$\Big\| v_\epsilon - U_\lambda\Big(\frac{\cdot - x_0}{\epsilon}\Big)\Big\|_{H^1(\mathbb{R}^3)} \to 0 \quad as \quad \epsilon \to 0, \tag{2.5}$$

where $\lambda^2 = W(x_0) + \omega$. Moreover, there exists $\xi_\epsilon \in \mathbb{R}^3$, $w_\epsilon \in H^1(\mathbb{R}^3)$, such that

$$v_\epsilon = U_\lambda\Big(\frac{\cdot - x_0}{\epsilon} - \xi_\epsilon\Big) + w_\epsilon\Big(\frac{\cdot - x_0}{\epsilon}\Big), \tag{2.6}$$

$$\xi_\epsilon \to 0 \quad in \quad \mathbb{R}^3, \tag{2.7}$$

$$\|w_\epsilon\|_{H^1(\mathbb{R}^3)} \leq C\epsilon^2.$$

From now on, it is assumed that $\lambda^2 := W(x_0) + \omega$. For the proof of Theorem 1, Theorem 2 and Theorem 3, it is convenient to rescale the time and space variables by $t = \epsilon s$ and $x = \epsilon y + x_0 = y_\epsilon$. Setting $\Phi(y, s) := \Psi(y_\epsilon, \epsilon s)$, we get the rescaled equation

$$-i\Phi_s - \Delta_y \Phi + W(y_\epsilon)\Phi + \epsilon^2 K(y_\epsilon)\left(|y|^{-1} * K(y_\epsilon)|\Phi|^2\right)\Phi - |\Phi|^{p-1}\Phi = 0. \tag{2.8}$$

A standing wave $\Psi_\epsilon(x,t) = \exp(\frac{i\omega}{\epsilon}t)v_\epsilon(x)$ for (1.1) becomes, in the new time and space variables, the following standing wave for (2.8): $\Phi_\epsilon(y, s) = \exp(i\omega s)u_\epsilon(y)$, where $u_\epsilon(y) := v_\epsilon(y_\epsilon)$ is a solution of

$$-\Delta u + [W(y_\epsilon) + \omega]\, u + \epsilon^2 K(y_\epsilon)\left(|y|^{-1} * K(y_\epsilon)u^2\right) u - |u|^{p-1}u = 0. \tag{2.9}$$

It is clear that Ψ_ϵ is stable/unstable if and only if Φ_ϵ is stable/unstable.

We point out that, in terms of the rescaled function $u_\epsilon(x) := v_\epsilon(x_\epsilon)$, from Proposition 2.1 it follows that, for ϵ sufficiently small, u_ϵ is a positive solution of equation (2.9), and that $\|u_\epsilon - U_\lambda\|_{H^1(\mathbb{R}^3)} \to 0$ as $\epsilon \to 0$. Moreover, (2.6) becomes

$$u_\epsilon = U_\lambda(\cdot - \xi_\epsilon) + w_\epsilon. \tag{2.10}$$

We consider the linearized operator of (2.9) in u_ϵ

$$\begin{aligned} L_\epsilon v := & -\Delta v + [W(x_\epsilon) + \omega]\, v - p u_\epsilon^{p-1} v + \epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right) v \\ & + 2\epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon v\right) u_\epsilon \end{aligned} \tag{2.11}$$

and the function

$$D(\omega) := \frac{\partial}{\partial\omega}\|u_\epsilon\|^2_{L^2(\mathbb{R}^3)}.$$

As announced in the Introduction, the number of eigenvalues of the operator L_ϵ and the sign of the function $D(\omega)$ allow us to determine whether there is stability or instability for the standing wave Ψ_ϵ. Hence, we need to study the spectral properties of L_ϵ and to determine the sign of $D(\omega)$. In order to do that we derive asymptotic expansion formulas for the operator L_ϵ and the function $D(\omega)$ as the parameter ϵ goes to zero. This is obtained, in both cases, starting from an expansion in ϵ of the solution u_ϵ (see Proposition 2.5).

Before doing the asymptotic expansion for u_ϵ, we derive some useful properties of the solution u_ϵ such as regularity and exponential decay.

Lemma 2.2. *One has $u_\epsilon \in H^1(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$. In particular, it follows that $u_\epsilon \in L^\infty(\mathbb{R}^3)$ and $\lim_{|x|\to+\infty} u_\epsilon(x) \to 0$.*

Proof. The function u_ϵ satisfies (2.9); namely

$$-\Delta u_\epsilon + \omega u_\epsilon = f_\epsilon,$$

where

$$f_\epsilon := -W(x_\epsilon)u_\epsilon + u_\epsilon^p - \epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right)u_\epsilon.$$

It is easy to see, using Sobolev embeddings, that $f_\epsilon \in L^m_{loc}(\mathbb{R}^3)$, where $m := \min\{3, \frac{6}{p}\}$. The result follows by a classical bootstrap argument and we omit the details. ∎

Lemma 2.3. *There exist $\delta > 0$ and $C_1, C_2 > 0$ independent of ϵ such that*

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}^3)} \le C_1, \tag{2.12}$$

$$|u_\epsilon(x)| \le C_2 e^{-\delta|x|} \quad \textit{for all } x \in \mathbb{R}^3. \tag{2.13}$$

Proof. First we prove (2.12). Let ζ_ϵ be the maximum point of u_ϵ (it exists because $u_\epsilon \in C^0(\mathbb{R}^3)$ and $\lim_{|x|\to\infty} u_\epsilon = 0$). We define the auxiliary function

$$\tilde{u}_\epsilon := u_\epsilon(\cdot + \zeta_\epsilon).$$

By definition, $\tilde{u}_\epsilon(0) = u_\epsilon(\zeta_\epsilon) = \|u_\epsilon\|_{L^\infty(\mathbb{R}^3)}$, $\|\tilde{u}_\epsilon\|_{L^\infty(\mathbb{R}^3)} = \|u_\epsilon\|_{L^\infty(\mathbb{R}^3)}$, and $\tilde{u}_\epsilon$ satisfies

$$-\Delta\tilde{u}_\epsilon + \omega\tilde{u}_\epsilon = g_\epsilon \quad \text{in} \quad \mathbb{R}^3, \tag{2.14}$$

where

$$g_\epsilon := -W(x_\epsilon + \epsilon\zeta_\epsilon)\tilde{u}_\epsilon + \tilde{u}_\epsilon^p - \epsilon^2 K(x_\epsilon + \epsilon\zeta_\epsilon)\left(|x|^{-1} * K(x_\epsilon + \epsilon\zeta_\epsilon)\tilde{u}_\epsilon^2\right)\tilde{u}_\epsilon.$$

Let $R>0$; then $\tilde{u}_\epsilon$ satisfies (2.14) in B_R. It is easy to see that $g_\epsilon \in L^m(B_R)$, where $m := \min\{\frac{6}{p}, 3\}$, and that, moreover, there exists $C>0$, independent of ϵ, such that $\|g_\epsilon\|_{L^m(B_R)} \leq C$. Thus, by a bootstrap argument, we have $\|\tilde{u}_\epsilon\|_{L^\infty(B_R)} \leq C$, independently of ϵ. The conclusion follows observing that, by definition,

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}^3)} = \|\tilde{u}_\epsilon\|_{L^\infty(\mathbb{R}^3)} = \|\tilde{u}_\epsilon\|_{L^\infty(B_R)}.$$

We turn now to the proof of (2.13). We define

$$H(x) := [W(x_\epsilon)+\omega] + \epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right) - u_\epsilon^{p-1}.$$

Then u_ϵ satisfies

$$-\Delta u_\epsilon + H(x)u_\epsilon = 0.$$

We claim that $H \in L^\infty(\mathbb{R}^3)$. Indeed $W \in L^\infty(\mathbb{R}^3)$, $K \in L^\infty(\mathbb{R}^3)$, $u_\epsilon^{p-1} \in L^\infty(\mathbb{R}^3)$ and $\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right) \in L^\infty(\mathbb{R}^3)$ because it is in $C^0(\mathbb{R}^3)$ ($u_\epsilon, K \in C^0(\mathbb{R}^3)$) and in $L^6(\mathbb{R}^3)$. Moreover, since $u_\epsilon(x) \to 0$ as $|x| \to \infty$, we have $l := \lim_{R\to\infty} \operatorname{ess\,inf}_{|x|\geq R} H(x) \geq \inf_{\mathbb{R}^3}\{\omega + W\} > 0$. Hence, 0 is below the essential spectrum of the Schrödinger operator $-\Delta + H(x)$. As a consequence it follows (see e.g. [41, page 281]) that the eigenfunction u_ϵ of $-\Delta + H(x)$ decays exponentially. Precisely, there exist $\delta > 0$ and $C > 0$ (independent of ϵ) such that

$$|u_\epsilon(x)| \leq C\|u_\epsilon\|_{L^\infty(\mathbb{R}^3)} e^{-\delta|x|}.$$

The conclusion follows from (2.12). □

Lemma 2.4. *We have $u_\epsilon \longrightarrow U_\lambda$ in $L^\infty(\mathbb{R}^3)$ as $\epsilon \to 0$.*

Proof. Let $\delta > 0$. Since u_ϵ and U_λ decay exponentially independently of ϵ, there exists R such that

$$\|u_\epsilon - U_\lambda\|_{L^\infty(\mathbb{R}^3/B_R)} \leq \|u_\epsilon\|_{L^\infty(\mathbb{R}^3/B_R)} + \|U_\lambda\|_{L^\infty(\mathbb{R}^3/B_R)} \leq \frac{\delta}{2}.$$

Moreover, $u_\epsilon \to U_\lambda$ in $H^1(B_R)$ as $\epsilon \to 0$ and $u_\epsilon, U_\lambda \in C^0(\overline{B_R})$ hence $u_\epsilon(x) \to U_\lambda(x)$ for all $x \in \overline{B_R}$ and so for ϵ small we also have

$$\|u_\epsilon - U_\lambda\|_{L^\infty(B_R)} \leq \frac{\delta}{2}.$$

Combining this with the previous inequality and letting δ go to zero we get the conclusion. □

We are now in position to perform the asymptotic expansion of u_ϵ. Recall that $\xi_\epsilon \to 0$ as $\epsilon \to 0$ and that $U_{\lambda,\epsilon}$ is defined by $U_{\lambda,\epsilon} := U_\lambda(\cdot - \xi_\epsilon)$.

Proposition 2.5. *There exists* $w_0 \in H^1(\mathbb{R}^3)$ *such that*

$$u_\epsilon = U_{\lambda,\epsilon} + \epsilon^2 w_0 + o(\epsilon^2)$$

(*with* $o(\epsilon^2) \in H^1(\mathbb{R}^3)$) *and*

$$L_0 w_0 = -K(x_0)^2 \left(|x|^{-1} * U_\lambda^2\right) U_\lambda - \tfrac{1}{2} < \mathrm{Hess} W(x_0)x, x > U_\lambda.$$

Proof. By (2.10) we have $u_\epsilon = U_{\lambda,\epsilon} + w_\epsilon$ and $\|w_\epsilon\|_{H^1(\mathbb{R}^3)} \leq C\epsilon^2$. Substituting into (2.9), and dividing by ϵ^2, we get

$$\sum_{k=1}^{8} A_k = 0,$$

where

$$\begin{aligned}
&A_1 := \epsilon^{-2}\big[-\Delta U_{\lambda,\epsilon} + \lambda^2 U_{\lambda,\epsilon} - U_{\lambda,\epsilon}^p\big], \quad A_2 := -\Delta \tilde{w}_\epsilon + \lambda^2 \tilde{w}_\epsilon - pU_\lambda^{p-1}\tilde{w}_\epsilon,\\
&A_3 := \epsilon^{-2}\left[W(x_\epsilon) - W(x_0)\right] U_{\lambda,\epsilon}, \quad A_4 := \left[W(x_\epsilon) - W(x_0)\right] \tilde{w}_\epsilon,\\
&A_5 := \epsilon^{-2}\big[U_{\lambda,\epsilon}^p - (U_{\lambda,\epsilon} + w_\epsilon)^p + pU_\lambda^{p-1} w_\epsilon\big],\\
&A_6 := K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon) U_{\lambda,\epsilon}^2\right) U_{\lambda,\epsilon},\\
&A_7 := K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)\left(2U_{\lambda,\epsilon} w_\epsilon + w_\epsilon^2\right)\right)\left(U_{\lambda,\epsilon} + w_\epsilon\right),\\
&A_8 := K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon) U_{\lambda,\epsilon}^2\right) w_\epsilon,
\end{aligned}$$

and where we have defined $\tilde{w}_\epsilon := \frac{w_\epsilon}{\epsilon^2}$. Obviously, $A_1 = 0$. Moreover, $A_2 \to L_0 w_0$ in $H^{-1}(\mathbb{R}^3)$ as $\epsilon \to 0$. In fact $\|\tilde{w}_\epsilon\|_{H^1(\mathbb{R}^3)} \leq C$, therefore there exists $w_0 \in H^1(\mathbb{R}^3)$ such that $\tilde{w}_\epsilon \to w_0$ weakly in $H^1(\mathbb{R}^3)$.

In addition $A_3 \to \frac{1}{2} < \mathrm{Hess} W(x_0)x, x > U_\lambda(x)$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0$. In fact, since x_0 is a non-degenerate critical point for W (and we also assumed that the derivatives of W are bounded), we have

$$W(x_\epsilon) = W(x_0) + \frac{\epsilon^2}{2} < \mathrm{Hess} W(x_0)x, x > +O(\epsilon^3)|x|^3,$$

thus in $H^1(\mathbb{R}^3)$

$$\begin{aligned}
A_3 &= \tfrac{1}{2} < \mathrm{Hess} W(x_0)x, x > U_{\lambda,\epsilon} + O(\epsilon)|x|^3 U_{\lambda,\epsilon}\\
&\to \tfrac{1}{2} < \mathrm{Hess} W(x_0)x, x > U_\lambda(x).
\end{aligned}$$

We show that $A_4 \to 0$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0$. Observe that $w_\epsilon = u_\epsilon - U_{\lambda,\epsilon}$, and so, from (2.13), it follows that w_ϵ is exponentially decaying (independently of

ϵ). Let $\delta > 0$ and let R be large enough to have $\|O(\epsilon)|x|^3 w_\epsilon\|_{H^1(\mathbb{R}^3/B_R)} < \frac{\delta}{2}$ and $\|\frac{1}{2} < \mathrm{Hess}W(x_0)x, x > w_\epsilon\|_{H^1(\mathbb{R}^3/B_R)} < \frac{\delta}{2}$. As before

$$[W(x_\epsilon) - W(x_0)]\, \tilde{w}_\epsilon = \tfrac{1}{2} < \mathrm{Hess}W(x_0)x, x > w_\epsilon + O(\epsilon)|x|^3 w_\epsilon.$$

Therefore, the conclusion follows observing that, for ϵ small enough, we have

$$\left\|\tfrac{1}{2} < \mathrm{Hess}W(x_0)x, x > w_\epsilon\right\|_{H^1(B_R)} \leq \frac{\delta}{2},$$

and also

$$\|O(\epsilon|x|^3)\|_{H^1(B_R)} \leq \frac{\delta}{2}.$$

We show that $A_5 \to 0$ in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$. Define

$$N(w_\epsilon) := \left[U^p_{\lambda,\epsilon} - (U_{\lambda,\epsilon} + w_\epsilon)^p + pU_\lambda^{p-1} w_\epsilon\right],$$

so we have to show that $\epsilon^{-2} N(w_\epsilon) \to 0$ in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$. Observe that

$$\|N(w_\epsilon)\|^2_{L^2(\mathbb{R}^3)} \leq \|N(w_\epsilon)\|_{L^\infty(\mathbb{R}^3)} \|N(w_\epsilon)\|_{L^1(\mathbb{R}^3)},$$

and that (see [2, page 132]) also

$$\|N(w_\epsilon)\|_{L^1(\mathbb{R}^3)} \leq C\left(\|w_\epsilon\|^2_{H^1(\mathbb{R}^3)} + \|w_\epsilon\|^{p+1}_{H^1(\mathbb{R}^3)}\right).$$

Therefore, since $\|w_\epsilon\|_{H^1(\mathbb{R}^3)} = O(\epsilon^2)$,

$$\|N(w_\epsilon)\|_{L^1(\mathbb{R}^3)} = O(\epsilon^4).$$

On the other hand, by Lemma 2.4, we have

$$\|w_\epsilon\|_{L^\infty(\mathbb{R}^3)} = \|u_\epsilon - U_\lambda\|_{L^\infty(\mathbb{R}^3)} + \|U_\lambda - U_{\lambda,\epsilon}\|_{L^\infty(\mathbb{R}^3)} = o(1),$$

therefore, $\|N(w_\epsilon)\|_{L^\infty(\mathbb{R}^3)} = o(1)$, indeed,

$$\|p|U_\lambda|^{p-1} w_\epsilon\|_{L^\infty(\mathbb{R}^3)} \leq C\|w_\epsilon\|_{L^\infty(\mathbb{R}^3)} = o(1)$$

and

$$\|U^p_{\lambda,\epsilon} - (U_{\lambda,\epsilon} + w_\epsilon)^p\|_{L^\infty(\mathbb{R}^3)} \leq 2^{p-1} \|w_\epsilon\|^p_{L^\infty(\mathbb{R}^3)} = o(1).$$

We now prove that $A_6 \to K(x_0)^2 \left(|x|^{-1} * U_\lambda^2\right) U_\lambda$ in $L^2(\mathbb{R}^3)$, as $\epsilon \to 0$.

$$\begin{aligned}
&\left\|K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)U^2_{\lambda,\epsilon}\right) U_{\lambda,\epsilon} - K(x_0)^2\left(|x|^{-1} * U^2_\lambda\right) U_\lambda\right\|_{L^2(\mathbb{R}^3)} \qquad (2.15)\\
&\leq \left\|\left(K(x_\epsilon) - K(x_0)\right)\left(|x|^{-1} * K(x_\epsilon)U^2_{\lambda,\epsilon}\right) U_{\lambda,\epsilon}\right\|_{L^2(\mathbb{R}^3)}\\
&\quad + \left\|K(x_0)\left(|x|^{-1} * K(x_\epsilon)U^2_{\lambda,\epsilon}\right)\left(U_{\lambda,\epsilon} - U_\lambda\right)\right\|_{L^2(\mathbb{R}^3)}\\
&\quad + \left\|K(x_0)\left(|x|^{-1} * \left(K(x_\epsilon) - K(x_0)\right) U^2_{\lambda,\epsilon}\right) U_\lambda\right\|_{L^2(\mathbb{R}^3)}\\
&\quad + \left\|K(x_0)^2\left(|x|^{-1} * \left(U^2_{\lambda,\epsilon} - U^2_\lambda\right)\right) U_\lambda\right\|_{L^2(\mathbb{R}^3)}
\end{aligned}$$

$$=: I + II + III + IV.$$

Observe that

$$\begin{aligned}
I &\le C\epsilon \left\| \left(|x|^{-1} * K(x_\epsilon) U_{\lambda,\epsilon}^2\right) U_{\lambda,\epsilon} |x| \right\|_{L^2(\mathbb{R}^3)} \\
&\le C\epsilon \left\| |x|^{-1} * K(x_\epsilon) U_{\lambda,\epsilon}^2 \right\|_{L^6(\mathbb{R}^3)} \| U_{\lambda,\epsilon} |x| \|_{L^3(\mathbb{R}^3)} \\
&\le C\epsilon \| U_{\lambda,\epsilon} \|_{H^1(\mathbb{R}^3)}^2 \| |x| U_{\lambda,\epsilon} \|_{L^3(\mathbb{R}^3)} \\
&= C\epsilon \| U_\lambda \|_{H^1(\mathbb{R}^3)}^2 \| |x + |\xi_\epsilon| U_\lambda \|_{L^3(\mathbb{R}^3)} \le C\epsilon,
\end{aligned}$$

where we used the fact that $\xi_\epsilon \to 0$ as $\epsilon \to 0$. Moreover,

$$\begin{aligned}
II &\le C \| U_{\lambda,\epsilon} \|_{H^1(\mathbb{R}^3)}^2 \| U_{\lambda,\epsilon} - U_\lambda \|_{L^3(\mathbb{R}^3)} = o(1), \\
III &\le C\epsilon \left\| \left(|x|^{-1} * |x| U_{\lambda,\epsilon}^2\right) U_\lambda \right\|_{L^2(\mathbb{R}^3)} \le C\epsilon, \\
IV &\le C \| U_{\lambda,\epsilon}^2 - U_\lambda^2 \|_{H^1(\mathbb{R}^3)} \| U_\lambda \|_{L^3(\mathbb{R}^3)} = o(1).
\end{aligned}$$

Finally, putting together the four estimates, we obtain the conclusion.

We prove that $A_7 \to 0$ in $H^{-1}(\mathbb{R}^3)$ as $\epsilon \to 0$. Take $\phi \in H^1(\mathbb{R}^3)$, then

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) \left(2U_{\lambda,\epsilon} w_\epsilon + w_\epsilon^2\right)\right) \left(U_{\lambda,\epsilon} + w_\epsilon\right) \phi dx \\
&\quad \le \left\| K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) \left(2U_{\lambda,\epsilon} w_\epsilon + w_\epsilon^2\right)\right) \left(U_{\lambda,\epsilon} + w_\epsilon\right) \right\|_{L^2(\mathbb{R}^3)} \|\phi\|_{L^2(\mathbb{R}^3)} \\
&\quad \le C \left\| \left(|x|^{-1} * \left(2U_{\lambda,\epsilon} w_\epsilon + w_\epsilon^2\right)\right) \right\|_{L^6(\mathbb{R}^3)} \| (U_{\lambda,\epsilon} + w_\epsilon) \|_{L^6(\mathbb{R}^3)} \|\phi\|_{L^2(\mathbb{R}^3)} \\
&\quad \le C \| w_\epsilon \|_{H^1(\mathbb{R}^3)} \| 2U_{\lambda,\epsilon} + w_\epsilon \|_{H^1(\mathbb{R}^3)} \| (U_{\lambda,\epsilon} + w_\epsilon) \|_{L^6(\mathbb{R}^3)} \|\phi\|_{L^2(\mathbb{R}^3)} \\
&\quad \le C \| w_\epsilon \|_{H^1(\mathbb{R}^3)} = O(\epsilon^2).
\end{aligned}$$

Last we prove that $A_8 \to 0$ in $H^{-1}(\mathbb{R}^3)$ as $\epsilon \to 0$. Take $\phi \in H^1(\mathbb{R}^3)$, then

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) U_{\lambda,\epsilon}^2\right) w_\epsilon \phi dx \\
&\quad \le C \left\| \left(|x|^{-1} * U_{\lambda,\epsilon}^2\right) w_\epsilon \right\|_{L^2(\mathbb{R}^3)} \|\phi\|_{L^2(\mathbb{R}^3)} \\
&\quad \le C \left\| |x|^{-1} * U_{\lambda,\epsilon}^2 \right\|_{L^6(\mathbb{R}^3)} \| w_\epsilon \|_{L^3(\mathbb{R}^3)} \|\phi\|_{L^2(\mathbb{R}^3)} \le C \| w_\epsilon \|_{H^1(\mathbb{R}^3)} = O(\epsilon^2).
\end{aligned}$$

This concludes the proof. □

3. The spectral information

In this section, we study the spectral properties of the operator L_ϵ, as ϵ goes to zero. In doing so, the well-known properties of the spectrum of the operator L_0 (Lemma 3.1 below) will be useful (for the proof see e.g [2]).

Lemma 3.1. *The spectrum of* $L_0 v = -\Delta v + \lambda^2 v - pU_\lambda^{p-1} v$ *consists of essential spectrum in* $[\lambda^2, +\infty)$ *and of a finite number of eigenvalues in* $(-\infty, \frac{\lambda^2}{2})$. *The first eigenvalue* μ_1 *of* L_0 *is negative and simple. The second eigenvalue is* 0 *and is of multiplicity* 3. *The kernel of* L_0 *is spanned by* $(U_\lambda)_j$, $j = 1, 2, 3$, *where* $(U_\lambda)_j = \frac{\partial U_\lambda}{\partial x_j}$.

The general perturbation result is the following.

Proposition 3.2. *The spectrum of* L_ϵ *consists of essential spectrum in* $[C, +\infty)$, *for a certain* $C > 0$ *and a finite number of eigenvalues in* $(-\infty, C')$ *for any* $C' < C$. *In particular, there exists a set of simple eigenvalues* $\{\mu_{\epsilon,1}, \mu_{\epsilon,2}, \mu_{\epsilon,3}, \mu_{\epsilon,4}\}$ *such that* $\mu_{\epsilon,1} < \mu_{\epsilon,2} \leq \mu_{\epsilon,3} \leq \mu_{\epsilon,4}$ *and satisfying as* $\epsilon \to 0$, $\mu_{\epsilon,1} \to \mu_1 < 0$, $\mu_{\epsilon,h} \to 0$, $h = 2, 3, 4$. *Moreover, letting* $\psi_{\epsilon,h}$ *be such that* $L_\epsilon \psi_{\epsilon,h} = \mu_{\epsilon,h} \psi_{\epsilon,h}$, *for* $h = 2, 3, 4$, *one has*

$$\psi_{\epsilon,h} \longrightarrow \sum_{j=1}^{3} \alpha_j^h (U_\lambda)_j \quad \text{as } \epsilon \to 0 \text{ in } L^2(\mathbb{R}^3),\ \alpha_j^h \in \mathbb{R}.$$

Proof. Since L_ϵ is a self-adjoint operator, its spectrum lies on the real line. From (V1)-(V3), (K1)-(K2) and (2.5), we infer that the operator L_ϵ is a compact perturbation of $-\Delta + C$ for some $C > 0$. Hence, by Weyl's theorem, the essential spectrum of L_ϵ lies in $[C, +\infty)$. Since L_ϵ is bounded from below, for any $C' < C$ there exists only a finite number of eigenvalues of L_ϵ in $(-\infty, C']$. The existence and properties of $\{\mu_{\epsilon,h}\}$ and $\{\psi_{\epsilon,h}\}$ follow from the classical perturbation theory for linear operators (see e.g. [30, page 213]). $\square$

Proposition 3.2 is not sufficient to count the number of negative eigenvalues of L_ϵ. Indeed, when $h = 2, 3, 4$, we only know that the eigenvalues $\{\mu_{\epsilon,h}\}$ are close to 0 without having information on their sign. Hence, in the following proposition, we derive an asymptotic expansion formula for the eigenvalues of L_ϵ. Note that the eigenvalues of L_ϵ close to 0 are intimately related with the eigenvalues of the Hessian matrix $\mathrm{Hess}W(x_0)$.

Proposition 3.3. *The eigenvalues* $(\mu_{\epsilon,h})$ *of* L_ϵ *can be expanded in the following way:*

$$\mu_{\epsilon,h} = c_h \epsilon^2 + o(\epsilon^2), \quad h = 2, 3, 4,$$

where $c_h := \frac{1}{2} \frac{\|U_\lambda\|_{L^2}^2}{\|(U_\lambda)_h\|_{L^2}^2} a_h$ *and* $\{a_i\}_{i=1,2,3}$ *are the eigenvalues of the matrix* $\mathrm{Hess}W(x_0)$.

Before proving Proposition 3.3, we need some preparation. We first observe that, since $\mathrm{Hess}W(x_0)$ is a symmetric real matrix, it can be diagonalized through an orthogonal matrix. Hence, without loss of generality, we assume in the rest of the paper that $\mathrm{Hess}W(x_0) = \mathrm{diag}\{a_1, a_2, a_3\}$.

Lemma 3.4. *For ϵ close to 0, we have*

$$\begin{aligned} L_\epsilon \left(U_{\lambda,\epsilon}\right)_j &= \epsilon^2 \left[\tfrac{1}{2} < \mathrm{Hess}W(x_0)x, x > -p(p-1)U_{\lambda,\epsilon}^{p-2} w_0\right] \left(U_{\lambda,\epsilon}\right)_j \\ &+ 2\epsilon^2 K(x_0)^2 \left(|x|^{-1} * U_\lambda (U_\lambda)_j\right) U_\lambda \\ &+ \epsilon^2 K(x_0)^2 \left(|x|^{-1} * U_\lambda^2\right) (U_\lambda)_j + o(\epsilon^2) \quad \text{in } L^2(\mathbb{R}^3). \end{aligned}$$

Proof. By definition of L_ϵ (see (2.11)), we have

$$\begin{aligned} L_\epsilon \left(U_{\lambda,\epsilon}\right)_j &= -\Delta \left(U_{\lambda,\epsilon}\right)_j + \left[W(x_\epsilon) + \omega\right] \left(U_{\lambda,\epsilon}\right)_j - p u_\epsilon^{p-1} \left(U_{\lambda,\epsilon}\right)_j \\ &+ \epsilon^2 K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon^2\right) \left(U_{\lambda,\epsilon}\right)_j \\ &+ 2\epsilon^2 K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon \left(U_{\lambda,\epsilon}\right)_j\right) u_\epsilon. \end{aligned}$$

We decompose $L_\epsilon \left(U_{\lambda,\epsilon}\right)_j$ in the following way:

$$L_\epsilon \left(U_{\lambda,\epsilon}\right)_j = \sum_{k=1}^{5} A_k,$$

where

$$\begin{aligned} &A_1 := -\Delta \left(U_{\lambda,\epsilon}\right)_j + \left[W(x_0) + \omega\right] \left(U_{\lambda,\epsilon}\right)_j - p U_{\lambda,\epsilon}^{p-1} \left(U_{\lambda,\epsilon}\right)_j, \\ &A_2 := \left[W(x_\epsilon) - W(x_0)\right] \left(U_{\lambda,\epsilon}\right)_j, \quad A_3 := -p \left[u_\epsilon^{p-1} - U_{\lambda,\epsilon}^{p-1}\right] \left(U_{\lambda,\epsilon}\right)_j, \\ &A_4 := \epsilon^2 K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon^2\right) \left(U_{\lambda,\epsilon}\right)_j, \\ &A_5 := 2\epsilon^2 K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon \left(U_{\lambda,\epsilon}\right)_j\right) u_\epsilon. \end{aligned}$$

Since $U_{\lambda,\epsilon}$ satisfies (2.1), by deriving with respect to x_j we see that $A_1 = 0$. Remembering that x_0 is a critical point of W, a Taylor expansion gives

$$A_2 = \frac{\epsilon^2}{2} < \mathrm{Hess}W(x_0)x, x > \left(U_{\lambda,\epsilon}\right)_j + O(\epsilon^3)|x|^3 \left(U_{\lambda,\epsilon}\right)_j.$$

By Proposition 2.5 we have

$$\begin{aligned} A_3 &= -p \left[\left(U_{\lambda,\epsilon} + \epsilon^2 w_0 + o(\epsilon^2)\right)^{p-1} - U_{\lambda,\epsilon}^{p-1}\right] (U_{\lambda,\epsilon})_j \\ &= -p(p-1) U_{\lambda,\epsilon}^{p-2} w_0 \epsilon^2 \left(U_{\lambda,\epsilon}\right)_j + o(\epsilon^2). \end{aligned}$$

For A_4 and A_5, it is easy to see that we have in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$

$$K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right)(U_{\lambda,\epsilon})_j \longrightarrow K(x_0)^2\left(|x|^{-1} * U_\lambda^2\right)(U_\lambda)_j,$$

$$K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon\,(U_{\lambda,\epsilon})_j\right)u_\epsilon \longrightarrow K(x_0)^2\left(|x|^{-1} * U_\lambda(U_\lambda)_j\right)U_\lambda,$$

which concludes the proof. □

Lemma 3.5. *For ϵ close to 0, we have*

$$\int_{\mathbb{R}^3}\left(L_\epsilon\,(U_{\lambda,\epsilon})_j\right)(U_{\lambda,\epsilon})_k = \frac{\epsilon^2}{2}a_k\|U_\lambda\|^2_{L^2(\mathbb{R}^3)}\delta_{jk} + o(\epsilon^2).$$

Proof. From Lemma 3.4, we get

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(L_\epsilon\,(U_{\lambda,\epsilon})_j\right)(U_{\lambda,\epsilon})_k\\
&= \epsilon^2\int_{\mathbb{R}^3}\left[\frac{1}{2} < \mathrm{Hess}W(x_0)x, x > -p(p-1)U_{\lambda,\epsilon}^{p-2}w_0\right](U_{\lambda,\epsilon})_j\,(U_{\lambda,\epsilon})_k\\
&\quad + 2\epsilon^2 K(x_0)^2\int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda(U_\lambda)_j\right)U_\lambda\,(U_{\lambda,\epsilon})_k\\
&\quad + \epsilon^2 K(x_0)^2\int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda^2\right)(U_\lambda)_j\,(U_{\lambda,\epsilon})_k + o(\epsilon^2).
\end{aligned}$$

We first remark that

$$(U_{\lambda,\epsilon})_j = (U_\lambda)_j\,(\cdot - \xi_\epsilon) = (U_\lambda)_j + O(|\xi_\epsilon|) = (U_\lambda)_j + o(1),$$

where the last equality follows from (2.7). Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(L_\epsilon\,(U_{\lambda,\epsilon})_j\right)(U_{\lambda,\epsilon})_k \qquad (3.1)\\
&= \epsilon^2\int_{\mathbb{R}^3}\left[\frac{1}{2} < \mathrm{Hess}W(x_0)x, x > -p(p-1)U_{\lambda}^{p-2}w_0\right](U_{\lambda})_j\,(U_{\lambda})_k\\
&\quad + 2\epsilon^2 K(x_0)^2\int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda(U_\lambda)_j\right)U_\lambda\,(U_{\lambda})_k\\
&\quad + \epsilon^2 K(x_0)^2\int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda^2\right)(U_\lambda)_j\,(U_{\lambda})_k + o(\epsilon^2).
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
&2\int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda(U_\lambda)_j\right)U_\lambda\,(U_\lambda)_k + \int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda^2\right)(U_\lambda)_j\,(U_\lambda)_k\\
&= -\int_{\mathbb{R}^3}\left(|x|^{-1} * U_\lambda^2\right)U_\lambda\,(U_\lambda)_{jk}\,,
\end{aligned}$$

and substituting into (3.1) we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(L_\epsilon\left(U_{\lambda,\epsilon}\right)_j\right)\left(U_{\lambda,\epsilon}\right)_k\\&=\epsilon^2\int_{\mathbb{R}^3}\left[\frac{1}{2}<\mathrm{Hess}W(x_0)x,x>-p(p-1)U_\lambda^{p-2}w_0\right]\left(U_\lambda\right)_j\left(U_\lambda\right)_k\\&\quad-\epsilon^2K(x_0)^2\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda\left(U_\lambda\right)_{jk}+o(\epsilon^2).\end{aligned}\tag{3.2}$$

From (2.3), we get

$$\begin{aligned}&-\epsilon^2\int_{\mathbb{R}^3}p(p-1)U_\lambda^{p-2}w_0\left(U_\lambda\right)_j\left(U_\lambda\right)_k\\&=-\epsilon^2\int_{\mathbb{R}^3}w_0\left(L_0\left(U_\lambda\right)_{jk}\right)=-\epsilon^2\int_{\mathbb{R}^3}\left(L_0w_0\right)\left(U_\lambda\right)_{jk}.\end{aligned}$$

By Proposition 2.5 this gives

$$\begin{aligned}&-\epsilon^2\int_{\mathbb{R}^3}p(p-1)U_\lambda^{p-2}w_0\left(U_\lambda\right)_j\left(U_\lambda\right)_k\\&=\epsilon^2K(x_0)^2\int_{\mathbb{R}^3}(|x|^{-1}*U_\lambda^2)U_\lambda(U_\lambda)_{jk}+\frac{\epsilon^2}{2}\int_{\mathbb{R}^3}<\mathrm{Hess}W(x_0)x,x>U_\lambda(U_\lambda)_{jk}.\end{aligned}$$

Substituting into (3.2) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(L_\epsilon\left(U_{\lambda,\epsilon}\right)_j\right)\left(U_{\lambda,\epsilon}\right)_k\\&=\frac{\epsilon^2}{2}\int_{\mathbb{R}^3}<\mathrm{Hess}W(x_0)x,x>\left[\left(U_\lambda\right)_j\left(U_\lambda\right)_k+U_\lambda\left(U_\lambda\right)_{jk}\right]+o(\epsilon^2).\end{aligned}$$

Recalling that $\mathrm{Hess}W(x_0)=\mathrm{diag}\{a_1,a_2,a_3\}$ and integrating by parts, we find

$$\begin{aligned}&\int_{\mathbb{R}^3}<\mathrm{Hess}W(x_0)x,x>U_\lambda\left(U_\lambda\right)_{jk}\\&=-\int_{\mathbb{R}^3}\sum_{i=1}^3a_ix_i^2\left(U_\lambda\right)_k\left(U_\lambda\right)_j-2a_k\int_{\mathbb{R}^3}x_kU_\lambda\left(U_\lambda\right)_j.\end{aligned}$$

Therefore, integrating by parts once more, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(L_\epsilon\left(U_{\lambda,\epsilon}\right)_j\right)\left(U_{\lambda,\epsilon}\right)_k=-\epsilon^2a_k\int_{\mathbb{R}^3}x_kU_\lambda\left(U_\lambda\right)_j+o(\epsilon^2)\\&=-\frac{\epsilon^2}{2}a_k\int_{\mathbb{R}^3}x_k\frac{\partial}{\partial x_j}\left(U_\lambda^2\right)+o(\epsilon^2)=\frac{\epsilon^2}{2}\delta_{kj}a_k\int_{\mathbb{R}^3}U_\lambda^2+o(\epsilon^2),\end{aligned}$$

which concludes the proof. □

Lemma 3.6. *Let $\psi_{\epsilon,h}$ be given by Proposition 3.2. There exist $\{c_j^{\epsilon,h}\}$ and $\psi_{\epsilon,h}^{\perp} \in \big(\mathrm{span}\{(U_{\lambda,\epsilon})_j,\ j=1,2,3\}\big)^{\perp_{L^2}}$ such that*

$$\psi_{\epsilon,h} = \sum_{j=1}^{3} c_j^{\epsilon,h}\,(U_{\lambda,\epsilon})_j + \psi_{\epsilon,h}^{\perp}. \tag{3.3}$$

As $\epsilon \to 0$, we have

$$\|\psi_{\epsilon,h}^{\perp}\|_{L^2(\mathbb{R}^3)} \longrightarrow 0 \tag{3.4}$$

and

$$\sum_{j=1}^{3} c_j^{\epsilon,h}\,(U_{\lambda,\epsilon})_j \longrightarrow \sum_{j=1}^{3} \alpha_j^h (U_\lambda)_j \quad \text{in } L^2(\mathbb{R}^3). \tag{3.5}$$

Moreover, $c_j^{\epsilon,h}$ is bounded and $c_j^{\epsilon,h} \to \alpha_j^h$ as $\epsilon \to 0$ for $j=1,2,3$.

Proof. Fix $h \in \{2,3,4\}$. For the sake of simplicity, we drop the dependency in h in the notation. From Proposition 3.2 we already know that

$$\Big\|\psi_\epsilon - \sum_{j=1}^{3} \alpha_j (U_\lambda)_j\Big\|_{L^2(\mathbb{R}^3)}^2 \to 0 \ \text{ as } \ \epsilon \to 0.$$

Observe now that

$$\begin{aligned}
\Big\|\psi_\epsilon - \sum_{j=1}^{3} \alpha_j (U_\lambda)_j\Big\|_{L^2(\mathbb{R}^3)}^2 &= \Big\| \sum_{j=1}^{3} c_j^{\epsilon}\,(U_{\lambda,\epsilon})_j - \sum_{j=1}^{3} \alpha_j (U_\lambda)_j + \psi_\epsilon^{\perp}\Big\|_{L^2(\mathbb{R}^3)}^2 \\
= \Big\| \sum_{j=1}^{3} c_j^{\epsilon}(U_{\lambda,\epsilon})_j - \sum_{j=1}^{3} \alpha_j (U_\lambda)_j\Big\|_{L^2(\mathbb{R}^3)}^2 &+ \|\psi_\epsilon^{\perp}\|_{L^2(\mathbb{R}^3)}^2 \\
&- 2\sum_{j=1}^{3} \alpha_j \big((U_\lambda)_j, \psi_\epsilon^{\perp}\big)_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Since ψ_ϵ is bounded in $L^2(\mathbb{R}^3)$, $\psi_\epsilon^{\perp}$ is also bounded in $L^2(\mathbb{R}^3)$ and there exists ψ_0 such that $\psi_\epsilon^{\perp} \to \psi_0$ weakly in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$. Therefore,

$$\Big((U_\lambda)_j, \psi_\epsilon^{\perp}\Big)_{L^2(\mathbb{R}^3)} \to 0 \qquad \text{as} \qquad \epsilon \to 0.$$

Consequently,

$$\Big\| \sum_{j=1}^{3} c_j^{\epsilon}\,(U_{\lambda,\epsilon})_j - \sum_{j=1}^{3} \alpha_j (U_\lambda)_j\Big\|_{L^2(\mathbb{R}^3)}^2 + \|\psi_\epsilon^{\perp}\|_{L^2(\mathbb{R}^3)}^2 \to 0 \ \text{ as } \ \epsilon \to 0$$

and this proves (3.4) and (3.5).

We now prove that c_j^ϵ is bounded. Suppose by contradiction that there exists j such that $|c_j^\epsilon| \to +\infty$, as $\epsilon \to 0$. Then, since $(U_{\lambda,\epsilon})_j \perp_{L^2} (U_{\lambda,\epsilon})_h$ for $j \neq h$ and $\| (U_{\lambda,\epsilon})_j \|_{L^2(\mathbb{R}^3)} \to \| (U_\lambda)_j \|_{L^2(\mathbb{R}^3)}$ as $\epsilon \to 0$, we obtain

$$\Big\| \sum_{j=1}^{3} c_j^\epsilon (U_{\lambda,\epsilon})_j \Big\|_{L^2(\mathbb{R}^3)} = \sum_{j=1}^{3} |c_j^\epsilon| \| (U_{\lambda,\epsilon})_j \|_{L^2(\mathbb{R}^3)} \to +\infty, \quad \text{as} \quad \epsilon \to 0. \tag{3.6}$$

This is impossible because (3.5) implies

$$\Big\| \sum_{j=1}^{3} c_j^\epsilon (U_{\lambda,\epsilon})_j \Big\|_{L^2(\mathbb{R}^3)} \to \Big\| \sum_{j=1}^{3} \alpha_j (U_\lambda)_j \Big\|_{L^2(\mathbb{R}^3)} < +\infty.$$

It remains to show that $c_j^\epsilon \to \alpha_j$, as $\epsilon \to 0$. We already know that

$$\Big\| \sum_{j=1}^{3} \Big(c_j^\epsilon (U_{\lambda,\epsilon})_j - \alpha_j (U_\lambda)_j \Big) \Big\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

By (3.5), since $(U_\lambda)_j \perp_{L^2} (U_\lambda)_h$ for $j \neq h$ and $(U_{\lambda,\epsilon})_j \perp_{L^2} (U_{\lambda,\epsilon})_h$ for $j \neq h$, we also have

$$\begin{aligned}
\Big\| \sum_{j=1}^{3} \Big(c_j^\epsilon (U_{\lambda,\epsilon})_j - \alpha_j (U_\lambda)_j \Big) \Big\|^2_{L^2(\mathbb{R}^3)} &= \sum_{j=1}^{3} \| c_j^\epsilon (U_{\lambda,\epsilon})_j - \alpha_j (U_\lambda)_j \|^2_{L^2(\mathbb{R}^3)} \\
&+ \sum_{\substack{j,h=1 \\ j \neq h}}^{3} \int_{\mathbb{R}^3} \Big(c_j^\epsilon (U_{\lambda,\epsilon})_j - \alpha_j (U_\lambda)_j \Big) \Big(c_h^\epsilon (U_{\lambda,\epsilon})_h - \alpha_h (U_\lambda)_h \Big) \\
&= \sum_{j=1}^{3} \| c_j^\epsilon (U_{\lambda,\epsilon})_j - \alpha_j (U_\lambda)_j \|^2_{L^2(\mathbb{R}^3)} - 2 \sum_{\substack{j,h=1 \\ j \neq h}}^{3} c_h^\epsilon \alpha_j \int_{\mathbb{R}^3} (U_{\lambda,\epsilon})_h (U_\lambda)_j .
\end{aligned}$$

Since c_j^ϵ is bounded, $(U_{\lambda,\epsilon})_h \to (U_\lambda)_h$ in $L^2(\mathbb{R}^3)$ and $(U_\lambda)_j \perp_{L^2} (U_\lambda)_h$ if $j \neq h$, it follows also that

$$\sum_{\substack{j,h=1 \\ j \neq h}}^{3} c_h^\epsilon \alpha_j \int_{\mathbb{R}^3} (U_{\lambda,\epsilon})_h (U_\lambda)_j \to 0 \quad \text{as} \quad \epsilon \to 0.$$

As a consequence

$$\|c_j^\epsilon \left(U_{\lambda,\epsilon}\right)_j - \alpha_j \left(U_\lambda\right)_j \|_{L^2(\mathbb{R}^3)} \to 0 \text{ as } \epsilon \to 0, \ \forall j = 1,2,3.$$

Recalling that $\left(U_{\lambda,\epsilon}\right)_j \to \left(U_\lambda\right)_j$ in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$, the conclusion follows. $\square$

Proof of Proposition 3.3. Fix $h \in \{2,3,4\}$. As before, we drop the dependence on h in the notation and write

$$\psi_\epsilon := \psi_{\epsilon,h} \text{ and } \mu_\epsilon := \mu_{\epsilon,h}.$$

From $L_\epsilon \psi_\epsilon = \mu_\epsilon \psi_\epsilon$ and (3.3) we obtain

$$\sum_{j=1}^{3} c_j^\epsilon L_\epsilon \left(U_{\lambda,\epsilon}\right)_j + L_\epsilon \psi_\epsilon^\perp = \mu_\epsilon \sum_{j=1}^{3} c_j^\epsilon \left(U_{\lambda,\epsilon}\right)_j + \mu_\epsilon \psi_\epsilon^\perp.$$

We multiply by $\left(U_{\lambda,\epsilon}\right)_k$ and integrate over $\mathbb{R}^3$ to get

$$\begin{aligned} &\sum_{j=1}^{3} c_j^\epsilon \int_{\mathbb{R}^3} \left(L_\epsilon \left(U_{\lambda,\epsilon}\right)_j\right) \left(U_{\lambda,\epsilon}\right)_k + \int_{\mathbb{R}^3} \left(L_\epsilon \psi_\epsilon^\perp\right) \left(U_{\lambda,\epsilon}\right)_k \\ &= \mu_\epsilon \sum_{j=1}^{3} c_j^\epsilon \int_{\mathbb{R}^3} \left(U_{\lambda,\epsilon}\right)_j \left(U_{\lambda,\epsilon}\right)_k + \mu_\epsilon \int_{\mathbb{R}^3} \psi_\epsilon^\perp \left(U_{\lambda,\epsilon}\right)_k . \end{aligned} \tag{3.7}$$

Observe that by construction

$$\int_{\mathbb{R}^3} \psi_\epsilon^\perp \left(U_{\lambda,\epsilon}\right)_k = 0$$

and that

$$\int_{\mathbb{R}^3} \left(L_\epsilon \psi_\epsilon^\perp\right) \left(U_{\lambda,\epsilon}\right)_k = \int_{\mathbb{R}^3} \psi_\epsilon^\perp \left(L_\epsilon \left(U_{\lambda,\epsilon}\right)_k\right).$$

Moreover,

$$\int_{\mathbb{R}^3} \left(U_{\lambda,\epsilon}\right)_j \left(U_{\lambda,\epsilon}\right)_k = \delta_{jk} \| \left(U_\lambda\right)_k \|_{L^2(\mathbb{R}^3)}^2,$$

so (3.7) becomes

$$\sum_{j=1}^{3} c_j^\epsilon \int_{\mathbb{R}^3} \left(L_\epsilon \left(U_{\lambda,\epsilon}\right)_j\right) \left(U_{\lambda,\epsilon}\right)_k + \int_{\mathbb{R}^3} \psi_\epsilon^\perp \left(L_\epsilon \left(U_{\lambda,\epsilon}\right)_k\right) = \mu_\epsilon c_k^\epsilon \| \left(U_\lambda\right)_k \|_{L^2(\mathbb{R}^3)}^2. \tag{3.8}$$

Using Lemma 3.4, Lemma 3.5 and Lemma 3.6, (3.8) becomes

$$\frac{\epsilon^2}{2}c_k^\epsilon a_k\|U_\lambda\|^2_{L^2(\mathbb{R}^3)}+o(\epsilon^2)\sum_{j=1}^3 c_j^\epsilon+o(\epsilon^2)=\mu_\epsilon c_k^\epsilon\|\left(U_\lambda\right)_k\|^2_{L^2(\mathbb{R}^3)}.$$

Since by Lemma 3.6 $c_k^\epsilon\to\alpha_k$ as $\epsilon\to 0$, there exists at least an index k such that for ϵ small enough $c_k^\epsilon\neq 0$ (because for such a k we have $\alpha_k\neq 0$). Dividing by $c_k^\epsilon\|\left(U_\lambda\right)_k\|^2_{L^2(\mathbb{R}^3)}$ we get

$$\mu_\epsilon=\frac{\epsilon^2}{2}a_k\frac{\|U_\lambda\|^2_{L^2(\mathbb{R}^3)}}{\|\left(U_\lambda\right)_k\|^2_{L^2(\mathbb{R}^3)}}+o(\epsilon^2).$$

Observe now that in general (if $a_1\neq a_2\neq a_3$) we necessarily have $\alpha_k\neq 0$ for one and only one k (otherwise our proof would lead to different expansions for the same eigenvalue, which is of course impossible). Without loss of generality we can take $k=h$ and this finishes the proof. □

4. The slope information

This section is devoted to the study of the sign of $D(\omega)$. We have split our result into the following two propositions.

Proposition 4.1. *For ϵ small enough we have*

$$\begin{array}{ll} D(\omega)<0 & \text{if } p>1+\frac{4}{3},\\ D(\omega)>0 & \text{if } p<1+\frac{4}{3}.\end{array}$$

Proposition 4.2. *Suppose that $p=1+\frac{4}{3}$. Then for ϵ small enough we have*

$$\begin{array}{lll} D(\omega)>0, & \text{if} & \Delta W(x_0)>K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}}C,\\ D(\omega)<0, & \text{if} & \Delta W(x_0)<K(x_0)^2\left[W(x_0)+\omega\right]^{\frac{2}{p-1}}C,\end{array}$$

where the constant $C>0$ (independent of x_0,K,W) is explicitly known.

Before proving Propositions 4.1 and 4.2, some preliminaries are in order.

Lemma 4.3. *Let R_ω^ϵ be defined by $R_\omega^\epsilon:=\frac{\partial}{\partial\omega}u_\epsilon$. Then*

$$L_\epsilon R_\omega^\epsilon=-u_\epsilon. \tag{4.1}$$

Moreover,

$$R_\omega^\epsilon=\sum_{j=1}^3 d_j^\epsilon\left(U_\lambda\right)_j+\frac{1}{W(x_0)+\omega}R_0+o(1), \tag{4.2}$$

where $d_j^\epsilon=O(1)$ and R_0 is given by (2.2).

Remark 4.4. The decomposition (4.2) is used only in the case $p = 1 + \frac{4}{3}$.

We recall the following result (see e.g. [2]).

Lemma 4.5. *For each $\xi \in \mathbb{R}^3$, the map*

$$L\phi := -\Delta\phi + [W(x_0) + \omega]\phi - pU_\lambda(x-\xi)^{p-1}\phi$$

is invertible from $K_\xi^\perp$ to $C_\xi^\perp$, where

$$K_\xi^\perp := \left\{\phi \in H^2(\mathbb{R}^3) : \phi \perp_{L^2} (U_\lambda(\cdot - \xi))_j\,,\ j = 1,2,3\right\} \subset H^2(\mathbb{R}^3),$$

$$C_\xi^\perp := \left\{\phi \in L^2(\mathbb{R}^3) : \phi \perp_{L^2} (U_\lambda(\cdot - \xi))_j\,,\ j = 1,2,3\right\} \subset L^2(\mathbb{R}^3).$$

Proof of Lemma 4.3. We derive

$$-\Delta u_\epsilon + \omega u_\epsilon + W(x_\epsilon)u_\epsilon - u_\epsilon^p + \epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right)u_\epsilon = 0,$$

with respect to ω to obtain

$$\begin{aligned}&-\Delta R_\omega^\epsilon + [\omega + W(x_\epsilon)]\,R_\omega^\epsilon - pu_\epsilon^{p-1}R_\omega^\epsilon + 2\epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon R_\omega^\epsilon\right)u_\epsilon\\ &+ \epsilon^2 K(x_\epsilon)\left(|x|^{-1} * K(x_\epsilon)u_\epsilon^2\right)R_\omega^\epsilon = -u_\epsilon.\end{aligned}$$

This gives immediately

$$L_\epsilon R_\omega^\epsilon = -u_\epsilon.$$

As a consequence we have $L_\epsilon R_\omega^\epsilon \longrightarrow -U_\lambda$ in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$. Since u_ϵ is uniformly differentiable in ω, R_ω^ϵ is bounded in $H^1(\mathbb{R}^3)$, therefore,

$$(L_0 - L_\epsilon)\,R_\omega^\epsilon \longrightarrow 0 \ \text{ in } L^2(\mathbb{R}^3) \ \text{ as } \ \epsilon \to 0.$$

Consequently,

$$L_0 R_\omega^\epsilon = (L_0 - L_\epsilon)\,R_\omega^\epsilon + L_\epsilon R_\omega^\epsilon \longrightarrow -U_\lambda \ \text{ in } L^2(\mathbb{R}^3) \ \text{ as } \ \epsilon \to 0.$$

We decompose

$$R_\omega^\epsilon = \sum_{j=1}^{3} d_j^\epsilon\,(U_{\lambda,\epsilon})_j + \frac{1}{W(x_0)+\omega}R_{0,\epsilon} + R_\omega^{\epsilon\,\perp}, \tag{4.3}$$

with

$$R_{0,\epsilon} := R_0(\cdot - \xi_\epsilon) \quad \text{ and } \quad R_\omega^{\epsilon\,\perp} \in \left(\operatorname{span}\left\{(U_{\lambda,\epsilon})_j\right\}\right)^{\perp_{L^2}}.$$

We remark that $(U_{\lambda,\epsilon})_j = (U_\lambda)_j + o(1)$ and $R_{0,\epsilon} = R_0 + o(1)$. Using the decomposition we have

$$L_{0\epsilon}R_\omega^\epsilon = \sum_{j=1}^{3} d_j^\epsilon L_{0\epsilon}\,(U_{\lambda,\epsilon})_j + \frac{1}{W(x_0)+\omega}L_{0\epsilon}R_{0,\epsilon} + L_{0\epsilon}R_\omega^{\epsilon\,\perp},$$

where $L_{0\epsilon} := -\Delta + [W(x_0)+\omega] - pU_{\lambda,\epsilon}^{p-1}$. Therefore,

$$L_{0\epsilon} R_\omega^{\epsilon\,\perp} = L_{0\epsilon} R_\omega^\epsilon + U_{\lambda,\epsilon},$$

and so $L_{0\epsilon} R_\omega^{\epsilon\,\perp} \longrightarrow 0$ in $L^2(\mathbb{R}^3)$ as $\epsilon \to 0$. Since $L_{0\epsilon}$ is invertible from $H^2(\mathbb{R}^3)/\ker L_{0\epsilon}$ to $L^2(\mathbb{R}^3)/\ker L_{0\epsilon}$ (see Lemma 4.5) and $R_\omega^{\epsilon\,\perp} \in (\ker L_{0\epsilon})^{\perp_{L^2}}$, we get $R_\omega^{\epsilon\,\perp} \to 0$ in $H^2(\mathbb{R}^3)$ as $\epsilon \to 0$. It remains to show that $d_j^\epsilon = O(1)$. From (4.1) and (4.3) we get

$$\sum_{j=1}^3 d_j^\epsilon L_\epsilon (U_{\lambda,\epsilon})_j + \frac{1}{W(x_0)+\omega} L_\epsilon R_{0,\epsilon} + L_\epsilon R_\omega^{\epsilon\,\perp} = -u_\epsilon.$$

Multiplying by $(U_{\lambda,\epsilon})_k$ and integrating we obtain

$$\begin{aligned} -\int_{\mathbb{R}^3} u_\epsilon (U_{\lambda,\epsilon})_k &= \sum_{j=1}^3 d_j^\epsilon \int_{\mathbb{R}^3} L_\epsilon (U_{\lambda,\epsilon})_j (U_{\lambda,\epsilon})_k \\ &\quad + \frac{1}{W(x_0)+\omega} \int_{\mathbb{R}^3} L_\epsilon R_{0,\epsilon} (U_{\lambda,\epsilon})_k + \int_{\mathbb{R}^3} L_\epsilon R_\omega^{\epsilon\,\perp} (U_{\lambda,\epsilon})_k . \end{aligned} \tag{4.4}$$

Let us analyze each term separately. From Lemma 3.5 we know that

$$\sum_{j=1}^3 d_j^\epsilon \int_{\mathbb{R}^3} L_\epsilon (U_{\lambda,\epsilon})_j (U_{\lambda,\epsilon})_k = \frac{\epsilon^2}{2} d_k^\epsilon a_k \|U_\lambda\|^2_{L^2(\mathbb{R}^3)} + o(\epsilon^2) \sum_{j=1}^3 d_j^\epsilon.$$

Moreover, since from Lemma 3.4 we know that $L_\epsilon (U_{\lambda,\epsilon})_k = O(\epsilon^2)$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} L_\epsilon R_{0,\epsilon} (U_{\lambda,\epsilon})_k &= \int_{\mathbb{R}^3} R_{0,\epsilon} L_\epsilon (U_{\lambda,\epsilon})_k \\ &= \int_{\mathbb{R}^3} R_0 L_\epsilon (U_{\lambda,\epsilon})_k + o(1) \int_{\mathbb{R}^3} L_\epsilon (U_{\lambda,\epsilon})_k = O(\epsilon^2). \end{aligned}$$

Recalling that $R_\omega^{\epsilon\,\perp} = o(1)$, we also have

$$\int_{\mathbb{R}^3} L_\epsilon R_\omega^{\epsilon\,\perp} (U_{\lambda,\epsilon})_k = \int_{\mathbb{R}^3} R_\omega^{\epsilon\,\perp} L_\epsilon (U_{\lambda,\epsilon})_k = o(\epsilon^2).$$

Finally, from Proposition 2.5 we have

$$\begin{aligned} \int_{\mathbb{R}^3} u_\epsilon (U_{\lambda,\epsilon})_k &= \int_{\mathbb{R}^3} U_{\lambda,\epsilon} (U_{\lambda,\epsilon})_k + \epsilon^2 \int_{\mathbb{R}^3} w_0 (U_{\lambda,\epsilon})_k + o(\epsilon^2) \int_{\mathbb{R}^3} (U_{\lambda,\epsilon})_k \\ &= \epsilon^2 \int_{\mathbb{R}^3} w_0 (U_{\lambda,\epsilon})_k + o(\epsilon^2) = O(\epsilon^2). \end{aligned}$$

So (4.4) becomes

$$\frac{\epsilon^2}{2} d_k^\epsilon a_k \|U_\lambda\|^2_{L^2(\mathbb{R}^3)} + o(\epsilon^2) \sum_{j=1}^{3} d_j^\epsilon = O(\epsilon^2).$$

Dividing by ϵ^2 we get $d_k^\epsilon C + o(1) \sum_{j=1}^3 d_j^\epsilon = O(1)$ and therefore it is clear that $d_k^\epsilon = O(1)$, which concludes the proof. □

We now derive two useful identities.

Lemma 4.6. *The following equalities hold:*

$$\int_{\mathbb{R}^3} R_\omega^\epsilon L_\epsilon \Big(\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon \Big) = \Big(\frac{3}{4} - \frac{1}{p-1} \Big) \|u_\epsilon\|^2_{L^2(\mathbb{R}^3)}. \tag{4.5}$$

$$\begin{aligned} [W(x_\epsilon) + \omega]\, u_\epsilon = {} & -L_\epsilon \Big(\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon \Big) - \frac{\epsilon}{2} x \cdot \nabla W(x_\epsilon) u_\epsilon \\ & + \epsilon^2 \frac{4-2p}{p-1} K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon^2 \right) u_\epsilon + O(\epsilon^3). \end{aligned} \tag{4.6}$$

Proof. We start with the proof of (4.5). By symmetry of L_ϵ, we have

$$\int_{\mathbb{R}^3} R_\omega^\epsilon L_\epsilon \Big(\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon \Big) = \int_{\mathbb{R}^3} L_\epsilon R_\omega^\epsilon \Big(\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon \Big).$$

By Lemma 4.3, we have $L_\epsilon R_\omega^\epsilon = -u_\epsilon$, thus,

$$\begin{aligned} \int_{\mathbb{R}^3} R_\omega^\epsilon L_\epsilon \Big(\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon \Big) &= - \int_{\mathbb{R}^3} u_\epsilon \Big(\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon \Big) \\ &= - \frac{1}{p-1} \|u_\epsilon\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{2} \int_{\mathbb{R}^3} u_\epsilon x \cdot \nabla u_\epsilon. \end{aligned}$$

Integrating by parts it is easy to see that

$$\int_{\mathbb{R}^3} u_\epsilon x \cdot \nabla u_\epsilon = -3 \|u_\epsilon\|^2_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} u_\epsilon x \cdot \nabla u_\epsilon.$$

The conclusion follows for (4.5).

We turn now to the proof of (4.6). First we remark that

$$\frac{1}{p-1} u_\epsilon + \frac{1}{2} x \cdot \nabla u_\epsilon = \frac{\partial}{\partial \alpha} u_\epsilon^\alpha \big|_{\omega=1},$$

where $u_\epsilon^\alpha = \alpha^{1/(p-1)} u_\epsilon(\alpha^{1/2}\, \cdot\,)$. We define by I_ϵ the functional whose critical points are solutions of (2.9):

$$I_\epsilon(v) \;\; := \;\; \int_{\mathbb{R}^3} \Big[\frac{1}{2} |\nabla v|^2 + \frac{1}{2} [W(x_\epsilon) + \omega]\, v^2 - \frac{1}{p+1} |v|^{p+1} \Big]$$

$$+\int_{\mathbb{R}^3}\frac{\epsilon^2}{4}K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)v^2\right)v^2.$$

Then

$$\begin{aligned}L_\epsilon\Big(\frac{1}{p-1}u_\epsilon+\frac{1}{2}x\cdot\nabla u_\epsilon\Big)&=L_\epsilon\Big(\frac{\partial}{\partial\alpha}u_\epsilon^\alpha\big|_{\alpha=1}\Big)\\&=I_\epsilon''(u_\epsilon)\Big(\frac{\partial}{\partial\alpha}u_\epsilon^\alpha\big|_{\alpha=1}\Big)=\frac{\partial}{\partial\alpha}\left(I_\epsilon'(u_\epsilon^\alpha)\right)\big|_{\alpha=1}.\end{aligned}$$

Now, it is easy to see that

$$\begin{aligned}&I_\epsilon'(u_\epsilon^\alpha)=-\alpha^{\frac{1}{p-1}+1}\Delta u_\epsilon+\left[W(x_{\epsilon,\alpha})+\omega\right]\alpha^{\frac{1}{p-1}}u_\epsilon-\alpha^{\frac{1}{p-1}+1}u_\epsilon^p\\&+\epsilon^2\alpha^{\frac{4-p}{p-1}}K(x_{\epsilon,\alpha})\left(|x|^{-1}*K(x_{\epsilon,\alpha})u_\epsilon^2\right)u_\epsilon,\end{aligned}$$

where we have set $x_{\epsilon,\alpha}:=\epsilon x\alpha^{-1/2}+x_0$. Consequently,

$$\begin{aligned}\frac{\partial}{\partial\alpha}\left(I_\epsilon'(u_\epsilon^\alpha)\right)&=-\alpha^{\frac{1}{p-1}}\frac{p}{p-1}\Delta u_\epsilon+\alpha^{\frac{1}{p-1}-1}\frac{1}{p-1}\left[W(x_{\epsilon,\alpha})+\omega\right]u_\epsilon\\&-\frac{\epsilon}{2}\alpha^{\frac{1}{p-1}-\frac{3}{2}}x\cdot\nabla W(x_{\epsilon,\alpha})u_\epsilon-\alpha^{\frac{1}{p-1}}\frac{p}{p-1}u_\epsilon^p\\&+\epsilon^2\frac{4-p}{p-1}\alpha^{\frac{4-p}{p-1}-1}K(x_{\epsilon,\alpha})\left(|x|^{-1}*K(x_{\epsilon,\alpha})u_\epsilon^2\right)u_\epsilon\\&-\frac{\epsilon^3}{2}\alpha^{\frac{4-p}{p-1}-\frac{3}{2}}x\cdot\nabla K(x_{\epsilon,\alpha})\left(|x|^{-1}*K(x_{\epsilon,\alpha})u_\epsilon^2\right)u_\epsilon\\&-\frac{\epsilon^3}{2}\alpha^{\frac{4-p}{p-1}-\frac{3}{2}}K(x_{\epsilon,\alpha})\left(|x|^{-1}*x\cdot\nabla K(x_{\epsilon,\alpha})u_\epsilon^2\right)u_\epsilon.\end{aligned}$$

For $\alpha=1$ we get

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\left(I_\epsilon'(u_\epsilon^\alpha)\right)\big|_{\alpha=1}-\frac{p}{p-1}\Delta u_\epsilon+\frac{1}{p-1}\left[W(x_\epsilon)+\omega\right]u_\epsilon\\&-\frac{\epsilon}{2}x\cdot\nabla W(x_\epsilon)u_\epsilon-\frac{p}{p-1}u_\epsilon^p+\epsilon^2\frac{4-p}{p-1}K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)u_\epsilon^2\right)u_\epsilon+O(\epsilon^3).\end{aligned}$$

Recalling that u_ϵ satisfies (2.9), we get

$$\begin{aligned}\frac{\partial}{\partial\alpha}\left(I_\epsilon'(u_\epsilon^\alpha)\right)\big|_{\alpha=1}&=\frac{p}{p-1}\Big(\Delta u_\epsilon+\left[W(x_\epsilon)+\omega\right]u_\epsilon-u_\epsilon^p\\&+\epsilon^2K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)u_\epsilon^2\right)u_\epsilon\Big)\\&+\Big(\frac{1}{p-1}-\frac{p}{p-1}\Big)\left[W(x_\epsilon)+\omega\right]u_\epsilon-\frac{\epsilon}{2}x\cdot\nabla W(x_\epsilon)u_\epsilon\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^2\Big(\frac{4-p}{p-1}-\frac{p}{p-1}\Big)K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)u_\epsilon^2\right)u_\epsilon+O(\epsilon^3)\\
&=-\left[W(x_\epsilon)+\omega\right]u_\epsilon-\frac{\epsilon}{2}x\cdot\nabla W(x_\epsilon)u_\epsilon\\
&+\epsilon^2\frac{4-2p}{p-1}K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)u_\epsilon^2\right)u_\epsilon+O(\epsilon^3),
\end{aligned}
$$

which concludes the proof. □

Proof of Proposition 4.1. The proof consists in deriving an asymptotic expansion formula for the function $D(\omega)$ as ϵ goes to zero. First observe that

$$
D(\omega)=\frac{\partial}{\partial\omega}\|u_\epsilon\|^2_{L^2(\mathbb{R}^3)}=2\int_{\mathbb{R}^3}\Big(\frac{\partial}{\partial\omega}u_\epsilon\Big)u_\epsilon=2\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon.
$$

Then

$$
[W(x_0)+\omega]\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon=\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon\left[W(x_0)-W(x_\epsilon)\right]+\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon\left[W(x_\epsilon)+\omega\right].
$$

By (4.6), we have

$$
\begin{aligned}
[W(x_0)+\omega]\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon&=\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon\Big[W(x_0)-W(x_\epsilon)-\frac{\epsilon}{2}x\cdot\nabla W(x_\epsilon)\Big]\\
&-\int_{\mathbb{R}^3}R^\epsilon_\omega L_\epsilon\Big(\frac{1}{p-1}u_\epsilon+\frac{1}{2}x\cdot\nabla u_\epsilon\Big)\\
&+\epsilon^2\frac{4-2p}{p-1}\int_{\mathbb{R}^3}R^\epsilon_\omega K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)u_\epsilon^2\right)u_\epsilon+O(\epsilon^3)\int_{\mathbb{R}^3}R^\epsilon_\omega.
\end{aligned}
$$

By (4.5), we have

$$
\begin{aligned}
&[W(x_0)+\omega]\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon=\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon\Big[W(x_0)-W(x_\epsilon)-\frac{\epsilon}{2}x\cdot\nabla W(x_\epsilon)\Big]\\
&+\Big(\frac{1}{p-1}-\frac{3}{4}\Big)\|u_\epsilon\|^2_{L^2(\mathbb{R}^3)}+\epsilon^2\frac{4-2p}{p-1}\int_{\mathbb{R}^3}R^\epsilon_\omega K(x_\epsilon)\left(|x|^{-1}*K(x_\epsilon)u_\epsilon^2\right)u_\epsilon\\
&\quad+O(\epsilon^3).
\end{aligned}
$$

Moreover, it is easy to see that

$$
\Big[W(x_0)-W(x_\epsilon)-\frac{\epsilon}{2}x\cdot\nabla W(x_\epsilon)\Big]=-\epsilon^2<\mathrm{Hess}W(x_0)x,x>+O(\epsilon^3|x|^3).
$$

Thus, we get

$$
[W(x_0)+\omega]\int_{\mathbb{R}^3}R^\epsilon_\omega u_\epsilon=\Big(\frac{1}{p-1}-\frac{3}{4}\Big)\|u_\epsilon\|^2_{L^2(\mathbb{R}^3)}\tag{4.7}
$$

$$
\begin{aligned}
&- \epsilon^2 \int_{\mathbb{R}^3} R^\epsilon_\omega u_\epsilon < \mathrm{Hess} W(x_0)x, x > + \int_{\mathbb{R}^3} O(\epsilon^3 |x|^3) R^\epsilon_\omega u_\epsilon \\
&+ \epsilon^2 \frac{4-2p}{p-1} \int_{\mathbb{R}^3} R^\epsilon_\omega K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon^2 \right) u_\epsilon + O(\epsilon^3) \\
&= \Big(\frac{1}{p-1} - \frac{3}{4} \Big) \|u_\epsilon\|^2_{L^2(\mathbb{R}^3)} + O(\epsilon^2).
\end{aligned}
$$

In conclusion, we have obtained

$$
\frac{\partial}{\partial \omega} \|u_\epsilon\|^2_{L^2(\mathbb{R}^3)} = \Big(\frac{1}{p-1} - \frac{3}{4} \Big) \frac{2}{W(x_0) + \omega} \|u_\epsilon\|^2_{L^2(\mathbb{R}^3)} + O(\epsilon^2), \quad (4.8)
$$

and this finishes the proof. □

Proof of Proposition 4.2. If $p = 1 + \frac{4}{3}$ then (4.8) is not sufficient to determine the sign of $D(\omega)$ for ϵ small. We derive now a more accurate asymptotic expansion formula for $D(\omega)$. From (4.7) we have

$$
\begin{aligned}
D(\omega) = &- \frac{2}{[W(x_0) + \omega]} \epsilon^2 \int_{\mathbb{R}^3} R^\epsilon_\omega u_\epsilon < \mathrm{Hess} W(x_0)x, x > \\
&- \frac{1}{[W(x_0) + \omega]} \epsilon^2 \int_{\mathbb{R}^3} R^\epsilon_\omega K(x_\epsilon) \left(|x|^{-1} * K(x_\epsilon) u_\epsilon^2 \right) u_\epsilon + O(\epsilon^3).
\end{aligned}
$$

From Proposition 2.5 and the fact that $\xi_\epsilon \to 0$ and

$$
A_6 \to K(x_0)^2 \left(|x|^{-1} * U_\lambda^2 \right) U_\lambda,
$$

in $L^2(\mathbb{R}^3)$ (see the proof of Proposition 2.5), we have

$$
\begin{aligned}
D(\omega) = &- \frac{2}{[W(x_0) + \omega]} \epsilon^2 \int_{\mathbb{R}^3} R^\epsilon_\omega U_\lambda < \mathrm{Hess} W(x_0)x, x > \\
&- \frac{1}{[W(x_0) + \omega]} \epsilon^2 K(x_0)^2 \int_{\mathbb{R}^3} R^\epsilon_\omega \left(|x|^{-1} * U_\lambda^2 \right) U_\lambda + o(\epsilon^2). \quad (4.9)
\end{aligned}
$$

Recall from Lemma 4.3 that

$$
R^\epsilon_\omega = \sum_{j=1}^{3} d^\epsilon_j \left(U_\lambda \right)_j + \frac{1}{W(x_0) + \omega} R_0 + o(1).
$$

Thus,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} R^\epsilon_\omega U_\lambda < \mathrm{Hess} W(x_0)x, x >= \sum_{j=1}^{3} d^\epsilon_j \int_{\mathbb{R}^3} \left(U_\lambda \right)_j U_\lambda < \mathrm{Hess} W(x_0)x, x > \\
&+ \frac{1}{W(x_0) + \omega} \int_{\mathbb{R}^3} R_0 U_\lambda < \mathrm{Hess} W(x_0)x, x > + o(1)
\end{aligned}
$$

$$= \frac{1}{W(x_0)+\omega}\int_{\mathbb{R}^3} R_0 U_\lambda < \mathrm{Hess} W(x_0)x, x > +o(1), \tag{4.10}$$

because the first term is cancelled by parity. Similarly,

$$\int_{\mathbb{R}^3} R_\omega^\epsilon \left(|x|^{-1} * U_\lambda^2\right) U_\lambda = \sum_{j=1}^{3} d_j^\epsilon \int_{\mathbb{R}^3} (U_\lambda)_j \left(|x|^{-1} * U_\lambda^2\right) U_\lambda$$

$$+ \frac{1}{W(x_0)+\omega}\int_{\mathbb{R}^3} R_0 \left(|x|^{-1} * U_\lambda^2\right) U_\lambda + o(1)$$

$$= \frac{1}{W(x_0)+\omega}\int_{\mathbb{R}^3} R_0 \left(|x|^{-1} * U_\lambda^2\right) U_\lambda + o(1). \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9) we obtain

$$D(\omega) = -\frac{2}{[W(x_0)+\omega]^2}\epsilon^2 \int_{\mathbb{R}^3} R_0 U_\lambda < \mathrm{Hess} W(x_0)x, x >$$

$$- \frac{1}{[W(x_0)+\omega]^2}\epsilon^2 K(x_0)^2 \int_{\mathbb{R}^3} R_0 \left(|x|^{-1} * U_\lambda^2\right) U_\lambda + o(\epsilon^2). \tag{4.12}$$

Now, recall that from our choice of p it follows that $R_0 = \frac{3}{4}U_\lambda + \frac{1}{2}x \cdot \nabla U_\lambda$, and that we have assumed that $\mathrm{Hess} W(x_0) = \mathrm{diag}\{a_1, a_2, a_3\}$. Thus,

$$\int_{\mathbb{R}^3} R_0 U_\lambda < \mathrm{Hess} W(x_0)x, x >= \frac{3}{4}\sum_{i=1}^{3} a_i \int_{\mathbb{R}^3} U_\lambda^2 x_i^2 + \frac{1}{2}\sum_{i=1}^{3} a_i \int_{\mathbb{R}^3} U_\lambda x \cdot \nabla U_\lambda x_i^2.$$

Remarking that

$$\int_{\mathbb{R}^3} U_\lambda x \cdot \nabla U_\lambda x_i^2 = \sum_{k=1}^{3} \int_{\mathbb{R}^3} U_\lambda x_k \frac{\partial}{\partial x_k} U_\lambda x_i^2$$

$$= \frac{1}{2}\sum_{k=1}^{3} \int_{\mathbb{R}^3} x_k \frac{\partial}{\partial x_k}\left(U_\lambda^2\right) x_i^2 = -\frac{1}{2}\sum_{k=1}^{3} \int_{\mathbb{R}^3} \frac{\partial}{\partial x_k}\left(x_k x_i^2\right) U_\lambda^2$$

$$= -\frac{1}{2}\sum_{k=1}^{3} \int_{\mathbb{R}^3} \left(x_i^2 + 2x_i x_k \delta_{ik}\right) U_\lambda^2 = -\frac{5}{2}\int_{\mathbb{R}^3} x_i^2 U_\lambda^2,$$

we get

$$\int_{\mathbb{R}^3} R_0 U_\lambda < \mathrm{Hess} W(x_0)x, x >= -\frac{1}{2}\sum_{i=1}^{3} a_i \int_{\mathbb{R}^3} U_\lambda^2 x_i^2 = -\frac{1}{2}\Delta W(x_0) \int_{\mathbb{R}^3} U_\lambda^2 x_i^2. \tag{4.13}$$

On the other hand

$$\int_{\mathbb{R}^3} R_0\left(|x|^{-1}*U_\lambda^2\right)U_\lambda = \frac{3}{4}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda^2 + \frac{1}{2}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda x\cdot\nabla U_\lambda.$$

Remarking that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda x\cdot\nabla U_\lambda = \sum_{k=1}^{3}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda x_k\frac{\partial}{\partial x_k}U_\lambda\\
&= \frac{1}{2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)x_k\frac{\partial}{\partial x_k}(U_\lambda^2) = -\frac{1}{2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\frac{\partial}{\partial x_k}\left[\left(|x|^{-1}*U_\lambda^2\right)x_k\right]U_\lambda^2\\
&= -\frac{3}{2}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda^2 - \frac{1}{2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\frac{\partial}{\partial x_k}\left[\left(|x|^{-1}*U_\lambda^2\right)\right]x_kU_\lambda^2\\
&= -\frac{3}{2}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda^2 - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda\frac{\partial}{\partial x_k}U_\lambda\right)x_kU_\lambda^2\\
&= -\frac{3}{2}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda^2\right)U_\lambda^2 - \int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda\nabla U_\lambda\right)\cdot xU_\lambda^2,
\end{aligned}$$

we obtain

$$\int_{\mathbb{R}^3} R_0\left(|x|^{-1}*U_\lambda^2\right)U_\lambda = -\frac{1}{2}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda\nabla U_\lambda\right)\cdot xU_\lambda^2. \tag{4.14}$$

Finally, substituting (4.13) and (4.14) in (4.12), we obtain the following expression for $D(\omega)$:

$$\begin{aligned}
D(\omega) &= \frac{1}{[W(x_0)+\omega]^2}\epsilon^2\Delta W(x_0)\int_{\mathbb{R}^3}U_\lambda^2x_i^2\\
&\quad+\frac{1}{2\,[W(x_0)+\omega]^2}\epsilon^2K(x_0)^2\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda\nabla U_\lambda\right)\cdot xU_\lambda^2+o(\epsilon^2)\\
&= \epsilon^2\left[\Delta W(x_0)C_1+K(x_0)^2C_2\right]+o(\epsilon^2),
\end{aligned}$$

where

$$C_1 := \frac{1}{[W(x_0)+\omega]^2}\int_{\mathbb{R}^3}U_\lambda^2x_i^2 = \lambda^{\frac{4}{p-1}-9}\int_{\mathbb{R}^3}U_1^2x_i^2 > 0,$$

and

$$\begin{aligned}
C_2 &:= \frac{1}{2}\frac{1}{[W(x_0)+\omega]^2}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_\lambda\nabla U_\lambda\right)\cdot xU_\lambda^2\\
&= \frac{1}{2}\lambda^{\frac{8}{p-1}-9}\int_{\mathbb{R}^3}\left(|x|^{-1}*U_1\nabla U_1\right)\cdot xU_1^2.
\end{aligned}$$

The conclusion follows by taking

$$C := -\frac{1}{2}\frac{\int_{\mathbb{R}^3}\left(|x|^{-1} * U_1\nabla U_1\right)\cdot xU_1^2}{\int_{\mathbb{R}^3} U_1^2 x_i},$$

and recalling that $\lambda^2 = W(x_0) + \omega$. Let us observe that the sign of the constant C is positive. Indeed, we can prove that

$$\int_{\mathbb{R}^3}\left(|x|^{-1} * U_1\nabla U_1\right)\cdot xU_1^2 < 0,$$

in the following way. For $k = 1, 2, 3$, we define the function

$$g_k(x) := |x|^{-1} * U_1\frac{\partial}{\partial x_k}U_1 = \int_{\mathbb{R}^3}\frac{U_1(y)\frac{\partial}{\partial x_k}U_1(y)}{|x-y|}dy.$$

Then

$$\int_{\mathbb{R}^3}\left(|x|^{-1} * U_1\nabla U_1\right)\cdot xU_1^2 = \sum_{k=1}^{3}\int_{\mathbb{R}^3} g_k(x)x_kU_1^2.$$

Now, we show that

$$\begin{cases} g_k(x) < 0 & \text{if } \ x_k > 0,\\ g_k(x) > 0 & \text{if } \ x_k < 0.\end{cases}$$

Let $x \in \mathbb{R}^3$ and $k = 1, 2, 3$ be fixed and assume that $x_k > 0$. We define two half-spaces by $\Gamma_+ := \{y \in \mathbb{R}^3 : y_k > 0\}$, $\Gamma_- := \{y \in \mathbb{R}^3 : y_k < 0\}$. Since U_1 is radially decreasing, we clearly have

$$U_1(y)\frac{\partial}{\partial x_k}U_1(y) < 0 \text{ for } y \in \Gamma_+ \text{ and } U_1(y)\frac{\partial}{\partial x_k}U_1(y) > 0 \text{ for } y \in \Gamma_-. \tag{4.15}$$

For $y \in \mathbb{R}^3$, we denote by $\tilde{y}$ the reflection of y with respect to the hyperplane $\{z \in \mathbb{R}^3 : z_k = 0\}$. Since $x \in \Gamma_+$, it is easy to see that for all $y \in \Gamma_+$ we have

$$\left|\frac{U_1(\tilde{y})\frac{\partial}{\partial x_k}U_1(\tilde{y})}{|x-\tilde{y}|}\right| < \frac{U_1(y)\frac{\partial}{\partial x_k}U_1(y)}{|x-y|}.$$

Consequently,

$$\left|\int_{\Gamma_-}\frac{U_1(y)\frac{\partial}{\partial x_k}U_1(y)}{|x-y|}dy\right| < \int_{\Gamma_+}\frac{U_1(y)\frac{\partial}{\partial x_k}U_1(y)}{|x-y|}.$$

Combined with (4.15), this implies

$$g_k(x) = \int_{\mathbb{R}^3}\frac{U_1(y)\frac{\partial}{\partial x_k}U_1(y)}{|x-y|}dy < 0.$$

The case $x_k < 0$ follows from similar arguments, hence the conclusion. $\square$

5. Conclusion

From Proposition 3.2 and Proposition 3.3 it follows that L_ϵ has $m+1$ negative eigenvalues and no zero eigenvalue, where m is the number of negative eigenvalues of the matrix $\mathrm{Hess}W(x_0)$. In particular $m=0$ if x_0 is a local minimum, while $1 \leq m \leq 3$ otherwise. Hence, indicating by $n(L_\epsilon)$ the number of negative eigenvalues of L_ϵ, it follows that

$$n(L_\epsilon) = \begin{cases} 1 & \text{if } x_0 \text{ is a minimum for } W, \\ m+1 \geq 2 & \text{otherwise .} \end{cases}$$

Moreover, we define

$$p(D) := \begin{cases} 0 & \text{if } D(\omega) < 0, \\ 1 & \text{if } D(\omega) > 0. \end{cases}$$

Proposition 4.1 implies that for $p \neq 1 + \frac{4}{3}$

$$p(D) = \begin{cases} 0 & \text{if } p > 1 + \frac{4}{3}, \\ 1 & \text{if } p < 1 + \frac{4}{3}; \end{cases}$$

while for $p = 1 + \frac{4}{3}$ it follows by Proposition 4.2 that

$$p(D) = \frac{1}{2}\Big(1 + \frac{\Delta W(x_0) - K(x_0)^2\,[W(x_0)+\omega]^{\frac{2}{p-1}}\,C}{\big|\Delta W(x_0) - K(x_0)^2\,[W(x_0)+\omega]^{\frac{2}{p-1}}\,C\big|}\Big).$$

Combining these results, by the orbital stability criteria of [22, 23], we obtain Theorem 1, Theorem 2 and Theorem 3 respectively.

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