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# Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential

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**Abstract**

We study analytically and numerically the stability of the standing waves for a nonlinear Schrödinger equation with a point defect and a power type nonlinearity. A major difficulty is to compute the number of negative eigenvalues of the linearized operator around the standing waves. This is overcome by a perturbation method and continuation arguments. Among others, in the case of a repulsive defect, we show that the standing-wave solution is stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ and unstable in $H^1(\mathbb{R})$ under subcritical nonlinearity. Further we investigate the nature of instability: under critical or supercritical nonlinear interaction, we prove the instability by blowup in the repulsive case by showing a virial theorem and using a minimization method involving two constraints. In the subcritical radial case, unstable bound states cannot collapse, but rather narrow down until they reach the stable regime (a *finite-width instability*). In the nonradial repulsive case, all bound states are unstable, and the instability is manifested by a lateral drift away from the defect, sometimes in combination with a finite-width instability or a blowup instability.


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## 1. Introduction

Solitary waves are localized waves that propagate in nonlinear media where dispersion and/or diffraction are present. They appear in various fields of physics such as nonlinear optics, Bose–Einstein Condensates (BEC), plasma physics, solid state physics, water waves etc. The dynamics of solitons are modeled by the Nonlinear Schrödinger equation (NLS) in the context of nonlinear optics or the Gross–Pitaevskii (GP) equation in the context of BEC.

By now, the stability and dynamics of solitons in homogeneous media have been well understood. However, stability and dynamics of solitons in inhomogeneous media are still a matter of intense research, both theoretically and experimentally. Of particular interest is the NLS equation with a linear potential (or lattice)

$$\begin{cases} i\partial_t u(t,x) = -\partial_x^2 u - V(x)u - |u|^{p-1}u, \\ u(0,x) = u_0. \end{cases} \tag{1}$$

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In nonlinear optics, the potential $V(x)$ describes the variation of the linear refractive index in space. In BEC, it describes an external potential applied to the condensate. The potential can be localized (e.g., a single waveguide in nonlinear optics [39,32]), parabolic (e.g., a magnetic trap in BEC [1,31]) or periodic (e.g., a waveguide array or photonic crystal lattice in nonlinear optics [42]).

In the presence of a potential, a key parameter is the relative width of the solitary wave, compared with the characteristic length-scale of the potential. For example, in the case of a periodic lattice, narrow solitary waves are affected, to leading order, by the local changes of the potential near the soliton center [16,14,15,43], whereas wide solitary waves are affected by the potential average over a single period [14,15].

In this paper we consider a NLS/GP equation with a delta-function potential

$$\begin{cases} i\partial_t u(t,x) = -\partial_x^2 u - \gamma\delta(x)u - |u|^{p-1}u, \\ u(0,x) = u_0, \end{cases} \tag{2}$$

where $\gamma \in \mathbb{R}$, $1 < p < +\infty$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Here, $\delta$ is the Dirac distribution at the origin, namely, $\langle \delta, v\rangle = v(0)$ for $v \in H^1(\mathbb{R})$. Eq. (2) can be viewed as a prototype model for the interaction of a wide soliton with a narrow potential. The main advantage of using the delta-function potential rather than a finite-width potential is the existence of an explicit expression for the soliton profile. This allows one to prove results, the proofs of which are considerably harder for a general linear potential.

In nonlinear optics, Eq. (2) models a soliton propagating in a medium with a point defect [22,36] or the interaction of a wide soliton with a much narrower one in a bimodal fiber [6]. In BEC, this equation models the dynamics of a condensate in the presence of an impurity of a length-scale much smaller than the healing length. Such an impurity can be realized by a tightly focused beam, by another spin state of the same atom or by another alkali atom confined in an optical trap [41]. In contrast to wide solitons in a periodic potential, in Eq. (2) the (wide) soliton profile is affected only by the local variation of the potential rather than by its average. Moreover, since the potential is localized, there is no band structure and gap solitons characteristic of a periodic potential, see e.g., [10].

Eq. (2) was studied previously by several authors. In [6,22,25–27,40,41], the phenomenon of soliton scattering by the effect of the defect was observed, namely, interactions between the defect and the homogeneous medium soliton. For example, varying amplitude and velocity of the soliton, they studied how the defect is separating the soliton into two parts: one part is transmitted past the defect, the other one is captured at the defect. Holmer, Marzuola and Zworski [25,26] gave numerical simulations and theoretical arguments on this subject. Recently, these results were observed experimentally for a single waveguide potential [32].

In this paper, we study the stability and instability of the standing-wave solution of (2) of the form $u(t,x) = e^{i\omega t}\varphi(x)$ where $\varphi$ is required to satisfy

$$\begin{cases} -\partial_x^2\varphi + \omega\varphi - \gamma\delta(x)\varphi - |\varphi|^{p-1}\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases} \tag{3}$$

Stability under radial (symmetric) perturbations was studied analytically in [22,13,12]. In this paper, we study stability under *nonradial* perturbations. We also show that the instability associated with momentum-nonconserving perturbations is excited only for a repulsive defect ($\gamma < 0$), and is manifested by a lateral movement of the wave away from the defect.

In the numerical part of this study we combine some recent ideas such as a quantitative approach to (in)stability and characterization of the instability type (width or drift instability) in order to provide a systematic description of the standing-wave dynamics. We emphasize that both our approach and results are relevant to standing waves of the NLS (1) with a general linear potential, and also to NLS with a nonlinear potential [14,15].

## 2. Review of previous results

*Notations*: The space $L^r(\mathbb{R},\mathbb{C})$ will be denoted by $L^r(\mathbb{R})$ and its norm by $\|\cdot\|_r$. When $r = 2$, the space $L^2(\mathbb{R})$ will be endowed with the scalar product

$$(u,v)_2 = \operatorname{Re}\int_{\mathbb{R}} u\bar{v}\,\mathrm{d}x \quad \text{for } u, v \in L^2(\mathbb{R}).$$

The space $H^1(\mathbb{R},\mathbb{C})$ will be denoted by $H^1(\mathbb{R})$, its norm by $\|\cdot\|_{H^1(\mathbb{R})}$ and the duality product between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$ by $\langle\cdot,\cdot\rangle$. We write $H^1_{\mathrm{rad}}(\mathbb{R})$ for the space of radial (even) functions of $H^1(\mathbb{R})$:

$$H^1_{\mathrm{rad}}(\mathbb{R}) = \{v \in H^1(\mathbb{R});\ v(x) = v(-x),\ x \in \mathbb{R}\}.$$

For $\gamma = 0$, the set of solutions of (3) has been known for a long time. In particular, modulo translation and phase, there exists a unique positive solution, which is explicitly known. This solution is even and is a ground state (see, for example, [4,7,29] for such results). For $\gamma \neq 0$, an explicit solution of (3) was presented in [13,22] and the following was proved in [12,13].

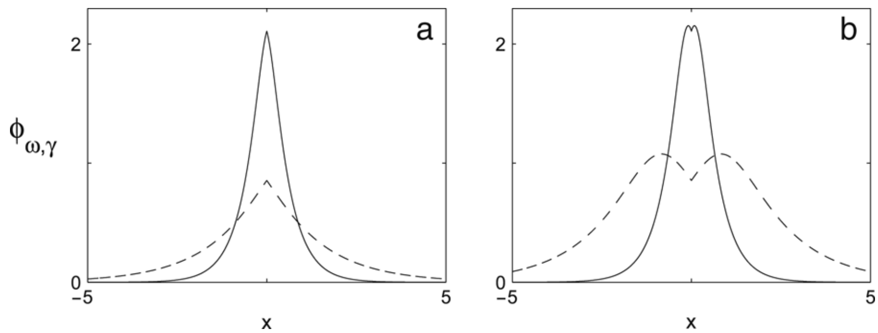


Fig. 1. $\varphi_{\omega,\gamma}$ as a function of $x$ for $\omega = 4$ (solid line) and $\omega = 0.5$ (dashed line). (a) $\gamma = 1$; (b) $\gamma = -1$. Here, $p = 4$.

**Proposition 1.** *Let $\omega > \gamma^2/4$. Then there exists a unique positive solution $\varphi_{\omega,\gamma}$ of (3). This solution is the unique positive minimizer of*

$$d(\omega) = \begin{cases} \inf\{S_{\omega,\gamma}(v);\, v \in H^1(\mathbb{R}) \setminus \{0\},\, I_{\omega,\gamma}(v) = 0\} & \text{if } \gamma \geqslant 0, \\ \inf\{S_{\omega,\gamma}(v);\, v \in H^1_{\mathrm{rad}}(\mathbb{R}) \setminus \{0\},\, I_{\omega,\gamma}(v) = 0\} & \text{if } \gamma < 0, \end{cases}$$

*where $S_{\omega,\gamma}$ and $I_{\omega,\gamma}$ are defined for $v \in H^1(\mathbb{R})$ by*

$$S_{\omega,\gamma}(v) = \frac{1}{2}\|\partial_x v\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1},$$
$$I_{\omega,\gamma}(v) = \|\partial_x v\|_2^2 + \omega\|v\|_2^2 - \gamma|v(0)|^2 - \|v\|_{p+1}^{p+1}.$$

*Furthermore, we have an explicit formula for $\varphi_{\omega,\gamma}$*

$$\varphi_{\omega,\gamma}(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}. \tag{4}$$

The dependence of $\varphi_{\omega,\gamma}$ on $\omega$ and $\gamma$ can be seen in Fig. 1. The parameter $\omega$ affects the width and height of $\varphi_{\omega,\gamma}$: the larger $\omega$ is, the narrower and higher $\varphi_{\omega,\gamma}$ becomes, and vice versa. The sign of $\gamma$ determines the profile of $\varphi_{\omega,\gamma}$ near $x = 0$: It has a "$\vee$" shape when $\gamma < 0$, and a "$\wedge$" shape when $\gamma > 0$.

**Remark 1.** (i) As was stated in [12, Remark 8 and Lemma 26], the set of solutions of (3)

$$\{v \in H^1(\mathbb{R}) \setminus \{0\} \text{ such that } -\partial_x^2 v + \omega v - \gamma v\delta - |v|^{p-1}v = 0\}$$

is explicitly given by $\{e^{i\theta}\varphi_{\omega,\gamma} \mid \theta \in \mathbb{R}\}$.

(ii) There is no nontrivial solution in $H^1(\mathbb{R})$ for $\omega \leqslant \gamma^2/4$.

The local well-posedness of the Cauchy problem for (2) is ensured by [7, Theorem 4.6.1]. Indeed, the operator $-\partial_x^2 - \gamma\delta$ is a self-adjoint operator on $L^2(\mathbb{R})$ (see [2, Chapter I.3.1] and Section 2 for details). Precisely, we have

**Proposition 2.** *For any $u_0 \in H^1(\mathbb{R})$, there exist $T_{u_0} > 0$ and a unique solution $u \in \mathcal{C}([0, T_{u_0}), H^1(\mathbb{R})) \cap \mathcal{C}^1([0, T_{u_0}), H^{-1}(\mathbb{R}))$ of (2) such that $\lim_{t\uparrow T_{u_0}} \|\partial_x u\|_2 = +\infty$ if $T_{u_0} < +\infty$. Furthermore, the conservation of energy and charge hold, that is, for any $t \in [0, T_{u_0})$ we have*

$$E(u(t)) = E(u_0), \tag{5}$$
$$\|u(t)\|_2^2 = \|u_0\|_2^2, \tag{6}$$

*where the energy $E$ is defined by*

$$E(v) = \frac{1}{2}\|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \quad \text{for } v \in H^1(\mathbb{R}).$$

(See also a verification of this proposition in [13, Proposition 1].)

**Remark 2.** From the uniqueness result of Proposition 2 it follows that if an initial data $u_0$ belongs to $H^1_{\mathrm{rad}}(\mathbb{R})$ then $u(t)$ also belongs to $H^1_{\mathrm{rad}}(\mathbb{R})$ for all $t \in [0, T_{u_0})$.

We consider the stability in the following sense.

**Definition 3.** Let $\varphi$ be a solution of (3). We say that the standing wave $u(x,t) = e^{i\omega t}\varphi(x)$ is *(orbitally) stable* in $H^1(\mathbb{R})$ (resp. $H^1_{\mathrm{rad}}(\mathbb{R})$) if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property : if $u_0 \in H^1(\mathbb{R})$ (resp. $H^1_{\mathrm{rad}}(\mathbb{R})$) satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R})} < \eta$, then the solution $u(t)$ of (2) with $u(0) = u_0$ exists for any $t \geq 0$ and

$$\sup_{t\in[0,+\infty)} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\varphi\|_{H^1(\mathbb{R})} < \varepsilon.$$

Otherwise, the standing wave $u(x,t) = e^{i\omega t}\varphi(x)$ is said to be *(orbitally) unstable* in $H^1(\mathbb{R})$ (resp. $H^1_{\mathrm{rad}}(\mathbb{R})$).

**Remark 4.** With this definition and Remark 2, it is clear that stability in $H^1(\mathbb{R})$ implies stability in $H^1_{\mathrm{rad}}(\mathbb{R})$ and conversely that instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ implies instability in $H^1(\mathbb{R})$.

For $\gamma = 0$, the orbital stability for (2) has been extensively studied (see [3,7,8,44,45] and the references therein). In particular, from [8] we know that $e^{i\omega t}\varphi_{\omega,0}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega > 0$ if $1 < p < 5$. On the other hand, it was shown that $e^{i\omega t}\varphi_{\omega,0}(x)$ is unstable in $H^1(\mathbb{R})$ for any $\omega > 0$ if $p \geqslant 5$ (see [3] for $p > 5$ and [45] for $p = 5$).

In [22], Goodman, Holmes and Weinstein focused on the special case $p = 3$, $\gamma > 0$ and proved that the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is orbitally stable in $H^1(\mathbb{R})$. For $\gamma > 0$, the orbital stability and instability were completely studied in [13]: the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega > \gamma^2/4$ if $1 < p \leqslant 5$, and if $p > 5$, there exists a critical frequency $\omega_1 > \gamma^2/4$ such that $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \omega_1)$ and unstable in $H^1(\mathbb{R})$ for any $\omega > \omega_1$.

For $\gamma < 0$, Fukuizumi and Jeanjean showed the following result in [12].

**Proposition 3.** *Let $\gamma < 0$ and $\omega > \gamma^2/4$.*

(i) *If $1 < p \leqslant 3$ the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1_{\mathrm{rad}}(\mathbb{R})$.*
(ii) *If $3 < p < 5$, there exists $\omega_2 > \gamma^2/4$ such that the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ when $\omega > \omega_2$ and unstable in $H^1(\mathbb{R})$ when $\gamma^2/4 < \omega < \omega_2$.*
(iii) *If $p \geqslant 5$, then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is unstable in $H^1(\mathbb{R})$.*

*The critical frequency $\omega_2$ is given by*

$$\frac{J(\omega_2)(p-5)}{p-1} = \frac{\gamma}{2\sqrt{\omega_2}}\left(1 - \frac{\gamma^2}{4\omega_2}\right)^{-(p-3)/(p-1)},$$

$$J(\omega_2) = \int_{A(\omega_2,\gamma)}^{+\infty} \operatorname{sech}^{4/(p-1)}(y)\,dy, \qquad A(\omega_2,\gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_2}}\right).$$

## 3. Summary of results

The results of stability of [12] recalled in Proposition 3 assert only on stability under radial perturbations. Furthermore, the nature of instability is not revealed. In this paper, we prove that there is instability in the whole space when stability holds under radial perturbation (see Theorem 4), and that, when $p \geqslant 5$, the instability established in [12] is strong instability (see Definition 6 and Theorem 5).

Our first main result is the following.

**Theorem 4.** *Let $\gamma < 0$ and $\omega > \gamma^2/4$.*

(i) *If $1 < p \leqslant 3$ the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is unstable in $H^1(\mathbb{R})$.*
(ii) *If $3 < p < 5$, the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is unstable in $H^1(\mathbb{R})$ for any $\omega > \omega_2$, where $\omega_2$ is defined in Proposition 3.*

As in [12,13], our stability analysis relies on the abstract theory by Grillakis, Shatah and Strauss [23,24] for a Hamiltonian system which is invariant under a one-parameter group of operators. In trying to follow this approach the main point is to check the following two conditions:

(1) The *slope condition*: The sign of $\partial_\omega\|\varphi_{\omega,\gamma}\|_2^2$.
(2) The *spectral condition:* The number of negative eigenvalues of the linearized operator

$$L^\gamma_{1,\omega}v = -\partial_x^2 v + \omega v - \gamma\delta v - p\varphi_{\omega,\gamma}^{p-1}v.$$

We refer the reader to Section 4 for the precise criterion and a detailed explanation on how $L^\gamma_{1,\omega}$ appears in this stability analysis. Making use of the explicit form (4) for $\varphi_{\omega,\gamma}$, the sign of $\partial_\omega\|\varphi_{\omega,\gamma}\|_2^2$ was explicitly computed in [12,13].

In [12], a spectral analysis is performed to count the number of negative eigenvalues, and it is proved that the number of negative eigenvalues of $L^\gamma_{1,\omega}$ in $H^1_{\mathrm{rad}}(\mathbb{R})$ is one. This spectral analysis of $L^\gamma_{1,\omega}$ relies on the variational characterization of $\varphi_{\omega,\gamma}$. However, since $\varphi_{\omega,\gamma}$ is a minimizer only in the space of radial (even) functions $H^1_{\mathrm{rad}}(\mathbb{R})$, the result on the spectrum holds only in $H^1_{\mathrm{rad}}(\mathbb{R})$, namely for even eigenfunctions. Therefore the number of negative eigenvalues is known only for $L^\gamma_{1,\omega}$ considered in $H^1_{\mathrm{rad}}(\mathbb{R})$. With this approach, it is not possible to see whether other negative eigenvalues appear when the problem is considered on the whole space $H^1(\mathbb{R})$.

To overcome this difficulty, we develop a perturbation method. In the case $\gamma = 0$, the spectrum of $L^0_{1,\omega}$ is well known by the work of Weinstein [46] (see Lemma 14): there is only one negative eigenvalue, and 0 is a simple isolated eigenvalue (to see that, one proves that the kernel of $L^0_{1,\omega}$ is spanned by $\partial_x\varphi_{\omega,0}$, that $\partial_x\varphi_{\omega,0}$ has only one zero, and applies the Sturm Oscillation Theorem). When $\gamma$ is small, $L^\gamma_{1,\omega}$ can be considered as a holomorphic perturbation of $L^0_{1,\omega}$. Using the theory of holomorphic perturbations for linear operators, we prove that the spectrum of $L^\gamma_{1,\omega}$ depends holomorphically on the spectrum of $L^0_{1,\omega}$ (see Lemma 15). Then the use of Taylor expansion for the second eigenvalue of $L^\gamma_{1,\omega}$ allows us to get the sign of the second eigenvalue when $\gamma$ is small (see Lemma 16). A continuity argument combined with the fact that if $\gamma \neq 0$ the nullspace of $L^\gamma_{1,\omega}$ is zero extends the result to all $\gamma \in \mathbb{R}$ (see the proof of Lemma 12). See Section 4.2 for details. We will see that there are two negative eigenvalues of $L^\gamma_{1,\omega}$ in $H^1(\mathbb{R})$ if $\gamma < 0$.

**Remark 5.** (i) Our method can be applied as well in $H^1(\mathbb{R})$ or in $H^1_{\mathrm{rad}}(\mathbb{R})$, and for $\gamma$ negative or positive (see Sections 4.3 and 4.4). Thus we can give another proof of the result of [13] in the case $\gamma > 0$ and of Proposition 3.

(ii) The study of the spectrum of linearized operators is often a central point when one wants to use the abstract theory of [23,24]. See [14,17–19,28] among many others for related results.

The results of instability given in Theorem 4 and Proposition 3 show only that a certain solution which starts close to $\varphi_{\omega,\gamma}$ will exit from a tubular neighborhood of the orbit of the standing wave in finite time. However, as this might be of importance for the applications, we want to understand further the nature of instability. For that, we recall the concept of strong instability.

**Definition 6.** A standing wave $e^{i\omega t}\varphi(x)$ of (2) is said to be *strongly unstable in* $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exist $u_\varepsilon \in H^1(\mathbb{R})$ with $\|u_\varepsilon - \varphi\|_{H^1(\mathbb{R})} < \varepsilon$ and $T_{u_\varepsilon} < +\infty$ such that $\lim_{t\uparrow T_{u_\varepsilon}} \|\partial_x u(t)\|_2 = +\infty$, where $u(t)$ is the solution of (2) with $u(0) = u_\varepsilon$.

Our second main result is the following.

**Theorem 5.** *Let $\gamma \leqslant 0$, $\omega > \gamma^2/4$ and $p \geqslant 5$. Then the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is strongly unstable in $H^1(\mathbb{R})$.*

Whether the perturbed standing wave blows up or not depends on the perturbation. Indeed, in Remark 10 we define an invariant set of solutions and show that if we consider an initial data in this set, then the solution exists globally even when the standing wave $e^{i\omega t}\varphi_{\omega,\gamma}(x)$ is strongly unstable.

We also point out that when $1 < p < 5$, it is easy to prove using the conservation laws and the Gagliardo–Nirenberg inequality that the Cauchy problem in $H^1(\mathbb{R})$ associated with (2) is globally well posed. Accordingly, even if the standing wave may be unstable when $1 < p < 5$ (see Theorem 4), a strong instability cannot occur.

As in [3,45], which deal with the classical case $\gamma = 0$, we use the virial identity for the proof of Theorem 5. However, even if the formal calculations are similar to those of the case $\gamma = 0$, a rigorous proof of the virial theorem does not immediately follow from the approximation by regular solutions (e.g. see [7, Proposition 6.4.2], or [20]). Indeed, the argument in [7] relies on the $H^2(\mathbb{R})$ regularity of the solutions of (2). Because of the defect term, we do not know if this $H^2(\mathbb{R})$ regularity still holds when $\gamma \neq 0$. Thus we need another approach. We approximate the solutions of (2) by solutions of the same equation where the defect is approximated by a Gaussian potential for which it is easy to have the virial theorem. Then we pass to the limit in the virial identity to obtain:

**Proposition 6.** *Let $u_0 \in H^1(\mathbb{R})$ such that $xu_0 \in L^2(\mathbb{R})$ and $u(t)$ be the solution of (2). Then the function $f : t \mapsto \|xu(t)\|_2^2$ is $\mathcal{C}^2$ and*

$$\partial_t f(t) = 4\mathrm{Im}\int_{\mathbb{R}} \bar{u}x\partial_x u\,\mathrm{d}x, \tag{7}$$

$$\partial_t^2 f(t) = 8Q_\gamma(u(t)), \tag{8}$$

*where $Q_\gamma$ is defined for $v \in H^1(\mathbb{R})$ by*

$$Q_\gamma(v) = \|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Even if we benefit from the virial identity, the proofs given in [3,45] for the case $\gamma = 0$ do not apply to the case $\gamma < 0$. For example, the method of Weinstein [45] in the case $p = 5$ requires in a crucial way an equality between $2E$ and $Q$ which does not hold anymore when $\gamma < 0$. Moreover, the heart of the proof of [3] consists in minimizing the functional $S_{\omega,\gamma}$ on the constraint $Q_\gamma(v) = 0$, but the standard variational methods to prove such results are not so easily applied to the case of $\gamma \neq 0$. To get over these difficulties we introduce an approach based on a minimization problem involving two constraints. Using this minimization problem, we identify some invariant properties under the flow of (2). The combination of these invariant properties and the conservation of energy and charge allows us to prove strong instability. We mention that some related techniques have been introduced in [33–35,37,47]. In conclusion, we can give a simpler method to prove Theorem 5 than that of [3] even though we have a term of delta potential.

**Remark 7.** The case $\gamma < 0$, $\omega = \omega_2$ and $3 < p < 5$ cannot be treated with our approach and is left open (see Remark 8). In light of Theorem 4, we believe that the standing wave is unstable in this case, at least in $H^1(\mathbb{R})$ (see also [12, Remark 12]). When $\gamma > 0$, the case $\omega = \omega_1$ and $p > 5$ is also open (see [13, Remark 1.5]).

Let us summarize the previously known and our new rigorous results on stability in (2):

(i) For both positive and negative $\gamma$, there is always only one negative eigenvalue of linearized operator in $H^1_{\mathrm{rad}}(\mathbb{R})$ ([12], Section 2.5). Hence, the standing wave is stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ if the slope is positive, and unstable if the slope is negative.

(ii) $\gamma > 0$. In this case the number of the negative eigenvalues of linearized operator is always one in $H^1(\mathbb{R})$. Stability is determined by the slope condition, and the standing wave is stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ if and only if it is stable in $H^1(\mathbb{R})$. Specifically ([12,13], Section 2.4),
  (a) $1 < p \leq 5$: Stability in $H^1(\mathbb{R})$ for any $\omega > \gamma^2/4$.
  (b) $5 < p$: Stability in $H^1(\mathbb{R})$ for $\gamma^2/4 < \omega < \omega_1$, instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ for $\omega > \omega_1$.

(iii) $\gamma < 0$. In this case the number of negative eigenvalues is always two (Lemma 12) and all standing waves are unstable in $H^1(\mathbb{R})$ (Theorems 4 and 5). Stability in $H^1_{\mathrm{rad}}(\mathbb{R})$ is determined by the slope condition and is as follows [12]:
  (a) $1 < p \leq 3$: Stability in $H^1_{\mathrm{rad}}(\mathbb{R})$ for any $\omega > \gamma^2/4$.
  (b) $3 < p < 5$: Stability in $H^1_{\mathrm{rad}}(\mathbb{R})$ for $\omega > \omega_2$, instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ for $\gamma^2/4 < \omega < \omega_2$.
  (c) $5 \leq p$: Strong instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ (and in $H^1(\mathbb{R})$) for any $\gamma^2/4 < \omega$ (Theorem 5).

There are, however, several important questions which are still open, and which we explore using numerical simulations. Our simulations suggest the following:

(i) Although an attractive defect ($\gamma > 0$) stabilizes the standing waves in the critical case ($p = 5$), their stability is weaker than in the subcritical case, in particular for $0 < \gamma \ll 1$.

(ii) Theorem 5 shows that instability occurs by blowup when $\gamma < 0$ and $p \geqslant 5$. In all other cases, however, it remains to understand the nature of instability. Our simulations suggest the following:
  (a) When $\gamma > 0$, $p > 5$, and $\omega > \omega_1$, instability can occur by blowup.
  (b) When $\gamma < 0$, $3 < p < 5$, and $\gamma^2/4 < \omega < \omega_2$, the instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ is a *finite-width instability*, i.e., the solution initially narrows down along a curve $\phi_{\omega*(t),\gamma}$, where $\omega^*(t)$ can be defined by the relation
  $$\max_x \phi_{\omega^*(t),\gamma}(x) = \max_x |u(x,t)|.$$
  As the solution narrows down, $\omega^*(t)$ increases and crosses from the unstable region $\omega < \omega_2$ to the stable region $\omega > \omega_2$. Subsequently, collapse is arrested at some finite width.
  (c) When $\gamma < 0$, the standing waves undergo a *drift instability*, away from the (repulsive) defect, sometimes in combination with finite-width or blowup instability. Specifically,
    (c.i) When $1 < p \leq 3$ and when $3 < p < 5$ and $\omega > \omega_2$ (i.e., when the standing waves are stable in $H^1_{\mathrm{rad}}(\mathbb{R})$), the standing waves undergo a *drift instability*.
    (c.ii) When $3 < p < 5$ and $\gamma^2/4 < \omega < \omega_2$, the instability in $H^1(\mathbb{R})$ is a combination of a drift instability and a finite-width instability.
    (c.iii) When $p \geq 5$, the instability in $H^1(\mathbb{R})$ is a combination of a drift instability and a blowup instability.

(iii) Although when $p = 5$ and $\gamma > 0$, and when $p > 5$, $\gamma > 0$, and $\gamma^2/4 < \omega < \omega_1$ the standing wave is stable, it can collapse under a sufficiently large perturbation.

We note that all of the above hold more generally for NLS equations with a nonlinear potential [14,15] and for narrow solitons of a linear potential [43].

The rest of the paper is organized as follows. In Section 4, we prove Theorem 4 and explain how our method allows us to recover the results of [12,13]. In Section 5, we establish Theorem 5. Numerical results are given in Section 6.

Throughout the paper the letter $C$ will denote various positive constants whose exact values may change from line to line but are not essential for the analysis of the problem.

## 4. Instability with respect to nonradial perturbations

We use the general theory of Grillakis, Shatah and Strauss [24] to prove Theorem 4.

First, we explain how we derive a criterion for stability or instability for our case from the theory of Grillakis, Shatah and Strauss. In our case, it is clear that *Assumption* 1 and *Assumption* 2 of [24] are satisfied. The last assumption, *Assumption* 3, should be checked. We consider the sesquilinear form $S''_{\omega,\gamma}(\varphi_{\omega,\gamma}) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ as a linear operator $H^\gamma_\omega : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$. The spectrum of $H^\gamma_\omega$ is the set $\{\lambda \in \mathbb{C}$ such that $H^\gamma_\omega - \lambda I$ is not invertible$\}$, where $I$ denotes the usual $H^1(\mathbb{R}) - H^{-1}(\mathbb{R})$ isomorphism, and we denote by $n(H^\gamma_\omega)$ the number of negative eigenvalues of $H^\gamma_\omega$. Having established the assumptions of [24], the next proposition follows from [24, Instability Theorem and Stability Theorem].

**Proposition 7.** (1) *The standing wave* $e^{i\omega_0 t}\varphi_{\omega_0,\gamma}(x)$ *is unstable if the integer* $n(H^\gamma_{\omega_0}) - p(d''(\omega_0))$ *is odd, where*

$$p(d''(\omega_0)) = \begin{cases} 1 & \text{if } \partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0 \text{ at } \omega = \omega_0, \\ 0 & \text{if } \partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0 \text{ at } \omega = \omega_0. \end{cases}$$

(2) *The standing wave* $e^{i\omega_0 t}\varphi_{\omega_0,\gamma}(x)$ *is stable if* $n(H^\gamma_{\omega_0}) - p(d''(\omega_0)) = 0$.

Let us now consider the case $\gamma < 0$. It was proved in [12] that

**Lemma 8.** *Let* $\gamma < 0$ *and* $\omega > \gamma^2/4$. *We have :*

(i) *If* $1 < p \leqslant 3$ *and* $\omega > \gamma^2/4$ *then* $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,
(ii) *If* $3 < p < 5$ *and* $\omega > \omega_2$ *then* $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,
(iii) *If* $3 < p < 5$ *and* $\gamma^2/4 < \omega < \omega_2$ *then* $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$,
(iv) *If* $p \geqslant 5$ *and* $\omega > \gamma^2/4$ *then* $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$.

Thus Theorem 4 follows from Proposition 7, Lemma 8 and

**Lemma 9.** *If* $\gamma < 0$, *then* $n(H^\gamma_\omega) = 2$.

**Remark 8.** (1) Let $\gamma < 0$. In the cases $3 < p < 5$ and $\omega < \omega_2$ or $p \geq 5$ it was proved in [12] that $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$. From Lemma 9, we know that the number of negative eigenvalues of $H^\gamma_\omega$ is $n(H^\gamma_\omega) = 2$ when $H^\gamma_\omega$ is considered on the whole space $H^1(\mathbb{R})$. Therefore $n(H^\gamma_\omega) - p(d''(\omega)) = 2$ and this corresponds to a case where the assumption of [24] may not be applied. However, if we consider $H^\gamma_\omega$ in $H^1_{\mathrm{rad}}(\mathbb{R})$, then it follows from [12] that $n(H^\gamma_\omega) = 1$, thus $n(H^\gamma_\omega) - p(d''(\omega)) = 1$. Then, we can apply Proposition 7 to this case and it allows us to conclude instability in $H^1_{\mathrm{rad}}(\mathbb{R})$ (as it was done in [12]). But, with Remark 4, we can conclude that instability holds on the whole space $H^1(\mathbb{R})$.
(2) Note that the case $\omega = \omega_2$ corresponds to $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 = 0$ $(3 < p < 5)$ and will not be treated here. In view of Theorem 4, we believe that the standing wave is unstable in this case, at least in $H^1(\mathbb{R})$.

We divide the rest of this section into four parts. In Section 4.1 we introduce the general setting to perform our proof, and we study whether *Assumption* 3 of [24] is satisfied. Lemma 9 will be proved in Section 4.2. Finally, we discuss the positive case and the radial case in Sections 4.3 and 4.4.

### *4.1. Setting for the spectral problem*

To express $H^\gamma_\omega$, it is convenient to split $u$ into real and imaginary parts: for $u \in H^1(\mathbb{R},\mathbb{C})$ we write $u = u_1 + iu_2$ where $u_1 = \mathrm{Re}(u) \in H^1(\mathbb{R},\mathbb{R})$ and $u_2 = \mathrm{Im}(u) \in H^1(\mathbb{R},\mathbb{R})$. Now we set

$$H^\gamma_\omega u = L^\gamma_{1,\omega} u_1 + i L^\gamma_{2,\omega} u_2,$$

where the operators $L^\gamma_{1,\omega}, L^\gamma_{2,\omega} : H^1(\mathbb{R},\mathbb{R}) \to H^{-1}(\mathbb{R})$ are defined for $v \in H^1(\mathbb{R})$ by

$$L^\gamma_{1,\omega} v = -\partial_x^2 v + \omega v - \gamma v\delta - p\varphi^{p-1}_{\omega,\gamma} v,$$
$$L^\gamma_{2,\omega} v = -\partial_x^2 v + \omega v - \gamma v\delta - \varphi^{p-1}_{\omega,\gamma} v.$$

When we work with $L^\gamma_{1,\omega}, L^\gamma_{2,\omega}$, the functions considered are understood to be real valued.

For the spectral study of $H^\gamma_\omega$, it is convenient to view $H^\gamma_\omega$ as an unbounded operator on $L^2(\mathbb{R})$, thus we rewrite our spectral problem in this setting. First, we redefine the two operators $L^\gamma_{1,\omega}$ and $L^\gamma_{2,\omega}$ as unbounded operators on $L^2(\mathbb{R})$. We begin by considering the bilinear forms on $H^1(\mathbb{R})$ associated with $L^\gamma_{1,\omega}$ and $L^\gamma_{2,\omega}$ by setting for $v, w \in H^1(\mathbb{R})$

$$B^\gamma_{1,\omega}(v,w) := \langle L^\gamma_{1,\omega} v, w\rangle \quad \text{and} \quad B^\gamma_{2,\omega}(v,w) := \langle L^\gamma_{2,\omega} v, w\rangle,$$

which are explicitly given by

$$\begin{aligned} B^{\gamma}_{1,\omega}(v,w) &= \int_{\mathbb{R}} \partial_x v \partial_x w \mathrm{d}x + \omega \int_{\mathbb{R}} vw \mathrm{d}x - \gamma v(0)w(0) - \int_{\mathbb{R}} p\varphi^{p-1}_{\omega,\gamma} vw\mathrm{d}x, \\ B^{\gamma}_{2,\omega}(v,w) &= \int_{\mathbb{R}} \partial_x v \partial_x w \mathrm{d}x + \omega \int_{\mathbb{R}} vw \mathrm{d}x - \gamma v(0)w(0) - \int_{\mathbb{R}} \varphi^{p-1}_{\omega,\gamma} vw\mathrm{d}x. \end{aligned} \tag{9}$$

Let us now consider $B^{\gamma}_{1,\omega}$ and $B^{\gamma}_{2,\omega}$ as bilinear forms on $L^2(\mathbb{R})$ with domain $D(B^{\gamma}_{1,\omega}) = D(B^{\gamma}_{2,\omega}) := H^1(\mathbb{R})$. It is clear that these forms are bounded from below and closed. Then the theory of representation of forms by operators (see [30, VI. Section 2.1]) implies that we define two self-adjoint operators $\widetilde{L^{\gamma}_{1,\omega}} : D(\widetilde{L^{\gamma}_{1,\omega}}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\widetilde{L^{\gamma}_{2,\omega}} : D(\widetilde{L^{\gamma}_{2,\omega}}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by setting

$$\begin{aligned} D(\widetilde{L^{\gamma}_{1,\omega}}) &:= \{v \in H^1(\mathbb{R}) | \exists w \in L^2(\mathbb{R}) \text{ s.t. } \forall z \in H^1(\mathbb{R}),\ B^{\gamma}_{1,\omega}(v,z) = (w,z)_2\}, \\ D(\widetilde{L^{\gamma}_{2,\omega}}) &:= \{v \in H^1(\mathbb{R}) | \exists w \in L^2(\mathbb{R}) \text{ s.t. } \forall z \in H^1(\mathbb{R}),\ B^{\gamma}_{2,\omega}(v,z) = (w,z)_2\} \end{aligned}$$

and setting for $v \in D(\widetilde{L^{\gamma}_{1,\omega}})$ (resp. $v \in D(\widetilde{L^{\gamma}_{2,\omega}})$) that $\widetilde{L^{\gamma}_{1,\omega}} v := w$ (resp. $\widetilde{L^{\gamma}_{2,\omega}} v := w$), where $w$ is the (unique) function of $L^2(\mathbb{R})$ which satisfies $B^{\gamma}_{1,\omega}(v,z) = (w,z)_2$ (resp. $B^{\gamma}_{2,\omega}(v,z) = (w,z)_2$) for all $z \in H^1(\mathbb{R})$.

For notational simplicity, we drop the tilde over $\widetilde{L^{\gamma}_{1,\omega}}$ and $\widetilde{L^{\gamma}_{2,\omega}}$. It turns out that we are able to describe explicitly $L^{\gamma}_{1,\omega}$ and $L^{\gamma}_{2,\omega}$.

**Lemma 10.** *The domain of $L^{\gamma}_{1,\omega}$ and of $L^{\gamma}_{2,\omega}$ in $L^2(\mathbb{R})$ is*

$$D_{\gamma} = \{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\});\ \partial_x v(0^+) - \partial_x v(0^-) = -\gamma v(0)\}$$

*and for $v \in D_{\gamma}$ the operators are given by*

$$\begin{aligned} L^{\gamma}_{1,\omega} v &= -\partial_x^2 v + \omega v - p\varphi^{p-1}_{\omega,\gamma} v, \\ L^{\gamma}_{2,\omega} v &= -\partial_x^2 v + \omega v - \varphi^{p-1}_{\omega,\gamma} v. \end{aligned} \tag{10}$$

The proof of this lemma is given in Appendix A. We conclude this subsection mentioning some basic properties of the spectrum of $H^{\gamma}_{\omega}$. To be precise, checking [24, Assumption 3] is equivalent to checking the following lemma.

**Lemma 11.** *Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega > \gamma^2/4$.*

(i) *The operator $H^{\gamma}_{\omega}$ has only a finite number of negative eigenvalues,*
(ii) *The kernel of $H^{\gamma}_{\omega}$ is* $\mathrm{span}\{\mathrm{i}\varphi_{\omega,\gamma}\}$,
(iii) *The rest of the spectrum of $H^{\gamma}_{\omega}$ is positive and bounded away from 0.*

Our proof of Lemma 11 borrows some elements of [12]. In particular, (ii) in Lemma 11 is shown in [12, Lemma 28 and Lemma 31]. For the sake of completeness, we provide a proof in Appendix B.

### 4.2. *Count of the number of negative eigenvalues*

In this subsection, we prove Lemma 9. First, we remark that, as was shown in the proof of Lemma 11, 0 is the first eigenvalue of $L^{\gamma}_{2,\omega}$. Thus $n(H^{\gamma}_{\omega}) = n(L^{\gamma}_{1,\omega})$, where $n(L^{\gamma}_{1,\omega})$ is the number of negative eigenvalues of $L^{\gamma}_{1,\omega}$. Therefore, Lemma 9 follows from

**Lemma 12.** *Let $\gamma < 0$ and $\omega > \gamma^2/4$. Then $n(L^{\gamma}_{1,\omega}) = 2$.*

Our proof of Lemma 12 is divided into two steps. First, we use a perturbative approach to prove that, if $\gamma$ is close to 0 and negative, $L^{\gamma}_{1,\omega}$ has two negative eigenvalues (Lemma 16). To do this, we have to ensure that the eigenvalues and the eigenvectors are regular enough with respect to $\gamma$ (Lemma 15) to make use of the Taylor formula. This follows from the use of the analytic perturbation theory of operators (see [30,38]). The second step consists in extending the result of the first step to any values of $\gamma < 0$. Our argument relies on the continuity of the spectral projections with respect to $\gamma$ and it is crucial, as it was proved in Lemma 11, that 0 cannot be an eigenvalue of $L^{\gamma}_{1,\omega}$ (see [17,18] for related arguments).

We fix $\omega > \gamma^2/4$. For the sake of simplicity we denote $L^{\gamma}_{1,\omega}$ by $L^{\gamma}_1$ and $\varphi_{\omega,\gamma}$ by $\varphi_{\gamma}$, and so on in this Section 2. The following lemma verifies the holomorphicity of the operator $L^{\gamma}_{1,\omega}$, see the proof in Appendix B.

**Lemma 13.** *As a function of $\gamma$, $(L^{\gamma}_1)$ is a real-holomorphic family of self-adjoint operators (of type (B) in the sense of Kato).*

The following classical result of Weinstein [46] gives a precise description of the spectrum of the operator we want to perturb.

**Lemma 14.** *The operator $L_1^0$ has exactly one negative simple isolated first eigenvalue. The second eigenvalue is* 0*, and it is simple and isolated. The nullspace is* $\mathrm{span}\{\partial_x \varphi_0\}$*, and the rest of the spectrum is positive.*

Combining Lemmas 13 and 14, we can apply the theory of analytic perturbations for linear operators (see [30, VII. Section 1.3]) to get the following lemma. Actually, the perturbed eigenvalues are holomorphic since they are simple.

**Lemma 15.** *There exist $\gamma_0 > 0$ and two functions $\lambda : (-\gamma_0, \gamma_0) \mapsto \mathbb{R}$ and $f : (-\gamma_0, \gamma_0) \mapsto L^2(\mathbb{R})$ such that*

(i) $\lambda(0) = 0$ *and* $f(0) = \partial_x \varphi_0$,
(ii) *For all $\gamma \in (-\gamma_0, \gamma_0)$, $\lambda(\gamma)$ is the simple isolated second eigenvalue of $L_1^\gamma$ and $f(\gamma)$ is an associated eigenvector,*
(iii) *$\lambda(\gamma)$ and $f(\gamma)$ are holomorphic in $(-\gamma_0, \gamma_0)$.*

*Furthermore, $\gamma_0 > 0$ can be chosen small enough to ensure that, except the two first eigenvalues, the spectrum of $L_1^\gamma$ is positive.*

Now we investigate how the perturbed second eigenvalue moves depending on the sign of $\gamma$.

**Lemma 16.** *There exists $0 < \gamma_1 < \gamma_0$ such that $\lambda(\gamma) < 0$ for any $-\gamma_1 < \gamma < 0$ and $\lambda(\gamma) > 0$ for any $0 < \gamma < \gamma_1$.*

**Proof of Lemma 16.** We develop the functions $\lambda(\gamma)$ and $f(\gamma)$ of Lemma 15. There exist $\lambda_0 \in \mathbb{R}$ and $f_0 \in L^2(\mathbb{R})$ such that for $\gamma$ close to 0 we have

$$\lambda(\gamma) = \gamma\lambda_0 + O(\gamma^2), \tag{11}$$

$$f(\gamma) = \partial_x\varphi_0 + \gamma f_0 + O(\gamma^2). \tag{12}$$

From the explicit expression (4) of $\varphi_\gamma$, we deduce that there exists $g_0 \in H^1(\mathbb{R})$ such that for $\gamma$ close to 0 we have

$$\varphi_\gamma = \varphi_0 + \gamma g_0 + O(\gamma^2). \tag{13}$$

Furthermore, using (13) to substitute into (3) and differentiating (3) with respect to $\gamma$, we obtain

$$\langle L_1^0 g_0, \psi\rangle = \varphi_0(0)\psi(0) + O(\gamma), \tag{14}$$

for any $\psi \in H^1(\mathbb{R})$.

To develop $\lambda_0$ with respect to $\gamma$, we compute $(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2$ in two different ways.

On the one hand, using $L_1^\gamma f(\gamma) = \lambda(\gamma) f(\gamma)$, (11) and (12) leads us to

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = \lambda_0\gamma\|\partial_x\varphi_0\|_2^2 + O(\gamma^2). \tag{15}$$

On the other hand, since $L_1^\gamma$ is self-adjoint, we get

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = (f(\gamma), L_1^\gamma\partial_x\varphi_0)_2. \tag{16}$$

Here we note that $\partial_x\varphi_0 \in D(L_1^\gamma)$: indeed, $\partial_x\varphi_0 \in H^2(\mathbb{R})$ and $\partial_x\varphi_0(0) = 0$. We compute the right-hand side of (16). We use (10), $L_1^0\partial_x\varphi_0 = 0$, and (13) to obtain

$$\begin{aligned} L_1^\gamma\partial_x\varphi_0 &= p(\varphi_0^{p-1} - \varphi_\gamma^{p-1})\partial_x\varphi_0, \\ &= -\gamma p(p-1)\varphi_0^{p-2} g_0\partial_x\varphi_0 + O(\gamma^2). \end{aligned} \tag{17}$$

Hence, it follows from (12) that

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = -(\partial_x\varphi_0, \gamma g_0 p(p-1)\varphi_0^{p-2}\partial_x\varphi_0)_2 + O(\gamma^2). \tag{18}$$

Now, as was remarked in [14, Lemma 28], it is easy to see that using (3) with $\gamma = 0$ we get

$$L_1^0(\varphi_0 - \varphi_0^{p-1}) = p(p-1)\varphi_0^{p-2}\partial_x\varphi_0^2, \tag{19}$$

which combined with (18) gives

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = -\gamma\langle L_1^0 g_0, \varphi_0 - \varphi_0^p\rangle + O(\gamma^2). \tag{20}$$

Finally, with (14) we obtain from (20)

$$(L_1^\gamma f(\gamma), \partial_x\varphi_0)_2 = -\gamma(\varphi_0(0)^2 - \varphi_0(0)^{p+1}) + O(\gamma^2). \tag{21}$$

Combining (21) and (15) we obtain

$$\lambda_0 = -\frac{\varphi_0(0)^2 - \varphi_0(0)^{p+1}}{\|\partial_x\varphi_0\|_2^2} + O(\gamma).$$

It follows that $\lambda_0$ is positive for sufficiently small $|\gamma|$, which in view of (11) ends the proof. □

We are now in a position to prove Lemma 12.

**Proof of Lemma 12.** Let $\gamma_\infty$ be defined by

$$\gamma_\infty = \inf\{\tilde{\gamma} < 0;\ L_1^\gamma \text{ has exactly two negative eigenvalues for all } \gamma \in (\tilde{\gamma}, 0]\}.$$

From Lemma 16, we know that $\gamma_\infty$ is well defined and $\gamma_\infty \in [-\infty, 0)$. Arguing by contradiction, we suppose $\gamma_\infty > -\infty$.

Let $N$ be the number of negative eigenvalues of $L_1^{\gamma_\infty}$. Denote the first eigenvalue of $L_1^{\gamma_\infty}$ by $\Lambda_{\gamma_\infty}$. Let $\Gamma$ be defined by

$$\Gamma = \{z \in \mathbb{C};\ z = z_1 + \mathrm{i}z_2,\ (z_1, z_2) \in [-b, 0] \times [-a, a], \text{ for some } a > 0, b > |\Lambda_{\gamma_\infty}|\}.$$

From Lemma 11, we know that $L_1^{\gamma_\infty}$ does not admit zero as an eigenvalue. Thus $\Gamma$ defines a contour in $\mathbb{C}$ of the segment $[\Lambda_{\gamma_\infty}, 0]$ containing no positive part of the spectrum of $L_1^{\gamma_\infty}$, and without any intersection with the spectrum of $L_1^{\gamma_\infty}$. It is easily seen (for example, along the lines of the proof of [30, Theorem VII-1.7]) that there exists a small $\gamma_* > 0$ such that for any $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$, we can define a holomorphic projection on the negative part of the spectrum of $L_1^\gamma$ contained in $\Gamma$ by

$$\Pi(\gamma) = \frac{-1}{2\pi \mathrm{i}} \int_\Gamma (L_1^\gamma - z)^{-1} \mathrm{d}z.$$

Let us insist on the fact that we can choose $\Gamma$ independently of the parameter $\gamma$ because 0 is not an eigenvalue of $L_1^\gamma$ for all $\gamma$.

Since $\Pi$ is holomorphic, $\Pi$ is continuous in $\gamma$, then by a classical connectedness argument (for example, see [30, Lemma I-4.10]), we know that $\dim(\mathrm{Ran}\ \Pi(\gamma)) = N$ for any $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$. Furthermore, $N$ is exactly the number of negative eigenvalues of $L_1^\gamma$ when $\gamma \in [\gamma_\infty - \gamma_*, \gamma_\infty + \gamma_*]$: indeed, if $L_1^\gamma$ has a negative eigenvalue outside of $\Gamma$ it suffices to enlarge $\Gamma$ (i.e., enlarge $b$) until it contains this eigenvalue to raise a contradiction since then $L_1^{\gamma_\infty}$ would have, at least, $N + 1$ eigenvalues. Now by the definition of $\gamma_\infty$, $L_1^{\gamma_\infty + \gamma_*}$ has two negative eigenvalues and thus we see that $L_1^\gamma$ has two negative eigenvalues for all $\gamma \in [\gamma_\infty - \gamma^*, 0[$ contradicting the definition of $\gamma_\infty$.

Therefore $\gamma_\infty = -\infty$. □

**Remark 9.** In [12, Lemma 32], the authors proved that there are *at most* two negative eigenvalues of $L_1^\gamma$ in $H^1(\mathbb{R})$ using variational methods. In our present proof, we can directly show that there are exactly two negative eigenvalues without such variational techniques.

### *4.3. The case $\gamma > 0$*

The proof of Lemma 12 can be easily adapted to the case $\gamma > 0$, and with Lemma 16 we can infer that $L_1^\gamma$ has only one simple negative eigenvalue when $\gamma > 0$. Since $n(H^\gamma) = n(L_1^\gamma)$, it follows that (in the following Lemmas 17 and 18 and Proposition 19, there is no omission of parameter $\omega$ to understand the dependence clearly)

**Lemma 17.** *Let $\gamma > 0$ and $\omega > \gamma^2/4$. Then the operator $H_\omega^\gamma$ has only one negative eigenvalue, that is $n(H_\omega^\gamma) = 1$.*

When $\gamma > 0$, the sign of $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2$ was computed in [13]. Precisely:

**Lemma 18.** *Let $\gamma > 0$ and $\omega > \gamma^2/4$. We have :*

(i) *If $1 < p \leqslant 5$ and $\omega > \gamma^2/4$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,*
(ii) *If $p > 5$ and $\gamma^2/4 < \omega < \omega_1$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 > 0$,*
(iii) *If $p > 5$ and $\omega > \omega_1$ then $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$.*

*Here $\omega_1$ is defined as follows:*

$$\frac{p-5}{p-1} J(\omega_1) = \frac{\gamma}{2\sqrt{\omega_1}} \left(1 - \frac{\gamma^2}{4\omega_1}\right)^{-(p-3)/(p-1)},$$

$$J(\omega_1) = \int_{A(\omega_1,\gamma)}^\infty \mathrm{sech}^{4/(p-1)} y \mathrm{d}y, \qquad A(\omega_1, \gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_1}}\right).$$

Then, using Lemmas 17 and 18 and Proposition 7, we can give an alternative proof of [13, Theorem 1] (see also [12, Remark 33]). Precisely, we obtain:

**Proposition 19.** *Let $\gamma > 0$ and $\omega > \gamma^2/4$.*

(i) *Let $1 < p \leqslant 5$. Then $\mathrm{e}^{\mathrm{i}\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, +\infty)$.*
(ii) *Let $p > 5$. Then $\mathrm{e}^{\mathrm{i}\omega t}\varphi_{\omega,\gamma}(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \omega_1)$, and unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1, +\infty)$.*

*4.4. The radial case*

Before we start to discuss the stability in the radial case, we mention the following remarkable fact.

**Lemma 20.** *The function $f(\gamma)$ defined in Lemma 15 and corresponding to the second negative eigenvalue of $L_1^\gamma$ can be extended to $(-\infty,+\infty)$. Furthermore, $f(\gamma)\in H^1(\mathbb{R})$ is an odd function, for each $\gamma\in(-\infty,+\infty)$.*

The proof uses an idea similar to that of Lemma 12, see Appendix C.

We can deduce the number of negative eigenvalues of $L_1^\gamma$ in the radial case from the result on the eigenvalues of $L_1^\gamma$ considered in the whole space $L^2(\mathbb{R})$. Indeed, Lemma 20 ensures that the second eigenvalue of $L_1^\gamma$ considered in the whole space $L^2(\mathbb{R})$ is associated with an odd eigenvector, and thus disappears when the problem is restricted to subspace of radial functions. Furthermore, since $\varphi_\gamma\in H^1_{\text{rad}}(\mathbb{R})$ and $\langle L_1^\gamma\varphi_\gamma,\varphi_\gamma\rangle<0$, we can infer that the first negative eigenvalue of $L_1^\gamma$ is still present when the problem is restricted to sets of radial functions. Recalling that $n(H^\gamma)=n(L_1^\gamma)$, we obtain.

**Lemma 21.** *Let $\gamma<0$. Then the operator $H^\gamma$ considered on $H^1_{\text{rad}}(\mathbb{R})$ has only one negative eigenvalue, that is $n(H^\gamma)=1$.*

Combining Lemmas 21 and 8 and Proposition 7, we recover the results of [12] recalled in Proposition 3.

Alternatively, Section 4.2 can be adapted to the radial case. All the function spaces should be reduced to spaces of even functions, and Lemma 21 can also be proved in this way.

## 5. Strong instability

This section is devoted to the proof of Theorem 5. We use the virial theorem (Proposition 6) whose verification will be given in Appendix D.

We begin by introducing some notations

$$\mathscr{M}=\{v\in H^1_{\text{rad}}(\mathbb{R})\setminus\{0\};\ Q_\gamma(v)=0,\ I_{\omega,\gamma}(v)\leqslant 0\},$$
$$d_{\mathscr{M}}=\inf\{S_{\omega,\gamma}(v);\ v\in\mathscr{M}\},$$

where $S_{\omega,\gamma}$ and $I_{\omega,\gamma}$ are defined in Proposition 1 and $Q_\gamma$ in Proposition 6.

Our proof is divided into three steps.

*Step* 1. We prove that $\varphi_{\omega,\gamma}$ is also a minimizer of $d_{\mathscr{M}}$.

Because of Pohozaev identity $Q_\gamma(\varphi_{\omega,\gamma})=0$ (see [4]), it is clear that $d_{\mathscr{M}}\leqslant d(\omega)$, thus we only have to show $d_{\mathscr{M}}\geqslant d(\omega)$. Let $v\in\mathscr{M}$. If $I_{\omega,\gamma}(v)=0$, we have $S_{\omega,\gamma}(v)\geqslant d(\omega)$, therefore we suppose $I_{\omega,\gamma}(v)<0$. For $\alpha>0$, let $v^\alpha$ be such that $v^\alpha(x)=\alpha^{1/2}v(\alpha x)$. We have

$$I_{\omega,\gamma}(v^\alpha)=\alpha^2\|\partial_x v\|_2^2+\omega\|v\|_2^2-\gamma\alpha|v(0)|^2-\alpha^{(p-1)/2}\|v\|_{p+1}^{p+1},$$

thus $\lim_{\alpha\to0}I_{\omega,\gamma}(v^\alpha)=\omega\|v\|_2^2>0$, and by continuity there exists $0<\alpha_0<1$ such that $I_{\omega,\gamma}(v^{\alpha_0})=0$. Therefore

$$S_{\omega,\gamma}(v^{\alpha_0})\geqslant d(\omega). \tag{22}$$

Consider now $\frac{\partial}{\partial\alpha}S_{\omega,\gamma}(v^\alpha)=\alpha\|\partial_x v\|_2^2-\frac{\gamma}{2}|v(0)|^2-\frac{p-1}{2(p+1)}\alpha^{(p-3)/2}\|v\|_{p+1}^{p+1}$. Since $p\geqslant5$ and $Q_\gamma(v)=0$, we have for $\alpha\in[0,1]$

$$\frac{\partial}{\partial\alpha}S_{\omega,\gamma}(v^\alpha)\geqslant\alpha Q_\gamma(v)-\frac{\gamma}{2}(1-\alpha)|v(0)|^2=-\frac{\gamma}{2}(1-\alpha)|v(0)|^2$$

and thus $\frac{\partial}{\partial\alpha}S_{\omega,\gamma}(v^\alpha)\geqslant0$ for all $\alpha\in[0,1]$, which leads to $S_{\omega,\gamma}(v)\geqslant S_{\omega,\gamma}(v^{\alpha_0})$. It follows by (22) that $S_{\omega,\gamma}(v)\geqslant d(\omega)$, which concludes $d_{\mathscr{M}}=d(\omega)$.

*Step* 2. We construct a sequence of initial data $\varphi_\omega^\alpha$ satisfying the following properties:

$$S_{\omega,\gamma}(\varphi_\omega^\alpha)<d(\omega),\qquad I_{\omega,\gamma}(\varphi_\omega^\alpha)<0\quad\text{and}\quad Q_\gamma(\varphi_\omega^\alpha)<0.$$

These properties are invariant under the flow of (2).

For $\alpha>0$, we define $\varphi_\omega^\alpha$ by $\varphi_\omega^\alpha(x)=\alpha^{1/2}\varphi_{\omega,\gamma}(\alpha x)$. Since $p\geqslant5$, $\gamma<0$ and $Q_\gamma(\varphi_{\omega,\gamma})=0$, easy computations permit us to obtain

$$\frac{\partial^2}{\partial\alpha^2}S_{\omega,\gamma}(\varphi_\omega^\alpha)_{|\alpha=1}<0,\qquad\frac{\partial}{\partial\alpha}I_{\omega,\gamma}(\varphi_\omega^\alpha)_{|\alpha=1}<0\quad\text{and}\quad\frac{\partial}{\partial\alpha}Q_\gamma(\varphi_\omega^\alpha)_{|\alpha=1}<0,$$

and thus for any $\alpha>1$ close enough to 1 we have

$$S_{\omega,\gamma}(\varphi_\omega^\alpha)<S_{\omega,\gamma}(\varphi_{\omega,\gamma}),\qquad I_{\omega,\gamma}(\varphi_\omega^\alpha)<0\quad\text{and}\quad Q_\gamma(\varphi_\omega^\alpha)<0. \tag{23}$$

Now fix a $\alpha > 1$ such that (23) is satisfied, and let $u^\alpha(t,x)$ be the solution of (2) with $u^\alpha(0) = \varphi_\omega^\alpha$. Since $\varphi_\omega^\alpha$ is radial, $u^\alpha(t)$ is also radial for all $t > 0$ (see Remark 2). We claim that the properties described in (23) are invariant under the flow of (2). Indeed, since from (5) and (6) we have for all $t > 0$

$$S_{\omega,\gamma}(u^\alpha(t)) = S_{\omega,\gamma}(\varphi_\omega^\alpha) < S_{\omega,\gamma}(\varphi_{\omega,\gamma}), \tag{24}$$

we infer that $I_{\omega,\gamma}(u^\alpha(t)) \neq 0$ for any $t \geqslant 0$, and by continuity we have $I_{\omega,\gamma}(u^\alpha(t)) < 0$ for all $t \geqslant 0$. It follows that $Q_\gamma(u^\alpha(t)) \neq 0$ for any $t \geqslant 0$ (if not $u^\alpha(t) \in \mathscr{M}$ and thus $S_{\omega,\gamma}(u^\alpha(t)) > S_{\omega,\gamma}(\varphi_{\omega,\gamma})$ which contradicts (24)), and by continuity we have $Q_\gamma(u^\alpha(t)) < 0$ for all $t \geqslant 0$.

*Step* 3. We prove that $Q_\gamma(u^\alpha)$ remains negative and away from 0 for all $t \geqslant 0$.

Let $t > 0$ be arbitrarily chosen, define $v = u^\alpha(t)$ and for $\beta > 0$ let $v^\beta$ be such that $v^\beta(x) = v(\beta x)$. Then we have

$$Q_\gamma(v^\beta) = \beta\|\partial_x v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \beta^{-1}\frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1},$$

thus $\lim_{\beta\to+\infty} Q_\gamma(v^\beta) = +\infty$, and by continuity there exists $\beta_0$ such that $Q_\gamma(v^{\beta_0}) = 0$. If $I_{\omega,\gamma}(v^{\beta_0}) \leqslant 0$, we keep $\beta_0$ unchanged; otherwise, we replace it by $\tilde\beta_0$ such that $1 < \tilde\beta_0 < \beta_0$, $I_{\omega,\gamma}(v^{\tilde\beta_0}) = 0$ and $Q_\gamma(v^{\tilde\beta_0}) \leqslant 0$. Thus in any case we have $S_{\omega,\gamma}(v^{\beta_0}) \geqslant d(\omega)$. Now, we have

$$S_{\omega,\gamma}(v) - S_{\omega,\gamma}(v^{\beta_0}) = \frac{1-\beta_0}{2}\|\partial_x v\|_2^2 + (1-\beta_0{}^{-1})\left(\frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}\right),$$

from the expression of $Q_\gamma$ and $\beta_0 > 1$ it follows that

$$S_{\omega,\gamma}(v) - S_{\omega,\gamma}(v^{\beta_0}) \geqslant \frac{1}{2}(Q_\gamma(v) - Q_\gamma(v^{\beta_0})). \tag{25}$$

Therefore, from (25), $Q_\gamma(v^{\beta_0}) \leqslant 0$ and $S_{\omega,\gamma}(v^{\beta_0}) \geqslant d(\omega)$ we have

$$Q_\gamma(v) \leqslant -m = 2(S_{\omega,\gamma}(v) - d(\omega)) < 0, \tag{26}$$

where $m$ is independent of $t$ since $S_{\omega,\gamma}$ is a conserved quantity.

*Conclusion*. Finally, thanks to (26) and Proposition 6, we have

$$\|xu^\alpha(t)\|_2^2 \leqslant -mt^2 + Ct + \|x\varphi_\omega^\alpha\|_2^2. \tag{27}$$

For $t$ large, the right member of (27) becomes negative, thus there exists $T^\alpha < +\infty$ such that

$$\lim_{t\to T^\alpha}\|\partial_x u^\alpha(t)\|_2^2 = +\infty.$$

Since it is clear that $\varphi_\omega^\alpha \to \varphi_{\omega,\gamma}$ in $H^1(\mathbb{R})$ when $\alpha \to 1$, Theorem 5 is proved. □

**Remark 10.** It is not hard to see that the set

$$\mathcal{I} = \{v \in H^1(\mathbb{R});\ S_{\omega,\gamma}(v) < d(\omega),\ I_{\omega,\gamma}(v) > 0\}$$

is invariant under the flow of (2), and that a solution with initial data belonging to $\mathcal{I}$ is global. Thus using the minimizing character of $\varphi_{\omega,\gamma}$ and performing an analysis in the same way as in [23], it is possible to find a family of initial data in $\mathcal{I}$ approaching $\varphi_{\omega,\gamma}$ in $H^1(\mathbb{R})$ and such that the associated solution of (2) exists globally but escapes in finite time from a tubular neighborhood of $\varphi_{\omega,\gamma}$ (see also [11,21] for an illustration of this approach on a related problem).

## 6. Numerical results

In this section, we use numerical simulations to complement the rigorous theory on stability and instability of the standing waves of (2). Our approach here is similar to the one in [14]. In order to study stability under radial perturbations, we use the initial condition

$$u_0(x) = (1+\delta_p)\varphi_{\omega,\gamma}(x). \tag{28}$$

In order to study stability under nonradial (asymmetric) perturbations, we use the initial condition

$$u_0(x) = \varphi_{\omega,\gamma}(x - \delta_c), \tag{29}$$

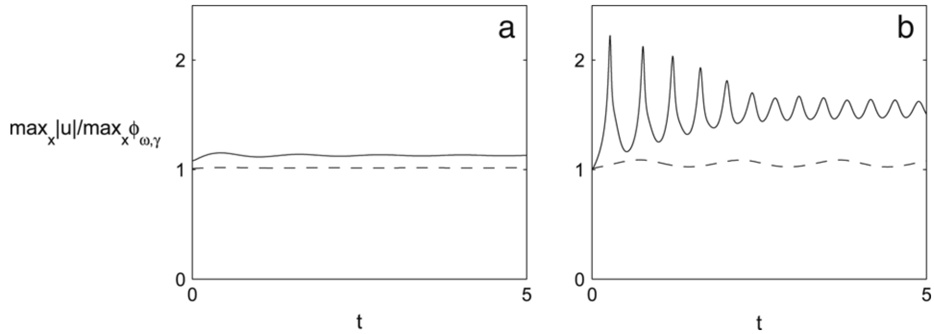


Fig. 2. $\max_x |u|/\max_x \varphi_{\omega,\gamma}$ as a function of $t$ for $\omega = 4$, $\gamma = 1$, $\delta_p = 0.01$ (dashed line) and $\delta_p = 0.08$ (solid line). (a) $p = 3$ (b) $p = 5$.

when $\delta_c$ is the lateral shift of the initial condition. Since the evolution of the momentum for solutions of Eq. (1) is given by

$$\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = -2\int |u|^2 \nabla V(x)\mathrm{d}x, \tag{30}$$

one can see that symmetry-breaking perturbations (29) do not conserve the momentum and thus, may give rise to drift instabilities. In some cases (when the standing wave has a negative slope and the linearized problem has two negative eigenvalues), we use the initial condition

$$u_0(x) = (1+\delta_p)\varphi_{\omega,\gamma}(x-\delta_c). \tag{31}$$

In order to demonstrate the agreement of the numerics with the rigorous stability theory, one needs to observe that $\|u-\varphi_{\omega,\gamma}\|_{H^1}$ remains "small" in the case of stability but increases in the case of instability. In the latter case, however, observing numerically that $\|u-\varphi_{\omega,\gamma}\|_{H^1}$ increases does not enable us to distinguish between the different types of instabilities such as total diffraction (i.e., when $\lim_{t\to\infty}\|u\|_\infty = 0$), finite-width instability, strong instability or drift instability. Therefore, instead of presenting the $H^1$ norm, we plot the dynamics of the maximal amplitude of the solution and of the location of the maximal amplitude. Together, these two quantities give a more informative description of the dynamics, while also showing whether the soliton is stable.

### 6.1. Stability in $H^1_{\mathrm{rad}}(\mathbb{R})$

#### 6.1.1. Strength of radial stability

When $\gamma > 0$, the standing waves are known to be stable in $H^1_{\mathrm{rad}}(\mathbb{R})$ for $1 < p \le 5$. The rigorous theory, however, does not address the issue of the *strength of radial stability*. This issue is of most interest in the case $p = 5$, which is unstable when $\gamma = 0$.

For $\delta_p > 0$, it is useful to define

$$F(\delta_p) = \max_{t\ge 0}\left\{\frac{\max\limits_x |u(x,t)| - \max\limits_x \varphi_{\omega,\gamma}}{\max\limits_x \varphi_{\omega,\gamma}}\right\} \tag{32}$$

as a measure of the strength of radial stability. Fig. 2 shows the normalized values $\max_x |u|/\max_x \varphi_{\omega,\gamma}$ as a function of $t$, for the initial condition (28) with $\omega = 4$ and $\gamma = 1$. When $p = 3$, a perturbation of $\delta_p = 0.01$ induces small oscillations and $F(0.01) = 1.9\%$. Therefore, roughly speaking, a 1% perturbation of the initial condition leads to a maximal deviation of 2%. A larger perturbation of $\delta_p = 0.08$ causes the magnitude of the oscillations to increase approximately by the same ratio, so that $F(0.08) = 15\%$. Using the same perturbations with $p = 5$, however, leads to significantly larger deviations. Thus, $F(0.01) = 8.8\%$, i.e., more than 4 times bigger than for $p = 3$, and $F(0.08) = 122\%$, i.e., more than 8 times than for $p = 3$.

In [14,15], Fibich, Sivan and Weinstein observed that the strength of radial stability is related to the magnitude of slope $\partial_\omega\|\varphi_{\omega,\gamma}\|_2^2$, so that larger the $\partial_\omega\|\varphi_{\omega,\gamma}\|_2^2$, the "more stable" the solution. Indeed, numerically we found that when $\omega = 4$, $\partial_\omega\|\varphi_{\omega,\gamma}\|_2^2$ is equal to 1.0 for $p = 3$ and 0.056 for $p = 5$.

Since when $\gamma = 0$, the slope is positive for $p < 5$ but zero for $p = 5$, for $\gamma > 0$ the slope is smaller in the critical case than in the subcritical case. Therefore, we make the following informal observation:

**Observation 11.** *Radial stability of the standing waves of (2) with $\gamma > 0$ is "weaker" in the critical case $p = 5$ than in the subcritical case $p < 5$.*

Clearly, this difference would be more dramatic at smaller (positive) values of $\gamma$. Indeed, if in the simulation of Fig. 2 with $\delta_p = 0.01$ we reduce $\gamma$ from 1 to 0.5 and then to 0.1, this has almost no effect when $p = 3$, where the value of $F$ slightly increases from 1.9% to 2.1% and to 2.5%, respectively, see Fig. 3(a). However, if we repeat the same simulations with $p = 5$, then reducing the value of

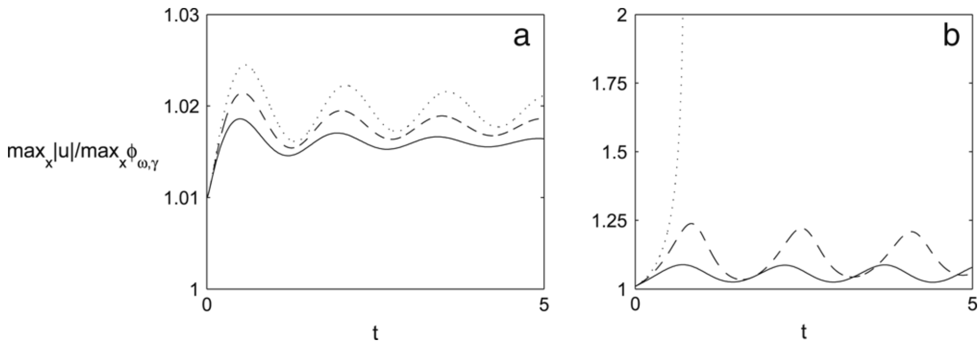


Fig. 3. $\max_x |u|/\max_x \varphi_{\omega,\gamma}$ as a function of $t$ for $\omega = 4$, $\delta_p = 0.01$, and $\gamma = 1$ (solid line), $\gamma = 0.5$ (dashed line) and $\gamma = 0.1$ (dots). (a) $p = 3$ (b) $p = 5$.

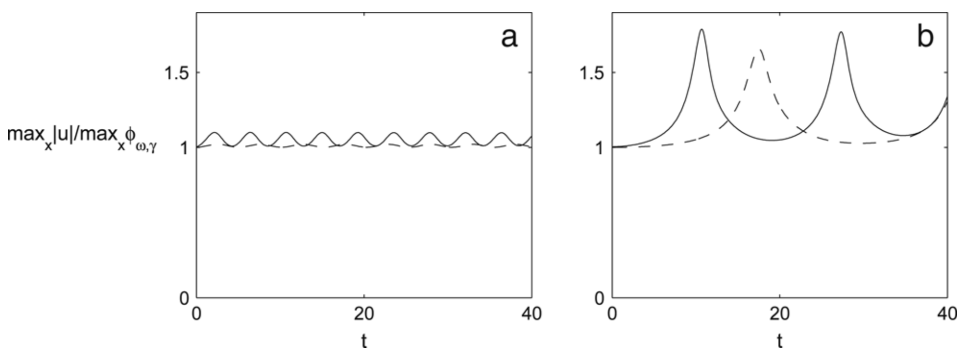


Fig. 4. $\max_x |u|/\max_x \varphi_{\omega,\gamma}$ as a function of $t$ for $p = 4$, $\gamma = -1$, $\delta_p = 0.001$ (dashed line) and $\delta_p = 0.005$ (solid line). (a) $\omega = 2$; (b) $\omega = 0.5$.

$\gamma$ has a much larger effect, see Fig. 3(b), where $F$ increases from 8.9% for $\gamma = 1$ to 24% for $\gamma = 0.5$. Moreover, when we further reduced $\gamma$ to 0.1, the solution seems to undergo collapse.[1] This implies that when $p = 5$ and $\gamma > 0$, the standing wave is stable, yet it can collapse under a sufficiently large perturbation.

### 6.1.2. *Characterization of radial instability for* $3 < p < 5$ *and* $\gamma < 0$

We consider the subcritical repulsive case $p = 4$ and $\gamma = -1$. In this case, there is threshold $\omega_2$ such that $\varphi_{\omega,\gamma}$ is stable for $\omega > \omega_2$ and unstable for $\omega < \omega_2$. By numerical calculation we found that $\omega_2(p = 4, \gamma = -1) \approx 0.82$. Accordingly, we chose two representative values of $\omega$: $\omega = 0.5$ in the unstable regime, and $\omega = 2$ in the stable regime.

Fig. 4(a) demonstrates the stability for $\omega = 2$. Indeed, reducing the perturbation from $\delta_p = 0.005$ to 0.001 results in reduction of the relative magnitude of the oscillations by roughly five times, from $F(0.005) \approx 10\%$ to $F(0.001) \approx 2\%$. The dynamics in the unstable case $\omega = 0.5$ are also oscillatory, see Fig. 4(b). However, in this case $F(0.005) = 79\%$, i.e., eight times larger than for $\omega = 2$. More importantly, unlike the stable case, a perturbation of $\delta_p = 0.001$ does not result in a reduction of the relative magnitude of the oscillations by $\approx 5$. In fact, the relative magnitude of the oscillations decreases only to $F(0.001) = 66\%$.

In the homogeneous NLS, unstable standing waves perturbed with $\delta_p > 0$ always undergo collapse. Since, however, for $p = 4$ it is impossible to have collapse, an interesting question is the nature of the instability in the unstable region $\omega < \omega_2$. In Fig. 4(b) we already saw that $\max |u(x,t)|$ undergoes oscillations. In order to better understand the nature of this unstable oscillatory dynamics, we plot in Fig. 5 the spatial profile of $|u(x,t)|$ at various values of $t$. In addition, at each $t$ we plot $\phi_{\omega^*(t),\gamma}(x)$, where $\omega^*(t)$ is determined from the relation

$$\max_x \phi_{\omega^*(t),\gamma}(x) = \max_x |u(x,t)|.$$

Since the two curves are nearly indistinguishable (especially in the central region), this shows that the unstable dynamics correspond to “movement along the curve $\phi_{\omega*(t)}$”.

In Fig. 6 we see that $\omega^*(t)$ undergoes oscillations, in accordance with the oscillations of $\max_x |u|$. Furthermore, as one may expect, collapse is arrested only when $\omega^*(t)$ reaches a value ($\approx 2.86$) which is in the stability region (i.e., above $\omega_2$).

**Observation 12.** *When $\gamma < 0$ and $3 < p < 5$, the instability in $H^1_{\text{rad}}(\mathbb{R})$ is a “finite-width instability”, i.e., the solution narrows down along the curve $\phi_{\omega*(t),\gamma}$ until it “reaches” a finite width in the stable region $\omega > \omega_2$, at which point collapse is arrested.*

[1] Clearly, one cannot use numerics to determine that a solution becomes singular, as it is always possible that collapse would be arrested at some higher focusing levels.

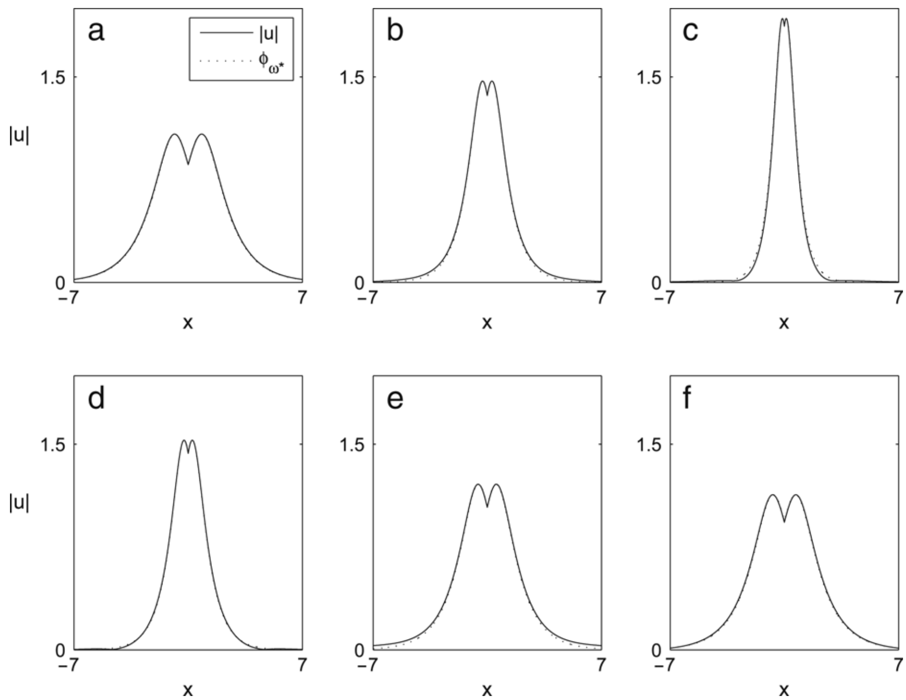


Fig. 5. $|u(x,t)|$ (solid line) and $\phi_{\omega*(t)}(x)$ (dots) as functions of $x$ for the simulation of Fig. 4(b) with $\delta_p = 0.005$. (a) $t = 0$ ($\omega^* = 0.508$) (b) $t = 9$ ($\omega^* = 1.27$) (c) $t = 10.69$ ($\omega^* = 2.86$) (d) $t = 12$ ($\omega^* = 1.43$) (e) $t = 15$ ($\omega^* = 0.706$) (f) $t = 20$ ($\omega^* = 0.58$).

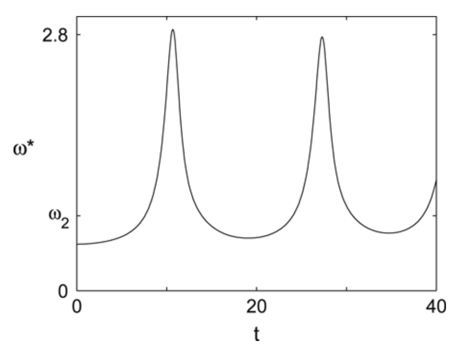


Fig. 6. $\omega^*$ as a function of $t$ for the simulation of Fig. 5.

Note that this behavior was already observed in [14], Fig 19. Therefore, more generally, we conjecture that

**Observation 13.** *When the slope is negative (i.e., $\partial_\omega \|\varphi_{\omega,\gamma}\|_2^2 < 0$), then the symmetric perturbation (28) with $0 < \delta_p \ll 1$ leads to a finite-width instability in the subcritical case, and to a finite-time collapse in the critical and supercritical cases.*

*6.1.3. Supercritical case* ($p > 5$)

We recall that when $\gamma > 0$ and $p > 5$, the standing wave is stable for $\gamma^2/4 < \omega < \omega_1$ and unstable for $\omega_1 < \omega$. When $\gamma < 0$ and $p > 5$ the standing wave is strongly unstable under radial perturbations for any $\omega$, i.e., an infinitesimal perturbation can lead to collapse.

Fig. 7 shows the behavior of perturbed solutions for $p = 6$ and $\omega = 1$. As predicted by the theory, when $\delta_p = 0.001$, the solution blows up for $\gamma = -1$ and $\gamma = 0$, but undergoes small oscillations (i.e., is stable) for $\gamma = 1$. Indeed, we found numerically that $\omega_1(p = 6, \gamma = 1) \approx 2.9$, so that the standing wave is stable for $\omega = 1$. However, when we increase the perturbation to $\delta_p = 0.1$, the solution with $\gamma = 1$ also seems to undergo collapse. This implies that when $p > 5, \gamma > 0$ and $\omega < \omega_1$ the standing wave is stable, yet it can collapse under a sufficiently large perturbation. In order to find the type of instability for $\gamma > 0$ and $\omega > \omega_1$, we solve the NLS (2) with $p = 6$, $\gamma = 1$ and $\omega = 4$. In this case, $\delta_p = 0.001$ seems to lead to collapse, see Fig. 8, suggesting a strong instability for $p > 5, \gamma > 0$ and $\omega > \omega_1$. Therefore, we make the following informal observation:

**Observation 14.** *If a standing wave of (2) with $p > 5$ is unstable in $H^1_{\mathrm{rad}}(\mathbb{R})$, then the instability is strong.*

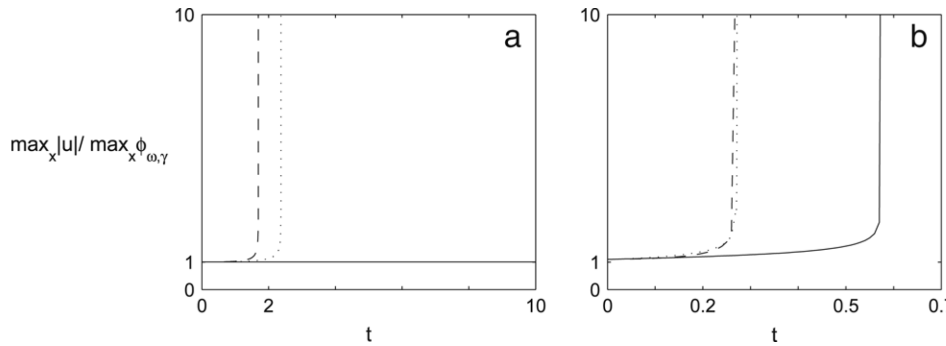


Fig. 7. $\max_x |u(x,t)|/\max_x \varphi_{\omega,\gamma}$ as a function of $t$ for $p = 6$, $\omega = 1$ and $\gamma = -1$ (dashed line), $\gamma = 0$ (dots), $\gamma = +1$ (solid line). (a) $\delta_p = 0.001$ (b) $\delta_p = 0.1$.

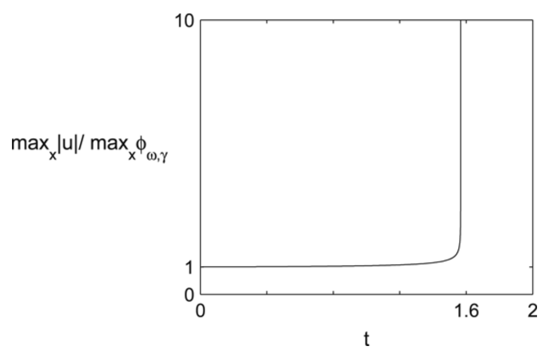


Fig. 8. $\max_x |u(x,t)|/\max_x \varphi_{\omega,\gamma}$ as a function of $t$ for $p = 6$, $\omega = 4$, $\gamma = 1$ and $\delta_p = 0.001$.

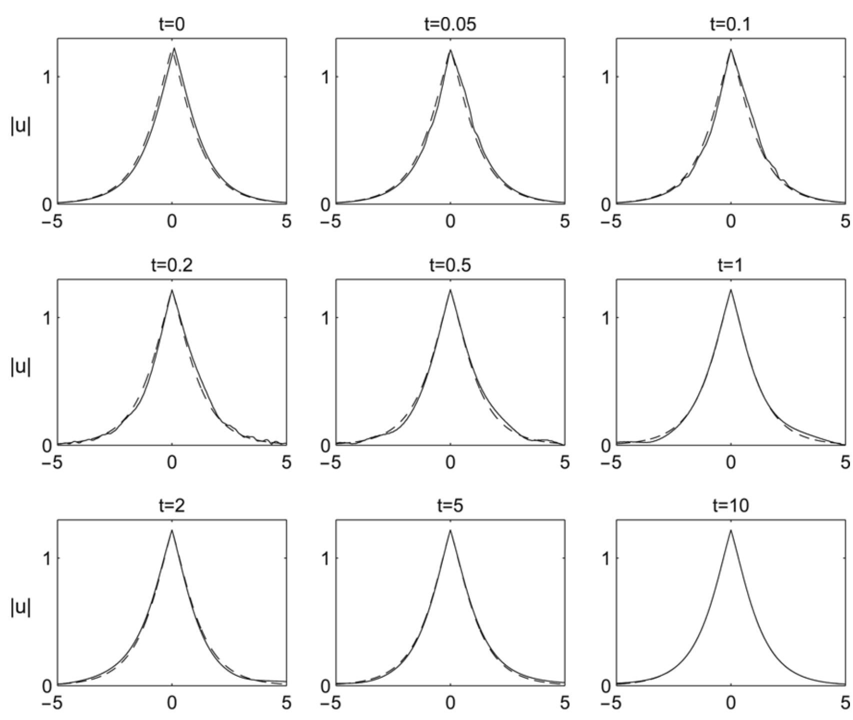


Fig. 9. $|u(x,t)|$ (solid line) and $\phi_{\omega*=0.995}(x)$ (dashed line) as functions of $x$. Here, $p = 3$, $\omega = 1$, $\gamma = 1$ and $\delta_c = 0.1$.

### 6.2. *Stability under nonradial perturbations*

#### 6.2.1. *Stability for* $1 < p < 5$ *and* $\gamma > 0$

Fig. 9 shows the evolution of the solution when $p = 3, \gamma = 1, \omega = 1$ and $\delta_c = 0.1$. The peak of the solution moves back towards $x = 0$ very quickly (around $t \approx 0.003$) and stays there at later times. Subsequently, the solution converges to

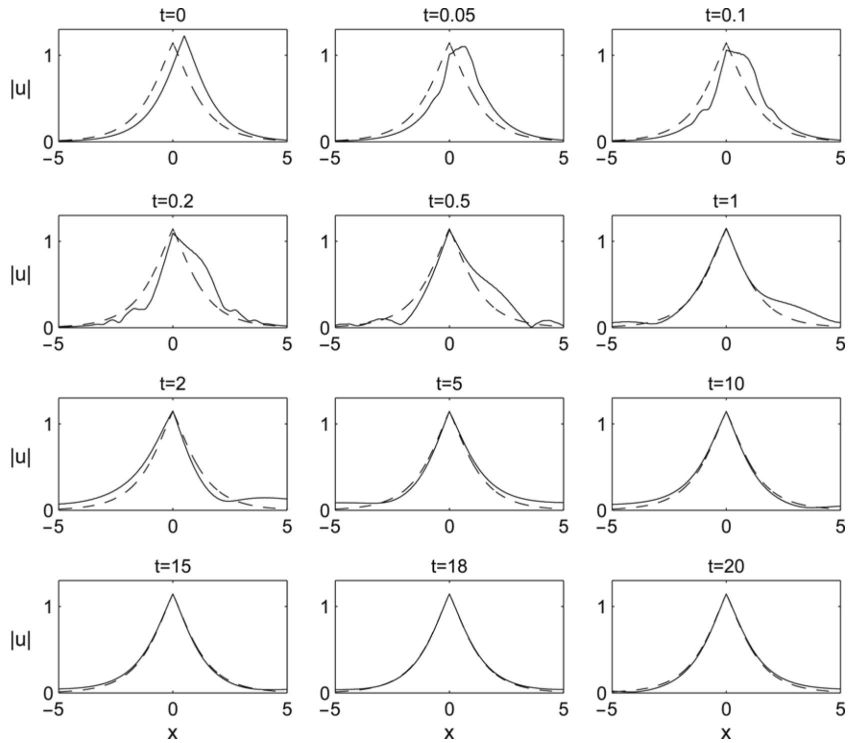


Fig. 10. Same as Fig. 9 with $\delta_c = 0.5$ and $\omega^* = 0.905$.

the bound state $\phi_{\omega^*=0.995}$. This convergence starts near $x = 0$ and spreads sideways, accompanied by radiation of the excess power $\|u_0\|_2^2 - \|\phi_{\omega^*=0.995}\|_2^2 \cong 2.00 - 1.99 = 0.01$. In Fig. 10 we repeat this simulation with a larger shift of $\delta_c = 0.5$. The overall dynamics are similar: The solution peak moves back to $x = 0$, and the solution converges (from the center outwards) to $\phi_{\omega^*=0.905}$. In this case, it takes longer for the maximum to return to $x = 0$ (at $t \approx 0.11$), and more power is radiated in the process $(\|u_0\|_2^2 - \|\phi_{\omega^*=0.905}\|_2^2 \cong 2.00 - 1.81 = 0.19$. We verified that the "nonsmooth" profiles (e.g., at $t = 0.2$) are not numerical artifacts.

### 6.2.2. *Drift instability for* $1 < p \le 3$ *and* $\gamma < 0$

Fig. 11 shows the evolution of the solution for $p = 3$, $\gamma = -1$, $\omega = 1$ and $\delta_c = 0.1$. Unlike the attractive case with the same parameters (Fig. 9), as a result of this small initial shift to the right, nearly all the power flows from the left side of the defect ($x < 0$) to the right side ($x > 0$), see Fig. 12(a), so that by $t \approx 3$, $\approx 90\%$ of the power is in the right side. Subsequently, the right component moves to the right at a constant speed (see Fig. 12(b) while assuming the sech profile of the homogeneous NLS bound state (see Fig. 11 at t = 8); the left component also drifts away from the defect.

We thus see that

**Observation 15.** *When* $1 < p \le 3$, *the standing waves are stable under shifts in the attractive case, but undergo a drift instability away from the defect in the repulsive case.*

We note that a similar behavior was observed in the subcritical NLS with a periodic nonlinearity, see [14], Section 5.1.

### 6.2.3. *Drift and finite-width instability for* $3 < p < 5$ *and* $\gamma < 0$

In Figs. 4(b), 5 and 6 we saw that when $p = 4$, $\gamma = -1$, $\omega = 0.5$, and $\delta_p = 0.005$, the solution undergoes a finite-width instability in $H^1_{\mathrm{rad}}(\mathbb{R})$. In Figs. 13 and 14 we show the dynamics (in $H^1(\mathbb{R})$) when we add a small shift of $\delta_c = 0.1$. In this case, the (larger) right component undergoes a combination of a drift instability and a finite-width instability, whereas the (smaller) left component undergoes a drift instability. Therefore, we make the following observation

**Observation 16.** *When* $3 < p < 5$, $\gamma^2/4 < \omega < \omega_2$ *and* $\gamma < 0$, *the standing waves undergo a combined drift and finite-width instability.*

### 6.2.4. *Drift and strong instability for* $5 \le p$ *and* $\gamma < 0$

In Figs. 15 and 16 we show the solution of the NLS (2) with $p = 6$, $\gamma = -1$ and $\omega = 1$, for the initial condition (31) with $\delta_c = 0.2$ and $\delta_p = 0.001$. As predicted by the theory, this strongly unstable solution undergoes collapse. Note, however, that, in parallel, the solution also undergoes a drift instability. We thus see that

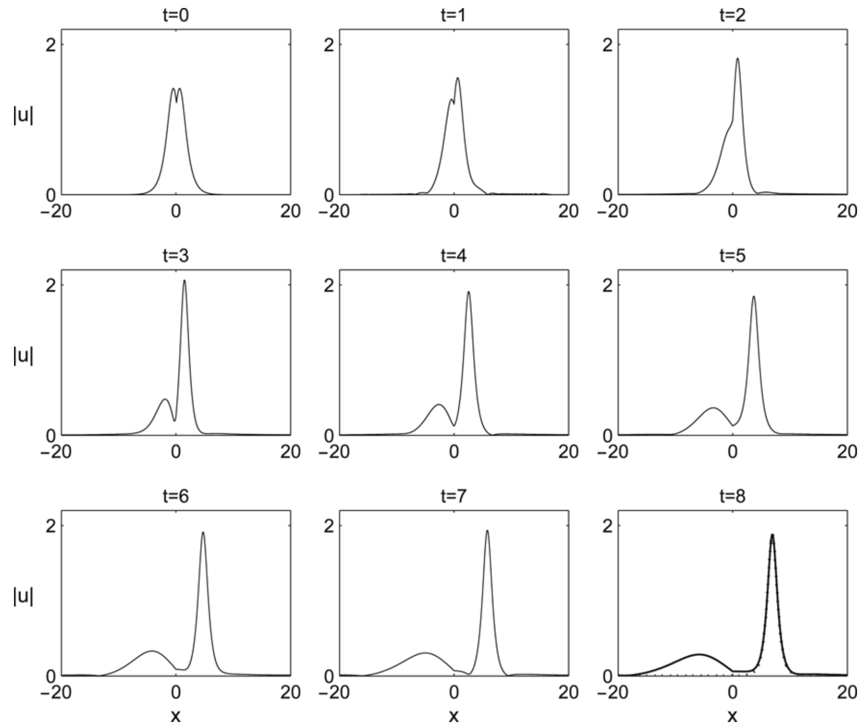


Fig. 11. $|u(x,t)|$ (solid line) as a function of $x$. Here $p=3$, $\gamma=-1$, $\omega=1$ and $\delta_c=0.1$. Dotted line at $t=8$ is $\sqrt{2\omega^*}\operatorname{sech}(\sqrt{\omega^*}(x-x^*))$ with $\omega^*=1.768$ and $x^*\approx 7$.

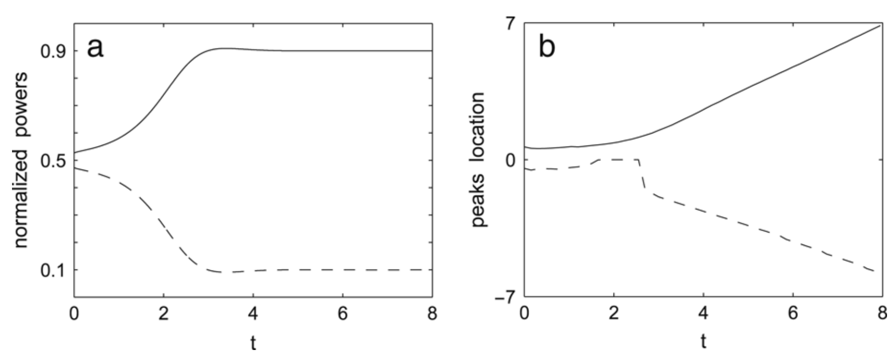


Fig. 12. (a) The normalized powers $\int_0^\infty |u|^2\mathrm{d}x/\int_{-\infty}^\infty |u_0|^2\mathrm{d}x$ (solid line) and $\int_{-\infty}^0 |u|^2\mathrm{d}x/\int_{-\infty}^\infty |u_0|^2\mathrm{d}x$ (dashed line), and (b) location of $\max_{0\le x}|u(x,t)|$ (solid line) and of $\max_{x\le 0}|u(x,t)|$ (dashed line), for the simulation of Fig. 11.

**Observation 17.** *In the critical and supercritical repulsive case, the standing waves collapse while undergoing a drift instability away from the defect.*

Note that a similar behavior was observed in [14], Section 5.2.

### 6.3. Numerical methods

We solve the NLS (2) using fourth-order finite differences in $x$ and the second-order implicit Crack–Nicholson scheme in time. Clearly, the main question is how to discretize the delta potential at $x=0$. Recall that in the continuous case

$$\lim_{x\to 0^+}\partial_x u(x)-\lim_{x\to 0^-}\partial_x u(x)=-\gamma u(0).$$

Discretizing this relation with $\mathcal{O}(h^2)$ accuracy gives

$$\frac{u(2h)-4u(h)+3u(0)}{2h}-\frac{-u(-2h)+4u(-h)-3u(0)}{2h}=-\gamma u(0),$$

when $h$ is the spatial grid size. By rearrangement of the terms we get the equation

$$-u(2h)+4u(h)+[2h\gamma-6]u(0)+4u(-h)-u(-2h)=0. \tag{33}$$

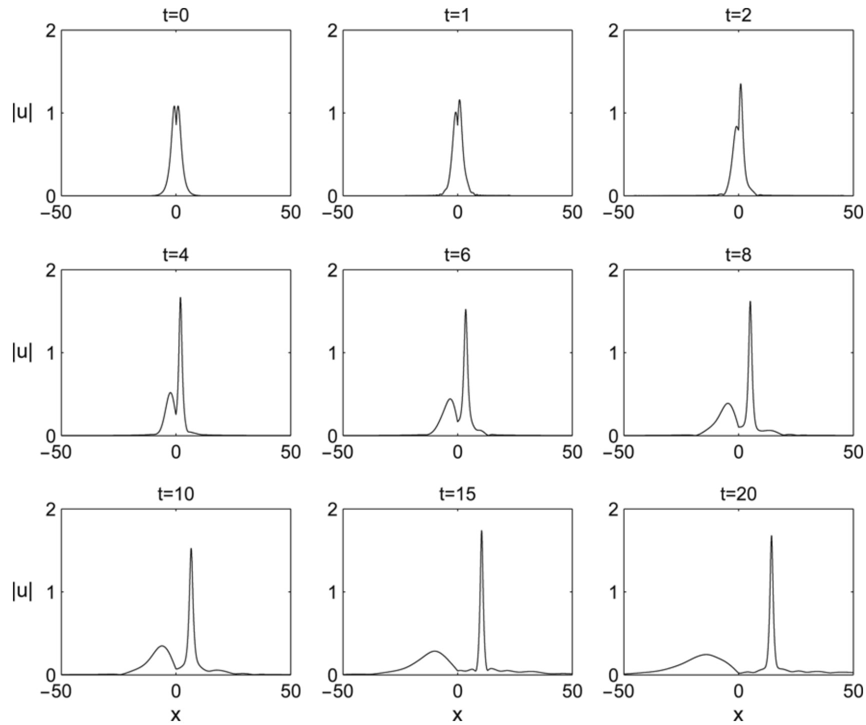


Fig. 13. $u(x,t)$ as a function of $x$. Here $p=4$, $\gamma=-1$, $\omega=0.5$, $\delta_p=0.005$, and $\delta_c=0.1$.

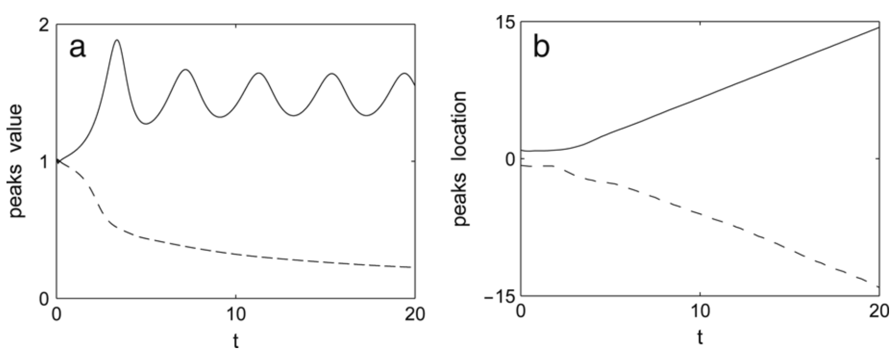


Fig. 14. (a) The value, and (b) the location, of the right peak $\max_{0\le x}|u(x,t)|$ (solid line) and left peak $\max_{x\le 0}|u(x,t)|$ (dashed line), for the simulation of Fig. 13.

When we simulate symmetric perturbations (Section 6.1), we enforce symmetry by solving only on half space $[0,+\infty)$. In this case, because of the symmetry condition $u(-x)=u(x)$, (33) becomes

$$[2h\gamma-6]u(0)+8u(h)-2u(2h)=0.$$

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## Appendix A. Proof of Lemma 10

Since the proof of $L^{\gamma}_{2,\omega}$ is similar to the one of $L^{\gamma}_{1,\omega}$, we only deal with $L^{\gamma}_{1,\omega}$. The form $B^{\gamma}_{1,\omega}$ can be decomposed into $B^{\gamma}_{1,\omega}=B^{\gamma}_{1,1}+B^{\gamma}_{1,2,\omega}$ with $B^{\gamma}_{1,1}:H^1(\mathbb{R})\times H^1(\mathbb{R})\to\mathbb{R}$ and $B^{\gamma}_{1,2,\omega}:L^2(\mathbb{R})\times L^2(\mathbb{R})\to\mathbb{R}$ defined by

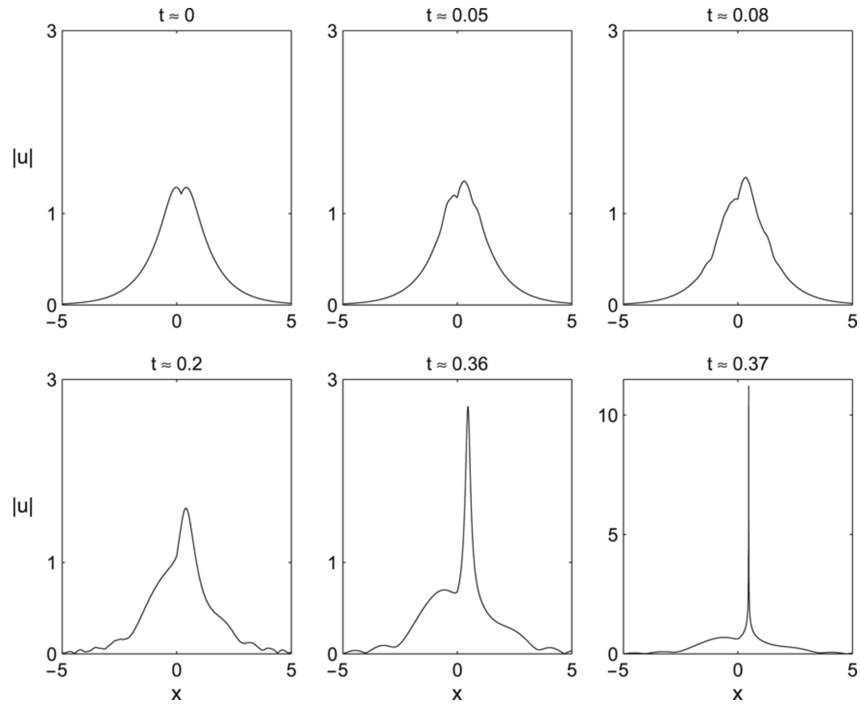


Fig. 15. $|u(x,t)|$ as a function of $x$, at various values of $t$. Here, $p = 6$, $\gamma = -1$, $\omega = 1$, $\delta_c = 0.2$ and $\delta_p = 0.001$.

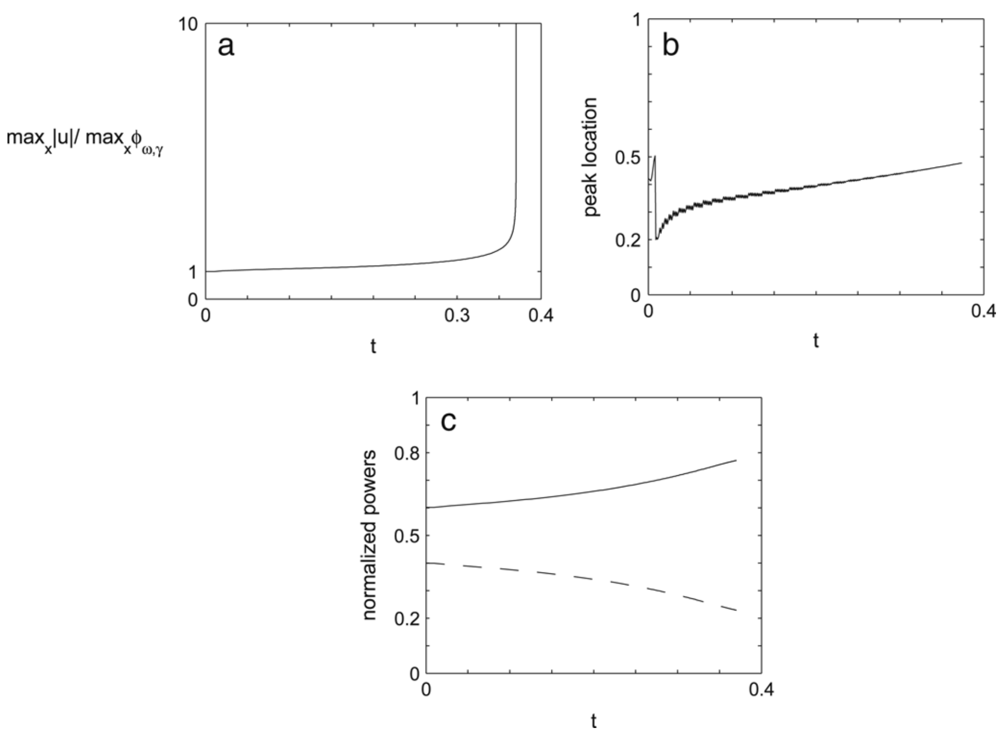


Fig. 16. (a) $\max_x |u(x,t)|/\max_x \varphi_{\omega,\gamma}$ (b) location of $\max_x |u(x,t)|$ and (c) The normalized powers $\int_0^{\infty} |u|^2 \mathrm{d}x / \int_{-\infty}^{\infty} |u_0|^2 \mathrm{d}x$ (solid line) and $\int_{-\infty}^{0} |u|^2 \mathrm{d}x / \int_{-\infty}^{\infty} |u_0|^2 \mathrm{d}x$ (dashed line), for the solution of Fig. 15.

$$\begin{aligned} B_{1,1}^{\gamma}(v,z) &= \int_{\mathbb{R}} \partial_x v \partial_x z \mathrm{d}x - \gamma v(0) z(0), \\ B_{1,2,\omega}^{\gamma}(v,z) &= \omega \int_{\mathbb{R}} v z \mathrm{d}x - \int_{\mathbb{R}} p \varphi_{\omega,\gamma}^{p-1} v z \mathrm{d}x. \end{aligned} \tag{A.1}$$

If we denote by $T_1$ (resp. $T_2$) the self-adjoint operator on $L^2(\mathbb{R})$ associated with $B^\gamma_{1,1}$ (resp $B^\gamma_{1,2,\omega}$), it is clear that $D(T_2) = L^2(\mathbb{R})$ and

$$D(L^\gamma_{1,\omega}) = D(T_1).$$

If we take $v \in H^2(\mathbb{R})$ such that $v(0) = 0$, and put $w = -\partial_x^2 v \in L^2(\mathbb{R})$, it follows that for any $z \in H^1(\mathbb{R})$ we have

$$B^\gamma_{1,1}(v, z) = \int_{\mathbb{R}} \partial_x v \partial_x z \mathrm{d}x = (w, z)_2.$$

Thus $v \in D(T_1)$, and we can deduce that $T_1$ is a self-adjoint extension of the operator $T$ defined by

$$T = -\partial_x^2, \qquad D(T) = \{v \in H^2(\mathbb{R}); v(0) = 0\}.$$

On the other hand, using the theory of self-adjoint extensions of symmetric operators, one can see (see [2, Theorem I-3.1.1]) that there exists $\alpha \in \mathbb{R}$ such that

$$D(T_1) = \{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}); \partial_x v(0^+) - \partial_x v(0^-) = -\alpha v(0)\}.$$

Now, take $v \in D(T_1)$ with $v(0) \neq 0$. Then

$$\begin{aligned}(T_1 v, v)_2 &= \int_{-\infty}^0 (-\partial_x^2 v) v \mathrm{d}x + \int_0^{+\infty} (-\partial_x^2 v) v \mathrm{d}x \\ &= -v(0)\partial_x v(0-) + \int_{-\infty}^0 |\partial_x v|^2 \mathrm{d}x + v(0)\partial_x v(0+) + \int_0^{+\infty} |\partial_x v|^2 \mathrm{d}x \\ &= \int_{\mathbb{R}} |\partial_x v|^2 \mathrm{d}x - \alpha v(0)^2\end{aligned}$$

which should be equal to

$$B^\gamma_{1,1}(v, v) = \int_{\mathbb{R}} |\partial_x v|^2 \mathrm{d}x - \gamma v(0)^2.$$

Thus $\gamma = \alpha$, and the lemma is proved. □

## Appendix B. Proofs of Lemmas 11 and 13

**Proof of Lemma 11.** We start by showing that (i) and (iii) are satisfied. We work on $L^\gamma_{1,\omega}$ and $L^\gamma_{2,\omega}$. The essential spectrum of $T_1$ (see the proof of Lemma 10) is $\sigma_{\text{ess}}(T_1) = [0, +\infty)$. This is standard when $\gamma = 0$ and a proof for $\gamma \neq 0$ can be found in [2, Theorem I-3.1.4]. From Weyl's theorem (see [30, Theorem IV-5.35]), the essential spectrum of both operators $L^\gamma_{1,\omega}$ and $L^\gamma_{2,\omega}$ is $[\omega, +\infty)$. Since both operators are bounded from below, there can be only finitely many isolated eigenvalues (of finite multiplicity) in $(-\infty, \omega')$ for any $\omega' < \omega$. Then (i) and (iii) follow easily.

Next, we consider (ii). Since $\varphi_{\omega,\gamma}$ satisfies $L^\gamma_{2,\omega}\varphi_{\omega,\gamma} = 0$ and $\varphi_{\omega,\gamma} > 0$, the first eigenvalue of $L^\gamma_{2,\omega}$ is 0 and the rest of the spectrum is positive. This is classical for $\gamma = 0$ and can be easily proved for $\gamma \neq 0$, see [5, Chapter 2, Section 2.3, Paragraph 3]. Thus to ensure that the kernel of $H^\gamma_\omega$ is reduced to span$\{i\varphi_{\omega,\gamma}\}$ it is enough to prove that the kernel of $L^\gamma_{1,\omega}$ is $\{0\}$. This is equivalent to proving that 0 is the unique solution of

$$L^\gamma_{1,\omega} u = 0, \qquad u \in D(L^\gamma_{1,\omega}). \tag{B.1}$$

To be more precise, the solutions of (B.1) satisfy

$$u \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}), \tag{B.2}$$

$$-\partial_x^2 u + \omega u - p\varphi^{p-1}_{\omega,\gamma} u = 0, \tag{B.3}$$

$$\partial_x u(0+) - \partial_x u(0-) = -\gamma u(0). \tag{B.4}$$

Consider first (B.3) on $(0, +\infty)$. If we look at (3) only on $(0, +\infty)$, we see that $\varphi_{\omega,\gamma}$ satisfies

$$-\partial_x^2 \varphi_{\omega,\gamma} + \omega\varphi_{\omega,\gamma} - \varphi^p_{\omega,\gamma} = 0 \quad \text{on } (0, +\infty). \tag{B.5}$$

If we differentiate (B.5) with respect to $x$ (which is possible because $\varphi_{\omega,\gamma}$ is smooth on $(0, +\infty)$), we see that $\partial_x\varphi_{\omega,\gamma}$ satisfies (B.3) on $(0, +\infty)$. Since we look for solutions in $L^2(\mathbb{R})$ (in fact solutions going to 0 at infinity), it is standard that every solution of (B.3) in $(0, +\infty)$ is of the form $\mu\partial_x\varphi_{\omega,\gamma}$, $\mu \in \mathbb{R}$ (see, for example, [5, Chapter 2, Theorem 3.3]). A similar argument can be applied to (B.3) on $(-\infty, 0)$, thus every solution of (B.3) in $(-\infty, 0)$ is of the form $\nu\partial_x\varphi_{\omega,\gamma}$, $\nu \in \mathbb{R}$.

Now, let $u$ be a solution of (B.2)–(B.4). Then there exist $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}$ such that

$$u = \nu \partial_x \varphi_{\omega,\gamma} \quad \text{on } (-\infty, 0),$$
$$u = \mu \partial_x \varphi_{\omega,\gamma} \quad \text{on } (0, +\infty).$$

Since $u \in H^1(\mathbb{R})$, $u$ is continuous at 0, thus we must have $\mu = -\nu$, that is $u$ is of the form

$$u = -\mu \partial_x \varphi_{\omega,\gamma} \quad \text{on } (-\infty, 0),$$
$$u = \mu \partial_x \varphi_{\omega,\gamma} \quad \text{on } (0, +\infty),$$
$$u(0) = -\mu \partial_x \varphi_{\omega,\gamma}(0-) = \mu \partial_x \varphi_{\omega,\gamma}(0+) = \frac{-\mu}{2} \gamma \varphi_{\omega,\gamma}(0).$$

Furthermore, $u$ should satisfy the jump condition (B.4). Since $\varphi_{\omega,\gamma}$ satisfies

$$\partial_x^2 \varphi_{\omega,\gamma}(0-) = \partial_x^2 \varphi_{\omega,\gamma}(0+) = \omega \varphi_{\omega,\gamma}(0) - \varphi_{\omega,\gamma}^p(0),$$

if we suppose $\mu \neq 0$ then (B.4) reduces to

$$\varphi_{\omega,\gamma}^{p-1}(0) = \frac{4\omega - \gamma^2}{4}.$$

But from (4) we know that

$$\varphi_{\omega,\gamma}^{p-1}(0) = \frac{p+1}{8}(4\omega - \gamma^2).$$

This is a contradiction, therefore $\mu = 0$. In conclusion, $u \equiv 0$ on $\mathbb{R}$, and the lemma is proved. □

**Proof of Lemma 13.** We recall that $L_1^\gamma$ is defined with the help of a bilinear form $B_1^\gamma$ (see (9)). To prove the holomorphicity of $(L_1^\gamma)$ it is enough to prove that $(B_1^\gamma)$ is bounded from below and closed, and that for any $v \in H^1(\mathbb{R})$ the function $B_1^\gamma(v) : \gamma \mapsto B_1^\gamma(v, v)$ is holomorphic (see [30, Theorem VII-4.2]). It is clear that $B_1^\gamma$ is bounded from below and closed on the same domain $H^1(\mathbb{R})$ for all $\gamma$, thus we just have to check the holomorphicity of $B_1^\gamma(v) : \gamma \mapsto B_1^\gamma(v, v)$ for any $v \in H^1(\mathbb{R})$. We recall the decomposition of $B_1^\gamma$ into $B_{1,1}^\gamma$ and $B_{1,2}^\gamma$ (see (A.1)). We see that $B_{1,1}^\gamma(v)$ is clearly holomorphic in $\gamma$. From the explicit form of $\varphi_\gamma$ (see (4)) it is clear that $\gamma \mapsto \varphi_\gamma^{p-1}(x)$ is holomorphic in $\gamma$ for any $x \in \mathbb{R}$. It then also follows that $\gamma \mapsto B_{1,2}^\gamma(v)$ is holomorphic. □

**Remark 18.** There exists another way to show that $(L_1^\gamma)$ is a real-holomorphic family with respect to $\gamma \in \mathbb{R}$. We can use the explicit resolvent formula in [2],

$$(T_1 - k^2)^{-1} = (-\partial_x^2 - k^2)^{-1} + 2\gamma k(-\mathrm{i}\gamma + 2k)^{-1}(\overline{G_k(\cdot)}, \cdot) G_k(\cdot),$$

where $k^2 \in \rho(T_1)$, $\operatorname{Im} k > 0$, $G_k(x) = (\mathrm{i}/2k)\mathrm{e}^{\mathrm{i}k|x|}$, to verify the holomorphicity.

## Appendix C. Proof of Lemma 20

First, we indicate how the extension of $f$ to $(-\infty, +\infty)$ can be done. We see by the proof in [38, Theorem XII.8] that the functions $f(\gamma)$ and $\lambda(\gamma)$ defined in Lemma 15 exist, are holomorphic and represent an eigenvector and an eigenvalue for all $\gamma \in \mathbb{R}$, since $(L_1^\gamma)$ is a real-holomorphic family in $\gamma \in \mathbb{R}$. Namely we can repeat the argument of Lemma 15 at each point $\gamma$ and on each neighborhood of $\gamma$. This is possible because the set $\{(\gamma, \lambda); \gamma \in \mathbb{R}, \lambda \in \rho(L_1^\gamma)\}$ is open and the function $(L_1^\gamma - \lambda)^{-1}$ defined on this set is a holomorphic function of two variables ([38, Theorem XII.7]).

Secondly, as it was observed in [12,14], the eigenvectors of $L_1^\gamma$ are even or odd. Indeed, let $\xi$ be an eigenvalue of $L_1^\gamma$ with eigenvector $v \in D(L_1^\gamma)$. Then clearly $\tilde{v}$ with $\tilde{v}(x) = v(-x)$ is also an eigenvector associated to $\xi$. In particular, $v$ and $\tilde{v}$ both satisfy

$$-\partial_x^2 v + (\omega - \xi) v - p \varphi_\gamma^{p-1} v = 0 \quad \text{on } [0, +\infty),$$

thus there exists $\eta \in \mathbb{R}$ such that $v = \eta \tilde{v}$ on $[0, +\infty)$ (this is standard, see, for example, [5, Chapter 2, Theorem 3.3]). If $v(0) \neq 0$, it is immediate that $\eta = 1$. If $v(0) = 0$, then $\partial_x v(0+) \neq 0$ (otherwise the Cauchy–Lipschitz Theorem leads to $v \equiv 0$), and it is also immediate that $\eta = -1$. Arguing in the same way on $(-\infty, 0]$, we conclude that $v$ is even or odd, and in particular $v$ is even if and only if $v(0) \neq 0$.

Finally, we prove the last statement only for the case $\gamma < 0$ since the case $\gamma > 0$ is similar. We remark that $\partial_x \varphi_0$ is odd. Since $\lim_{\gamma \to 0}(f(\gamma), \partial_x \varphi_0)_2 = \|\partial_x \varphi_0\|_2^2 \neq 0$, we have $(f(\gamma), \partial_x \varphi_0)_2 \neq 0$ for $\gamma$ close to 0, thus $f(\gamma)$ cannot be even, and therefore $f(\gamma)$ is odd. Let $\tilde{\gamma}_\infty$ be

$$\tilde{\gamma}_\infty = \inf\{\tilde{\gamma} < 0;\ f(\gamma) \text{ is odd for any } \gamma \in (\tilde{\gamma}, 0]\}.$$

We suppose that $\tilde{\gamma}_\infty > -\infty$. If $f(\tilde{\gamma}_\infty)$ is odd, by continuity in $\gamma$ of $f(\gamma)$ with $L^2$ value, there exists $\varepsilon > 0$ such that $f(\tilde{\gamma}_\infty - \varepsilon)$ is odd which is a contradiction with the definition of $\tilde{\gamma}_\infty$, thus $f(\tilde{\gamma}_\infty)$ is even. Now, $f(\tilde{\gamma}_\infty)$ is the limit of odd functions, thus is odd. The only possibility to have $f(\tilde{\gamma}_\infty)$ both even and odd is $f(\tilde{\gamma}_\infty) \equiv 0$, which is impossible because $f(\tilde{\gamma}_\infty)$ is an eigenvector. □

## Appendix D. Proof of Proposition 6

For $a \in \mathbb{N}\setminus\{0\}$, we define $V^a(x) = \gamma a e^{-\pi a^2 x^2}$. It is clear that $\int_{\mathbb{R}} V^a(x) = \gamma$ and $V^a \rightharpoonup \gamma\delta$ weak-$\star$ in $H^{-1}(\mathbb{R})$ when $a \to +\infty$. We begin by the construction of approximate solutions: for

$$\begin{cases} i\partial_t u = -\partial_x^2 u - V^a u - |u|^{p-1}u, \\ u(0) = u_0, \end{cases} \tag{D.1}$$

and thanks to [7, Proposition 6.4.1], for every $a \in \mathbb{N}\setminus\{0\}$ there exists $T^a > 0$ and a unique maximal solution $u^a \in \mathcal{C}([0,T^a), H^1(\mathbb{R})) \cap \mathcal{C}^1([0,T^a), H^{-1}(\mathbb{R}))$ of (D.1) which satisfies for all $t \in [0, T^a)$

$$E^a(u^a(t)) = E^a(u_0), \tag{D.2}$$

$$\|u^a(t)\|_2 = \|u_0\|_2, \tag{D.3}$$

where $E^a(v) = \frac{1}{2}\|\partial_x v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}} V^a|v|^2 dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1}$. Moreover, the function $f^a : t \mapsto \int_{\mathbb{R}} x^2|u^a(t,x)|^2 dx$ is $\mathcal{C}^2$ by [7, Proposition 6.4.2], and

$$\partial_t f^a = 4\mathrm{Im}\int_{\mathbb{R}} \overline{u^a} x \partial_x u^a dx, \tag{D.4}$$

$$\partial_t^2 f^a = 8Q_\gamma^a(u^a), \tag{D.5}$$

where $Q_\gamma^a$ is defined for $v \in H^1(\mathbb{R})$ by

$$Q_\gamma^a(v) = \|\partial_x v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}} x(\partial_x V^a)|v|^2 dx - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Then, we find estimates on $(u^a)$. Let $M \geqslant \|u_0\|_{H^1(\mathbb{R})}$ (an exact value of $M$ will be obtained precisely later). We define

$$t^a = \sup\{t > 0;\ \|u^a(s)\|_{H^1(\mathbb{R})} \leqslant 2M \text{ for all } s \in [0,t)\}. \tag{D.6}$$

Since $u^a$ satisfies (D.1), we have

$$\sup_{a\in\mathbb{N}\setminus\{0\}} \|\partial_t u^a\|_{L^\infty([0,t^a),H^{-1}(\mathbb{R}))} \leqslant C,$$

and thus for all $t \in [0,t^a)$ and for all $a \in \mathbb{N}\setminus\{0\}$ we get

$$\|u^a(t) - u_0\|_2^2 = 2\int_0^t (u^a(s) - u_0, \partial_t u^a(s))_2 ds \leqslant Ct, \tag{D.7}$$

where $C$ depends only on $M$. Now we have

$$\frac{1}{p+1}(\|u^a\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1}) = \int_0^1\int_{\mathbb{R}} (u^a - u_0)|su^a + (1-s)u_0|^p dx ds$$

which combined with Hölder inequality, Sobolev embeddings, (D.6) and (D.7) gives

$$\frac{1}{p+1}(\|u^a\|_{p+1}^{p+1} - \|u_0\|_{p+1}^{p+1}) \leqslant Ct^{1/2}. \tag{D.8}$$

Moreover, using (D.6), Sobolev embeddings, the Gagliardo–Nirenberg inequality and (D.7) we obtain

$$\left|\int_{\mathbb{R}} V^a(|u^a|^2 - |u_0|^2)\right| \leqslant Ct^{1/4}. \tag{D.9}$$

Combining (D.2), (D.3), (D.8) and (D.9) leads to

$$\|u^a(t)\|_{H^1(\mathbb{R})}^2 \leqslant M^2 + C(t^{1/4} + t^{1/2}) \quad \text{for all } t \in [0,t^a) \text{ and for all } a \in \mathbb{N}\setminus\{0\},$$

and choosing $T_M$ (depending only on $M$) such that $C(T_M^{1/4} + T_M^{1/2}) = 3M^2$ we obtain for all $a \in \mathbb{N} \setminus \{0\}$ the estimates

$$\begin{aligned} &\|u^a\|_{L^\infty([0,T_M),H^1(\mathbb{R}))} \leqslant 2M, \\ &\|\partial_t u^a\|_{L^\infty([0,T_M),H^{-1}(\mathbb{R}))} \leqslant C. \end{aligned} \tag{D.10}$$

In particular, it follows from (D.10) that $T_M \leqslant t^a$ for all $a \in \mathbb{N} \setminus \{0\}$.

Now we can pass to the limit: thanks to (D.10) there exists $u \in L^\infty([0, T_M), H^1(\mathbb{R}))$ such that for all $t \in [0, T_M)$ we have

$$u^a(t) \rightharpoonup u(t) \quad \text{weakly in } H^1(\mathbb{R}) \text{ when } a \to +\infty, \tag{D.11}$$

which immediately induces that when $a \to +\infty$,

$$|u^a(t)|^{p-1}u^a(t) \rightharpoonup |u(t)|^{p-1}u(t) \quad \text{weakly in } H^{-1}(\mathbb{R}). \tag{D.12}$$

In particular, thanks to Sobolev embeddings, we have

$$u^a(t,x) \to u(t,x) \quad \text{a.e. and uniformly on the compact sets of } \mathbb{R},$$

and it is not hard to see that this permits us to show

$$V^a u^a \rightharpoonup u\gamma\delta \quad \text{weak-}\star \text{ in } H^{-1}(\mathbb{R}). \tag{D.13}$$

Since $u^a$ satisfies (D.1), it follows from (D.11)–(D.13) that $u$ satisfies (2). Finally, by (6) and (D.3), we have

$$u^a \to u \quad \text{in } \mathcal{C}([0,T_M), L^2(\mathbb{R})),$$

thus, from the Gagliardo–Nirenberg inequality and (D.10), we have

$$u^a \to u \quad \text{in } \mathcal{C}([0,T_M), L^{p+1}(\mathbb{R})),$$

and by (5) and (D.2) it follows that

$$u^a \to u \quad \text{in } \mathcal{C}([0,T_M), H^1(\mathbb{R})). \tag{D.14}$$

We have to prove that the time interval $[0, T_M)$ can be extended to be as large as we need. Let $0 < T < T_{u_0}$ and

$$M = \sup\{\|u(t)\|_{H^1(\mathbb{R})}, t \in [0,T]\}.$$

If $T_M \geqslant T$, there is nothing left to do, thus we suppose $T_M < T$. From (D.14) we have $\|u^a(T_M)\|_{H^1(\mathbb{R})} \leqslant M$ for $a$ large enough. By performing a shift of time of length $T_M$ in (2) and (D.1) and repeating the first steps of the proof we obtain

$$u^a \to u \quad \text{in } \mathcal{C}([T_M, 2T_M), H^1(\mathbb{R})).$$

Now we reiterate this procedure a finite number of times until we covered the interval $[0, T]$ to obtain

$$u^a \to u \quad \text{in } \mathcal{C}([0,T], H^1(\mathbb{R})). \tag{D.15}$$

To conclude, we remark that (7) follows from the same proof as in [7, Lemma 6.4.3] (computing with $\|e^{\varepsilon|x|^2}xu(t)\|_2^2$ and passing to the limit $\varepsilon \to 0$), thus we have

$$\|xu(t)\|_2^2 = \|xu_0\|_2^2 + 4\int_0^t \operatorname{Im}\int_{\mathbb{R}} \overline{u(s)}x\partial_x u(s)\mathrm{d}x\mathrm{d}s. \tag{D.16}$$

From (D.4), the Cauchy–Schwartz inequality and (D.10) we have

$$\partial_t\left(\|xu^a(t)\|_2^2\right) \leqslant C\|xu^a(t)\|_2,$$

which implies that

$$\|xu^a(t)\|_2 \leqslant \|xu_0\|_2 + Ct.$$

Since in addition we have

$$xu^a(t,x) \to xu(t,x) \quad \text{a.e.},$$

we infer that

$$xu^a(t,x) \rightharpoonup xu(t,x) \quad \text{weakly in } L^2(\mathbb{R}).$$

Recalling that

$$\partial_x u^a \to \partial_x u \quad \text{strongly in } L^2(\mathbb{R})$$

we can pass to the limit in (D.16) to obtain

$$\|xu^a(t)\|_2 \to \|xu(t)\|_2.$$

On the other hand, since we have (D.5) and (D.15), we get (8). □

**Remark 19.** Our method of approximation is inspired by the one developed in [9] by Cazenave and Weissler to prove the local well-posedness of the Cauchy problem for nonlinear Schrödinger equations. Actually, slight modifications in our proof of Proposition 6 would permit one to give an alternative proof of Proposition 2.

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