Multisolitons for NLS

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Plan

NLS

$$\text{(NLS)} \qquad \begin{cases} iu_t + \Delta u + g(|u|^2)u = 0 \\ u\big|_{t=0} = u_0 \end{cases} \qquad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$$

(A0) (regular) $g \in \mathcal{C}^1((0,+\infty),\mathbb{R})$

(A1) (superlinear) $g(0) = 0, \ \lim_{s\to 0} sg'(s) = 0$

(A2) (H^1-subcritical) $\exists p \in (1, 2^\star - 1)$ s.t. $\left|s^2 g'(s^2)\right| \lesssim s^{p-1}$ $(s > 1)$

(A3) (focusing) $\exists s_0$ s.t. $F(s_0) > \frac{s_0^2}{2}$; $F(z) := \int_0^{|z|} g(s^2)s\,ds.$

Cauchy Problem

$$(NLS) \quad iu_t + \Delta u + g(|u|^2)u = 0$$

Proposition

(NLS) is locally well-posed in H^1 under (A0)-(A3), i.e. there exists a unique maximal solution $u \in \mathcal{C}((-T_\star, T^\star), H^1)$.

Conserved Quantities

$$\begin{cases} E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^d} F(u)dx & \textit{(Energy)} \\ M(u) = \frac{1}{2}\|u\|_{L^2}^2 & \textit{(Mass)} \\ P(u) = \frac{1}{2} Im \int_{\mathbb{R}^d} \nabla \bar{u} u dx & \textit{(Momentum)} \end{cases}$$

Blow Up Alternative : If $T^\star < +\infty$, then $\lim_{t \to T^\star} \|u(t)\|_{H^1} = +\infty$

Large Time Behavior - Heuristic

$$u \text{ solution of } (NLS) \quad iu_t + \Delta u + g(|u|^2)u = 0$$

3 possible behaviors at large time

- Scattering $u(t) \sim e^{-i\Delta t}u_+$ as $t \to +\infty$
- Focusing/Blow Up $T^\star < +\infty$
- Soliton fixed profile in a moving frame

$\to$ Solitons Resolution Conjecture

Modest Goal

Study solutions of NLS composed of several solitons

Solitons

Definition of a soliton

Take a frequency $\omega > 0$, a speed $v \in \mathbb{R}^d$, an initial phase $\gamma \in \mathbb{R}$, an initial position $x_0 \in \mathbb{R}^d$, and a bound state solution $\Phi \in H^1$ to

$$\text{(SNLS)} \qquad -\Delta\Phi + \omega\Phi - g(|\Phi|^2)\Phi = 0, \qquad \Phi \in H^1.$$

A soliton is a solution of (NLS) traveling on the line $x = x_0 + vt$ and given by

$$R(t,x) := \Phi(x - vt - x_0)e^{i(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t + \omega t + \gamma)}.$$

Bound states

(SNLS) $$-\Delta\Phi + \omega\Phi - g(|\Phi|^2)\Phi = 0, \qquad \Phi \in H^1$$

Proposition

- *(Ground State) There exists a solution[a] Q of* (SNLS)*, which minimizes the action S, i.e.*

$$S(Q) = \min\{S(w) | w \text{ sol of (SNLS)}\}$$

where $S(w) := E(w) + \omega M(w)$.

- *(Excited States) If $d \geq 2$, there exists infinitely many other solutions to* (SNLS)
- *(Exponential Decay) Any solution Φ to* (SNLS) *verifies* $|\Phi(x)| \lesssim e^{i\frac{\sqrt{\omega}}{2}|x|}$.

[a] unique if $g(|u|^2)u = |u|^{p-1}u$

Dynamical properties of the solitons

In the case $g(|u|^2)u = |u|^{p-1}u$.

- with ground states
  - $1 < p < 1 + \frac{4}{d}$: stability (Cazenave-Lions, 1982)
  - $1 + \frac{4}{d} \leq p$: instability (Berestycki-Cazenave, 1981; Weinstein, 1983)
- with excited states
  - $1 < p < 1 + \frac{4}{d}$: partial results of instability (Grillakis 1990, Mizumachi 2007, Chang, Gustafson, Nakanishi et Tsai 2007).
  - $1 + \frac{4}{d} \leq p$: instability for real and radial excited states (Grillakis 1988, Jones 1988) or for some vortices (Mizumachi 2005).

Set ans sum of solitons

Definition of a set of solitons

A set of solitons is the data of $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$, where $N \in \mathbb{N}$, $N \geq 1$, and for $j = 1, ..., N$, we have $\omega_j > 0$, $x_j, v_j \in \mathbb{R}^d$ with $v_j \neq v_k$ if $j \neq k$, $\gamma_j \in \mathbb{R}$, and Φ_j is a sol of (SNLS) (with ω_j instead of ω).

Given a set of solitons, we define the sum of solitons

$$R(t,x) = \sum_{j=1}^{N} R_j(t,x),$$

where $R_j(t,x) := \Phi_j(x - v_j t - x_j) e^{i(\frac{1}{2} v_j \cdot x - \frac{1}{4}|v_j|^2 t + \omega_j t + \gamma_j)}$.

NB : R is *not* a solution to (NLS)

Multi-solitons

Definition of a multi-soliton

A multisoliton is a solution u of (NLS) for which there exist T_0 and R defined as above such that u exists on $[T_0, +\infty)$ and

$$\lim_{t \to +\infty} \|u(t) - R(t)\|_{H^1} = 0.$$

Natural questions about multisolitons

- Existence
- Uniqueness
- Stability

1972

SOVIET PHYSICS JETP VOLUME 34, NUMBER 1 JANUARY 1972

EXACT THEORY OF TWO-DIMENSIONAL SELF-FOCUSING AND ONE-DIMENSIONAL SELF-MODULATION OF WAVES IN NONLINEAR MEDIA

V. E. ZAKHAROV and A. B. SHABAT
Institute of Hydrodynamics, Siberian Division, U.S.S.R. Academy of Sciences

Submitted December 22, 1970
Zh. Eksp. Teor. Fiz. 61, 118–134 (July, 1971)

... the equation $i\partial\psi/\partial t + \psi_{xx} + \kappa|\psi|^2\psi = 0$, which describes plane self-focusing and ... the inverse scattering problem

Theorem (Zakharov and Shabat 1972)

If $d = 1$, $g(|u|^2)u = |u|^2 u$ (completely integrable case), then for any set of solitons $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$, there exists a solution u of (NLS) such that

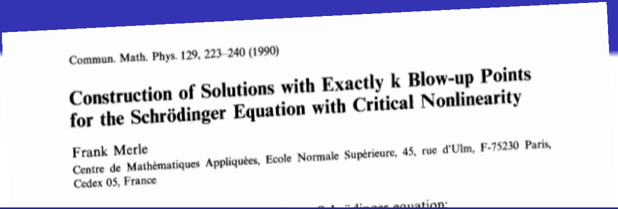
- u exists on $\mathbb{R}$
- there is an explicit construction for u
- $$\begin{aligned} u(t) - \tilde{R}(t) &\to 0 \text{ as } t \to -\infty \\ u(t) - R(t) &\to 0 \text{ as } t \to +\infty, \end{aligned}$$ where $\tilde{R} \sim R$ with $(\tilde{x}_j, \tilde{\gamma}_j)$.

It is convenient to assign the variable t the meaning of time.

In the present paper we shall investigate Eq. (3) with $\kappa > 0$. As applied to Eq. (1), this means $\delta n_{nl} > 0$. Under this condition, Eq. (1) describes stationary two-dimensional self-focusing and the associated transverse instability of a plane monochromatic wave.[5] For ...

... equation. In the case considered by them, the role of the operator $\hat{L}$ was played by the one-dimensional Schrödinger operator. They revealed the fundamental role played by the particular solutions of the KDV equation—solitons, which are directly connected with the discrete spectrum of the operator $\hat{L}$, namely, it was established that the asymptotic state as $t \to \pm\infty$ of any initial condition is a finite set of solitons. In our prob-...

1990


Commun. Math. Phys. 129, 223–240 (1990)

Construction of Solutions with Exactly k Blow-up Points for the Schrödinger Equation with Critical Nonlinearity

Frank Merle
Centre de Mathématiques Appliquées, Ecole Normale Supérieure, 45, rue d'Ulm, F-75230 Paris, Cedex 05, France

∼Merle 1990

Let $d \geq 1$ and take a set of solitons $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$.
Assume $g(|u|^2)u = |u|^{4/d}u$ (L^2-critical case) and a large frequencies assumption.
Then there exists $T_0 > 0$ and a multisoliton solution u of (NLS) s.t.

$$\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0.$$

$\forall t\in[0,T)$,

Remark

It is the dual version of the result of existence of a multiple blow-up points solutions (by conformal invariance).

Ginibre and Velo [4,5], Kato [7]). Furthermore, we have $\forall t\in[0,T)$, (2)

2006

Theorem (Martel-Merle 2006)

Let $d \geq 1$ and take a set of solitons $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$.
Assume $g(|u|^2)u$ verifies a stability assumption (L^2-subcritical) and (Φ_j) are ground states.
Then there exist $T_0 \in \mathbb{R}$ and a solution u of (NLS) on $[T_0, +\infty)$ s.t.

$$\|u(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star} v_\star t} \qquad \text{on } [T_0, +\infty)$$

where $\alpha > 0$ and

$v_\star := \min\{|v_j - v_k|, j, k = 1, ..., N, j \neq k\}$ (minimal relative speed)
$\omega_\star := \min\{\omega_j, j = 1, ..., N\}$ (minimal frequency)

2011

Rev. Mat. Iberoamericana **27** (2011), no. 1, 273–302

Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations

Raphaël Côte, Yvan Martel and Frank Merle

Theorem (Côte-Martel-Merle 2011)

Let $d \geq 1$ and take a set of solitons $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$.
Assume $g(|u|^2)u = |u|^{p-1}u$, $p > 1 + 4/d$ (L^2–supercritical) and (Φ_j) are ground states.
Then there exist $T_0 \in \mathbb{R}$ and a solution u of (NLS) on $[T_0, +\infty)$ s.t.

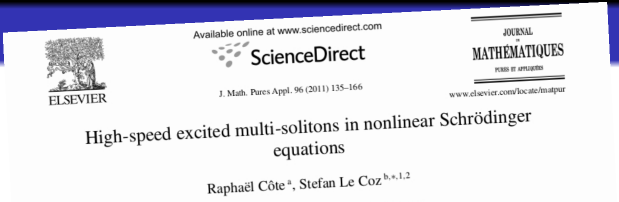
$$\|u(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star} v_\star t} \qquad \text{on } [T_0, +\infty)$$

where $\alpha > 0$ and

$$v_\star := \min\{|v_j - v_k|, j, k = 1, ..., N, j \neq k\} \quad \text{(minimal relative speed)}$$
$$\omega_\star := \min\{\omega_j, j = 1, ..., N\} \quad \text{(minimal frequency)}$$

2000 *Mathematics Subject Classification*: 35Q51, 35Q53, 35Q55.
Multi-solitons, generalized Korteweg-de Vries equation, nonlinear Schrödinger

2011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J. Math. Pures Appl. 96 (2011) 135–166
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

High-speed excited multi-solitons in nonlinear Schrödinger equations

Raphaël Côte[a], Stefan Le Coz[b,*,1,2]

Theorem (Côte-L.C. 2011)

Let $d \geq 1$ and take a set of solitons $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$.
Assume g is generic (satisfies (A0)-(A3)) and
(Φ_j) are ground states or excited states.
There exists $v_\sharp$ such that if $v_\star > v_\sharp$ (high relative speeds)
then there exist $T_0 \in \mathbb{R}$ and a solution u of (NLS) on $[T_0, +\infty)$ s.t.

$$\|u(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star} v_\star t} \qquad \text{on } [T_0, +\infty)$$

where $\alpha > 0$ and

$$v_\star := \min\{|v_j - v_k|, j, k = 1, \ldots, N, j \neq k\} \quad \text{(minimal relative speed)}$$
$$\omega_\star := \min\{\omega_j, j = 1, \ldots, N\} \qquad \text{(minimal frequency)}$$

Summary for existence

NLS

- 1972 Zakharov and Shabat, integrable case
- 1990 Merle, L^2-critical case, high frequencies
- 2006 Martel-Merle, L^2 subcritical with ground states
- 2011 Côte-Martel-Merle, L^2 supercritical with ground states
- 2011 Côte-L.C., any nonlinearity, ground and excited states, high speeds

Open problem

- Small speeds for excited states

Scheme of proof : Backward resolution of (NLS)

Take a set of solitons $(N, \omega_j, v_j, x_j, \gamma_j, \Phi_j)$.
Take $(T_n) \uparrow +\infty$ and (u_n) solutions to (NLS)
with final data $u_n(T_n) = R(T_n)$.

Goal

- Approximate Multisolitons. Show that for each n, u_n exists on $[T_0, T_n]$ with T_0 independent of n
- Convergence. Show that (u_n) converges to a multi-soliton

Tools

- Uniform Estimates
- Compactness Argument

Tools

Proposition (Uniform Estimates)

There exists $v_\sharp > 0$ s.t. if $v_\star > v_\sharp$ there exists T_0 independent of n s.t. for n large u_n exist on $[T_0, T_n]$ and for all $t \in [T_0, T_n]$

$$\|u_n(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star}v_\star t}.$$

Proposition (Compactness)

There exists $u_0 \in H^1$ s.t.

$$\lim_{n\to+\infty} \|u_n(T_0) - u_0\|_{L^2} = 0$$

Rk: u_0 will be the initial data of the multisoliton.

Proof of the existence of a multisoliton

Assume the uniform estimates and the compactness result.
Take the solution u of (NLS) with at T_0 initial data $u(T_0) = u_0$.
Then for $t > T_0$

$$u_n(t) \underset{H^1}{\overset{L^2}{\rightrightarrows}} u(t).$$

Therefore

$$\|u(t) - R(t)\|_{H^1} \leq \liminf_{n \to +\infty} \|u_n(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star} v_\star t}$$

Hence u is a multi-soliton.

Proof of Uniform Estimates - Scheme

- A bootstrap argument
- Energy control of one soliton up to L^2 directions
- Localization procedure
- Energy control of N-solitons up to L^2 directions
- Control of the L^2 directions

The Bootstrap argument

Proposition (Bootstrap)

There exists $v_\sharp > 0$ *s.t. if* $v_\star > v_\sharp$ *there exists* T_0 *independent of* n *s.t. for* n *large* u_n *exist on* $[T_0, T_n]$ *and verifies:*
for all $t^\dagger \in [T_0, T_n]$, *if for all* $t \in [t^\dagger, T_n]$ *we have*

$$\text{(Bootstrap-1)} \qquad \|u_n(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star} v_\star t},$$

then for all $t \in [t^\dagger, T_n]$ *we have*

$$\text{(Bootstrap-1/2)} \qquad \|u_n(t) - R(t)\|_{H^1} \leq \frac{1}{2} e^{-\alpha\sqrt{\omega_\star} v_\star t}.$$

Energy control of one soliton

$$R_0(t,x) := \Phi_0(x - v_0 t - x_0)e^{i(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)}.$$

R_0 is a critical point of

$$S_0 = E + \left(\omega_0 + \frac{|v_0|^2}{4}\right) M + v_0 \cdot P$$

Define $H_0(t,w) := \langle S_0''(R_0)w, w\rangle$.

Proposition (Coercivity)

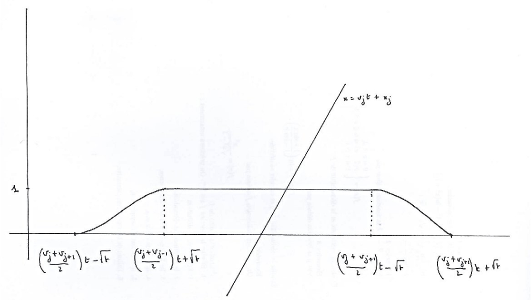
There exist $\nu_0 \in \mathbb{N}\setminus\{0\}$ and $X_0^1, ..., X_0^{\nu_0} \in L^2$ s.t. for all $w \in H^1$

$$\|w\|_{H^1}^2 \lesssim H_0(t,w) + \sum_{k=1}^{\nu_0} (w, X_0^k(t))_2^2.$$

where $X_0^k(t)(x) = X_0^k(x - v_0 t - x_0)e^{i(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)}.$

Localization of the conservation laws - 1

For each $j = 1, ..., N$, we define a cut-off function ϕ_j :



Localization of the conservation laws - 2

and we set

$$
\begin{aligned}
E_j(t,w) =& \frac{1}{2}\int_{\mathbb{R}^d} |\nabla w|^2 \phi_j(t,x)dx - \int_{\mathbb{R}^d} F(w)\phi_j(t,x)dx \\
M_j(t,w) =& \frac{1}{2}\int_{\mathbb{R}^d} |w|^2 \phi_j(t,x)dx \\
P_j(t,w) =& \frac{1}{2}\Im \int_{\mathbb{R}^d} \bar{w}\nabla w \phi_j(t,x)dx \\
S_j(t,w) =& E_j(t,w) + \left(\omega_j + \frac{|v_j|^2}{4}\right) M_j(t,w) - v_j \cdot P(t,w) \\
H_j(t,w) =& \left\langle S_j''(R_j(t)) w\phi_j(t), w \right\rangle
\end{aligned}
$$

Energy control for N solitons

Proposition

For any t large enough

$$\|w_n\|_{H^1}^2 \lesssim \mathcal{H}(t, w_n) + \sum_{j=1}^{N} \sum_{l=1}^{\nu_j} (w_n, X_j^l(t))_2^2,$$

where $\mathcal{H}(t, w) = \sum_{j=1}^{N} H_j(t, w)$.

Control of the linearized action

Take $w_n = u_n - R$ and recall (Bootstrap-1)

$$\|w_n(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star}v_\star t} \qquad \text{on } [t^\dagger, T_n].$$

Control of the linearized action

$$\mathcal{H}(t, w_n) \lesssim \frac{1}{\sqrt{t}} e^{-2\alpha v_\star t} + o(\|w_n\|_{H^1}^2) \qquad \text{on } [t^\dagger, T_n].$$

Almost conservation of the localized quantities

For T_0 large enough, we have on $[t^\dagger, T_n]$ and for any j

$$|M_j(t, w_n) - M_j(T_n, w_n)| + |P_j(t, w_n) - P_j(T_n, w_n)| \lesssim \frac{1}{\sqrt{t}} e^{-2\alpha\sqrt{\omega_\star}v_\star t}.$$

Key ingredient : exponential decay of the solitons.

Control of the L^2-directions

Recall that $w_n = u_n - R$ and (Bootstrap-1):

$$\|w_n(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_\star}v_\star t} \qquad \text{on } [t^\dagger, T_n].$$

Then

$$i\partial_t w_n + \mathcal{L}w_n + \mathcal{N}(w_n) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|w_n\|_{L^2}^2 = (i\mathcal{L}w_n, w_n)_2 + (i\mathcal{N}(w_n), w_n)_2.$$

It is easy to see that

$$(i\mathcal{L}w_n, w_n)_2 \leq \frac{C_{\mathcal{L}}}{2}\|w_n\|_{L^2}^2 \qquad ; \qquad (i\mathcal{N}(w_n), w_n)_2 = o(\|w_n\|_{H^1}^2),$$

Then $\left|\frac{d}{dt}\|w_n\|_{L^2}^2 t\right| \leq Ce^{-2\alpha v_\star t}$, donc $\|w_n\|_{L^2}^2 \leq \frac{C}{2\alpha v_\star}e^{-2\alpha v_\star t}$.

Summary of the proof

- Backward resolution of (NLS)
- Uniform estimates
  - Energy control
  - Localization procedure
  - Deal with L^2-directions
- Compactness argument for the initial data

The notion of stability

If a solution of (NLS) starts close to a sum of solitons, then it

- Orbital Stability: remains close for all time to the sum of solitons, possibly modified by translations or phase shifts.
- Asymptotic Stability: converges as $t \to +\infty$ to the sum of solitons, possibly modified by translations or phase shifts.
- (Forward) Instability: leaves in finite time the neighborhood of the sum of solitons, possibly modified by translations or phase shifts.

2004

Asymptotic stability of N-soliton states of NLS

I. Rodnianski, W. Schlag and A. Soffer *

October 10, 2003

Abstract

The asymptotic stability and asymptotic completeness of NLS solitons is
turbations of arbitrary number of non-colliding solitons.

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
Vol. 29, Nos. 7 & 8, pp. 1051–1095, 2004

Asymptotic Stability of Multi-soliton Solutions for Nonlinear Schrödinger Equations

Galina Perelman*

Perelman 1997,2004/Rodnianski, Schlag, and Soffer 2003

If

- Flatness of the nonlinearity at 0,
- Spectral Assumptions (linear stability)
- High relative speeds.

Then asymptotic stability of well-ordered multi-solitons in strong norms.

2006

STABILITY IN H^1 OF THE SUM OF K SOLITARY WAVES FOR SOME NONLINEAR SCHRÖDINGER EQUATIONS

YVAN MARTEL, FRANK MERLE, and TAI-PENG TSAI

Abstract

Martel-Merle-Tsai 2006

If

- Flatness of the nonlinearity at 0,
- Orbital Stability of each solitons,
- High relative speeds.

Then orbital stability of well-ordered multi-solitons in the energy space H^1.

A new construction of a multisoliton...

Theoreme (Côte-L.C.)

Take $a \in \mathbb{R}$ and a sum of solitons $R = \sum_{j=1}^{N} R_j$.
Assume $g \in \mathcal{C}^\infty$ and one soliton (e.g. R_1) is linearly unstable , i.e. there exists an eigenvalue $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ of the linearization L of (NLS) around R_1. Then there exists $v_\natural > 0$ s.t. if $v_\star > v_\natural$ (large relative speeds) there exist T_0 and u solution of (NLS) on $[T_0, +\infty)$ s.t. for all $t \geq T_0$

$$\|u(t) - R(t) - aY(t)\|_{H^1} \leq Ce^{-2\Re(\lambda)t}$$

where Y is of the form

$$Y(t) = e^{-\Re(\lambda)t}(\cos(\Im(\lambda)t)Y_1 + \sin(\Im(\lambda)t)Y_2).$$

... which turns out to be unstable

Corollary (Orbital instability of the multisoliton)

Same hypotheses.
There exists $\varepsilon > 0$, such that for all $n \in \mathbb{N} \setminus \{0\}$ and for all $T \in \mathbb{R}$ the following holds. There exists $I_n, J_n \in \mathbb{R}$, $T \leq I_n < J_n$ and a solution $w_n \in \mathcal{C}([I_n, J_n], H^1(\mathbb{R}^d))$ to (NLS) such that

$$\lim_{n \to +\infty} \|w_n(I_n) - R(I_n)\|_{H^1} = 0,$$

$$\inf_{\substack{y_j \in \mathbb{R}^d, \vartheta_j \in \mathbb{R}, \\ j=1,\dots,N}} \left\| w_n(J_n) - \sum_{i=1}^{N} \Phi_j(x - y_j) e^{i(\frac{1}{2} v_j \cdot x + \vartheta_j)} \right\|_{L^2} \geq \varepsilon.$$

Proof of the Theorem

By fixed point around a good approximation of u.

Proposition

Take $N_0 \in \mathbb{N}$ and $a \in \mathbb{R}$. Then there exists $W^{N_0} \in C^\infty([0,+\infty), \mathscr{H})$ s.t. $U = R + W^{N_0}$ is a solution to (NLS) up to an order $O(e^{-\rho(N_0+1)t})$ when $t \to +\infty$, i.e.

$$U_t + \Delta U + g(|U|^2)U = Err = O(e^{-\rho(N_0+1)t})$$

Define the map

$$w \mapsto \Psi(w) = -i \int_t^{+\infty} e^{i\Delta(t-\tau)}(f((U+w)(\tau)) - f(U(\tau)) - Err(\tau))d\tau.$$

Fixed point argument in

$$X^{\sigma}_{T_0,N_0}(B) := \left\{ w \in \mathcal{C}((T_0,+\infty), H^\sigma) \,\middle|\, \sup_{t \geq T_0} e^{(N_0+1)\rho t} \|w(t)\|_{H^\sigma} < B \right\}.$$

Construction of the profile - preliminaries

Inspired by works of Duyckaerts, Merle, Roudenko.
Look at the linearization of (NLS) around $e^{i\omega t}\Phi(x)$.
If u is a solution of (NLS) and $u = e^{i\omega t}(\Phi(x) + w)$, then v is a solution of

$$w_t + L_{\mathbb{C}} w = \mathscr{M}_{\mathbb{C}}(w),$$

where

$$L_{\mathbb{C}} w = -i\Delta w + i\omega w - idf(\Phi).w,$$
$$\mathscr{M}_{\mathbb{C}}(w) = if(\Phi + w) - if(\Phi) - idf(\Phi).w,$$

Construction of the profil

Proposition

Take $N_0 \in \mathbb{N}$ *and* $a \in \mathbb{R}$*. Then there exists* $W^{N_0} \in C^\infty([0,+\infty),\mathscr{H})$ *s.t. when* $t \to +\infty$,

$$\partial_t W^{N_0} + L_{\mathbb{R}^2} W^{N_0} = \mathscr{M}_{\mathbb{R}^2}(W^{N_0}) + O(e^{-\rho(N_0+1)t}),$$

Separate $L_{\mathbb{C}}$ into real and imaginary parts:

$$L_{\mathbb{R}^2}\begin{pmatrix} w_R \\ w_I \end{pmatrix} = \begin{pmatrix} J & \Delta - \omega + I_1 \\ -\Delta + \omega - I_2 & -J \end{pmatrix}\begin{pmatrix} w_R \\ w_I \end{pmatrix}.$$

with I and J real valued potentials

Construction of the profile - order 1

- Complexify $L_{\mathbb{R}^2}$ into $L_{\mathbb{C}^2}$
- There exists an eigenvalue $\lambda = \rho + i\theta \in \mathbb{C}$ of $L_{\mathbb{C}^2}$ with maximal real part $\rho > 0$.
- Set $Z = \begin{pmatrix} Z^+ \\ Z^- \end{pmatrix}$ an eigenvector and denote $Y_1 = \Re(Z)$, $Y_2 = \Im(Z)$.
- Denote $Y(t) = e^{-\rho t}(\cos(\theta t)Y_1 + \sin(\theta t)Y_2)$.
- Then $\partial_t Y + L_{\mathbb{R}^2} Y = 0$.

Construction of the profile - order N_0

- Look for W^{N_0} in the form

$$W(t,x) = \sum_{k=1}^{N_0} e^{-\rho k t} \left(\sum_{j=0}^{k} A_{j,k}(x)\cos(j\theta t) + B_{j,k}(x)\sin(j\theta t) \right)$$

- Remark that

$$\mathscr{M}_{\mathbb{R}^2}(W) = \sum_{\kappa=2}^{N_0} e^{-\kappa\rho t} \sum_{j=0}^{\kappa} \left(\tilde{A}_{j,\kappa}(x)\cos(j\theta t) + \tilde{B}_{j,\kappa}(x)\sin(j\theta t) \right) + HOT$$

- In addition

$$(\partial_t W + L_{\mathbb{R}^2} W) = \sum_{k=1}^{N_0} e^{-\rho k t} \left(\sum_{j=0}^{k} (L_{\mathbb{R}^2} A_{j,k} + j\theta B_{j,k} - k\rho A_{j,k})\cos(j\theta t) \right.$$

$$\left. + (L_{\mathbb{R}^2} B_{j,k} - j\theta A_{j,k} - k\rho B_{j,k})\sin(j\theta t) \right).$$

Construction of the profile - order N_0

•Therefore to find a satisfying W^{N_0} it is enought to solve for $k \geq 2$

$$\begin{cases} L_{\mathbb{R}^2} A_{j,k} + j\theta B_{j,k} - k\rho A_{j,k} = \tilde{A}_{j,k}, \\ L_{\mathbb{R}^2} B_{j,k} - j\theta A_{j,k} - k\rho B_{j,k} = \tilde{B}_{j,k}, \end{cases}$$

which is possible because λ is of maximal real part.

Open problems

- Existence for small speeds for excited states
- Better stability/instability results
- Uniqueness/Classification

Other equations

- KdV : Martel, Merle, Muñoz, Combet
- Camassa-Holm : El Dika, Molinet
- Schrödinger systems : Ianni, L.C.
- Hartree : Krieger, Martel, Raphaël
- GP : Béthuel, Gravejat, Smets
- etc.