Parametric Curves

duration : 30 minutes

No documents, no calculator, no mobile phone.

Exercise 1 (20 points)**.** We consider the parametric curve defined by

$$\begin{cases} x(t) = 3\cos(t) + \cos(3t)\,, \\ y(t) = 3\sin(t) - \sin(3t)\,. \end{cases}$$

1. (1 point) For which values of t are $x(t)$ and $y(t)$ well defined ?

Solution: The curve is defined for all $t \in \mathbb{R}$.

2. (1 point) What is the minimal period of the curve ?

Solution: The functions sin and cos are periodic with minimal period 2π, therefore the functions $t \mapsto \cos(3t)$, $t \mapsto \sin(3t)$ are periodic with minimial period $\frac{2\pi}{3}$. The least common multiple of 2π and $\frac{2\pi}{3}$ is 2π, therefore x et y are periodic with minimal period 2π.

3. (3 points) Calculate $x(-t)$ and $y(-t)$ in terms of $x(t)$ and $y(t)$.
What is the corresponding symmetry for the curve ?

Solution: Since $\cos(-t) = \cos(t)$ and $\sin(-t) = \sin(t)$ for any $t \in \mathbb{R}$, we have

$$x(-t) = x(t), \quad y(-t) = -y(t).$$

The curve is therefore symmetric with respect to the x-axis.

4. (3 points) Same question with $x(\pi - t)$ and $y(\pi - t)$.

Solution: Since $\cos(\pi - t) = -\cos(t)$ and $\sin(\pi - t) = \sin(t)$ for any $t \in \mathbb{R}$ and the functions sin et cos are periodic with period 2π, we have

$$\cos(3(\pi - t)) = \cos(\pi - 3t + 2\pi) = \cos(\pi - 3t) = -\cos(3t), \quad \sin(3(\pi - t)) = \sin(3t),$$

and therefore

$$x(\pi - t) = -x(t), \quad y(\pi - t) = y(t).$$

Thus, the curve is symmetric with respect to the y-axis.

5. (3 points) Same question with $x(\pi + t)$ and $y(\pi + t)$.

Solution: Since $\cos(\pi + t) = -\cos(t)$ and $\sin(\pi + t) = -\sin(t)$ for any $t \in \mathbb{R}$ and the functions sin et cos are periodic with period 2π, we have

$$\cos(3(\pi + t)) = \cos(\pi + 3t + 2\pi) = \cos(\pi + 3t) = -\cos(3t), \quad \sin(3(\pi - t)) = -\sin(3t),$$

and therefore

$$x(\pi - t) = -x(t), \quad y(\pi - t) = -y(t).$$

Thus, the curve is symmetric with respect to the origin.

6. (1 point) Show that we can restrict the domain of study to $\left[0, \frac{\pi}{2}\right]$.

Solution: Since the curve is periodic, we can restrict the domain of study to $[-\pi, \pi]$. From 3, we can restrict further to $[0, \pi]$ and from 4 we can restrict to $\left[0, \frac{\pi}{2}\right]$. No further restriction occurs from 5.

7. (2 points) After some simplifications, the derivatives of x and y are given by

$$x'(t) = -12\sin(t)\cos^2(t), \quad y'(t) = 12\cos(t)\sin^2(t),$$

Find the values of $t \in \left[0, \frac{\pi}{2}\right]$ such that $x'(t) = 0$ or $y'(t) = 0$.

Solution: For $t \in \left[0, \frac{\pi}{2}\right]$ we have $x'(t) = 0$ when $t = 0$ and $t = \frac{\pi}{2}$ and $y'(t) = 0$ also when $t = 0$ and $t = \frac{\pi}{2}$.

8. (3 points) Construct the *tabeau de variation* of the curve (do not forget the last line for the slopes of the tangents).

Solution: The *tabeau de variation* is the following

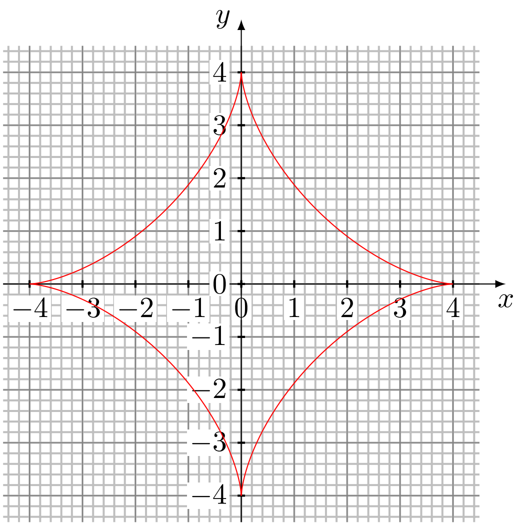
t	0		$\frac{\pi}{2}$
$x'(t)$	0	$-$	0
$x(t)$	4	$\searrow$	0
$y'(t)$	0	$+$	0
$y(t)$	0	$\nearrow$	4
$\frac{y'(t)}{x'(t)}$	0		$-\infty$

Notice that

$$\frac{y'(t)}{x'(t)} = -\frac{\sin(t)}{\cos(t)} = -\tan(t).$$

9. (3 points) Sketch the curve.

Solution:



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$$\begin{cases} x(t) = 3\cos(t) + \cos(3t), \\ y(t) = 3\sin(t) - \sin(3t). \end{cases}$$

1. (1 point) For which values of t are $x(t)$ and $y(t)$ well defined ?

> **Solution:** The curve is defined for all $t \in \mathbb{R}$.

2. (1 point) What is the minimal period of the curve ?

> **Solution:** The functions sin and cos are periodic with minimal period 2π, therefore the functions $t \mapsto \cos(3t)$, $t \mapsto \sin(3t)$ are periodic with minimial period $\frac{2\pi}{3}$. The least common multiple of 2π and $\frac{2\pi}{3}$ is 2π, therefore x et y are periodic with minimal period 2π.

3. (3 points) Calculate $x(-t)$ and $y(-t)$ in terms of $x(t)$ and $y(t)$.
What is the corresponding symmetry for the curve ?

> **Solution:** Since $\cos(-t) = \cos(t)$ and $\sin(-t) = \sin(t)$ for any $t \in \mathbb{R}$, we have
>
> $$x(-t) = x(t), \quad y(-t) = -y(t).$$
>
> The curve is therefore symmetric with respect to the x-axis.

4. (3 points) Same question with $x(\pi - t)$ and $y(\pi - t)$.

> **Solution:** Since $\cos(\pi - t) = -\cos(t)$ and $\sin(\pi - t) = \sin(t)$ for any $t \in \mathbb{R}$ and the functions sin et cos are periodic with period 2π, we have
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> $$\cos(3(\pi - t)) = \cos(\pi - 3t + 2\pi) = \cos(\pi - 3t) = -\cos(3t), \quad \sin(3(\pi - t)) = \sin(3t),$$
>
> and therefore
>
> $$x(\pi - t) = -x(t), \quad y(\pi - t) = y(t).$$
>
> Thus, the curve is symmetric with respect to the y-axis.

5. (3 points) Same question with $x(\pi + t)$ and $y(\pi + t)$.

> **Solution:** Since $\cos(\pi + t) = -\cos(t)$ and $\sin(\pi + t) = -\sin(t)$ for any $t \in \mathbb{R}$ and the functions sin et cos are periodic with period 2π, we have
>
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> and therefore
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> $$x(\pi - t) = -x(t), \quad y(\pi - t) = -y(t).$$
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> Thus, the curve is symmetric with respect to the origin.

6. (1 point) Show that we can restrict the domain of study to $\left[0, \frac{\pi}{2}\right]$.

> **Solution:** Since the curve is periodic, we can restrict the domain of study to $[-\pi, \pi]$. From 3, we can restrict further to $[0, \pi]$ and from 4 we can restrict to $\left[0, \frac{\pi}{2}\right]$. No further restriction occurs from 5.

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Notice that

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Solution: