Parametric curves, Polar curves

Exercise 1 (A parametric curve)**.** We consider the following parametric curve

$$\begin{aligned}\gamma : [0,\pi] &\to \mathbb{R}^2\\ t &\mapsto (x(t),y(t)) = (2\cos(t), 3\sin(t)).\end{aligned}$$

1. By evaluating $\gamma(t)$ for a certain number of well-chosen values of t, give a preliminary sketch of the curve parametrized by γ.
2. Show that the function $t \mapsto 9x(t)^2 + 4y(t)^2$ is constant.
3. What curve is represented by γ ?

Exercise 2 (Folium)**.** We consider the parametric curve defined by the following equations

$$\begin{cases} x(t) = \sin(2t), \\ y(t) = \sin(3t), \end{cases} \qquad t \in \mathbb{R}.$$

1. Using the symmetry properties of the curve, show that we can restrict the domain of study first to $t \in [-\pi, \pi]$, then to $t \in [0, \pi]$.
2. Express $x(\pi - t)$ and $y(\pi - t)$ in terms of $x(t)$ and $y(t)$. Show that the curve has an additional symmetry and that we can restrict the domain of study to $t \in \left[0, \frac{\pi}{2}\right]$.
3. Construct the *tableau de variation*[1] of x and y on $\left[0, \frac{\pi}{2}\right]$. Indicate the values of x, x', y, y' at $t = 0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$.
4. Sketch the curve first for $t \in \left[0, \frac{\pi}{2}\right]$, then sketch the whole curve.

Exercise 3 (Astroid)**.** We consider the parametric curve defined by the following equations

$$\begin{cases} x(t) = \cos^3(t), \\ y(t) = \sin^3(t), \end{cases} \qquad t \in \mathbb{R}.$$

1. Using the symmetry properties of the curve, restrict the domain of study of the curve to an interval of $\mathbb{R}$.
2. Construct the *tableau de variation* for x and y.
3. Give the coordinates of the curve when $t = 0$, $\frac{\pi}{2}$, π and give the direction vectors of the tangents at these points.
4. Sketch the curve.
5. Calculate the length and curvature of the astroid.

Exercise 4 (Infinite branches)**.** We consider the parametric curve defined by the following equations

$$\begin{cases} x(t) = \frac{1}{t(t-1)}, \\ y(t) = \frac{t^2}{1-t}, \end{cases} \qquad t \in \mathbb{R}.$$

1. Express $x\left(\frac{1}{t}\right)$ and $y\left(\frac{1}{t}\right)$ in terms of $x(t)$ and $y(t)$. Show that the curve has a symmetry and that we can restrict the domain of study to $I = (-1, 1) \setminus \{0\}$.
2. Construct the *tableau de variation* on I.
3. Study the infinite branches on I.
4. Sketch the curve.

Exercise 5. We consider the polar curve defined by

$$\rho(\theta) = \sin(3\theta), \quad \theta \in \mathbb{R}.$$

1. What is the period of ρ ?
2. Express $\rho(-\theta)$ and $\rho(\pi - \theta)$ in terms of $\rho(\theta)$. What are the symmetries of the curve ? Show that we can restrict the domain of study to $\theta \in \left[0, \frac{\pi}{3}\right]$.
3. Construct the *tableau de variation* on I. Give the equations for the tangents at the curve when $\theta = 0$ and $\theta = \frac{\pi}{3}$.
4. Sketch the curve.

Exercise 6. Study the polar curves defined for $\theta \in \mathbb{R}$ by

$$(1)\ \rho(\theta) = \cos(\theta) + 2, \quad (2)\ \rho(\theta) = \cos^2\left(\frac{\theta}{3}\right), \quad (3)\ \rho(\theta) = 1 + \sin(3\theta).$$

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1. The *tableau de variation* is a specifically French concept. For lack of an appropriate translation, we have chosen to keep the French expression in this document.