Parametric curves, Polar curves

Exercise 1 (A parametric curve)**.** We consider the following parametric curve

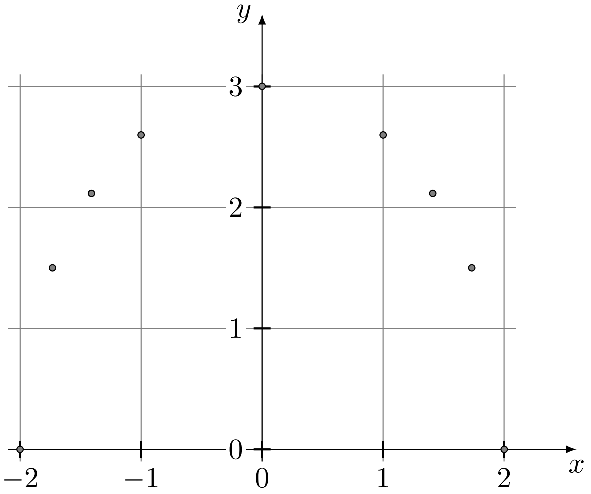
$$\gamma : [0,\pi] \to \mathbb{R}^2$$
$$t \mapsto (x(t), y(t)) = (2\cos(t), 3\sin(t)).$$

1. By evaluating $\gamma(t)$ for a certain number of well-chosen values of t, give a preliminary sketch of the curve parametrized by γ.

Solution: The curve described by γ being based on sin and cos, we can use the particular values of these functions to construct points of the curve. We summarize the results in the following table.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$x(t)$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2
$y(t)$	0	$\frac{3}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{3}}{2}$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3}{2}$	0

We obtain the following figure. The curve could be a parabola or an half-ellipse.



2. Show that the function $t \mapsto 9x(t)^2 + 4y(t)^2$ is constant.

Solution: We know from the Pythagorean theorem that

$$\cos^2(t) + \sin^2(t) = 1$$

for any $t \in \mathbb{R}$. As a consequence, for any $t \in \mathbb{R}$ we have the equality

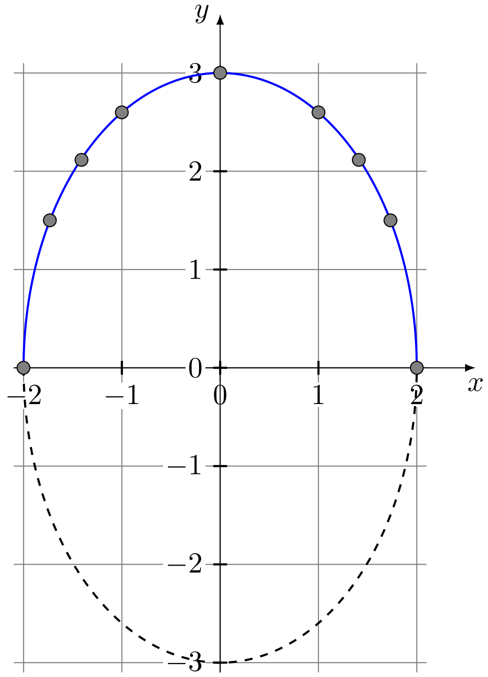
$$9x^2(t) + 4y^2(t) = 36\cos^2(t) + 36\sin^2(t) = 36.$$

3. What curve is represented by γ?

Solution: In the equation

$$9x^2 + 4y^2 = 36$$

we recognize the equation of an ellipse centered at the origin. We already know that for any $t \in [0, \pi]$ the point $(x(t), y(t))$ is on this ellipse. However, since $t \in [0, \pi]$, the ellipse is not entirely covered. At $t = 0$, we start from the point $(2, 0)$ on the ellipse, then we run over the upper half of the ellipse to arrive at the point $(-2, 0)$. The inferior part of the ellipse is not part of the curve (it would be if t had been taken in $[0, 2\pi]$. The curve γ is in blue in the following curve.



Exercise 2 (Folium)**.** We consider the parametric curve defined by the following equations

$$\begin{cases} x(t) = \sin(2t), \\ y(t) = \sin(3t), \end{cases} \qquad t \in \mathbb{R}.$$

1. Using the symmetry properties of the curve, show that we can restrict the domain of study first to $t \in [-\pi, \pi]$, then to $t \in [0, \pi]$.

Solution: We start by reminding that the sin function is periodic with period 2π. Therefore, the function x is also periodic with a period given by $T_x = \frac{2\pi}{2} = \pi$. Moreover, the function y is also periodic, with period $T_y = \frac{2\pi}{3}$. The ratio between these two periods is

$$\frac{T_y}{T_x} = \frac{\frac{2\pi}{3}}{\pi} = \frac{2}{3}.$$

It is a rational number, therefore there exists a common period T between x and y, which is given by

$$T = 3T_y = 2T_x = 2\pi.$$

We can therefore restrict the study of the curve to an interval of length 2π, which is usually chosen to be either $[0, 2\pi]$ or $[-\pi, \pi]$. In the present case, $[-\pi, \pi]$ is the best choice.
We consider now the parity of the curve. The sin function is odd, hence we have

$$x(-t) = -x(t), \quad y(-t) = -y(t).$$

As a consequence, the curve for $t < 0$ can be obtained by symmetry with respect to the origin from the curve for $t > 0$. We may thus restrict the study to the positive part of the interval previously selected, i.e. restrict the study to $[0, \pi]$.

2. Express $x(\pi - t)$ and $y(\pi - t)$ in terms of $x(t)$ and $y(t)$. Show that the curve has an additional symmetry and that we can restrict the domain of study to $t \in \left[0, \frac{\pi}{2}\right]$.

> **Solution:** We have
>
> $$x(\pi - t) = \sin(2(\pi - t)) = \sin(2\pi - 2t) = \sin(-2t) = -\sin(2t) = -x(t),$$
> $$y(\pi - t) = \sin(3(\pi - t)) = \sin(3\pi - 3t) = \sin(\pi - 3t) = \sin(3t) = y(t).$$
>
> Therefore, we may deduce the curve for $t \in \left[\frac{\pi}{2}, \pi\right]$ from the curve on $\left[0, \frac{\pi}{2}\right]$ by symmetry with respect to the y-axis. Thus, it is enough to study the curve on the interval $\left[0, \frac{\pi}{2}\right]$.

3. Construct the *tableau de variation*[1] of x and y on $\left[0, \frac{\pi}{2}\right]$. Indicate the values of x, x', y, y' at $t = 0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$.

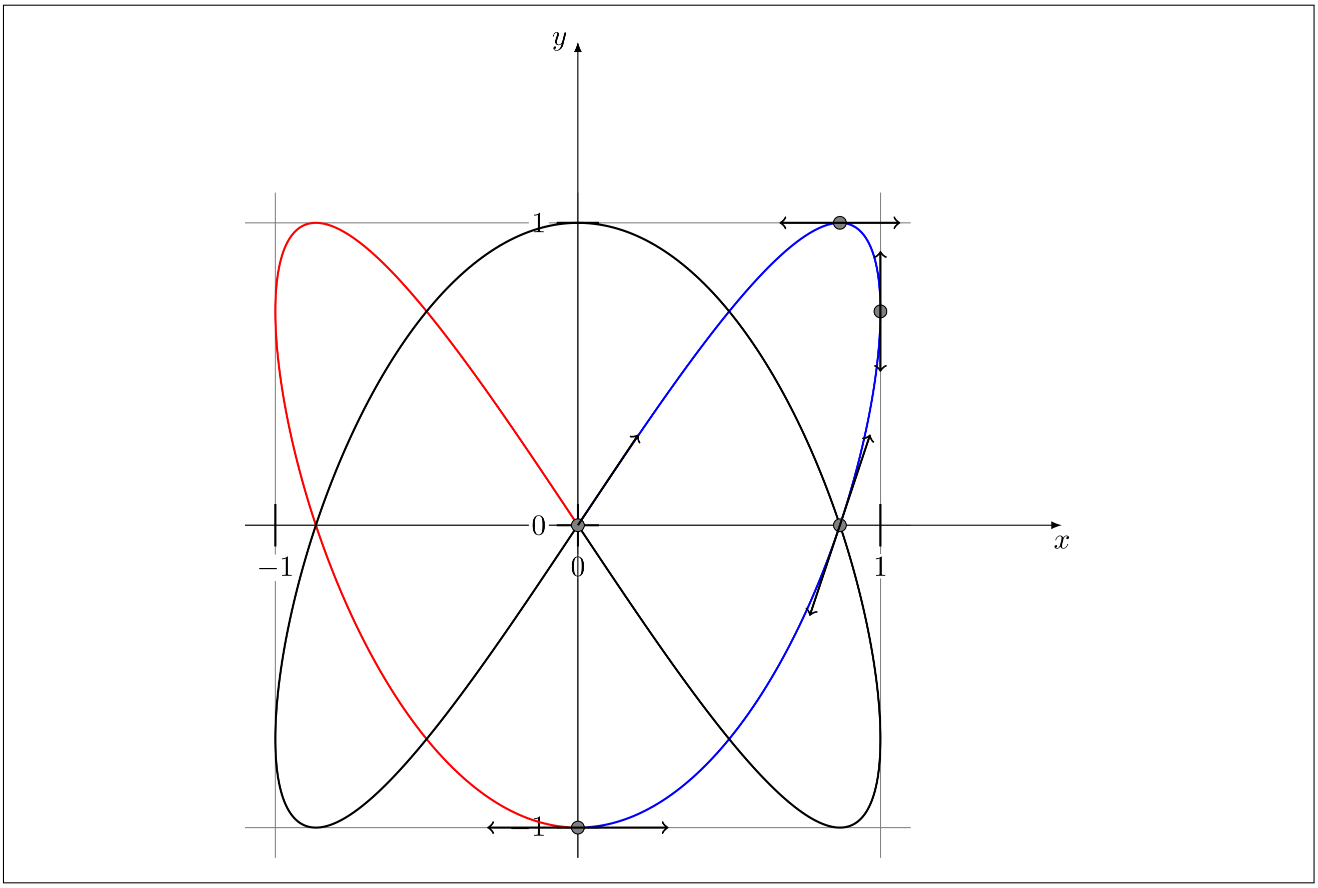
> **Solution:** The derivatives of x and y are given by
>
> $$x'(t) = 2\cos(2t), \quad y'(t) = 3\cos(3t).$$
>
> The *tableau de variation* of the curve is the following.

t	0		$\frac{\pi}{6}$		$\frac{\pi}{4}$		$\frac{\pi}{3}$		$\frac{\pi}{2}$
$x'(t)$	2	$+$	1	$+$	0	$-$	-1	$-$	-2
$x(t)$	0	↗	$\frac{\sqrt{3}}{2}$	↗	1	↘	$\frac{\sqrt{3}}{2}$	↘	0
$y'(t)$	3	$+$	0	$-$	$-\frac{3\sqrt{2}}{2}$	$-$	-3	$-$	0
$y(t)$	0	↗	1	↘	$\frac{\sqrt{2}}{2}$	↘	0	↘	-1
$\frac{y'(t)}{x'(t)}$	$\frac{3}{2}$		0		$-\infty$		3		0

4. Sketch the curve first for $t \in \left[0, \frac{\pi}{2}\right]$, then sketch the whole curve.

> **Solution:** Combining the results previously obtained, we get the following figure. The blue part corresponds to the curve on $\left[0, \frac{\pi}{2}\right]$. The red part is obtained by symmetry with respect to the y-axis and the black part by symmetry with respect to the origin.

1. The *tableau de variation* is a specifically French concept. For lack of an appropriate translation, we have chosen to keep the French expression in this document.



Exercise 3 (Astroid)**.** We consider the parametric curve defined by the following equations

$$\begin{cases} x(t) = \cos^3(t), \\ y(t) = \sin^3(t), \end{cases} \qquad t \in \mathbb{R}.$$

1. Using the symmetry properties of the curve, restrict the domain of study of the curve to an interval of $\mathbb{R}$.

Solution:
The functions sin and cos are periodic with period 2π and so are x and y. We may therefore restrict the interval of study to an interval of length 2π, which we choose to be $[-\pi, \pi]$.
From the parity properties of sin and cos, we may further restrict the interval of study. Indeed, we

$$x(-t) = \cos^3(-t) = \cos^3(t) = x(t), \quad y(-t) = \sin^3(-t) = -\sin^3(t) = -y(t).$$

We may therefore deduce the curve on the interval $[-\pi, 0]$ from the curve on the interval $[0, \pi]$ by symmetry with respect to the x-axis and therefore restrict the study to the interval $[0, \pi]$.
Observe now that

$$x(\pi - t) = \cos^3(\pi - t) = -\cos^3(t) = -x(t), \quad y(\pi - t) = \sin^3(\pi - t) = \sin^3(t) = y(t).$$

Hence the curve on the interval $\left[\frac{\pi}{2}, \pi\right]$ might be deduce from the curve on the interval $\left[0, \frac{\pi}{2}\right]$ by symmetry with respect to the y-axis and we may therefore restrict the study to the interval $\left[0, \frac{\pi}{2}\right]$.
Finally, we remark that

$$x\left(\frac{\pi}{2} - t\right) = \cos^3\left(\frac{\pi}{2} - t\right) = \sin^3(t) = y(t), \quad y\left(\frac{\pi}{2} - t\right) = \sin^3\left(\frac{\pi}{2} - t\right) = \cos^3(t) = x(t).$$

We may therefore deduce the curve on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ from the curve on the interval $\left[0, \frac{\pi}{4}\right]$ by symmetry with respect to the line described by the equation $y = x$ and therefore we may restrict the study to the interval $\left[0, \frac{\pi}{4}\right]$.

2. Construct the *tableau de variation* for x and y.

Solution: Before being able to construct the *tableau de variation*, we calculate the derivatives of x and y :

$$x'(t) = -3\sin(t)\cos^2(t), \quad y'(t) = 3\cos(t)\sin^2(t).$$

Note that

$$\frac{y'(t)}{x'(t)} = \frac{-3\sin(t)\cos^2(t)}{3\cos(t)\sin^2(t)} = -\frac{\sin(t)}{\cos(t)} = -\tan(t)$$

and we may therefore calculate the slope of the tangent at every point, including the point $t = 0$. Indeed, even though we have $x'(0) = 0$ and $y'(0) = 0$, we nevertheless find

$$\lim_{t\to 0} \frac{y'(t)}{x'(t)} = \tan(0) = 0.$$

The *tableau de variation* is the following.

t	0		$\frac{\pi}{4}$
$x'(t)$	0	$-$	$-\frac{3\sqrt{2}}{4}$
$x(t)$	1	$\searrow$	$\frac{\sqrt{2}}{4}$
$y'(t)$	0	$+$	$\frac{3\sqrt{2}}{4}$
$y(t)$	0	$\nearrow$	$\frac{\sqrt{2}}{4}$
$\frac{y'(t)}{x'(t)}$	0		-1

3. Give the coordinates of the curve when $t = 0, \frac{\pi}{2}, \pi$ and give the direction vectors of the tangents at these points.

Solution: We might proceed by direct calculations or use the symmetries of the curve previously obtained. In both cases, we obtain the following coordinates and slopes for the tangent :

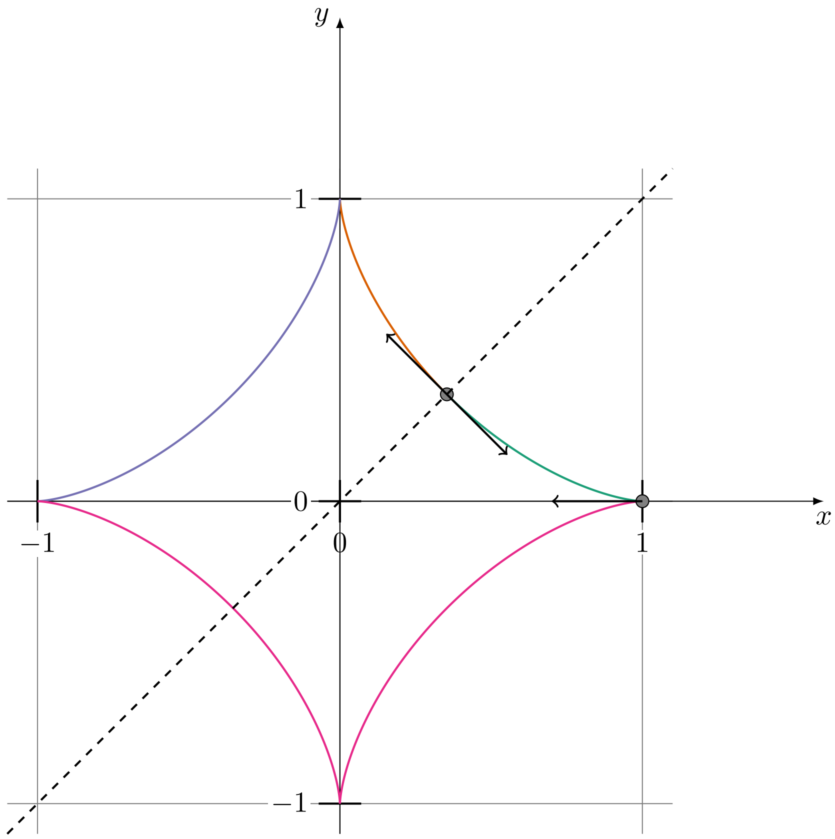
$$x(0) = 1, \qquad y(0) = 0, \qquad \lim_{t\to 0} \frac{y'(t)}{x'(t)} = 0,$$

$$x\left(\frac{\pi}{2}\right) = 0, \qquad y\left(\frac{\pi}{2}\right) = 1, \qquad \lim_{t\to \frac{\pi}{2}} \frac{y'(t)}{x'(t)} = -\infty,$$

$$x(\pi) = 1, \qquad y(\pi) = 0, \qquad \lim_{t\to \pi} \frac{y'(t)}{x'(t)} = 0.$$

4. Sketch the curve.

Solution: Combining the results previously obtained, we get the following figure. The green part corresponds to the curve for $t \in \left[0, \frac{\pi}{4}\right]$. The orange part is obtained by symmetry with respect to the line $y = x$, the blue part by symmetry with respect to the y-axis and finally the pink part by symmetry with respect to the x-axis.



5. Calculate the length and curvature of the astroid.

Solution: We first recall that the length of a parametric curve is given on interval $[a, b]$ by the formula

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

In the present case, we find, for the total length of the curve (on one period)

$$\int_{-\pi}^{\pi} \sqrt{(3\sin(t)\cos^2(t))^2 + (3\cos(t)\sin^2(t))^2}dt = 3\int_{-\pi}^{\pi} |\sin(t)\cos(t)|\sqrt{\cos^2(t) + \sin^2(t)}dt =$$
$$\frac{3}{2}\int_{-\pi}^{\pi} |\sin(2t)|dt = 6\int_0^{\frac{\pi}{2}} \sin(2t)dt = 3\left[-\cos(2t)\right]_0^{\frac{\pi}{2}} = 6.$$

We also recall that the curvature is given at each t by the formula

$$\frac{x'y'' - y'x''}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

In the present case, we have

$$x''(t) = 6\sin(t)^2\cos(t) - 3\cos(t)^3, \quad y''(t) = 6\cos(t)^2\sin(t) - 3\sin(t)^3,$$

and therefore the curvature is given by the formula :

$$\frac{-3\sin(t)\cos^2(t)(6\cos(t)^2\sin(t) - 3\sin(t)^3) - 3\cos(t)\sin^2(t)(6\sin(t)^2\cos(t) - 3\cos(t)^3)}{((3\sin(t)\cos^2(t))^2 + (3\cos(t)\sin^2(t))^2)^{\frac{3}{2}}}$$
$$= \frac{\left(\frac{9}{2}\sin^2(2t)\right)\left(-2\cos(t)^2 - \sin^2(t) - 2\sin(t)^2 - \cos^2(t)\right)}{\left(\frac{9}{2}\sin^2(2t)\right)^{\frac{3}{2}}} = -3|\sin(2t)|.$$

Exercise 4 (Infinite branches)**.** We consider the parametric curve defined by the following equations

$$\begin{cases} x(t) = \frac{1}{t(t-1)}, \\ y(t) = \frac{t^2}{1-t}, \end{cases} \qquad t \in \mathbb{R}.$$

1. Express $x\left(\frac{1}{t}\right)$ and $y\left(\frac{1}{t}\right)$ in terms of $x(t)$ and $y(t)$. Show that the curve has a symmetry and that we can restrict the domain of study to $I = (-1,1) \setminus \{0\}$.

Solution: We have

$$x\left(\frac{1}{t}\right) = \frac{1}{\frac{1}{t}\left(\frac{1}{t}-1\right)} = \frac{1}{\frac{1}{t}\left(\frac{1-t}{t}\right)} = \frac{1}{\frac{1-t}{t^2}} = \frac{t^2}{1-t} = y(t).$$

Similarly, we obtain

$$y\left(\frac{1}{t}\right) = x(t).$$

Therefore, the curve is symmetric with respect to the line $y = x$. Moreover, the function $t \to \frac{1}{t}$ maps $(-1,1) \setminus \{0\}$ to $(-\infty,-1) \cup (1,\infty)$. Therefore the curve for $t \in (-\infty,1) \cup (1,\infty)$ can be deduced by symmetry from the curve for $t \in (-1,1)\setminus\{0\}$ and we may restrict the interval of study to $(-1,1)\setminus\{0\}$.

2. Construct the *tableau de variation* on I.

Solution: We have

$$x'(t) = \frac{-2t+1}{t^2(t-1)^2}, \quad y'(t) = -\frac{t(t-2)}{(t-1)^2}.$$

Therefore, the *tableau de variation* is the following.

t	-1		0		$\frac{1}{2}$		1
$x'(t)$	$\frac{3}{4}$	$+$	‖	$+$	0	$-$	‖
$x(t)$	$\frac{1}{2}$	↗	$+\infty$ ‖ $-\infty$	↗	-4	↘	$-\infty$ ‖
$y'(t)$	$-\frac{3}{4}$	$-$	‖	$+$	3	$+$	‖
$y(t)$	$\frac{1}{2}$	↘	0 ‖ 0	↗	$\frac{1}{2}$	↗	$+\infty$ ‖
$\frac{y'(t)}{x'(t)}$	-1		‖		$\pm\infty$		‖

3. Study the infinite branches on I.

Solution: We have an infinite branch when there exists t_0 such that

$$\lim_{t \to t_0} x(t) = \pm\infty \text{ or } \lim_{t \to t_0} y(t) = \pm\infty.$$

In the present case, this happens when $t = 0$ and $t = 1$. At $t = 0$, we have

$$\lim_{t \to 0, t<0} x(t) = +\infty, \qquad \lim_{x \to 0, t>0} x(t) = -\infty,$$

and in any case

$$\lim_{t\to 0} y(t) = 0.$$

The x-axis is therefore asymptote of the curve to the left and to the right.

At $t = 1$, we have

$$\lim_{t\to 1, t<1} x(t) = -\infty, \quad \lim_{t\to 1, t<1} y(t) = +\infty.$$

Thus the curve possibly admits a slant (oblique) asymptote. To verify it, we calculate

$$\lim_{t\to 1, t<1} \frac{y(t)}{x(t)} = \lim_{t\to 1, t<1} (-t^3) = -1.$$

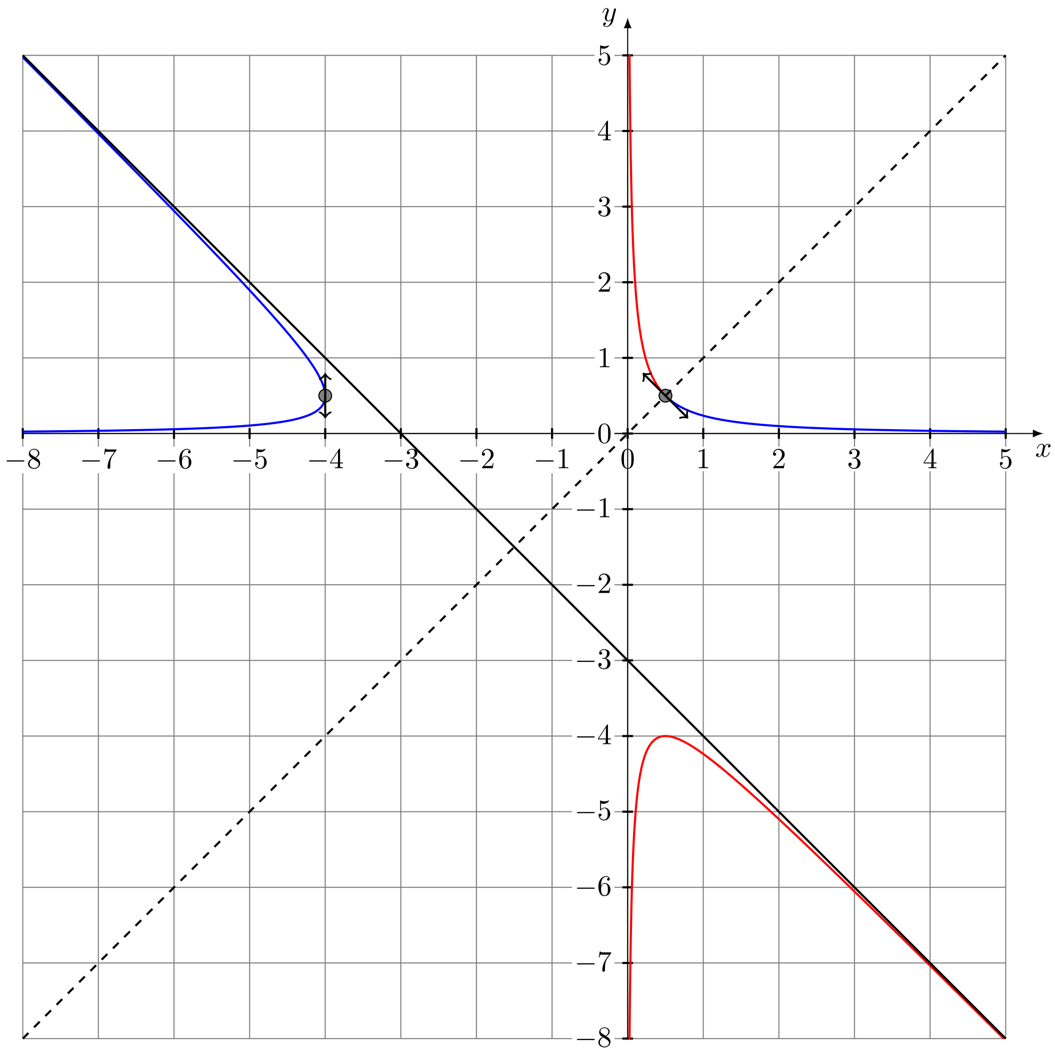
Therefore the curve admits a slant asymptote, of slope -1. We calculate the y-intercept in the following way

$$\lim_{t\to 1, t<1} (y(t) - (-1)x(t)) = \lim_{t\to 1} \left(-\frac{t^2+t+1}{t} \right) = -3.$$

Therefore the line of equation $y = -x - 3$ is a slant asymptote to the curve.

4. Sketch the curve.

Solution: Combining the informations previously obtained, we get the following figure. The blue part corresponds to $t \in I$. The red part corresponds to $t \in \mathbb{R} \setminus [-1, 1]$ and is obtained from the blue part by symmetry with respect to the line $y = x$ (dotted line). The asymptote of equation $y = -x-3$ is also represented in the figure, as well as the tangents when $t = -1$ and $t = \frac{1}{2}$.



Exercise 5. We consider the polar curve defined by

$$\rho(\theta) = \sin(3\theta), \quad \theta \in \mathbb{R}.$$

1. What is the period of ρ ?

> **Solution:** The sin function is periodic with period 2π, therefore ρ is periodic with period $\frac{2\pi}{3}$. As a consequence, the curve is invariant by rotation of $\frac{2\pi}{3}$. Hence, to obtain the curve, we may start by representating it for θ on an interval of length $\frac{2\pi}{3}$, and then proceed to 2 rotations of angle $\frac{2\pi}{3}$ of the obtained curve to get the entirety of the curve. We therefore chose to restrict the study of the curve to the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

2. Express $\rho(-\theta)$ and $\rho(\pi - \theta)$ in terms of $\rho(\theta)$. What are the symmetries of the curve ? Show that we can restrict the domain of study to $\theta \in \left[0, \frac{\pi}{3}\right]$.

> **Solution:** We have
> $$\rho(-\theta) = \sin(-3\theta) = -\sin(3\theta) = -\rho(\theta).$$
> Therefore ρ is odd and θ is symmetric with respect to the y-axis. We may therefore further restrict the domain of study from $\theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ to $\theta \in \left[0, \frac{\pi}{3}\right]$. Indeed, the curve for $\theta \in \left[-\frac{\pi}{3}, 0\right]$ will be obtain from the one for $\theta \in \left[0, \frac{\pi}{3}\right]$ by symmetry with respect to the y-axis.

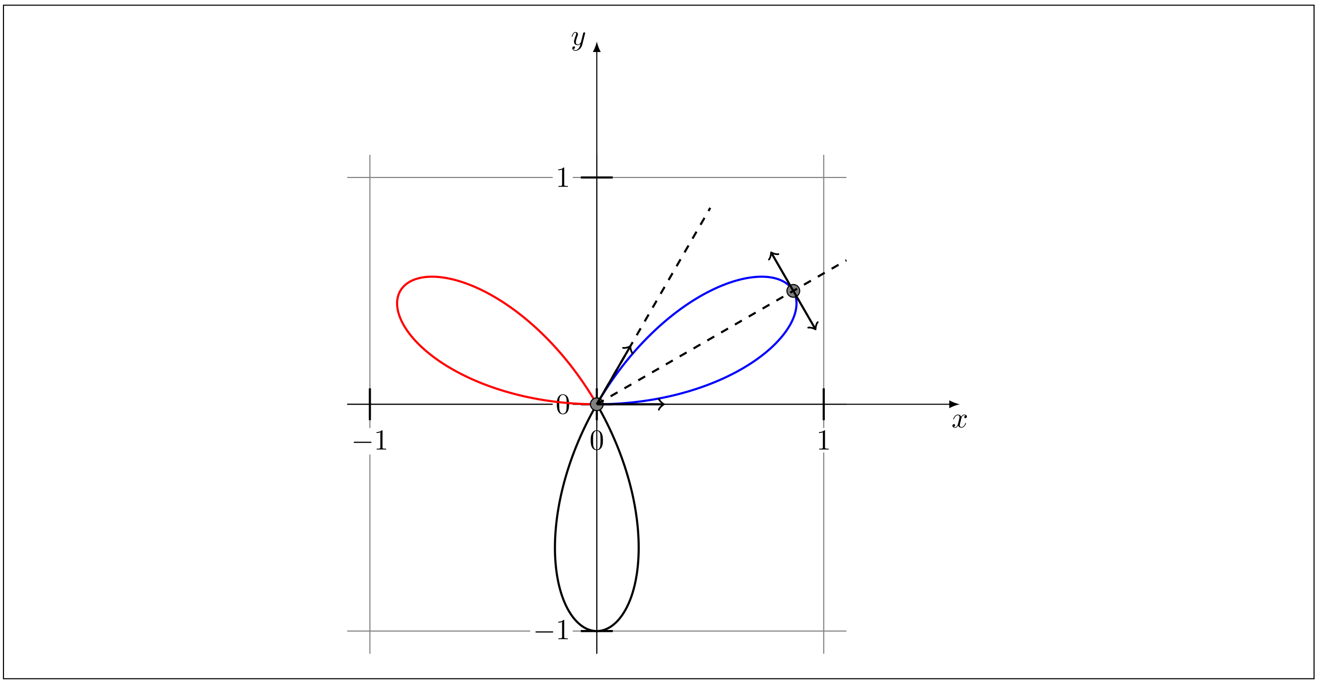
3. Construct the *tableau de variation* on I. Give the equations for the tangents at the curve when $\theta = 0$ and $\theta = \frac{\pi}{3}$.

> **Solution:** We have
> $$\rho'(\theta) = 3\cos(3\theta).$$
> In $\left[0, \frac{\pi}{3}\right]$ the only value for which $\rho'(\theta) = 0$ is $\theta = \frac{\pi}{6}$. The *tableau de variation* is given as follows.

θ	0		$\frac{\pi}{6}$		$\frac{\pi}{3}$
$\rho'(\theta)$	3	$+$	0	$-$	-3
$\rho(\theta)$	0	$\nearrow$	1	$\searrow$	0
$\frac{\rho(\theta)}{\rho'(\theta)}$	0		$+\infty$		0

4. Sketch the curve.

> **Solution:** We may finally sketch the curve in the following way. The blue part of the curve corresponds to the curve for $\theta \in \left[0, \frac{\pi}{3}\right]$. Observe that three tangents at three remarkable points are given (the line $\frac{\rho(\theta)}{\rho'(\theta)}$ represents the slope of the tangent at $M(\theta) = (\theta, \rho(\theta)$ *in the orthogonal frame centered at* $M(\theta)$ *and with "horizontal" axis the line with angle* θ. The red part corresponds to the curve for $\theta \in \left[-\frac{\pi}{3}, 0\right]$ and is obtained by symmetry with respect to the y-axis. Finally, the black part of the curve is obtained via a rotation of angle $\frac{2\pi}{3}$ of the red and blue curves (the blue curve then becoming the red one and the red one the black one). Any further rotation of the curve with the same angle would recover one of the possible combination of the three parts of the curve previously obtained.



Exercise 6. Study the polar curves defined for $\theta \in \mathbb{R}$ by

$$(1)\, \rho(\theta) = \cos(\theta) + 2, \quad (2)\, \rho(\theta) = \cos^2\left(\frac{\theta}{3}\right), \quad (3)\, \rho(\theta) = 1 + \sin(3\theta).$$

Solution: (1)

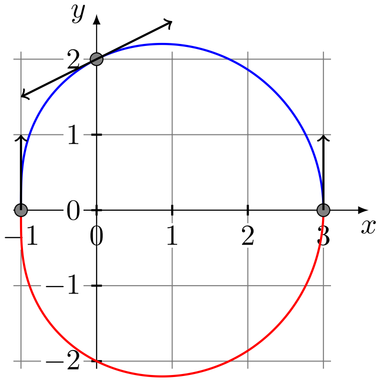
The function ρ is periodic with period 2π. We may therefore restrict the study of the curve to an interval of length 2π and it is convenient here to choose $[-\pi, \pi]$. The curve is entirely described by ρ for θ in $[-\pi, \pi]$, as the image of a curve from a rotation of angle 2π would exactly cover the same curve. Since the function cos is even, the function is also even and we may further restrict the interval of study to $[0, \pi]$ (the curve will be symmetric with respect to the x-axis). We have

$$\rho'(\theta) = -\sin(\theta).$$

The *tableau de variations* on $[0, \pi]$ is the following. For future convenience, we have decided to add the non-remarkable value $\theta = \frac{\pi}{2}$ in the tableau de variation.

t	0		$\frac{\pi}{2}$		π
$\rho'(t)$	0	$-$	-1	$-$	0
$\rho(t)$	3	↘	2	↘	1
$\frac{\rho(t)}{\rho'(t)}$	$\pm\infty$		-2		$\pm\infty$

Combining the above informations, we may now sketch the curve. The blue part corresponds to the curve on $[0, \pi]$, the red part corresponds to the curve on $[-\pi, 0]$ and is obtained from the blue one by symmetry with respect to the x-axis.



Solution: (2)

The function ρ is periodic with period 3π. We may therefore restrict the study of the curve to an interval of length 3π and it is convenient here to choose $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$. The full image of the curve will be obtained after rotation of the curve on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$ by an angle 3π (or, equivalently, by an angle π or a symmetry with respect to the origin).

Since the function cos is even, the function ρ is also even and we may further restrict the interval of study to $\left[0, \frac{3\pi}{2}\right]$ (the curve will be symmetric with respect to the x-axis). We have

$$\rho'(\theta) = -\frac{2}{3}\cos\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right) = -\frac{1}{3}\sin\left(\frac{2\theta}{3}\right),$$

where we have used the double angle formula for the last equality.

The *tableau de variations* on $\left[0, \frac{3\pi}{2}\right]$ is given as follows. For future convenience, we have decided to add the non-remarkable values $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ in the tableau de variation.

θ	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$
$\rho'(\theta)$	0	$-$	$-\frac{1}{2\sqrt{3}}$	$-$	$-\frac{1}{2\sqrt{3}}$	$-$	0
$\rho(\theta)$	1	↘	$\frac{3}{4}$	↘	$\frac{1}{4}$	↘	0
$\frac{\rho(\theta)}{\rho'(\theta)}$	$\pm\infty$		$-\frac{3\sqrt{3}}{2}$		$-\frac{\sqrt{3}}{2}$		0

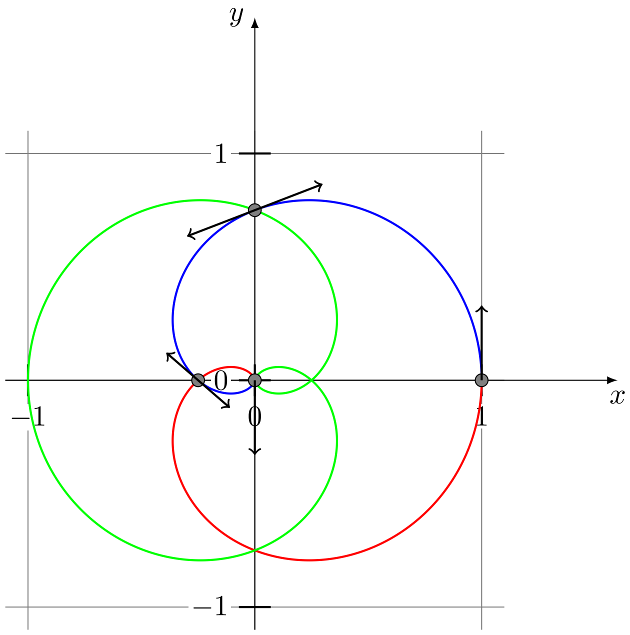
The calculations are straightforward, except for the bottom right value which is obtained as follows :

$$\frac{\rho(\theta)}{\rho'(\theta)} = \frac{\cos^2\left(\frac{\theta}{3}\right)}{-\frac{2}{3}\cos\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right)} = -\frac{3\cos\left(\frac{\theta}{3}\right)}{2\sin\left(\frac{\theta}{3}\right)} = -\frac{3}{\tan\left(\frac{\theta}{3}\right)}.$$

Since $\lim_{t\to\frac{\pi}{2}}\tan(t) = \infty$, we have

$$\lim_{t\to\frac{3\pi}{2}}\frac{\rho(\theta)}{\rho'(\theta)} = 0.$$

Combining the above informations, we may now sketch the curve. The blue part corresponds to the curve on $\left[0, \frac{3\pi}{2}\right]$, the red part corresponds to the curve on $\left[-\frac{3\pi}{2}, 0\right]$ and is obtained from the blue one by symmetry with respect to the x-axis. The green part corresponds to the curve obtained by a rotation of angle 3π. Any further rotation of same angle would cover the existing parts of the curve.



Solution: (3)

The function ρ is periodic with period $\frac{2\pi}{3}$. We may therefore restrict the study of the curve to an interval of length $\frac{2\pi}{3}$. The canonical choices for such an interval are $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ or $\left[0, \frac{2\pi}{3}\right]$. We will however here for future convenience make a different choice and select the interval $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$ (which is also of length $\frac{2\pi}{3}$). The reasons for such a choice will appear in a few lines. The full image of the curve will be obtained after 2 rotations of the curve on $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$ by an angle $\frac{2\pi}{3}$.

Observe that, since the function sin is odd, we have

$$\rho(-\theta) = 1 + \sin(-3\theta) = 1 - \sin(3\theta) \neq \pm\rho(\theta),$$

and no further domain reduction can be obtained by parity.

Note than one may find additional symmetry in the curve by investigating $\rho(\frac{\pi}{3} - \theta)$, but we will not pursue in this direction here.

We have

$$\rho'(\theta) = -3\cos(3\theta).$$

Observe now that the interval of study $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$ is chosen in such a way that $\rho'(\theta)$ changes signs only once over the interval.

The *tableau de variations* on $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$ is given as follows.

θ	$-\frac{\pi}{6}$		$\frac{\pi}{6}$		$\frac{\pi}{2}$
$\rho'(\theta)$	0	+	0	−	0
$\rho(\theta)$	0	↗	2	↘	0
$\frac{\rho(\theta)}{\rho'(\theta)}$	0		$\pm\infty$		0

To remove the $\frac{0}{0}$ indeterminations on the last line, we may apply L'Hospital's rule to get

$$\lim_{\theta\to-\frac{\pi}{6}} \frac{\rho(\theta)}{\rho'(\theta)} = \lim_{\theta\to-\frac{\pi}{6}} \frac{\rho'(\theta)}{\rho''(\theta)} = \lim_{\theta\to-\frac{\pi}{6}} \frac{-3\cos(3\theta)}{9\sin(3\theta)} = \lim_{\theta\to-\frac{\pi}{6}} \frac{1}{3\tan(3\theta)} = 0.$$

Similarly, we may prove that

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\rho(\theta)}{\rho'(\theta)} = 0.$$

Combining the above informations, we may now sketch the curve. The blue part corresponds to the curve on $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$. The red part corresponds to the curve obtained by a rotation of angle $\frac{2\pi}{3}$ and the green part corresponds to the curve obtained by another rotation of angle $\frac{2\pi}{3}$. Any further rotation of same angle would cover the already existing parts of the curve.