M2 MAT-RI
EIMAR4E1
Advanced Course B3
Theoretical and numerical analysis of dispersive PDEs.

S. LE COZ's Exam.

Duration : 1.5 hours. No documents, no calculator, no cell-phone.

In this exam, $S(t) = e^{it\Delta}$ will denote the Schrödinger group, i.e. $S(t)u_0$ is the solution of

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0) = u_0, \end{cases} \qquad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}.$$

Exercise 1. Let $u_0 \in H^1(\mathbb{R}^d)$ and $\lambda > 0$. We define u_0^λ by $u_0^\lambda(x) = u_0(\lambda x)$.

1. Show that $S(t)u_0^\lambda$ is given by
$$(S(t)u_0^\lambda)(x) = (S(\lambda^2 t)u_0)(\lambda x).$$
2. Express $\|u_0^\lambda\|_{L^2_x}$ in terms of $\|u_0\|_{L^2_x}$.
3. Given $q, r \in [2, \infty]$, express $\|S(t)u_0^\lambda\|_{L^q_t L^r_x}$ in terms of $\|S(t)u_0\|_{L^q_t L^r_x}$.
4. Give a necessary condition (the answer should be justified) on q, r and d for the following property to hold: there exists $C > 0$ such that for any $u_0 \in H^1(\mathbb{R}^d)$ we have
$$\|S(t)u_0\|_{L^q_t L^r_x} \leq C\|u_0\|_{L^2_x}.$$

Exercise 2. In this exercise, the space dimension is $d = 2$. We recall the dispersive estimate: there exists $C > 0$ such that for all $q \in [2, \infty]$, for $u_0 \in L^{q'}(\mathbb{R}^2)$ (where $1/q + 1/q' = 1$) and for all $t > 0$ we have

$$\|S(t)u_0\|_{L^q_x} \leq C|t|^{-1+\frac{2}{q}}\|u_0\|_{L^{q'}_x}.$$

We define the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = |z|^2 z.$$

Let $\lambda > 0$. Let $W, H : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$ be such that

$$\|W\|_{L^\infty_t L^4_x} + \left\| e^{\lambda t} \|H(t)\|_{L^{\frac{4}{3}}_x} \right\|_{L^\infty_t} \leq 1.$$

Define the functional Φ for $\eta : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$ by

$$\Phi(\eta) = -i \int_t^\infty S(t-s) \left(f(W(s) + \eta(s)) - f(W(s)) + H(s) \right) ds.$$

Define the ball

$$B = \left\{ \eta : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C} : \left\| e^{\lambda t} \|\eta(t)\|_{L^4_x} \right\|_{L^\infty_t} \leq 1 \right\}.$$

Endowed with the norm

$$\|\cdot\|_B = \left\| e^{\lambda t} \|\cdot\|_{L^4_x} \right\|_{L^\infty_t},$$

the ball B is a complete metric space.

1. Preliminary: Show that there exists $C > 0$ such that for any $z_1, z_2 \in \mathbb{C}$ we have

$$|f(z_1) - f(z_2)| \leq C|z_1 - z_2| \left(|z_1|^2 + |z_2|^2\right).$$

2. Let $\eta \in B$ and $t \geq 0$. Show that there exists $C > 0$ such that

$$\|\Phi(\eta)(t)\|_{L^4_x} \leq C \int_t^\infty |t-s|^{-\frac{1}{2}} e^{-\lambda s} ds.$$

3. Given any $C > 0$, show that there exists λ^* sufficiently large such that if $\lambda > \lambda^*$, then for all $t \geq 0$ we have

$$C \int_t^\infty |t-s|^{-\frac{1}{2}} e^{-\lambda s} ds \leq e^{-\lambda t}.$$

4. Show that Φ maps B into B for $\lambda > \lambda^*$.
5. Show that there exists $\lambda^{**} > 0$ such that Φ is a contraction mapping on B for $\lambda > \lambda^{**}$.
6. Let $\eta \in B$ be such that $\Phi(\eta) = \eta$. Assume that W verifies the equation

$$i\partial_t W + \Delta W + f(W) = H.$$

What is the equation verified by u defined by $u = W + \eta$?

Exercise 3 (Optional exercise, to be treated only if time permits)**.** We consider the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0, \quad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}, \quad 1 < p < \infty. \tag{1}$$

in dimension $d = 2$. Given $x \in \mathbb{R}^2$, we use the notation $x = (x_1, x_2)$ and the partial derivatives with respect to x_1 and x_2 are denoted by ∂_1 and ∂_2. We define the *angular momentum* by

$$X(u) = \text{Im} \int_{\mathbb{R}^2} (x_1 \partial_2 \bar{u} - x_2 \partial_1 \bar{u}) u dx.$$

We denote by $\mathcal{S}(\mathbb{R}^2)$ the Schwartz space of functions $v : \mathbb{R}^2 \to \mathbb{C}$ smooth and rapidly decaying.

1. Let $v \in \mathcal{S}(\mathbb{R}^2)$.
   (a) Express $\partial_1 \left(|v|^{p+1}\right)$ in terms of $\partial_1 v$ and v (do not forget that v is complex valued !).
   (b) What is the value of
   $$\int_{\mathbb{R}^2} x_2 \partial_1 \left(|v|^{p+1}\right) dx \ ?$$
   (c) Show that
   $$\text{Re} \int_{\mathbb{R}^2} x_1 \partial_2 \bar{v} \Delta v dx = -\text{Re} \int_{\mathbb{R}^2} \partial_2 \bar{v} \partial_1 v dx.$$
2. Let $u \in \mathcal{C}^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^2))$ be a solution of (1). Show that the angular momentum X is a constant of motion for u, i.e. that
   $$X(u(t)) = X(u(0)) \quad \text{for all } t \in \mathbb{R}.$$