M2 MAT-RI
EIMAR4E1
Advanced Course B3
Theoretical and numerical analysis of dispersive PDEs.
S. LE COZ's Exam.

Duration : 1.5 hours. No documents, no calculator, no cell-phone.

In this exam, $S(t) = e^{it\Delta}$ will denote the Schrödinger group, i.e. $S(t)u_0$ is the solution of

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0) = u_0, \end{cases} \qquad u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}.$$

Exercise 1. We consider the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}, \quad 1 < p < \infty. \tag{1}$$

in dimension $d = 2$. Given $x \in \mathbb{R}^2$, we use the notation $x = (x_1, x_2)$ and the partial derivatives with respect to x_1 and x_2 are denoted by ∂_1 and ∂_2. We define the *angular momentum* by

$$X(u) = \operatorname{Im} \int_{\mathbb{R}^2} (x_1 \partial_2 \bar{u} - x_2 \partial_1 \bar{u}) u dx.$$

We denote by $\mathcal{S}(\mathbb{R}^2)$ the Schwartz space of functions $v : \mathbb{R}^2 \to \mathbb{C}$ smooth and rapidly decaying.

1. Let $v \in \mathcal{S}(\mathbb{R}^2)$.
   (a) Express $\partial_1 \left(|v|^{p+1}\right)$ in terms of $\partial_1 v$ and v (do not forget that v is complex valued !).

   Solution: To avoid the possible singularities at 0, we write

   $$|v|^{p+1} = (v\bar{v})^{\frac{p+1}{2}}.$$

   Differentiating using the usual rules, we get

   $$\partial_1 \left(|v|^{p+1}\right) = \frac{p+1}{2} \partial_1 (v\bar{v}) (v\bar{v})^{\frac{p-1}{2}} = \frac{p+1}{2} (\partial_1 v \bar{v} + v \partial_1 \bar{v}) |v|^{p-1} = (p+1) \operatorname{Re} \left(|v|^{p-1} v \partial_1 \bar{v}\right).$$

   (b) What is the value of

   $$\int_{\mathbb{R}^2} x_2 \partial_1 \left(|v|^{p+1}\right) dx \ ?$$

   Solution: Since $v \in \mathcal{S}(\mathbb{R}^2)$, we can integrate by part without boundary terms and obtain

   $$\int_{\mathbb{R}^2} x_2 \partial_1 \left(|v|^{p+1}\right) dx = -\int_{\mathbb{R}^2} (\partial_1 x_2) |v|^{p+1} dx = 0.$$

(c) Show that

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\Delta v dx = -\mathrm{Re}\int_{\mathbb{R}^2} \partial_2\bar{v}\partial_1 v dx.$$

Solution: Expanding the Laplace operator, we have

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\Delta v dx = \mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\partial_{11} v dx + \mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\partial_{22} v dx. \tag{2}$$

We treat each part in the second member separately, starting by the first one. Integrating by part in x_1, we obtain

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\partial_{11} v dx = -\mathrm{Re}\int_{\mathbb{R}^2} \partial_2\bar{v}\partial_1 v dx - \mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_{12}\bar{v}\partial_1 v dx.$$

Remark that

$$\partial_{12}\bar{v}\partial_1 v = \frac{1}{2}\partial_2\left(|\partial_1 v|^2\right),$$

Therefore, after integrating by part in x_2 we obtain

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_{12}\bar{v}\partial_1 v dx = \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\left(|v|^2\right)dx = -\frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^2}\left(\partial_2 x_1\right)|v|^2 dx = 0.$$

As a consequence, we have

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\partial_{11} v dx = -\mathrm{Re}\int_{\mathbb{R}^2} \partial_2\bar{v}\partial_1 v dx.$$

We now show that the second part of the right hand side of (**??**) is 0. Indeed, integrating by part in x_2, we obtain

$$\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\partial_{22} v dx = -\int_{\mathbb{R}^2} x_1\partial_{22}\bar{v}\partial_2 v dx$$

which implies that

$$2\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{v}\partial_{22} v dx = 0.$$

This ends the proof.

2. Let $u \in \mathcal{C}^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^2))$ be a solution of (1). Show that the angular momentum X is a constant of motion for u, i.e. that

$$X(u(t)) = X(u(0)) \quad \text{for all } t \in \mathbb{R}.$$

Solution: To obtain the result, we simply show that $X(u(t))$ is constant by differentiating with respect to t. The solution u being smooth and rapidly decaying, all the following calculations are justified. We have

$$\begin{aligned}\frac{\partial}{\partial t}X(u(t)) &= \mathrm{Im}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u}_t - x_2\partial_1\bar{u}_t)udx + \mathrm{Im}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u} - x_2\partial_1\bar{u})u_t dx\\
&= \mathrm{Im}\int_{\mathbb{R}^2}(-x_1\bar{u}_t\partial_2 u + x_2\bar{u}_t\partial_1 u)dx + \mathrm{Im}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u} - x_2\partial_1\bar{u})u_t dx\\
&= 2\mathrm{Im}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u} - x_2\partial_1\bar{u})u_t dx\\
&= -2\mathrm{Re}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u} - x_2\partial_1\bar{u})iu_t dx\\
&= 2\mathrm{Re}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u} - x_2\partial_1\bar{u})(\Delta u + |u|^{p-1}u)dx.\end{aligned}$$

We remark now that

$$\mathrm{Re}(\partial_1\bar{u}|u|^{p-1}u) = \frac{1}{p+1}\partial_1\left(|u|^{p+1}\right),$$

Therefore

$$\mathrm{Re}\int_{\mathbb{R}^2} x_2\partial_1\bar{u}|u|^{p-1}udx = \frac{1}{p+1}\int_{\mathbb{R}^2} x_2\partial_1\left(|u|^{p+1}\right)dx = 0.$$

And similarly we also have

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{u}|u|^{p-1}udx = 0.$$

We have already seen that

$$\mathrm{Re}\int_{\mathbb{R}^2} x_1\partial_2\bar{u}\Delta udx = -\mathrm{Re}\int_{\mathbb{R}^2}\partial_2\bar{u}\partial_1 udx,$$

and from similar arguments we have

$$\mathrm{Re}\int_{\mathbb{R}^2} x_2\partial_1\bar{u}\Delta udx = -\mathrm{Re}\int_{\mathbb{R}^2}\partial_1\bar{u}\partial_2 udx.$$

Therefore

$$\mathrm{Re}\int_{\mathbb{R}^2}(x_1\partial_2\bar{u} - x_2\partial_1\bar{u})\Delta udx = 0.$$

Combining all previous equalities, we infer that

$$\frac{\partial}{\partial t}X(u(t)) = 0,$$

which gives the desired result.

Exercise 2. Let $u_0 \in H^1(\mathbb{R}^d)$ and $\lambda > 0$. We define u_0^λ by $u_0^\lambda(x) = u_0(\lambda x)$.

1. Show that $S(t)u_0^\lambda$ is given by

$$(S(t)u_0^\lambda)(x) = (S(\lambda^2 t)u_0)(\lambda x).$$

Solution: Several approaches are possible. Here, we show that the function u^λ defined by

$$u^\lambda(t,x) = (S(\lambda^2 t)u_0)(\lambda x)$$

indeed solves the linear Schrödinger equation with initial data u_0^λ. First, when $t=0$, we have

$$u^\lambda(0,x) = (S(0)u_0)(\lambda x) = u_0(\lambda x) = u_0^\lambda(x).$$

Moreover, differentiating in t, we obtain

$$\partial_t u^\lambda(t,x) = \partial_t(S(\lambda^2 t)u_0)(\lambda x) = \lambda^2(\partial_t S(\lambda^2 t)u_0)(\lambda x) = \lambda^2(i\Delta S(\lambda^2 t)u_0)(\lambda x).$$

Now, differentiating in x we have

$$\Delta u^\lambda(t,x) = \Delta(S(\lambda^2 t)u_0)(\lambda x) = \lambda^2(\Delta S(\lambda^2 t)u_0)(\lambda x) = -i\partial_t u^\lambda(t,x),$$

which concludes the proof.

2. Express $\|u_0^\lambda\|_{L^2_x}$ in terms of $\|u_0\|_{L^2_x}$.

Solution: By a change of variable, we obtain

$$\|u_0^\lambda\|_{L^2_x} = \lambda^{-\frac{d}{2}}\|u_0\|_{L^2_x}.$$

3. Given $q, r \in [2,\infty]$, express $\|S(t)u_0^\lambda\|_{L^q_t L^r_x}$ in terms of $\|S(t)u_0\|_{L^q_t L^r_x}$.

Solution: Using the expression of $S(t)u_0^\lambda$ previously given, by a change of variable (in time and in space), we obtain

$$\|S(t)u_0^\lambda\|_{L^q_t L^r_x} = \lambda^{-\frac{2}{q}-\frac{d}{r}}\|S(t)u_0\|_{L^q_t L^r_x}.$$

4. Give a necessary condition (the answer should be justified) on q, r and d for the following property to hold: there exists $C>0$ such that for any $u_0 \in H^1(\mathbb{R}^d)$ we have

$$\|S(t)u_0\|_{L^q_t L^r_x} \le C\|u_0\|_{L^2_x}.$$

Solution: Since the inequality should be valid for any u_0, it should in particular be valid for any u_0^λ, hence necessarily

$$\frac{2}{q} - \frac{d}{r} = \frac{d}{2}.$$

Exercise 3. In this exercise, the space dimension is $d=2$. We recall the dispersive estimate: there exists $C>0$ such that for all $q \in [2,\infty]$, for $u_0 \in L^{q'}(\mathbb{R}^2)$ (where $1/q + 1/q' = 1$) and for all $t>0$ we have

$$\|S(t)u_0\|_{L^q_x} \le C|t|^{-1+\frac{2}{q}}\|u_0\|_{L^{q'}_x}.$$

We define the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = |z|^2 z.$$

Let $\lambda > 0$. Let $W, H : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$ be such that

$$\|W\|_{L_t^\infty L_x^4} + \left\| e^{\lambda t} \|H(t)\|_{L_x^{\frac{4}{3}}} \right\|_{L_t^\infty} \leq 1.$$

Define the functional Φ for $\eta : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$ by

$$\Phi(\eta) = -i \int_t^\infty S(t-s) \left(f(W(s) + \eta(s)) - f(W(s)) + H(s) \right) ds.$$

Define the ball

$$B = \left\{ \eta : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C} : \left\| e^{\lambda t} \|\eta(t)\|_{L_x^4} \right\|_{L_t^\infty} \leq 1 \right\}.$$

Endowed with the norm

$$\|\cdot\|_B = \left\| e^{\lambda t} \|\cdot\|_{L_x^4} \right\|_{L_t^\infty},$$

the ball B is a complete metric space.

1. Preliminary: Show that there exists $C > 0$ such that for any $z_1, z_2 \in \mathbb{C}$ we have

$$|f(z_1) - f(z_2)| \leq C |z_1 - z_2| \left(|z_1|^2 + |z_2|^2 \right).$$

Solution: We have

$$\left| |z_1|^2 z_1 - |z_2|^2 z_2 \right| \leq \left| |z_1|^2 (z_1 - z_2) \right| + \left| (|z_1|^2 - |z_2|^2) z_2 \right|.$$

Moreover

$$\left| |z_1|^2 - |z_2|^2 \right| = .||z_1| - |z_2|| \cdot ||z_1| + |z_2||.$$

and

$$||z_1| - |z_2|| \leq |z_1 - z_2|.$$

Thus

$$\left| |z_1|^2 z_1 - |z_2|^2 z_2 \right| \leq |z_1 - z_2| (|z_1|^2 + |z_2|(|z_1| + |z_2|)) \leq 3|z_1 - z_2|(|z_1|^2 + |z_2|^2).$$

2. Let $\eta \in B$ and $t \geq 0$. Show that there exists $C > 0$ such that

$$\|\Phi(\eta)(t)\|_{L_x^4} \leq C \int_t^\infty |t-s|^{-\frac{1}{2}} e^{-\lambda s} ds.$$

Solution: We have, using the dispersive estimate

$$\|\Phi(\eta)(t)\|_{L_x^4} \leq \int_t^\infty \|S(t-s) \left(f(W(s) + \eta(s)) - f(W(s)) + H(s) \right)\|_{L_x^4} ds$$

$$\leq C \int_t^\infty |t-s|^{-\frac{1}{2}} \|f(W(s) + \eta(s)) - f(W(s)) + H(s)\|_{L_x^{\frac{4}{3}}} ds.$$

As we have seen,

$$|f(W(s)+\eta(s))-f(W(s))| \leq C|\eta(s)|(|W(s)|^2+|\eta(s)|^2).$$

Therefore

$$\|f(W(s)+\eta(s))-f(W(s))\|_{L_x^{\frac{4}{3}}} \leq C\||\eta(s)||W(s)|^2\|_{L_x^{\frac{4}{3}}} + \||\eta(s)|^3\|_{L_x^{\frac{4}{3}}}.$$

We have $\||\eta(s)|^3\|_{L_x^{\frac{4}{3}}} = \|\eta(s)\|_{L_x^4}^3$ and by Hölder inequality

$$\||\eta(s)||W(s)|^2\|_{L_x^{\frac{4}{3}}} \leq C\|\eta(s)\|_{L_x^4}\||W(s)|^2\|_{L_x^2} \leq C\|\eta(s)\|_{L_x^4}\|W(s)\|_{L_x^4}^2.$$

Gathering the previous inequalities and using the assumptions on W and H we get

$$\begin{aligned}\|\Phi(\eta)(t)\|_{L_x^4} &\leq C\int_t^\infty |t-s|^{-\frac{1}{2}}(\|\eta(s)\|_{L_x^4}\|W(s)\|_{L_x^4}^2 + \|\eta(s)\|_{L_x^4}^3 + \|H(s)\|_{L_x^{\frac{4}{3}}})ds \\ &\leq C\int_t^\infty |t-s|^{-\frac{1}{2}}(e^{-\lambda s}+e^{-3\lambda s}+e^{-\lambda s})ds \leq C\int_t^\infty |t-s|^{-\frac{1}{2}}e^{-\lambda s}ds.\end{aligned}$$

3. Given any $C>0$, show that there exists λ^* sufficiently large such that if $\lambda > \lambda^*$, then for all $t \geq 0$ we have

$$C\int_t^\infty |t-s|^{-\frac{1}{2}}e^{-\lambda s}ds \leq e^{-\lambda t}.$$

Solution: By changes of variables, we have

$$\int_t^\infty |t-s|^{-\frac{1}{2}}e^{-\lambda s}ds = \int_0^\infty |\sigma|^{-\frac{1}{2}}e^{-\lambda\sigma-\lambda t}d\sigma = e^{-\lambda t}\lambda^{-\frac{1}{2}}\int_0^\infty |z|^{-\frac{1}{2}}e^{-z}dz.$$

Hence choosing λ large enough gives the desired result.

4. Show that Φ maps B into B for $\lambda > \lambda^*$.

Solution: Combining the previous results, if $\lambda > \lambda^*$, then for any $\eta \in B$ we have

$$\|\Phi(\eta)(t)\|_{L_x^4} \leq C\int_t^\infty |t-s|^{-\frac{1}{2}}e^{-\lambda s}ds \leq e^{-\lambda t}.$$

Thus

$$\|e^{\lambda t}\|\Phi(\eta)\|_{L_x^4}\|_{L_t^\infty} \leq 1$$

and $\Phi(\eta) \in B$.

5. Show that there exists $\lambda^{**} > 0$ such that Φ is a contraction mapping on B for $\lambda > \lambda^{**}$.

Solution: The proof that Φ is a contraction follows similar lines. Let $\eta_1, \eta_2 \in B$. Then, at fixed t, we have

$$\begin{aligned}\|\Phi(\eta_1) - \Phi(\eta_2)\|_{L^4_x} &\leq C \int_t^\infty |t-s|^{-\frac{1}{2}} \|f(W+\eta_1) - f(W+\eta_2)\|_{L_x^{\frac{4}{3}}} ds \\ &\leq C \int_t^\infty |t-s|^{-\frac{1}{2}} \||\eta_1 - \eta_2|(|W|^2 + |\eta_1|^2 + |\eta_2|^2)\|_{L_x^{\frac{4}{3}}} ds \\ &\leq C \int_t^\infty |t-s|^{-\frac{1}{2}} \|\eta_1 - \eta_2\|_{L^4_x} (\|W\|^2_{L^4_x} + \|\eta_1\|^2_{L^4_x} + \|\eta_2\|^2_{L^4_x}) ds \\ &\leq C \|e^{\lambda t} \|\eta_1 - \eta_2\|_{L^4_x}\|_{L^\infty_t} e^{\lambda t} \int_t^\infty |t-s|^{-\frac{1}{2}} e^{-2\lambda s} ds \\ &\leq \|e^{\lambda t} \|\eta_1 - \eta_2\|_{L^4_x}\|_{L^\infty_t} e^{-\lambda t}.\end{aligned}$$

6. Let $\eta \in B$ be such that $\Phi(\eta) = \eta$. Assume that W verifies the equation

$$i\partial_t W + \Delta W + f(W) = H.$$

What is the equation verified by u defined by $u = W + \eta$?

Solution: Differentiating η in time, we observe that

$$\partial_t \eta = i(f(W+\eta) - f(W) + H) + i\Delta\eta.$$

Therefore, η verifies the Schrödinger equation

$$i\partial_t \eta + \Delta\eta + f(W+\eta) - f(W) = -H.$$

Summing up with the equation of W and using $u = W + \eta$, we obtain for u the equation

$$iu_t + \Delta u + f(u) = 0.$$