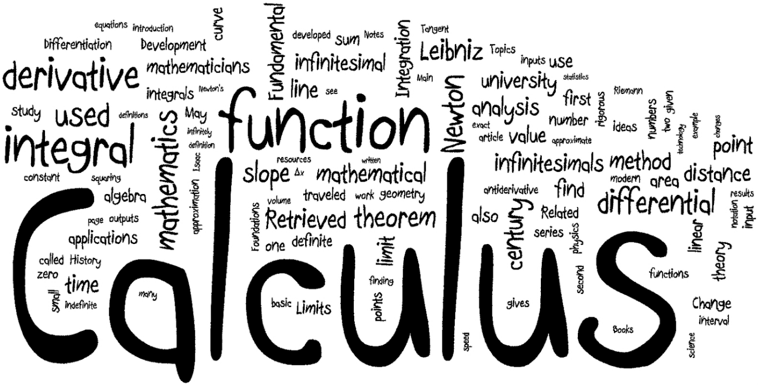
Calculus
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method
Newton
Leibniz
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mathematicians
university
analysis
slope
distance
point
line
Fundamental
Integration
integrals
Development
Differentiation
equations
introduction
curve
developed
sum
Notes
Tangent
Topics
inputs
use
statistics
Main
first
Riemann
see
Newton's
study
definitions
May
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definition
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exact
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physics
notation
input
linear
theory
called
History
zero
time
small
indefinite
many
limit
finding
basic
Limits
points
gives
second
functions
Change
interval
Books
speed
science

MATHS180-102

Intructor: Stefan LE COZ

November 28, 2014

Common webpage:
`http://www.math.ubc.ca/~andrewr/maths100180/maths100_180_common.html`
Section webpage:
`http://www.math.univ-toulouse.fr/~slecoz/MATHS180-102.html`

Survival Guide

- Come to class and be active: take notes, participate, ask questions !
- Nevertheless: Most of the work will be done outside the class
  - Study the textbook
  - Do WeBWorK assignements
  - Do the suggested homework problems
- Seek for help:
  - Discuss with your classmates
  - Go to the Maths Learning Centre (see common webpage for infos)
  - Look at AMS tutoring
  - See the Mathematics Department website (lots of ressources)
  - Come to Office hours

Foreword

Calculus is the mathematical study of change

Gottfried Leibniz (1646-1716) and Isaac Newton (1643-1727)

3. Differentiation rules

The Tangent Problem

Problem

Given the graph of a function, find the equation of the tangent at a point on the graph.

The Velocity Problem

Problem

Given the position at each time of an object moving on a straight line, find the *instantaneous* speed of the object at a given time.

Definition

Suppose that f is a function defined close to a number a. Then we write

$$\lim_{x \to a} f(x) = L$$

and say *"the limit of $f(x)$, as x approaches a, equals L"* if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a (but different from a)

Limits on the left and on the right

Definition (limit on the left)

We write

$$\lim_{x \to a^-} f(x) = L$$

and say *"the limit of $f(x)$, as x approaches a from the left, equals L"* if we can make the values of $f(x)$ arbitrarily close to L by taking x (different from a) sufficiently close to a and less than a.

Remark

Another notation: $\lim\limits_{\substack{x \to a \\ x < a}} f(x) = L$

Remark (limit on the right)

We define

$$\lim_{x \to a^+} f(x) = L$$

in a similar manner

Infinite Limits

Definition (Infinite limit)

We write

$$\lim_{x \to a} f(x) = \infty$$

and if we can make the values of $f(x)$ arbitrarily large by taking x (different from a) sufficiently close to a.

Remark (negative infinite limit)

We say

$$\lim_{x \to a} f(x) = -\infty$$

if we can make the values of $f(x)$ arbitrarily small by taking x (different from a) sufficiently close to a.

Remarks

Remark (Uniqueness of the limit)

A function can have only one limit at a point.

Remark

We have

$$\lim_{x \to a} f(x) = L$$

if and only the limits on the left and on the right exist and

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Limit laws

Limit laws

Let c be a constant. Assume that $\lim\limits_{x \to a} f(x)$, and $\lim_{x \to a} g(x)$ exist. Then

- $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
- $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$

Direct substitution property

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Comparison of limits and the Squeeze Theorem

Theorem (Comparison Theorem)

If $f(x) \leq g(x)$ for x close to a (but different from a), then, provided the limits exist, we have

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x),$$

Theorem (The Squeeze Theorem or the Sandwich Theorem)

If for x close to a (but different from a)

$$f(x) \leq g(x) \leq h(x), \quad \textit{and} \ \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

A useful Substitution Property

Theorem (Substitution Property)

If $f(x) = g(x)$ for x close to a (but different from a), then, provided the limit exist, we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$$

Continuity

Definition
A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

Definition
A function f is continuous on an interval I if it is continuous at every point of I.

Operations and classical functions

Proposition

Take two functions f and g both continuous at a number a and c a constant. Then the following functions are also continuous at a:

$$f+g, \quad f \cdot g, \quad c \cdot f, \quad \frac{f}{g} \text{ if } g(a) \neq 0.$$

Theorem

The following classical types of functions are continuous at every number in their domain:

> *polynomials, rational functions, root functions,*
> *trigonometric functions, inverse trigonometric functions,*
> *exponential functions, logarithmic functions.*

Intermediate Value Theorem (IVT)

Theorem

Assume that f is continuous on $[a, b]$ and $f(a) \neq f(b)$. Let N be any number between $f(a)$ and $f(b)$. Then there exists c in $[a, b]$ such that $f(c) = N$.

Composition

Theorem
Assume that f is continuous at b and $\lim_{x \to a} g(x) = b$. *Then*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

Theorem
Assume that f is continuous at b, g is continuous at a and $g(a) = b$. *Then the composite function* $f \circ g$ *given by* $(f \circ g)(x) = f(g(x))$ *is continuous at a.*

Definition

Let f be defined on $(a, +\infty)$. We say that

$$\lim_{x \to +\infty} f(x) = L$$

if the value of $f(x)$ becomes arbitrarily close to L as x is taken sufficiently large.

Remark

We define $\lim_{x \to -\infty} f(x) = L$ *in a similar way.*

Definition

The line $y = L$ is called an horizontal asymptote for the graph of f if either

$$\lim_{x \to +\infty} f(x) = L, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

Important rule

Let r be a positive rational number, then

$$\lim_{x \to +\infty} \frac{1}{x^r} = 0.$$

If x^r is defined for x negative, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Definition

The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We say that f is differentiable at a.

Remark

Equivalently, we have $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$*.*

Definition

Assume that f is differentiable at each point of an interval (a, b). Then the derivative f' is a function:

$$\begin{aligned} f' : (a, b) &\to \mathbb{R} \\ x &\mapsto f'(x) \end{aligned}$$

Remark (Other notations)

If $y = f(x)$, then $f'(x)$ can also be denoted

$$y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

Theorem

If f is differentiable at a, then f is continuous at a.

Definition

We denote by f'' and call the second derivative of f the function obtained as the derivative of the derivative of f.

Remark

We can define define analogously the third derivative f''', etc.

Derivative of a constant

$$\frac{d}{dx}(c) = 0$$

Derivative of a power

For any r,

$$\frac{d}{dx}(x^r) = r \cdot x^{r-1}$$

Operations compatibles with derivations

Sum, Difference, product with a constant

Let f and g be differentiable and c a constant. Then

$$\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g,$$
$$\frac{d}{dx}(f-g) = \frac{d}{dx}f - \frac{d}{dx}g,$$
$$\frac{d}{dx}(cf) = c \cdot \frac{d}{dx}f.$$

The exponential function

Derivative of the exponential

$$\frac{d}{dx} e^x = e^x$$

Remark

The number e is such that $e = e^1 = 2.711828$.

The Product rule

If f and g are differentiable, then

$$\frac{d}{dx}(f(x) \cdot g(x)) = \left(\frac{d}{dx}f(x)\right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx}g(x)\right)$$

Remark

In short:

$$(f \cdot g)' = f' \cdot g + f \cdot g'.$$

The Quotient Rule

If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx}f(x)\right)\cdot g(x) - f(x)\cdot\left(\frac{d}{dx}g(x)\right)}{\left(g(x)\right)^2}$$

Remark

In short:

$$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}.$$

Remark

Note that we need $g(x) \neq 0$ *for* $\frac{f(x)}{g(x)}$ *to be differentiable.*

Proposition

$$\lim_{h\to 0} \frac{sin(h)}{h} = 1, \quad \lim_{h\to 0} \frac{cos(h)-1}{h} = 0.$$

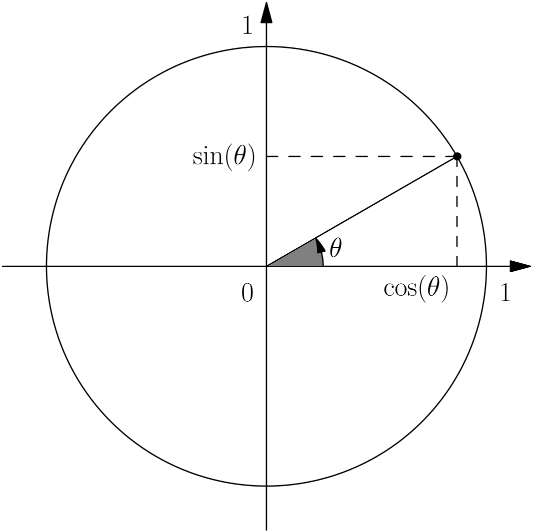
Theorem

$$\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x).$$

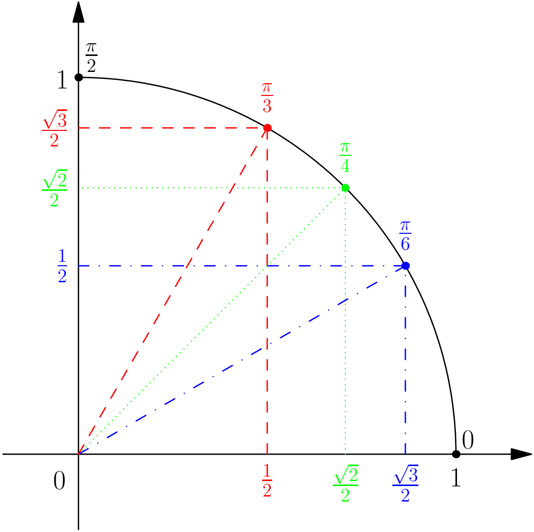
Rule of thumb

Differentiating cos and sin is like making a quarter turn on the trigonometric circle.

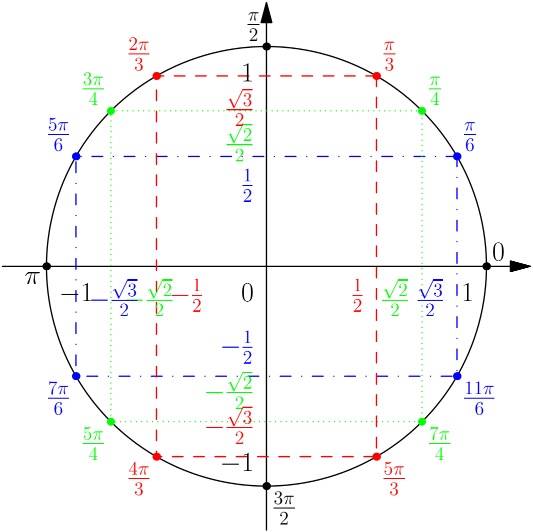
Trigonometric circle



Remarkable values



More remarkable values



The Chain Rule

Assume g is differentiable at x and f is differentiable at $g(x)$. Then the composite function $h = f \circ g$ is differentiable at x and h' is

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Other Notation

Set $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Applications of the Chain Rule

$f(x)$	$f'(x)$
$u^{\alpha}(x),\ \alpha \in \mathbb{R}^*$	$\alpha u'(x) u^{\alpha-1}(x)$
$e^{u(x)}$	$u'(x) e^{u(x)}$
$\sin(u(x))$	$u'(x)\cos(u(x))$
$\cos(u(x))$	$-u'(x)\sin(u(x))$

2. Limits and Derivatives

3. Differentiation rules

1. Functions and models
1.6 Inverse Functions and Logarithms

3. Differentiation rules

4. Applications of Differentiation

A note on the definition of a function

Definition

A function is a relation between a set of input numbers (the *domain*) and a set of permissible output numbers (the *codomain*) with the property that each input is related to exactly one output.

Remark

- *The domain and codomain are often implicit*
- *Usually the function is given by a formula*
- *Do not confuse the codomain and the range (or image).*

Definition

We say that a function is one to one if it never take the same value twice. That is:

$$\text{If } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

Horizontal line test

A function is one-to-one if and only if no horizontal line $x = c$ intersects the graph $y = f(x)$ more than once.

Definition

Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any $y \in B$.

Definition (Logarithmic functions)

Let $a > 0$. The function $\log_a$ is the inverse of the function $y \mapsto a^y$ and is defined by

$$\log_a(x) = y \Leftrightarrow a^y = x.$$

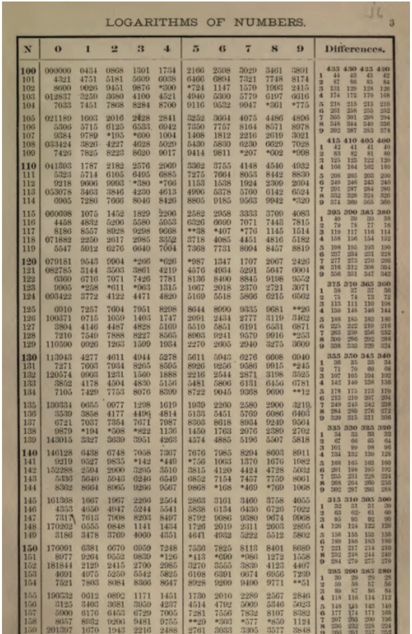
Remark

$\ln(x) = \log_e(x)$ *(natural or Naperian logarithm) and* $\log_{10}$ *(common or decimal logarithm) are the most used.*

Laws of Logarithms

- $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$,
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$,
- $\log_a(x^r) = r\log_a(x)$,
- $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

How did people compute logarithms not so long ago ?



N	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	*300	*724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	*361	*775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	*195	*600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	*207	*602	*998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	*380	*766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	**38	*407	*776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	079181	9543	9904	*266	*626	*987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	*258	*611	*963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	**26
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	**12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	*194	*508	*822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	*142	*449	*756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	*168	*469	*769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	*126	*413	*699	*985	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	**51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	**29	*303	*577	*850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848

Differences.

	435	**430**	**425**	**420**
1	44	43	43	42
2	87	86	85	84
3	131	129	128	126
4	174	172	170	168
5	218	215	213	210
6	261	258	255	252
7	305	301	298	294
8	348	344	340	336
9	392	387	383	378
	415	**410**	**405**	**400**
1	42	41	41	40
2	83	82	81	80
3	125	123	122	120
4	166	164	162	160
5	208	205	203	200
6	249	246	243	240
7	291	287	284	280
8	332	328	324	320
9	374	369	365	360
	395	**390**	**385**	**380**
1	40	39	39	38
2	79	78	77	76
3	119	117	116	114
4	158	156	154	152
5	198	195	193	190
6	237	234	231	228
7	277	273	270	266
8	316	312	308	304
9	356	351	347	342
	375	**370**	**365**	**360**
1	38	37	37	36
2	75	74	73	72
3	113	111	110	108
4	150	148	146	144
5	188	185	183	180
6	225	222	219	216
7	263	259	256	252
8	300	296	292	288
9	338	333	329	324
	355	**350**	**345**	**340**
1	36	35	35	34
2	71	70	69	68
3	107	105	104	102
4	142	140	138	136
5	178	175	173	170
6	213	210	207	204
7	249	245	242	238
8	284	280	276	272
9	320	315	311	306
	335	**330**	**325**	**320**
1	34	33	33	32
2	67	66	65	64
3	101	99	98	96
4	134	132	130	128
5	168	165	163	160
6	201	198	195	192
7	235	231	228	224
8	268	264	260	256
9	302	297	293	288
	315	**310**	**305**	**300**
1	32	31	31	30
2	63	62	61	60
3	95	93	92	90
4	126	124	122	120
5	158	155	153	150
6	189	186	183	180
7	221	217	214	210
8	252	248	244	240
9	284	279	275	270
	295	**290**	**285**	**280**
1	30	29	29	28
2	59	58	57	56
3	89	87	86	84
4	118	116	114	112
5	148	145	143	140
6	177	174	171	168
7	207	203	200	196
8	236	232	228	224
9	[illegible]	[illegible]	[illegible]	[illegible]

Definition

The inverse sine function denoted by arcsin or $\sin^{-1}$ is defined by

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$\arcsin(x) = y \Leftrightarrow \sin(y) = x$$

Definition

The inverse cosine function denoted by arccos or $\cos^{-1}$ is defined by

$$\arccos : [-1, 1] \to [0, \pi]$$
$$\arccos(x) = y \Leftrightarrow \cos(y) = x$$

Definition

The inverse tangent is denoted by arctan or $\tan^{-1}$ is defined by

$$\arctan : (-\infty, +\infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$\arctan(x) = y \Leftrightarrow \tan(y) = x$$

Definition

$$\begin{aligned} y = f(x) &\quad y \text{ is explicit} \\ g(x, y) = 0 &\quad y \text{ is implicit} \end{aligned}$$

Rule

If y is implicitly defined as one or more functions of x, it is possible to compute y' by differentiating the implicit relation.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}},$$
$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}},$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Derivative of the natural logarithm

$$\frac{d}{dx}\log(x) = \frac{1}{x}$$

Remark

Derivative of others logarithms

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\log(a)}.$$

Useful trick: logarithmic differentiation

Calculate $\frac{d}{dx}\left(\frac{x^2\sqrt{1+x}}{(2+\sin(x))^7}\right)$.

Definition

If a quantity y depends explicitely on a quantity x, meaning $y = f(x)$, the

- average rate of change of y with respect to x over $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x}, \quad \text{where } \begin{cases} \Delta y = f(x_2) - f(x_1) \\ \Delta x = x_2 - x_1 \end{cases}$$

- instantaneous rate of change of y with respect to x at x_1 is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Some examples

In Physics

With $f(t)$ the position at time t of a particle moving in a straight line (e.g. a photon in a laser beam).

In Chemistry

With $V(P)$ the volume of balloon of gas with respect to the pressure.

In Biology

With $f(t)$ the number at time t of individual of an animal or plant population.

Theorem

The only solutions to the differential equation

$$\frac{dy}{dt} = ky$$

are the exponential functions

$$y = y(0)e^{kt}$$

Population growth

The rate of change in the population is proportional to the size of the population:

$$\frac{dP}{dt} = kP.$$

Newton's law of cooling

The rate of change in temperature of an object is proportional to the difference between its temperature and that of its surroundings:

$$\frac{dT}{dt} = k(T - T_{\text{surroundings}}).$$

Coffee at home

- temperature when the coffee comes out of the machine: 93°C
- Neighbor comes at door to borrow suggar: 1 minute
- Temperature of the coffee after I get rid of neighbor: 88°C

Questions:

- When can I drink my coffee without burning myslelf (i.e. at 63°C) ?
- What happens if I have to leave my coffee for a long time ?

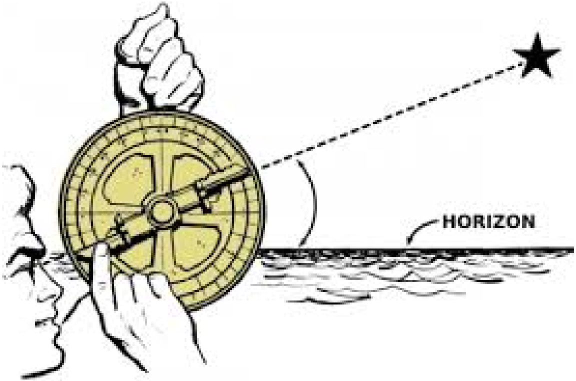
Problem solving strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

1re EXPÉRIENCE AEROSTATIQUE A ANNONAY, le 4 Juin 1783

COLLECTION 476. 1re Série. (N° 2) ROMANET & Cie IMP. EDIT. PARIS

Astrolabe/Protractor



Definition

Let f be a differentiable function. We say that

$$f(x) \simeq f(a) + f'(a)(x - a)$$

is the linear approximation of f at a. We call

$$L(x) = f(a) + f'(a)(x - a)$$

the linearization of f at a.

Midterm teaching survey

- Writing/Pronunciation
- Office hours
- Webwork
- Class atmosphere
- Hard Problems during the lectures

Goal

Approximate functions by a polynomial.

Definition

Let f be n time differentiable at a point a. The Taylor Polynomial of degree n for f at a is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor Polynomials for Classical Functions at $a = 0$

$$e^x : \qquad T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

$$\sin(x) : \qquad T_n(x) = x - \frac{x^3}{3!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$(2k+1$ greatest odd integer $\leq n)$

$$\cos(x) : \qquad T_n(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$$

$(2k$ greatest even integer $\leq n)$

$$\ln(1+x) : \qquad T_n(x) = x - \frac{x^2}{2} + \cdots + (-1)^{n+1} \frac{x^n}{n}$$

$$\frac{1}{1-x} : \qquad T_n(x) = 1 + x + x^2 + \cdots + x^n$$

2. Limits and Derivatives

3. Differentiation rules

1. Functions and models

3. Differentiation rules

3.5 Implicit Differentiation
3.6 Derivatives of Logarithmic Functions
3.7 Rates of Change in the Natural and Social Sciences
3.8 Exponential growth and decay
3.9 Related rates
3.10 Linear approximation and differentials
3.11 Taylor Polynomials
3.12 Taylor's Formula with Remainder

4. Applications of Differentiation

Goal

Evaluate the difference between a function and its Taylor polynomial at a point

The Taylor-Lagrange Formula

Let f be $n+1$ time differentiable at a. Then for each x there exists c between a and x such that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Definition

Let c be in the domain D of a function f. We say that $f(c)$ is an

- absolute maximum value for f if $f(c) \geq f(x)$ for all $x \in D$.
- absolute minimum value for f if $f(c) \leq f(x)$ for all $x \in D$.

Definition

We say that $f(c)$ is a

- local maximum value for f if $f(c) \geq f(x)$ for x close to c.
- local minimum value for f if $f(c) \leq f(x)$ for x close to c.

Remark

In general, we speak about an extremum for a maximum or a minimum.

Extreme value theorem

If f is continuous on the closed interval $[a, b]$, then f has a global maximum value $f(c)$ and a global minimum value $f(d)$ for some c, d in $[a, b]$.

Definition

A critical number of f is c such that

$$f'(c) = 0 \text{ or } f'(c) \text{ DNE.}$$

Theorem (Fermat's theorem)

If f has a local extremum at c, then c is a critical number for f.

Closed interval method

To find the global extrema of f on $[a, b]$ closed interval:

- Find all critical numbers and the values of f at that points
- Find $f(a)$, $f(b)$.
- The largest and smallest values give the extrema.

Theorem (Rolle's theorem)

Let f be such that

- f *continuous on* $[a, b]$
- f *differentiable on* (a, b)
- $f(a) = f(b)$

Then there exists $c \in [a, b]$ *such that* $f'(c) = 0$*.*

The mean value Theorem

Let f be such that

- f continuous on $[a, b]$
- f differentiable on (a, b)

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem

If $f'(x) = 0$ *for all* $x \in (a, b)$*, then* f *is constant on* (a, b)

Corollary

If $f'(x) = g'(x)$ *for all* $x \in (a, b)$*, then there exists* $K \in \mathbb{R}$ *such that*

$$f(x) = g(x) + K.$$

Increasing/Decreasing test

- If $f'(x) > 0$ on (a, b), then f is increasing on (a, b).
- If $f'(x) < 0$ on (a, b), then f is decreasing on (a, b)

First derivative test

Suppose that c is a critical number of a continuous function f.

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' does not change sign at c, then f has no local maximum or minimum at c.

Definition

If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Concavity Test

- If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes at P from concave upward to concave downward or vice-versa.

The Second Derivative Test

Suppose f is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Informations to gather before sketching a graph

- Domain
- Intercepts
- Symmetries
- Asymptotes
- Increasing / decreasing
- Local max / min
- Concavity

Trick: Make a *table of changes*.

Slant Asymptotes

Definition

A line $y = mx + b$ is a slant asymptote if

$$\lim_{\substack{x \to +\infty \\ \text{or } x \to -\infty}} (f(x) - (mx + b)) = 0$$

Proposition

The graph of f admits a slant asymptote at $+\infty$ if and only if

$$\lim_{x\to+\infty} \frac{f(x)}{x} = m, \text{ with } m \text{ finite real number.}$$

In that case, the equation of the asymptote is

$$y = mx + b,$$

where $b = \lim_{x\to+\infty}(f(x) - mx)$.

Problem solving strategy

1. Read the problem carefully.
2. Draw a diagram.
3. Introduce notation. E.g. call Q the quantity to maximize.
4. Express Q in terms of the other symbols.
5. If Q has been expressed as a function of more than one variable, use the given information to find relationships among these variables. Eliminate all but one variable.
6. Find the maximum value.
7. Verify that the value is consistent with the problem.

Problem 1

You need to make a box. You are given a square of cardboard (12cm by 12cm) and you need to cut out squares from the the corners of your sheet so that you may fold it into a box. How large should these cut-out squares be so as to maximise the volume of the box?

Problem 2

You need to cross a small canal to get from point A to point B. The canal is 300m wide and point B is 800m from the closest point on the other side. You can row at 6km/h and run at 10km/h. To which point on the opposite side of the canal should you row to in order to minimise your travel time from A to B?

Student Evaluation of Teaching

- The feedback is used to assess and improve my teaching.
- Heads and Deans look at evaluation results as an important component of decisions about reappointment, tenure, promotion and merit for faculty members.
- Evaluations are used to shape departmental curriculum.
- We take 15 minutes of class to complete the survey (use your mobile devices)

More precise definitions

Definition

Let $f : [a, b] \to \mathbb{R}$. A critical number of f is $c \in (a, b)$ (thus $c \neq a$ and $c \neq b$) such that

$$f'(c) = 0 \text{ or } f'(c) \text{ DNE.}$$

Definition

Let $f : [a, b] \to \mathbb{R}$. Let $c \in (a, b)$ (thus $c \neq a$ and $c \neq b$). We say that $f(c)$ is a

- local maximum value for f if $f(c) \geq f(x)$ for x close to c.
- local minimum value for f if $f(c) \leq f(x)$ for x close to c.

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ for x close to a, $x \neq a$. Assume that

$$\begin{array}{lccc} & \lim_{x\to a} f(x) = 0 & \text{and} & \lim_{x\to a} g(x) = 0 \\ \text{or} & \lim_{x\to a} f(x) = \pm\infty & \text{and} & \lim_{x\to a} g(x) = \pm\infty \end{array}$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Definition

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

Let f be a function and F be an antiderivative of F. Then all antiderivatives of F are of the form

$$F(x) + C, \quad C \in \mathbb{R}.$$

Rectilinear Motion, an example

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6cm/s$ and its displacement is $s(0) = 9cm$. Find its position function $s(t)$.