First order linear diff syst.

2) first order linear diff syst with const. coef.

They are systems of the form

$$\frac{dY}{dt} = AY + B(t)$$

where the matrix $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ is independante of t.

2.1) Elementary exponential solution of $\frac{dY}{dt} = AY$

We look for a solution of the form $Y(t) = e^{\lambda t} V$

with $\lambda \in \mathbb{R}$, $V \in \mathbb{R}^N$. This function is a solution if and only if

$$\lambda e^{\lambda t} V = e^{\lambda t} AV$$

That is

$$AV = \lambda V.$$

This is the case ~~iff~~ if λ is an eigenvalue of A and V is an eigenvector of A associated to λ.

- 1^{st} case : A is diagonalizable

In this case there exists a base $(V_1, \dots, V_N)$ of $\mathbb{R}^N$ of eigenvectors of A with eigenvalues $(\lambda_j)_{j=1 \dots N}$. We obtain therefore N solutions linearly independant

$$t \mapsto e^{\lambda_j t} V_j \quad 1 \le j \le N.$$

Since the space of solutions S of dimension N, we have obtained a basis of the space of solutions. The general sol is given by

$$Y(t) = \alpha_1 e^{\lambda_1 t} V_1 + \dots + \alpha_N e^{\lambda_N t} V_N \,, \quad (\alpha_j) \in \mathbb{R}.$$

<u>Example</u>: We look for the solutions to

$$\frac{dY}{dt} = AY$$

with $Y : \mathbb{R} \to \mathbb{R}^2$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- 2^{nd} case: A is not diagonalizable.

In this case, we need to introduce the notion of exponential of a matrix.

2) Example:

$$\frac{dY}{dt} = AY \quad , \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for the e.v. 1.

sol $e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

missing $\rightarrow$ sols.

1.2) Exponential of a matrix

We recall that for a real $x \in \mathbb{R}$, we have $e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$

<u>Def</u>: For $A \in \mathcal{M}_N(\mathbb{R})$, we define e^A or $\exp(A)$ by

$$e^A := \sum_{k=0}^{+\infty} \frac{1}{k!} A^k$$

Assume that $\mathcal{M}_m(\mathbb{R})$ is endowed with the norm

$$|||A||| = \sup_{\|v\|=1} \|Av|$$

Then
$$|||\frac{1}{m!}A^m||| = \sup_{|v|=1} |\frac{1}{m!}A^m v|$$
$$\leq \frac{1}{m!} |||A|||^m$$

Therefore, the ~~sum~~ series $\sum \frac{1}{m!} A^m$ is absolutely converging.

Moreover, $|||e^A||| \leq e^{|||A|||}$

<u>Proposition</u>: If $A, B \in \mathcal{M}_N(\mathbb{R})$ commute (ie $A.B = BA$), then

$$e^{A+B} = e^A e^B.$$

Proof : admit.

Rk 1: e^A is an invertible matrix with inverse e^{-A}.

Rk 2: If A and B do not commute, then the property is wrong.

ex: $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.

...

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute e^A.

$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, ..., $A^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow e^A = \sum_{m=0}^{\infty} \frac{1}{m!} A^m = \sum_{m=0}^{\infty} \begin{pmatrix} \frac{1}{m!} & \frac{m}{m!} \\ 0 & \frac{1}{m!} \end{pmatrix} \qquad I+N$$

$$= \begin{pmatrix} e^1 & e^1 \\ 0 & e^1 \end{pmatrix}$$

Formula: $\det(e^A) = \exp(\mathrm{tr}(A))$.

2.3 General solution of the syst with second member.

$$\frac{dY}{dt} = AY.$$

<u>Th</u>: ~~Ve~~ The solution Y s.t. $Y(t_0) = Y_0$ is given by

$$Y(t) = e^{(t-t_0)A} V_0 \qquad \forall t \in \mathbb{R}.$$

<u>Proof</u>: We have $Y(t_0) = e^{0} V_0 = V_0$.

On the other hand, we can derive

$$e^{tA} = \sum_{m=0}^{+\infty} \frac{1}{m!} A^m$$

term by term for $t \in \mathbb{R}$ do get

$$\frac{d}{dt}\left(e^{tA}\right) = \sum_{m=1}^{+\infty} \frac{1}{(m-1)!} t^{m-1} A^m = \sum_{p=0}^{+\infty} \frac{1}{p!} t^p A^{p+1}$$

$$= A \cdot e^{tA} = e^{tA} \cdot A$$

Thus, we have

$$\frac{dY}{dt} = \frac{d}{dt}\left(e^{(t-t_0)A} V_0 \right) = A e^{(t-t_0)A} V_0 = A Y(t) \qquad \blacksquare$$

<u>Example</u>: $\dfrac{dY}{dt} = AY$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then it is easy to see that $e^{(t-t_0)A} = \begin{pmatrix} e^{t-t_0} & e^{t-t_0} \\ 0 & e^{t-t_0} \end{pmatrix}$

~~Then~~ The solution with $Y(t_0) = V_0$ is then

$$Y(t) = \begin{pmatrix} e^{t-t_0} & e^{t-t_0} \\ 0 & e^{t-t_0} \end{pmatrix} V_0.$$

2.4) General solution of $\dfrac{dY}{dt} = AY + B(t)$

As before, the gen sol ~~for~~ $\dfrac{dY}{dt} = AY + B(t)$ is given by

$$Y(t) = Y_g(t) + Y_p(t)$$

where Y_g is the general sol of the homogeneous syst

$$\frac{dY}{dt} = AY$$

and Y_p is a particular sol of $\dfrac{dY}{dt} = AY + B(t)$.

<u>Method of the variation of constants.</u>

We look for a particular solution of the form

$$Y(t) = e^{tA} V(t).$$

It comes

$$Y'(t) = A e^{tA} V(t) + e^{tA} V'(t)$$

$$= A Y(t) + e^{tA} V'(t)$$

To have a solution, we just have to choose V s.t.

$e^{tA} V'(t) = B(t)$.

For example $V(t) = \int_{t_0}^{t} e^{-sA} B(s)\, ds$, $t_0 \in Z$

Thus we get the particular sol

$$Y(t) = e^{tA} \int_{t_0}^{t} e^{-sA} B(s)\, ds = \int_{t_0}^{t} e^{(t-s)A} B(s)\, ds$$

This is the sol s.t. $Y(t_0) = 0$.

Therefore ~~To have~~ the solution of ~~the~~ $\frac{dY}{dt} = AY + B(t)$ with $Y(t_0) = V_0$ is given by

$$Y(t) = e^{(t-t_0)A} . V_0 + \int_{t_0}^{t} e^{(t-s)A} B(s)\, ds .$$

Example: Find the solution of

$$\begin{cases} x(t) = y(t) \\ y(t) = x(t) + e^{t} \end{cases}$$

Such that $x(0) =$ $\quad$ $y(0) =$

Sol:

3) Linear differential equations of order p with constant coefficients

We consider the homogeneous eq

(E) $\quad a_p y^{(p)} + \dots + a_1 y' + a_0 y = 0$

where $y : \mathbb{R} \to \mathbb{C}$, $t \mapsto y(t)$ is the unknown functions

and $(a_j) \subset \mathbb{C}$ are constants, $a_p \neq 0$

Equation (E) is equivalent to a first order differential system in $\mathbb{C}^p$.

Indeed, define p functions $\{y_0, \dots, y_{p-1}\}$ by

$$y_0 = y$$
$$y_1 = y'$$
$$\vdots$$
$$y_{p-1} = y^{(p-1)}$$

Then these fct verify

$$\begin{cases} y_0' = y_1 \\ \vdots \\ y_{p-2}' = y_{p-1} \\ y_{p-1}' = -\dfrac{1}{a_p}\left(a_0 y_0 + \dots + a_{p-1} y_{p-1}\right). \end{cases}$$

what we can rewrite, using the matrix,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{p-1} \end{pmatrix}, \quad c_j = -\frac{a_j}{a_p} \quad \text{and } Y = \begin{pmatrix} y_0 \\ \vdots \\ y_{p-1} \end{pmatrix}$$

by $\quad Y' = AY$.

<u>Thm</u>: The set $\mathcal{S}$ of solutions of (E) is a vector-space of dimension p

Now, we look for the solutions of (E).

We first search for solutions of the exponential form $y(t) = e^{\lambda t}$.

Since $y^{(j)}(t) = \lambda^j e^{\lambda t}$, we see that y is a solution of (E) if and only if λ is a root ~~zero~~ of the characteristic polynomial

$$P(\lambda) = a_p \lambda^p + \dots + a_1 \lambda + a_0$$

3.1) <u>Case where P has only simple roots</u>

If P has p ~~distinct~~ different roots, $\lambda_1, \dots, \lambda_p$, then we obtain p ~~distinct~~ different solutions $t \mapsto e^{\lambda_j t}$, $1 \leq j \leq p$.

We will see that these fct are linearly independent.

Hence, the set of solutions to (E) is the vector space of dimension p of functions of the form

$$y(t) = \alpha_1 e^{\lambda_1 t} + \dots + \alpha_p e^{\lambda_p t} \quad , \quad \alpha_j \in \mathbb{C}$$

3.2) <u>Case ~~d~~ where P has multiple roots</u>

Assume that P has s roots $\lambda_1, \dots, \lambda_s$ with multiplicity $m_1, \dots, m_s$, ~~i.e~~,

$$P(\lambda) = a_p \prod_{j=1}^{s} (\lambda - \lambda_j)^{m_j} \quad , \quad \sum_{j=1}^{s} m_j = p.$$

Consider the differential operator

$$P\left(\frac{d}{dt}\right) = \sum_{k=1}^{p} a_k \frac{d^k}{dt^k}$$

We can rewrite (E) as

$$(E) \qquad P\left(\frac{d}{dt}\right) y = 0$$

Moreover, we have $P\left(\frac{d}{dt}\right) e^{\lambda t} = P(\lambda) e^{\lambda t} \ \forall \lambda \in \mathbb{C}$.

The idea is now to look for solutions of the form $t \mapsto t^q e^{\lambda t}$.

We observe that, by Schwarz Theorem,

$$P\left(\frac{d}{dt}\right)\left(t^q e^{\lambda t}\right) = P\left(\frac{d}{dt}\right)\left(\frac{d^q}{d\lambda^q} e^{\lambda t}\right) = \frac{d^q}{d\lambda^q}\left(P\left(\frac{d}{dt}\right) e^{\lambda t}\right) = \frac{d^q}{d\lambda^q}\left(P(\lambda) e^{\lambda t}\right).$$

By Leibnitz formula, we get

$$P\left(\frac{d}{dt}\right)\left(t^q e^{\lambda t}\right) = \sum_{k=0}^{q} \binom{q}{k} P^{(k)}(\lambda) t^{(q-k)} e^{\lambda t}$$

Since λ_j is a root of multiplicity m_j, we have

$$P^{(k)}(\lambda_j) = 0 \quad \forall \quad 0 \leq k \leq m_j - 1 \text{ and}$$

$$P^{(m_j)}(\lambda_j) = 0 \, .$$

We infer that

$$P\left(\frac{d}{dt}\right)\left(t^q e^{\lambda_j t}\right) = 0 \quad \forall \quad 0 \leq q \leq m_j - 1 .$$

Hence (E) admit ~~for~~ the solutions

$$y(t) = t^q e^{\lambda_j t}, \quad 0 \leq q \leq m_j - 1, \quad 1 \leq j \leq s.$$

That is in total $m_1 + \dots + m_s = p$ solutions

<u>Lemma</u>: If the (λ_j), $1 \leq j \leq s$ are differents, then the functions $y_{j,q}(t) := t^q e^{\lambda_j t}$ $1 \leq j \leq s$ $q \in \mathbb{N}$ are linearly independant.

<u>Proof</u>: Assume there exists $(\alpha_{j,q}) \subset \mathbb{C}$ a finite family s.t.

$$(R) \qquad \sum \alpha_{j,q} \, y_{j,q} = 0 ,$$

If the $\alpha_{j,q}$ are not all 0, let N be the maximum of the integer q s.t there exists j with $\alpha_{j,q} \neq 0$.

$$N := \max \{ q \in \mathbb{N} \,/\, \exists\, j \in \mathbb{N},\ \alpha_{j,q} \neq 0 \}.$$

Assume, for example, that $\alpha_{1,N} \neq 0$. We set

$$Q(\lambda) = (\lambda - \lambda_1)^N (\lambda - \lambda_2)^{N+1} \dots (\lambda - \lambda_s)^{N+1}$$

Then $Q^{(h)}(\lambda_j) = 0$ for $j \geq 2$, $0 \leq h \leq N$.

whereas $Q^{(h)}(\lambda_1) = 0$ for $0 \leq h \leq N-1$, and $Q^{(N)}(\lambda_1) \neq 0$.

We infer that

$$Q\left(\frac{d}{dt}\right)\left(t^q e^{\lambda_j t}\right) = \sum_{h=0}^{q} \binom{q}{h} Q^{(h)} \lambda_j\, t^{q-h} e^{\lambda_j t}$$

$$= 0 \quad \forall\ 0 \leq q \leq N, \quad 1 \leq j \leq s, \text{ ~~except~~ } \underline{\text{except if}}$$

$q = N$, $j = 1$, where

$$Q\left(\frac{d}{dt}\right)\left(t^N e^{\lambda_1 t}\right) = Q^{(N)}(\lambda_1)\, e^{\lambda_1 t}$$

Applying $Q\left(\frac{d}{dt}\right)$ to (R) we obtain

$$\alpha_{1,N}\, Q^{N}(\lambda_1)\, e^{\lambda_1 t} = 0,$$

which is impossible since $\alpha_{1,N} \neq 0$ and $Q^{(N)}(\lambda_1) \neq 0$. ∎

<u>Thm</u>: When the characteristic polynom $P(\lambda)$ has complex roots $\lambda_1, \dots, \lambda_s$ of multiplicity $m_1, \dots, m_s$, the set of solutions $\mathcal{S}$ is the $\mathbb{C}$-vector space of dimension p with basis

$$\left\{ t \mapsto t^q e^{\lambda_j t}, \ 1 \leq j \leq s, \ 0 \leq q \leq m_j - 1 \right\}$$

<u>Examples</u>: • (E) $y'' - y = 0$

Characteristic polynomial : $P(\lambda) = \lambda^2 - 1$

Roots : $1, -1$

Sol : e^t, e^{-t}

• (E) $y'' + y = 0$

Char Pol $P(\lambda) = \lambda^2 + 1$

Roots : $i, -i$

Sol e^{it}, e^{-it}

<u>Rk</u>: real solutions are given taking the real part ...

• (E) $y'' - 2y' + y = 0$

Char Pol = $P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

Root : 1 multiplicity 2.

Sol e^t, te^t.

3.3) ~~Eq~~ lin diff eq of order p with second member

We consider

$$(E) \qquad a_p y^{(p)} + \dots + a_1 y' + a_0 y = b(t)$$

where $b : I \to \mathbb{C}$ is a given function. The general

As usual, the general ~~solutions~~ solution of (E) ~~are~~ is given by the sum of ~~solutions~~ solution of ~~We start by finding~~ (E) without second member and a particular sol of (E).

- Some particular cases.

→ If b is a polynomial:

$$y'' - y = t^2$$

we look for a solution of the form $y(t) = \alpha t^2 + \beta t + \gamma$:

$$y'(t) = 2\alpha t + \beta$$
$$y''(t) = 2\alpha$$

$$\Rightarrow \quad y'' - y = 2\alpha - (\alpha t^2 + \beta t + \gamma)$$

$$= -\alpha t^2 + (2\alpha - \gamma) = t^2$$

$$\Rightarrow \quad \alpha = -1, \quad \gamma = -2$$

→ If $b(t) = \alpha e^{\lambda t}$ and λ is not a root of the char poly.

$$y'' - y = e^{2t}$$

we look for a sol $y(t) = \beta e^{2t}$: $y''(t) = 4\beta e^{2t}$

$$y'' - y = (4\beta - \beta) e^{2t} \quad \Rightarrow \beta = \frac{1}{3}.$$

- In general: variation of constant

As before, we write (E) as a system:

$$(S) \quad Y' = AY + B(t)$$

with $Y = \begin{pmatrix} y_0 \\ \vdots \\ y_{p-1} \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(p-1)} \end{pmatrix}$ and $B(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{a_p} b(t) \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 1 & 0 & \text{---} & 0 \\ 0 & 0 & 1 & \text{---} & 0 \\ \vdots & & & & \\ & & & & 1 \\ \frac{a_0}{a_p} & \frac{a_1}{a_p} & & \cdots & \frac{a_{p-1}}{a_p} \end{pmatrix}$$

The homogeneous system

$$(S_0) \quad Y' = AY$$

admit a basis of solutions

$$V_1 = \begin{pmatrix} v_1 \\ v_1' \\ \vdots \\ v_1^{(p-1)} \end{pmatrix}, \dots, V_p = \begin{pmatrix} v_p \\ v_p' \\ \vdots \\ v_p^{(p-1)} \end{pmatrix}$$

We look for a particular sol of (S) of the form

$$Y(t) = \alpha_1(t) V_1(t) + \dots + \alpha_p(t) V_p(t)$$

Since $V_j' = AV_j$, we get

$$Y'(t) = \sum \alpha_j(t) V_j'(t) + \sum \alpha_j'(t) V_j(t) = AY(t) + \sum \alpha_j'(t) V_j(t)$$

To have a solution of (S), it is enough to choose the α_j s.t. $\sum \alpha_j'(t)\, v_j(t) = B(t)$.

i.e.:

$$\begin{cases} \alpha_1'(t)\, v_1(t) + \dots + \alpha_p'(t)\, v_p(t) = 0 \\ \dots \\ \alpha_1'(t)\, v_1^{(p-2)}(t) + \dots + \alpha_p'(t)\, v_p^{(p-2)}(t) = 0 \\ \alpha_1'(t)\, v_1^{(p-1)}(t) + \dots + \alpha_p'(t)\, v_p^{(p-1)}(t) = \frac{1}{a_p}\, b(t) \end{cases}$$

<u>Example</u>: (E) $\quad y'' + 4y = \tan t \quad , \quad t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

- We start by solving

(E₀) $\quad y'' + 4y = 0$

$P(\lambda) = \lambda^2 + 4 \rightarrow 2i, -2i$

$t \mapsto e^{2it}, \quad t \mapsto e^{-2it}$

For real fct° $\quad t \mapsto \cos 2t, \quad t \mapsto \sin 2t$

- We look for a particular sol of (E) by setting

$y(t) = \alpha_1(t) \cos 2t + \alpha_2(t) \sin 2t$.

This leads us to finding the solution of

$$\begin{cases} \alpha_1'(t) \cos 2t + \alpha_2'(t) \sin(2t) = 0 \\ \alpha_1'(t)(-2) \sin 2t + \alpha_2'(t)\, 2 \cos(2t) = \tan t \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix} \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan t \end{pmatrix}$$

$$\det \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix} = 2$$

Kramer:
$$\begin{cases} \alpha_1' = \dfrac{\begin{vmatrix} 0 & \sin 2t \\ \tan t & 2\cos 2t \end{vmatrix}}{2} \\ \alpha_2' = \dfrac{\begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \tan t \end{vmatrix}}{2} \end{cases}$$

i.e.
$$\begin{cases} \alpha_1' = -\frac{1}{2} \sin 2t \tan t = \dfrac{\sin t \cos t \sin t}{\cos t} = \sin^2 t \\ \quad = -\frac{1}{2}(1-\cos 2t) \\ \alpha_2' = \frac{1}{2} \tan t \cos 2t = \frac{1}{2} \sin 2t - \frac{1}{2} \tan t \end{cases}$$

$$\Rightarrow \quad \alpha_1 = -\frac{t}{2} + \frac{1}{4} \sin 2t$$

$$\alpha_2 = -\frac{1}{4} \cos 2t + \frac{1}{2} \ln(\cos t).$$

Therefore, the particular solution is

$$y(t) = -\frac{t}{2} \cos 2t + \frac{1}{2} \sin 2t \ln(\cos t).$$

and the general solution

$$\boxed{y(t) = -\frac{t}{2} \cos 2t + \frac{1}{2} \sin 2t \ln(\cos t) + \alpha_1 \cos 2t + \alpha_2 \sin 2t.}$$

4) Linear differential systems with non constant coefficients

4.1) Resolvant of a linear system

We consider a homogeneous linear system

$$(E_0) \qquad Y' = A(t)Y$$

where $A : I \to \mathcal{M}_m(\mathbb{R})$ is a matrix $m \times m$ with coefficients in $\mathbb{R}$ continuous.

Let $\mathcal{S}$ be the set of maximal solutions to (E_0).

For all $t_0 \in I$, we know that

$$\Phi_{t_0} : \mathcal{S} \to \mathbb{R}^m$$
$$Y \mapsto Y(t_0)$$

is an linear isomorphisme

For all couple $(t, t_0) \in I^2$, we define

$$R(t,t_0) = \Phi_t \circ \Phi_{t_0}^{-1} \qquad \mathbb{R}^m \xrightarrow{\Phi_{t_0}^{-1}} \mathcal{S} \xrightarrow{\Phi_t} \mathbb{R}^m$$
$$V \longmapsto Y \longmapsto Y(t)$$

In this way, we have

$$R(t,t_0).V = Y(t),$$

where Y is the solution s.t. $Y(t_0) = V$.

Since $R(t,t_0)$ is an isomorphism for $\mathbb{R}^m$ ~~to~~, we identify it with its associated matrix

<u>Definition</u> : $R(t,t_0)$ is called the <u>resolvant</u> of the linear system (E_0).

Let $V \in \mathbb{R}^m$. We remark that

$$\left(\frac{d}{dt} R(t,t_0)\right).V = \frac{d}{dt}\left(R(t,t_0).V\right) = \frac{dY}{dt} = A(t)\,Y(t) = A(t)\,R(t,t_0)\,V$$

Hence
$$\frac{d}{dt} R(t,t_0) = A(t)\,R(t,t_0) .$$

<u>Proposition</u>: Properties of the resolvant

(i) $\forall t \in I$, $R(t,t) = Id_{\mathbb{R}^m}$

(ii) $\forall (t_0,t_1,t_2) \in I^3$, $R(t_2,t_1)\,R(t_1,t_0) = R(t_2,t_0)$

(iii) $R(t,t_0)$ is the solution in $\mathcal{M}_m(\mathbb{R})$ of the diff syst

$$\frac{d\Pi}{dt} = A(t)\,\Pi(t)$$

where $\Pi(t) \in \mathcal{M}_m(\mathbb{R})$ verify the initial condition $\Pi(t_0) = Id_{\mathbb{R}^m}$

<u>Proof</u>: (i) $R(t,t) = \Phi_t \circ \Phi_t^{-1} = Id_{\mathbb{R}^m}$

(ii) $R(t_2,t_1).R(t_1,t_0) = \left(\Phi_{t_2} \circ \Phi_{t_1}^{-1}\right) \circ \left(\Phi_{t_1} \circ \Phi_{t_0}^{-1}\right)$
$= \Phi_{t_2} \circ \Phi_{t_0}^{-1} = R(t_2,t_0)$.

(iii) See above.

Rk: It seems that we are complicating things by introducing $\frac{d\Pi}{dt} = A(t)\,\Pi(t)$ (

~~How~~ Because m^2 eq instead of m. However, Algebra str of $M_m(\mathbb{R})$.

Example: Suppose that $A(t)\,A(s) = A(s)\,A(t) \quad \forall\, t, s \in I$.

Then $R(t, t_0) = \exp\left(\int_{t_0}^{t} A(s)\, ds\right)$.

proof: ~~It~~ is enough to prove that

$$\Pi(t) = \exp\left(\int_{t_0}^{t} A(s)\, ds\right)$$

verifies

$$\begin{cases} \frac{d\Pi}{dt} = A(t)\,\Pi(t) \\ \Pi(t_0) = Id_{\mathbb{R}^m} \end{cases}$$

We clearly have $\Pi(t_0) = \exp\left(0_{\mathbb{R}^m}\right) = Id_{\mathbb{R}^m}$

~~Thus~~ Since $A(s)$ commute with $A(t)$ for any s, t, we have

$$\int_a^b A(s)\, ds \times \int_c^d A(t)\, dt = \iint_{[a,b]\times[c,d]} A(s)\,A(t)\, ds\, dt \qquad \forall\, a,b,c,d \in I,\ a<b,\ c<d$$

Therefore

$$\Pi(t+h) = \exp\left(\int_{t_0}^{t} A(s)\, ds + \int_t^{t+h} A(s)\, ds\right)$$

$$= \exp\left(\int_t^{t+h} A(s)\, ds\right) \Pi(t)$$

By the formula for exp, we have, since $\int_t^{t+h} A(s)\, ds = h\, A(t) + o(h)$

$$\text{If } \Pi(t+h) = \left(I_{\mathbb{R}^m} + h\, A(t) + o(h) \right) \Pi(t)$$

$$= \Pi(t) + h\, A(t)\, \Pi(t) + o(h).$$

which means $\dfrac{d\Pi}{dt} = A(t)\,\Pi(t)$

Example: Let U and V be constant matrices s.t. $U.V = VU$. and define $A(t) = f(t)\,U + g(t)\,V$ for scalar functions f, g. Then we have

$$R(t,t_0) = \exp\left(\int_{t_0}^{t} f(s)\,ds \,.\, U + \int_{t_0}^{t} g(s)\,ds \,.\, V \right)$$

$$= \exp\left(\int_{t_0}^{t} f(s)\,ds \,.\, U \right) . \exp\left(\int_{t_0}^{t} g(s)\,ds \; V \right).$$

Exercise 1: ~~Use~~ ~~this~~ Calculate the resolvent associated with

$$A(t) = \begin{pmatrix} a(t) & -b(t) \\ b(t) & a(t) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} 1 & 0 & \cos^2 t \\ 0 & 1 & \cos^2 t \\ 0 & 0 & \sin^2 t \end{pmatrix}$$

Exercise 2: Solve the linear system

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{t} x + t y \\[2mm] \dfrac{dy}{dt} = y \end{cases} \qquad \text{where } A(t) = \begin{pmatrix} 1/t & t \\ 0 & 1 \end{pmatrix}$$

Deduce the formula of the resolvent.

Show that

$$R(t,t_0) \neq \exp\left(\int_{t_0}^{t} A(s)\,ds\right).$$

Rk: Here, we see that, as in most of the time, solving the system gives the resolvent and not the converse.

4.2) Wronskian

Def: The wronskian of a system of solutions $Y_1, \dots, Y_m$ of (E_0) is

$$W(t) = \det\left(Y_1(t), \dots, Y_m(t)\right).$$

Let $V_j = Y_j(t_0)$.

Then $Y_j(t) = R(t,t_0) \cdot V_j$

Thus $W(t) = \det\left(\left(R(t,t_0) \cdot V_j\right)\right) = \det R(t,t_0) \det\left(V_1, \dots, V_m\right)$

Computing the wronskian is equivalent to compute the quantity

$$D(t) = \det\left(R(t,t_0)\right).$$

For that, we will show that $D(t)$ verify ~~an ordinary diff~~ simple ODE. We have

$$D(t+h) = \det\left(R(t+h,t_0)\right) = \det\left(R(t+h,t)\,R(t,t_0)\right)$$

$$= \det\left(R(t+h,t)\right) D(t).$$

Since $R(t,t) = I_{\mathbb{R}^m}$ and

$$\frac{d}{ds} R(s,t)\Big|_{s=t} = A(t)\, R(t,t) = A(t),$$

a Taylor expansion gives us

$$R(t+h,t) = I_m + h A(t) + o(h)$$

$$\det\big(R(t+h,t)\big) = \det\big(I_m + h A(t)\big) + o(h).$$

<u>Lemma</u>: Let $A = (a_{ij}) \in \mathcal{M}_m(\mathbb{R})$. Then

$$\det(I_m + hA) = 1 + \alpha_1 h + \dots + \alpha_m h^m$$

where $\alpha_1 = \operatorname{tr} A = \sum_{1 \le i \le m} a_{ii}$

<u>Proof</u>: Indeed, in $\det(I_m + hA)$ the diagonal term is

$$(1 + h a_{11})(\dots)(1 + h a_{mm}) = 1 + h \sum a_{ii} + h^2 \dots$$

and the non diagonal terms are multiple of h^2 ∎

This implies

$$\det\big(R(t+h,t)\big) = 1 + h \operatorname{tr}\big(A(t)\big) + o(h).$$

$$D(t+h) = D(t) + h \operatorname{tr}\big(A(t)\big) D(t) + o(h).$$

Hence $\boxed{D'(t) = \operatorname{tr}\big(A(t)\big) D(t)}$

Since $D(t_0) = \det(R(t_0,t_0)) = \det(I_m) = 1$, we get

$$\det R(t,t_0) = D(t) = \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\, ds\right)$$

$$W(t) = \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right) \det(V_1, \dots, V_m).$$

4.3) Wronskian in dimension 2

Knowing a solution for an 2^{nd} order ODE, the other one can be computed with the wronskian.

<u>Example</u>: Solve $t^2 y'' - t y' + y = 0$ on $]0, \infty[$

An obvious solution is $y_1(t) = t$

Let y_2 be another sol, ~~not~~ linearly ind of $y(t)$.

Then the wronskian is given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = t y_2' - y_2$$

On the other hand,

$$W'(t) = \frac{1}{t} W(t) \quad \text{and so} \quad W(t) = W_0 t$$

Thus y_2 satisfy the equation

$$t y_2' - y_2 = W_0 t$$

Or in the reduced form

$$y_2' - \frac{1}{t} y_2 = W_0$$

Hence $y_2 = W_0 t e^{mt} + Ct$.

4.4) <u>Variation of constant</u>

Let

$$(E) \qquad Y' = A(t)Y + B(t)$$

and $R(t,t_0)$ the resolvant of the linear system without second member

$$(E_0) \qquad Y' = A(t)Y$$

We look for a particular solution of (E) of the form

$$Y(t) = R(t,t_0)\, V(t)$$

It comes

$$\frac{dY}{dt} = \left(\frac{d}{dt} R(t,t_0)\right) \cdot V(t) + R(t,t_0)\, V'(t)$$

$$= A(t)\, R(t,t_0)\, V(t) + R(t,t_0)\, V'(t)$$

$$= A(t)\, Y(t) + R(t,t_0)\, V'(t)$$

We choose V s.t.

$$R(t,t_0)\, V'(t) = B(t).$$

That is

$$V'(t) = \left(R(t,t_0)\right)^{-1} B(t) = R(t_0,t)\, B(t)$$

$$V(t) = \int_{t_0}^{t} R(t_0,s)\, B(s)\, ds.$$

Hence $Y(t) = R(t,t_0)\, V(t) = \int_{t_0}^{t} R(t,t_0) R(t_0, s) B(s)\, ds$

$$Y(t) = \int_{t_0}^{t} R(t,s)\, B(s)\, ds.$$

is the solution such that $Y(t_0) = 0$.

The solution s.t $Y(t_0) = V_0$ is given by

$$Y(t) = R(t,t_0) . V_0 + \int_{t_0}^{t} R(t,s) B(s)\, ds$$

Rk: If $A(t) \equiv A$ is with constant coefficients, we have $R(t,t_0) = e^{(t-t_0)A}$ and we recover the formula

$$Y(t) = e^{(t-t_0)A} + \int_{t_0}^{t} e^{(t-s)} B(s)\, ds.$$

Stability of solutions and singular points of a vector field

1) Stability of solutions.

We consider the Cauchy problem associated with a diff eq

(E) $\quad y' = f(t,y)$

with initial condition $y(t_0) = z_0$.

We suppose that the sol of this pb exists on $[t_0, +\infty[$,

1.1) Definition.

<u>Def</u>: • Let $y(t,z)$ be the maximal solution of (E) with $y(t_0, z) = z$. We say that the solution $y(t,z_0)$ is $\boxed{\text{stable}}$ if there exists a ball $\overline{B}(z_0, r)$ and a constant $C \geq 0$ s.t.

(i) $\forall z \in \overline{B}(z_0, r)$, $t \mapsto y(t,z)$ is defined on $[t_0, +\infty)$

(ii) $\forall z \in \overline{B}(z_0, r)$ and $t \geq t_0$, we have

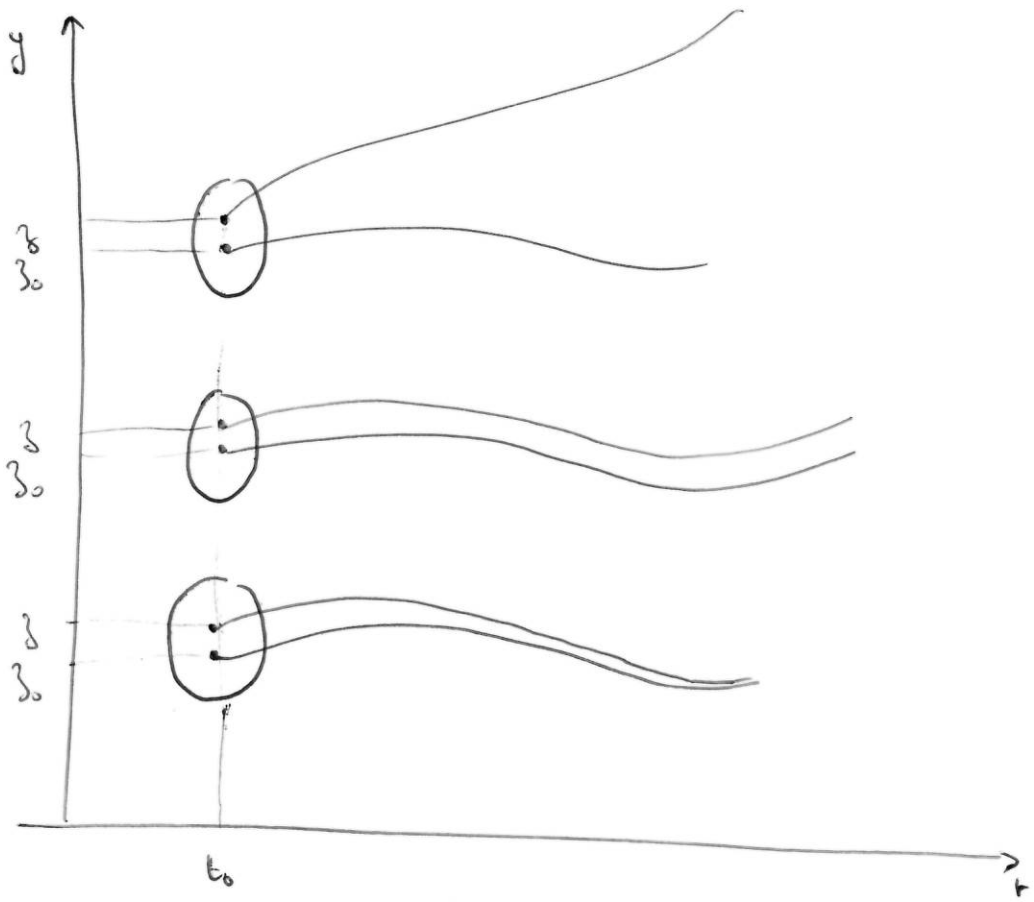
$$\| y(t,z) - y(t,z_0) \| \leq C \| z - z_0 \|$$

• The solution $y(t,z_0)$ is said to be $\boxed{\text{asymptotically stable}}$ if it is stable and (ii') is satisfied

(ii') $\exists\, \overline{B}(z_0, r)$ and $\gamma : [t_0, +\infty) \to \mathbb{R}_+$ continuous with $\lim_{t \to +\infty} \gamma(t) = 0$

s.t. $\forall z \in \overline{B}(z_0, r)$ and $t \geq t_0$ we have

$$\| y(t,z) - y(t,z_0) \| \leq \gamma(t) \, \| z - z_0 \| .$$



1.2) Case of a linear system with constant coefficient

We start with a syst without second member

$$(E) \quad Y' = AY \quad , \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} , \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}$$

with $y_j, a_{ij} \in \mathbb{C}$.

The solution of the Cauchy problem with initial condition $Y(t_0) = Z$ is given by

$$Y(t,z) = e^{(t-t_0)A} \cdot Z$$

Therefore, we have

$$Y(t,z) - Y(t,z_0) = e^{(t-t_0)A}(z-z_0).$$

Hence, the stability is linked to the behavior of $e^{(t-t_0)A}$ when t goes to $+\infty$. In particular, $|||e^{(t-t_0)A}|||$ must stay bounded.

• __If $m=1$__

$A=(a)$. That is, we have

$$\left| e^{(t-t_0)a} \right| = e^{(t-t_0)\,\mathrm{Re}(a)}.$$

Solutions are stable if and only if this quantity stays bdd, that is if $\mathrm{Re}(a) \leq 0$. Beside, solutions are asymptotically stables if $\mathrm{Re}(a) < 0$ ~~and then~~ (we take $\gamma(t) = e^{(t-t_0)\,\mathrm{Re}(a)}$).

• __If $m>1$ and A is diagonalizable__

After a change of coordinate, we can ~~have~~ replace A by

$$\tilde{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A.

The system is then reduced to a family of independent eq $\quad y_j' = \lambda_j y_j$ and admit for sol

$$y_j(t, z) = z_j e^{\lambda_j (t-t_0)} \qquad 1 \le j \le m.$$

Hence solutions are stable if and only if $Re(\lambda_j) \le 0$ for all j and asymptotically stable if and only if $Re(\lambda_j) < 0 \quad \forall j$.

• if $m > 1$ and A is not diagonalizable.

We observe what happen for each block of a triangulation of A. We assume

$$A = \begin{pmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \lambda I + N$$

where N is a nilpotent (triangular superior)

We have

$$e^{(t-t_0)A} = e^{(t-t_0)\lambda I} \cdot e^{(t-t_0)N}$$

$$= e^{\lambda(t-t_0)} \sum_{k=0}^{m-1} \frac{(t-t_0)^k}{k!} N^k$$

Therefore, the coefficients of $e^{(t-t_0)A}$ are of the form $P_{ij}(t)\, e^{\lambda(t-t_0)}$

where at least one of the P_{ij} is ~~not~~ of degree ≥ 1.

If $\mathrm{Re}\,\lambda < 0$, the coefficients tend to 0 and if

$\mathrm{Re}(\lambda) > 0$ they tend to ∞.

If $\mathrm{Re}\,\lambda = 0$, then $|e^{\lambda(t-t_0)}| = 1$ and thus $e^{(t-t_0)A}$

is ~~not~~ unbounded (because at least one of the P_{ij}

is of degree ≥ 1).

Therefore either solutions are asymptotically stable (when $\mathrm{Re}(\lambda)<0$)

or they are unstable.

=

To sum up

<u>Theorem</u>: Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of the matrix A

Then the solutions of the linear syst $Y' = AY$ are

- asymptotically stable iff $\mathrm{Re}(\lambda_j) < 0 \quad \forall j = 1, \dots, m$.
- stable iff $\forall j$ either $\mathrm{Re}(\lambda_j) < 0$ or $\mathrm{Re}(\lambda_j)$ and the corresponding block is diagonalizable

1.3) Perturbation of a linear syst

We consider in $\mathbb{C}^m$ a system of the form

(E) $\quad Y' = AY + g(t, Y)$

where $g : [t_0, +\infty[\times \mathbb{C}^m \to \mathbb{C}^m$

is a continuous function.

<u>Theorem</u>: We assume that the eigenvalues λ_j of A have ~~area~~ all a negative real part $\mathrm{Re}(\lambda_j) < 0$.

(a) If there exists a function $k : [t_0, +\infty) \to \mathbb{R}_+$ s.t.

$$\lim_{t \to +\infty} k(t) = 0$$

and

$$\forall t \in (t_0, +\infty),\ \forall Y_1, Y_2 \in \mathbb{C}^m,\quad \|g(t, Y_1) - g(t, Y_2)\| \leq k(t) \|Y_1 - Y_2\|$$

then all solutions of (E) are asymptotically stable

(b) If $g(t, 0) = 0$ and there exists $r_0 > 0$ and a continuous fct $k : [0, r_0] \to \mathbb{R}_+$ s.t. $\lim_{r \to 0} k(r) = 0$ and

$\forall t \in [t_0, +\infty)\ \forall\ 0 < r \leq r_0\ \forall Y_1, Y_2 \in \overline{B}(0, r)$,

$$\|g(t, Y_1) - g(t, Y_2)\| \leq k(r) \|Y_1 - Y_2\|$$

Then there exists a ball $\overline{B}(0, r_1) \subset \overline{B}(0, r_0)$ s.t. all solution $Y(t, Z_0)$ with $Z_0 \in \overline{B}(0, r_1)$ is asymptotically stable.

<u>Proof</u>: There exists a basis $(e_1, \dots, e_m)$ in which A has the form

$$A = \begin{pmatrix} \lambda_1 & a_{12} & \dots & a_{1m} \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & a_{m-1,m} \\ 0 & \dots & & \lambda_m \end{pmatrix}$$

Set $\tilde{e}_j = \varepsilon^j e_j$ with $\varepsilon > 0$ small.

We have

$$A \tilde{e}_j = \varepsilon^j \left(a_{1j} e_1 + \dots + a_{j-1,j} e_{j-1} + \lambda_j e_j \right)$$

$$= \varepsilon^{j-1} a_{1j} \tilde{e}_1 + \dots + \varepsilon a_{j-1,j} \tilde{e}_{j-1} + \lambda_j \tilde{e}_j$$

Hence, in the basis $\tilde{e}_j$, the coefficient of the matrix A are

$$\tilde{a}_{ij} = \varepsilon^{j-i} a_{ij}$$

In particular, we can make the coefficient off diagonal as small as we want.

Therefore, we will assume $|a_{ij}| < \varepsilon$ if $i \neq j$.

We consider two solutions $Y(t, z)$ and $Y(t, z_0)$:

$$Y'(t, z) = A\, Y(t, z) + g\,(t, Y(t, z))$$

$$Y'(t, z_0) = A\, Y(t, z_0) + g\,(t, Y(t, z_0))$$

and we evaluate the difference

$$\Delta(t) = Y(t, z) - Y(t, z_0)$$

(a) In this case, $f(t,Y) = AY + g(t,Y)$ is Lipschitz in Y with Lipschitz constant $|||A||| + k(t)$.

This implies that the solutions are globally defined on $[t_0, +\infty)$. We have

$$\Delta'(t) = A\Delta(t) + g(t, Y(t,Z)) - g(t, Y(t,Z_0))$$

$$\| g(t, Y(t,Z)) - g(t, Y(t,Z_0)) \| \leq k(t) \|\Delta(t)\|.$$

Write $\Delta(t) = \begin{pmatrix} \delta_1(t) \\ \vdots \\ \delta_m(t) \end{pmatrix}$

and $\rho(t) = \|\Delta(t)\|^2 = \sum_{j=1}^{m} \delta_j(t) \overline{\delta_j(t)}$.

We have

$$\rho'(t) = \sum_{j=1}^{m} \delta_j'(t)\overline{\delta_j(t)} + \delta_j(t)\overline{\delta_j'(t)}$$

$$= 2 \sum_{j=1}^{m} \delta_j'(t)\overline{\delta_j(t)}$$

$$= 2 \operatorname{Re}\left({}^t\overline{\Delta(t)}\, \Delta'(t) \right)$$

$$= 2 \operatorname{Re}\left({}^t\overline{\Delta(t)}\, A\Delta(t) \right) + \underbrace{2 \operatorname{Re}\, {}^t\overline{\Delta(t)} \left(g(t, Y(t,Z)) - g(t, Y(t,Z_0)) \right)}$$

$$= 2(\alpha + \beta)$$

$$\beta \leq \|\Delta(t)\| \, \| g(t, Y(t,Z)) - g(t, Y(t,Z))\| \leq k(t) \|\Delta(t)\|^2 = 2k(t)\, \rho(t)$$

On the other hand

$$ {}^t\overline{\Delta(t)}\, A\, \Delta(t) = \sum_{j=1}^{m} \lambda_j\, |\delta_j(t)|^2 + \sum_{i>j} a_{ij}\, \overline{\delta_i(t)}\, \delta_j(t) $$

Thus

$$ \mathrm{Re}\left({}^t\overline{\Delta(t)}\, A\, \Delta(t)\right) \leq \sum_{j=1}^{m} (\mathrm{Re}\,\lambda_j)\, |\delta_j(t)|^2 + \left(\sum_{i<j} |a_{ij}|\right) \|\Delta(t)\|^2 $$

Since $\mathrm{Re}\,\lambda_j < 0$ and $|a_{ij}| \leq \varepsilon$ if $i \neq j$, there exists a choice of ε s.t.

$$ \mathrm{Re}\left({}^t\overline{\Delta(t)}\, A\, \Delta(t)\right) \leq -\alpha \sum_{j=1}^{m} |\delta_j(t)|^2 = -\alpha\, \rho(t) $$

for some $\alpha > 0$.

We obtain

$$ \rho'(t) \leq -2\alpha\rho(t) + 2h(t)\rho(t) $$

$$ \frac{\rho'(t)}{\rho(t)} \leq 2(-\alpha + h(t)) $$

$$ \ln \frac{\rho(t)}{\rho(t_0)} \leq -2\int_{t_0}^{t} (\alpha - h(s))\, ds $$

$$ \rho(t) \leq \|z - z_0\|^2 \exp\left(-2\int_{t_0}^{t} (\alpha - h(s))\, ds\right) $$

since $\rho(t_0) = \|z - z_0\|^2$

Note that $\ell(t) = \|\Delta(t)\|^2$ vanishes only if the two solutions coincide.

Taking the square root, we get

$$\|y(t,z) - y(t,z_0)\| \leq \gamma(t)\, \|z - z_0\|$$

with $\gamma(t) = \exp\left(-\int_{t_0}^{t} (\alpha - k(s))\, ds\right)$.

Since $\lim_{s \to +\infty} (\alpha - k(s)) = \alpha > 0$, $\int_{t_0}^{t} (\alpha - k(s))\, ds \underset{t \to \infty}{\longrightarrow} +\infty$

and $\lim_{t \to +\infty} \gamma(t) = 0$.

Hence sol are asymptotically stable

→ Exercice (6).

2) Singular points of a vector field

Let $V : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$M = \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto V(M) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

be a $\mathcal{C}^1$ vector field and consider the associated system

$$(E) \qquad \frac{dM}{dt} = V(M) \iff \begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{cases}$$

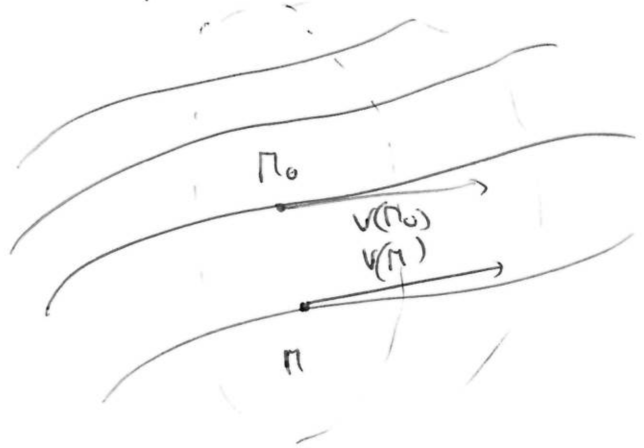
<u>Def</u>: An integral curve of ~~[illegible]~~ V is a curve $(I, \gamma) \in \mathbb{R}$ such that $\frac{\partial}{\partial t} \gamma(t) = V(\gamma(t))$.
It is said to be maximal if I is maximal.

<u>Rk</u>: By Cauchy-Lipschitz theorem, ~~[illegible]~~ for each point of $\mathbb{R}^2$ there exists one and only one integral curve of V passing through this point.

<u>Pb</u>: Given M_0, describe the family of integral curves passing ~~close to~~ close to M_0.

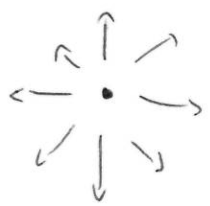
<u>First case</u>: $V(M_0) \neq 0$.

In this case, the angle between $V(M)$ and $V(M_0)$ goes to 0 when M goes to M_0. Consequently, the tangent to the integral curves are almost parallel in a neighborhood of M_0. Such a path M_0 is said to be regular

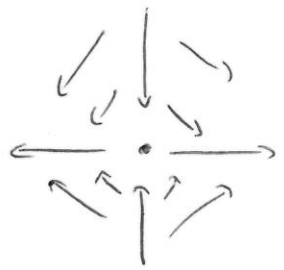


Second case: $V(M_0) = 0$

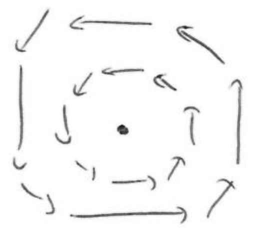
Then, many configurations are possible for the ~~tangent~~ field of tangents



$V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$



$V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$



$V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

Def: If $V(M_0) = 0$, we say that M_0 is a singular point of the vector field V.

Such a point gives clearly a constant solution of (E), or equilibrium $M(t) = M(t_0)$.

In what follows, we suppose $M_0 = 0$.
Then, $f(0,0) = g(0,0) = 0$, so that the system can be rewritten as

$$\frac{dx}{dt} = f(x,y) = ax + by + o(|x| + |y|)$$

$$\frac{dy}{dt} = g(x,y) = cx + dy + o(|x| + |y|)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f'_x(0,0) & f'_y(0,0) \\ g'_x(0,0) & g'_y(0,0) \end{pmatrix}$

Then the system can be written as

$$\frac{dM}{dt} = AM + G(M)$$

with $G(0,0) = G'_x(0,0) = G'_y(0,0) = 0$.

The function

$$k(r) = \sup_{M \in \overline{B}(0,r)} ||| G'(M) |||$$

goes to 0 when $r \to 0$ and the mean value theorem gives

$$\| G(M_1) - G(M_2) \| \leq k(r) \, \| M_1 - M_2 \|$$

for all $M_1, M_2 \in \overline{B(0,r)}$.

By the previous thm we get

Prop: ~~Lem~~ For a singular point M_0 to be asymptotically stable (~~i.e. all in~~ it is enough that the eigenvalues of the jacobian matrix A

$$A = \begin{pmatrix} f'_x(x_0, y_0) & f'_y(x_0, y_0) \\ g'_x(x_0, y_0) & g'_y(x_0, y_0) \end{pmatrix}$$

are with negative real part.

Def: We say that a singular point of a ~~matrix~~ vector field V is ~~isolated~~ non degenerate if $\det A \neq 0$.

2.2. Case of a linear vector field

$$\frac{dM}{dt} = AM \iff \begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we supp $\det A \neq 0$.

(a) The eigenvalues λ_1, λ_2 of A are real

- supp $\lambda_1 \neq \lambda_2$. Then A is diagonalizable. After a change of basis, we can suppose

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and the system reduce to

$$\begin{cases} \dfrac{dx}{dt} = \lambda_1 x \\ \dfrac{dy}{dt} = \lambda_2 y \end{cases}$$

The sol of the C-P with $M(0) = (x_0, y_0)$ is

$$\begin{cases} x(t) = x_0 e^{\lambda_1 t} \\ y(t) = y_0 e^{\lambda_2 t} \end{cases}$$

Hence, integral curves are given by

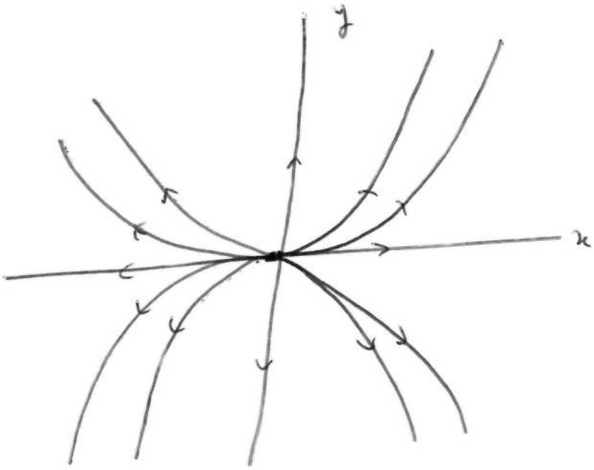
$$y = \frac{y_0}{|x_0|^{\lambda_2/\lambda_1}} \left(|x_0| e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} = \frac{y_0}{|x_0|^{\lambda_2/\lambda_1}} |x(t)|^{\lambda_2/\lambda_1} \quad \text{if } x_0 \neq 0$$

$$= C\,|x|^{\lambda_2/\lambda_1} \qquad C \in \mathbb{R}$$

and $x = 0$ if $x_0 = 0$.

* Assume λ_1, λ_2 have the same sign and, for example, $|\lambda_1| < |\lambda_2|$

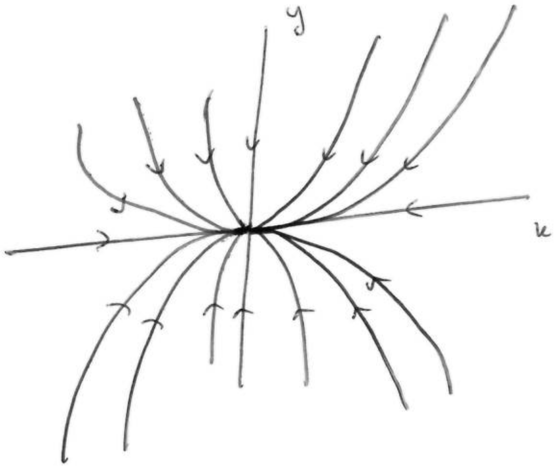
Then $\lambda_2/\lambda_1 > 1$. We say we have a node



$0 < \lambda_1 < \lambda_2$

unstable node

node source



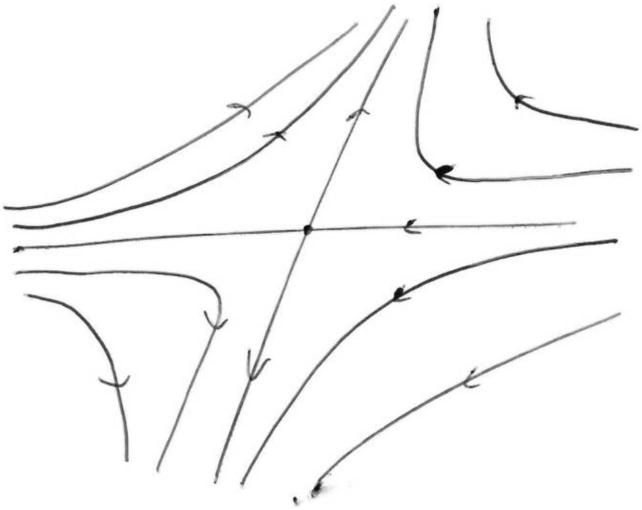
$\lambda_2 < \lambda_1 < 0$

node sink

stable node

* λ_1, λ_2 have opposite signs, e.g. $\lambda_1 < 0 < \lambda_2$.

Then O is a saddle point.



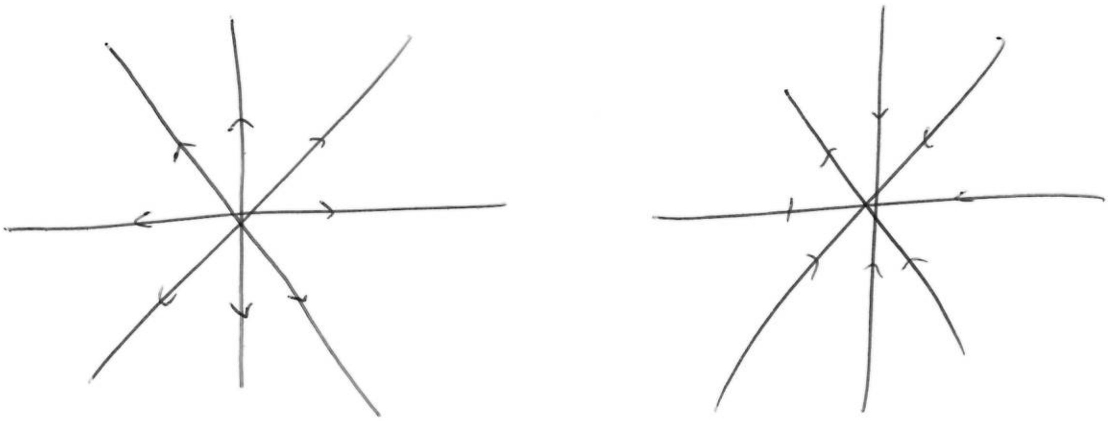
- $\lambda_1 = \lambda_2 = \lambda$.

* A is diagonalizable. Then A is diagonal and the integral curve are given by

$$\begin{cases} x(t) = x_0 e^{\lambda t} \\ y(t) = y_0 e^{\lambda t} \end{cases}$$

That is

$$\begin{cases} y = \alpha x & \alpha \in \mathbb{R} \\ x = 0 \end{cases}$$



$\lambda > 0$ $\qquad\qquad$ $\lambda < 0$

* A is not diagonalizable.

Then there exists a basis in which A and the system write

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \begin{cases} \dfrac{dx}{dt} = \lambda x \\ \dfrac{dy}{dt} = x + \lambda y \end{cases}$$

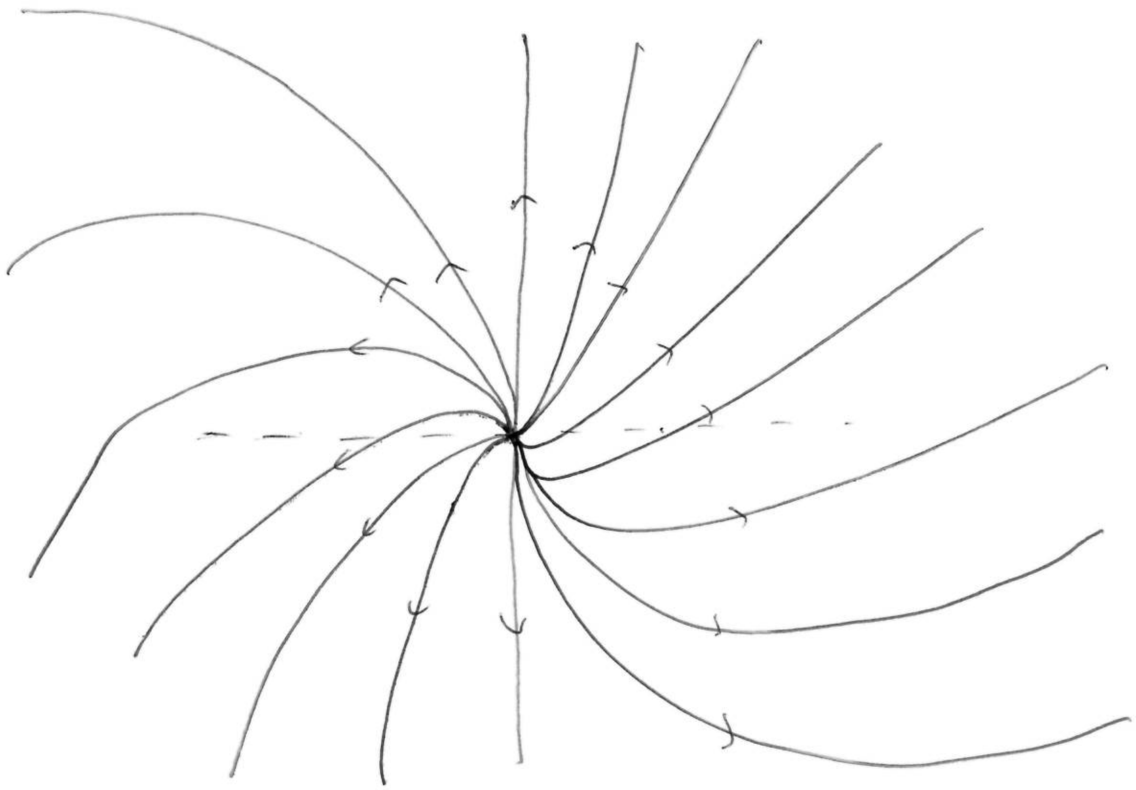
Integral curves are given by

$$\begin{cases} x(t) = x_0 e^{\lambda t} \\ y(t) = (y_0 + x_0 t) e^{\lambda t} \end{cases}$$

Since any integral curve with $x_0 \neq 0$ pass through a point s.t. $|x(t)| = 1$, we obtain all integral other than $x = 0$ by taking $x_0 = \pm 1$. Then

$$t = \frac{1}{\lambda} \ln |x|$$

and hence $$y = y_0 |x| + \frac{x}{\lambda} \ln |x|$$



$\lambda > 0$.

(b) <u>eigenvalues of A are complex non real</u>

~~The~~ In this case, we have conjugated eigenvalues $\alpha + i\beta$, $\alpha - i\beta$, with $\beta > 0$.

There exists a basis in which

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \qquad \begin{cases} \dfrac{dx}{dt} = \alpha x - \beta y \\[2mm] \dfrac{dy}{dt} = \beta x + \alpha y \end{cases}$$

To solve this system, we set $z = x + iy$.

We find

$$\frac{dz}{dt} = (\alpha + i\beta)\, x + (-\beta + \alpha i)\, y = (\alpha + i\beta)(x + iy)$$

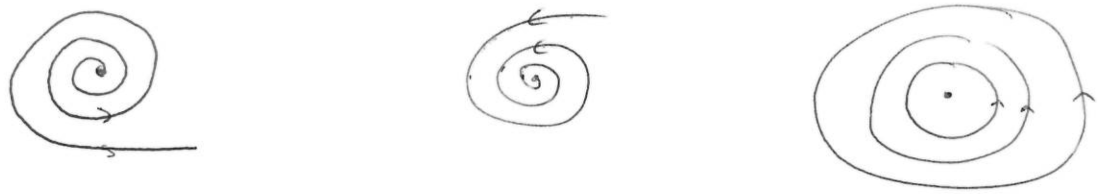
$$= (\alpha + i\beta) z$$

Hence, $z(t) = z_0 e^{(\alpha + i\beta)t} = z_0 e^{\alpha t} e^{i\beta t}$

In polar coordinate $z = re^{i\theta}$

$$\begin{cases} r = r_0 e^{\alpha t} \\ \theta = \theta_0 + \beta \end{cases} \Rightarrow \quad r = r_0 e^{\frac{\alpha}{\beta}(\theta - \theta_0)}$$

It is a logarithmic spiral if $\alpha \neq 0$ and a circle if $\alpha = 0$.



$\alpha > 0$ $\alpha < 0$ $\alpha = 0$

2.3) Singularity of nonlinear vector fields

Example 1:

$$(S) \quad \begin{cases} \dfrac{dx}{dt} = -y - x\,(x^2+y^2) \\[2mm] \dfrac{dy}{dt} = x - y\,(x^2+y^2) \end{cases}$$

The associated linear syst is

$$\begin{cases} \dfrac{dx}{dt} = -y \\[2mm] \dfrac{dy}{dt} = x \end{cases}$$

e.v. i and $-i$ $\Rightarrow \alpha = 0 \Rightarrow$ origin is a center.

In polar coordinates, (S) becomes

$$\begin{cases} r\dfrac{dr}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = -(x^2+y^2)^2 \\[2mm] \dfrac{d\theta}{dt} = \dfrac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2+y^2} = 1 \end{cases}$$

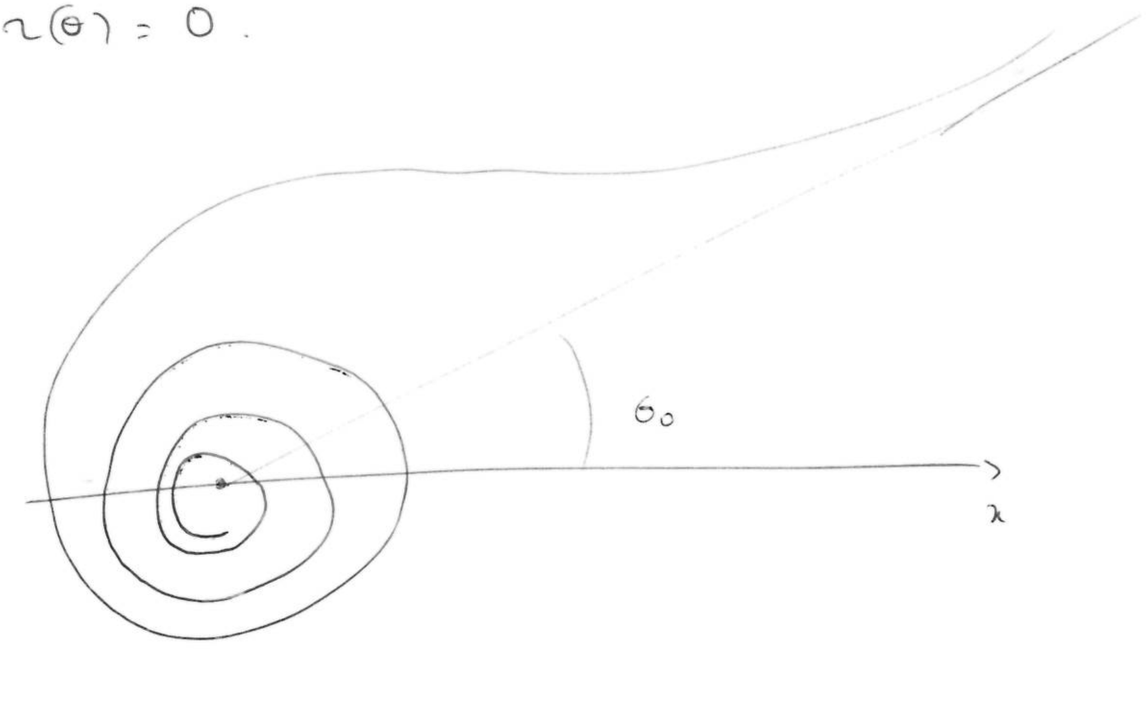
$$\Leftrightarrow \begin{cases} \dfrac{1}{r^3} \dfrac{dr}{dt} = 1 \\ \dfrac{d\theta}{dt} = 1 \end{cases} \rightsquigarrow \begin{cases} -\dfrac{1}{2r^2} = t + c_1 \\ \theta = t + \theta_0 \end{cases} \Leftrightarrow \begin{cases} r^2 = -\dfrac{1}{2(t+c)} \\ \theta = t + c_2 \end{cases}$$

$$\Leftrightarrow \frac{1}{2r^2} = \theta - \theta_0$$

$$\Rightarrow r = \left(2(\theta - \theta_0)\right)^{-\frac{1}{2}} \qquad \theta > \theta_0$$

$$\theta = t + c$$

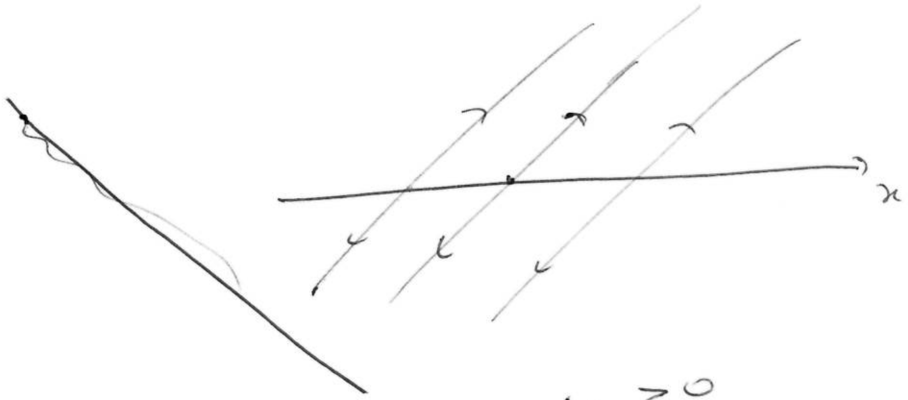
$$\lim_{\theta \to +\infty} r(\theta) = 0 .$$



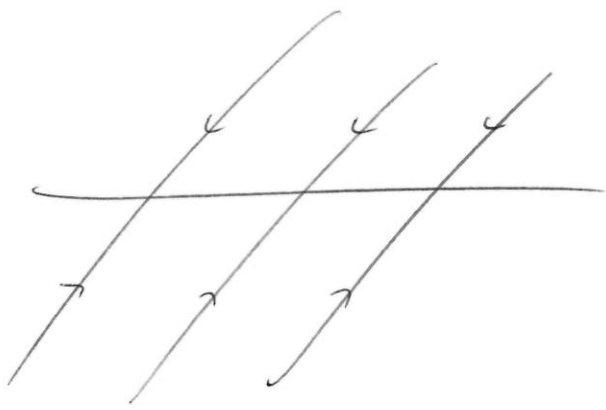
Suppose now $\det A = 0$. Then at least one of the e.v of A is 0, say $\lambda_1 = 0$

- assume $\lambda_2 \neq 0$, then the matrix reads, in some basis

$$\begin{cases} \dfrac{dx}{dt} = 0 \\[2ex] \dfrac{dy}{dt} = \lambda_2 y \end{cases} \Rightarrow \begin{cases} x = x_0 \\[2ex] y = y_0 e^{\lambda_2 t} \end{cases}$$



$\lambda_2 > 0$



$\lambda_2 < 0.$

- $\lambda_1 = \lambda_2 = 0$

$$\begin{cases} \dfrac{dx}{dt} = 0 \\[1ex] \dfrac{dy}{dt} = 0 \end{cases} \Leftrightarrow \begin{cases} x = x_0 \\[2ex] y = y_0 \end{cases}$$

Example:

Consider the syst

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and observe how the variations of α affects the trajectories

$\alpha < -2$ node sink

$-2 < \alpha < 0$ spiral sink

$\alpha = 0$ center

$0 < \alpha < 2$ spiral source

$\alpha > 0$ node source

$\alpha = \pm 2$ deg node.