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I Laplace equation

1) The Laplacian of a function

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in C^2(\Omega)$.
The Laplacian of u is defined by

$$\Delta u := \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2}$$

<u>Proposition 1</u> (elementary properties of Δ). Let $u \in C^2(\Omega)$

1) $\Delta u = \operatorname{div}(\nabla u)$

2) Polar coordinates : $\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta}$

3) For radial functions : $\Delta u = v'' + \frac{N-1}{r} v'$ if $u(x) = v(|x|)$.

4) $\mathbb{R}^2 \simeq \mathbb{C}$ $\partial_z = 1\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\partial_{\bar z} = 1\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. ~~Please~~ $\Delta = \partial_z \partial_{\bar z}$

<u>Proof</u> : exercise

<u>Exercise</u> : ~~Retour~~ Suppose $N=3$ and (r, θ, ϕ) are spherical coordinates.

prove that

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

2) Some calculus theorems

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^1 boundary.
We denote the unit outward normal vector at a point $x \in \partial\Omega$ by ν.
and $\frac{\partial u}{\partial \nu} := \nu . \nabla u$ is the normal derivative of $u \in C^1(\bar\Omega)$

Theorem: Let $u, v \in \mathcal{C}^2(\overline{\Omega})$. Then

1) $$\int_{\Omega} \operatorname{div}(u)\,dx = \int_{\partial\Omega} u.\nu\,d\sigma$$ (divergence theorem also called Gauss-Green, Gauss, Ostrogradsky, ...)

2) $$\int_{\Omega} \operatorname{div}(u) v\,dx = -\int_{\Omega} u\,\operatorname{div}(v)\,dx + \int_{\partial\Omega} uv.\nu\,d\sigma$$

3)
- $$\int_{\Omega} \Delta u\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,d\sigma$$
- $$\int_{\Omega} \nabla v \nabla u\,dx = -\int_{\Omega} u\Delta v + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u\,d\sigma$$
- $$\int_{\Omega} u\Delta v - v\Delta u\,dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\,d\sigma$$

Green's formulas

Exercise: Admit 1) and prove 2) and 3).

3) Harmonic functions

Def: A function $u \in \mathcal{C}^2(\Omega)$ is said to be harmonic in Ω if

$$\Delta u = 0 \quad \text{in } \Omega.$$

Rq: $\mathbb{R}^2 \simeq \mathbb{C}$. Every holomorphic function is harmonic.

a) The mean value property.

Theorem: Let $u \in \mathcal{C}^2(\Omega)$ be an harmonic function. Then for all ball $B = B_R(x) \subset\subset \Omega$ we have

$$u(x) = \frac{1}{N\omega_N R^{N-1}} \int_{\partial B} u\,d\sigma = \frac{1}{\omega_N R^N} \int_B u\,dx$$

(here, ω_N is the volume of the unit sphere $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(\frac{N}{2})}$)

Proof: Consider the function

$$\phi(r) := \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y)\, d\sigma$$

1) Remark that $\displaystyle \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y)\, d\sigma = \frac{1}{\omega_N} \int_{\partial B_1(0)} u(x + rz)\, d\sigma$

2) Compute the derivative of ϕ:

$$\phi'(r) = \frac{1}{\omega_N} \int_{\partial B_1(0)} \nabla u(x + rz) \cdot z \, d\sigma$$

$$= \frac{1}{\omega_N} \int_{\partial B_1(0)} \frac{\partial u}{\partial \nu}(x + rz)\, d\sigma$$

$$= \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}(y)\, d\sigma$$

Apply Green formula, we get

$$\phi'(r) = \frac{1}{N\omega_N r^{N-1}} \int_{B_r(x)} \Delta u(y)\, dy = 0$$

Hence ϕ is constant.

3) Prove that $\lim_{r \to 0} \phi(r) = u(x)$ and conclude for the first ~~eq~~ equality

4) Using that $\displaystyle \int_{B(x,r)} u\, dy = \int_0^r \left(\int_{\partial B(x,s)} u\, d\sigma \right) ds$

prove the second equality.

Theorem (converse to mean value property)

Let $u \in C^2(\Omega)$ satisfy $u(x) = \frac{1}{N \omega_N r^{N-1}} \int_{\partial B_r(x)} u(y)\, d\sigma$

for each ball $B(x,r) \subset \Omega$. Then u is harmonic

Proof: Choose r s.t. $\Delta u > 0$ in $B(x,r)$ and find a contradiction with ϕ as above.

b) Maximum principles

Theorem (strong maximum principle): Suppose that Ω is connected. Let $u \in C^2(\Omega)$ be harmonic and suppose that there exists $x \in \Omega$ s.t.

$$u(x) = \max_{y \in \Omega} u(y).$$

Then u is constant

Proof: Let r be such that $B_r(x) \subset \Omega$.

Then $u(x) = \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y)\,dy \leq u(x)$,

It is possible only if $u \equiv u(x)$ in $B_r(x)$.

Consider the set $\{ y \in \Omega ;\ u(y) = u(x) \}$.

The above argument show that it is ~~closed~~ open in Ω.

On the other hand, since u is C^0, it is also closed.

Hence it is Ω.

Theorem (weak maximum principle) Ω ~~connected~~ bounded

Let $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ be an harmonic function.

Then
$$\max_{y \in \bar\Omega} u(y) = \max_{y \in \partial\Omega} u(y).$$

Proof: Apply the strong max principle.

Rk: Similar results are available with max replaced by min.

Exercise: Let Ω be connected and bounded and $u \in C^2(\Omega) \cap C(\bar\Omega)$ be a solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Where $g \geq 0$ and there exists $x \in \partial\Omega$ s.t. $g(x) > 0$. Prove that $u > 0$.

c) Uniqueness

Theorem: Let $g \in C(\partial\Omega)$. There exists at most one solution $u \in C^2(\Omega) \cap C(\bar\Omega)$ of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

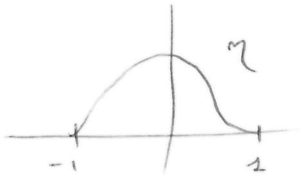
Proof: Supp that u, v satisfy the eq and apply the max principle to $u - v$.

d) Regularity.

Theorem: Let $u \in C(\Omega)$ satisfy the mean value property in Ω.
Then $u \in C^\infty(\Omega)$.

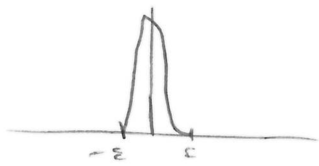
<u>Proof</u>: Consider a standard mollifier: define $\eta \in C^\infty(\mathbb{R}^N)$ by

$$\eta(x) := \begin{cases} C \exp\left(\dfrac{1}{|x|^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



with C s.t. $\displaystyle\int_{\mathbb{R}^N} \eta \, dx = 1$

For all $\varepsilon>0$, set $\eta_\varepsilon(x) := \dfrac{1}{\varepsilon^N}\,\eta\left(\dfrac{x}{\varepsilon}\right)$



Let $u_\varepsilon := \eta_\varepsilon * u$ in $\Omega_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$.

Since u_ε is a convolution product, $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$.

* <u>Prove that $u \equiv u_\varepsilon^\varepsilon$ on Ω_ε</u>

Let $x \in \Omega_\varepsilon$.

$$u_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)\,u(y)\,dy = \frac{1}{\varepsilon^N}\int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\,dy$$

$$= \frac{1}{\varepsilon^N}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\left(\int_{\partial B(x,r)} u\,d\sigma\right) dr$$

$$= \frac{1}{\varepsilon^N}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) N\omega_N\, r^{N-1}\, dr$$

$$= \frac{u(x)}{\varepsilon^N}\int_{B(0,\varepsilon)} \eta_\varepsilon\, dy = u(x)$$

<u>Rk</u>: Deduce from the last th. an improvement in the Th on fct° + mean v. $\Rightarrow$ harmonicity

e) estimates on derivatives

Theorem: Assume u is harmonic in Ω. Then

$$| D^\alpha u(x) | \leq \frac{C_k}{r^{N+k}} \|u\|_{L^1(B(x,r))}$$

for each ball $B(x,r) \subset \Omega$ and each multiindex α of length $|\alpha| = k$.

(Moreover, the constants C_k are explicitely known:

$$C_0 = \frac{1}{\omega_N}, \quad C_k = \frac{(2^{N+1} N k)^k}{\omega_N} \quad)$$

Proof: By induction.

- $k=0$ ok by mean value formula
- $k=1$. Recall $u \in C^\infty(\Omega)$, so we can differentiate $\Delta u = 0$ to get

$$\Delta \frac{\partial u}{\partial x_j} = 0$$

Hence $\frac{\partial u}{\partial x_j}$ is harmonic and verifies

$$\left| \frac{\partial u}{\partial x_j}(x) \right| = \left| \frac{1}{(\frac{r}{2})^N \omega_N} \int_{B(x, \frac{r}{2})} \frac{\partial u}{\partial x_j} \, dy \right|$$

$$= \left| \left(\frac{2}{r}\right)^N \frac{1}{\omega_N} \int_{\partial B(x, \frac{r}{2})} u \nu_i \, d\sigma \right| \leq \frac{2N}{r} \|u\|_{L^\infty(\partial B(x, \frac{r}{2}))}$$

For any $y \in \partial B(x, \frac{r}{2})$, we have $B(y, \frac{r}{2}) \subset B(x, r)$ and so

$$|u(y)| \leq \frac{1}{\omega_N} \left(\frac{2}{r}\right)^N \|u\|_{L^1(B(x_0, r))}$$

- for $k \geq 1$ exercise

f) Liouville's theorem

Theorem: Suppose $u : \mathbb{R}^N \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof: Fix $x \in \mathbb{R}^N$, $r > 0$. Then

$$|\nabla u(x)| \leq \frac{C}{r^{N+1}} \|u\|_{L^1(B(x,r))} \leq \frac{C}{r} \|u\|_{L^\infty(\mathbb{R}^N)} \xrightarrow[r \to \infty]{} 0$$

Hence $\nabla u \equiv 0$ and u is constant.

g) Analyticity.

Theorem: Assume u is harmonic in Ω. Then u is analytic in Ω.

Proof: admitted (see Evans).

h) Harnack's inequality

Theorem: For each connected open set $V \subset\subset \Omega$, there exists C depending only on V, such that for all non-negative harmonic functions u in Ω we have

$$\sup_V u \leq C \inf_V u.$$

Proof: Let $r := \frac{1}{4} \operatorname{dist}(V, \partial\Omega)$. Let $x_0 \in V$ and $x, y \in B(x_0, r)$ ~~s.t. $|x-y| \leq r$~~. Then

$$u(x) = \frac{1}{\omega_N r^N} \int_{B(x,r)} u \, dz \leq \frac{1}{\omega_N r^N} \int_{B(x_0, 2r)} u \, dz = 2^N u(x_0)$$

$$\cancel{u(y) = \frac{1}{\omega_N (3r)^N} \int_{B(y,3r)} u \, dz \geq \frac{1}{}}$$

and similarly

$$\cancel{u(y) \leq 2^N u(x)}$$

$$u(y) = \frac{1}{\omega_N (2r)^N} \int_{B(y,2r)} u \, dz \geq \frac{1}{\omega_N (2r)^N} \int_{B(x_0, r)} u \, dz = 2^N u(x_0).$$

Covering V with a finite number of ball gives the result.

4) The fundamental solution of the Laplacian

Aim: find an "elementary" explicit solution the ~~Laplace~~ Laplace equation in $\mathbb{R}^N$

We look for a solution to

$$\Delta u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

Idea: we ~~reduce~~ restrict ourselves to the research of a radial function, i.e. to some $v: \mathbb{R}^+ \to \mathbb{R}$ s.t. $u(x) = v(|x|)$.

Recall: $\Delta u = v'' + \frac{N-1}{r} v'$

Hence we search v s.t.

$$v'' + \frac{N-1}{r} v' = 0$$

Multiplying by r^{N-1}, we get

$$r^{N-1} v'' + (N-1) r^{N-2} v' = 0$$

Therefore, $\left(r^{N-1} v'\right)' = 0$

Thus $r^{N-1} v' = C_1$

i.e. $v' = \frac{C_1}{r^{N-1}}$ for $r \neq 0$.

If $N \geq 3$, we obtain $v(r) = \dfrac{c_1}{r^{N-2}} + c_2$

If $N = 2$, we get $v(r) = c_1 \log(r) + c_2$

For reasons that will appear later, we fix $c_2 = 0$ and $c_1 = -\frac{1}{2\pi}$ if $N=2$, $\dfrac{1}{N(N-2)\,\omega_N}$ if $N \geq 3$.

Definition: The function

$$\Phi : \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}$$

$$x \longmapsto \Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } N = 2 \\ \dfrac{1}{N(N-2)\omega_N} \dfrac{1}{|x|^{N-2}} & \text{if } N \geq 3 \end{cases}$$

is called the fundamental solution of the Laplacian.

II Poisson's equation

1) Poisson's equation in $\mathbb{R}^N$

Theorem: Let $f \in C_c^2(\mathbb{R}^N)$. Then the function

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^N} \Phi(y) f(x-y)\, dy$$

is $C^2(\mathbb{R}^N)$ and satisfies the Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^N$$

Proof: admitted (see the course).

Representation formula for bounded solutions.

Theorem: Let $N \geq 3$ and $f \in C^2_c(\mathbb{R}^N)$. ~~Then~~ If u is a bounded solution of

$$-\Delta u = f \quad \text{in } \mathbb{R}^N$$

then u is of the form

$$u = \Phi * f + C$$

for some constant $C \in \mathbb{R}$.

Proof: for $N \geq 3$, $\Phi(x) \to 0$ as $|x| \to +\infty$, hence $\Phi * f$ is a bounded solution. Let u be another bounded solution. Then $u - \Phi * f$ verifies

$$\Delta u = 0 \quad \text{in } \mathbb{R}^N$$

and $u - \Phi * f$ is bounded. From Liouville theorem, we conclude that

$$u - \Phi * f \equiv C.$$

2) Green's functions:

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded ~~set~~ with a C^1 boundary.
We are looking for solution to the equation

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad \text{where } f \in C(\Omega) \text{ and } g \in C(\partial\Omega).$$

a) derivation of Green's functions

Let $u \in C^2(\bar{\Omega})$ be ~~a~~ solution of (P). Fix $x \in \Omega$, $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subset \Omega$ and apply Green's formula ~~on~~ the region $V_\varepsilon := \Omega - B(x,\varepsilon)$ ~~and~~ to $u(y)$ and $\Phi(y-x)$

$$\int_{V_\varepsilon} u(y)\, \Delta \Phi(y-x) - \Phi(y-x)\, \Delta u(y)\, dy$$

$$= \int_{\partial V_\varepsilon} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, d\sigma \qquad (*)$$

Observe that:

- since Φ is harmonic in $\mathbb{R}^N \setminus \{0\}$, we have $\Delta \Phi(x-y) = 0$ if $x \neq y$.

- $$\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, d\sigma = -\Phi'(\varepsilon) \int_{\partial B(x,\varepsilon)} u(y)\, d\sigma$$

$$= \frac{-1}{N\omega_N \varepsilon^{N-1}} \int_{\partial B(x,\varepsilon)} u(y)\, d\sigma \longrightarrow -u(x) \text{ as } \varepsilon \to 0.$$

- $$\int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, d\sigma = \Phi(\varepsilon) \int_{\partial B(x,\varepsilon)} \frac{\partial u}{\partial \nu}(y)\, d\sigma$$

$$\leq \Phi(\varepsilon)\, N\omega_N \varepsilon^{N-1} \, \|\nabla u\|_{L^\infty(\partial B(x,\varepsilon))} \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

Sending $\varepsilon \to 0$ in $(*)$, we get

$$\int_{\partial \Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, d\sigma - u(x)$$

$$= -\int_{\Omega} \Phi(y-x)\, \Delta u(y)\, dy$$

And therefore

$$u(x) = \int_\Omega \Phi(y-x)\,\Delta u(y)\,dy + \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,d\sigma . \quad (**)$$

This would give us a good representation formula for u, except that we do not know the value of $\frac{\partial u}{\partial \nu}$ along $\partial\Omega$.

To overcome this difficulty, we introduce the function $\phi(x,y)$ that for fixed x solves

$$\begin{cases} +\Delta \phi(x,\cdot) = 0 & \text{in } \Omega \\ \phi(x,\cdot) = \Phi(\cdot - x) & \text{on } \partial\Omega \end{cases}$$

Applying again Green's formula, we get

$$\int_\Omega \underbrace{u(y)\Delta\phi(x,y)}_{\overset{\shortparallel}{0}} - \phi(x,y)\,\Delta u(y)\,dy = \int_{\partial\Omega} u(y)\frac{\partial\phi(x,y)}{\partial\nu}\,d\sigma - \int_{\partial\Omega}\phi(x,y)\frac{\partial u}{\partial\nu}(y)\,d\sigma$$

Hence

$$(***) \quad -\int_\Omega \phi(x,y)\,\Delta u(y)\,dy = \int_{\partial\Omega} u(y)\frac{\partial\phi(x,y)}{\partial\nu}\,d\sigma - \int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial\nu}(y)\,d\sigma$$

Adding $(**)$ and $(***)$, we get

$$u(x) = \int_\Omega \left(\Phi(y-x) - \phi(x,y)\right)\Delta u(y)\,dy + \int_{\partial\Omega} u(y)\left(\frac{\partial\Phi}{\partial\nu}(y-x) - \frac{\partial\phi(x,y)}{\partial\nu}\right)d\sigma$$

Definition: The Green's function for the domain Ω is

$$G(x,y) = \Phi(y-x) - \phi(x,y) \qquad (x,y \in \Omega,\ x \neq y).$$

Theorem: If $u \in C^2(\bar{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Then

$$u(x) = \int_{\Omega} f(y)\, G(x,y)\, dy + \int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}(x,y)\, d\sigma.$$

Rk: G is the solution of

$$\begin{cases} -\Delta G = \delta_x & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases}$$

Exercise: Prove the symmetry of the Green function:

$$\forall x,y \in \Omega, \quad G(x,y) = G(y,x)$$

b) Green's function for the half-space

$\mathbb{R}^N_+ := \{ (x_1, \dots, x_N) \in \mathbb{R}^N ; x_N > 0 \}$

Def: For $x = (x_1, \dots, x_N) \in \mathbb{R}^N_+$, it reflection according to the plane $\partial \mathbb{R}^N_+$ is $\tilde{x} = (x_1, \dots, x_{N-1}, -x_N)$.

Fix $x \in \mathbb{R}^N$

We look for $\phi(x,y)$ s.t.

$$(1) \begin{cases} \Delta \phi(x,y) = 0 & \text{in } \mathbb{R}^N_+ \\ \phi(x,y) = \Phi(y-x) & \text{on } \partial \mathbb{R}^N_+ \end{cases}$$

$\Phi(y-x)$ would be a good candidate to solve this equation, except that there is a singularity at $y = x$. To avoid that, we reflect the singularity outside $\mathbb{R}^N_+$:

Let $\phi(x,y) := \Phi(y - \tilde{x})$.

Then ϕ satisfies (1).

Def: The Green function for the half space $\mathbb{R}^N_+$ is

$$G(x,y) := \Phi(y-x) - \Phi(y-\tilde{x})$$

Exercise: Compute $\frac{\partial G}{\partial \nu}(x,y)$.

$$\frac{\partial G}{\partial y_N}(x,y) = \frac{-1}{N \omega_N} \left[\frac{y_N - x_N}{|y-x|^N} - \frac{y_N + x_N}{|y-\tilde{x}|^N} \right]$$

$$\Rightarrow \frac{\partial G}{\partial \nu}(x,y) = -\frac{\partial G}{\partial y_N}(x,y) = \frac{-2x_N}{N\omega_N |x-y|^N}$$

We want now to solve the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+ \\ u = g & \text{on } \partial \mathbb{R}^N_+ \end{cases}$$

We ~~except~~ expect that the solution will be

$$(2) \quad u(x) := \frac{2 x_N}{N \omega_N} \int_{\partial \mathbb{R}^N_+} \frac{g(y)}{|x-y|^N} \, dy$$

Vocabulary: the function $K(x,y) := \dfrac{2 x_N}{N \omega_N} \dfrac{1}{|x-y|^N}$

is called the Poisson kernel and the above formula the Poisson formula

Thm: Assume $g \in \mathcal{C}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ and let u be defined by (2). Then

(i) $u \in \mathcal{C}^\infty(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$

(ii) $\Delta u = 0$ in $\mathbb{R}^N_+$

and (iii) $\displaystyle\lim_{\substack{y \to x \\ y \in \mathbb{R}^N_+}} u(y) = g(x) \quad \forall x \in \partial \mathbb{R}^N_+$.

Proof: 1) For fixed x, the function $y \mapsto G(x,y)$ is harmonic if $y \neq x$ and for fixed y, ——— $x \mapsto G(x,y)$ ——— if $x \neq y$.

Thus $x \mapsto -\dfrac{\partial G}{\partial y_N}(x,y) = K(x,y)$ is harmonic for $x \in \mathbb{R}^N_+$, $y \in \partial \mathbb{R}^N_+$

- We have $\displaystyle\int_{\partial \mathbb{R}^N_+} K(x,y)\,dy = 1 \quad \forall x \in \mathbb{R}^N_+.$

Since g is bounded, this implies u bd.

Since $x \mapsto K(x,y)$ is smooth, u is smooth

and $\displaystyle \Delta u(x) = \int_{\partial \mathbb{R}^N_+} \Delta_x K(x,y)\, g(y)\,dy = 0 \quad \forall x \in \mathbb{R}^N_+$

- Fix $x_0 \in \partial \mathbb{R}^N_+$, $\varepsilon > 0$, let $\delta > 0$ s.t.

$$|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \varepsilon \quad \forall y \in \partial \mathbb{R}^N_+.$$

If $|x - x_0| < \frac{\delta}{2}$ $(x \in \mathbb{R}^N_+)$.

$$|u(x) - g(x_0)| = \left| \int_{\partial \mathbb{R}^N_+} K(x,y)\left[g(y) - g(x_0)\right] dy \right|$$

$$\leq \underbrace{\int_{\partial \mathbb{R}^N_+ \cap B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\,dy}_{I} + \underbrace{\int_{\partial \mathbb{R}^N_+ \setminus B(x_0,\delta)}}_{J}$$

Clear that $I \leq \varepsilon$.

For J, rq that if $|x - x_0| < \frac{\delta}{2}$ and $|y - x_0| \geq \delta$ then

$$|y - x_0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x_0|$$

$$\Rightarrow \quad |y - x| \geq \frac{1}{2}|y - x_0|$$

Thus

$$J \leq 2\|g\|_{L^\infty} \int_{\partial \mathbb{R}^N_+ \setminus B(x_0,\delta)} K(x,y)\, dy.$$

$$\leq \frac{2^{N+2}\|g\|_\infty}{N\omega_N} x_N \int_{\_\_} |y-x_0|^{-N}\, dy \longrightarrow 0 \quad \text{as } x_N \to 0$$

c) Green's function for a ball

We define Green function on the unit ball $B(0,1)$ of $\mathbb{R}^N$.

Def: If $x \in \mathbb{R}^N \setminus \{0\}$, $\tilde{x} = \frac{x}{|x|^2}$ is the dual point of x.

We look for

$$\begin{cases} \Delta \phi(x,y) = 0 \\ \phi(x,y) = \Phi(y-x) \quad \text{on } \partial B(0,1) \end{cases}$$

We define $\phi(x,y) := \Phi\left(|x|\,(y-\tilde{x})\right)$.

Exercise: • Check that ϕ is the good fct°.

• Define the Green function G.

• Check that $\frac{\partial G}{\partial \nu}(x,y) = \frac{-(1-|x|^2)}{N\omega_N |x-y|^N}$

3) Dirichlet's principle

Ω open, bounded with $\mathcal{C}^1$ boundary.

$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

We define the energy functional I by

$$I(v) := \int_\Omega \frac{1}{2} |\nabla v|^2 - f v \, dx$$

for any v belonging to the admissible set

$$\mathcal{A} := \left\{ v \in \mathcal{C}^2(\bar{\Omega}) \,;\; v = g \text{ on } \partial\Omega \right\}$$

Thm: ~~Assume that $u \in \mathcal{C}^2(\bar{\Omega})$ solves (P)~~.

~~Th~~ Let $u \in \mathcal{A}$. The following assertions are equivalent:

(i) $I(u) = \min_{v \in \mathcal{A}} I(v)$,

(ii) u solves (P).

~~proof~~: • (i) $\Rightarrow$ (ii).

fix any $v \in \mathcal{C}_c^\infty(\Omega)$ and write

$$i(t) := I(u + tv) \quad \text{for } t \in \mathbb{R}.$$

Rk that $i(t)$ is well defined since $u + tv \in \mathcal{A}$ for any $t \in \mathbb{R}$.

By (i), the function i has a minimum at 0, and thus

$$i'(0) = 0.$$

Note that

$$i(t) = \int_\Omega \frac{1}{2}|\nabla u + t\nabla v|^2 - (u+tv) f \, dx$$

$$= \int_\Omega \frac{1}{2}|\nabla u|^2 + t\nabla u\nabla v + \frac{t^2}{2}|\nabla v|^2 - uf - tvf \, dx.$$

Thus i is differentiable and

$$i'(t) = \int_\Omega \nabla u \nabla v + t|\nabla v|^2 - vf \, dx$$

In particular

$$0 = i'(0) = \int_\Omega \nabla u\nabla v - vf \, dx.$$

Integrating by part, we get

$$\int_\Omega \left(-\Delta u - f\right) v \, dx = 0.$$

This is valid for any $v \in \mathcal{C}_c^\infty(\Omega)$, thus $-\Delta u = f$ in Ω and u verifies (ii).

- (ii) $\Rightarrow$ (i).

Let $v \in A$. Since u verifies (P), we have

$$0 = \int_\Omega (-\Delta u - f)(u-v) \, dx$$

By parts:

$$0 = \int_\Omega \nabla u \nabla(u-v) - f(u-v) \, dx$$

No boundary term since $u - v = 0$ on $\partial\Omega$.

Hence

$$\int_\Omega |\nabla u|^2 - uf = \int_\Omega \nabla u \nabla v - vf \, dx$$

$$\leq \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 - vf \, dx$$

Thus $I(u) \leq I(v)$.

4) A variational principle for the first eigenvalue of Laplace operator.

Let Ω be open and bounded.

We consider the unbounded operator $-\Delta : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ with domain $H_0^1(\Omega)$.

Thm: ~~The eigenvalues of $-\Delta$~~

The spectrum of $-\Delta$ is a sequence of real, positive eigenvalues $\{\lambda_k\}$ such that $\lambda_k \to +\infty$ (if the eigenvalues are ordered by increasing values), and there exist an orthonormal basis $\{e_k\}$ of $L^2(\Omega)$ ~~constat~~ formed by eigenvectors of $-\Delta$. Here $e_k \in H_0^1(\Omega) \cap C^\infty(\Omega)$ for each k.

Def: λ_1 is called the principal or first e.v. of $-\Delta$.

Thm: (i) $\lambda_1 = \min \left\{ \int |\nabla u|^2 dx ;\ u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} = 1 \right\}$

(ii) λ_1 is simple and e_1 can be chosen to be positive

Proof: • Remark first that

$$\int_\Omega |\nabla e_k|^2 = -\int_\Omega (\Delta e_k)\, e_k = \lambda_k \|e_k\|^2_{L^2(\Omega)} = \lambda_k, \qquad (1)$$

and

$$\int_\Omega \nabla e_k \nabla e_\ell = \lambda_k \int e_k e_\ell = 0 \text{ if } k \neq \ell. \qquad (2)$$

• Take $u \in H^1_0(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$.

Since $\{e_k\}$ is an orthonormal basis of $L^2(\Omega)$, we have

$$u = \sum_{k=1}^{\infty} \alpha_k e_k \quad \text{for } \alpha_k = (u, e_k)_2$$

and $\|u\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \alpha_k^2 = 1$

• Let $w_k := \frac{e_k}{\sqrt{\lambda_k}}$. We claim that $\{w_k\}$ is an orthonormal basis of $H^1_0(\Omega)$

for the scalar product $\int_\Omega \nabla . \nabla .$

From (1) and (2), it is clearly an orthonormal subset.

Thus, it is enough to prove that

If $\int \nabla e_k \nabla u = 0 \quad \forall k$, then $u \equiv 0$

Since $\int \nabla e_k \nabla u = \lambda_k \int e_k u$, it is clearly the case.

• We have thus

$$\int_\Omega |\nabla u|^2 = \sum_{k=1}^{\infty} \int \nabla e_k \nabla u = \sum_{k=1}^{\infty} \lambda_k \alpha_k^2 \geq \lambda_1 \sum_{k=1}^{\infty} \alpha_k^2 = \lambda_1$$

since $\int_\Omega |\nabla e_1|^2 = \lambda_1$, (i) is proved.

- Suppose now that $\|u\|_2 = 1$, $u \in H^1_0(\Omega)$ and $\int |\nabla u|^2 = \lambda_1$

We have

$$\int |\nabla u|^2 = \sum_{k=1}^{\infty} \lambda_k \alpha_k^2 = \lambda_1 = \sum_{k=1}^{\infty} \lambda_1 \alpha_k^2$$

Thus $\sum_{k=1}^{\infty} (\lambda_k - \lambda_1)\, \alpha_k^2 = 0$

Wich implies $\alpha_k = 0$ if $\lambda_k > \lambda_1$, and ~~then~~ therefore ~~(u, e_k)~~

$$u = \sum_{k=1}^{m} \alpha_k e_k \quad \text{with } -\Delta e_k = \lambda_1 e_k \text{ for } k = 1, \dots, m$$

which implies $-\Delta u = \lambda_1 u$.

- Recall that $|u| \in H^1_0(\Omega)$. ~~Moreover~~, and $\int |\nabla u|^2 \geq \int |\nabla |u||^2$.

Then $\int |\nabla |u||^2 = \lambda_1$ and $\begin{cases} -\Delta |u| = \lambda_1 |u| & \text{in } \Omega \\ |u| = 0 & \text{on } \partial\Omega \end{cases}$

Thus, by the maximum principle, $|u| > 0$ in Ω.

So $u > 0$ or $u < 0$ in Ω.

- Finally, given two solutions u, v of $\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

we have, up to a sign change,

$$\int u > 0, \quad \int v > 0 \quad \text{and there exist } C \text{ s.t.}$$

$$\int_\Omega u - Cv = 0$$

$\Rightarrow$ $u = Cv$ in Ω, hence λ_1 is simple.

III Wave equation

1) In one ~~dim~~ space dimension.

$$(W)\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = g,\quad u_t = h & \text{on } \mathbb{R}\times\{t=0\} \end{cases}$$

We want a representation formula for u in terms of g and h.

Rk: factorisation of (W):

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx}$$

Write $v = u_t - u_x$.

Then $v_t + v_x = 0$.

This is a 1-D transport equation whose solutions ~~are~~ given by

$$v(x,t) = a(x-t) \quad \text{with} \quad a(x) = v(x,0).$$

Therefore,

$$u_t - u_x = a(x-t).$$

This is a non homogeneous transport eq whose solution is given by

$$u(x,t) = \int_0^t a\big(x + (t-s) - s\big)\,ds + b(x+t)$$

$$= \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy + b(x+t)$$

whith $b(x) = u(x,0)$.

It remains to compute a and b.

First, it's clear that $b(x) = u(x,0) = g(x)$.

Also, $a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$.

Hence

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y)\,dy + g(x+t)$$

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \qquad (d'A)$$

This is d'Alembert formula.

Thm: Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and assume u is given by $(d'A)$. Then $u \in C^2(\mathbb{R} \times [0,\infty))$ and verifies (W).

Exercise:

(a) Show that the general solution of $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y).$$

for any F and G

(b) Using $\xi = x+t$, $\eta = x-t$, show $u_{tt} - u_{xx} = 0 \iff u_{\xi\eta} = 0$

(c) Rederive d'Alembert formula

On the half line

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g \ , \ u_t = h & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on } \{x=0\} \times (0,\infty) \end{cases}$$

$\boxed{*}$ $g(0) = h(0) = 0$.

We adapt (d'A) by odd reflection

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x \leq 0 \end{cases}$$

$$\tilde{g}(x) := \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases}$$

$$\tilde{h}(x) := \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x \leq 0 \end{cases}$$

Then
$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g} \ , \ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

Then
$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy$$

Thus

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & x \geq t \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & x \leq t. \end{cases}$$

2) Spherical means

Let $N \geq 2$, $k \geq 2$ and $u \in C^k(\mathbb{R}^N \times [0,\infty))$ solves

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^N \times \{t=0\} \end{cases}$$

Def for $x \in \mathbb{R}^N$, $t > 0$, $r > 0$.

Define $U(x,r,t) := \dfrac{1}{\mathrm{vol}(\partial B(x,r))} \displaystyle\int_{\partial B(x,r)} u(y,t)\, d\sigma$

$$G(x,r) := \fint_{\partial B(x,r)} g(y)\, d\sigma$$

$$H(x,r) := \fint_{\partial B(x,r)} h(y)\, d\sigma$$

Lemma: Fix $x \in \mathbb{R}^N$. Then $U \in C^k(\overline{\mathbb{R}_+} \times [0,\infty))$ and

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{N-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G, \quad U_t = H & \text{on } \mathbb{R}_+ \times \{0\} \end{cases}$$

proof: (Ex).

1) $U_r(x,r,t) = \dfrac{r}{N} \displaystyle\fint_{B(x,r)} \Delta u(y,t)\, dy$

$$U_{rr}(x,r,t) = \fint_{\partial B(x,r)} \Delta u\, d\sigma + \left(\frac{1}{N} - 1\right) \fint_{B(x,r)} \Delta u\, dy$$

$\ldots \Rightarrow U \in C^k$

2) $$⨍ U_r = \frac{r}{N} ⨍_{B(x,r)} u_{tt}\, dy$$

$$= \frac{1}{N\omega_N r^{N-1}} \int_{B(x,r)} u_{tt}\, dy$$

$$\Rightarrow \quad r^{N-1} U_r = \frac{1}{N\omega_N} \int_{B(x,r)} u_{tt}\, dy$$

$$\left(r^{N-1} U_r\right)_r = \frac{1}{N\omega_N} \int_{\partial B(x,r)} u_{tt}\, d\sigma$$

$$= r^{N-1} ⨍_{\partial B(x,r)} u_{tt}\, d\sigma = r^{N-1} U_{tt}.$$

3) Solution for $N=3$

Let $\tilde{U} := rU$, $\tilde{G} = rG$, $\tilde{H} := rH$

Then
$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{on } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G}, \ \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{0\} \\ \tilde{U} = 0 & \text{on } \{0\} \times (0,\infty). \end{cases}$$

From the result on the half line, we get

$$\tilde{U}(x,r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\, dy \quad \text{for } r \le t.$$

On the other hand, since

$$u(x,t) = \lim_{r \to 0} U(x,r,t) \quad , \text{ we get}$$

$$u(x,t) = \lim_{r\to 0} \frac{\tilde{U}}{r}$$

$$= \lim_{r\to 0} \frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

Thus

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g\,d\sigma \right) + t \int_{\partial B(x,t)} h\,d\sigma$$

We have $\displaystyle \fint_{\partial B(x,t)} g(y)\,d\sigma = \fint_{\partial B(0,1)} g(x+tz)\,d\sigma$

Thus $\displaystyle \frac{\partial}{\partial t}\left(\fint_{\partial B(x,t)} g(y)\,d\sigma \right) = \fint_{\partial B(0,1)} \nabla g(x+tz) z\,d\sigma$

$$= \fint_{\partial B(x,t)} \nabla g(y) \cdot \frac{y-x}{t}\,d\sigma$$

Therefore,

$$u(x,t) = \fint_{\partial B(x,t)} t\, h(y) + g(y) + \nabla g(y) \cdot (y-x)\, d\sigma$$

Kirchoff formula.

2) Solution for $N=2$

Look for

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^2 \times \{0 \end{cases}$$

Idea:

$\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$

$\bar{g}, \bar{h}$ ...

Implies $\bar{u}_{tt} - \Delta \bar{u} = 0$ in $\mathbb{R}^3 \times (0,\infty)$

$\bar{u} = \bar{g}, \quad \bar{u}_t = \bar{h}$ on $\mathbb{R}^3 \times \{0\}$

then $u(x,t) = \bar{u}(x,0,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B((x,0),t)} \bar{g}\, d\sigma \right) + t \fint_{\partial B((x,0),t)} \bar{h}\, d\sigma.$

We remark that

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{\sigma} = \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{\sigma}$$

$$= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y) \left(1 + |D\gamma(y)|^2\right)^{\frac{1}{2}} dy$$

for $\gamma(y) = \left(t^2 - |y-x|^2\right)^{\frac{1}{2}}$.

Note that $\left(1 + |D\gamma|^2\right)^{\frac{1}{2}} = t\left(t^2 - |y-x|^2\right)^{-\frac{1}{2}}$

Thus

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{\sigma} = \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y)}{\left(t^2 - |y-x|^2\right)^{\frac{1}{2}}} dy$$

$$= \frac{t}{2} \fint_{B(x,t)} \frac{g(y)}{\left(t^2 - |y-x|^2\right)^{\frac{1}{2}}} dy$$

Now,

$$t^2 \fint_{B(x,t)} \frac{g(y)}{\left(t^2 - |y-x|^2\right)^{1/2}} dy = t \fint_{B(0,1)} \frac{g(x+tz)}{\left(1 - |z|^2\right)^{\frac{1}{2}}} dz$$

Thus

$$\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{\left(t^2 - |y-x|^2\right)^{\frac{1}{2}}} dy\right)$$

$$= \fint_{B(0,1)} \frac{g(x+tz)}{\left(1 - |z|^2\right)^{\frac{1}{2}}} dz + t \fint_{B(0,1)} \frac{\nabla g(x+tz)\cdot z}{\left(1 - |z|^2\right)^{1/2}} dz$$

$$= t \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy + t \fint_{B(x,t)} \frac{\nabla g(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy$$

Consequently, we can rewrite the formula to get

$$u(x,t) = \frac{1}{2} \fint_{B(x,t)} \frac{t\,g(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy$$

5) <u>The nonhomogeneous problem</u>

We look for solutions of

$$(NH)\begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^N \times (0,\infty) \\ u=0, \quad u_t = 0 & \text{on } \mathbb{R}^N \times \{0\} \end{cases}$$

We introduce the auxiliary problem

$$\begin{cases} u_{tt}(\cdot\,;s) - \Delta u(\cdot\,;s) = 0 & \text{in } \mathbb{R}^N \times (s,\infty) \\ u(\cdot,s) = 0, \quad u_t(\cdot\,;s) = f(\cdot,s) & \text{on } \mathbb{R}^N \times \{t=s\} \end{cases}$$

And we define

$$u(x,t) := \int_0^t u(x,t;s)\,ds \qquad x\in\mathbb{R}^N,\ t\geq 0 \qquad (1)$$

<u>Thm</u>: Assume $n=1,2,3$ and $f\in C^3(\mathbb{R}^N\times[0,\infty))$. Define u by (1)

Then (i) $u\in C^2(\mathbb{R}^N\times[0,\infty))$

(ii) u solves (NH).

<u>Proof</u>: (i) Clear

(ii) $u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds$

$$u_{tt} = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds$$

Moreover,

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)\,ds = \int_0^t u_{tt}(x,t;s)\,ds$$

Thus $\quad u_{tt}(x,t) - \Delta u(x,t) = f(x,t).$

<u>Exercise</u>: Write ~~d'Alembert~~ Duhamel formula for $N=1$ and $N=3$

- $N=1$: $\quad u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dy\,ds \qquad (x\in\mathbb{R}, t\geq 0.$

- $N=3$: $\quad u(x,t) = \int_0^t (t-s) ⨍_{\partial B(x,t-s)} f(y,s)\,d\sigma\,ds$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{(t-s)}\,d\sigma\,ds$$

$$= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,d\sigma\,dr$$

$$= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy.$$

6) Uniqueness

For $\Omega \subset \mathbb{R}^N$, bounded open with smooth boundary $\partial\Omega$, we set

$\Omega_T = \Omega \times (0,T)$, $\Gamma_T = \overline{\Omega_T} - \Omega_T$ for $T>0$

We are interested in

$$(2) \begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } \Omega \times (t=0) \end{cases}$$

Thm: There exists at most one function $u \in C^2(\overline{\Omega_T})$ solving (2).

Proof: Let u and $\tilde{u}$ be two solutions of (2) and set $v = u - \tilde{u}$.
Then v solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \Omega_T \\ v = 0 & \text{on } \Gamma_T \\ v_t = 0 & \text{on } \Omega \times \{t=0\} \end{cases}$$

We define the energy by

$$E(v(t)) = \frac{1}{2} \int_\Omega v_t^2 + |\nabla v|^2 \, dx$$

Then

$$\frac{d}{dt} E(v(t)) = \int_\Omega v_t v_{tt} + \nabla v \nabla v_t \, dx$$

$$= \int_\Omega v_t (v_{tt} - \Delta v) \, dx = 0.$$

Thus $E(v(t)) = E(v(0)) = 0$ for all t.
This gives $v_t = \nabla v = 0$ in Ω_T and thus $v = 0$ in Ω_T.

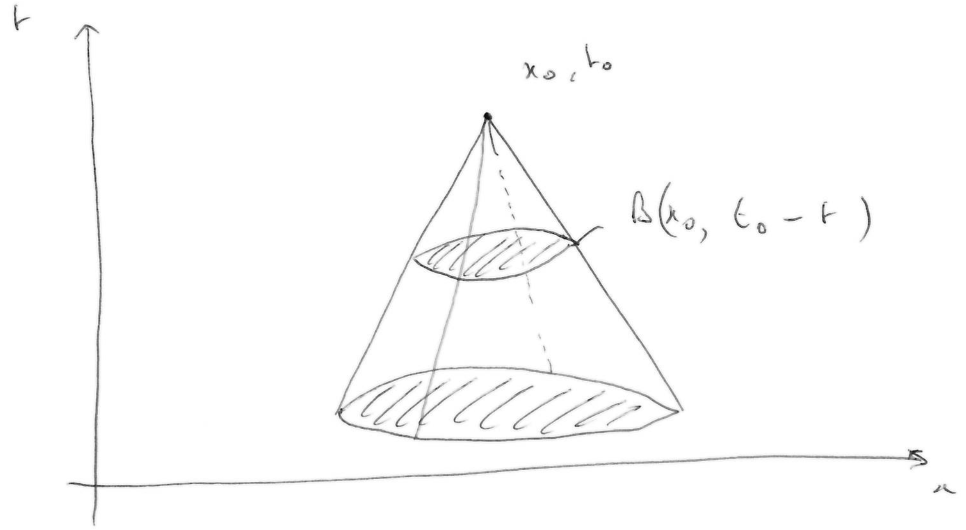
7) Domain of dependence, finite propagation of speed, causality principle

Suppose $u \in C^2$ solves

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

Fix $x_0 \in \mathbb{R}^N$, $t_0 > 0$, and consider the cone

$$C = \{ (x,t) \mid 0 \le t \le t_0, \ |x - x_0| \le t_0 - t \}$$



Thm: If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t=0\}$, then $u \equiv 0$ within the cone C.

Proof: Define

$$E(t) := \frac{1}{2} \int_{B(x_0, t_0 - t)} u_t^2(x,t) + |\nabla u(x,t)|^2 \, dx \qquad 0 \le t \le t_0.$$

Then

$$\frac{d}{dt} E(t) = \int_{B(x_0, t_0-t)} u_t u_{tt} + \nabla u \nabla u_t \, dx - \frac{1}{2} \int_{\partial B(x_0, t_0-t)} u_t^2 + |\nabla u|^2 \, d\sigma$$

$$= \int_{B(x_0, t_0 - t)} u_t (u_{tt} - \Delta u)\, dx + \int_{\partial B(x_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t \, d\sigma - \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} u_t^2 + |\nabla u|^2 \, d\sigma$$

$$= \int_{\partial B(x_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \, d\sigma$$

Now, $\left| \frac{\partial u}{\partial \nu} u_t \right| \leq |u_t| |\nabla u| \leq \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2$, which implies

$\frac{\partial}{\partial t} E(t) \leq 0$ and thus $E(t) \leq E(0) = 0$. $\forall\ 0 \leq t \leq t_0$.

Therefore, $u_t = \nabla u = 0$ and $u \equiv 0$ in C.

- <u>Exercises</u>: Klein-Gordon equation.

$$u_{tt} - \Delta u + m^2 u = 0, \quad m > 0.$$

1) What is the energy? Show it is constant.

2) Prove the causality principle.

- Huygen's principle.

- <u>Exercise</u>: Let g, h be smooth with compact support and u solves

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Prove ~~that~~ there exists $C > 0$ s.t.

$$|u(x,t)| \leq \frac{C}{t} \quad \forall x \in \mathbb{R}^3, \ \forall t > 0. \quad \left(⨍_{\partial B} = \frac{1}{N \omega_N r^{N-1}} \int_{\partial B} \right)$$

Exercise: Same hypothesis, but in dimension 2.

1) Fix $x \in \mathbb{R}^2$. Show

$$|u(x,t)| \leq \frac{C}{t} \quad \forall t > 0$$

2) Show that ~~for all $x \in \mathbb{R}^2$~~

$$|u(x,t)| \leq \frac{C}{t^{1/2}} \quad \forall x \in \mathbb{R}^2 \ \forall t > 0. \qquad \left(\fint_B = \frac{1}{\omega_N r^N} \int_B \right)$$

Exercise: Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = g$, $u_t(x,0) = h$.

Exercise: For u sol of $u_{tt} = u_{xx}$, we define the energy density $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density $p = u_t u_x$

1) Show $\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}$

2) Show $e(x,t)$, $p(x,t)$ solve the wave eq.

Exercise: Show that the wave equation has the following invariance properties.

1) Any translate $u(x-y,t)$ for fixed y is also a solution

2) u_x is also a solution

3) the dilated function $u(ax, at)$ is also ~~a~~ solution

IV Heat equation

$$u_t - \Delta u = 0$$

1) fundamental solution.

Def: The function

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^N \quad t > 0 \\ 0 & x \in \mathbb{R}^N \quad t < 0 \end{cases}$$

is called the fundamental solution of the heat equation.

Exercise: Show that for $t > 0$, $\int_{\mathbb{R}^N} \Phi(x,t)\,dx = 1$.

2) Initial value problem.

$$(\text{IVP}) \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^N \times \{t=0\} \end{cases}$$

Set $u(x,t) := \int_{\mathbb{R}^N} \Phi(x-y,t)\, g(y)\, dy$

Thm: Assume $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and let ~~u~~ be defined as ~~a~~bove. Then

(i) $u \in C^\infty(\mathbb{R}^N \times (0,\infty))$

(ii) ~~$u_t(x,t) - \Delta u\ x$~~ u satisfies (IVP).

Rk: Infinite propagation of speed.

We remark if $g \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ verifies $g \not\equiv 0$, $g \geq 0$, then

$u(x,t) > 0$ for all $x \in \mathbb{R}^N$ and $t > 0$.

3) Nonhomogeneous problem.

$$(NH) \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^N \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}^N \times \{t=0\} \end{cases}$$

As for the wave equation, we introduce an auxiliary problem

$$\begin{cases} v_t(x,t;s) - \Delta v(x,t;s) = 0 & \mathbb{R}^N \times (s,\infty) \\ v(x,t;s) = f(x,s) & \mathbb{R}^N \times \{t=s\} \end{cases}$$

The solution is

$$v(x,t;s) = \int_{\mathbb{R}^N} \Phi(x-y, t-s) f(y,s)\, dy$$

Thm: let $f \in \mathcal{C}^2(\mathbb{R}^N \times [0,\infty))$ with compact support.

Define $u(x,t) := \int_0^t v(x,t;s)\, ds$

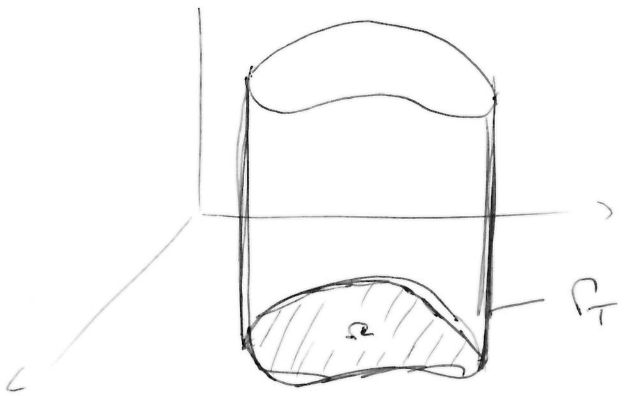
$$= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy\, ds.$$

Then $u \in \mathcal{C}^2(\mathbb{R}^N \times (0,\infty))$ and verifies (NH).

4) Mean-value formula

For $\Omega \subset \mathbb{R}^N$ open and bounded, $T > 0$, we define $\Omega_T = \Omega \times]0,T]$

and $\Gamma_T := \overline{\Omega_T} \setminus \Omega_T$.



Def: Fix $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, $r > 0$ and define the Heat ball

$$E(x,t,r) := \left\{ (y,s) \in \mathbb{R}^{N+1} \;\middle|\; s \leq t, \quad \Phi(x-y, t-s) \geq \frac{1}{r^N} \right\}.$$

Thm: Let $u \in C^2(\Omega_T)$ solve the heat equation. Then

$$u(x,t) = \frac{1}{4 r^N} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds$$

for each $E(x,t,r) \subset \Omega_T$.

Proof: Assume that $x=0$, $t=0$ and u is smooth (it is reasonable, as we shall see it in the sequel).

We write $E(0,0;r) = E(r)$ and

$$\phi(r) := \frac{1}{r^N} \int_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds.$$

$$= \int_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dy\, ds$$

We compute

$$\phi'(r) = \int_{E(1)} \sum_{j=1}^{N} u_{y_i} \cdot y_i \frac{|y|^2}{s^2} + 2 r u_s \frac{|y|^2}{s^2}\, dy\, ds$$

$$= \frac{1}{r^{N+1}} \int_{E(r)} \left(\nabla_y u \cdot y + 2 u_s \right) \frac{|y|^2}{s}\, dy\, ds.$$

To do the computations, it is easier to introduce

$$\psi := -\frac{N}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + N \log(r).$$

Rk: $\psi = 0$ on $\partial E(r)$ (why?).

We have

$$\frac{1}{r^{N+1}} \int_{E(r)} 2 u_s \frac{|y|^2}{s}\, dy\, ds = \frac{1}{r^{N+1}} \int_{E(r)} 4 u_s \sum_{i=1}^{N} y_i \psi_{y_i}\, dy\, ds.$$

$$= -\frac{1}{r^{N+1}} \int_{E(r)} 4 n u_s \psi + 4 \sum_{i=1}^{N} u_{s y_i} y_i \psi\, dy\, ds$$

$$= \frac{1}{r^{N+1}} \int_{E(r)} -4N u_s \psi + 4 \sum_{i=1}^{N} u_{y_i} y_i \psi_s \, dy\, ds.$$

$$= \frac{1}{r^{N+1}} \int_{E(r)} -4N u_s \psi + 4 \sum_{i=1}^{N} u_{y_i} y_i \left(-\frac{N}{2s} - \frac{|y|^2}{4s^2} \right) dy\, ds$$

$$= \frac{1}{r^{N+1}} \int_{E(r)} -4N u_s \psi - \frac{2N}{s} \sum_{i=1}^{N} u_{y_i} y_i \, dy\, ds \; - A .$$

Since u solves the heat equation.

$$\phi'(r) = \frac{1}{r^{N+1}} \int_{E(r)} -4N \Delta u \, \psi - \frac{2N}{s} \sum_{i=1}^{N} u_{y_i} y_i \, dy\, ds$$

$$= \sum_{i=1}^{N} \frac{1}{r^{N+1}} \int_{E(r)} 4N u_{y_i} \psi_{y_i} - \frac{2N}{s} u_{y_i} y_i \, dy\, ds.$$

$$= 0 .$$

$\Rightarrow \phi$ cst.

$$\phi(r) = \lim_{t \to 0} \phi(t) = u(0,0) \left(\lim_{t \to 0^+} \iint_{E(t)} \frac{|y|^2}{s^2} \, dy\, ds \right)$$

$$= 4\, u(0,0) \quad \text{(admit)}.$$

5) Maximum principle

Thm: Assume $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat eq in Ω_T. Then

(i) $\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$

(ii) if Ω is connected, and there exists $(x_0, t_0) \in \Omega_T$ s.t.

$u(x_0, t_0) = \max_{\overline{\Omega_T}} u$ then u is constant in $\overline{\Omega}_{t_0}$

Proof: 1) Supp $\exists\ (x_0, t_0) \in \Omega_T$ with $u(x_0, t_0) = M := \max_{\overline{\Omega_T}} u$.

for r small, $E(x_0, t_0, r) \subset \Omega_T$ and

$$M = u(x_0, t_0) = \frac{1}{4r^N} \iint_{E(x_0, t_0, r)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dy\, ds \leq M.$$

(because $\iint_{E(x_0, t_0, r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dy\, ds$

It is possible only if $u \equiv M$ in $E(x_0, t_0, r)$.

Consider a line segment connecting (x_0, t_0) to some $(y_0, s_0) \in \Omega_T$, $s_0 < t_0$.

Let $r_0 := \min \{ s \geq s_0 \mid u(x, t) = M \ \forall (x, t) \in L, \ s \leq t \leq t_0 \}$.

Since u is continuous, the min is attained. Assume $r_0 > s_0$. Then

$u(z_0, r_0) = M$ for some (z_0, r_0) on $L \cap \Omega_T$, and so $u \equiv M$ in $E(z_0, r_0, r)$

for r small. Since $E(z_0, r_0; r)$ contains $L \cap \{ r_0 - \sigma \leq t \leq r_0 \}$ for $\sigma > 0$ small,

we reach a contradiction. Thus $r_0 = s_0$ and $u \equiv M$ on L.

2) by connectedness

Thm: $g \in C(\mathbb{R}^N)$, $f \in C(\mathbb{R}^N \times [0, T])$. There exists at most one sol $u \in C^2(\mathbb{R}^N \times (0, T]) \cap C(\mathbb{R}^N \times [0, T])$ of $\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^N \times (0, T) \\ u = g & \text{on } \mathbb{R}^N \times \{t = 0\} \end{cases}$

satisfying $|u(x, t)| \leq A e^{a|x|^2}$ $\forall x \in \mathbb{R}^N, \forall\ 0 \leq t \leq T$ for $a, A > 0$.

6) Regularity:

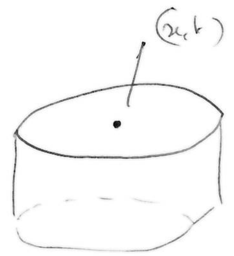
Thm: Assume $u: \Omega_T \to \mathbb{R}$ satisfies

- $u, D_x u, D_x^2 u, u_t \in C(\Omega_T)$
- u satisfies the heat equation.

Then $u \in C^\infty(\Omega_T)$.

Proof 1) We write

$C(x,t;r) = \{ (y,s) \mid |x-y| \leq r, \ t-r^2 \leq s \leq t$



Fix $(x_0,t_0) \in \Omega_T$ and choose $r>0$ s.t. $C = C(x_0,t_0,r) \subset \Omega_T$.

Let also $C' := C(x_0, t_0; \frac{3}{4} r)$, $C'' := C(x_0, t_0, \frac{1}{2} r)$.

Let $\zeta = \zeta(x,t)$ s.t.

$$\begin{cases} 0 \leq \zeta \leq 1, \ \zeta \equiv 1 & \text{on } C' \\ \zeta \equiv 0 & \text{on } \partial C \end{cases}$$

Extend $\zeta \equiv 0$ in $(\mathbb{R}^n \times [0,t_0]) \setminus C$.

2) Assume temporarily that $u \in C^\infty(\Omega_T)$ and set

$v(x,t) := \zeta(x,t)\, u(x,t) \quad (x \in \mathbb{R}^n, \ 0 \leq t \leq t_0)$

Then $v_t = \zeta u_t + \zeta_t u$, $\Delta v = \zeta \Delta u + 2 \nabla\zeta \nabla u + u \Delta \zeta$

Consequently, $v = 0$ on $\mathbb{R}^n \times \{t=0\}$

and $v_t - \Delta v = \zeta_t u - 2\nabla\zeta \cdot \nabla u - \Delta\zeta =: \tilde{f}$ in $\mathbb{R}^n \times (0,t_0)$

Now set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^N} \Phi(x-y, t-s)\, \tilde{f}(y,s)\, dy\, ds.$$

By unicity, $v \equiv \tilde{v}$

Now, suppose $(x,t) \in C''$. Since $\zeta \equiv 0$ out C, we have

$$u(x,t) = \iint_C \Phi(x-y, t-s)\Big[\big(\zeta_s(y,s) - \Delta\zeta(y,s)\big)\, u(y,s) - 2\nabla\zeta(y,s)\nabla u(y,s)\Big]\, dy\, ds$$

$$= \iint_C \Big[\Phi(x-y,t-s)\big(\zeta_s(y,s) + \Delta\zeta(y,s)\big) + 2\nabla_y \Phi(x-y,t-s).\nabla\zeta(y,s)\Big] u(y,s)\, dy\, ds$$

If u is not smooth, this result can be obtained by mollifying u.

3) The formula has the form

$$u(x,t) = \iint_C K(x,t,y,s)\, u(y,s)\, dy\, ds$$

with $K(x,t,y,s) \equiv 0$ in C'

and K smooth on $C \setminus C'$. This implies $u \in C^\infty$.

7) <u>Uniqueness</u>.

$$\begin{cases} u_t - \Delta u = f & \text{on } \Omega_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Ω open, bdd, $\partial\Omega$ C^1.

<u>Thm</u>: There exists at most one solution $u \in C^1_t(\Omega_T) \cap C^2_x(\Omega_T)$.

Proof: Let $u, \tilde{u}$ be two solutions, and set $w := u - \tilde{u}$. Then

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Set $e(t) := \int_\Omega w^2(x,t)\,dx \quad (0 \le t \le T)$

Then $\frac{\partial}{\partial t} e(t) = 2\int_\Omega w\, w_t\,dx$

$$= 2\int_\Omega w\,\Delta w\,dx$$

$$= -2\int_\Omega |\nabla w|^2\,dx \le 0$$

Therefore, $e(t) \le e(0) = 0 \quad (0 \le t \le T)$

and $w \equiv 0$ in Ω_T.

8) Backward uniqueness

Suppose
(BU) $\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial\Omega \times [0,T] \end{cases}$

and $\tilde{u}$ also a sol.

Thm: Suppose $u, \tilde{u} \in C^2(\overline{\Omega_T})$ solve (BU). If

$$u(x,T) = \tilde{u}(x,T) \quad (x \in \Omega),$$

then $u \equiv \tilde{u}$ in Ω_T.

Proof: $w = u - \tilde{u}$, $e(t) := \int_\Omega w^2\,dx$. $(0 \le t \le T.)$

$$\frac{\partial e}{\partial t} = -2\int_\Omega |\nabla w|^2\,dx$$

Moreover,
$$\frac{\partial^2}{\partial t^2} e(t) = -4 \int_\Omega \nabla w \nabla w_t \, dx$$
$$= 4 \int_\Omega \Delta w \, w_t \, dx$$
$$= 4 \int_\Omega (\Delta w)^2 \, dx$$

Now, since $w = 0$ on $\partial\Omega$

$$\int_\Omega |\nabla w|^2 = - \int_\Omega w \Delta w \, dx \leq \left(\int_\Omega w^2 \, dx \right)^{1/2} \left(\int_\Omega (\Delta w)^2 \, dx \right)^{1/2}$$

Thus $\left(\dot{e}(t)\right)^2 \leq e(t) \, \ddot{e}(t)$

2) if $e(t) = 0 \quad \forall \, 0 \leq t \leq T$, we are done.

otherwise, there exists $(t_1, t_2) \subset [0, T]$ s.t.

$e(t) > 0$ for $t_1 \leq t < t_2$, $e(t_2) = 0$.

3) write $f(t) := \log e(t) \quad (t_1 \leq t \leq t_2)$

$$\frac{\partial^2}{\partial t^2} f(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0$$

$\Rightarrow f$ convex on t_1, t_2: for $0 < \tau < 1$, $t_1 < t < t_2$,

$$f\left((1-\tau) t_1 + \tau t\right) \leq (1-\tau) f(t_1) + \tau f(t)$$

thus $e\left((1-\tau) t_1 + \tau t\right) \leq e(t_1)^{1-\tau} e(t)^\tau$

So $0 \leq e\left((1-\tau) t + \tau t_2\right) \leq e(t_1)^{1-\tau} \underbrace{e(t_2)^\tau}_{=0}$

$\Rightarrow e(t) = 0 \quad t_1 \leq t \leq t_2$ ↯.

Exercise: u smooth, $u_t - \Delta u = 0$ in $\mathbb{R}^N \times (0,\infty)$.

(i) $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ solve heat for $\lambda \in \mathbb{R}$

(ii) Prove $v(x,t) := x \cdot \nabla u(x,t) + 2t\, u_t(x,t)$ solves heat.

Exercise: $N = 1$, $u(x,t) = v\left(\frac{x^2}{t}\right)$

1) Show $u_t = u_{xx} \iff 4z\, v''(z) + (2+z)\, v'(z) = 0 \quad (z > 0)\ (*)$

2) Show gen sol of $(*)$ is $v(z) = c \int_0^z e^{-s/4} s^{-\frac{1}{2}}\, ds + d$

3) Differentiate $v\left(\frac{x^2}{t}\right)$ in x and t and select c to obtain the fundamental sol Φ for $N = 1$.

Exercise 3 (Exams):

Let u satisfy
$$\begin{cases} -\Delta u = f & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}$$

Prove that for $N \geq 3$ we have

$$u(0) = \fint_{\partial B(0,r)} g \, d\sigma + \frac{1}{N(N-2)\omega_N} \int_{B(0,r)} \left(\frac{1}{|x|^{N-2}} - \frac{1}{r^{N-2}} \right) f \, dx$$

Hint: modify the proof of the mean value formula.

Exercise: KG

$$u_{tt} - \Delta u + m^2 u = 0 \quad , \quad m > 0$$

1) What is the energy? Show it is constant.

2) Prove the causality principle.

Exercise 17 Evans Equipartition of energy

Let $u \in C^2(\mathbb{R} \times [0,\infty))$ solve the IVP for the wave eq in 1D

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

Supp g, h have compact supports. The kinetic energy is

$k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the potential energy is

$p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t)\,dx$.

(i) $k(t) + p(t)$ is constant in t

(ii) $k(t) = p(t)$ for all large enough times t.

Exercise 6 (Strauss)

Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation.

This means the following.

A spherical wave in N-D space satisfies

$$u_{tt} = u_{rr} + \frac{N-1}{r} u_r$$

Consider such a wave of the form $u(r,t) = \alpha(r)\, f(t - \beta(r))$

$\alpha(r)$ = attenuation

$\beta(r)$ = delay

Does such solution exists for arbitrary f

(a) Plug into the PDE to get an ODE for f

(b) Set the coeff of f'', f' and f eq to 0

(c) Solve the ODEs to see $N = 1$ or $N = 3$

(d) If $N = 1$, show $\alpha(r)$ is constant

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