ALMOST A BUILDING FOR THE TAME AUTOMORPHISM GROUP

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Abstract. Inspired by the Bruhat–Tits building of $\mathrm{SL}_n(\mathbb{F})$, for $\mathbb{F}$ a field with a valuation, we construct a complete metric space $\mathbf{X}$ with an action of the tame automorphism group of the affine space $\mathrm{Tame}(\Bbbk^n)$. The points in $\mathbf{X}$ are certain monomial valuations, and $\mathbf{X}$ admits a natural structure of Euclidean CW-complex of dimension $n-1$. When $n=3$, and for $\Bbbk$ of characteristic zero, we prove that $\mathbf{X}$ is locally CAT(0) and simply connected, hence $\mathbf{X}$ is a CAT(0) space. As an application we obtain the linearizability of finite subgroups in $\mathrm{Tame}(\Bbbk^3)$.

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1. Introduction

Let $\Bbbk$ be a field, and $n \geqslant 2$. We denote by $\mathrm{Aut}(\Bbbk^n)$ the group of polynomial automorphisms of the affine space of dimension n over $\Bbbk$. A coordinate system $(x_1, \dots, x_n)$ being fixed, an element $g \in \mathrm{Aut}(\Bbbk^n)$ is written

$$g\colon (x_1, \dots, x_n) \mapsto (g_1, \dots, g_n)$$

where the g_i are polynomials in the x_i, and the inverse of g admits a writing of the same form.

The group $\mathrm{Tame}(\Bbbk^n)$ of *tame automorphisms* of the affine space is the subgroup of $\mathrm{Aut}(\Bbbk^n)$ generated by the linear group $\mathrm{GL}_n(\Bbbk)$ and by the elementary automorphisms, or "polynomial transvections", of the form

$$(x_1, \dots, x_n) \mapsto (x_1 + P(x_2, \dots, x_n), x_2, \dots, x_n).$$

We study the structure of the tame group, in particular in dimension $n=3$ and over a ground field $\Bbbk$ of characteristic zero. After having obtained the existence of many normal subgroups in [LP19], we are interested in the present work in the question of the classification of its finite subgroups up to conjugation. Our long-term motivation is to establish a panorama of the properties that can be expected from the entire group of polynomial automorphisms, or even from the Cremona group, which is the group of birational (and no longer only bipolynomial) transformations of the affine space. We refer to the introductions of [BFL14] and [LP19] for more details on the articulation between these different groups, and a history of recent developments. We focus in this introduction on the specific aspects of this work.

A subgroup G of $\mathrm{Aut}(\Bbbk^n)$ is said to be *linearizable* if there exists a *linearizing map* $\varphi \in \mathrm{Aut}(\Bbbk^n)$ such that $\varphi G \varphi^{-1} \subset \mathrm{GL}_n(\Bbbk)$. It is known that, for a field $\Bbbk$ of characteristic zero, any finite subgroup G of $\mathrm{Aut}(\Bbbk^2)$ is linearizable. We recall the argument, which can be summed up simply, and which will serve as a model to go into dimension 3. We consider the action of G on the Bass–Serre tree associated with the amalgamated product structure for $\mathrm{Tame}(\Bbbk^2) = \mathrm{Aut}(\Bbbk^2)$ (this is Jung–van der Kulk's theorem, see for example [Lam02]). This always admits at least one fixed point, so G is conjugated to a subgroup of one of the factors of the amalgamated product. Finally, the finite subgroups of these two factors are linearizable by a general averaging criterion, see Lemma 8.1. In higher dimension, the Bass–Serre tree generalizes to a simplicial complex $\mathcal{C}_n$ of dimension $n-1$ on which the tame group acts with a

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simplex as fundamental domain. It is this complex $\mathcal{C}_3$ that we used in our previous work [LP19] to obtain acylindrical hyperbolicity, which implies in particular the existence of normal subgroups in $\mathrm{Tame}(\Bbbk^3)$ [DGO17]. It is also with a variant of this construction that was established in [BFL14] an analogous result of linearization of finite subgroups for the group $\mathrm{Tame}(V)$ of tame automorphisms of an affine quadric $V \subset \Bbbk^4$. In the latter context, the complex on which $\mathrm{Tame}(V)$ acts naturally is a CAT(0) square complex, this property of non-positive curvature ensuring the existence of a global fixed point for any finite group action. To extend these results to the group $\mathrm{Tame}(\Bbbk^3)$, the problem that we had to circumvent is that the study of the links of the vertices shows that the triangles of $\mathcal{C}_3$ cannot be equipped with an Euclidean structure which makes them a CAT(0) space. Note also that although $\mathcal{C}_3$ is a 2-dimensional contractile complex [LP19, Theorem A], to our knowledge it remains an open question whether it implies the existence of a fixed point for any action of a finite group (see [OS02, p.205]).

In this paper, we introduce a new space $\mathbf{X}_n$ on which acts the tame group $\mathrm{Tame}(\Bbbk^n)$ (Section 2). The space $\mathbf{X}_n$, whose points are certain monomial valuations modulo dilation, is also a CW-complex whose cells are Euclidean regions (but in general not polyhedra) of dimension at most $n-1$, and its construction is inspired by the Bruhat–Tits building of $\mathrm{SL}_n(\mathbb{F})$, for $\mathbb{F}$ a valued field. A nice property of the space $\mathbf{X}_n$ is that the stabilizers for the action of $\mathrm{Tame}(\Bbbk^n)$ are conjugate to semi-direct products $M \rtimes L$ with M a subgroup of triangular automorphisms stable under taking mean, and L a subgroup of the linear group (Proposition 3.2). By the general linearization criterion already mentioned this allows to obtain that any finite subgroup of such a stabilizer is linearizable. It remains to show that any finite group acts on the space $\mathbf{X}_n$ with a global fixed point. As in the examples mentioned above, this follows automatically once one has been able to endow $\mathbf{X}_n$ with a metric which makes it a complete CAT(0) space, considering the “circumcenter” of any orbit. The construction of such a metric is the main result of this article.

First of all in arbitrary dimension $n \geqslant 2$ and over an arbitrary field $\Bbbk$ we construct $\mathbf{X}_n$ as the quotient of a union of copies of the Euclidean space $\mathbb{R}^{n-1}$, which we call apartments. We show that the induced quotient pseudo-metric is a metric, which makes $\mathbf{X}_n$ a complete length space (Proposition 5.4 and Lemma 5.8). Then, restricting the study to the case $n = 3$ and to the case of a field $\Bbbk$ of characteristic zero, we show that $\mathbf{X}_3$ is simply connected (Proposition 6.3). Here the restriction on the characteristic of $\Bbbk$ comes from the reduction theory of Shestakov–Umirbaev and Kuroda, whose famous consequence is the strict inclusion $\mathrm{Tame}(\Bbbk^3) \subsetneq \mathrm{Aut}(\Bbbk^3)$, but which we use here only through the description of $\mathrm{Tame}(\Bbbk^3)$ as an amalgamated product of three factors along their pairwise intersections. Finally, based on the study of intersections between apartments (Section 4) we show that $\mathbf{X}_3$ is locally CAT(0) (Proposition 7.1). Finally, by the Cartan–Hadamard theorem we conclude (see Section 8):

Theorem A. *Over a field $\Bbbk$ of characteristic zero, the space $\mathbf{X}_3$ is a* CAT(0) *complete metric space.*

As a corollary, following the strategy described above, we obtain:

Corollary B. *Over a field $\Bbbk$ of characteristic zero, any finite subgroup of* $\mathrm{Tame}(\Bbbk^3)$ *is linearizable.*

Note that in this corollary the linearizing map will be in $\mathrm{Tame}(\Bbbk^3)$.

In dimension $n = 2$ the previous construction produces a tree $\mathbf{X}_2$ which is non-isometric (and even not equivariantly isomorphic) to the Bass–Serre tree of $\mathrm{Aut}(\Bbbk^2)$, which was a first indication that this complex potentially contains new information about the tame group. Concerning the possibility of generalizing our results in higher dimension, let us first recall that there exist finite non-linearizable subgroups in $\mathrm{Aut}(\Bbbk^4)$. For example, following [FMJ02], if $\Bbbk$ contains 3 cubic roots of unity, 1, ω and ω^{-1}, then the action on $\Bbbk^4$ of the symmetric group $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\sigma\tau)^2 = 1 \rangle$ defined as follows is non-linearizable:

$$\begin{aligned} \sigma(a,b,x,y) &= (\omega a, \omega^{-1} b, x, y), \\ \tau(a,b,x,y) &= (b, a, -b^3 x + (1 + ab + a^2 b^2) y, (1 - ab) x + a^3 y). \end{aligned}$$

One may suspect that τ is an non-tame automorphism, but this is an open question whether the inclusion $\mathrm{Tame}(\Bbbk^n) \subset \mathrm{Aut}(\Bbbk^n)$ is strict in dimension $n \geqslant 4$. On the other hand, as τ after composition by the involution $(a, b, x, y) \mapsto (b, a, y, x)$ is identified with an element of $\mathrm{SL}_2(\Bbbk[a, b])$, it is known from [Sus77] that this example becomes tame after extension to $\Bbbk^5$, trivially extending the action on the additional variable. Indeed, Suslin proves that for all $r \geqslant 3$, the group $\mathrm{SL}_r(\Bbbk[x_1, \dots, x_n])$, which can be thought of as a subgroup of $\mathrm{Aut}(\Bbbk^{r+n})$, is generated by the elementary matrices, which are particular tame automorphisms. We mention that this property of being "stably tame", i.e. tame after trivial extension to some additional variables, is not valid only for automorphisms depending linearly on a part of the variables: by [BEW12, Theorem 4.10], any polynomial automorphism in the two variables x, y and with coefficients in the ring $\Bbbk[a, b]$ becomes tame after extension to $\Bbbk^6$. Moreover, still following [FMJ02], the previous example remains non-linearizable in $\mathrm{Aut}(\Bbbk^{4+m})$, if the action of S_3 is trivially extended to any number m of additional variables. By combining with Suslin's result, for all $n \geqslant 5$ we thus obtain a finite non-linearizable subgroup of $\mathrm{Tame}(\Bbbk^n)$; in other words Corollary B, and therefore also the Theorem A, are no longer valid in dimension $n \geqslant 5$. It would however be interesting to study if one of the two ingredients of the property CAT(0) persists, namely the simple connectedness or the local CAT(0) property. Finally, the case of the dimension $n = 4$ remains open, but seems difficult in the absence of a theory of reductions.

The space $\mathbf{X}_n$ is constructed by considering the orbit under the action of $\mathrm{Tame}(\Bbbk^n)$ of the set of monomial valuations associated with the coordinate system $x_1, \dots, x_n$. One could just as well consider the action of the group $\mathrm{Aut}(\Bbbk^n)$, and obtain a larger space on which the entire group of polynomial automorphisms acts. The point is that this new space is no longer connected, and the connected component containing the initial monomial valuations is precisely our space $\mathbf{X}_n$. We have made the choice to restrict ourselves from the start to the action of the tame group, however the interested reader will be able to verify that most of the general statements (in particular in sections 2 and 3) would remain valid for the group $\mathrm{Aut}(\Bbbk^n)$, with unchanged proofs.

We see the space $\mathbf{X}_n$ endowed with the action of $\mathrm{Tame}(\Bbbk^n)$ as an analogue of the affine Bruhat–Tits building associated with $\mathrm{SL}_n(\mathbb{F})$, for $\mathbb{F}$ a valued field. We have in mind the construction via ultrametric norms (see [BT84, Par00]), where an apartment is associated with each base of $\mathbb{F}^n$ by varying the weights of the associated ultrametric norms. In our situation we can see each $f \in \mathrm{Tame}(\Bbbk^n)$ as giving a $\Bbbk$-algebra basis of $\Bbbk[x_1, \dots, x_n] = \Bbbk[f_1, \dots, f_n]$, and associate to it a collection $\mathbf{E}_f$ of monomial valuations naturally parametrized by a Euclidean space $\mathbb{R}^{n-1}$, which we call l apartment associated with f. Another similarity is that each such apartment contains a particular valuation (one whose weights are all equal), a neighborhood of which in $\mathbf{X}_n$ is isometric to the cone over the spherical building associated with $\mathrm{GL}_n(\Bbbk)$. However, as we show in Appendix 9.3, for $n \geqslant 3$ the space $\mathbf{X}_n$ is not the Davis realization of a building [AB08, Definition 12.65], because the crucial property "by two points passes an apartment" [AB08, Definition 4.1(B1)] is not satisfied, even locally. We also mention that any element in the stabilizer of an apartment $\mathbf{E}_f$ acts on it like a permutation matrix, and never like a translation as one might have anticipated. In particular, a fundamental domain for the action of $\mathrm{Tame}(\Bbbk^n)$ on $\mathbf{X}_n$ is an entire Weyl chamber, and therefore is not compact (Corollary 2.5(ii)). It is not clear to us whether certain weakened notions of building (for example the "masures" of Gaussent and Rousseau [GR08]) could encompass our construction.

By definition $\mathbf{X}_n$ is included in the space of all valuations over the ring $\Bbbk[x_1, \dots, x_n]$, studied in particular in previous works by Boucksom, Favre and Jonsson [FJ07, BFJ08]. These authors mainly study the space of valuations centered at the origin, but the case of valuations centered at infinity is similar, and is treated in detail in the case $n = 2$ in [FJ07, Appendix A]. In particular our tree $\mathbf{X}_2$ is a subtree of the tree $\mathcal{V}_1$ introduced in [FJ07, §A.3], but with different normalization choices. The case of arbitrary dimension n is treated in [BFJ08], and a simplicial affine structure is discussed which in restriction to the space $\mathbf{X}_n$ corresponds to our apartments. It would be interesting to explore the relations between the properties of the space of all valuations and those of the subspace $\mathbf{X}_n$ that we consider in this article.

Another natural question would be to understand the axes of the loxodromic isometries on $\mathbf{X}_3$. Such a study could lead to a Tits alternative for $\mathrm{Tame}(\Bbbk^3)$, or to an understanding of the dynamical properties of the elements of $\mathrm{Tame}(\Bbbk^3)$, the growth of degrees under iteration.

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2. Valuation space

2.1. Valuations. A *valuation* on the ring $\Bbbk[x_1,\dots,x_n]$ is a function

$$\nu\colon \Bbbk[x_1,\dots,x_n] \to \mathbb{R}\cup\{+\infty\}$$

such that, for all polynomials P_1, P_2:

- $\nu(P_1+P_2) \geqslant \min\{\nu(P_1),\nu(P_2)\}$;
- $\nu(P_1P_2) = \nu(P_1)+\nu(P_2)$;
- $\nu(P) = 0$ for any constant non-zero polynomial P;
- $\nu(P) = +\infty$ if and only if $P = 0$.

We denote by $\mathcal{V}_n$ the set of classes of such valuations, modulo homothety by a positive real number. If $P \in \Bbbk[x_1,\dots,x_n]$ is a polynomial and $g = (g_1,\dots,g_n) \in \mathrm{Aut}(\Bbbk^n)$ is an automorphism, we denote g^*P the polynomial $P(g_1,\dots,g_n)$. The group $\mathrm{Aut}(\Bbbk^n)$ acts (on the left) on all the valuations via the formula:

$$(g\cdot\nu)(P) := \nu(g^*(P)).$$

This action commutes with the homotheties, we thus obtain an action of $\mathrm{Aut}(\Bbbk^n)$ on $\mathcal{V}_n$.

We now describe some *monomial valuations* forming a subset of $\mathcal{V}_n$. We denote by Π the positive quadrant of $\mathbb{R}^n$, i.e.

$$\Pi = \{\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n;\ \alpha_i > 0 \text{ for all } i\}.$$

We will say that Π is the *space of weights*. We also define the subspace of well-ordered weights

$$\Pi^+ = \{\alpha = (\alpha_1,\dots,\alpha_n);\ \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n > 0\} \subset \Pi.$$

For all $\alpha = (\alpha_1,\dots,\alpha_n) \in \Pi$, we denote $[\alpha_1,\dots,\alpha_n]$, or simply $[\alpha]$, the class of α modulo homothety by a positive real number, and we denote ∇ the projectivization of Π, which is an open simplex of dimension $n-1$. Similarly $\nabla^+ \subset \nabla$ will denote the half-open simplex which is the projectivization of Π^+. We equip ∇ and ∇^+ with the topology induced by $\mathbb{P}^{n-1}(\mathbb{R})$. In particular, we will say that a weight $[\alpha] \in \nabla^+$ satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ is inside ∇^+, and on the contrary that $[\alpha]$ is in the boundary of ∇^+ if there are two indices $i > j$ such that $\alpha_i = \alpha_j$. It will often be useful to extend these definitions to include weights with some (but not all) zero coefficients, which will be called the *boundary at infinity* of ∇.

Given a weight $\alpha = (\alpha_1,\dots,\alpha_n) \in \Pi$, we define a monomial valuation $\nu_{\mathrm{id},\alpha}$ as follows. For every

$$P = \sum_{I=(i_1,\dots,i_n)} c_I x_1^{i_1}\dots x_n^{i_n} \in \Bbbk[x_1,\dots,x_n],$$

we call support of P, denoted $\operatorname{Supp} P$, the set of multi-indices I such that $c_I \neq 0$, and we set:

$$\nu_{\mathrm{id},\alpha}(P) = \min_{I\in\operatorname{Supp} P}\left(-\sum_{k=1}^n \alpha_k i_k\right).$$

In other words $-\nu_{\mathrm{id},\alpha}(P)$ is the weighted degree of P, where each variable x_i is assigned the weight α_i. We have adopted the point of view of valuations for the sake of compatibility with existing works, in particular [FJ07].

Observe that for any weight $\alpha \in \Pi$ and any $t > 0$, we have $t\nu_{\mathrm{id},\alpha} = \nu_{\mathrm{id},t\alpha}$. Thus the homothety class of $\nu_{\mathrm{id},\alpha}$ depends only on $[\alpha] \in \nabla$, and we denote it $\nu_{\mathrm{id},[\alpha]} \in \mathcal{V}_n$. We denote $\mathbf{E}_{\mathrm{id}}$ (resp. $\mathbf{E}^+_{\mathrm{id}}$) the set of all classes $\nu_{\mathrm{id},[\alpha]}$ with $[\alpha] \in \nabla$ (resp. $[\alpha] \in \nabla^+$). We then define the space $\mathbf{X}_n \subset \mathcal{V}_n$ (or simply $\mathbf{X}$ if the dimension is clear by context) as the orbit of $\mathbf{E}_{\mathrm{id}}$ under the action of $\mathrm{Tame}(\Bbbk^n)$. If $f \in \mathrm{Tame}(\Bbbk^n)$, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \Pi$, we will note

$$\nu_{f,\alpha} := f \cdot \nu_{\mathrm{id},\alpha}.$$

Explicitly, by noting $f^{-1} = (g_1, \dots, g_n)$, the valuation $\nu_{f,\alpha}$ is characterized by the property: $\nu_{f,\alpha}(g_i) = \alpha_i$ for all $1 \leqslant i \leqslant n$. Indeed $\nu_{f,\alpha}(g_i) = f \cdot \nu_{\mathrm{id},\alpha}(g_i) = \nu_{\mathrm{id},\alpha}(g_i \circ f) = \nu_{\mathrm{id},\alpha}(x_i)$. Again, the homothety class of $\nu_{f,\alpha}$ depends only on $[\alpha] \in \nabla$, and we denote it $\nu_{f,[\alpha]}$.

We identify the symmetric group S_n with a subgroup of $\mathrm{GL}_n(\Bbbk)$, and therefore also of $\mathrm{Tame}(\Bbbk^n)$, via the action by permutation on the vectors of the canonical basis of $\Bbbk^n$. In terms of coordinates, this is equivalent to setting, for all $\sigma \in S_n$:

$$\sigma = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) \in \mathrm{Tame}(\Bbbk^n).$$

Thus, for all $f \in \mathrm{Tame}(\Bbbk^n)$ and $P \in \Bbbk[x_1, \dots, x_n]$, we have

$$\sigma^* P = P(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) \in \Bbbk[x_1, \dots, x_n].$$

Similarly, the symmetric group acts on Π by

$$\sigma(\alpha) = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(n)}).$$

Lemma 2.1. *For all $f \in \mathrm{Tame}(\Bbbk^n)$, $\alpha \in \Pi$, $\sigma \in S_n$, we have*

$$\nu_{f\sigma,\alpha} = \nu_{f,\sigma(\alpha)}.$$

Proof. The verification is straightforward:

$$\begin{aligned}\nu_{f\sigma,\alpha}(P) = \nu_{\mathrm{id},\alpha}((f\sigma)^*(P)) = \nu_{\mathrm{id},\alpha}(\sigma^*(f^*P)) &= \nu_{\mathrm{id},\alpha}(f^*P(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})) \\ &= \nu_{\mathrm{id},\sigma(\alpha)}(f^*P(x_1, \dots, x_n)) = \nu_{f,\sigma(\alpha)}(P).\end{aligned}$$

□

Remark 2.2. The previous proof is elementary, but in practice the presence of σ^{-1} can easily lead to errors. We can in particular convince ourselves of the fourth equality by considering the example $f = \mathrm{id}, \sigma = (123)$. It's about checking

$$\nu_{\mathrm{id},(\alpha_1,\alpha_2,\alpha_3)}(P(x_3, x_1, x_2)) = \nu_{\mathrm{id},(\alpha_3,\alpha_1,\alpha_2)}(P(x_1, x_2, x_3)).$$

For $P = x_1$, $P = x_2$ and $P = x_3$ these monomial valuations take respectively the values α_3, α_1 and α_2, we deduce that they coincide on all $\Bbbk[x_1, x_2, x_3]$.

2.2. **Apartments.** For all $f \in \mathrm{Tame}(\Bbbk^n)$, we call *appartment* associated with f the set

$$\mathbf{E}_f := f(\mathbf{E}_{\mathrm{id}}) = \{\nu_{f,[\alpha]};\ [\alpha] \in \nabla\} \subset \mathbf{X}.$$

In particular $\mathbf{E}_{\mathrm{id}}$ will be called the *standard appartment.* Similarly we call *chamber* associated with f the set

$$\mathbf{E}^+_f := f(\mathbf{E}^+_{\mathrm{id}}) = \{\nu_{f,[\alpha]};\ [\alpha] \in \nabla^+\} \subset \mathbf{E}_f,$$

and we call $\mathbf{E}^+_{\mathrm{id}}$ the *standard chamber.* Lemma 2.1 implies that the apartment $\mathbf{E}_f$ associated with f is the union of the $n!$ chambers $\mathbf{E}^+_{f\sigma}$, where $\sigma \in S_n$ is a permutation.

For any weight $\alpha \in \Pi$, there exists a unique $\alpha^+ \in \Pi^+$ obtained by reordering the α_i. This application descends to the projectivization as an application

$$[\alpha] \in \nabla \mapsto [\alpha]^+ := [\alpha^+] \in \nabla^+.$$

We will note $\alpha^+ = (\alpha^+_1, \dots, \alpha^+_n)$ the coordinates of α^+, in particular

$$\alpha^+_1 = \max_{1 \leqslant i \leqslant n} \alpha_i \ \text{ et } \ \alpha^+_n = \min_{1 \leqslant i \leqslant n} \alpha_i$$

Lemma 2.3. *Let $\ell_i, \dots, \ell_n$ be independent linear forms on $\Bbbk^n$, and $\nu = \nu_{\mathrm{id},\alpha}$ for a weight $\alpha \in \Pi$. Then*

$$-\nu(\ell_i) - \cdots - \nu(\ell_n) \geqslant \alpha^+_i + \cdots + \alpha^+_n.$$

Proof. Let $\sigma \in S_n$ such that $\alpha_j^+ = \alpha_{\sigma(j)}$ for all $1 \leqslant j \leqslant n$. For a linear form $\ell = \sum_{j=1}^n a_{\sigma(j)} x_{\sigma(j)}$, we have $-\nu(\ell) = \max\{\alpha_j^+ \mid a_{\sigma(j)} \neq 0\}$. So if $-\nu(\ell) = \alpha_k^+$, ℓ only depends on the variables $x_{\sigma(k)}, \dots, x_{\sigma(n)}$. Now if $\ell_i, \dots, \ell_n$ are independent, they cannot all depend only on the variables $x_{\sigma(i+1)}, \dots, x_{\sigma(n)}$, and therefore there is an index $i \leqslant j \leqslant n$ such that $-\nu(\ell_j) \geqslant \alpha_i^+$. We conclude by induction on the number of linear forms. $\square$

Observe that the inequality can be strict in Lemma 2.3. For example in dimension 2, if $\ell_1 = x_1$, $\ell_2 = x_1 + x_2$ and $\alpha = (2, 1)$, we have

$$-\nu(\ell_1) - \nu(\ell_2) = 2 + 2 > 2 + 1 = \alpha_1^+ + \alpha_2^+.$$

Proposition 2.4. *Let $f, g \in \mathrm{Tame}(\Bbbk^n)$, and $\alpha, \beta \in \Pi$. If $\nu_{f,\alpha} = \nu_{g,\beta}$, then $\alpha^+ = \beta^+$.*

Proof. Given a valuation $\nu = \nu_{f,\alpha} \in \mathbf{X}$, we define by descending induction a sequence γ_i by setting, for $i = n, n-1, \dots, 1$:

$$\gamma_i := \inf\left(-\nu(h_i h_{i+1} \dots h_n)\right) - \sum_{j=i+1}^{n} \gamma_j,$$

where the infimum is taken over the $(h_i, \dots, h_n)$ which are $n - i + 1$ distinct components of some $h = (h_1, \dots, h_n) \in \mathrm{Tame}(\Bbbk^n)$. We will see that $(\gamma_1, \dots, \gamma_n) = (\alpha_1^+, \dots, \alpha_n^+)$, which will show that $\alpha^+ = \gamma$ is intrinsically defined.

First observe that we can assume $f = \mathrm{id}$, because $\nu_{\mathrm{id},\alpha} = f^{-1} \cdot \nu_{f,\alpha}$ and by construction the definition of γ is invariant under the action of $\mathrm{Tame}(\Bbbk^n)$. Suppose that $\gamma_j = \alpha_j^+$ for $j = i+1, \dots, n$ (which corresponds to an empty condition in the case $i = n$), and show that it is also the case for $j = i$. By definition

$$\gamma_i = \inf\left(-\nu(h_i h_{i+1} \dots h_n)\right) - \alpha_{i+1}^+ - \cdots - \alpha_n^+.$$

For every $h = (h_1, \dots, h_n) \in \mathrm{Tame}(\Bbbk^n)$ the linear parts of the polynomials $h_i, \dots, h_n$ are independent forms, because otherwise h would not be invertible. By Lemma 2.3, we have

$$-\nu(h_i h_{i+1} \dots h_n) = -\nu(h_i) - \cdots - \nu(h_n) \geqslant \alpha_i^+ + \cdots + \alpha_n^+$$

and the equality is achieved for $h \in S_n$ a permutation such that $h(\alpha) = \alpha^+$. We conclude that $\gamma_i = \alpha_i^+$. $\square$

We point out three immediate consequences:

Corollary 2.5.

(i) *There is a well-defined map $\rho_+ \colon \mathbf{X} \to \nabla^+$, which sends $\nu_{f,[\alpha]}$ to $[\alpha]^+$, and which is therefore a bijection in restriction to each chamber $\mathbf{E}_f^+$.*

(ii) *The standard chamber $\mathbf{E}_{\mathrm{id}}^+$ is a fundamental domain for the action of* $\mathrm{Tame}(\Bbbk^n)$ *on* $\mathbf{X}$*, i.e. any orbit meets $\mathbf{E}_{\mathrm{id}}^+$ at exactly one point.*

(iii) *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \Pi^+$ and $f = (f_1, \dots, f_n) \in \mathrm{Tame}(\Bbbk^n)$. If $f \cdot \nu_{\mathrm{id},[\alpha]} = \nu_{\mathrm{id},[\alpha]}$, then $f \cdot \nu_{\mathrm{id},\alpha} = \nu_{\mathrm{id},\alpha}$.*

Proof. We explain the argument for (iii): if $f \cdot \nu_{\mathrm{id},[\alpha]} = \nu_{\mathrm{id},[\alpha]}$ then $\nu_{f,\alpha} = f \cdot \nu_{\mathrm{id},\alpha} = \nu_{\mathrm{id},t\alpha}$ for some real $t > 0$. Proposition 2.4 gives $\alpha^+ = t\alpha^+$, hence $t = 1$. $\square$

Remark 2.6. For each apartment $\mathbf{E}_f$ and each choice of $\sigma \in S_n$, we have a map

$$\mathbf{E}_f \mapsto \nabla$$
$$\nu_{f\sigma,[\alpha]} \to [\alpha]$$

However, it is not possible to make coherent choices that would extend this application to $\mathbf{X}$.

For example, in dimension $n = 2$, it is already not possible to extend this application to the union of the three apartments $\mathbf{E}_{\mathrm{id}} \cup \mathbf{E}_f \cup \mathbf{E}_g$ for $f = (x_2, x_1 + x_2)$ and $g = (x_1 + x_2, x_1)$. Intuitively, this comes from the fact that these apartments form a “tripod”. Formally, each of these apartments contains two of the following three valuations:

$$\nu_{\mathrm{id},[1,2]} = \nu_{f,[2,1]}, \qquad \nu_{f,[1,2]} = \nu_{g,[2,1]}, \qquad \nu_{g,[1,2]} = \nu_{\mathrm{id},[2,1]},$$

and each should be sent to $[1,2]$ or $[2,1]$. However, there is no mapping from a set of cardinality 3 to a set of cardinality 2 which is injective on each pair.

We will propose in Lemma 3.5 a partial remedy to this fact.

We will denote by $\mathrm{Fix}(f)$ the subset of $\mathbf{X}$ fixed by f. Since we never consider the fixed points of f as an automorphism of the affine space $\Bbbk^n$, this notation should not lead to confusion.

Corollary 2.7. *Let $f \in \mathrm{Tame}(\Bbbk^n)$ and $\mathrm{Fix}(f) \subset \mathbf{X}$ be its set of fixed points. Then*

$$\mathbf{E}^+_f \cap \mathbf{E}^+_{\mathrm{id}} = \mathrm{Fix}(f) \cap \mathbf{E}^+_{\mathrm{id}}.$$

Proof. If $\nu_{\mathrm{id},[\alpha]} \in \mathrm{Fix}(f) \cap \mathbf{E}^+_{\mathrm{id}}$, we have

$$\nu_{f,[\alpha]} = f(\nu_{\mathrm{id},[\alpha]}) = \nu_{\mathrm{id},[\alpha]} \in \mathbf{E}^+_f \cap \mathbf{E}^+_{\mathrm{id}}.$$

Conversely any point $\nu_{f,[\alpha]} = \nu_{\mathrm{id},[\beta]} \in \mathbf{E}^+_f \cap \mathbf{E}^+_{\mathrm{id}}$ verifies $[\alpha] = [\alpha]^+ = [\beta]^+ = [\beta]$, where the second equality is Proposition 2.4. We therefore have $f(\nu_{\mathrm{id},[\alpha]}) = \nu_{f,[\alpha]} = \nu_{\mathrm{id},[\alpha]}$, in other words $\nu_{\mathrm{id},[\alpha]} \in \mathrm{Fix}(f) \cap \mathbf{E}^+_{\mathrm{id}}$ as expected. □

According to Corollary 2.7, in order to describe the intersections between chambers it is necessary to understand the places $\mathrm{Fix}(f) \cap \mathbf{E}^+_{\mathrm{id}}$, which we will do in the two following sections.

3. Stabilizers

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \Pi^+$. We are going to determine the stabilizer $\mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$ of the class $\nu_{\mathrm{id},[\alpha]}$ for the action of $\mathrm{Tame}(\Bbbk^n)$ on $\mathbf{X}$. The group of *triangular automorphisms*, which is a subgroup of $\mathrm{Tame}(\Bbbk^n)$, will play an important role here. Recall that an automorphism is said to be triangular if it has the form

$$(x_1, \dots, x_n) \mapsto (a_1 x_1 + P_1(x_2, \dots, x_n), \dots, a_i x_i + P_i(x_{i+1}, \dots, x_n), \dots, a_n x_n + c).$$

We write

$$\alpha = (\underbrace{\gamma_1, \dots, \gamma_1}_{m_1 \text{ fois}}, \dots, \underbrace{\gamma_r, \dots, \gamma_r}_{m_r \text{ fois}})$$

with $\gamma_1 > \dots > \gamma_r > 0$, so that $\sum_{i=1}^r m_i = n$. We now define two subgroups L_α and M_α of $\mathrm{Tame}(\Bbbk^n)$, which are also clearly subgroups of $\mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$.

First of all

$$L_\alpha \simeq \mathrm{GL}_{m_1}(\Bbbk) \times \dots \times \mathrm{GL}_{m_r}(\Bbbk)$$

is the subgroup of $\mathrm{GL}_n(\Bbbk)$ of block diagonal matrices of size m_i. In particular L_α contains the subgroup of diagonal matrices.

For any index $i = 1, \dots, n$, note $1 \leqslant b(i) \leqslant r$ the number of the block to which the index i belongs, i.e. such that $\alpha_i = \gamma_{b(i)}$. We define M_α as the subgroup of triangular automorphisms of the form

$$(x_1, \dots, x_n) \mapsto (x_1 + P_1, \dots, x_i + P_i, \dots, x_n + c),$$

where each P_i satisfies $\alpha_i \geqslant -\nu_\alpha(P_i)$ and only depends on variables x_j with $b(j) > b(i)$. In particular M_α contains the subgroup of translations. Note that the definitions of L_α and M_α only depend on the dilation class of α. Observe also that the group $\langle L_\alpha, M_\alpha \rangle$ always contains the subgroup of upper triangular matrices.

Example 3.1.

(i) If $\alpha = (1, \dots, 1)$, then $L_\alpha = \mathrm{GL}_n(\Bbbk)$ and M_α is the group of translations.

(ii) If $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > \dots > \alpha_n$, then L_α is the group of diagonal matrices, and M_α is the group of triangular automorphisms

$$(x_1, \dots, x_n) \mapsto (x_1 + P_1, \dots, x_i + P_i, \dots)$$

where the P_i satisfy $\alpha_i \geqslant -\nu_\alpha(P_i)$.

Note that in dimension $n = 2$, the two previous examples cover all possible cases. In dimension $n = 3$, there are two other possibilities (here we use the notation $\deg P$ for the ordinary degree of a polynomial P, i.e. each variable is assigned the weight 1):

(iii) If $\alpha = (\alpha_1, 1, 1)$ with $\alpha_1 > 1$, then

$$L_\alpha = \{(ax_1, bx_2 + cx_3, b'x_2 + c'x_3)\} \subset \mathrm{GL}_3(\Bbbk),$$
$$M_\alpha = \{(x_1 + P(x_2, x_3), x_2 + d, x_3 + d');\ \alpha_1 \geqslant \deg P\}.$$

(iv) If $\alpha = (\alpha_1, \alpha_1, 1)$ with $\alpha_1 > 1$, then

$$L_\alpha = \{(ax_1 + bx_2, a'x_1 + b'x_2, cx_3)\} \subset \mathrm{GL}_3(\Bbbk),$$
$$M_\alpha = \{(x_1 + P(x_3), x_2 + Q(x_3), x_3 + d);\ \alpha_1 \geqslant \deg P, \deg Q\}.$$

Proposition 3.2. *Or $\nu_{\mathrm{id},[\alpha]} \in \mathbf{E}^+_{\mathrm{id}}$. Then the stabilizer of $\nu_{\mathrm{id},[\alpha]}$ for the action of* $\mathrm{Tame}(\Bbbk^n)$ *on* $\mathbf{X}$ *is the semi-direct product of M_α and L_α:*

$$\mathrm{Stab}(\nu_{\mathrm{id},[\alpha]}) = M_\alpha \rtimes L_\alpha.$$

Proof. Consider $f = (f_1, \dots, f_n) \in \mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$. Thanks to Corollary 2.5(iii), we have $f \in \mathrm{Stab}(\nu_{\mathrm{id},\alpha})$. By composing by a translation, which is an element of M_α, we can assume that the f_i have no constant term. For each index i, we write $f_i = \ell_i + P_i$, with ℓ_i linear and P_i whose monomials are of degree at least 2. The condition

$$\alpha_i = -\nu_{\mathrm{id},\alpha}(x_i) = -(f \cdot \nu_{\mathrm{id},\alpha})(x_i) = -\nu_{\mathrm{id},\alpha}(\ell_i + P_i) \geqslant -\nu_{\mathrm{id},\alpha}(P_i)$$

implies that P_i only depends on the variables x_j satisfying $\alpha_i > \alpha_j$. So by composing by an element of M_α we can reduce ourselves to the case where all the P_i are zero. We thus obtain an element of $\mathrm{GL}_n(\Bbbk) \cap \mathrm{Stab}(\nu_{\mathrm{id},\alpha})$, which must be triangular by blocks of size $m_i \times m_j$. In particular, we can write such a matrix as the composite of an element of L_α and of $\mathrm{GL}_n(\Bbbk) \cap M_\alpha$.

We conclude that $\mathrm{Stab}(\nu_{\mathrm{id},[\alpha]}) = \langle M_\alpha, L_\alpha \rangle$. By construction $M_\alpha \cap L_\alpha = \{\mathrm{id}\}$, and the fact that M_α is normalized by L_α is an immediate computation. □

Corollary 3.3. *The stabilizer in* $\mathrm{Tame}(\Bbbk^n)$ *of the standard chamber $\mathbf{E}^+_{\mathrm{id}}$ is the semi-direct product of the group of translations and the group of upper triangular matrices. In particular, $f \in$* $\mathrm{Tame}(\Bbbk^n)$ *fixes every point of $\mathbf{E}_{\mathrm{id}}$ if and only if f is of the form*

$$f = (c_1 x_1 + t_1, c_2 x_2 + t_2, \dots).$$

Proof. Let $f \in \mathrm{Stab}(\mathbf{E}^+_{\mathrm{id}})$, and $\alpha \in \Pi^+$. By Corollary 2.5(iii), we have $\nu_{f,\alpha} = f(\nu_{\mathrm{id},\alpha}) = \nu_{\mathrm{id},\beta}$ for some $\beta \in \Pi^+$, and by Proposition 2.4, $\alpha = \alpha^+ = \beta^+ = \beta$. In other words f fixes pointwise the elements of $\mathbf{E}^+_{\mathrm{id}}$. In particular f fixes the monomial valuation of weight $(1, \dots, 1)$, which corresponds to Example 3.1(i). By composing by a translation, we can therefore assume $f \in \mathrm{GL}_n(\Bbbk)$. If we then consider α such that $\alpha_1 > \dots > \alpha_n > 0$, then $f \in \mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$ and falls under Example 3.1(ii). As we have just seen that f is linear, we obtain that f is upper triangular. Thus $\mathrm{Stab}(\mathbf{E}^+_{\mathrm{id}})$ is the triangular affine group, and therefore equal to the expected semi-direct product.

The second assertion is obtained by conjugating by elements of the symmetric group. □

We can use Corollary 3.3 to prove the faithfullness of the action of $\mathrm{Tame}(\Bbbk^n)$ on $\mathbf{X}$. We defer the proof to the appendix (Proposition 9.1) because this statement is not necessary for the proof of Theorem A.

We call *face* of ∇^+ any subset of ∇^+ defined by some equalities of the form $\alpha_k = \alpha_{k+1}$. A *face* of $\mathbf{X}$ is a set of valuations $\nu_{f,[\alpha]}$, for some fixed $f \in \mathrm{Tame}(\Bbbk^n)$ and $[\alpha]$ in a fixed face of ∇^+.

Let Δ be the poset of the left cosets of $\mathrm{GL}_n(\Bbbk)$ with respect to the standard parabolic subgroups. We recall following [AB08, Definition 6.32] that Δ is the spherical building $\Delta(G, B)$ for $G = \mathrm{GL}_n(\Bbbk)$ and B the subgroup of upper triangular matrices.

Lemma 3.4. *The poset of the faces of $\mathbf{X}$ containing $\nu_{\mathrm{id},[1,\dots,1]}$ is isomorphic to Δ.*

Proof. By Corollary 3.3 the subgroup $\mathrm{GL}_n(\Bbbk) \subset \mathrm{Stab}(\nu_{\mathrm{id},[1,\dots,1]})$ acts transitively on the faces of $\mathbf{X}$ containing $\nu_{\mathrm{id},[1,\dots,1]}$. Moreover, by Proposition 3.2, the stabilizers in $\mathrm{GL}_n(\Bbbk)$ of the faces of $\mathbf{X}$ contained in $\mathbf{E}^+_{\mathrm{id}}$ are the standard parabolic subgroups of $\mathrm{GL}_n(\Bbbk)$. Thus the faces of $\mathbf{X}$ containing $\nu_{\mathrm{id},[1,\dots,1]}$ correspond to the left cosets of $\mathrm{GL}_n(\Bbbk)$ with respect to the standard parabolic subgroups. □

We denote by $\mathrm{Tame}_0(\mathbb{k}^n) \subset \mathrm{Tame}(\mathbb{k}^n)$ the subgroup of tame automorphisms fixing the origin of $\mathbb{k}^n$. Any automorphism $f \in \mathrm{Tame}(\mathbb{k}^n)$ is written in the form $f = f_0 \circ t$, where $f_0 \in \mathrm{Tame}_0(\mathbb{k}^n)$ and t is the translation which sends $f^{-1}(0)$ to 0. By Corollary 3.3, the translations fix pointwise the standard appartment $\mathbf{E}_{\mathrm{id}}$, so we have $\mathbf{E}_f = \mathbf{E}_{f_0}$. Thus $\mathbf{X}$ is covered by the $\mathbf{E}_{f_0}$ with $f_0 \in \mathrm{Tame}_0(\mathbb{k}^n)$.

Let $\mathrm{Diff}\colon \mathrm{Tame}_0(\mathbb{k}^n) \to \mathrm{GL}_n(\mathbb{k})$ be the differential at the origin, or in other words the homomorphism which forgets all the terms of degree > 1 in $x_1, \dots, x_n$. Moreover, for each $a \in \mathrm{GL}_n(\mathbb{k})$ the Bruhat decomposition [Bou68, IV.2] gives a unique $\sigma_a \in S_n$ such that $a \in B_n \sigma_a B_n$, where $B_n \subset \mathrm{GL}_n(\mathbb{k})$ is the subgroup of upper triangular matrices. For each $f \in \mathrm{Tame}_0(\mathbb{k}^n)$, denote $\sigma_f := \sigma_{\mathrm{Diff}(f)}$.

Lemma 3.5.

(i) *The map* $\rho\colon \mathbf{X} \to \nabla$ *which to any* $\nu = \nu_{f,[\alpha]} \in \mathbf{X}$ *with* $f \in \mathrm{Tame}_0(\mathbb{k}^n)$ *and* $\alpha \in \Pi^+$ *associates* $\rho(\nu) = [\sigma_f(\alpha)] \in \nabla$ *is well defined.*

(ii) *The application* ρ_+ *of Corollary 2.5*(i) *factors by* ρ*, in the sense that the following diagram commutes* (*we always assume* $f \in \mathrm{Tame}_0(\mathbb{k}^n)$ *and* $\alpha \in \Pi^+$)*:*

$$\begin{tikzcd} \nu_{f,[\alpha]} \in \mathbf{X} \arrow[rr, "\rho_+"] \arrow[dr, "\rho"'] & & \nabla^+ \ni [\alpha]^+ = [\sigma_f(\alpha)]^+ \\ & {[\sigma_f(\alpha)] \in \nabla} \arrow[ur] & \end{tikzcd}$$

(iii) *Let* $f \in \mathrm{Tame}_0(\mathbb{k}^n)$ *such that* $\mathrm{Diff}(f) \in B_n$*. Then for all* $\alpha \in \Pi$ *we have* $\rho(\nu_{f,[\alpha]}) = [\alpha]$.

Proof. (i) Let $f, g \in \mathrm{Tame}_0(\mathbb{k}^n)$ and $\alpha, \beta \in \Pi^+$ such that $\nu_{f,[\alpha]} = \nu_{g,[\beta]}$. By Proposition 2.4 we have $[\alpha] = [\beta]$. We now have to show that $\sigma_g = \sigma_f \circ \sigma$ for a permutation $\sigma \in S_n$ satisfying $\sigma(\alpha) = \alpha$. We therefore denote by $\Sigma \subset S_n$ the subgroup of permutations fixing α. Observe that since $\alpha \in \Pi^+$, the group Σ is generated by a subset of the standard generators $\sigma_i = (i, i{+}1) \in S_n$.

By Proposition 3.2, $f^{-1}g \in M_\alpha \rtimes L_\alpha$. Note that $\mathrm{Diff}(M_\alpha \cap \mathrm{Tame}_0(\mathbb{k}^n)) \subset B_n$. Moreover, by considering a Bruhat decomposition for each block we obtain $L_\alpha \subset B_n \Sigma B_n$. Thus $\mathrm{Diff}(g) \in \mathrm{Diff}(f) B_n \Sigma B_n$, therefore $\mathrm{Diff}(g) \in B_n \sigma_f B_n \sigma_{i_1} \cdots \sigma_{i_k} B_n$, where the $\sigma_{i_j} \in \Sigma$ are standard generators. By one of the axioms of Tits systems (the axiom (T3) in [Bou68, IV.2]), there exists a sub-product σ of the product $\sigma_{i_1} \cdots \sigma_{i_k}$ such that $\mathrm{Diff}(g) \in B_n \sigma_f \sigma B_n$. Thus $\sigma_g = \sigma_f \sigma$, as expected.

(ii) This point is immediate, since by definition for any weight α and any permutation σ we have $[\alpha]^+ = [\sigma(\alpha)]^+$.

(iii) The assumption $\mathrm{Diff}(f) \in B_n$ implies $\sigma_f = \mathrm{id}$. Given $\alpha \in \Pi$, write $\alpha = \sigma(\alpha^+)$ with $\alpha^+ \in \Pi^+, \sigma \in S_n$.

We have $\mathrm{Diff}(f\sigma) = \mathrm{Diff}(f)\sigma \in B_n \sigma$, and therefore $\sigma_{f\sigma} = \sigma$. Thus by Lemma 2.1,

$$\rho(\nu_{f,[\alpha]}) = \rho(\nu_{f,[\sigma(\alpha^+)]}) = \rho(\nu_{f\sigma,[\alpha^+]}) = [\sigma(\alpha^+)] = [\alpha],$$

as expected. □

Corollary 3.6. *Let* $f \in \mathrm{Tame}(\mathbb{k}^n)$ *such that* $\mathbf{E}_f$ *shares a point with the interior of* $\mathbf{E}^+_{\mathrm{id}}$*. Then there exists* $\sigma \in S_n$ *such that* $\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}} = \mathrm{Fix}(f\sigma) \cap \mathbf{E}_{\mathrm{id}}$

Proof. Up to composing f on the right by a translation, we can assume $f \in \mathrm{Tame}_0(\mathbb{k}^n)$. For each $\sigma \in S_n$ we have $\mathbf{E}_f = \mathbf{E}_{f\sigma}$, hence the inclusion $\mathrm{Fix}(f\sigma) \cap \mathbf{E}_{\mathrm{id}} \subset \mathbf{E}_{f\sigma} \cap \mathbf{E}_{\mathrm{id}} = \mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}}$. For the converse, by hypothesis there exists $[\alpha']$ inside ∇^+ with $\nu_{\mathrm{id},[\alpha']} \in \mathbf{E}_f$. By Proposition 2.4 and Lemma 2.1,

$$\nu_{\mathrm{id},[\alpha']} = \nu_{f,[\sigma(\alpha')]} = \nu_{f\sigma,[\alpha']}$$

for some $\sigma \in S_n$. By Proposition 3.2 and Example 3.1(ii), $\mathrm{Diff}(f\sigma) \in B_n$, we can therefore apply Lemma 3.5(iii) to $f\sigma$. So for each $\nu = \nu_{f\sigma,[\alpha]} \in \mathbf{E}_f$ we have $\rho(\nu) = [\alpha]$. If moreover $\nu \in \mathbf{E}_{\mathrm{id}}$, again by Lemma 3.5(iii) applied to id we have $\nu = \nu_{\mathrm{id},[\alpha]}$. Thus $\nu \in \mathrm{Fix}(f\sigma)$ as expected. □

4. Fixed Points

4.1. **Admissible equations.** We will say that a weight $\alpha \in \Pi$ satisfies an *admissible equation* if there exist integers $m_j \geqslant 0$ not all zero and an index i such that

$$\alpha_i = \sum_{j \neq i} m_j \alpha_j. \tag{1}$$

We call *admissible hyperplane* associated with such an equation the set of $\alpha \in \Pi$ satisfying the equation, and *admissible halfspace* the set of α satisfying the inequality

$$\alpha_i \geqslant \sum_{j \neq i} m_j \alpha_j.$$

In particular, if the admissible equation is not of the form $\alpha_i = \alpha_k$, then the admissible half-space is the half-space not containing $(1, \ldots, 1)$. We will also use the terminology of *hyperplanes* and *admissible halfspaces* for their projectivizations in ∇.

For example, when $n = 2$ the admissible hyperplanes of ∇ are the points of the form $[p, 1]$ and $[1, p]$, where $p \geqslant 1$ is an integer, and the admissible half-spaces are the half-open intervals of $[p, 1]$ towards the point at infinity $[1, 0]$, or from $[1, p]$ to $[0, 1]$. When $n = 3$ the admissible hyperplanes in ∇ are line segments that we will call *admissible lines* (see Figure 1).

Among the admissible hyperplanes, in Π as well as in ∇, we call *principal hyperplane* any hyperplane with an equation of the form $\alpha_i = m_k \alpha_k$, for some indices $i \neq k$. In other words in the equation (1) we ask that all m_j except one to be zero. Geometrically the principal hyperplanes in ∇ are exactly the admissible hyperplanes passing through $n - 2$ vertices of the simplex.

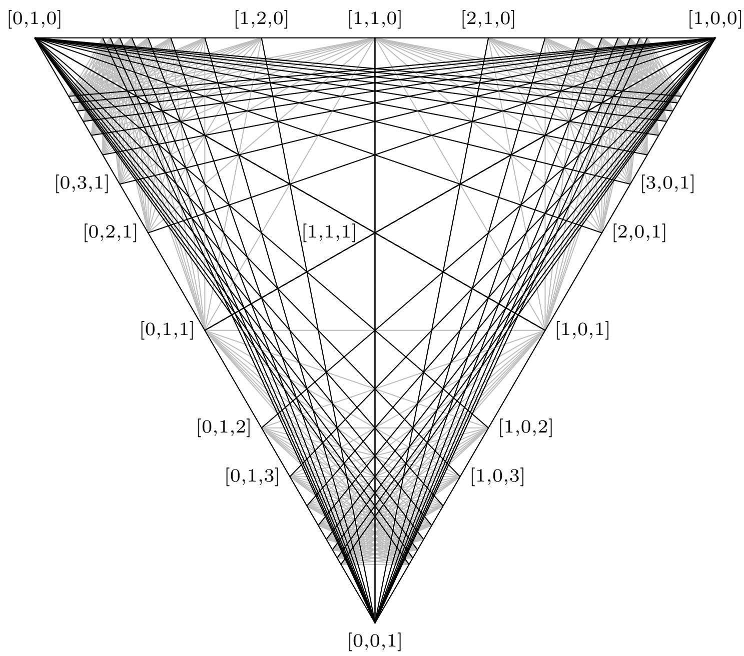

Figure 1. A few admissible lines, principal or not, in the case $n = 3$.

Lemma 4.1. *For any compact $K \subset \nabla$, the set of admissible hyperplanes meeting K is finite.*

Proof. For any integer $p \geqslant 1$, let

$$K_p = \left\{ [\alpha];\ p \geqslant \frac{\alpha_1^+}{\alpha_n^+} \right\}.$$

As K is compact, the function $[\alpha] \mapsto \frac{\alpha_1^+}{\alpha_n^+}$ is bounded on K, so it is sufficient to prove the lemma for $K = K_p$. Now this follows from the observation that if $\sum_{j \neq i} m_j > p$ in (1), then for all $[\alpha] \in K_p$

$$\sum_{j \neq i} m_j \alpha_j > p\alpha_n^+ \geqslant \alpha_1^+ \geqslant \alpha_i,$$

and therefore the corresponding admissible hyperplane does not meet K_p. □

For $\alpha \in \Pi$ we define the *multiplicity* $\operatorname{mult}(\alpha) = \operatorname{mult}([\alpha])$ as the number of admissible equations satisfied by α. As an immediate consequence of Lemma 4.1, we have $\operatorname{mult}([\alpha]) < \infty$ for all $[\alpha] \in \nabla$. We also get:

Remark 4.2. Let $[\alpha] \in \nabla$, and denote by $\mathcal{H}_{[\alpha]} \subset \nabla$ the union of all admissible hyperplanes not passing through $[\alpha]$. Let U denote the connected component containing $[\alpha]$ of $\nabla \smallsetminus \mathcal{H}_{[\alpha]}$. Then by Lemma 4.1 U is a neighborhood of $[\alpha]$ in ∇, which by construction intersects no other admissible hyperplane than those passing through $[\alpha]$.

Remark 4.3. For any $L \subset \nabla$ an admissible half-space, there exists $g \in \operatorname{Tame}(\mathbb{k}^n)$ such that $g(\nu_{\mathrm{id},[\alpha]}) = \nu_{\mathrm{id},[\alpha]}$ if and only if $[\alpha] \in L$. Indeed, up to composing by a permutation, we can assume that the inequality defining L is of the form $\alpha_1 \geqslant \sum_{i \geqslant 2} m_i \alpha_i$, and then $g = (x_1 + x_2^{m_2} \dots x_n^{m_n}, x_2, \dots, x_n)$ works.

More generally, the admissible half-spaces allow to characterize the locus fixed by an automorphism.

Proposition 4.4. *Let $f = (f_1, \dots, f_n) \in \operatorname{Tame}(\mathbb{k}^n)$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \Pi$. Then $\nu_{\mathrm{id},[\alpha]} \in \operatorname{Fix}(f) \cap \mathbf{E}_{\mathrm{id}}$ if and only if α satisfies every inequality of the form*

$$\alpha_i \geqslant \sum_{k=1}^{r} m_{j_k} \alpha_{j_k},$$

where $i = 1, \dots, n$ and $x_{j_1}^{m_{j_1}} \dots x_{j_r}^{m_{j_r}}$ is a monomial (distinct from x_i) occurring in the polynomial f_i.

Proof. We are interested in the homothety classes $\nu_{\mathrm{id},[\alpha]}$ fixed by f, which by Corollary 2.5(iii) amounts to finding the valuations $\nu_{\mathrm{id},\alpha}$ fixed by f.

Suppose the valuation $\nu = \nu_{\mathrm{id},\alpha}$ is fixed by f, which means that for each i we have $-\nu(f_i) = \alpha_i$. In particular for each monomial $x_{j_1}^{m_{j_1}} \dots x_{j_r}^{m_{j_r}}$ distinct from x_i appearing in f_i we must have

$$\alpha_i = -\nu(f_i) \geqslant -\nu(x_{j_1}^{m_{j_1}} \dots x_{j_r}^{m_{j_r}}) = \sum_{k=1}^{r} m_{j_k} \alpha_{j_k}.$$

This implies that $i \notin \{j_1, \dots, j_r\}$, and gives the expected list of admissible equations.

Conversely, suppose that α satisfies all these inequalities. Note ℓ_i the linear part of the component f_i, so we have $(\ell_1, \dots, \ell_n) \in \mathrm{GL}_n(\mathbb{k})$ and $-\nu(f_i) \geqslant -\nu(\ell_i)$ for each i. By Lemma 2.3, for any valuation $\nu = \nu_{\mathrm{id},\alpha}$ we have

$$-\nu(\ell_1) - \dots - \nu(\ell_n) \geqslant \alpha_1 + \dots + \alpha_n.$$

The fact that α_i satisfies the inequalities given by the monomials of f_i gives $\alpha_i \geqslant -\nu(f_i)$ for all i. We get $\alpha_i \geqslant -\nu(\ell_i)$ for all i, and if one of these inequalities were strict, then we would get a contradiction

$$\alpha_1 + \dots + \alpha_n > -\nu(\ell_1) - \dots - \nu(\ell_n) \geqslant \alpha_1 + \dots + \alpha_n.$$

We conclude that $\alpha_i = -\nu(f_i) = -\nu(\ell_i)$ for all i, and therefore f fixes $\nu_{\mathrm{id},\alpha}$ as expected. □

By Proposition 4.4, for each linear or elementary automorphism $f \in \operatorname{Tame}(\mathbb{k}^n)$ we have $\mathbf{E}_f \cap \mathbf{E}_{\mathrm{id}} \neq \emptyset$. Since $\operatorname{Tame}(\mathbb{k}^n)$ is generated by the linear and elementary automorphisms, we immediately obtain:

Corollary 4.5. *For all $f, g \in \operatorname{Tame}(\mathbb{k}^n)$, there is a sequence $f_0, \dots, f_k \in \operatorname{Tame}(\mathbb{k}^n)$ such that $f = f_0$, $g = f_k$, and $\mathbf{E}_{f_{i-1}} \cap \mathbf{E}_{f_i} \neq \emptyset$ for each $i = 1, \dots, k$.*

Example 4.6. If $g = (x_1 + x_2^3, x_2)$ and $h = (x_1, x_2 + x_1^2)$, then the apartments $\mathbf{E}_h$, $\mathbf{E}_{\mathrm{id}}$, $\mathbf{E}_g$ and $\mathbf{E}_{gh}$ form a sequence of apartments of $\mathbf{X}_2$ such that each pair of consecutive apartments intersect, see Figure 2. We will establish later in Section 6.2 that, as suggested by the figure, $\mathbf{X}_2$ admits a tree structure.

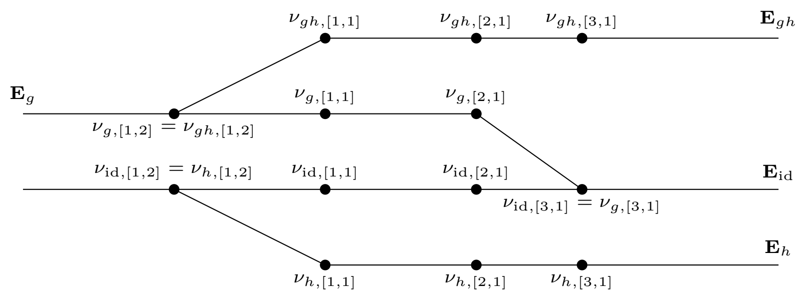

FIGURE 2. Four appartments in $\mathbf{X}_2$

Proposition 4.4 also has the following consequences:

Corollary 4.7. *Let $\alpha', \alpha'' \in \Pi$. Suppose that every admissible half-space containing α' contains α''. Then the stabilizer in* $\mathrm{Tame}(\Bbbk^n)$ *of $\nu_{\mathrm{id},[\alpha']}$ is contained in the stabilizer of $\nu_{\mathrm{id},[\alpha'']}$.*

Proof. Let $f \in \mathrm{Tame}(\Bbbk^n)$. By Proposition 4.4, $\mathrm{Fix}(f) \cap \mathbf{E}_{\mathrm{id}} = \{\nu_{\mathrm{id},[\alpha]}\}$ where $[\alpha] \in L_1 \cap \ldots \cap L_k$, intersection of admissible half-spaces of ∇. If $\nu_{\mathrm{id},[\alpha']} \in \mathrm{Fix}(f)$, then $[\alpha'] \in L_i$ for each $i = 1, \ldots, k$. By assumption, $[\alpha''] \in L_i$ for each $i = 1, \ldots, k$. So $\nu_{\mathrm{id},[\alpha'']} \in \mathrm{Fix}(f)$, as expected. □

Corollary 4.8. *Let $n = 3$ and $\alpha\colon (t_0, \infty) \to \mathrm{int}(\Pi^+)$ be a curve along which $\frac{\alpha_1}{\alpha_2}$ and $\frac{\alpha_2}{\alpha_3}$ are non-decreasing. Then the stabilizers in* $\mathrm{Tame}(\Bbbk^n)$ *of the points $\nu_{\mathrm{id},[\alpha(t)]}$ form an increasing family of groups.*

Proof. Let $t \in (t_0, \infty)$ and $f \in \mathrm{Tame}(\Bbbk^n)$ be such that $\nu_{\mathrm{id},[\alpha(t)]} \in \mathrm{Fix}(f)$. If L is an admissible half-space in ∇ containing $[\alpha(t)]$, since $\alpha_1(t) > \alpha_2(t) > \alpha_3(t)$, the inequality defining L is of the form $\alpha_2 \geqslant m_3\alpha_3$ or $\alpha_1 \geqslant m_2\alpha_2 + m_3\alpha_3$. Since $\frac{\alpha_1}{\alpha_2}$ and $\frac{\alpha_2}{\alpha_3}$ are non-decreasing along $\alpha(t)$, in all cases $[\alpha(t')] \in L$ for $t' \geqslant t$. By Corollary 4.7, we obtain $\nu_{\mathrm{id},[\alpha(t')]} \in \mathrm{Fix}(f)$. □

4.2. **Around** $[m, p, 1]$**.** We now place ourselves in dimension $n = 3$, and we will study more precisely the intersections of apartments around a valuation $\nu_{\mathrm{id},[\alpha]}$ of weight $\alpha = (m, p, 1)$ with $m \geqslant p \geqslant 1$ two integers.

Remark 4.9. Let $p \geqslant 1$ be an integer and $\alpha \in \Pi^+$ with $\alpha_2 = p\alpha_3$ and $\mathrm{mult}(\alpha) \geqslant 2$. Every other admissible equation for α is of the form $\alpha_1 = m_2\alpha_2 + m_3\alpha_3$, which results in $\alpha_1 = m_2p\alpha_3 + m_3\alpha_3 = (m_2p + m_3)\alpha_3$. So $\alpha = [m, p, 1]$ for $m = m_2p + m_3$.

Note that there are weights $\alpha \in \Pi^+$ with $\mathrm{mult}(\alpha) \geqslant 2$ which are not of this form, for example $(6, 3, 2)$ satisfying $\alpha_1 = 2\alpha_2$ and $\alpha_1 = 3\alpha_3$, or $(11, 3, 2)$ satisfying $\alpha_1 = 3\alpha_2 + \alpha_3$ and $\alpha_1 = \alpha_2 + 4\alpha_3$.

Remark 4.10. Fix $m \geqslant p \geqslant 1$ two integers, and denote $m = pq + r$ the Euclidean division of m by p. The weight $\alpha = (m, p, 1)$ satisfies exactly $q + 2$ admissible equations, which are $\alpha_2 = p\alpha_3$ and $\alpha_1 = a\alpha_2 + (m - pa)\alpha_3$, $a = 0, \ldots, q$. These equations correspond respectively to the directions from $[m, p, 1]$ to $[1, 0, 0]$ (or $[0, p, 1]$) and to $[m - pa, 0, 1]$ (or $[a, 1, 0]$).

Let α in the interior of Π^+ and $U \subset \nabla^+$ be the neighborhood of $[\alpha]$ as in Remark 4.2. Let $f, g \in \mathrm{Tame}(\Bbbk^n)$ with $\nu := \nu_{f,[\alpha]} = \nu_{g,[\alpha]}$. Note $U_{f,g} := \{[\alpha'] \in U;\ \nu_{f,[\alpha']} = \nu_{g,[\alpha']}\} \subset \nabla^+$. We will say that $\mathbf{E}_f$ and $\mathbf{E}_g$ are *locally equivalent* in ν, denoted $f \sim_\nu g$, if $U_{f,g} = U$.

If $\nu = \nu_{\mathrm{id},[\alpha]}$ with $\alpha = (m, p, 1)$, $m > p > 1$, recall that $M_\alpha \subset \mathrm{Stab}(\nu)$ is the subgroup of automorphisms

$$(x_1 + P(x_2, x_3), x_2 + Q(x_3), x_3 + d)$$

with $-\nu(P) \leqslant m$, $-\nu(Q) = \deg Q \leqslant p$. We then define $N_\alpha \subset M_\alpha$ as the subgroup of automorphisms with $-\nu(P) < m$, $\deg Q < p$.

Lemma 4.11. *Let $\nu = \nu_{\mathrm{id},[\alpha]}$ of weight $\alpha = (m,p,1)$ with $m > p > 1$.*

(i) *For each $f \in \mathrm{Stab}(\nu)$ there exists $g \in M_\alpha$ such that $\mathbf{E}^+_f = \mathbf{E}^+_g$, and therefore in particular $f \sim_\nu g$.*
(ii) *Let $h \in M_\alpha$. Then $\mathrm{id} \sim_\nu h$ if and only if $h \in N_\alpha$.*
(iii) *N_α is normal in M_α. In particular for $f, g \in M_\alpha$ we have $f \sim_\nu g$ if and only if f and g become equal in the quotient M_α/N_α.*

Proof. (i) We are in the context of Example 3.1(ii), so the Proposition 3.2 gives

$$f = (x_1 + P(x_2,x_3), x_2 + Q(x_3), x_3 + d) \circ (a_1x_1, a_2x_2, a_3x_3) \in M_\alpha \rtimes L_\alpha,$$

with $-\nu(P) \leqslant m$, $\deg Q \leqslant p$. By Corollary 3.3 we have $\mathbf{E}^+_{\mathrm{id}} = \mathbf{E}^+_a$ where $a = (a_1x_1, a_2x_2, a_3x_3)$. We conclude by taking $g = (x_1 + P(x_2,x_3), x_2 + Q(x_3), x_3 + d)$.

(ii) This point follows from Proposition 4.4, noting that $h \in N_\alpha$ if and only if the weight α is contained in the interior of each admissible half-space associated with a monomial of one of the components of h.

(iii) consider

$$\begin{aligned} f &= \big(x_1 + P(x_2,x_3), x_2 + Q(x_3), x_3 + d\big) \in M_\alpha, \\ g &= \big(x_1 + P'(x_2,x_3), x_2 + Q'(x_3), x_3 + d'\big) \in N_\alpha, \\ f^{-1} &= \Big(x_1 - P\big(x_2 - Q(x_3 - d), x_3 - d\big), x_2 - Q(x_3 - d), x_3 - d\Big). \end{aligned}$$

We set

$$\begin{aligned} Q''(x_3) &= Q(x_3) - Q(x_3 + d'), \\ P''(x_2,x_3) &= P(x_2,x_3) - P\big(x_2 + Q''(x_3) + Q'(x_3 + d), x_3 + d'\big). \end{aligned}$$

So

$$\begin{aligned} &f^{-1}gf \\ &= f^{-1}\Big(x_1 + P(x_2,x_3) + P'\big(x_2 + Q(x_3), x_3 + d\big), x_2 + Q(x_3) + Q'(x_3 + d), x_3 + d + d'\Big) \\ &= \Big(x_1 + P''(x_2,x_3) + P'\big(x_2 + Q(x_3), x_3 + d\big), x_2 + Q''(x_3) + Q'(x_3 + d), x_3 + d'\Big). \end{aligned}$$

Since $\deg Q' < p$ and $\deg Q'' < \deg Q \leqslant p$ the second component of $f^{-1}gf$ is as expected. Likewise, $-\nu(P') < m$ and $-\nu(P'') < -\nu(P) \leqslant m$, so the first component of $f^{-1}gf$ is as expected, and $f^{-1}gf \in N_\alpha$. □

A *sector centered at* $[\alpha]$ in ∇ is the intersection of two admissible half-spaces whose boundary lines pass through $[\alpha]$. The boundary of a sector is thus the union of two half-lines from $[\alpha]$, each directed towards a point at the boundary at infinity of ∇.

Lemma 4.12. *Let $\nu = \nu_{\mathrm{id},[\alpha]}$ of weight $\alpha = (m,p,1)$ with $m > p > 1$ and $f, g \in M_\alpha$. If $U_{f,g} \neq U$, then $U_{f,g}$ is the intersection of U with a sector $S_{f,g} \subset \nabla$ centered in $[\alpha]$, and $g \sim_\nu fh$ for a triangular automorphism h that we can choose of the following form, depending on the two half-lines of the sector $S_{f,g}$:*

(i) *If one of the ray from $S_{f,g}$ points to $[m - pa, 0, 1]$, then the other ray points to $[b, 1, 0]$ with $\lfloor \frac{m}{p} \rfloor \geqslant b \geqslant a \geqslant 0$, and*

$$h = (x_1 + \sum_{i=a}^{b} c_i x_2^i x_3^{m-pi}, x_2, x_3), \quad c_a \neq 0, c_b \neq 0.$$

(ii) *If one of the half-lines of $S_{f,g}$ points to $[0, p, 1]$, then $S_{f,g}$ is the half-space $\alpha_2/\alpha_3 \geqslant p$ and*

$$h = (x_1, x_2 + cx_3^p, x_3), \quad c \neq 0.$$

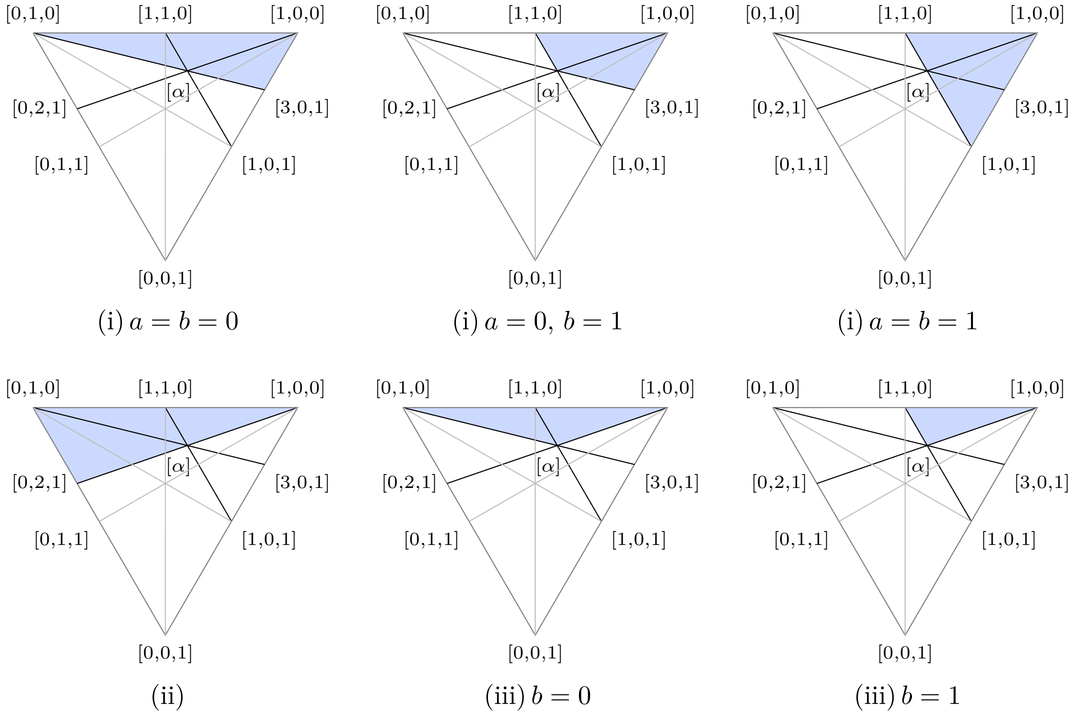

FIGURE 3. The 3 admissible lines in ∇ passing through $[\alpha] = [3,2,1]$, and the various possible sectors $S_{f,g}$.

(iii) *If one of the rays of* $S_{f,g}$ *points to* $[1,0,0]$ *and* $S_{f,g}$ *is not a half-space, then the other half-line points to* $[b,1,0]$ *with* $\lfloor \frac{m}{p} \rfloor \geqslant b \geqslant 0$ *and*

$$h = (x_1 + \sum_{i=0}^{b} c_i x_2^i x_3^{m-pi}, x_2 + c x_3^p, x_3), \quad c_b \neq 0, c \neq 0.$$

Proof. By Corollary 2.5(ii) we can assume $f = \mathrm{id}$. We write

$$g = (x_1 + P(x_2, x_3), x_2 + Q(x_3), x_3 + d),$$

with $-\nu(P) \leqslant m$, $\deg Q \leqslant p$. By Lemma 4.11, up to composing by an element of N_α we can suppose $d = 0$, Q homogeneous of degree p, and P homogeneous of degree m with the variables x_2, x_3 of respective weights p, 1. This gives the expected h: by Proposition 4.4 the three cases of the statement correspond respectively to (i) $Q = 0$, (ii)$P = 0$, and (iii) P and Q both nonzero. The case $Q = P = 0$ is excluded by the hypothesis $U_{\mathrm{id},g} \neq U$. □

Example 4.13 (Figure 3)**.** If $\alpha = (3,2,1)$, there are exactly 3 admissible lines passing through $[\alpha]$, corresponding to the equations:

$$\alpha_2 = 2\alpha_3; \qquad \alpha_1 = 3\alpha_3; \qquad \alpha_1 = \alpha_2 + \alpha_3.$$

Observe that equation $2\alpha_1 = 3\alpha_2$ is also satisfied by α, but by definition is not an admissible equation. For any choice of the coefficients c_i, the following automorphism is an element from $\mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$:

$$(x_1 + c_0 x_3^3 + c_1 x_2 x_3, x_2 + c_2 x_3^2, x_3).$$

By canceling some coefficients among c_i, we can realize each of the six sectors $S_{f,g}$ centered in $[\alpha]$ predicted by Lemma 4.12.

5. Metric

5.1. **Length space.** According to Corollary 2.5(ii), $\mathbf{X}$ is the union of the translates of $\mathbf{E}_{\mathrm{id}}^{+}$ under the action of Tame($\mathbb{k}^n$). We can therefore identify $\mathbf{X}$ with $\left(\bigsqcup \mathbf{E}_f^{+}\right)/\sim$, a disjoint union of copies of $\mathbf{E}_{\mathrm{id}}^{+}$ indexed by Tame($\mathbb{k}^n$), quotiented by an equivalence relation $\sim$. Here we commit the writing abuse of noting $\mathbf{E}_f^{+}$ both the chamber in $\mathbf{X}$ and its copy which we use in the abstract construction by disjoint union and quotient.

In order to endow $\mathbf{X}$ with a metric, we first focus on (each copy of) $\mathbf{E}_{\mathrm{id}}^{+}$ and more precisely, via the application ρ_+ of Corollary 2.5(i), on the simplex of projectivized weights ∇^+. We identify each $[\alpha] \in \nabla^+$ with its representative in Π^+ contained in the hyperboloid $\prod \alpha_i = 1$. Passing to logarithms $\beta_i = \log \alpha_i$, we get

$$\nabla^+ = \left\{\alpha = (\exp\beta_1, \dots, \exp\beta_n); \beta_1 \geqslant \dots \geqslant \beta_n, \sum \beta_i = 0\right\}.$$

We then endow ∇^+ with the metric $|\cdot,\cdot|$ induced by the Euclidean metric of $\mathbb{R}^n = \{(\beta_1, \dots, \beta_n)\}$. This makes it isometric to the Weyl chamber defined by the inequalities $\beta_1 \geqslant \dots \geqslant \beta_n$ in

$$\mathbb{R}^{n-1} = \left\{(\beta_1, \dots, \beta_n) \in \mathbb{R}^n;\ \sum \beta_i = 0\right\}.$$

Note that the same application of passage to logarithms identifies the entire simplex ∇ with $(\mathbb{R}^{n-1}, |\cdot,\cdot|)$, where $\mathbb{R}^{n-1}$ is the hyperplane of equation $\sum \beta_i = 0$ and $|\cdot,\cdot|$ still denotes the metric induced by the Euclidean metric of $\mathbb{R}^n$. This method of defining a metric by passing to the logarithms of the weights is the one used in [BT84, Par00] in the context of the building of $\mathrm{SL}_n(\mathbb{F})$. If we did not switch to logarithms, the total space obtained would no longer be locally CAT(0), see Example 7.13 below shows. Another natural choice *a priori* would be to equip the simplex ∇ with the Hilbert metric. However as soon as the boundary of a convex domain contains two line segments generating a plane, the Hilbert metric is not uniquely geodesic [Pap14, Theorem 5.6.8], and in particular is not CAT(0).

Lemma 5.1. *Every admissible half-space of ∇ is a convex domain under the identification with $(\mathbb{R}^{n-1}, |\cdot,\cdot|)$.*

Proof. Consider an admissible equation $\alpha_1 = \sum_{j>1} m_j\alpha_j$. The domain of the solutions of the inequality $\alpha_1 \geqslant \sum_{j>1} m_j\alpha_j$ in $(\mathbb{R}^{n-1}, |\cdot,\cdot|)$ being closed, to obtain its convexity it suffices to show that for each β, β' at the boundary of the domain, the point $\frac{\beta+\beta'}{2}$ is in the domain. For such $(\beta_i) = (\log\alpha_i)$, $(\beta'_i) = (\log\alpha'_i)$ we have $\alpha_1 = \sum_{j>1} m_j\alpha_j$ and $\alpha'_1 = \sum_{j>1} m_j\alpha'_j$. We must therefore check

$$\exp\left(\frac{\log(m_2\alpha_2 + \dots + m_n\alpha_n) + \log(m_2\alpha'_2 + \dots + m_n\alpha'_n)}{2}\right) \geqslant \\ m_2 \exp\left(\frac{\log\alpha_2 + \log\alpha'_2}{2}\right) + \dots + m_n \exp\left(\frac{\log\alpha_n + \log\alpha'_n}{2}\right),$$

that we can also write

$$\sqrt{(m_2\alpha_2 + \dots + m_n\alpha_n)(m_2\alpha'_2 + \dots + m_n\alpha'_n)} \geqslant \\ \sqrt{(m_2\alpha_2)(m_2\alpha'_2)} + \dots + \sqrt{(m_n\alpha_n)(m_n\alpha'_n)}.$$

By squaring, we get the equivalent inequality:

$$\frac{1}{2}\sum_{j,k}\left((m_j\alpha_j)(m_k\alpha'_k) + (m_k\alpha_k)(m_j\alpha'_j)\right) \geqslant \sum_{j,k}\sqrt{(m_j\alpha_j)(m_j\alpha'_j)(m_k\alpha_k)(m_k\alpha'_k)}.$$

The latter follows directly from the classic inequality of arithmetico-geometric metric means $\frac{x+y}{2} \geqslant \sqrt{xy}$. $\square$

Remark 5.2. Principal hyperplanes are again hyperplanes for the metric $|\cdot,\cdot|$. For $n = 3$ the principal lines in ∇ form 3 families of parallel lines intersecting with angles $\pi/3$. We represent on Figure 4 the principal straight lines in ∇ with on the left the metric of the simplex, and on the right the metric $(\mathbb{R}^2, |\cdot,\cdot|)$. Also represented, in red and blue, are two admissible but non-principal lines of ∇: their images in $\mathbb{R}^2$ become curves, and the admissible half-space becomes a convex set.

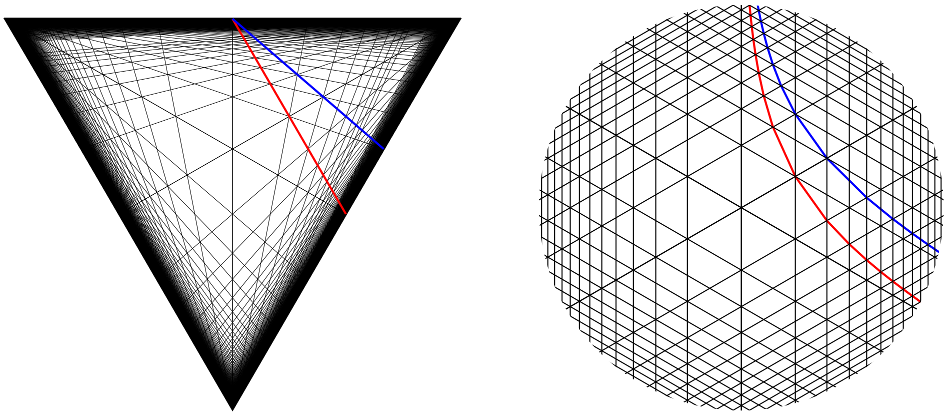

FIGURE 4. Homeomorphism between ∇ and $\mathbb{R}^2$.

Through the identification $[\alpha] \mapsto \nu_{f,[\alpha]}$ of ∇^+ with $\mathbf{E}_f^+$, we provide each chamber $\mathbf{E}_f^+$ with a metric that we also denote $|\cdot,\cdot|$. Now consider $\overline{x}, \overline{y} \in \mathbf{X} = \left(\bigsqcup \mathbf{E}_f^+\right)/\sim$. We define the metric $d_{\mathbf{X}}(\overline{x}, \overline{y})$ as follows:

Definition 5.3. A *chain* (of *length* k) is a sequence

$$(x_0', x_1 \sim x_1', x_2 \sim x_2', \dots, x_{k-1} \sim x_{k-1}', x_k)$$

in $\bigsqcup \mathbf{E}_f^+$ such that there is $f_0, \dots, f_{k-1} \in \mathrm{Tame}(\mathbb{k}^n)$ with $x_i', x_{i+1} \in \mathbf{E}_{f_i}^+$. The metric $d_{\mathbf{X}}(\overline{x}, \overline{y})$ is defined as the infimum of the sums $\sum_{i=0}^{k-1} |x_i', x_{i+1}|$, for all integers k and all chains of length k, with $x_0' \in \overline{x}$ (i.e. x_0' a representative of $\overline{x}$) and $x_k \in \overline{y}$. Following [BH99, I.5.19], we call $d_{\mathbf{X}}$ the *quotient pseudo-metric*.

Proposition 5.4. *The quotient pseudo-metric $d_{\mathbf{X}}$ is a metric, and $\mathbf{X}$ equipped with this metric is a length space.*

Proof. To show that $d_{\mathbf{X}}$ is a metric, according to [BH99, I.5.28] it suffices to verify the two properties below. Moreover, $\mathbf{X}$ will then be a length space by [BH99, I.5.20].

(i) For all $x, z \in \mathbf{E}_f^+, x', z' \in \mathbf{E}_{f'}^+$ such that $x \sim x'$ and $z \sim z'$, we have $|x, z| = |x', z'|$.
(ii) For any $\overline{x} \in \mathbf{X}$ there exists $\varepsilon = \varepsilon(\overline{x}) > 0$ such that $\bigcup_{x \in \overline{x}} B(x, \varepsilon)$ is a union of equivalence classes for the relation $\sim$, where $B(x, \varepsilon)$ denotes an open ball in each $(\mathbf{E}_f^+, |\cdot,\cdot|)$.

Property (i) is a direct consequence of Proposition 2.4.

To establish the property (ii), if $x = \nu_{f,[\alpha]} \in \mathbf{E}_f^+$, first choose $\varepsilon > 0$ small enough so that under the identification of $\mathbf{E}_f^+$ with $\nabla^+ \subset \nabla$, the neighborhood U of $[\alpha]$ from Remark 4.2 contains $B(x, \varepsilon)$. Thus $B(x, \varepsilon)$ does not intersect any other admissible hyperplane than those passing through $[\alpha]$. These are smooth in the coordinates of $(\mathbb{R}^{n-1}, |\cdot,\cdot|)$, and are finite in number again by Lemma 4.1, so up to decreasing ε, for each $y \in B(x, \varepsilon)$ there is a path from y to x in $B(x, \varepsilon)$ along which the multiplicity is constant, except perhaps at the end x where it can increase. Note that $\varepsilon = \varepsilon([\alpha])$ does not depend on f but only on $[\alpha]$, thanks to Proposition 2.4. Thus $\varepsilon(\overline{x})$ is well defined.

Now consider $y \in B(x, \varepsilon) \subset \mathbf{E}_f^+$, and let $y' \sim y$ with $y' \in \mathbf{E}_{f'}^+$. Using the action of $\mathrm{Tame}(\mathbb{k}^n)$ (and replacing f by $f'^{-1}f$) we can assume $f' = \mathrm{id}$. By Proposition 2.4, the relation $y' \sim y$

translates into $\overline{y} \in \mathrm{Fix}(f)$. By definition of ε, there is a path from y to x whose all points are contained in the same set of admissible hyperplanes (except perhaps x which is then contained in a larger set of hyperplanes). Then by Corollary 4.7, we have $\overline{x} \in \mathrm{Fix}(f)$, and therefore the point $x' \in \mathbf{E}^+_{\mathrm{id}}$ of weight $[\alpha]$ satisfies $x' \sim x$. Finally $y' \in \bigcup_{x\in\overline{x}} B(x,\varepsilon)$, and this set is therefore well saturated for the equivalence relation $\sim$. □

In the following lemma we keep the notations $B(z,\varepsilon)$ and $\overline{B}(z,\varepsilon)$ for the balls respectively open or closed in a chamber $\mathbf{E}^+_f$, and we use the notations $B_{\mathbf{X}}(\overline{z},\varepsilon)$, $\overline{B}_{\mathbf{X}}(\overline{z},\varepsilon)$ for the balls in $\mathbf{X}$ with respect to $d_{\mathbf{X}}$.

Lemma 5.5. *Let $\overline{z} \in \mathbf{X}$ and $\varepsilon = \varepsilon(\overline{z})$ satisfying property* (ii) *above. Let $\mathbf{V}(\overline{z},\varepsilon) = \bigsqcup_{z\in\overline{z}} B(z,\varepsilon)/\sim$ with the induced quotient pseudo-metric of $\bigsqcup_{z\in\overline{z}} B(z,\varepsilon)$. Similarly, let $\overline{\mathbf{V}}(\overline{z},\frac{\varepsilon}{4}) = \bigsqcup_{z\in\overline{z}} \overline{B}(z,\frac{\varepsilon}{4})/\sim$ with the induced quotient pseudo-metric of $\bigsqcup_{z\in\overline{z}} \overline{B}(z,\frac{\varepsilon}{4})$. So*

(a) *$\mathbf{V}(\overline{z},\varepsilon) \subset \mathbf{X}$ coincides as a set with $B_{\mathbf{X}}(\overline{z},\varepsilon)$, and*

(b) *$\overline{\mathbf{V}}(\overline{z},\frac{\varepsilon}{4})$ is isometric to the ball $\overline{B}_{\mathbf{X}}(\overline{z},\frac{\varepsilon}{4})$.*

Proof. According to [BH99, I.5.27(1,2)], the set $\mathbf{V}(\overline{z},\varepsilon) \subset \mathbf{X}$ coincides with $B_{\mathbf{X}}(\overline{z},\varepsilon)$ and the set $\overline{\mathbf{V}}(\overline{z},\frac{\varepsilon}{4}) \subset \mathbf{X}$ coincides with $\overline{B}_{\mathbf{X}}(\overline{z},\frac{\varepsilon}{4})$. Moreover, according to [BH99, I.5.27(3)], this last identification is an isometry for the metrics induced by the ambient spaces $\mathbf{V}(\overline{z},\varepsilon)$ and $\mathbf{X}$. Observe that we take radius $\varepsilon/4$, instead of $\varepsilon/2$ in [BH99], because we want to work here with closed balls.

It remains to notice that $\overline{\mathbf{V}}(\overline{z},\frac{\varepsilon}{4})$ is isometrically embedded in $\mathbf{V}(\overline{z},\varepsilon)$, because for all $\overline{x},\overline{y}$ in $\overline{\mathbf{V}}(\overline{z},\frac{\varepsilon}{4})$ and any chain in $\bigsqcup_{z\in\overline{z}} B(z,\varepsilon)$ of the form $(x'_0 \in \overline{x}, x_1 \sim x'_1, \dots, x_k \in \overline{y})$ as before, we can replace it with the following chain. The points x'_i, x_i belong to the balls $B(z_i,\varepsilon), B(z_{i-1},\varepsilon)$ in $\mathbf{E}^+_{f_i}, \mathbf{E}^+_{f_{i-1}}$, for z_i, z_{i-1} in $\mathbf{E}^+_{f_i}, \mathbf{E}^+_{f_{i-1}}$ representing $\overline{z}$. Call $p(x'_i)$ the radial projection (with respect to the center z_i) of x'_i onto $\overline{B}(z_i,\frac{\varepsilon}{4})$ in $\mathbf{E}^+_{f_i}$ and define $p(x_i) \in \overline{B}(z_{i-1},\frac{\varepsilon}{4})$ likewise. By the definition of a chain we have $x_i \sim x'_i$. By Proposition 2.3 this means, assuming for simplicity that $f_{i-1} = \mathrm{id}$, that f_i fixes $\overline{x}_i$. Since f_i also fixes $\overline{z}$, by the convexity of $\mathrm{Fix}(f_i)$ in $\mathbf{E}^+_{\mathrm{id}}$ which comes from Lemma 5.1 and Proposition 4.5, we obtain that f_i fixes $\overline{p(x_i)}$. Then $p(x_i) \sim p(x'_i)$ and the projections form a chain with $\sum_{i=0}^{k-1} |p(x'_i), p(x_{i+1})| \leqslant \sum_{i=0}^{k-1} |x'_i, x_{i+1}|$. □

Lemma 5.6. (i) *The application $\rho_+\colon (\mathbf{X}, d_{\mathbf{X}}) \to (\nabla^+, |\cdot,\cdot|)$ of Corollary* 2.5(i) *is an isometry in restriction to each $\mathbf{E}^+_f$.*

(ii) *The map $\rho\colon (\mathbf{X}, d_{\mathbf{X}}) \to (\nabla, |\cdot,\cdot|)$ of Lemma* 3.5 *is an isometry in restriction to each $\mathbf{E}^+_f$.*

(iii) *Applications $\rho_+\colon \mathbf{X} \to \nabla^+$ and $\rho\colon \mathbf{X} \to \nabla$ do not increase distances.*

Proof. (i) Or $\overline{x},\overline{y} \in \mathbf{E}^+_f$. Let $(x'_0, x_1 \sim x'_1, x_2 \sim x'_2, \dots, x_{k-1} \sim x'_{k-1}, x_k)$ be a chain with $x'_0 \in \overline{x}, x_k \in \overline{y}$. Then $\sum_{i=0}^{k-1} |x'_i, x_{i+1}| = \sum_{i=0}^{k-1} |\rho_+(\overline{x}_i), \rho_+(\overline{x}_{i+1})| \geqslant |\rho_+(\overline{x}), \rho_+(\overline{y})|$ with the equality for the trivial chain of length $k = 1$. Thus $d_{\mathbf{X}}(\overline{x},\overline{y}) = |\rho_+(\overline{x}), \rho_+(\overline{y})|$.

(ii) By Corollary 3.3 we can suppose $f \in \mathrm{Tame}_0(\mathbb{k}^n)$. By definition for $\alpha \in \Pi^+$ we have $\rho(\nu_{f,[\alpha]}) = [\sigma_f(\alpha)]$, where σ_f is the permutation associated with f defined just before lemma 3.5. Then the point (ii) follows from the point (i) and from the fact that the map $[\alpha] \to [\sigma_f(\alpha)]$ is an isometry of $(\nabla^+, |\cdot,\cdot|)$ on its image in $(\nabla, |\cdot,\cdot|)$.

(iii) Thanks to the previous points this is an immediate consequence of the definition of the metric $d_{\mathbf{X}}$. □

Remark 5.7. The application $e\colon (\nabla, |\cdot,\cdot|) \to (\mathbf{X}, d_{\mathbf{X}})$ defined by $e([\alpha]) = \nu_{\mathrm{id},[\alpha]}$ does not increase the distance. By Lemma 5.6(iii), ρ also does not increase the distance. Moreover, by Lemma 3.5(iii), $\rho \circ e$ is the identity. Thus, $\mathbf{E}_{\mathrm{id}}$ identified with $\mathbb{R}^{n-1}$ is isometrically embedded in $\mathbf{X}$. Using the action of $\mathrm{Tame}(\mathbb{k}^n)$ we get that each apartment $\mathbf{E}_f$ is an isometrically embedded $\mathbb{R}^{n-1}$, which justifies the notation $\mathbf{E}$ for "Euclidean".

Lemma 5.8. *The space $\mathbf{X}$ is complete.*

Proof. Let (ν_m) be a Cauchy sequence in $\mathbf{X}$. By Lemma 5.6(iii), $(\rho_+(\nu_m))$ is also a Cauchy sequence, and admits a limit $[\alpha] \in \nabla^+$ since $(\nabla^+, |\cdot,\cdot|)$ is closed in $\mathbb{R}^{n-1}$. Let $\varepsilon = \varepsilon([\alpha])$

correspond to property (ii) in the proof of Proposition 5.4. Let M be large enough such that for each $m \geqslant M$ we have

$$d_{\mathbf{X}}(\nu_M, \nu_m) < \frac{\varepsilon}{2} \text{ et } |\rho_+(\nu_m), [\alpha]| < \frac{\varepsilon}{2}.$$

If $\nu_M \in \mathbf{E}_f^+$, then $\overline{z} = \nu_{f,[\alpha]}$. Thus for $m \geqslant M$ we have $\nu_m \in B_{\mathbf{X}}(\overline{z}, \varepsilon) = \mathbf{V}(\overline{z}, \varepsilon)$ by Lemma 5.5(a). By definition of $\mathbf{V}(\overline{z}, \varepsilon)$, for every $m \geqslant M$ there is $z_m \in \overline{z}$ with $\nu_m \in B(z_m, \varepsilon)$. So $d_{\mathbf{X}}(\overline{z}, \nu_m) \leqslant |[\alpha], \rho_+(\nu_m)|$ which tends to 0 when m tends to infinity. Thus $\nu_m \to \overline{z}$. □

Finally, notice that since for all $f, g \in \mathrm{Tame}(\mathbb{k}^n)$ the action of f induces an isometry from $\mathbf{E}_g$ to $\mathbf{E}_{fg}$, the action of $\mathrm{Tame}(\mathbb{k}^n)$ on $\mathbf{X}$ is by isometries. Moreover, by Corollary 4.5, $\mathbf{X}$ is connected.

5.2. **An angular lemma.**

Lemma 5.9. *Let $\alpha = (m, p, 1) \in \Pi$ and $0 \leqslant k \leqslant m$. In ∇ consider the half-lines (see Figure 5):*

- *c_1 from $[\alpha]$ to $[0, 0, 1]$;*
- *c_2 from $[\alpha]$ to $[k, 0, 1]$;*
- *c_3 from $[\alpha]$ to $[m - k, 0, 1]$;*
- *c_4 from $[\alpha]$ to $[m, 0, 1]$.*

The c_i become smooth curves for the metric $|\cdot, \cdot|$ from $\mathbb{R}^2$. For $1 \leqslant i < j \leqslant 4$, denote θ_{ij} the angle at point $[\alpha]$ between the curves c_i and c_j, for the metric $|\cdot, \cdot|$. Then $\theta_{12} = \theta_{34}$, or equivalently $\theta_{12} + \theta_{13} = \pi/3$.

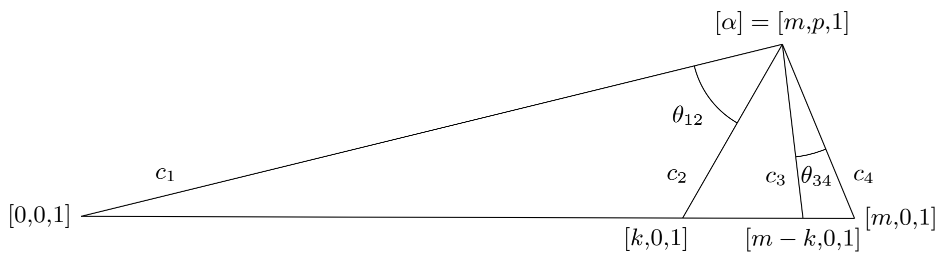

Figure 5. Angular Lemma 5.9. The picture is in the simplex ∇, but all angles must be understood with respect to the metric $|\cdot, \cdot|$.

Proof. Note that $[0, 1, 0]$ is collinear with $[\alpha]$ and $[m, 0, 1]$, so c_1 and c_4 are principal and $\theta_{14} = \frac{\pi}{3}$ by Remark 5.2. The fact that the two conclusions are equivalent then comes from the equality

$$\frac{\pi}{3} = \theta_{14} = \theta_{12} + \theta_{13} + (\theta_{34} - \theta_{12}).$$

Consider the involution on ∇ and its boundary at infinity given by:

$$\tau \colon [\alpha_1, \alpha_2, \alpha_3] \mapsto [\alpha_1, p\alpha_3, \alpha_2/p].$$

This involution fixes the line of weights $[t, p, 1]$, $t \geqslant 0$, which contains $[\alpha]$. Moreover, $\tau[0, 0, 1] = [0, 1, 0]$ which is collinear with $[\alpha]$ and $[m, 0, 1]$, and $\tau[k, 0, 1] = [k, p, 0]$ which is collinear with $[\alpha]$ and $[m - k, 0, 1]$. In particular, τ exchanges the lines containing c_1 and c_4, and also the lines containing c_2 and c_3, thus τ sends the two curves forming the angle θ_{12} onto those forming the angle θ_{34}.

At the level of $\beta_i = \log \alpha_i$, the involution τ becomes:

$$(\beta_1, \beta_2, \beta_3) \mapsto (\beta_1, \beta_3 + \log p, \beta_2 - \log p).$$

Thus for the metric $|\cdot, \cdot|$ the involution τ is an axial symmetry (with axis $\beta_2 - \beta_3 = \log p$). In particular it preserves the non-oriented angles and we conclude $\theta_{12} = \theta_{34}$. □

6. Simple connectedness

In this section we show that $\mathbf{X}_2$ is a tree and that $\mathbf{X}_3$ is simply connected. For this purpose we recall the formalism of simple complexes of groups, following [BH99, II.12].

6.1. **Group complexes.** A *complex of groups* $\mathcal{G}$ on a poset $(\Sigma, <)$ is a collection of groups $\{G_\sigma\}_{\sigma\in\Sigma}$, with for each pair $\sigma > \tau$ an injective homomorphism $\varphi_{\tau\sigma}\colon G_\sigma \to G_\tau$. We further require that each triple $\sigma > \tau > \rho$ gives rise to a compatibility relation $\varphi_{\rho\tau} \circ \varphi_{\tau\sigma} = \varphi_{\rho\sigma}$. A *subcomplex* of $\mathcal{G}$ is the data of a subset Σ' of Σ, endowed, for all $\sigma, \tau \in \Sigma'$, with the same $G_\sigma, \varphi_{\tau\sigma}$ as $\mathcal{G}$.

A *morphism* $\psi\colon \mathcal{G} \to G$ from a complex of groups to a group G is a collection of homomorphisms $\psi_\sigma\colon G_\sigma \to G$ satisfying $\psi_\sigma = \psi_\tau \circ \varphi_{\tau\sigma}$ for any pair $\sigma > \tau$. In the typical situation where Σ is the poset of the cells of a polyhedral complex D, to such a morphism we associate a *development* of D which is the disjoint union $D \times G$ of G copies of D quotient by the relation $(x, h) \sim (x, hg)$, where x belongs to a cell σ and g is contained in the image of ψ_σ. This notion of development can be seen as the inverse of a passage to the quotient:

Theorem 6.1 ([BH99, II.12.20(1)])**.** *Let $\mathcal{G}$ be a complex of groups on the poset of the cells of a polyhedral complex D. Suppose that D is a fundamental domain of the action of G on a polyhedral complex D', that each G_σ is the stabilizer of σ, and that the $\varphi_{\tau\sigma}\colon G_\sigma \to G_\tau, \psi_\sigma\colon G_\sigma \to G$ are the obvious inclusions. Then D' is isomorphic as a polyhedral complex, in an G-equivariant way, to the development associated with ψ.*

In the situation of Theorem 6.1 we say that $\mathcal{G}$ is *developable.*

Assume D simply connected. Then the *fundamental group* $F\mathcal{G}$ of $\mathcal{G}$ is the free product of all the G_σ quotient by the relations $g \sim \varphi_{\tau\sigma}(g)$. If $\mathcal{G}$ is developable, we define its *universal development* as the development associated with the natural morphism $\psi\colon \mathcal{G} \to F\mathcal{G}$. Each morphism $\psi\colon \mathcal{G} \to G$ induces a map $F\psi\colon F\mathcal{G} \to G$. If $F\psi$ is an isomorphism, then the development associated with ψ is isomorphic to the universal development.

Theorem 6.2 ([BH99, II.12.20(4)])**.** *The universal development is simply connected.*

6.2. **$\mathbf{X}_2$ is a tree.** When $n = 2$, ∇^+ endowed with the Euclidean metric $|\cdot,\cdot|$ is a half-line with end equal to $[1, 1]$, and each admissible hyperplane in ∇^+ is a point $[i, 1]$ for $i \geqslant 1$. The representative of $[i, 1]$ in the hyperboloid $\alpha_1\alpha_2 = 1$ is $(\alpha_1, \alpha_2) = \big(\sqrt{i}, \frac{1}{\sqrt{i}}\big)$, and therefore $\beta_1 = \frac{\log i}{2}, \beta_2 = -\frac{\log i}{2}$. We get that $[i + 1, 1]$ is at a distance $\frac{\log(i+1)-\log i}{\sqrt{2}}$ from $[i, 1]$. We then consider ∇^+ as a metric graph, with a vertex $s_i = [i, 1]$ for each $i \geqslant 1$, and an edge e_i of length $\frac{\log(i+1)-\log i}{\sqrt{2}}$ between each s_i and s_{i+1}. For each open cell σ of ∇^+, let $G_\sigma \subset \mathrm{Aut}(\Bbbk^2)$ denote the stabilizer of $\nu_{\mathrm{id},[\alpha]}$ for $[\alpha] \in \sigma$: by Proposition 4.4, this definition is indeed independent of the choice of $[\alpha]$.

According to Proposition 3.2, G_{s_1} is the affine group A_2 and G_{e_1} is the triangular affine subgroup in A_2. Moreover, for $i \geqslant 2$, $G_{s_i} = G_{e_i}$ is the group of automorphisms of the form $(ax_1+P(x_2), bx_2+c)$ with $\deg P \leqslant i$. In particular, for any $i \geqslant 2$ we have $G_{s_i} \subset G_{s_{i+1}}$. Let $\mathcal{G}$ be the graph of groups on the poset of the cells of ∇^+ endowed with these groups. The inductive limit $\lim_{i\geqslant 2} G_{s_i}$ is equal to the group E_2 of triangular automorphisms. Thus the fundamental group $F\mathcal{G}$ is the amalgamated product of A_2 and E_2 along the triangular affine group $A_2 \cap E_2$.

According to Corollary 2.5(ii), $\mathbf{E}^+_{\mathrm{id}}$ is a fundamental domain for the action of $\mathrm{Aut}(\Bbbk^2)$ on $\mathbf{X}_2$. By identifying $\mathbf{E}^+_{\mathrm{id}}$ with ∇^+, according to Theorem 6.1, the development associated with the natural morphism $\psi\colon \mathcal{G} \to \mathrm{Aut}(\Bbbk^2)$ is isomorphic to $\mathbf{X}_2$. Consider the morphism $F\psi\colon F\mathcal{G} \to \mathrm{Aut}(\Bbbk^2)$ induced by ψ. According to the Jung–van der Kulk's Theorem we have $\mathrm{Aut}(\Bbbk^2) = A_2 *_{A_2\cap E_2} E_2$, thus $F\psi$ is an isomorphism, and therefore $\mathbf{X}_2$ is isomorphic to the universal development of $\mathcal{G}$. This one being a simply connected graph by Theorem 6.2, we conclude as expected that $\mathbf{X}_2$ is a metric tree.

6.3. **$\mathbf{X}_3$ is simply connected.** We now use the same ideas as in the previous paragraph to show:

Proposition 6.3. *Over a field $\Bbbk$ of characteristic zero, $\mathbf{X}_3$ is simply connected.*

Instead of the Jung–van der Kulk's Theorem we use the following two results concerning the structure of the group $\mathrm{Tame}(\Bbbk^3)$. We note $A = A_3$ the affine group,

$$B = \{(ax_1 + P(x_2, x_3), bx_2 + cx_3 + d, b'x_2 + c'x_3 + d'); P \in \Bbbk[x_2, x_3], a \neq 0, bc' - b'c \neq 0\},$$

$$C = \{(f_1, f_2, f_3) \in \mathrm{Tame}(\Bbbk^3); f_3 = cx_3 + d, c \neq 0\},$$

$$H_1 = \{(ax_1 + P(x_2, x_3), bx_2 + R(x_3), cx_3 + d); P \in \Bbbk[x_2, x_3], R \in \Bbbk[x_3]\ a, b, c \neq 0\},$$

$$K_2 = \{(ax_1 + bx_2 + P(x_3), a'x_1 + b'x_2 + R(x_3), cx_3 + d); P, R \in \Bbbk[x_3], ab' - a'b \neq 0, c \neq 0\}.$$

Theorem 6.4 ([Wri15, Theorem 2], [Lam19, Corollary 5.8])**.** *Over a field $\Bbbk$ of characteristic zero, the group $\mathrm{Tame}(\Bbbk^3)$ is the amalgamated product of the groups A, B, C along their pairwise intersections. That is, $\mathrm{Tame}(\Bbbk^3)$ is the fundamental group of the triangle of groups where the vertex groups are A, B, C and the other groups are their adequate intersections.*

Proposition 6.5 ([LP19, Proposal 3.5])**.** *The group C is the amalgamated product of H_1 and K_2 along their intersection.*

Proof of Proposition 6.3. The admissible lines endow ∇^+ with a structure of polygonal complex (we temporarily forget the metric to keep only the combinatorial object). Precisely, the vertices are the points of intersection between different admissible lines, the open edges are the connected components of the complement of the vertices in the admissible lines, and the open cells of dimension 2 are the connected components in ∇^+ of the complement vertices and edges (i.e. admissible straight lines). For each open cell σ of ∇^+, let $G_\sigma \subset \mathrm{Tame}(\Bbbk^3)$ be the stabilizer of $\nu_{\mathrm{id},[\alpha]}$ for a weight $[\alpha] \in \sigma$. Let $\mathcal{G}$ be the complex of groups on the poset of the cells of ∇^+ endowed with these groups.

To obtain the simple connectedness of $\mathbf{X}_3$, according to the same argument as in the case $n = 2$, it suffices to prove that $F\mathcal{G} = \mathrm{Tame}(\Bbbk^3)$. Although Theorem 6.2 concerns the simple connectedness of the geometric realization of the poset of $\mathbf{X}_3$, the metric $d_{\mathbf{X}}$ is locally bilipschitz to it and this will therefore imply the simple connectedness of $(\mathbf{X}_3, d_{\mathbf{X}})$.

In order to analyze the fundamental group $F\mathcal{G}$, we consider the following partition of $\mathcal{G}$ into subcomplexes (Figure 6, left), where by definition the cells are equipped with the same groups as in $\mathcal{G}$. Let $\mathcal{A}$ be the vertex $[1, 1, 1]$, $\mathcal{B}$ the subcomplex on the poset of the cells in the ray $\frac{1}{2}\alpha_1 \geqslant \alpha_2 = \alpha_3$, and $\mathcal{C}$ the subcomplex on the poset of the cells in the region $\alpha_2 \geqslant 2\alpha_3$. Let $\mathcal{AB}$ be the edge between $[1, 1, 1]$ and $[2, 1, 1]$, $\mathcal{AC}$ the edge between $[1, 1, 1]$ and $[2, 2, 1]$, and $\mathcal{ABC}$ the triangle $([1, 1, 1], [2, 1, 1], [2, 2, 1])$. Finally, let $\mathcal{BC}$ be the subcomplex on the poset of the cells in the region $\alpha_1 \geqslant 2\alpha_3 > \alpha_2 > \alpha_3$.

Moreover, we further partition the subcomplex $\mathcal{C}$ as follows. Let $\mathcal{K}$ be the subcomplex on the poset of the cells in the half-line $\alpha_1 = \alpha_2 \geqslant 2\alpha_3$, $\mathcal{H}$ the subcomplex on the poset of the cells in the region defined by $\alpha_2 \geqslant 2\alpha_3$ and $\alpha_1 \geqslant \alpha_2 + \alpha_3$, and $\mathcal{H}K$ the subcomplex on the poset of cells in the region R determined by $\alpha_2 + \alpha_3 > \alpha_1 > \alpha_2 \geqslant 2\alpha_3$. Observe that here as above certain faces of the cells of these posets do not belong to them. We describe more precisely the poset Σ of the cells of $\mathcal{HK}$. The only admissible straight lines intersecting R are of equation $\alpha_1 = m\alpha_3$ or $\alpha_2 = m\alpha_3$, for $m > 2$, and they do not intersect in R. Then every 2-cell of Σ, except one, has exactly two edges in Σ (see Figure 6, right). The structure of the posets of $\mathcal{BC}$ and $\mathcal{H}$ is more complicated.

To prove $F\mathcal{G} = \mathrm{Tame}(\Bbbk^3)$, thanks to Theorem 6.4 it suffices to obtain:

(i) $F\mathcal{A} = A$
(ii) $F\mathcal{AB} = A \cap B$
(iii) $F\mathcal{AC} = A \cap C$
(iv) $F\mathcal{ABC} = A \cap B \cap C$
(v) $F\mathcal{B} = B$
(vi) $F\mathcal{BC} = B \cap C$
(vii) $F\mathcal{C} = C$
(viii) The groups of $\mathcal{BC}$ form an inductive system of groups; moreover for each element $g \in G_\sigma$ in $\mathcal{BC}$ there exists an element $g' \in G_{\sigma'}$ (resp. $g'' \in G_{\sigma''}$) in $\mathcal{BC}$ equivalent to g in the limit $F\mathcal{BC}$, such that σ' (resp. σ'') has a face in the poset of $\mathcal{B}$ (resp. $\mathcal{C}$).

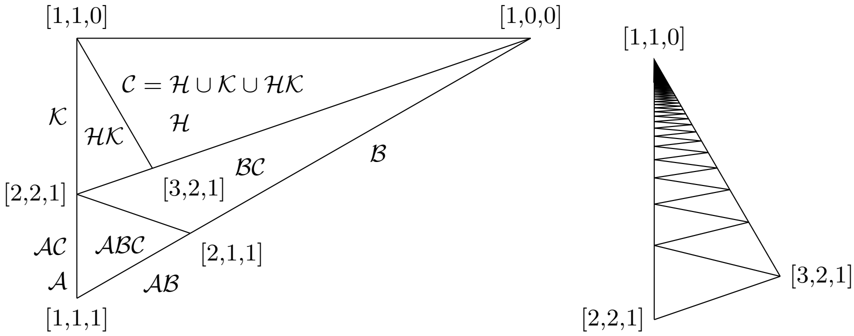

FIGURE 6. Partition of $\mathcal{G}$ into subcomplexes, and zoom on the cells in $\mathcal{HK}$.

Property (viii) is necessary to ensure that each element of $B \cap C$ is identified with its copy in B and C. The analogous properties for the other inclusions are immediate.

The properties (i)-(iv) follow immediately from Proposition 3.2, because in each of these cases the fundamental group is the stabilizer of a single point. Specifically, (i) matches Example 3.1(i), and (ii) matches Example 3.1(iii). Moreover, by Proposition 6.5 $A \cap C = A \cap K_2$ therefore $A \cap B \cap C = A \cap B \cap K_2$ which makes it possible to deduce (iii) and (iv) from Examples 3.1(iv) and 3.1(ii).

For (v), note that, also according to Proposition 3.2 and Example 3.1(iv), the groups of $\mathcal{B}$ form an increasing sequence whose union is B.

For (vi), consider a curve $\alpha\colon (t_0, \infty) \to \Pi^+$ satisfying the hypotheses of Corollary 4.8, such that its projection $[\alpha(t)] \subset \nabla^+$ is contained in the cells of $\mathcal{BC}$, and which tends towards the point $[1,0,0]$ while being an asymptote to the line $\alpha_2 = 2\alpha_3$. For example the curve $\alpha\colon (3,\infty) \to \Pi^+$ with $\alpha(t) = (t^2, 2t, t+1)$ is suitable (see Figure 7). For $\alpha \in \Pi^+$, let $\sigma(\alpha)$ be the cell of ∇^+ containing $[\alpha]$ in its interior. According to Corollary 4.8, the $G_{\sigma(\alpha(t))}$ form an increasing sequence of groups. By Proposition 3.2, the union of this sequence is equal to $B \cap C$. For any point in a cell of $\mathcal{BC}$, there is a line segment connecting this point to a point on the curve $[\alpha(t)]$ to which we can apply Corollary 4.8. We thus obtain that each group of $\mathcal{BC}$ is contained in a $G_{\sigma(\alpha(t))}$. Consequently the groups of $\mathcal{BC}$ form an inductive system of groups (which also gives the first part of (viii)) whose limit is $B \cap C$.

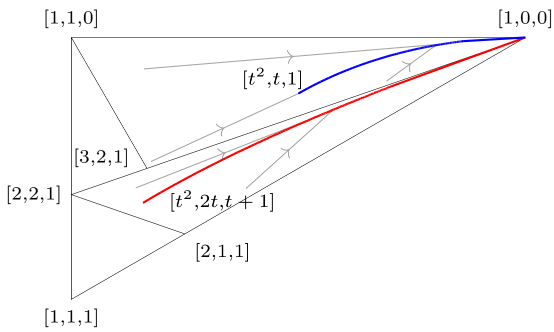

FIGURE 7. Curves and segments satisfying the assumptions of Corollary 4.8.

To show (vii) we compute $F\mathcal{C}$. We use the partition of $\mathcal{C}$ in $\mathcal{K}, \mathcal{H}$ and $\mathcal{HK}$. First, according to Proposition 3.2 and Example 3.1(iv), the groups of $\mathcal{K}$ form an increasing sequence whose union is K_2. Similarly, the groups of $\mathcal{HK}$ form an increasing sequence whose union is $H_1 \cap K_2$. Finally, in the cells of $\mathcal{H}$ consider the projection $[\alpha(t)]$ of the curve $\alpha\colon (2,\infty) \to \Pi^+$ with $\alpha(t) = (t^2, t, 1)$ (see Figure 7). According to Corollary 4.8, the $G_{\sigma(\alpha(t))}$ form an increasing sequence. By

Proposition 3.2 the union of this sequence is equal to H_1. According to Corollary 4.8 applied to well-chosen line segments, each group of $\mathcal{H}$ is contained in a $G_{\sigma(\alpha(t))}$. Thus the groups of $\mathcal{H}$ form an inductive system whose limit is H_1. The analog of the property (viii) with $H_1 \cap K_2$ instead of $B \cap C$ is immediate, because the groups of $\mathcal{HK}$ form an increasing sequence. We conclude thanks to Proposition 6.5 that $F\mathcal{C} = H_1 *_{H_1 \cap K_2} K_2 = C$.

Finally, for the second assertion of (viii), consider $g \in G_\sigma$ in $\mathcal{BC}$. By Lemma 4.1, σ is contained in a finite number of admissible half-spaces $L_1, \ldots, L_k$. Moreover, since σ is in the poset of $\mathcal{BC}$, the L_i are defined by inequalities of the form $\alpha_1 \geqslant m_2\alpha_2 + m_3\alpha_3$ or $\alpha_2 \geqslant \alpha_3$. Then each cell σ' (resp. σ'') having a face in the half-line $\alpha_2 = \alpha_3$ (resp. the half-line $\alpha_2 = 2\alpha_3$), and sufficiently far from $[1,1,1]$, is contained in all L_i. The groups of $\mathcal{BC}$ form an inductive system of groups, so there exists a G_ρ in $\mathcal{BC}$ containing G_σ and G'_σ simultaneously. By Corollary 4.7, $G_\sigma \subset G'_\sigma$ as subgroups of $\mathrm{Tame}(\Bbbk^3)$, and thus also for their copies in G_ρ. So there is $g' \in G_{\sigma'}$ equivalent to g, as expected. Similarly, there is $g'' \in G_{\sigma''}$ equivalent to g. □

7. Local CAT(0) property

This entire section is in dimension $n = 3$, over an arbitrary field $\Bbbk$. The aim is to show

Proposition 7.1. *Let $\alpha \in \Pi$, and $g \in \mathrm{Tame}(\Bbbk^3)$. Then $\nu_{g,[\alpha]}$ admits a neighborhood* CAT(0) *in $\mathbf{X}_3$.*

We will use the following fundamental results concerning cones and links.

Let Γ be a metric graph. The *cone over Γ of radius r* is the simplicial complex obtained as the simplicial cone over Γ, and considered as a length space as follows. For each edge of Γ of length l we endow the corresponding triangle in the cone with the metric of a sector of angle l of the Euclidean disk of radius r. Compare with [BH99, I.5.6] where is constructed a metric space called the 0-cone, in which our cone embeds like a closed ball of radius r.

Theorem 7.2 (Berestovskii, see [BH99, II.3.14])**.** *Let Γ be a metric graph and $r > 0$. Then the cone over Γ of radius r is* CAT(0) *if and only if Γ is* CAT(1).

Recall also that a metric graph Γ is CAT(1) if and only if any cycle embedded in Γ has length $\geqslant 2\pi$ [BH99, II.1.15(4)].

Now let X be a Euclidean polygonal complex. Let v be a vertex of X and assume that there are only finitely many isometry classes of pieces containing v. The *link de v dans X* is the metric graph whose vertices correspond to the edges of X containing v, and whose edges correspond to the polygons of X containing v. The length of such an edge is defined as the angle at v of the corresponding polygon. See [BH99, I.7.14].

Theorem 7.3 ([BH99, I.7.16])**.** *There exists $r > 0$ such that the closed ball of radius r around v in X is isometric to the cone of radius r over the link of v in X.*

In order to prove Proposition 7.1 we start by establishing an abstract criterion to verify that a space obtained by gluing is CAT(0).

7.1. CAT(0) limit spaces.

Let $r > 0$ and $\mathbf{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant r^2, y \geqslant 0\}$ be the upper closed half-disk of radius r in $(\mathbb{R}^2, |\cdot,\cdot|)$. We note O the center of $\mathbf{D}$. Let $l_0, l_1, \ldots, l_k, l_{k+1}$ be a sequence of rays pairwise distinct and ordered in the direct direction, with l_0 at end $(r,0)$ and l_{k+1} at end $(-r,0)$. For $j = 1, \ldots, k$, let c_j be the image of an embedded smooth curve $[0,1] \to \mathbf{D}$. We assume that c_j is distinct from the radius l_{j-1}, and is contained in the closed sector delimited by the rays l_{j-1} and l_j, with the center O of the half-disk and a point in the circular arc of the sector. Let C_j be the closure of the component of $\mathbf{D} \setminus c_j$ containing l_0, and assume each C_j to be convex. Let L_j be the closed sector delimited by the rays l_j and l_0: we therefore have $L_{j-1} \subsetneq C_j \subseteq L_j$. By convention, $C_0 = L_0 = c_0 = l_0, c_{k+1} = l_{k+1}, C_{k+1} = L_{k+1} = \mathbf{D}$, and $C_\infty = L_\infty = c_\infty = l_\infty$ is the diameter $l_0 \cup l_{k+1}$ of $\mathbf{D}$ (see Figure 8).

Let $(\mathbf{D}_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ be a family of half-disks indexed by a set Λ, where for each λ, $\varphi_\lambda \colon \mathbf{D} \to \mathbf{D}_\lambda$ is an isometry. Suppose there is an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{\lambda \in \Lambda} \mathbf{D}_\lambda$, which in restriction to each $\mathbf{D}_\lambda \sqcup \mathbf{D}_{\lambda'}$ (where $\lambda \neq \lambda'$) is given by $\varphi_\lambda(x) \sim \varphi_{\lambda'}(x)$ if and only

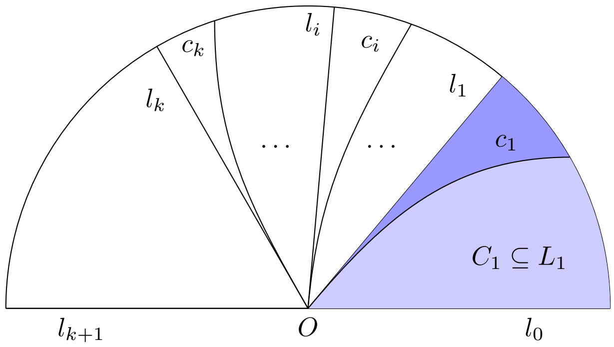

Figure 8. Notations pour la Section 7.1.

if $x \in C_{j(\lambda,\lambda')}$, for a function j which to each unordered pair λ, λ' of distinct elements of Λ associates an element of $\mathcal{I} = \{0, 1, \ldots, k, k+1, \infty\}$.

Remark 7.4. We claim that the relation $\sim_0$ on $\bigsqcup_{\lambda\in\Lambda} \mathbf{D}_\lambda$ which in restriction to each $\mathbf{D}_\lambda \sqcup \mathbf{D}_{\lambda'}$ is given by $\varphi_\lambda(x) \sim_0 \varphi_{\lambda'}(x)$ if and only if $x \in L_{j(\lambda,\lambda')}$, for the same function j as previously, is an equivalence relation.

To see this, notice that there is a symmetric map $\cap\colon \mathcal{I}\times\mathcal{I} \to \mathcal{I}$ such that for each $i, j \in \mathcal{I}$ we have $C_i \cap C_j = C_{i\cap j}$. Specifically, if $i, j < \infty$ then $i \cap j = \min\{i, j\}$, if $i < k+1$ then $i \cap \infty = 0$, and finally $\infty \cap \infty = (k+1) \cap \infty = \infty$. Moreover, $L_i \cap L_j = L_{i\cap j}$ for this same application $\cap$.

The transitivity of the relation $\sim$ amounts to saying that for any triplet of distinct indices $\lambda, \lambda', \lambda''$ we have

$$C_{j(\lambda,\lambda')\cap j(\lambda',\lambda'')} = C_{j(\lambda,\lambda')} \cap C_{j(\lambda',\lambda'')} \subset C_{j(\lambda,\lambda'')},$$

so

$$\big(j(\lambda,\lambda') \cap j(\lambda',\lambda'')\big) \cap j(\lambda,\lambda'') = j(\lambda,\lambda') \cap j(\lambda',\lambda'').$$

This gives a similar equality for L_i:

$$L_{j(\lambda,\lambda')} \cap L_{j(\lambda',\lambda'')} = L_{j(\lambda,\lambda')\cap j(\lambda',\lambda'')} \subset L_{j(\lambda,\lambda'')},$$

which shows that $\sim_0$ is transitive, and therefore is an equivalence relation.

Denote by $\mathbf{V}_0$, $\mathbf{V}$ the quotient spaces corresponding to $\sim_0$ and $\sim$ with their respective pseudo-metrics d_0, d, defined by lengths of chains in $\bigsqcup_{\lambda\in\Lambda} \mathbf{D}_\lambda$ as in the definition 5.3. In the same way as in the proof of Proposition 5.4, we prove that these pseudo-metrics are metrics, and the quotient spaces are spaces of length. Note that in each case the points $\varphi_\lambda(O)$ are identified with a point that we will call the *centre* of $\mathbf{V}_0$ or $\mathbf{V}$, and that we will also denote O. For example, if $\Lambda = \{\lambda, \lambda'\}$ and $j(\lambda,\lambda') = \infty$, then $\mathbf{V}_0 = \mathbf{V}$ is a closed disk.

The main result of this section is the following criterion.

Proposition 7.5. *Spaces* $\mathbf{V}_0$ *and* $\mathbf{V}$ *are complete. If* $\mathbf{V}_0$ *is* $\mathrm{CAT}(0)$*, then* $\mathbf{V}$ *is also* $\mathrm{CAT}(0)$.

Proof. Our strategy is to show that $\mathbf{V}$ is a limit of spaces $\mathrm{CAT}(0)$. For all integers $m \geqslant 1$ and $j = 0, 1, \ldots, k+1$ let us choose c_j^m a piecewise linear curve whose the extremities are the same as those of c_j, with all vertices on c_j, and each point of which is at distance $\leqslant \frac{1}{m}$ from c_j. We then denote C_j^m the closure of the component of $\mathbf{D} \setminus c_j^m$ containing l_0. As before, we agree that $C_\infty^m = c_\infty^m = c_\infty$. We note $(\mathbf{V}_m, d_m)$ the length space obtained via the equivalence relation $\sim_m$ on $\bigsqcup_{\lambda\in\Lambda} \mathbf{D}_\lambda$ which on each $\mathbf{D}_\lambda \sqcup \mathbf{D}_{\lambda'}$ is defined by $\varphi_\lambda(x) \sim \varphi_{\lambda'}(x)$ for $x \in C^m_{j(\lambda,\lambda')}$ (Remark 7.4 applies to $\sim_m$ in the same way as at $\sim_0$).

There is a bilipschitz map from $\mathbf{D}$ to $\mathbf{D}$ which maps each l_j to c_j (note that this is where we need $c_j \neq l_{j-1}$). Similarly, for each $m \geqslant 1$ there is a bilipschitz map from $\mathbf{D}$ to $\mathbf{D}$ which sends each l_j to c_j^m. Thus all the spaces $\mathbf{V}, \mathbf{V}_0, \mathbf{V}_m$ are pairwise bilipschitzian. The space $\mathbf{V}_0$ is completed by [BH99, I.7.13]. Precisely, it is necessary to replace each sector by an isosceles triangle, and therefore the half-disc by a polygon, in order to be reduced to the case of a polygonal complex with a finite number of isometry classes of pieces. This replacement is a bilipschitz application. The spaces $\mathbf{V}$, $\mathbf{V}_m$ are then also complete, because bilipschitzian at $\mathbf{V}_0$.

We now show that each $\mathbf{V}_m$ is CAT(0). As $\mathbf{V}_m$ is complete and simply connected (it retracts radially on the center), by [BH99, II.4.1(2)] it suffices to show that $\mathbf{V}_m$ is locally CAT(0). Note that there is a natural map $\mathrm{pr}\colon \mathbf{V}_m \to \mathbf{D}$ which is the inverse of each $\mathbf{D} \to \mathbf{D}_\lambda \subset \mathbf{V}_m$. Let $x \in \mathbf{V}_m$, and consider different possibilities for $\mathrm{pr}(x)$. If $\mathrm{pr}(x)$ does not belong to any of the curves c_j^m, then a sufficiently small neighborhood of x is a flat disk. If $\mathrm{pr}(x)$ belongs to a curve c_j^m but is distinct from the center O, then a sufficiently small neighborhood of x is obtained by gluing flat disks (or half-disks) along a subset common convex, and such a neighborhood is CAT(0) by [BH99, II.11.3]. Finally, consider the case $\mathrm{pr}(x) = O$, that is, x is the center of $\mathbf{V}_m$. By Theorem 7.3, applied to $X = X_m$ obtained as before by replacing in $\mathbf{V}_m$ the half-disks by polygons, there is a neighborhood of x which is a cone over the link Γ_m of x in $X_m \subset \mathbf{V}_m$. Similarly, $\mathbf{V}_0$ is a cone over a metric Γ_0 graph, and there is a natural map $g_m\colon \Gamma_m \to \Gamma_0$ contracting the distances which is a homotopy equivalence. As $\mathbf{V}_0$ is supposed to be CAT(0), by Theorem 7.2 the graph Γ_0 does not contain any cycle of length $< 2\pi$. Consequently, the graph Γ_m does not contain any either, and by Theorem 7.2, a neighborhood of x in $\mathbf{V}_m$ is CAT(0).

We now consider the subset $R_m \subset \mathbf{V}_m \times \mathbf{V}$ made up of the pairs $([x]_{\sim_m}, [x]_\sim)$ for $x \in \bigsqcup_{\lambda\in\Lambda} \mathbf{D}_\lambda$. In view of Lemma 7.6 below, R_m is a surjective M/m-relation in the sense of [BH99, I.5.33], for some constant $M > 0$. We deduce that the sequence $\mathbf{V}_m$ converges in the Gromov–Hausdorff sense to $\mathbf{V}$, and therefore $\mathbf{V}$ is CAT(0) by [BH99, II.3.10]. □

Lemma 7.6. *There exists a constant $M > 0$ such that for all $m \geqslant 1$, and all $x, y \in \bigsqcup_{\lambda\in\Lambda} \mathbf{D}_\lambda$, we have*

$$|d_m([x]_{\sim_m}, [y]_{\sim_m}) - d([x]_\sim, [y]_\sim)| \leqslant \frac{M}{m}.$$

Proof. Or $l = d([x]_\sim, [y]_\sim)$. Note first that as $C_j^m \subset C_j$, we have

$$d_m([x]_{\sim_m}, [y]_{\sim_m}) \geqslant l.$$

By definition of the (pseudo-)metric d, there is a chain: an integer N and for each $i = 0, 1, \dots, N-1$ an index $\lambda_i \in \Lambda$ and points $x_i', x_{i+1} \in \mathbf{D}_{\lambda_i}$ with $x_i \sim x_i'$ and $x = x_0', y = x_N$, such than $\sum_{i=0}^{N-1} |x_i', x_{i+1}| < l + \frac{1}{m}$. Observe that if $\mathrm{pr}(x_i) = \mathrm{pr}(x_i') = O$ for some i, then by perhaps decreasing $\sum |x_i', x_{i+1}|$ we can reduce to a subchain with $N = 2$ and $\mathrm{pr}(x_1) = \mathrm{pr}(x_1') = O$. By choosing an analogous chain in $\mathbf{V}_m$ we then obtain the expected estimate with $M = 1$. If on the contrary we have $\mathrm{pr}(x_i) = \mathrm{pr}(x_i') \neq O$ for all i, then for $i = 1, \dots, N-1$ we note $j(i) = j(\lambda_{i-1}, \lambda_i)$ and we operate the following reductions.

By passing to a subchain we can assume that for all $i = 1, \dots, N-1$ we have $\lambda_i \neq \lambda_{i-1}$ and for all $i = 2, \dots, N-1$ we have $x_{i-1}' \notin \varphi_{\lambda_{i-1}}(C_{j(i)})$, and $x_i \notin \varphi_{\lambda_{i-1}}(C_{j(i-1)})$. In particular, if $N > 2$ then $j(i) \neq k+1$. Also, if $j(i) \neq \infty$ then $j(i+1) = \infty$, and vice-versa, so without loss of generality we can assume that for every even i we have $j(i) = \infty$. Moreover, for all $i = 1, \dots, N-1$ the triangle inequality allows us to assume that $\mathrm{pr}(x_i) = \mathrm{pr}(x_i')$ is in $c_{j(i)}$.

Moreover, we claim that we can assume the chain *tight*, in the sense that for each $i = 2, \dots, N-2$, the union of the segments $[x_{i-1}', x_i]$ and $[x_i', x_{i+1}]$ is geodesic in $\big(\mathbf{D}_{\lambda_{i-1}} \sqcup \mathbf{D}_{\lambda_i}\big)/\sim$ (see [BH99, I. 7.20]). This means that the angle at point $x_i \in \mathbf{D}_{\lambda_{i-1}}$ between the line segment $[x_{i-1}', x_i]$ and the curve $\varphi_{\lambda_{i-1}}(c_{j(i)})$ is equal to the angle at point $x_i' \in \mathbf{D}_{\lambda_i}$ between $[x_i', x_{i+1}]$ and the curve $\varphi_{\lambda_i}(c_{j(i)})$. (It would be natural to ask for the same property for $i = 1, N-1$, but we don't need it and it shortens the proof a bit.)

The assertion follows from the following two observations. First, if the chain is not tight, and the previous definition fails for some integer $i = t$, then as $C_{j(t)}$ is convex and does not contain $\mathrm{pr}(x_{t-1}'), \mathrm{pr}(x_{t+1})$, the sum $\sum |x_i', x_{i+1}|$ is not minimal among sequences with given N and (λ_i), since we can decrease it by moving x_t along $\varphi_{\lambda_{t-1}}(c_{j(t)})$ (and simultaneously moving x_t'). On the other hand, as $\mathbf{D}_{\lambda_i}$ and $C_{j(i)}$ are compact, the minimum of $\sum |x_i', x_{i+1}|$ is reached. In summary, by replacing the chain by the one minimizing $\sum |x_i', x_{i+1}|$ among the chains with N and (λ_i) given, we produce a tight chain, which justifies our assertion.

We note θ_j the angle in $\mathbf{D}$ between c_j and l_{k+1}, in particular $\theta_j \geqslant \theta_k > 0$ for all $j = 1, \dots, k$. Since for the even i we have $j(i) = \infty$, the angle in the definition of a stretched chain decreases at each successive i by at least θ_k. Precisely, for i odd, if $c_{j(i)} = l_{j(i)}$ then working in the Euclidean triangle with vertices O, x_i, x_{i+1} we see that the angle decreases exactly by $\theta_{j(i)}$, and, by convexity of $C_{j(i)}$, in the general case the loss of angle is even greater (see Figure 9). The case of i even is similar. Thus $N - 4 < \frac{\pi}{\theta_k}$. For $i = 0, \dots, N-1$, there are $z_i \in \varphi_{\lambda_{i-1}}(c^m_{j(i)})$ with $|x_i, z_i| < \frac{1}{m}$. Using this chain of z_i instead of the x_i results in $d_m([x]_{\sim_m}, [y]_{\sim_m}) \leqslant l + \frac{1}{m} + 2N\frac{1}{m}$. So taking $M \geqslant 1 + 2(\frac{\pi}{\theta_k} + 4)$ we get the expected estimate. ☐

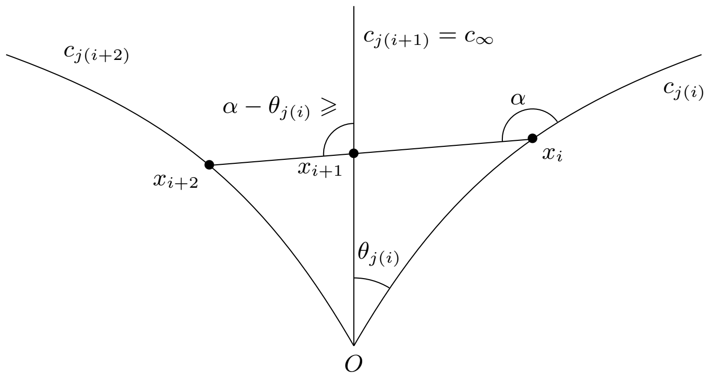

Figure 9. Decreasing angle in Lemma 7.6.

7.2. **Proof of the proposition.** The proof of Proposition 7.1 will be different depending on the form of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. By Corollary 2.5(ii), we can assume $g = \mathrm{id}$ and $\alpha \in \Pi^+$.

Note $\mathcal{F} = \mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$, which will serve as our set of indices throughout this section. Observe that $\mathcal{F}$ is the set of $f \in \mathrm{Tame}(\Bbbk^3)$ such that $\mathbf{E}^+_f$ contains $\nu_{\mathrm{id},[\alpha]}$. That is

$$\mathbf{V} := \bigsqcup_{f \in \mathcal{F}} \overline{B}(\nu_{f,[\alpha]}, \tfrac{\varepsilon}{4}) / \sim$$

the neighborhood from Lemma 5.5(b). Thus $\mathbf{V}$ is isometric to the ball $\overline{B}_{\mathbf{X}}(\nu_{\mathrm{id},[\alpha]}, \frac{\varepsilon}{4})$.

Note that if $\mathrm{mult}(\alpha) \leqslant 1$, and $\alpha_1 > \alpha_2 > \alpha_3$ (resp. $\alpha_1 = \alpha_2$ or $\alpha_2 = \alpha_3$), this neighborhood $\mathbf{V}$ is obtained by sticking together disks (resp. half-disks) along a same convex, given by Lemma 5.1 (resp. corresponding to the diameter of the half-disks). We conclude in this case that $\mathbf{V}$ is CAT(0) by [BH99, II.11.3]. We therefore assume in the following $\mathrm{mult}(\alpha) \geqslant 2$.

7.2.1. *First case:* $\alpha_1 = \alpha_2$. In this situation we have $[\alpha] = [p, p, 1]$ for an integer $p \geqslant 1$, and there are exactly three admissible lines going through $[\alpha]$. These straight lines are principal, so by Remark 5.2 they are still straight lines for the metric $|\cdot, \cdot|$, forming 6 sectors each of angle $\pi/3$. The neighborhood $\mathbf{V}$ is therefore a cone over a metric graph Γ whose each edge has length $\pi/3$. In particular, we do not have an isometric embedding $\Gamma \to \mathbf{V}$, but we still have a natural induced action of $\mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$ on Γ. To show that $\mathbf{V}$ is CAT(0), by Theorem 7.2 it suffices to show that Γ is CAT(1), which amounts to saying that any cycle embedded in Γ is formed of at least 6 edges.

For $p = 1$, by Lemma 3.4 the graph Γ is the spherical Bruhat–Tits building of $\mathrm{GL}_3(\Bbbk)$, in other words the graph of incidence of the points and lines of the projective plane $\mathbb{P}^2(\Bbbk)$, therefore each cycle embedded in Γ has at least 6 edges.

Now assume $p \geqslant 2$. Note that for each $f \in \mathcal{F}$ the ball $\overline{B}(\nu_{f,[\alpha]}, \frac{\varepsilon}{4}) \subset \mathbf{E}^+_f$ is a half-disk, or in other words a cone over a path of 3 edges $\Gamma^+_f \subset \Gamma$. The map ρ_+ of Corollary 2.5(i) induces a map (for which we keep the same notation) $\rho_+ \colon \Gamma \to I_3$, where I_3 is a path of 3 edges — the link of $[p, p, 1]$ in ∇^+. This projection ρ_+ is an isomorphism in restriction to each Γ^+_f. We note

s the extremity of I_3 corresponding to the direction towards $[1,1,1]$ and q the other extremity, corresponding to the direction towards $[1,1,0]$. Note $s_f = \rho_+^{-1}(s) \cap \Gamma_f^+$, $q_f = \rho_+^{-1}(q) \cap \Gamma_f^+$ and $\Gamma_f = \Gamma_f^+ \cup \Gamma_{f\sigma}^+$ for $\sigma = (1,2) \in S_3$. Thus Γ_f is a cycle of 6 edges corresponding to the directions in $\mathbf{E}_f$. We will say that $s_f \in \rho_+^{-1}(s)$ is the *base vertex* of Γ_f (or of Γ_f^+).

Lemma 7.7.

(i) *The fiber $\rho_+^{-1}(q) \subset \Gamma$ is a singleton.*

(ii) *For all $f, g \in \mathcal{F}$, the intersection $\Gamma_f \cap \Gamma_g$ is either connected of length $\leqslant \pi$ (that is to say formed of at most 3 edges), or equal to $\{q_f, s_f\} = \{q_g, s_g\}$. In particular any cycle embedded in $\Gamma_f \cup \Gamma_g$ consists of at least 6 edges.*

(iii) *Each vertex $u \in \Gamma \smallsetminus \rho_+^{-1}\{q, s\}$ is incident to exactly one edge e whose projection $\rho_+(e)$ separates $\rho_+(u)$ from q in I_3. Also, if $u \in \Gamma_f^+$, then e is also in Γ_f^+.*

Proof. (i) For $p' \geqslant p$, the weight $(p', p', 1)$ belongs to each admissible half-space containing $(p, p, 1)$. Thus, by Corollary 4.7, for all $f \in \mathcal{F}$ we have $q_{\mathrm{id}} = f(q_{\mathrm{id}}) = q_f$. So $q_f = q_g$ for each $f, g \in \mathcal{F}$, as expected.

(ii) Suppose $\Gamma_f \cap \Gamma_g$ is neither $\{q_f, s_f\}$ nor $\{q_f\}$. Then there is $\omega \in \{\mathrm{id}, \sigma\}$ with $\Gamma_f \cap \Gamma_{g\omega}^+ \neq \{q_f, s_f\}, \{q_f\}$. Thus we can apply Corollary 3.6 to $f' = \omega^{-1} g^{-1} f$, hence the existence of a permutation $\sigma' \in S_n$ such that $\mathbf{E}_{f'} \cap \mathbf{E}_{\mathrm{id}} = \mathrm{Fix}(f'\sigma') \cap \mathbf{E}_{\mathrm{id}}$. By Proposition 4.4, the set $\mathrm{Fix}(f'\sigma') \cap \mathbf{E}_{\mathrm{id}}$ is an intersection of admissible hyperplanes. So $\Gamma_{f'} \cap \Gamma_{\mathrm{id}} = \mathrm{Fix}(f'\sigma') \cap \Gamma_{\mathrm{id}}$ is an intersection of paths of length π. By acting on $g\omega$ we obtain the same property for $\Gamma_f \cap \Gamma_{g\omega} = \Gamma_f \cap \Gamma_g$. By (i) the intersection $\Gamma_f \cap \Gamma_g$ contains q_f and it was assumed to be distinct from $\{q_f, s_f\}$, so it is connected of length $\leqslant \pi$.

(iii) Let f be such that $u \in \Gamma_f^+$, and consider the edge incident to $\rho_+(u)$ which separates this vertex from q in I_3. Its preimage e in Γ_f^+ satisfies the announced condition. Suppose that e' is another edge in Γ_g^+ satisfying this condition. So $\Gamma_f \cap \Gamma_g$ contains q_f and u but does not contain e. This contradicts (ii). ☐

Lemma 7.8. *Assume $s_f = s_{f'}$ for some $f, f' \in \mathcal{F}$. Then there is $f'' \in \mathcal{F}$ such that $\Gamma_f^+, \Gamma_{f'}^+ \subset \Gamma_{f''}$.*

Proof. By Corollary 2.5(ii) we can assume $f = \mathrm{id}$. By Proposition 3.2, we can write $f' = hg$ with $h \in M_\alpha, g \in L_\alpha$. Observe that s_{id} is fixed by any element of L_α, and also by assumption by $f' = hg$. Thus we obtain that h fixes s_{id}, which amounts to saying that h fixes $\nu_{\mathrm{id},[p-1,p-1,1]}$. So, by Proposition 3.2, and given the description of M_α given in Example 3.1(iv), h is a triangular automorphism of the form $(x_1 + P(x_3), x_2 + Q(x_3), x_3 + d)$ with $p - 1 \geqslant \deg P, \deg Q$. Still by Proposition 3.2, h fixes any valuation of $\mathbf{E}_{\mathrm{id}}^+$ whose weight $(\alpha_1', \alpha_2', \alpha_3')$ satisfies the inequality $\alpha_2' \geqslant (p-1)\alpha_3'$. In particular h fixes a neighborhood of $\nu_{\mathrm{id},[p,p,1]}$ in $\mathbf{E}_{\mathrm{id}}^+$, and therefore h fixes the three edges of Γ_{id}^+.

Write $g = b_1 \sigma b_2$ the Bruhat decomposition of g, where b_1, b_2 are upper triangular, and $\sigma = (1,2)$ or id. Then $f'' = hb_1$ is indeed suitable:

$$\Gamma_{\mathrm{id}}^+ = h\Gamma_{\mathrm{id}}^+ = h\Gamma_{b_1}^+ = \Gamma_{hb_1}^+, \text{ and}$$
$$\Gamma_{hg}^+ = hb_1\sigma\Gamma_{b_2}^+ = hb_1\sigma\Gamma_{\mathrm{id}}^+ \subset hb_1\Gamma_{\mathrm{id}} = \Gamma_{hb_1}.$$ ☐

We can now complete the proof of the first case:

Proposition 7.9. *If $\alpha = (p, p, 1)$ with $p \geqslant 2$ an integer, then Γ is* CAT(1).

Proof. Let γ be a cycle embedded in Γ. We orient each edge e of γ so that its projection $\rho_+(e) \subset I_3$ is oriented towards s. By Lemma 7.7(iii), the cycle γ can be decomposed into an even number of oriented paths, each of which ends on a base vertex. Moreover, each such path is contained in a Γ_f^+. If γ consists of at most 5 edges, then it contains at most 2 base vertices. By Lemma 7.8, there exists $f, g \in \mathcal{F}$ such that these two vertices are s_f, s_g, and $\gamma \subset \Gamma_f \cup \Gamma_g$. This contradicts Lemma 7.7(ii). ☐

7.2.2. *Second case:* $\alpha_1 > \alpha_2 = \alpha_3$. In this situation the hypothesis $\mathrm{mult}(\alpha) \geqslant 2$ implies $[\alpha] = [p, 1, 1]$ for an integer $p \geqslant 2$. By Remark 4.10, $[\alpha]$ is contained in $p + 2 \geqslant 4$ admissible lines, of which only 3 are principal.

Let $\mathbf{D} \subset \nabla^+$ be the half-disk of center $[\alpha]$ and radius $\varepsilon/4$ for the metric $|\cdot,\cdot|$. For $f \in \mathcal{F}$, let φ_f be the isometry identifying $\mathbf{D}$ with the half-disk $\mathbf{D}_f := \overline{B}(\nu_{f,[\alpha]}, \frac{\varepsilon}{4})$ in $\mathbf{E}^+_f$. Thus the neighborhood $\mathbf{V}$ discussed above is written $\mathbf{V} = \bigsqcup_{f\in\mathcal{F}} \mathbf{D}_f/\sim$. Note that the non-principal admissible lines passing through $[\alpha]$ are no longer lines for the metric $|\cdot,\cdot|$. In particular $\mathbf{V}$ is no longer a cone, and to bring us back to this situation we will use the framework described in Section 7.1.

We note $c_1, \dots, c_{p+1}$ the trace on the half-disc $\mathbf{D}$ of the admissible lines, except the principal line $\alpha_2 = \alpha_3$ (which would correspond to the diameter of $\mathbf{D}$). Note that each of these curves c_i admits $[\alpha]$ for extremity. In the Euclidean metric $|\cdot,\cdot|$ these curves are no longer in general line segments (precisely, they are still straight lines if and only if they come from a main line), and we note l_i the radius of $\mathbf{D}$ tangent to c_i at point $[\alpha]$. We note $C_i \subset \mathbf{D}$ the convex delimited by c_i, trace on $\mathbf{D}$ of the admissible half-space associated with c_i, and $L_i \subset \mathbf{D}$ the sector delimited by the tangent ray l_i and containing C_i. We number the c_i so that the C_i form an increasing family of convexes. In particular, c_1 and c_{p+1} are the only two principal line segments among c_i. Up to decreasing ε, for each $i = 1, \dots, p+1$ we can assume $c_i \cap l_{i-1} = [\alpha]$, where by convention l_0 is the radius at the edge of $\mathbf{D}$ contained in each C_i. It will also be useful to note l_{p+2} the other ray bordering $\mathbf{D}$. Observe that all this corresponds to the conventions of Section 7.1 and Figure 8 by taking $k = p + 1$.

Let $f, g \in \mathcal{F}$. By Corollary 2.7 and Proposition 4.4, $\rho_+(\mathbf{D}_f \cap \mathbf{D}_g)$ is either one of the convexes C_i, or the whole half-disk $\mathbf{D}$, or its diameter. Thus we are brought back to the framework of Section 7.1, where we introduced the "linearized quotient" $\mathbf{V}_0$ of $\mathbf{V}$, which is a cone over a metric graph Γ_0. As a graph, Γ_0 is isomorphic to the link of $\nu_{\mathrm{id},[\alpha]}$ in $\mathbf{X}$ seen as a polygonal complex determined by the admissible lines. Let e be an edge of Γ_0 corresponding to a region delimited by two curves of $\mathbf{X}$ sent by ρ_+ on c_i, c_{i+1}. Then the length of e in Γ_0 is equal to the angle between c_i, c_{i+1} for the metric $|\cdot,\cdot|$. By Proposition 7.5 (and again by Theorem 7.2) to show that $\mathbf{V}$ is CAT(0) it suffices to show:

Proposition 7.10. *If $\alpha = (p, 1, 1)$ with $p \geqslant 2$ an integer, then Γ_0 is* CAT(1).

Let $\Gamma^+_f \subset \Gamma_0$ be the path corresponding to the half-disk $\mathbf{D}_f \subset \mathbf{E}^+_f$. We call *base vertex* of Γ^+_f, denoted s_f, the extremity of Γ^+_f corresponding to the direction towards $[1, 1, 1]$. Note $\Gamma_f = \Gamma^+_f \cup \Gamma^+_{f\sigma}$ for $\sigma = (2, 3) \in S_3$.

Remark 7.11. Let s_f be a base point. Any edge of Γ_0 resulting from s_f has length $\pi/3$, because after projection corresponds to the angle between the two main line segments $c_{p+1} = l_{p+1}$ and l_{p+2}.

Proof of the Proposition 7.10. Note that Lemma 7.8 remains true with Γ_0 instead of Γ: in the proof it suffices to replace $\sigma = (1, 2)$ by $(2, 3)$ and h by an automorphism of the form $(x_1 + P(x_2, x_3), x_2 + d, x_3 + d')$ where $p > \deg P$. Similarly, the analogue of Lemma 7.7 remains valid.

The proof is now identical to the proof of Proposition 7.9, except that it is now Remark 7.11 which allows us to assert that if a cycle $\gamma \subset \Gamma_0$ has length $< 2\pi$, then it contains at most 2 base vertices. □

7.2.3. *Third case:* $\alpha_1 > \alpha_2 > \alpha_3$. As before, let Γ_0 be the link of $[\alpha]$ in $\mathbf{X}$ with the polygonal structure determined by the admissible lines. We have a projection $\rho_+ \colon \Gamma_0 \to S^1$ towards the combinatorial circle corresponding to the link of $[\alpha]$ in ∇^+. We equip the edges of S^1 (and of Γ_0 via ρ_+^{-1}) with the lengths coming from the angles between the admissible straight lines (which have become curved) relatively to the metric $|\cdot,\cdot|$.

Let $q \in S^1$ (resp. $s \in S^1$) be the point corresponding to the direction towards $[1, 0, 0]$ (resp. to the opposite direction, i.e. towards $[0, \alpha_2, \alpha_3]$). Let $I \subset S^1$ be the path from q to s, of length π, corresponding to the half-space $\frac{\alpha_2'}{\alpha_3'} \geqslant \frac{\alpha_2}{\alpha_3}$. Finally, let us note $e^q \subset S^1$ (resp. $e^s \subset S^1$) the edge which contains the path of length $\frac{\pi}{3}$ in I starting in q (resp. in s): see Figure 10.

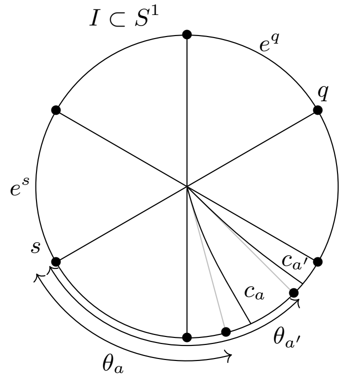

FIGURE 10. Notations for the case $\alpha_1 > \alpha_2 > \alpha_3$ (in the situation where s and q are vertices, and e^s, e^q have length exactly $\pi/3$).

For all $f \in \mathcal{F}$, let $\Gamma_f \subset \Gamma_0$ denote the length cycle 2π induced by $\mathbf{E}_f^+$, and let $s_f, q_f, e_f^q \subset \Gamma_f$ the respective preimages by ρ_+ of s, q, e^q. Thanks to Corollary 2.7 and the Proposition 4.4, for any pair $f, g \in \mathcal{F}$, either $\Gamma_f = \Gamma_g$, or else $\Gamma_f \cap \Gamma_g$ is a path of length $\leqslant \pi$ containing $e_f^q = e_g^q$. In particular, $\Gamma_f \cup \Gamma_g$ does not contain an embedded cycle of length $< 2\pi$. Note that q_f and s_f are not necessarily vertices (precisely, they are vertices if and only if α_2 is an integer multiple of α_3), so we call the s_f the *base points*. Similar to Remark 7.11, the path of length $\frac{2\pi}{3}$ in Γ_f centered at s_f does not contain any vertex distinct from s_f in its interior.

Note that by Proposition 7.5, the following proposition implies that $\mathbf{V}$ is CAT(0), which will end the proof of Proposition 7.1. Here we represent $\mathbf{V}$ as a union of half-discs by cutting each disc $\overline{B}(\nu_{f,[\alpha]}, \frac{\varepsilon}{4}) \subset \mathbf{E}_f^+$ along the diameter of direction $\frac{\alpha_2'}{\alpha_3'} = \frac{\alpha_2}{\alpha_3}$.

Proposition 7.12. *If $\alpha_1 > \alpha_2 > \alpha_3$, then Γ_0 is* CAT(1).

Proof. Let $\gamma \subset \Gamma_0$ be an embedded cycle of length $< 2\pi$. As in the proof of Proposition 7.9, we put an orientation on each edge of γ containing neither a base point nor a q_f so that its projection in $S^1 \smallsetminus q$ is oriented towards s. We divide each edge of γ containing a base point s_f (resp. a q_f) into two parts oriented towards s_f (resp. in the opposite direction to q_f). As in the previous cases, γ then decomposes into an even number of directed paths, each of which ends on a base point. Each such path is contained in a Γ_f and contributes at least $\frac{\pi}{3}$ to the length of γ. If there are only 2, $\gamma \subset \Gamma_f \cup \Gamma_g$ for some $f, g \in \mathcal{F}$ and we saw above that it was impossible. Thus we can assume that γ breaks down into 4 paths ending on 2 base points $s_f \neq s_g$, for some $f, g \in \mathcal{F}$. Also observe that $q \notin \rho_+(\gamma)$, because otherwise at least 2 of these paths would be of length π.

If s is a vertex, since $\rho_+^{-1}(e^q)$ is a single edge, similar to Lemma 7.7(iii) we prove the following statement: there is exactly one edge e_f^s (resp. e_g^s) incident to s_f (resp. s_g) whose projection separates s from q in I. In particular $e_f^s \subset \Gamma_f, e_g^s \subset \Gamma_g$. Thus, if γ goes through e_f^s, the projection $\gamma \to S^1$ is locally injective around s_f. Since $q \notin \pi(\gamma)$ this implies that e_g^s is also contained in γ. Therefore $\gamma \subset \Gamma_f \cup \Gamma_g$, contradiction. If s is not a vertex, the entire edges containing s_f, s_g are obviously borrowed by γ and we obtain a contradiction in the same way.

Thus we can assume that s is a vertex and that neither e_f^s nor e_g^s are contained in γ, in other words $\rho_+(\gamma) \subset S^1 \smallsetminus \operatorname{int} I$. Since s is a vertex, α satisfies an admissible equation of the form $\alpha_2 = p\alpha_3$, where s corresponds to the principal ray l to $[0, p, 1]$. By Remark 4.9, $[\alpha] = [m, p, 1]$ for integers $m > p > 1$. Let's write $m = pq + r$, with $0 \leqslant r < p$. So there exists $f_0, \ldots, f_3 \in \mathcal{F}$ such that

$$\gamma = \gamma_0 \cup \gamma_1^{-1} \cup \gamma_2 \cup \gamma_3^{-1}$$

where each γ_i is a path in Γ_{f_i}, oriented as before. In particular

$$\rho_+(\gamma_0 \cap \gamma_1) = \rho_+(\gamma_2 \cap \gamma_3) = s.$$

By Remark 4.10, the points $\rho_+(\gamma_1 \cap \gamma_2), \rho_+(\gamma_3 \cap \gamma_0)$ correspond to the directions towards $[m-pa, 0, 1]$ and $[m-pa', 0, 1]$, for some integers $0 \leqslant a, a' \leqslant q$. Let $c_a, c_{a'} \subset \nabla$ be the half-lines from $[\alpha]$ corresponding to these directions. Let θ_a (resp. $\theta_{a'}$) be the angle between l and c_a (resp. l and $c_{a'}$) relatively to the metric $|\cdot,\cdot|$ (for which $c_a, c_{a'}$ become curves). Up to a cyclic permutation of the γ_i, we can assume $a \geqslant a'$, which is equivalent to $\theta_a \leqslant \theta_{a'}$. The length of γ being $2(\theta_a + \theta_{a'})$, to obtain a contradiction it suffices to show $\theta_a + \theta_{a'} \geqslant \pi$. We will obtain this inequality in the form $(\theta_a - \pi/3) + (\theta_{a'} - \pi/3) \geqslant \pi/3$ (see Figure 10).

Using Corollary 2.5(ii) we can assume $f_0 = \mathrm{id}$. By Lemma 4.11(i), we can also assume $f_1, f_2, f_3 \in M_\alpha$. According to Lemma 4.12, we can write

$$f_1 = f_0^{-1} f_1 \sim_\nu h_0, \qquad f_1^{-1} f_2 \sim_\nu h_1,$$
$$f_2^{-1} f_3 \sim_\nu h_2, \qquad f_3^{-1} = f_3^{-1} f_0 \sim_\nu h_3,$$

where the h_i are triangular automorphisms of the following form:

$$h_0 = (x_1, x_2 + d' x_3^p, x_3), \qquad h_1 = (x_1 + P_1(x_2, x_3), x_2, x_3),$$
$$h_2 = (x_1, x_2 + d x_3^p, x_3), \qquad h_3 = (x_1 + P_3(x_2, x_3), x_2, x_3),$$

with P_1, P_3 homogeneous of degree m relatively to the variables x_2, x_3 of respective weights $p, 1$. By Lemma 4.11(iii), the group N_α is normal in M_α, hence $h_0 h_1 h_2 h_3 \sim_\nu \mathrm{id}$. Therefore

$$h_0 h_1 h_2 h_3 = (x_1 + P(x_2, x_3), x_2 + Q(x_3), x_3)$$

with $-\nu(P) < m, \deg Q = -\nu(Q) < p$. We have $Q(x_3) = (d + d') x_3^p$, therefore $d + d' = 0$. Since P_1, P_3 are homogeneous of degree m, so is P, hence $P = 0$ and $P_1(x_2 + d x_3^p, x_3) = -P_3(x_2, x_3)$. We write

$$P_3(x_2, x_3) = x_2^a (x_2 + d x_3^p)^b R_3(x_2, x_3)$$
$$P_1(x_2, x_3) = x_2^{a'} (x_2 - d x_3^p)^{b'} R_1(x_2, x_3)$$

where R_3 is prime with x_2 and $x_2 + d x_3^p$, and R_1 is prime with x_2 and $x_2 - d x_3^p$. Note that by Lemma 4.12(i), the exponents a and a' correspond to the integers defining the half-lines c_a and $c_{a'}$ introduced above. The equality $P_1(x_2 + d x_3^p, x_3) = -P_3(x_2, x_3)$ is equivalent to

$$(x_2 + d x_3^p)^{a'} x_2^{b'} R_1(x_2 + d x_3^p, x_3) = -x_2^a (x_2 + d x_3^p)^b R_3(x_2, x_3).$$

Finally $a' = b$, and $pa + pa' = p(a + b) \leqslant m$. We conclude thanks to Lemma 5.9 that $(\theta_a - \pi/3) + (\theta_{a'} - \pi/3) \geqslant \pi/3$, hence the expected contradiction. Precisely, in the case of equality $p(a + a') = m$, the radii c_2, c_3 of Lemma 5.9 correspond respectively to the half-lines $c_a, c_{a'}$, and we obtain $\theta_a = \pi/3 + \theta_{12}$, $\theta_{a'} = \pi/3 + \theta_{13}$, and $(\theta_a - \pi/3) + (\theta_{a'} - \pi/3) = \pi/3$. The case of a strict inequality $p(a + a') < m$ corresponds to larger angles θ_a, $\theta_{a'}$, and therefore is even more favorable. □

In the proof of Proposition 7.5 a good part of the complications comes from the fact that $d_{\mathbf{X}}$ is not the metric of the polygonal structure from $\mathbf{X}$, which forces us to first linearize the situation before we can conclude by studying the link of each point. It would seem *a priori* natural to want to keep the metric of the simplex, for which all the admissible lines, principal or not, are indeed lines. We conclude this section with an example that shows the problem with this approach.

Example 7.13 (Figure 11)**.** Let $p, q \geqslant 1$ be two integers, and denote $m = pq$, $\alpha = (m, p, 1)$, $\nu = \nu_{\mathrm{id},[\alpha]}$. Automorphisms

$$f = (x_1 + x_2^q, x_2, x_3) \quad \text{et} \quad g = (x_1 + x_3^m, x_2, x_3)$$

belong to the group $M_\alpha \subset \mathrm{Stab}(\nu)$, and commute. By following four arcs of circles centered in ν in the successive appartments $\mathbf{E}_{\mathrm{id}}$, $\mathbf{E}_f$, $\mathbf{E}_{fg}$ and $\mathbf{E}_g$, where the weights vary between the principal half-lines directed towards $[0, 0, 1]$ and $[0, 1, 0]$, we get a cycle. For the metric of the graph Γ_0 corresponding to the link of the linearized neighborhood $\mathbf{V}_0$, each of these arcs has length $2\pi/3$, and therefore the yaw has total length $8\pi/3 > 2\pi$. On the other hand, if we use the metric of the simplex, each of these arcs has length $\pi/3 + \varepsilon$ with ε tending towards 0 when q tends towards infinity, and thus we produce loops of length $4\pi/3 + 4\varepsilon < 2\pi$. We see that

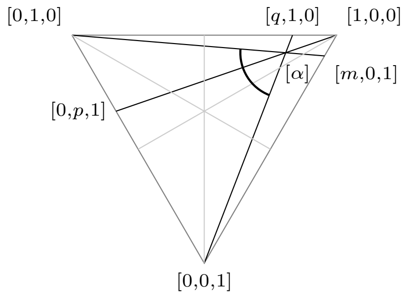

FIGURE 11. An arc of length $2\pi/3$ for the metric $|\cdot,\cdot|$, but of length $\pi/3+\varepsilon$ for the metric of the simplex.

equipping each apartment with the metric of the simplex would lead to a metric on $\mathbf{X}$ which would no longer be locally CAT(0).

8. Proof of main results

We first obtain Theorem A announced in the introduction:

Proof of Theorem A. We already know that the metric space $\mathbf{X}_3$ is complete by Lemma 5.8. Moreover $\mathbf{X}_3$ is simply connected by Proposition 6.3, and locally CAT(0) by Proposition 7.1. The result then follows from the Cartan–Hadamard Theorem [BH99, II.4.1(2)], which asserts that a complete, locally CAT(0) and simply connected metric space is CAT(0). □

To obtain the linearizability of finite subgroups, we use the following criterion, which relies on an averaging argument. Here we take advantage of the vector structure of $\Bbbk^n$ to define an endomorphism by average of automorphisms (in general such an average has no reason to be invertible).

Lemma 8.1 ([BFL14, Lemma 5.1])**.** *Let $\Bbbk$ be a field of characteristic zero, and G a subgroup of the group of bijections of $\Bbbk^n$ admitting a semi-direct product structure $G = M \rtimes L$ with $L \subseteq \mathrm{GL}_n(\Bbbk)$. Suppose that for any finite sequence $m_1, \dots, m_r$ of M, the mean $\frac{1}{r}\sum_{i=1}^r m_i$ is still in M. Then any finite subgroup of G is conjugate by an element of M to a subgroup of L.*

Proof of Corollary B. In fact, we will show that each finite subgroup F of $\mathrm{Tame}(\Bbbk^3)$ is conjugated by an element of $\mathrm{Tame}(\Bbbk^3)$ (and not only of $\mathrm{Aut}(\Bbbk^3)$) to a subgroup of $\mathrm{GL}_3(\Bbbk)$.

The group $F \subset \mathrm{Tame}(\Bbbk^3)$ acts by isometries on $\mathbf{X}_3$, which is complete CAT(0) by Theorem A. By [BH99, II.2.8(1)], F admits (at least) one global fixed point $\nu \in \mathbf{X}_3$, which can be constructed as the "circumcenter" of an arbitrary orbit. By Corollary 2.5(ii), by conjugating by an element of $\mathrm{Tame}(\Bbbk^3)$ we can assume that $\nu = \nu_{\mathrm{id},[\alpha]} \in \mathbf{E}^+_{\mathrm{id}}$. By Proposition 3.2, $F \subset \mathrm{Stab}(\nu) = M_\alpha \rtimes L_\alpha$. Since the group of triangular automorphisms M_α is stable by mean, we can therefore conclude by Lemma 8.1. □

9. Appendix: other properties of X

9.1. **Faithful action.**

Proposition 9.1. *The action of* $\mathrm{Tame}(\Bbbk^n)$ *on* $\mathbf{X}$ *is faithful.*

Proof. Let f be a non-trivial element of $\mathrm{Tame}(\Bbbk^n)$. We want to show that the map induced by f on $\mathbf{X}$ is not the identity. Suppose that f acts trivially on the standard appartment $\mathbf{E}_{\mathrm{id}}$ (otherwise there is nothing to show). Then by Corollary 3.3,

$$f = (c_1x_1 + t_1, c_2x_2 + t_2, \dots).$$

As f is non-trivial, by conjugating by an element of the symmetric group, we can also assume that we are not in the case of $c_1 = c_2 = 1$ and $t_1 = t_2 = 0$.

Let $r \geqslant 2$ be a prime integer with the characteristic of $\Bbbk$. Consider $g = (x_1 - x_2^r, x_2, \dots, x_n)$ and $\alpha \in \Pi$, and compare $\nu = \nu_{g,\alpha}$ and $f \cdot \nu$ by evaluating these valuations on the polynomial $P = x_1 + x_2^r$. We have

$$\begin{aligned}
-\nu(x_1 + x_2^r) &= \alpha_1; \\
-(f \cdot \nu)(x_1 + x_2^r) &= -\nu(c_1x_1 + t_1 + (c_2x_2 + t_2)^r) \\
&= -\nu_{\mathrm{id},\alpha}(c_1x_1 - c_1x_2^r + t_1 + c_2^r x_2^r + rc_2^{r-1}t_2x_2^{r-1} + \cdots + t_2^r) \\
&= \begin{cases} \alpha_1 & \text{si } c_2^r = c_1, t_2 = 0, \\ \max\{\alpha_1, (r-1)\alpha_2\} & \text{if } c_2^r = c_1, t_2 \neq 0 \\ \max\{\alpha_1, r\alpha_2\} & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, except in the first case, the two valuations are distinct as soon as $\alpha_2 > \alpha_1$, and in this case their homothety classes are distinct because otherwise $-\nu(x_2) = -(f \cdot \nu)(x_2) = \alpha_2$.

Finally, consider the case where $c_2^r = c_1$ and $t_2 = 0$. If $c_1^r \neq c_2$, or if $t_1 \neq 0$, we can reproduce the previous argument after having conjugated by the transposition which exchanges x_1 and x_2. Otherwise, we have $t_1 = t_2 = 0$, and $c_1^r = c_2$ and $c_2^r = c_1$ for any integer r prime with the characteristic of $\Bbbk$. But this implies $c_1 = c_2 = 1$: if $\operatorname{char} \Bbbk \neq 2$, the equations $c_1^2 = c_2$, $c_2^2 = c_1$ plus an additional equation $c_1^r = c_2$ with $r \equiv 1 \mod 3$ already suffice, and if $\operatorname{char} \Bbbk = 2$ it follows from the two equations $c_1^3 = c_2$, $c_1^5 = c_2$. Either way, we get a contradiction, which concludes the proof. □

9.2. **A Retraction Application.** The proof of Proposition 6.3 is an algebraic version of a first topological approach which finally turn out to be more delicate to implement. The idea is to build in any dimension n a natural map, candidate to be a homotopy equivalence, from $\mathbf{X}_n$ to a simplicial complex $\mathbf{Y}_n$ studied in [Lam19, LP19]. We briefly indicate the construction, which leads to interesting questions and which also sheds more light on the key role played by Shestakov–Umirbaev and Kuroda's theory of reductions (encapsulated above in Theorem 6.4, and here in the simple connectedness of $\mathbf{Y}_3$).

We first recall the construction of the simplicial complex which was denoted $\mathcal{C}_n$ in [Lam19], but which we will denote here $\mathbf{Y}_n$ in order to avoid any confusion with the group complex used above. For all $1 \leqslant r \leqslant n$, we call *$r$-uplet of components* a morphism

$$\begin{aligned} f \colon \Bbbk^n &\to \Bbbk^r \\ x = (x_1, \dots, x_n) &\mapsto (f_1(x), \dots, f_r(x)) \end{aligned}$$

which can extend to a tame automorphism $f = (f_1, \dots, f_n)$ of $\Bbbk^n$. We define n distinct types of vertices, considering r-tuples of components modulo composition by an affine automorphism on the range, $r = 1, \dots, n$:

$$[f_1, \dots, f_r] := A_r(f_1, \dots, f_r) = \{a \circ (f_1, \dots, f_r); a \in A_r\}$$

where $A_r = \mathrm{GL}_r(\Bbbk) \ltimes \Bbbk^r$ is the affine group in dimension r. We will say that $[f_1, \dots, f_r]$ is a vertex of *type* r. Then, for any tame automorphism $(f_1, \dots, f_n) \in \mathrm{Tame}(\Bbbk^n)$ we attach a $(n-1)$-simplex on the vertices $[f_n]$, $[f_{n-1}, f_n]$, …, $[f_1, \dots, f_n]$. This definition does not depend on a choice of representatives, and produces a simplicial complex $\mathbf{Y} = \mathbf{Y}_n$ of dimension $(n-1)$ on which the tame group acts by isometries, via

$$g \cdot [f_1, \dots, f_r] := [f_1 \circ g^{-1}, \dots, f_r \circ g^{-1}].$$

Example 9.2. The complex $\mathbf{Y}_2$ is the Bass–Serre tree of the splitting $\mathrm{Aut}(\Bbbk^2) = A_2 *_{A_2 \cap E_2} E_2$, where A_2 is the affine group and $E_2 = \{(ax_1 + P(x_2), bx_2 + d); a, b \neq 0\}$. In characteristic zero, the complex $\mathbf{Y}_3$ is the universal development of the triangle of groups A, B, C and their intersections (see Theorem 6.4).

We parameterize the simplex $[x_n], [x_{n-1}, x_n], \dots, [x_1, \dots, x_n]$ as follows. Let $\Delta \subset \Pi \subset \mathbb{R}^n$ be the $(n-1)$-simplex of vertices

$$v_1 = (2, \dots, 2, 1), v_2 = (2, \dots, 2, 1, 1), \dots, v_n = (1, \dots, 1, 1).$$

We define $\iota\colon \Delta \to \mathbf{Y}$ as the affine map sending each v_r on the vertex of type r $[x_{n-r+1}, \dots, x_n]$. Note that $\Delta \subset \Pi^+$ is identified by projectivization with a simplex of ∇^+ which is also denoted by Δ.

We now define a retraction $\nabla^+ \to \Delta$. Let $\alpha \in \Pi^+$. We define a new weight $\alpha' \in \Delta$ by setting, for all $i = 1, \dots, n$,

$$\alpha'_i = \min(2, \alpha_i/\alpha_n),$$

which only depends on $[\alpha] \in \nabla^+$. We define $\pi\colon \mathbf{X} \to \mathbf{Y}$ by $\pi(\nu_{f,[\alpha]}) = f \cdot \iota(\alpha')$. The application π sends the chamber $\mathbf{E}^+_{\mathrm{id}}$ on the simplex $[x_n], [x_{n-1}, x_n], \dots, [x_1, \dots, x_n]$. Figure 12 illustrates the effect of π in the case $n = 3$.

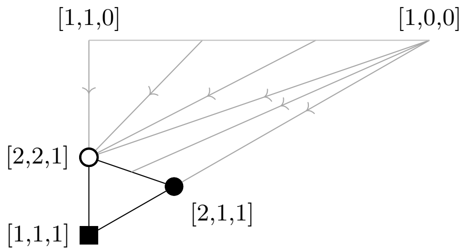

Figure 12. The map π in restriction to a chamber of $\mathbf{X}_3$.

Lemma 9.3. *If $\nu_{g,[\alpha]} = \nu_{f,[\beta]}$, then $f \cdot \iota(\alpha') = g \cdot \iota(\beta')$. In particular π is a well-defined application of $\mathbf{X}$ to $\mathbf{Y}$.*

Proof. By Proposition 2.4, we can assume $g = \mathrm{id}$ and $\beta = \alpha$. We therefore have $\nu_{\mathrm{id},[\alpha]} = \nu_{f,[\alpha]}$ where, by the proposition 3.2, $f \in M_\alpha \rtimes L_\alpha$. We normalize α by asking for $\alpha_n = 1$.

We need to show that $f \cdot \iota(\alpha') = \iota(\alpha')$. For that it suffices to show that f fixes each vertex of a simplex containing $\iota(\alpha')$. Consider $(i_1, \dots, i_k)$ the collection of indices $1 \leqslant i \leqslant n$ such that $\alpha'_{i-1} > \alpha'_i$, or by convention, $\alpha'_0 = 2$. Note that α' is a convex combination of v_{i_j}, so $\iota(\alpha')$ is contained in the vertex simplex v_{i_j}.

For each j the coordinates $f_{i_j}, \dots, f_n$ of f correspond to a automorphism in the variables $x_{i_j}, \dots, x_n$, since by definition of i_j we have $\alpha'_{i_j-1} > \alpha'_{i_j}$, and therefore also $\alpha_{i_j-1} > \alpha_{i_j}$. Moreover this automorphism is affine, because $2 > \alpha'_{i_j} = \alpha_{i_j}$. This gives the central equality in

$$f^{-1} \cdot \iota(v_{i_j}) = [f_{i_j}, \dots, f_n] = [x_{i_j}, \dots, x_n] = \iota(v_{i_j}). \qquad \square$$

Remark 9.4. For all $f = (f_1, \dots, f_n) \in \mathrm{Tame}(\Bbbk^n)$, the map π induces a local bijection between a neighborhood of the valuation $\nu_{f^{-1},[1,\dots,1]}$ in $\mathbf{X}$ and a neighborhood of the vertex of type n corresponding to $[f_1, \dots, f_n] \in \mathbf{Y}$.

Following [Lam19], recall that the vertices of type i at distance 1 of $[f_1, \dots, f_n]$ are in one-to-one relationship with the vector subspaces of dimension i of

$$\mathrm{vect}(f_1, \dots, f_n) \subset \Bbbk[x_1, \dots, x_n],$$

and similarly simplices containing $[f_1, \dots, f_n]$ match the flags in this space. We thus obtain an alternative proof that the link in $\mathbf{X}_n$ of a weight valuation $[1, \dots, 1]$ is isomorphic to the standard spherical building of $\mathrm{GL}_n(\Bbbk)$ (Lemma 3.4).

Question 9.5. Are the fibers of π contractile? If so, does this imply that π is a homotopy equivalence? A positive answer would give another proof of Proposition 6.3 thanks to the simple connectedness of $\mathbf{Y}_3$[Lam19, Proposition 5.7]. Moreover, once Theorem A is established, this would give another proof that $\mathbf{Y}_3$ is contractile (see [LP19, Theorem A]).

9.3. **Comparison with the classic notion of building.** We now justify the remark mentioned in the introduction that the apartment system $\{\mathbf{E}_f \mid f \in \mathrm{Tame}(\mathbb{k}^n)\}$ does not make $\mathbf{X}$ the Davis realization of a building [AB08, Definitions 4.1, 12.65].

Lemma 9.6. *Let $f \in \mathrm{Tame}(\mathbb{k}^n)$, and $[\alpha] \in \nabla$ distinct from $[1,\dots,1]$ and multiplicity at least 1. Then any neighborhood of $\nu_{f,[\alpha]}$ contains two points which do not belong to the same apartment.*

Proof. By Corollary 2.5(ii), without loss of generality we can assume $f = \mathrm{id}$. We note $\delta > 0$ the minimum of the metrics $|\sigma_1([\alpha]), \sigma_2([\alpha])|$ between two distinct points of the orbit of $[\alpha]$ for the action of S_n on ∇. Note that δ is well defined because the assumption $[\alpha] \neq [1,\dots,1]$ ensures that this orbit contains at least 2 points. A real $r > 0$ being fixed, let us consider a point α' of the open segment connecting α to $[1,\dots,1]$ such that $|[\alpha],[\alpha']| \leqslant r$. Then α, α' have the same symmetries, i.e. for all $\sigma \in S_n$, $\sigma(\alpha) = \alpha$ if and only if $\sigma(\alpha') = \alpha'$. Moreover there exists an admissible half-space containing α but not α', and thus by Remark 4.3 there exists $g \in \mathrm{Stab}(\nu_{\mathrm{id},[\alpha]})$ not fixing $\nu_{\mathrm{id},[\alpha']}$. Suppose there is an apartment $\mathbf{E}_h$ containing $\nu_{g,[\alpha']}$ and $\nu_{\mathrm{id},[\alpha']}$. Then we should have permutations $\sigma_1, \sigma_2 \in S_n$ such that

$$\nu_{h,[\sigma_1(\alpha')]} = \nu_{g,[\alpha']} \neq \nu_{\mathrm{id},[\alpha']} = \nu_{h,[\sigma_2(\alpha')]},$$

so $\sigma_1(\alpha') \neq \sigma_2(\alpha')$. This leads to a contradiction as soon as $r < \delta/4$, because using Remark 5.7 we get:

$$\begin{aligned}
2r &\geqslant d_{\mathbf{X}}(\nu_{h,[\sigma_1(\alpha')]}, \nu_{\mathrm{id},[\alpha]}) + d_{\mathbf{X}}(\nu_{\mathrm{id},[\alpha]}, \nu_{h,[\sigma_2(\alpha')]}) \geqslant d_{\mathbf{X}}(\nu_{h,[\sigma_1(\alpha')]}, \nu_{h,[\sigma_2(\alpha')]}) \\
&= |[\sigma_1(\alpha')],[\sigma_2(\alpha')]| \geqslant |[\sigma_1(\alpha)],[\sigma_2(\alpha)]| - |[\sigma_1(\alpha')],[\sigma_1(\alpha)]| - |[\sigma_2(\alpha')],[\sigma_2(\alpha)]| \\
&\geqslant \delta - 2r.
\end{aligned}$$

□

In contrast, by Lemma 3.4, the link of a weight point $[1,\dots,1]$ is a spherical building.

Similarly, for $n = 3$ one could demonstrate that the link of each valuation of weight $[p,p,1]$ is a “generalized hexagon”, i.e. a graph of diameter 3 and systole 6. In particular this link is a spherical building for the apartment system made up of all the hexagons (some of which are not the trace of an apartment $\mathbf{E}_f$). Our proof of this fact being a bit long and computational we omit it, but we wanted to mention it as a motivation for the following remark.

Remark 9.7. One could wonder if $\mathbf{X}$ could become Davis' realization of a building after expanding the apartment system. For $n = 2$ it is true, because $\mathbf{X}_2$ is a complete tree without leaves (see Section 6.2). On the other hand, the answer is always negative for $n \geqslant 3$. Indeed, in the Davis realization of a Euclidean building of dimension $\geqslant 2$, through two points passes a copy of $\mathbb{R}^2$ isometrically embedded. In particular, for all points ν, ν', ν'' and $\delta > 0$ with $|\nu,\nu'| = |\nu,\nu''| = \delta$, $|\nu',\nu''| = 2\delta$, there exists a point $\nu^\perp$ with $|\nu,\nu^\perp| = \delta$, $|\nu',\nu^\perp| = |\nu'',\nu^\perp| = \sqrt{2}\delta$.

However, take $\nu = \nu_{\mathrm{id},[\alpha]} \in \mathbf{X}$ with $[\alpha] \in \nabla$ of multiplicity 1 in an admissible hyperplane c which does not pass through any vertex of ∇ (and which therefore contains no line in the metric $|\cdot,\cdot|$). Let $\mathbf{n}$ be the vector normal to c in $[\alpha]$ directed towards the exterior of the admissible half-space L delimited by c, in the metric $|\cdot,\cdot|$. For $\beta = (\beta_i)$ with $\beta_i = \log \alpha_i$ let us put $\beta' = \beta + \delta\mathbf{n}$ and $\alpha'_i = \exp \beta'_i$, for $\delta = \frac{\varepsilon}{4}$ coming from Lemma 5.5(b). Let $\nu' = \nu_{\mathrm{id},[\alpha']}$ and $\nu'' = \nu_{g,[\alpha']}$ for a g which fixes ν but not ν' (see Remark 4.3). Since L is convex, $|\nu',\nu''| = 2\delta$. However, the intersection of the sphere of radius $\sqrt{2}\delta$ centered in ν' (resp. in ν'') with the sphere of radius δ centered in ν is contained in $\mathbf{E}_{\mathrm{id}} \smallsetminus \mathbf{E}_g$ (resp. $\mathbf{E}_g \smallsetminus \mathbf{E}_{\mathrm{id}}$). So a $\nu^\perp$ like above does not exist.

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