On the structure of the automorphism group of certain non compact surfaces

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This is a translation of the paper *Sur la structure du groupe d'automorphismes de certaines surfaces affines* [1] , Publ. Mat. 49, no. 1, p. 3-20, 2005.

Abstract

We describe the structure of the group of algebraic automorphisms of the following surfaces 1) $\mathbb{P}^1_k \times \mathbb{P}^1_k$ minus a diagonal; 2) $\mathbb{P}^1_k \times \mathbb{P}^1_k$ minus a fiber. The motivation is to get a new proof of two theorems proven respectively by L. Makar-Limanov and H. Nagao. We also discuss the structure of the semi-group of polynomial proper maps from $\mathbb{C}^2$ to $\mathbb{C}^2$.

Keywords: algebraic automorphisms, amalgamated products, polynomial proper maps.

1 Introduction

In [6] we explained how to get a geometric proof of the classical theorem of Jung-Van der Kulk which describes the structure of polynomial automorphisms of the affine plane $\mathbb{A}^2_k$ (k is an arbitrary field):

Theorem 1 (Jung - Van der Kulk) *The group of polynomial automorphisms of the affine plane $\mathbb{A}^2_k$ is the amalgamated product of the affine group*

$$A = \{(x,y) \mapsto (a_1x + b_1y + c_1, a_2x + b_2y + c_2);\ a_i, b_i, c_i \in k, a_1b_2 - a_2b_1 \neq 0\}$$

and the elementary group

$$E = \{(x,y) \mapsto (\alpha x + P(y), \beta y + \gamma);\ \alpha, \beta \in k^*, \gamma \in k, P \in k[X]\}$$

along their intersection.

In this article we apply the same line of argument to obtain a description of the automorphism group of some other classes of surfaces.

Precisely we are interested in the automorphism group of two surfaces distinct from $\mathbb{A}^2_k$: namely $\mathbb{P}^1_k \times \mathbb{P}^1_k$ minus either a diagonal or a fiber. Our motivation is to get an unified geometric proof of two known theorems due respectively to L. Makar-Limanov and H. Nagao. The result by Makar-Limanov is about the description of the automorphisms of an affine quadric surface. This result

[1] The choice of "affine" in the french title was infortunate, since one of the surface under study (namely $\mathbb{P}^1_k \times \mathbb{P}^1_k$ minus a fiber) is only quasi-projective...

(applied over the base field $k = \mathbb{C}(t)$) was used in [7] in the study of a certain class of automorphisms of $\mathbb{C}^3$. The result of Nagao is a description of the group $\mathrm{GL}(2, k[X])$. This theorem was given a new (and more general, see [9]) proof by Serre [11], using an action of $\mathrm{GL}_2(k[X])$ on a simplicial tree.

Note that V. Danilov and M. Gizatullin [4] studied the more general situation where G is the automorphism group of an affine surface which admits a projective compactification with exactly one rational curve at infinity; thus their results contained ours theorems 4 and 5. Their method is to show that G acts on a tree whose vertices are identified with certain compactifications of the surface, with fundamental domain an edge. Then applying the theory of Bass and Serre [11] they obtain a description of G as an amalgamated product. Our proof is also geometric bur nevertheless quite different. We use the fact that any birational maps between surfaces admits a factorization by a sequence of blow-ups to obtain generators for the group G, and then the question of the absence of relations (*i.e.* the structure of amalgamated product) is nothing more than an easy remark (see page 11).

In the last section we consider a different problem: the description of a class of endomorphisms (but no longer automorphisms) of the complex affine plane. Precisely we discuss the structure of the semi-group of proper polynomial maps from $\mathbb{C}^2$ to $\mathbb{C}^2$.

In this paper we use repetitively the following basic result, which is true over any algebraically closed field (see [5, th. 5.5]).

Theorem 2 (Zariski) *Any birational map between surfaces is a sequence of blow-ups and blow-downs; in other words if X, Y are (smooth) surfaces and*

$$g : X \dashrightarrow Y$$

is a birational map (which is not an isomorphism), then there exists a surface M and two sequences of blow-ups π_1 and π_2 such that the following diagram commutes:

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "g"'] & & Y \end{tikzcd}$$

We will refer to this diagram as the diagram of Zariski associated with g. Each one of the points blown-up in the sequence π_1 is called a base point of g; thus these points either belongs to X (one says that the base point is proper) or to a surface obtained after blowing-up X (one says that the base point is in an infinitely close neighborhood of a point of X [2]). We will note $\#\mathrm{ind}(g)$ the number of base points of g (proper or not). We will use certain compactifications of $\mathbb{A}^2$ called Hirzebruch surfaces, which are defined by $F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. All we need to know here is that such a surface, which is a fibration in $\mathbb{P}^1$ over $\mathbb{P}^1$, compactifies $\mathbb{A}^2$ by adding two transverse rational curves: one fiber at infinity f_∞ and a section s_∞ with self-intersection $-n$. We will use the notation $f_\infty(F_n)$ and $s_\infty(F_n)$ if there is a risk of ambiguity on the Hirzebruch surface at hand.

[2]if $\pi : \tilde{X} \to X$ is a blow-up sequence, if q belongs to one of the exceptional divisors and if $\pi(q) = p \in X$, one says that q is in the infinitely close neighborhood of p.

A basic example is F_1 which is obtained by blowing up a point p on the line at infinity in $\mathbb{P}^2$: in this case the fibration comes from the pencil of lines passing through p, f_∞ is the transform of the line at infinity and s_∞ is the exceptional divisor.

2 On the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ minus a rational curve

In the sequel k will be an arbitrary field (except in the context of the theorem 6 where we assume $\mathrm{char}(k) \neq 2$). We state two theorems describing the structure of the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ minus a diagonal or a fiber. Then we explain the links between these results and two theorems already found in the literature (due respectively to Makar-Limanov and Nagao), and finally we give the proofs in the last paragraph.

2.1 Statements of the theorems

Note $[t_0 : t_1], [u_0 : u_1]$ the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, and identify $\mathbb{A}^2$ with the open set $t_1 \neq 0, u_1 \neq 0$. We note $t = \frac{t_0}{t_1}, u = \frac{u_0}{u_1}$ the coordinates in $\mathbb{A}^2$. Let D be the diagonal of equation $t_0u_1 + t_1u_0 = 0$. We note A_D the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus D$ which extend as biregular automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$. This group is generated by $(t, u) \mapsto (u, t)$ and by the automorphisms of the form

$$(t, u) \mapsto \left(\frac{at + b}{ct + d}, \frac{-au + b}{cu - d}\right); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, k).$$

On the other hand we note E_D the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus D$ which preserve the pencil of curves $t + u = cte$. From the lemma 9 (see below) one can obtain an explicit description of the elements in E_D:

Lemma 3 *The group E_D is generated by $(t, u) \mapsto (u, t)$ and by the automorphisms of the form*

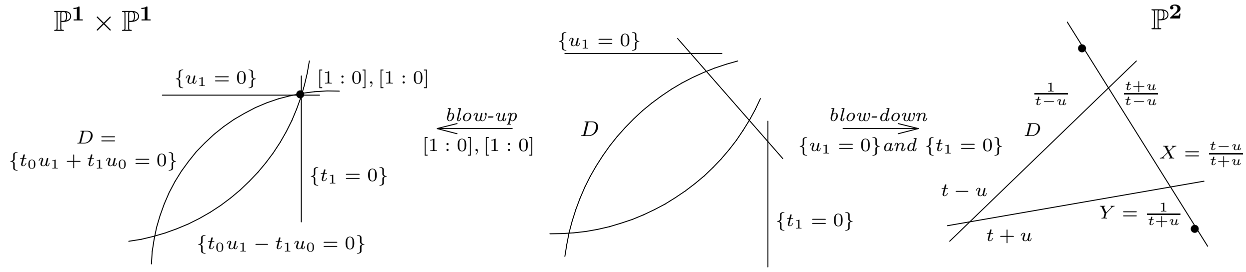
$$(t, u) \mapsto \left(\alpha t + P\left(\frac{1}{t + u}\right), \alpha u - P\left(\frac{1}{t + u}\right)\right); \ \alpha \in k^*, P \in k[X].$$

Proof. First note that

$$A_D \cap E_D = \{(t, u) \mapsto (at + b, au - b) \text{ ou } (au + b, at - b);\ a \in k^*, b \in k\}.$$

Now pick $g \in E_D \setminus A_D$, that we see as a birational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$. The pencil generated by the lines $t + u = cte$ contains a unique singular element, namely $\{t_1 = 0\} \cup \{u_1 = 0\}$. Thus these two are globally invariant under g, and up to composition by $(t, u) \mapsto (u, t)$ one can assume that each one of these lines is invariant under g. Furthermore the point $[1 : 0], [1 : 0]$, which is the unique base point of the pencil, is either a base point or a fixed point for g; but the later possibility is impossible since $g^{-1}(D)$ is the unique proper base point of g (the unicity is given by the lemma 9.1). Blow-up $[1 : 0], [1 : 0]$ on the initial and final $\mathbb{P}^1 \times \mathbb{P}^1$, and blow-down the strict transforms of $\{t_1 = 0\}$ and $\{u_1 = 0\}$ (*see* figure).

The map g is now a birational map from $\mathbb{P}^2$ to $\mathbb{P}^2$, which induces an automorphism of $\mathbb{A}^2 = \mathbb{P}^2 \setminus D$. Indeed the second base point of g is still located on the transform of D, because by the



assertion 5 of the lemma 9 we have to perform three successive blow-ups on D in order to get $D^2 = -1$ and so to be able to blow-down its transform. In the local chart $X = \frac{t-u}{t+u}, Y = \frac{1}{t+u}$ we can write g as

$$g : (X, Y) \mapsto (aX + P(Y), bY + c);\ a, b \in k^*, c \in k, P \in k[Y].$$

Since the line $\{Y = 0\}$ contains two fixed points for g (the points where we blew-down $\{t_1 = 0\}$ and $\{u_1 = 0\}$), we can in fact write $g : (X, Y) \mapsto (X + YP(Y), bY)$. Coming back to the coordinates t, u we obtain the statement. □

Theorem 4 *The automorphism group of* $\mathbb{P}^1 \times \mathbb{P}^1 \setminus D$ *is equal to the product of* A_D *and* E_D *amalgamated along their intersection.*

Now let $F \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the fiber $t_1 = 0$. Note A_F the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus F$ which extend as biregular automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$; this group corresponds to the following automorphisms:

$$(t, u) \mapsto \left(\alpha t + \beta, \frac{au + b}{cu + d}\right);\ \alpha \in k^*, \beta \in k, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, k).$$

On the other hand we note E_F the group of elementary automorphisms

$$(t, u) \mapsto (\alpha t + \beta, \gamma u + P(t));\ \alpha, \gamma \in k^*, \beta \in k, P \in k[X].$$

Theorem 5 *The automorphism group of* $\mathbb{P}^1 \times \mathbb{P}^1 \setminus F$ *is equal to the product of* A_F *and* E_F *amalgamated along their intersection.*

2.2 A theorem of Makar-Limanov

Here we explain the relation between the theorem 4 and the description of the automorphism group G of the smooth quadric surface $V_\lambda \subset \mathbb{A}^3$ with equation $y^2 + xz = \lambda$, where $\lambda \in k^*$ and $\mathrm{char}(k) \neq 2$. Note that if $\lambda = 0$ or if $\mathrm{char}(k) = 2$ this surface is no longer smooth. Makar-Limanov shows that in this case the automorphism group is slightly bigger: precisely we have to add the homotheties with center the singularity. We restrict ourselves to the smooth case.

There exist two natural subgroups of G. On one hand we have the orthogonal group $\mathrm{O}(3, k)$ associated with the quadratic form $y^2 + xz$, generated by $(x, y, z) \mapsto (x, -y, z)$ and by $\mathrm{SO}(3, k)$. The later group is composed of the matrices

$$\frac{1}{ad - bc} \begin{pmatrix} a^2 & 2ab & -b^2 \\ ac & ad + bc & -bd \\ -c^2 & -2cd & d^2 \end{pmatrix} \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, k).$$

Recall that the identification between $\mathrm{PGL}(2,k)$ and $\mathrm{SO}(3,k)$ can be obtained via the action by conjugacy of $\mathrm{PGL}(2,k)$ on the 2×2 matrices with trivial trace, where we identify k^3 to $(x,y,z)\mapsto \begin{pmatrix} -y & x \\ z & y \end{pmatrix}$. this action preserve the determinant which is equal, up to a sign, to the quadratic form y^2+xz. Note that since any matrix in $\mathrm{GL}(2,k)$ can be written as a composition of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we see that $\mathrm{O}(3,k)$ is generated by matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ -c^2 & -2c & 1 \end{pmatrix}, \quad \begin{pmatrix} a/d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d/a \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand we have E_G the group of automorphisms of the form[3]

$$(x,y,z)\mapsto \left(\alpha x+2\alpha yP(z)-\alpha zP^2(z), \pm\left(y-zP(z)\right), \frac{1}{\alpha}z\right);\ \alpha\in k^*, P\in k[X].$$

Note (as an attempt to justify the adjective 'natural' used above...) that the group E_G is exactly the subgroup of G obtained by intersecting G with the elementary automorphisms of $\mathbb{A}^3$.

Theorem 6 *Assume char$(k)\neq 2$. The group G is the product of $O(3,k)$ and E_G along their intersection.*

This result is contained in [8] (in fact Makar-Limanov obtains a more general result covering a larger class of surfaces, but he does not make the statement about the amalgamated product structure) and also in [4]. The statement above, applied over the field $k=\mathbb{C}(t)$, was crucial in [7] where we gave a description of the automorphisms of $\mathbb{C}^3$ which preserve the quadratic form y^2+xz.

Assume first that λ admits a square root δ in k. We now explain how to identify V_λ with an open set of $\mathbb{P}^1\times\mathbb{P}^1$. Note $[x:y:z:w]$ the homogeneous coordinates on $\mathbb{P}^3$, where we identify $\mathbb{A}^3$ with the open set $w\neq 0$. The homogeneous equation of V_λ is then $y^2-\lambda w^2+xz=0$. Consider the Segre embedding

$$[t_0:t_1],[u_0:u_1]\in\mathbb{P}^1\times\mathbb{P}^1\mapsto[t_0u_0:t_1u_0:t_0u_1:t_1u_1]\in\mathbb{P}^3$$

which identifies $\mathbb{P}^1\times\mathbb{P}^1$ with the smooth projective quadric $xw-yz=0$. By a linear change of coordinates we obtain the following parametrization of the quadric $(y-\delta w)(y+\delta w)+xz=0$ (where $\delta^2=\lambda$):

$$[t_0:t_1],[u_0:u_1]\in\mathbb{P}^1\times\mathbb{P}^1\mapsto\left[4\delta^2t_0u_0:\delta(t_0u_1-t_1u_0):t_1u_1:t_0u_1+t_1u_0\right]\in\mathbb{P}^3.$$

More simply, in the local charts (t,u) and (x,y,z) this parametrization becomes

$$(t,u)\to\left(4\delta^2\frac{tu}{t+u},\delta\frac{t-u}{t+u},\frac{1}{t+u}\right)$$

[3] We should mention that there was a sign missing in the definition of E_G given in [7], that we correct here.

Via this parametrization the quadric V_λ is identified to the open set in $\mathbb{P}^1 \times \mathbb{P}^1$ defined as the complement of the diagonal $D = \{t_0u_1 + t_1u_0 = 0\}$, and the fibration $z = cte$ corresponds to the fibration $t + u = cte$. Furthermore by a straightforward computation we can verify that through this parametrization the automorphism

$$(t,u) \mapsto \left(t + P\left(\frac{1}{t+u}\right), u - P\left(\frac{1}{t+u}\right)\right) \in E_D$$

is identified to

$$(x,y,z) \mapsto (x - 2y(2\delta P(z)) - z(2\delta P(z))^2, y + z(2\delta P(z)), z) \in E_G.$$

Similarly $(u,t) \mapsto \left(\frac{t}{ct+1}, \frac{-u}{cu-1}\right) \in A_D$ is identified to $\begin{pmatrix} 1 & 0 & 0 \\ -\frac{c}{2\delta} & 1 & 0 \\ -\frac{c^2}{4\delta^2} & \frac{c}{\delta} & 1 \end{pmatrix} \in \mathrm{O}(3,k)$; $(t,u) \mapsto (at, au)$ to $(x,y,z) \mapsto (ax, y, z/a)$, $(t,u) \mapsto (\frac{1}{4\delta^2 t}, \frac{1}{4\delta^2 u})$ to $(x,y,z) \mapsto (z,-y,x)$ and $(t,u) \mapsto (u,t)$ to $(x,y,z) \mapsto (x,-y,z)$. We conclude that the group $\mathrm{O}(3,k)$ is identified to A_D and the group E_G to E_D; thus the theorem 6 is equivalent to the theorem 4.

Suppose now that λ does not admit a square root in k. Note $\bar{k}$ the algebraic closure of k and $\delta \in \bar{k}$ a square root of λ. Let g be an automorphism of V_λ defined over k, *i.e.* g is given by an automorphism of the algebra $k[X,Y,Z]/(Y^2 + XZ - \lambda)$, that we can extend as an automorphism of the algebra $k(\delta)[X,Y,Z]/(Y^2 + XZ - \lambda)$. One can see g as an automorphism of V_λ defined over the field $k(\delta)$. The argument above says that g can be written as a composition of elements in $\mathrm{O}(3)$ and in E_G with coefficients in $k(\delta)$. We just have to prove that these coefficients are in fact in k; Let us recall briefly the argument already exposed p.313 of [6].

The main point is that we know that g admits a *unique* proper base point, which is the image by g^{-1} of the conic at infinity with equations $y^2 + xz = 0$ and $w = 0$. Pick a point p defined over k on this conic, such that p is not the base point of g^{-1} (one of the two points $[1:0:0:0]$ or $[0:0:1:0]$ will do). Then $g^{-1}(p)$ is the proper base point of g, and so is contained in $\mathbb{P}^3_k$. By a similar argument the proper base point of g^{-1} is also in $\mathbb{P}^3_k$. Composing g on the right and on the left with well chosen elements of $\mathrm{O}(3,k)$, we can reduce to the case where the base points of g and g^{-1} are both equal to $[1:0:0:0]$. This is equivalent to say that we are reduced to the case where the decomposition of g begins and ends with an element of E_G (a priori with coefficients in $k(\delta)$):

$$g = e_n \circ a_{n-1} \circ \cdots \circ a_1 \circ e_1 \text{ with } a_i \in O(3,k(\delta)) \setminus E_G, e_j \in E_G \setminus O(3,k(\delta)).$$

A straightforward induction then shows that g can be written

$$g : (x,y,z) \mapsto (\alpha^2\beta^{2d_1+1}z^{(2d_1+1)d_2} + \cdots, \alpha\beta^{d_1+1}z^{(d_1+1)d_2} + \cdots, \beta z^{d_2} + \cdots)$$

with $d_1, d_2 > 1$ (we have only written the homogeneous components of higher degree). Since by assumption f has coefficients in k we deduce that β, and so also α, are elements in k. Composing g on the left by the automorphism $(x,y,z) \mapsto (x - 2\alpha y z^{d_1} - \alpha^2 z^{2d_1+1}, y + \alpha z^{d_1+1}, z)$ which is an element in E_G with coefficients in k we obtain an element of $\mathrm{Aut}(V_\lambda)$ with degree strictly less than g. We conclude by induction on the degree.

2.3 A theorem of Nagao

The following theorem, due to Nagao [10], describes the structure of the GL_2 with coefficients in the polynomial ring $k[X]$:

Theorem 7 *The group $GL(2, k[X])$ is equal to the product of $GL(2, k)$ and the triangular subgroup $\left\{\begin{pmatrix} a & P(X) \\ 0 & d \end{pmatrix};\ a, d \in k^*, P \in k[X]\right\}$ amalgamated along their intersection.*

The group $\mathrm{PGL}(2, k[X])$ admits a natural identification with certain automorphisms of $\mathbb{A}^1 \times \mathbb{P}^1$:

$$\begin{pmatrix} a(X) & b(X) \\ c(X) & d(X) \end{pmatrix} : (t, u) \mapsto \left(t, \frac{a(t)u + b(t)}{c(t)u + d(t)}\right).$$

From this point of view the theorem of Nagao (or to be precise: the version of the theorem 7 for $\mathrm{PGL}(2, k[X])$, which is equivalent) is given by the following corollary of the theorem 5:

Corollary 8 *The group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus F$ which fix each line $t = cte$ is equal to the amalgamated product of*

$$\left\{(t, u) \mapsto \left(t, \frac{au + b}{cu + d}\right);\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, k)\right\}$$

and of

$$\{(t, u) \mapsto (t, \gamma u + P(t));\ \gamma \in k^*, P \in k[X]\}$$

along their intersection.

This statement comes by taking the quotient by $\{(t, u) \mapsto (\alpha t + \beta, u);\ \alpha \in k^*, \beta \in k\}$ of the two groups in the statement of theorem 5.

2.4 Proof of theorems 4 and 5

We want to show how we can adapt our proof of the theorem of Jung [6] to this new situation. First assume that k is an algebraically closed field. In the sequel the curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$, called divisor at infinity, will be either equal to the diagonal D or to the fiber F. We say that a birational map $g : X \dashrightarrow Y$ comes from an automorphism of $V = \mathbb{P}^1 \times \mathbb{P}^1 \setminus C$ if X and Y are compactifications of V and g induces an automorphism of V. A first step is to note that there are strong constraints on the possible configurations of the base points of g when $Y = \mathbb{P}^1 \times \mathbb{P}^1$ (the main point is that the divisor at infinity is an irreducible curve).

Lemma 9 *Let X be a surface and g a birational map from X to $\mathbb{P}^1 \times \mathbb{P}^1$ which comes from an automorphism of V. Assume that g is not a morphism. Then*

1. *g admits a single proper base point, located on the divisor at infinity of X;*
2. *g admits base points $p_1, \cdots, p_r$ $(r \geq 1)$ such that*
   (a) *p_1 is the proper base point;*
   (b) *For all $i = 2, \cdots, r$, the point p_i is located on the divisor produced by blowing-up p_{i-1};*

3. *Each irreducible curve contained in the divisor at infinity of X is contracted to a point by g;*

4. *Using the notations of the theorem of Zariski applied to g: the first contracted curve in the sequence π_2 is the strict transform of a curve contained in the divisor at infinity of X;*

5. *In particular, if $X = \mathbb{P}^1 \times \mathbb{P}^1$, the first curve contracted by π_2 is the transform of the curve $C \subset X$.*

Proof. It is exactly the same proof as the one of lemma 9 in [6]. □

Now pick g an automorphism of V, that we extend as a birational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$. By the lemma 9.1 g admits a unique proper base point, located on C. Composing g by an element in A_C we can reduce to the case where this point is $[1:0],[1:0]$. Assuming this, we are going to show that there exists a diagram

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dr, dashed, "g\circ\varphi^{-1}"] & \\ \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ur, dashed, "\varphi"] \arrow[rr, dashed, "g"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

where φ is the extension of an element in E_C, and such that

$$\#\mathrm{ind}(g \circ \varphi^{-1}) < \#\mathrm{ind}(g).$$

By induction on the number of base points of g, this will prove that g is contained in the group generated by E_C and A_C. We study the question of the amalgamated product structure at the end of this section.

Case $C = F$: context of the theorem 5. We identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the Hirzebruch surface F_0, and we note $f_\infty(F_0) = \{t_1 = 0\}$ and $s(F_0) = \{u_1 = 0\}$. Remark that here we note by s and not by s_∞ the section: indeed in this context the divisor at infinity is equal to the single curve f_∞. We can now apply the following lemma (ascending case, $n = 0$):

Lemma 10 *Assume that $h : F_n \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ which comes from an automorphism of V. Note p the proper base point of h, φ the birational map constructed by blowing-up p and blowing-down the transform of f_∞, and $h' = h \circ \varphi^{-1}$. Then $\#ind(h') = \#ind(h) - 1$, and we have two situations:*

- **ascending case**: *if $p = s \cap f_\infty$, then we obtain a map $h' : F_{n+1} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$;*

- **descending case**: *if $p \neq s \cap f_\infty$, then we obtain a map $h' : F_{n-1} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore the base point p' of h' (if h' is not a morphism) still satisfies $p' \neq s(F_{n-1}) \cap f_\infty(F_{n-1})$.*

Proof. Consider the Zariski decomposition of h. the lemma 9.4 tells us that the first curve contracted by π_2 is the transform of f_∞. Thus the point p, unique proper base point of h, is contained in f_∞. After the blow-up of p, f_∞ already has self-intersection -1; we deduce that all the other blow-ups in the sequence π_1 have centers outside f_∞. So it is equivalent to first realize all these blow-ups and then blow-down f_∞, or to blow-down f_∞ first and realize the blow-ups afterward. In other words we have a commutative diagram :

$$\begin{tikzcd}
 & & M \arrow[lldd, bend right, "\pi_1"'] \arrow[rrdd, bend left, "\pi_2"] \arrow[ld] \arrow[rd, "cont_{f_\infty}"] & & \\
 & {} \arrow[ld, "\pi_p"'] \arrow[d, "cont_{f_\infty}"] & & M' \arrow[lld] \arrow[rd] & \\
F_n \arrow[r, dashed, "\varphi"] \arrow[rrrr, dashed, bend right, "h"'] & F_{n'} \arrow[rrr, dashed, "h'"] & & & \mathbb{P}^2
\end{tikzcd}$$

This gives the equality $\#\text{ind}(h') = \#\text{ind}(h) - 1$; and the distinction between ascending and descending case is then straightforward. $\square$

After applying the lemma once (ascending case, $n = 0$), we are again in the conditions to apply the lemma, with $n = 1$. We apply the lemma as long as we are in the ascending case (say r times, with $r \geq 1$). We obtain the diagram

$$\begin{tikzcd}
 & F_r \arrow[rd, dashed, "g_1 = g\circ\varphi_1^{-1}"] & \\
F_0 = \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ru, dashed, "\substack{\varphi_1 = r \text{ times the} \\ \text{ascending lemma}}"] \arrow[rr, dashed, "g"'] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

where g_1 satisfies the conditions of the lemma (with $n = r$, descending case), and $\#\text{ind}(g_1) = \#\text{ind}(g) - r$. Then we can apply successively r times the lemma (descending case) until we obtain the diagram:

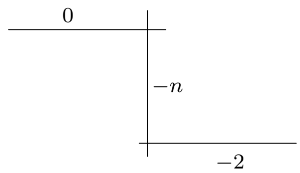
$$\begin{tikzcd}
 & F_0 = \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rd, dashed, "g_2"] & \\
F_r \arrow[ru, dashed, "\substack{\varphi_2 = r \text{ times the} \\ \text{descending lemma}}"] \arrow[rr, dashed, "g_1"'] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

In conclusion, we have a diagram

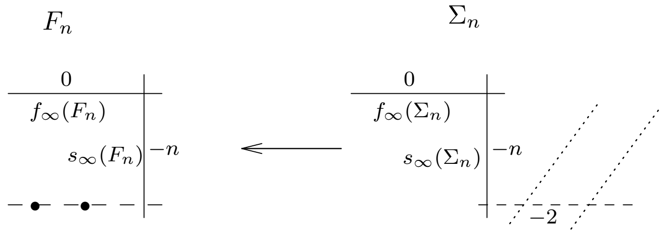
$$\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rd, dashed, "g_2 = g\circ\varphi^{-1}"] & \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[ru, dashed, "\varphi = \varphi_2\circ\varphi_1"] \arrow[rr, dashed, "g"'] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

with $\#\text{ind}(g \circ \varphi^{-1}) = \#\text{ind}(g) - 2r$. We still have to verify that $\varphi \in E_F$. But this is easy: remark that φ induces an automorphism of $\mathbb{A}^2 = \mathbb{P}^1 \times \mathbb{P}^1 \setminus (s \cup f_\infty)$ which preserves the fibration $t = cte$, and this property characterizes the elementary automorphisms.

Case $F = D$: context of the theorem 4. Consider again the Zariski diagram associated with g. By the lemma 9.5, we know that the first contracted curve by π_2 is the transform of D. But D has self-intersection $+2$ in $\mathbb{P}^1 \times \mathbb{P}^1$, thus we have to perform three blow-ups on D to bring down its self-intersection to -1. This implies that there is a unique possible configuration for the first 3 blow-ups of the sequence π_1, that we want to explain now. After the blow-up of the proper base point $[1:0],[1:0]$, the intersection point of the exceptional divisor and of the transform of D must be the second base point of g. Blowing-up this point, we obtain a surface Σ_1 and g factorizes through these two blow-ups as a map $g_1 : \Sigma_1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with $\#\mathrm{ind}(g_1) = \#\mathrm{ind}(g) - 2$. Note that Σ_1 can be seen as F_1 blown-up twice (contract the transforms of the two fibers of F_0 passing through $[1:0],[1:0]$): the fibration on F_1 corresponds to the fibration on Σ_1 given by the lines $t + u = cte$. More generally we will note Σ_n a compact surface containing an open set isomorphic to $V = \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$, and such that $\Sigma_n \setminus V$ is equal to three rational curves in the following configuration, and with self-intersection 0, $-n$ and -2:



We can see Σ_n as the surface obtained by blowing-up a Hirzebruch surface F_n in two points located on a same fiber (but not included in $f_\infty \cup s_\infty$). We still note by s_∞ and f_∞ the corresponding curves:



With these notations at hand, remark that the proper base point of g_1, which corresponds to the third base point of g, must be the intersection point of $s_\infty(\Sigma_1)$ and of $f_\infty(\Sigma_1)$. Thus we are in the conditions to apply the following lemma (with $n = 1$, ascending case):

Lemma 11 *Consider $h : \Sigma_n \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the extension of an automorphism of V. We note p the proper base point of h.*

- **ascending case:** *Suppose that $n \geq 1$ and $p = s_\infty \cap f_\infty$. Then by blowing-up p and blowing-down the transform of f_∞ we obtain a map $h' : \Sigma_{n+1} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.*
- **descending case:** *Suppose that $n \geq 2$ and $p \neq s_\infty \cap f_\infty$. Then $p \in f_\infty$, and by blowing-up p and blowing-down the transform of f_∞ we obtain a map $h' : \Sigma_{n-1} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.*

Furthermore in these two cases $\#ind(h') = \#ind(h) - 1$.

Proof. Apply the theorem of Zariski to h. By the lemma 9.4, the first curve contracted by π_2 is the transform of one of the three curves at infinity: thus it must be the transform of f_∞, because the two other have self-intersection strictly less than -1 in M. We conclude using exactly the same argument as in the proof of the lemma 10. $\square$

We apply the lemma, ascending case, as long as necessary (say r times) until we are in the conditions of the lemma, descending case. Then we can apply r times the lemma, descending case, and we obtain a map $g_2 : \Sigma_1 \mapsto \mathbb{P}^1 \times \mathbb{P}^1$, with $\#\mathrm{ind}(g_2) = \#\mathrm{ind}(g_1) - 2r$. Then we blow-down twice to get $\bar{g} : \mathbb{P}^1 \times \mathbb{P}^1 \mapsto \mathbb{P}^1 \times \mathbb{P}^1$. Clearly we have $\#\mathrm{ind}(\bar{g}) \leq \#\mathrm{ind}(g_2) + 2$; in fact it would be possible to prove that $\#\mathrm{ind}(\bar{g}) = \#\mathrm{ind}(g_2)$. Anyways even with this coarse estimation we get:

$$\#\mathrm{ind}(\bar{g}) \leq \#\mathrm{ind}(g) - 2r.$$

Now we just have to check that the map $\bar{g}^{-1} \circ g$ is an element in E_D. It is sufficient to remark that this map preserves the pencil of lines $t + u = cte$, which corresponds to the fibration on all the surfaces Σ_n that appear in the proof.

End of the proofs. Now we discuss the two missing points to complete the proofs of theorems 4 and 5: first the amalgamated product structure, and then the proof over an arbitrary field.

Amalgamated product structure (compare with [6, 4.2]). Note $p = [1 : 0], [1 : 0]$. Then for $C = D$ or $C = F$ we have:

1. Any element in $E_C \setminus A_C$, that we see as a map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, admits p as unique proper base point, and contracts C to p;
2. Any element in $A_C \setminus E_C$ does not fix the point p.

As a consequence any composition (non reduced to a single element in A_C) of elements in $E_C \setminus A_C$ and $A_C \setminus E_C$ contracts the curve C to a point, and thus cannot be equal to the identity map.

Proof over an arbitrary field. Let k be a field, and note $\bar{k}$ its algebraic closure. We want to check that the lemma 9 is still true over k, the key point being the assertion 9.1. Consider g a birational map from X_k to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ that comes from an algebraic automorphism of $\mathbb{P}^1_k \times \mathbb{P}^1_k \setminus C$. The base points of g, which a priori are points over $\bar{k}$, are invariant under the action of the Galois group $\mathrm{Gal}(\bar{k}/k)$. But we know (this is the lemma 9 applied over $\bar{k}$) that g admits a unique proper base point p_0. Thus this point is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$, which exactly means that p_0 is a point over k. All the other assertions of the lemma are then straightforward. In conclusion the proofs remains true because all the blow-ups that we realize concern points defined over k.

3 Proper polynomial maps of $\mathbb{C}^2$

In this section we discuss a possible generalisation of the result of Jung and Van der Kulk. A polynomial automorphism of $\mathbb{C}^2$ can be characterized as a proper morphism from $\mathbb{C}^2$ to $\mathbb{C}^2$ with topological degree equal to 1. Keeping the properness assumption but allowing maps with arbitrary topological degree we obtain the semi-group of proper polynomial maps from $\mathbb{C}^2$ to ce^2. It is known that a proper polynomial maps cannot be a candidate to be a counter-example to the Jacobian conjecture [1, th. 2.1]. This might explain the slight amount of interest this class of maps has received so far. However it seems to me that this class could present some dynamical behaviour different from the case of automorphisms, but still could be the subject of an almost exhaustive study (in this direction we mention the preprint [2] that study the dynamics of 'polynomial like' maps; however quoting the authors (p.18) 'in general a polynomial endomorphism of $\mathbb{C}^k$ with $k \geq 2$ is not polynomial like even if it is proper').

Thus we consider from now on polynomial maps from $\mathbb{C}^2$ to $\mathbb{C}^2$ with arbitrary topological degree which are proper (the preimage of a compact is compact). Of course elements in Aut$[\mathbb{C}^2]$ are basic examples. Another class of very simple examples are the maps of the form

$$(x, y) \mapsto (P(x), Q(y)).$$

An obvious counter-example is given by the map $(x, y) \mapsto (xy, y)$, indeed the preimage of any neighborhood of $(0, 0)$ contains the line $y = 0$. A more subtle counter-example (which was given to me by J.-P. Furter) is the map $(x, y) \mapsto (x + x^2 y, y)$. The preimage of any point contains one or two points but this maps is still not proper: The points $(-1/\varepsilon, \varepsilon)$ and $(0, \varepsilon)$ are the two preimages of $(0, \varepsilon)$, hence the preimage of a compact neighborhood of $(0, 0)$ is never compact. We do not know any other examples of proper polynomial maps apart from the previous examples (and composition of those): It is tempting to conjecture that there is indeed no other examples[4]. Let just say that we ask the

Question: Are all proper polynomial maps from $\mathbb{C}^2$ to $\mathbb{C}^2$ composition of polynomial automorphisms and maps of the form $(x, y) \mapsto (x, P(y))$?

We were able to answer the question only when the topological degree is equal to 2; this is what want to explain now. Note $g : \mathbb{C}^2 \mapsto \mathbb{C}^2$ a proper polynomial map with topological degree 2. From g we can construct an involution σ which exchange the two preimages of a point (possibly these two preimages are equal and give a fixed point for σ).

Lemma 12 *The involution σ is an element of Aut$[\mathbb{C}^2]$. In particular σ is conjugate in Aut$[\mathbb{C}^2]$ either to $(-x, -y)$ or to $(x, -y)$.*

Proof. Consider the birational extension $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Blowing-up sufficiently many times the final $\mathbb{P}^2$ we can assume that the line at infinity in the initial $\mathbb{P}^2$ is not contracted to a point, that is we have a diagram

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[ur, dashed, "\bar{g}"] \arrow[r, dashed, "g"'] & \mathbb{P}^2 \end{tikzcd}$$

where $\bar{g}$ does not contract any curve (not the line at infinity by construction, nor any other curve by the properness assumption). As above associate an involution to $\bar{g}$, this is a holomorphic map that extends σ and is well defined outside the finite number of base points of $\bar{g}$: By Hartogs' lemma σ is a birational map from $\mathbb{P}^2$ to $\mathbb{P}^2$. Since furthermore σ sends $\mathbb{C}^2$ on $\mathbb{C}^2$ (this is again by the properness assumption), we obtain $\sigma \in$ Aut$[\mathbb{C}^2]$.

It is known that an element of finite order in Aut$[\mathbb{C}^2]$, and so in particular an involution, is conjugate to an elementary automorphism. Furthermore (see [3]), any elementary automorphism is conjugate:

[4]It turns out that this guess was rather naive: see C. Bisi and F. Polizzi, "On Proper Polynomial Maps of $\mathbb{C}^2$", J. of Geom. Analysis, January 2010.

1. either to an element in the affine group;
2. or to an element of the form $(\beta^d x + \beta^d y^d q(y^r), \beta y)$ with $d \geq 1$, q non constant, β rth root of the unity.

Notice that in the second case the automorphism has always infinite order. Finally σ is conjugate to an affine automorphism. It is easy to verify that σ admits a fixed point; by conjugating by a translation and by diagonalizing we obtain the result. $\square$

Now we can prove the

Proposition 13 *Let $g : \mathbb{C}^2 \mapsto \mathbb{C}^2$ be a proper polynomial map with topological degree 2. Then there exist $f_1, f_2 \in Aut[\mathbb{C}^2]$ such that*

$$g = f_2 \circ (x, y^2) \circ f_1.$$

Proof. Consider the involution $\sigma \in \mathrm{Aut}[\mathbb{C}^2]$ associated with g, and note f_1 the automorphism given by the lemma 12 such that $\sigma' = f_1^{-1}\sigma f_1 = (x, -y)$ or $(-x, -y)$. Take $g' = g \circ f_1$, we have $g' \circ \sigma' = g'$. Suppose $\sigma' = (x, -y)$. This means that g' depends only on x and y^2, in other words there exists a polynomial map $f_2 : \mathbb{C}^2 \mapsto \mathbb{C}^2$ such that $g' = f_2 \circ (x, y^2)$. Furthermore f_2 is proper because g' is proper, and f_2 has topological degree 1 because g' and (x, y^2) have topological degree 2. Finally $f_2 \in \mathrm{Aut}[\mathbb{C}^2]$, and we have $g = f_2 \circ (x, y^2) \circ f_1$. It is easy to see that the case $\sigma' = (-x, -y)$ never happen: we would have $g' = f_2 \circ (x^2, y^2)$ but g' has topological degree 2 and (x^2, y^2) has degree 4. $\square$

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