The three-dimensional tame automorphism group

Action on a hyperbolic simplicial complex

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Tame automorphisms

We are interested in the group $\mathrm{Aut}(\mathbf{A}^n)$ of polynomial automorphisms of the affine space $\mathbf{A}^n$, over a given base field $\mathbf{k}$. The **tame automorphism group**

$$\mathrm{Tame}(\mathbf{A}^n) = \langle \mathrm{GL}_n, E_n \rangle \subseteq \mathrm{Aut}(\mathbf{A}^n)$$

is defined as the subgroup generated by linear and elementary automorphisms, where

$$E_n = \{(x_1, x_2, \dots, x_n) \mapsto (x_1 + P(x_2, \dots, x_n), x_2, \dots, x_n) \mid P \in \mathbf{k}[x_2, \dots, x_n]\}.$$

Reminder

Is the inclusion $\mathrm{Tame}(\mathbf{A}^n) \subseteq \mathrm{Aut}(\mathbf{A}^n)$ an equality or not ?
✓ Jung: *Yes!* when $n = 2$, over any base field;
✓ Shestakov & Umirbaev: *No!* when $n = 3$ and $\mathrm{char}\, \mathbf{k} = 0$;
✓ All other cases, $n \geq 4$, or $n = 3, \mathrm{char}\, \mathbf{k} > 0$, are open!

Simplicity ?

In this work we focus on the group $\mathrm{Tame}(\mathbf{A}^3)$ itself, and its subgroup $\mathrm{STame}(\mathbf{A}^3)$ of automorphisms with Jacobian 1.

Question

Is $\mathrm{STame}(\mathbf{A}^3)$ a simple group?

Our strategy is to use an action on a metric space with non-positive curvature properties. Specifically, we will answer the question by the negative by using an action of $\mathrm{Tame}(\mathbf{A}^3)$ on a Gromov hyperbolic simplicial complex. We hope that a similar strategy could provide a proof that all finite subgroups of $\mathrm{Tame}(\mathbf{A}^3)$ are linearizable, but that's another story...

Simplicial complex

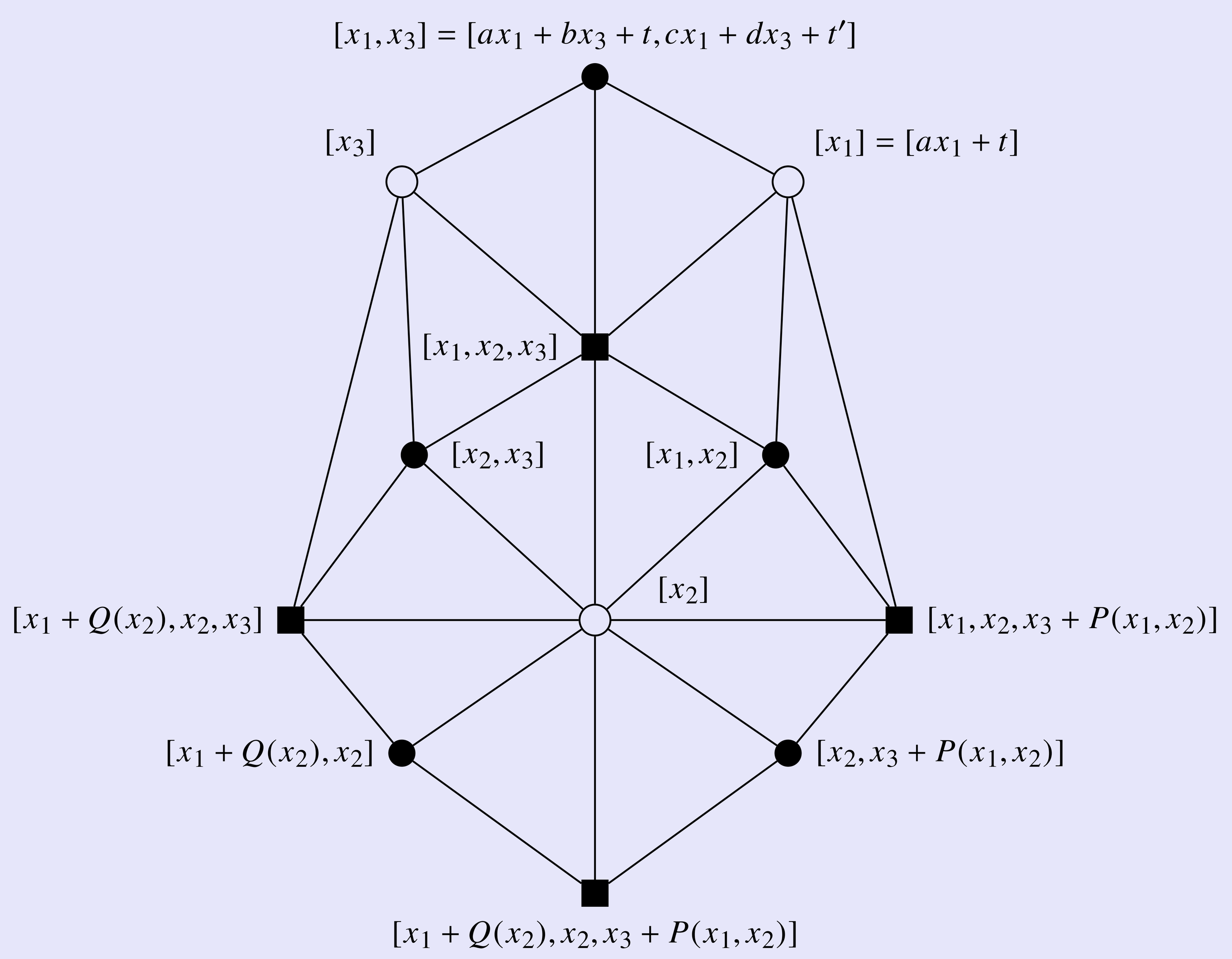

Figure 1: A few simplexes of the complex C

We construct a simplicial complex on which the group $\mathrm{Tame}(\mathbf{A}^n)$ acts naturally. First one defines n distinct types of vertices, by considering morphisms $f = (f_1, \dots, f_r)$ from $\mathbf{A}^n$ to $\mathbf{A}^r$ that can be extended as tame automorphisms $f = (f_1, \dots, f_n)$. A vertex of type r is the equivalence class of such a morphism, up to composition on the left by an affine automorphism:

$$[f_1, \dots, f_r] := \{a \circ (f_1, \dots, f_r) \mid a \in \mathrm{GL}_r(\mathbf{k}) \ltimes \mathbf{k}^r\}.$$

Now for any tame automorphism $(f_1, \dots, f_n) \in \mathrm{Tame}(\mathbf{A}^n)$ we attach a $(n-1)$-simplex on the vertices $[f_1]$, $[f_1, f_2]$, ..., $[f_1, \dots, f_n]$. This produces a connected, non-locally compact, $(n-1)$-dimensional simplicial complex C_n on which the tame group acts by isometries with fundamental a single simplex, by the formulas $g \cdot [f_1, \dots, f_r] := [f_1 \circ g^{-1}, \dots, f_r \circ g^{-1}]$.

We use this construction mostly in the case of $n = 3$, and we denote by C the 2-dimensional simplicial complex associated with $\mathrm{Tame}(\mathbf{A}^3)$. Observe however that in the case $n = 2$, one recover the classical Bass-Serre tree associated with the amalgamated product structure of $\mathrm{Aut}(\mathbf{A}^2)$.

Hyperbolicity

Definition

✓ A geodesic metric space X is **hyperbolic** if all triangles in X are δ-thin for some uniform $\delta \geq 0$.
✓ One says that $g \in \mathrm{Isom}(X)$ is **loxodromic** if for some $x \in X$ the limit $\lim \frac{1}{n} d(x, g^n x)$ is positive.
✓ A loxodromic element $g \in G \subseteq \mathrm{Isom}(X)$ has the **WPD property** if $\forall x \in X$, $\forall r > 0$, $\exists n \in \mathbf{N}$, such that the set of $f \in G$ satisfying $d(x, fx) < r$ and $d(g^n x, f g^n x) < r$ is finite.

The existence of such a WPD element is sufficient to ensure that the group G is not simple: in fact G contains free normal subgroups and is SQ-universal. Such elements were recently found in transformation groups, such as the Cremona group [Lon16], or the tame group of an affine quadric 3-fold [Mar15]. The main point of this work is to add $\mathrm{Tame}(\mathbf{A}^3)$ to this growing list.

Main Result

Theorem [LP16]

Assume the base field $\mathbf{k}$ has characteristic zero. Then
✓ The complex C has infinite diameter.
✓ The complex C is contractible.
✓ The complex C is hyperbolic.
✓ Their exist elements in $\mathrm{Tame}(\mathbf{A}^3)$ with the WPD property with respect to the action on C.
✓ The group $\mathrm{STame}(\mathbf{A}^3)$ is not simple.

Idea of proof

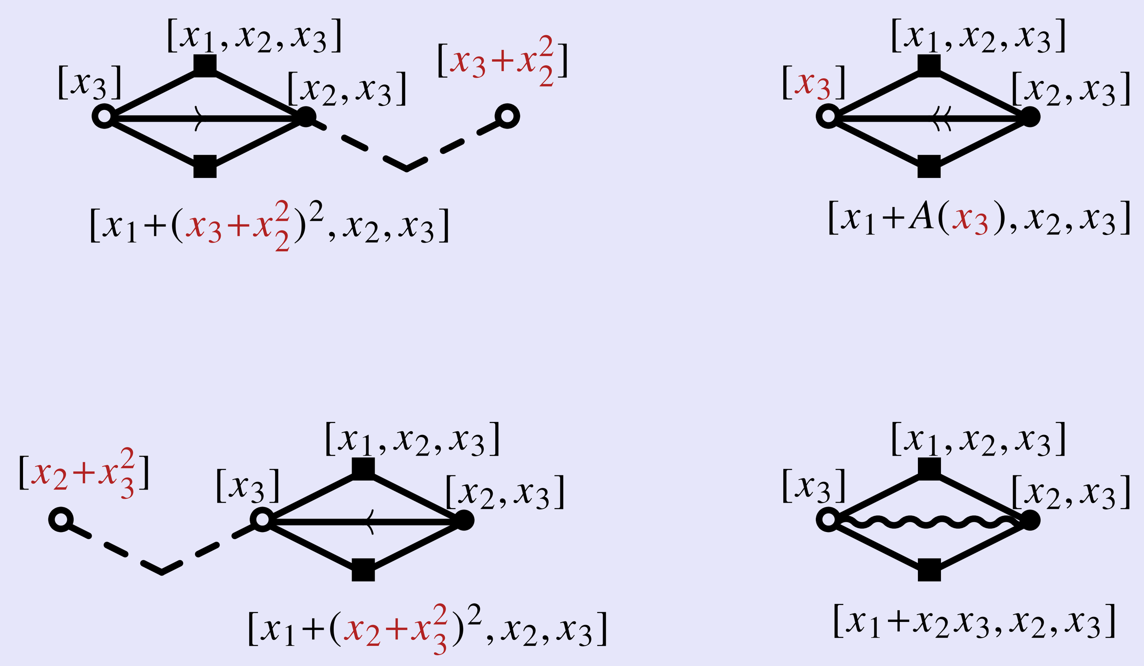

Figure 2: Examples of orientation

✓ The theory of reductions of Shestakov, Umirbaev and Kuroda amounts to understanding the relations in $\mathrm{Tame}(\mathbf{A}^3)$, and allows to prove that C is **simply connected**. This was first noticed by D. Wright [Wri15], see also [Lam15] for a self-contained proof.

✓ To each pair of adjacent triangles along an edge of type $1-2$ corresponds an elementary automorphism $(x_1 + P(x_2, x_3), x_2, x_3)$. If the polynomial $P(x_2, x_3)$ has the form $A(f)$ with $A(T) \in \mathbf{k}[T]$, and $f \in \mathbf{k}[x_2, x_3]$ a component of an element in $\mathrm{Aut}(\mathbf{A}^2)$, we put an **arrow** on the edge, pointing in the direction of $[f]$ inside the copy of the Bass-Serre tree associated with $\mathrm{Aut}(\mathbf{A}^2)$. A double arrow means that $[f]$ is the type one vertex of the edge, and a wavy edge that P is not of the form $A(f)$.

$$\begin{cases} K(v_1) = 1 - \dfrac{\deg v_1}{12} + \dfrac{\mathrm{out}(v_1) - \mathrm{in}(v_1)}{6} \\ K(v_2) = 1 - \dfrac{\deg v_2}{4} + \dfrac{\mathrm{out}(v_2) - \mathrm{in}(v_2)}{6} \\ K(v_3) = 1 - \dfrac{\deg v_3}{6} \end{cases}$$

✓ We define a **discrete curvature** at each interior vertex v_i of type i in a disc or sphere diagram, according to the formulas on the left. One should think that each triangle is endowed with a Euclidean metric with angles $\frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}$, with a correction coming from the arrows.

✓ One check that there is no local configuration in C with positive curvature. This yields that C does not contain any sphere diagram, from which the **contractibility** follows.

✓ Further analysis yields a list of exactly eight configurations with zero curvature: Figure 3. Then one shows that in a disc diagram, any vertex with zero curvature is at uniform distance of a vertex with negative curvature. Moreover one can also define a curvature at each boundary vertex of a disc diagram, which is bounded above by 1/2. Then we obtain the **hyperbolicity** of C via a classical criterion based on isoperimetric estimates.

✓ Finally, an explicit example of **WPD element** is given by

$$f = h \circ g^n, \quad \text{where } n \geq 12, \quad g = (x_2 + x_1 x_3, x_1, x_3), \quad h = (x_3, x_2, x_1).$$

Remark also that the existence of loxodromic elements insures that C has **infinite diameter**.

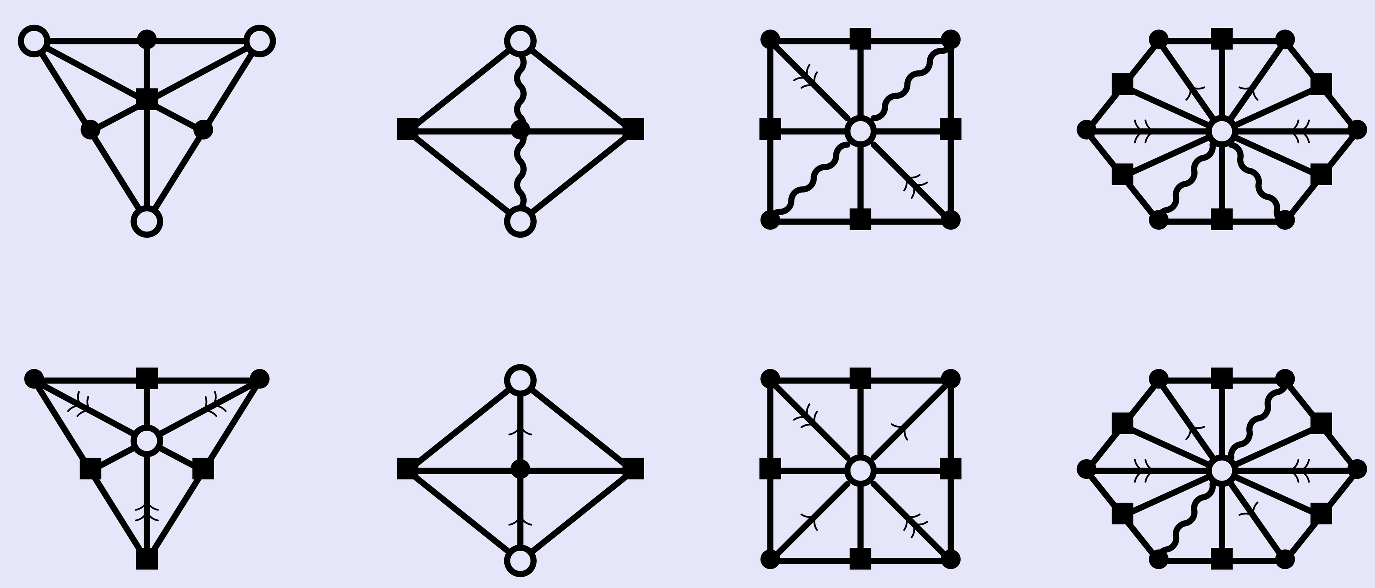

Figure 3: Eight local configurations with zero curvature

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