FIVE LECTURES ON SARKISOV PROGRAM AT LES DIABLERETS

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CONTENTS

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1. Preliminaries on blow-ups

1.1. **Canonical divisor.** If X is a smooth n-dimensional variety, the canonical divisor is the divisor of zeros and poles of a rational n-form: if $\omega = f du_1 \wedge \cdots \wedge du_n$ on an open set U with local parameters $u_1, \cdots, u_n$, then f is a local equation of the divisor (ω) on U. The following proposition says that this divisor K_X is uniquely defined up to linear equivalence

Proposition 1. *If ω and ω' are 2 n-forms on X, then $(\omega) \sim (\omega')$.*

Proof. Locally, the two n-forms are $\omega = f du_1 \wedge \cdots \wedge du_n$ and $\omega' = g du_1 \wedge \cdots \wedge du_n$ with $f, g \in \mathbb{C}[U]$. Thus $(\omega) = (\omega') + (f/g)$, indeed a change of chart induces the same Jacobian determinant in the expression of both ω and ω'. □

One can compute directly the canonical divisor of $\mathbb{P}^n$: the n-form $dx_1 \wedge \cdots \wedge dx_n$ has a pole of order $n+1$ on the hyperplane at infinity, as one verifies by computing in an affine chart around a point at infinity.

$$K_{\mathbb{P}^n} = -(n+1)H.$$

To compute the canonical divisor in general, we use a formula which gives the canonical divisor of a hypersurface starting from the canonical divisor of the ambient space: this is the adjunction formula.

Proposition 2 (Adjunction)**.** *If $Y \subset X$ is a hypersurface, with X and Y smooth, we have*

$$K_Y = (K_X + Y)|_Y.$$

The restriction here makes sense when we see divisors as line bundles (see next section, and [Sha94b]), but can also be interpreted as an intersection: to compute $D|_Y$, choose $D' \sim D$ intersecting transversely Y and then $D|_Y \sim D' \cap Y$.

Proof. We note $N_{X/Y}$ the normal bundle of Y in X. The proof proceeds by showing that

(1) $N_{X/Y} = Y|_Y$;
(2) $K_Y = (K_X + N_{X/Y})|_Y$.

1. If $\{f_\alpha\}$ are local equations of Y, then $f_\alpha = g_{\alpha\beta} f_\beta$ where the $g_{\alpha\beta}$ are the transition functions of the line bundle associated with Y. We have $d f_\alpha = g_{\alpha\beta} d f_\beta + d g_{\alpha\beta} f_\beta$, and this last term is zero in restriction to Y. One can see f_α as a section of the line bundle Y, and $d f_\alpha$ is then a 1-form with value in sections of Y, in other words a section of the bundle $N^*_{X/Y} \otimes Y$. Since this section is regular and without zero (because Y is smooth), the bundle $N^*_{X/Y} \otimes Y$ is trivial, that is $N_{X/Y} = Y|_Y$.

2. By definition $N_{X/Y} = T_X / T_Y$, hence $T^*_Y = T^*_X / N^*_{X/Y}$. So the transition matrix between open sets U_α and U_β for the vector bundle T^*_X has the form

$$\left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline * & A_{\alpha\beta} \end{array}\right)$$

where $A_{\alpha\beta}$ is the transition matrix of T^*_Y, and $g_{\alpha\beta}$ is the transition function of the line bundle $N^*_{X/Y}$. We see that $\det T^*_X = \det T^*_Y \otimes N^*_{X/Y}$, hence $K_Y = (K_X + N_{X/Y})|_Y$. □

1.2. **Digression: Weil vs Cartier divisors.**

Definition 3 (Weil divisor)**.** A (Weil) divisor D on a variety X is a finite collection of irreducible subvarieties C_i $(i = 1, \cdots, r)$ of codimension 1, each one labelled with a multiplicity $k_i \in \mathbb{Z}$.
We note $D = \sum_i k_i C_i$.

In particular if X is a curve, D is a collection of points with multiplicities; in this case we call $\deg D$ the sum of the k_i.

Definition 4 (Cartier divisor)**.** A (Cartier) divisor D on a variety X is given by an open covering (U_α) of X, and for each U_α a rational function f_α, with the compatibility condition:

On each intersection $U_\alpha \cap U_\beta$, the function f_α / f_β has values in $\mathbb{C}^*$ (in other words it has no zero and no pole).

A Cartier divisor corresponds to a line bundle, and the $g_{\alpha,\beta} = f_\alpha / f_\beta$ are then exactly the transition functions from $U_\alpha \times \mathbb{C}$ to $U_\beta \times \mathbb{C}$.

One can endow the set of (Weil or Cartier) divisors with a group law.

Proposition 5. *On a smooth variety the notions of Weil and Cartier divisors coincide.*

Standard counter-example on a singular variety: rule of the cone $y^2 = xz$ in $\mathbb{C}^3 \subset \mathbb{P}^3$.

The key property to show this equivalence is

Proposition 6. *Let X be a smooth variety, $Y \subset X$ be an irreducible subvariety of codimension 1, and $x \in Y$. Then in a neighborhood of x the subvariety Y can be defined by a single equation.*

Remark: the proposition is true even if Y is singular. On the other hand, the generalization in higher codimension is correct only if Y is smooth (see [Sha94a, p.111]).

Main point of Cartier divisors: one can define the pull-back by a morphism (pull-back the local equation).

On a singular variety the canonical divisor is only defined as a Weil divisor (divisor of poles and zeros of a n-form defined on the smooth part).

If a Weil divisor D is not Cartier, but a multiple nD is Cartier, one says that D is $\mathbb{Q}$-Cartier, and one can define the pull-back of D by a morphism π by the formula:

$$\pi^* D := \frac{1}{n} \pi^* nD.$$

This leads naturally to the notion of (Weil or Cartier) **$\mathbb{Q}$-divisors**, that is divisors with coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$.

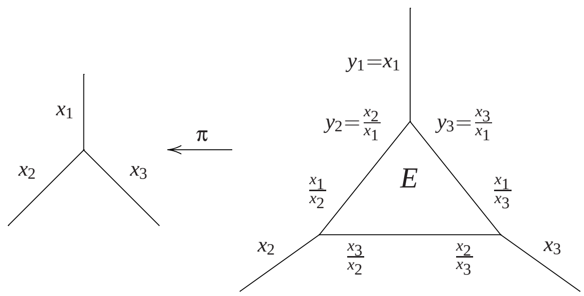
1.3. **Discrepancies.**

1.3.1. *Blow-up of a smooth point.* Consider p the origin of $X = \mathbb{A}^n$, with coordinates $x_1, \cdots, x_n$. For each $i = 1, \cdots, n$, define a map

$$U_i \simeq \mathbb{A}^n \to \mathbb{A}^n$$
$$(y_1, \cdots, y_n) \to (y_1 y_i, \cdots, y_i, \cdots, y_n y_i).$$

Apart from the hyperplane $y_i = 0$ which is contracted on p, this map is injective. The U_i are glued in a natural way (identify points in U_i and U_j with the same image), to produce a variety Y and a morphism $\pi\colon Y \to \mathbb{A}^n$ such that the preimage of p is isomorphic to $\mathbb{P}^{n-1}$.

In dimension 3 :



We compute now the **discrepancy** of E, i.e. the coefficient $a = a(E,X)$ in the ramification formula

$$K_Y = \pi^* K_X + aE.$$

In Y, consider the affine coordinates $y_1 = x_1$, $y_2 = x_2/x_1$, $y_3 = x_3/x_1$. So we have

$$\begin{aligned} x_1 &= y_1 & dx_1 &= dy_1 \\ x_2 &= y_1 y_2 & dx_2 &= y_2 dy_1 + y_1 dy_2 \\ x_3 &= y_1 y_3 & dx_3 &= y_3 dy_1 + y_1 dy_3 \end{aligned}$$

and finally

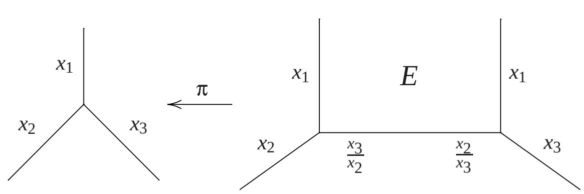
$$dx_1 \wedge dx_2 \wedge dx_3 = y_1^2 dy_1 \wedge dy_2 \wedge dy_3.$$

We obtain that the form $dx_1 \wedge dx_2 \wedge dx_3$, view on Y, has a zero of order 2 along $E = \{y_1 = 0\}$, in other word the discrepancy $a = 2$.

NB: In the previous argument you should be careful not to confuse (as I invariably do at some point) the pull-back of the 3-form with the pull-back of the divisor defined by the 3-form...

In general, the same argument shows that $a = n-1$ for the blow-up of a smooth point on a variety of dimension n.

1.3.2. *Blow-up of a curve.* First consider the case of the axis $x_2 = x_3 = 0$ in $\mathbb{A}^3$.



Similarly to the previous case we compute the **discrepancy** of E:

$$K_Y = \pi^* K_X + aE.$$

In Y, consider the affine coordinates $y_1 = x_1, y_2 = x_2, y_3 = x_3/x_2$. So we have

$$\begin{aligned} x_1 &= y_1 & dx_1 &= dy_1 \\ x_2 &= y_2 & dx_2 &= dy_2 \\ x_3 &= y_2y_3 & dx_3 &= y_3dy_2 + y_2dy_3 \end{aligned}$$

and finally

$$dx_1 \wedge dx_2 \wedge dx_3 = y_2 dy_1 \wedge dy_2 \wedge dy_3.$$

We obtain that the form $dx_1 \wedge dx_2 \wedge dx_3$, view on Y, has a zero of order 1 along E, in other word the discrepancy $a = 1$.

The same is true for the blow-up of any smooth curve in a smooth 3-fold: reduce to the previous case by working in a local analytic chart.

1.3.3. *Cone over a curve.* Now we compute the discrepancies of a simple class of singular points on surfaces.

Let $C \subset \mathbb{P}^n$ be a smooth curve of degree d. Let $p \in \mathbb{P}^{n+1}$, and X the cone with vertex p over C. Let $\pi \colon Y \to X$ the blow-up of p (that is, blow-up in the ambient projective space and restrict to the strict transform of X), and $E \subset Y$ the exceptional curve. The curve E is isomorphic to C (each rule intersects only once E and C). Note L a rule of X.

We want to compute the coefficients m and a in the ramification formulas

$$K_Y = \pi^* K_X + aE$$
$$L_Y = \pi^* L - mE.$$

The coefficient $a = a(E,X)$ is called the **discrepancy** of the divisor E over X (NB : a does not depend of a choice of resolution). The coefficient $m > 0$ is the multiplicity of L at the point p.

We have

$$0 = \pi^* L \cdot E = L_Y \cdot E + mE^2 = 1 + mE^2$$

hence

$$m = -\frac{1}{E^2}.$$

Similarly

$$0 = \pi^* K_X \cdot E = K_Y \cdot E - aE^2 = (2g-2) - (a+1)E^2$$

hence

$$a = \frac{2g-2}{E^2} - 1.$$

On X we have (consider hyperplane sections)

$$C^2 = d, \qquad\qquad C \sim dL$$

hence

$$L^2 = \frac{1}{d}.$$

We have

$$\frac{1}{d} = (\pi^* L)^2 = (L_Y + mE)^2 = 2m + m^2E^2 = -\frac{1}{E^2}$$

hence

$$E^2 = -d, \qquad m = \frac{1}{d}, \qquad a = \frac{2-2g}{d} - 1.$$

Digressions:

(1) The rational coefficients come from the fact that we are dealing with $\mathbb{Q}$-Cartier divisors.
(2) Case $g=0$: $a = \frac{2-d}{d}$. Hirzebruch surfaces. Weighted projective spaces $\mathbb{P}^2(n,1,1)$.

1.3.4. *Cone over a* $\mathbb{P}^2$. Let $S \simeq \mathbb{P}^2 \subset \mathbb{P}^n$ such that the image of each line in S is a curve of degree d in $\mathbb{P}^n$. Let $p \in \mathbb{P}^{n+1}$, and X the cone with vertex p over S. Let $\pi\colon Y \to X$ the blow-up of p, and $E \subset Y$ the exceptional divisor. The surface E is isomorphic to $\mathbb{P}^2$. Note L a rule of the cone X, and F the surface given by the rules over a line in E.

$$K_Y = \pi^* K_X + aE$$
$$F_Y = \pi^* F - mE.$$

We have

$$0 = \pi^* F \cdot L_E = F_Y \cdot L_E + mE \cdot L_E = 1 - md$$

hence

$$m = \frac{1}{d}.$$

Similarly

$$0 = \pi^* K_X \cdot L_E = K_Y \cdot L_E - aE \cdot L_E = -3 + (a+1)d$$

hence

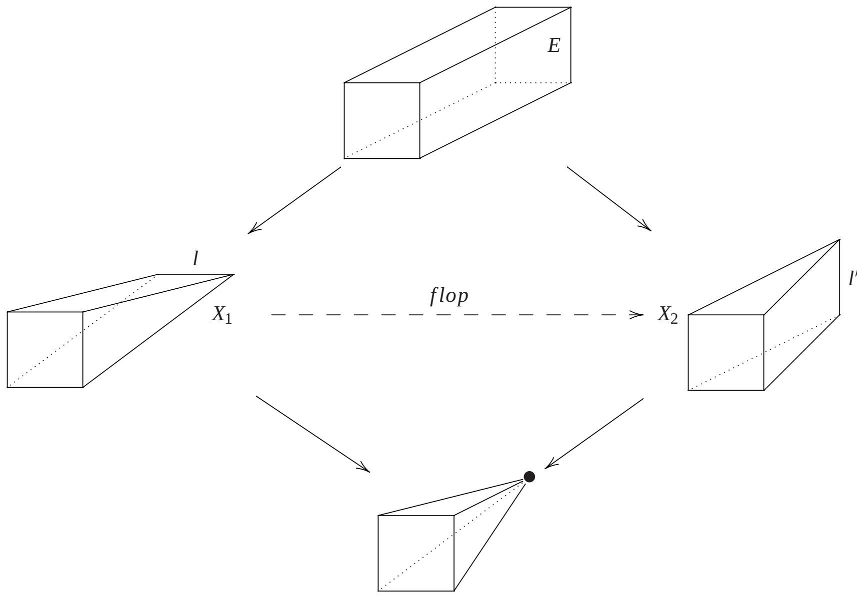
$$a = \frac{3-d}{d}.$$

The surprise is that $a = 1/2 > 0$ for $d = 2$: this gives our first example of terminal singularity (cone over the Veronese embedding of $\mathbb{P}^2$).

2. Flops, flips, contractions...

2.1. **Atiyah flop.** Consider $xy - zw = 0$ in $\mathbb{A}^4$: this is the cone over a smooth quadric surface in $\mathbb{P}^3$. Blow-up the vertex p of the cone. The exceptional divisor E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and for any line L of bidegree $(0,1)$ or $(1,0)$ in E we have $E \cdot L = -1$: indeed the fibered surface F over E is isomorphic to $\mathbb{F}_1$ by §1.3.3.

We can blow-down E on a $\mathbb{P}^1$ in both directions, and we obtain the diagram

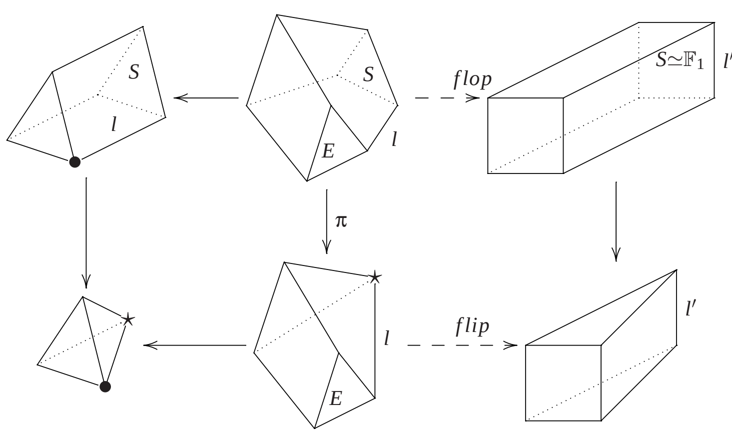


The birational map $X_1 \dashrightarrow X_2$ is an isomorphism in codimension 1, called a "flop". The flopped curves $l \subset X_1$ and $l' \subset X_2$ have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and so (see Exercise 6)

$$K_{X_1} \cdot l = K_{X_2} \cdot l' = 0.$$

Here is the same construction in a slightly different setting. Consider X the $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ with a section S of normal bundle $\mathcal{O}(-d)$ (see §1.3.4). Choose a fiber l, and $p \in l$ a point not contained in S. Let $Y \to X$ be the blow-up of p. Then the strict transform of l has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (this is easy to see since the transform of l is a complete intersection) and can be flopped. The resulting 3-fold is a $\mathbb{P}^1$-bundle over $\mathbb{F}_1$.

2.2. **Francia Flip.** In the last construction, consider now the contraction of S before and after flop: we obtain the following diagram (and the Francia "flip", when $d = 2$):



The left hand side of the diagram is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, with a section S with normal bundle $\mathcal{O}(-d)$, and the contraction of this section to a singular point $\star$: see §1.3.4.

The middle of the diagram is obtained by blowing-up the smooth point $\bullet$, producing an exceptional divisor $E \simeq \mathbb{P}^2$.

The right hand side of the diagram is obtained by flopping/flipping the curve l to the curve l'. The vertical arrow is the contraction of the divisor S on the curve l': note that all horizontal divisor are $\mathbb{F}_{d-1}$, and that the resulting 3-fold is smooth if $d = 2$, and singular along l' if $d \geq 3$.

The upper part of the diagram (blow-up of a point in a $\mathbb{P}^1$-bundle, followed by a flop) is an example of a Sarkisov link of type I (see next lecture).

NB: All this is mainly descriptive, and can be checked manually using local charts (or toric arguments, since all involved varieties are toric).

It is interesting to compute the intersection number $K \cdot l$ on the singular 3-fold before the flip. We have $\pi \colon Y \to X$ the contraction of S. Let us note by l_Y the flopped curve, and $l_X = \pi_*(l_Y)$ the flipped curve. We use the general property of intersection numbers, known as the "projection formula", that says that for any divisor D on X, and any curve C on Y, we have

$$\pi^* D \cdot C = D \cdot \pi_* C.$$

In particular, we have

$$K_X \cdot l_X = \pi^* K_X \cdot l_Y.$$

Recall from §1.3.4 that $\pi^* K_X = K_Y - \frac{3-d}{d} S$. Furthermore $K_Y \cdot l_Y = 0$ and $S \cdot l_Y = 1$, so we get

$$K_X \cdot l_X = \frac{d-3}{d}$$

which is negative iff $d = 2$.

By definition a **flip** replaces a curve of negative intersection with respect to the canonical divisor by a curve of positive intersection (whereas in the case of a **flop** both intersection numbers are zero). It turns out that the phenomenon of flip does not occur on smooth 3-fold. However it does occur on 3-folds with pretty mild singularities, as the example shows.

Final warning: This is not true that any smooth rational curve in a smooth 3-fold with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ can be flopped. In general it might happen that the blow-up/blow-down process leads to a non projective (but otherwise perfectly fine smooth algebraic) 3-fold.

2.3. **Divisorial contraction E_3.** Mori [Mor82] classified (among other things) all possible divisorial contractions from a smooth 3-fold. There are 5 types of these, often denoted by $E_1, \cdots, E_5$ (This notation seems to have been introduced first in the book [CKM88, p.30]. See also p. 33 for the Francia flip). E_1 is the blow-up of a smooth curve, E_2 the blow-up of a smooth point. E_5 is the blow-up of a singularity locally isomorphic to the cone over the Veronese embedding of $\mathbb{P}^2$ (the example we have seen in §1.3.4). An example of E_4 is given in Exercise 9. The case of E_3 is somewhat tricky, and the subject of this paragraph.

Warm-up : Let us consider first the blow-up $\pi_1 : X \to \mathbb{A}^3$ of $xy = z = 0$ in $\mathbb{A}^3$ (union of two lines). Recall that in general the blow-up of a subvariety of $\mathbb{A}^n$ defined by two equations $f = g = 0$ is given by

$$\left\{(x_1,\cdots,x_n),[u:v] \in \mathbb{A}^n \times \mathbb{P}^1; u f(x_1,\cdots,x_n) = vg(x_1,\cdots,x_n)\right\}$$

In our case the equation is $uxy = vz$, which gives in the affine chart $u = 1$

$$xy = vz.$$

This is exactly the equation of the singularity that was the starting point for our construction of the Atiyah flop:

In general, the blow-up of a singular curve in a smooth 3-fold produces a singular 3-fold.

The exceptional divisor E_1 of π_1 is (in the affine chart $u = 1$)

$$E_1 = \{(x,y,z,v); x = z = 0 \text{ or } y = z = 0\}.$$

In other words $E_1 = E_1^x \cup E_1^y$ is the cone over two intersecting rules of $\mathbb{P}^1 \times \mathbb{P}^1$, and has two irreducible components $E_1^x = \{x = z = 0\}$, $E_1^y = \{y = z = 0\}$.

We can blow-up the singular point (vertex of the cone), producing an exceptional divisor $E_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Denote by $L_x \subset E_2$ the rule in the closure of the strict transform of E_1^x, $L_y \subset E_2$ the rule in the closure of E_1^y, and G the strict transform of the line $E_1^x \cap E_1^y = \{x = y = z = 0\}$.

Note that π_1-exceptional curves in E_1^x tend to $G + L_x$. Similarly, π_1-exceptional curves in E_1^y tend to $G + L_y$.

Now for the real thing: consider a cubic nodal curve C in $\mathbb{P}^2 \subset \mathbb{P}^3$. The node is locally (analytically) isomorphic to $xy = z = 0$, so the blow-up π_1 of C produces a singularity as in the previous case. The blow-up π_2 of this singularity produces an exceptional divisor $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. But the point now is that E_1 is an irreducible divisor, and that we can deform π_1-exceptional curves both on $G + L_x$ or $G + L_y$. As a consequence, $L_X \equiv L_Y$, and so any *projective* morphism which contracts one of them has to contract the other.

In the vocabulary of the next lecture: L_X and L_Y correspond to a single extremal ray, and π_2 is a divisorial contraction: the so-called E_3 type.

3. Minimal model program and Sarkisov program

3.1. **Linear systems and rational maps.** Let $f : X \subset \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. One can express f using $m+1$ homogeneous polynomial f_i of the same degree and writing $f(x) = [f_0(x) : \cdots : f_m(x)]$. One can also write

$$f(x) = \left[1 : \frac{f_1(x)}{f_0(x)} : \cdots : \frac{f_m(x)}{f_0(x)}\right].$$

Here the f_i/f_0 are m rational functions on X, with poles bounded by the divisor $f_0 = 0$. Furthermore the divisor $f_0 = 0$ is the preimage by f of an hyperplane.

Conversely, let D be a divisor on X (say effective for now). We now construct a rational map such that D is the preimage of an hyperplane. Consider the set of rational functions on X with poles bounded by D, i.e.

$$\mathcal{L}(D) = H^0(X,D) = \{f \in \mathbb{C}(X); D + (f)_0 - (f)_\infty \geq 0\}.$$

By convention, $f \equiv 0$ is in $\mathcal{L}(D)$.

Theorem 7 (see [Sha94a, p. 173])**.** *$\mathcal{L}(D)$ is a finite dimensional vector space.*

Proof. Note first that if $D = D_1 - D_2$ with D_1, D_2 effective, then $\mathcal{L}(D) \subset \mathcal{L}(D_1)$, and so we can restrict ourselves to the case $D \geq 0$.

Suppose first that D is an effective divisor on a *curve* X. Let $x \in \operatorname{Supp} D$ be a point with multiplicity $r > 0$, and t a local parameter. Consider the linear function

$$\lambda : f \in \mathcal{L}(D) \to (t^r f)(x).$$

The kernel of λ is the effective divisor $D - x$ which has degree one less than D, thus after $\deg D$ such operations we obtain that $\mathcal{L}(0)$ is a subspace of $\mathcal{L}(D)$ defined by $\deg(D)$ linear forms. But $\mathcal{L}(0) = \mathbb{C}$ (regular functions on X are constant), and so $\dim \mathcal{L}(D) \leq \deg D + 1$.

The general case (higher dimensional X) follows by induction. Let us indicate briefly how to obtain the result when X is a surface. Let H be a general hyperplane section. Then H is a smooth curve and $D|_H$ is an effective divisor of degree $k > 0$. Thus if $f \in \mathcal{L}(D)$, the restriction $F|_H$ admits at most k poles, and also at most k zeros. Consider $H_1, \cdots, H_m$ general hyperplane sections, on each of them the restriction $\mathcal{L}(D)|_{H_i}$ is a finite dimensional space. So the subspace of $\mathcal{L}(D)$ containing functions identically zero on the union of H_i has finite codimension. But if $m > k$, this subspace is trivial. □

We have a bijection between $\mathbb{P}(\mathcal{L}(D))$ and the set $|D|$ of effective divisors linearly equivalent to D: simply note that such a divisor D' can be written $D' = D + (f)$ with $f \in \mathcal{L}(D)$. One says that $|D|$ is the (complete) linear system associated with D; a linear subspace of $|D|$ (corresponding to a vector subspace of $\mathcal{L}(D)$) is simply called a linear system. Note that the definitions of $\mathcal{L}(D)$ and $|D|$ can be easily generalized to the case of a non-effective divisor. We have:

Proposition 8. *If $D_1 \sim D_2$ then $|D_1| = |D_2|$ and $\dim \mathcal{L}(D_1) = \dim \mathcal{L}(D_2)$.*

Proof. Write $D_1 = D_2 + (g)$ and note that $f \in \mathcal{L}(D_1) \to fg \in \mathcal{L}(D_2)$ is an isomorphism. □

If $g_0, \cdots, g_r$ is an independent family in $\mathcal{L}(D)$, one can define a rational map $g\colon x \dashrightarrow [g_0(x) : g_1(x) : \cdots : g_r(x)]$ (the image of f is not contained in a hyperplane of $\mathbb{P}^r$ because the family is independent). If $D \geq 0$, one can always take $g_0 \equiv 1$.

If one take a basis of $\mathcal{L}(D)$, we obtain a rational map that we denote by $\phi_{|D|}$.

Definition 9. The **Kodaira dimension** $\kappa(D)$ of D is the maximum of the dimensions of the images $\phi_{|nD|}(X)$ when $n \in \mathbb{N}$. If $\mathcal{L}(nD) = \emptyset$ for all n, by convention $\kappa(D) = -\infty$.

Since $\dim \phi_{|nD|}(X)$ grows with n, it is clear that the max is reached for n sufficiently large and divisible. But if furthermore the algebra

$$R_D = \bigoplus_{n \geq 0} H^0(X, nD)$$

is finitely generated, then one can define $\mathrm{Proj}\, R_D$ and a rational map $\phi_D \colon X \dashrightarrow \mathrm{Proj}\, R_D$.

3.2. **Basics on Minimal Model Program.** Curves which are negative with respect to the canonical divisors are obstruction to the map ϕ_{K_X} to be well-defined. The general idea of the Minimal Model Program is to produce an algorithm which allows us to remove these undesirable curves.

Theorem 10 (Cone theorem, first version)**.** *Let X be a smooth complex projective variety. There exists a (minimal, at most countable) set of rational curves E_i with $K_X \cdot E_i < 0$ such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X) \cap K_{\geq 0} + \sum \mathbb{R}^+[E_i].$$

The rays $\mathbb{R}^+[E_i]$ in $\overline{\mathrm{NE}}(X)$ are called **extremal rays**.

To each extremal ray $\mathbb{R}^+[E_i]$ corresponds a morphism $X \to Y$ with connected fibers which contracts exactly the curves on X proportional to E_i: this is not trivial, and is often called the "contraction theorem".

There are 3 types of such contraction morphisms:

- Fibering type: $\dim Y < \dim X$. Then one says that X is a Mori fiber space. One can show that the general fiber is a smooth Fano variety. If X is a 3-fold, then we have del Pezzo fibrations, conic bundle and prime Fano (i.e the Picard number is 1, and Y is a point).
- Divisorial contraction: $X \to Y$ is birational, and the exceptional set is a divisor. When X is a smooth 3-fold, there are five types of these ($E_1, \cdots, E_5$, see §2.3). Note that Y might be singular (with terminal singularities).
- Small contraction: $X \to Y$ is birational, and the exceptional set has codimension at least 2. The big problem here is that intersection numbers are not well defined on Y (some Weil divisors are not Cartier, even up to a constant). The solution is to flip the curves in the exceptional set (even in the 3-fold case, this set does not have to be irreducible, even if this was the case in the simple example of the Francia flip). Existence of flip is by no mean easy(in contrast with unicity)...

There are several ways in which the cone and contraction theorems can be extended. In particular:

- singular version: one can allow terminal singularities, which is a class stable under divisorial contractions and flips.
- relative version: We consider varieties with a morphism to a fixed variety S. The relative effective cone $\mathrm{NE}(X/S)$ is the cone generated by contracted curves.
- logarithmic version: We replace K_X by a small perturbation $K_X + \Delta$. "Small" means a control on coefficients and singularities of Δ; there are several versions (Kawamata log terminal, divisorially log terminal, log canonical...)

We are now going to describe the Sarkisov program, which aims at factorizing a given birational map between Mori fiber spaces (such as $\mathbb{P}^n$) by mean of elementary links. The relative and logarithmic MMP play a key role in this story. First we look at the comparably easy case of surfaces.

3.3. **Sarkisov Program for surfaces.** Let $f\colon S \dashrightarrow S'$ be a birational map between 2-dimensional Mori fiber spaces S/T and S'/T'.

We start by taking a resolution $S \stackrel{\pi}{\leftarrow} X \stackrel{\pi'}{\rightarrow} S'$ of the base points of f, where X is a smooth projective surface and we choose an ample divisor H' on S'. We note $H_S \subset S$ (or $H_X \subset X$, etc...) the strict transform of a general member of the linear system $|H'|$ on S (or X, etc...), and $C_i \subset X$ the irreducible components of the exceptional locus of π. We write down the ramification formulas

$$K_X = \pi^* K_S + \sum c_i C_i \quad \text{and} \quad H_X = \pi^* H_S - \sum m_i C_i.$$

We define the maximal multiplicity λ as the maximum of the $\lambda_i = \frac{m_i}{c_i}$. It turns out that in the case of surfaces the maximum is realized by a divisor C_i with $c_i = 1$, that is C_i does not correspond to an infinitely near point. Note that in higher dimension the maximum of the λ_i does not coincide in general with the maximum of the m_i. Technically $1/\lambda$ is defined as a canonical threshold.

On the other hand we define the degree μ of f as the positive rational number $\frac{H_S \cdot C}{-K_S \cdot C}$ where C is any curve contained in a fiber of the log Mori fibration on S.

In the case $\lambda > \mu$, we consider $Z \to S$ the blow-up of a point realizing the maximal multiplicity. Then $Z \to T$ (recall that $S \to T$ was the Mori fibration on S) is a morphism and the relative Picard number is 2. In other word the relative cone $\overline{\text{NE}}(Z/T)$ is 2-dimensional, and so has exactly two extremal rays. One says (after Reid) that we are in position to play a 2-rays game. One of the rays corresponds to the maximal blow-up we just did. The other one corresponds to a ray which is strictly negative with respect to $K_Z + \frac{1}{\lambda} H_Z$: this comes from the assumption $\lambda > \mu$. The contraction of this second extremal ray gives either

- a Mori fiber structure on Z: we say we have a link of type I;
- a divisorial contraction to a surface S' with a Mori fiber structure $S' \to T$: we say that we have a link of type II.

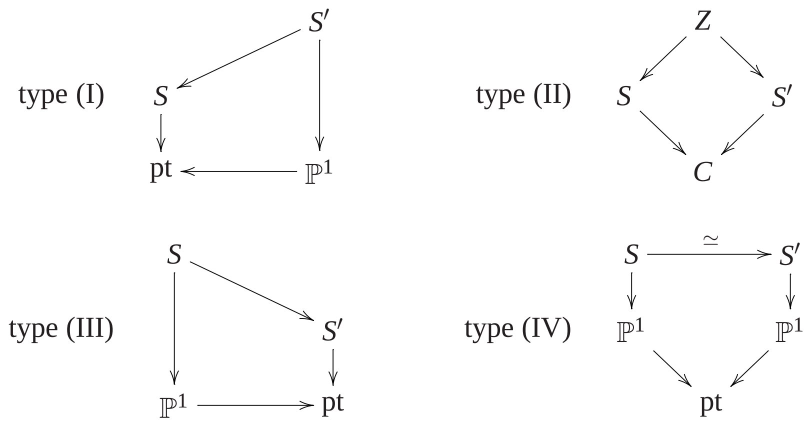
These operations done, one shows that we have simplified f in the sense that: either μ went down; or μ remained constant but λ went down; or μ and λ remained constant but the number of exceptional divisors in X realizing the multiplicity λ went down.

In the case $\lambda \le \mu$, it is possible to prove that $K_S + \frac{1}{\mu} H_S$ cannot be nef (otherwise f would be an isomorphism). We are again in position to play a 2-rays game, this time directly on S. Indeed $\text{NE}(S)$ is 2-dimensional, with one ray corresponding to the Mori fiber structure $S \to T$, and the other being a $K_S + \frac{1}{\mu} H_S$-negative ray. The contraction of this extremal rays gives either

- another Mori fiber structure on S: we say we have a link of type IV (S has to be $\mathbb{P}^1 \times \mathbb{P}^1$);

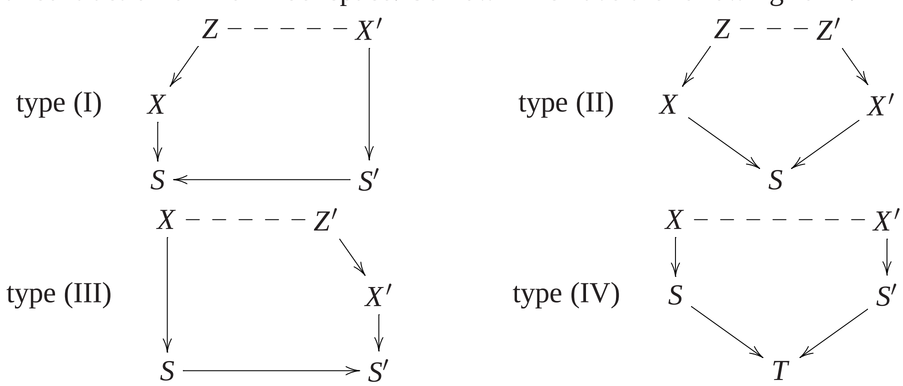
- a divisorial contraction to a surface S' with a Mori fiber structure $S' \to \text{pt}$: we say that we have a link of type III (S' has to be $\mathbb{P}^2$).

The four types of elementary links occurring in the factorization procedure are summarized in the following diagrams:



3.4. **Sarkisov Program after Corti.** One can try to apply the same strategy in higher dimension. Corti [Cor95] sorted it out in dimension 3. Here are the main distinctions from the 2-dimensional case:

- The 2-ray games might involves log-flips before producing the expected divisorial contraction or Mori fiber space. So now links have the following form:



- The maximal multiplicity might correspond to an infinitely near point or curve. In particular λ is in general rational, with no obvious control on the denominator.
- The varieties involved might in general be singular (terminal singularities); as a consequence the denominator of the degree μ is not easy to control either, especially in the case of Fano varieties with Picard number 1.
- From the previous observation it is not clear why the program should stop, since we are trying to make an induction argument using rational numbers with poor control on the denominator (the proof by Corti in the 3-dimensional case is quite tricky for this reason).
- As a final remark, Mori fiber spaces in higher dimension are quite complicated. There are plenty of smooth (or terminal...) Fano 3-folds (n-folds...)

with Picard number 1, and a $\mathbb{P}^1$-bundle over an arbitrary blow-up of $\mathbb{P}^2$ is a rational Mori fiber space(so the Picard number can be arbitrary large)... Thus the factorization is less satisfying than in dimension 2.

4. Mori dream spaces and Sarkisov links

4.1. **Cones.** Consider a projective toric (or Fano, or log Fano...) variety X: these are special cases of "Mori dream spaces" [HK00, McK10]. The finite dimensional vector space $N^1(X)$ of $\mathbb{Q}$-divisors modulo numerical equivalence contains the following interesting cones:

- Because of the assumption (X toric or Fano), the cone $\mathrm{Eff}(X)$ of **effective** divisors is closed and for any $D \in \mathrm{Eff}(X)$ we can define
$$\phi_D\colon X \dashrightarrow X_D = \mathrm{Proj} \bigoplus_{m\geq 0} H^0(X, \lfloor mD \rfloor).$$
- The **big** cone $\mathrm{Big}(X)$ is the cone of divisors D such that $\dim X_D = \dim X$. It is a general fact that $\mathrm{Big}(X)$ is the interior of $\mathrm{Eff}(X)$.
- the **nef** cone $\mathrm{Nef}(X)$ is the cone of divisors D such that $D \cdot C \geq 0$ for any curve $C \in X$.
- the **ample** cone $\mathrm{Amp}(X)$ is the cone of ample divisors; by Kleiman criterion it corresponds to divisors D such that $D \cdot C > 0$ for any 1-cycle $C \in \overline{\mathrm{NE}}(X)$. The ample cone is the interior of the nef cone, and the nef cone is the closure of the ample cone.
- The **movable** cone $\mathrm{Mov}(X)$ is the cone of divisors D without any divisor in the base locus of $|D|$.

4.2. **Chamber decomposition.**

Definition 11. We say that two divisors $D_1, D_2 \in \mathrm{Eff}(X)$ are **Mori equivalent** if there exists an isomorphisms between X_{D_1} and X_{D_2} which makes the following diagram commutes:

$$\begin{array}{ccc} & X & \\ {\scriptstyle \phi_{D_1}} \swarrow & & \searrow {\scriptstyle \phi_{D_2}} \\ \mathrm{Proj} \oplus H^0(X, \lfloor mD_1 \rfloor) & \xrightarrow{\simeq} & \mathrm{Proj} \oplus H^0(X, \lfloor mD_2 \rfloor) \end{array}$$

Taking equivalence classes we obtain a chamber decomposition of the cone $\mathrm{Eff}(X)$. For instance the cone of ample divisors is a (open, and maximal dimensional) chamber. If $\mathcal{C}_1, \mathcal{C}_2$ are the closure of chambers of maximal dimension, we say that:

- $\mathcal{W} = \mathcal{C}_1 \cap \mathcal{C}_2$ is an **internal wall** if $\dim \mathcal{W} = \dim \mathcal{C}_1 - 1$;
- $\mathcal{W} = \mathcal{C}_1 \smallsetminus \mathrm{Big}(X)$ is a **window** if $\mathcal{W}$ is a maximal convex subset of $\mathcal{C}_1 \smallsetminus \mathrm{Big}(X)$ (hence $\dim \mathcal{W} = \dim \mathcal{C}_1 - 1$). In this case I say that $\mathcal{C}_1$ is (the closure of) an **external** chamber with window $\mathcal{W}$.

The chambers admit the following description ([HK00]):

There exists a finite number of small birational maps $f_i\colon X \dashrightarrow X_i$ such that the $f_i^*(\mathrm{Nef}(X_i))$ are exactly the maximal dimensional chambers inside $\mathrm{Mov}(X)$. All others maximal dimensional chambers have the form $g_j^*(\mathrm{Nef}(X_j)) + \mathrm{Exc}(g_j)$ where $g_j\colon X \dashrightarrow$

X_j is a finite collection of birational contractions. All (closure of) chambers are polyhedral cones.

4.3. **Minimal Model Program.** Fix D a divisor on X (for instance the canonical divisor K_X). Choose A an ample generic divisor on X, and consider the segment $[A,D]\cap \mathrm{Eff}(X)$. The generic assumption on A is to be sure that this segment never cross two walls or windows simultaneously. Every wall-crossing (when going from A to D) corresponds to either a D-flip or a D-divisorial contraction. This MMP ends with a D-nef model if D is in the interior of $\mathrm{Eff}(X)$ (i.e. D big), or with a Mori Fiber space otherwise.

The key proposition here is (see proof next lecture, 16):

Proposition 12. *Consider $\mathcal{A}_i, \mathcal{A}_j$ two chambers, and $\mathcal{C}_i, \mathcal{C}_j$ their closures. If $\mathcal{C}_i \cap \mathcal{A}_j \neq \emptyset$, there exists $f_{i,j}\colon X_i \to X_j$ a morphism such that $f_j = f_{i,j}\circ f_i$. Furthermore the relative Picard number of X_i/X_j is equal to the difference in the dimensions of $\mathcal{C}_i$ and $\mathcal{C}_i \cap \mathcal{C}_j$.*

In particular, if $\mathcal{A}_i$ is an external chamber and $\mathcal{A}_j$ is its window, $f_{i,j}\colon X_i \to X_j$ is a Mori fiber space structure.

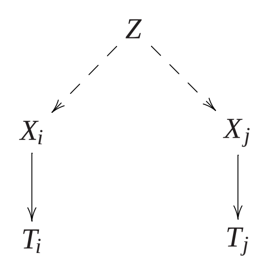
An internal wall corresponds to a flip if the corresponding X_i, X_j have the same Picard number. This is in particular the case when the wall is inside the movable cone.

When the Picard numbers differ by one, we have a divisorial contraction.

Thus on such a space (a "Mori dream space"), the MMP works with respect to any divisor D (whereas in general D as to be the canonical divisor K_X, or a small enough perturbation $K_X+\Delta$). In particular we can apply this to $D = K_X$. Note that the anticanonical divisor $-K_X$ is effective for a toric variety, so it is easier to picture $-K_X$. Depending on the position of $-K_X$ in the effective cone, it is not always possible to reach any external chamber by a K-MMP. These not-reachable chambers typically correspond to varieties X_i with singularities worst than terminal.

4.4. **Sarkisov program.** Suppose that $\mathcal{C}_i, \mathcal{C}_j$ are two external chambers with adjacent windows $\mathcal{W}_i, \mathcal{W}_j$ (meaning $\mathcal{W}_i \cap \mathcal{W}_j$ has codimension 1 in $\mathcal{W}_i$ and $\mathcal{W}_j$).

We have a diagram



I claim that the induced birational map $X_i \dashrightarrow X_j$ is a Sarkisov link. There are several cases to consider:

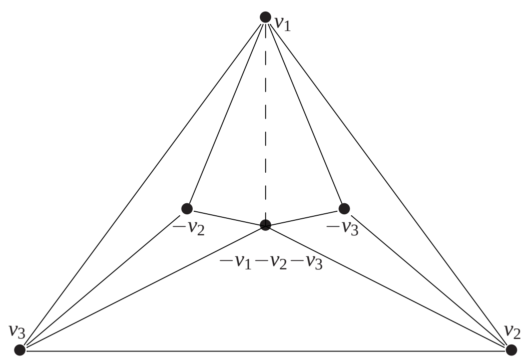
- If $\mathcal{W}_i \cap \mathcal{W}_j$, $\mathcal{W}_i$ and $\mathcal{W}_j$ all belong to distinct chambers. Then there is a rational map $Z \dashrightarrow T_{i,j}$ corresponding to the chamber $\mathcal{W}_i \cap \mathcal{W}_j$, and X_i, X_j have the same Picard number (because the relative Picard number over $T_{i,j}$ is 2 in both cases). This is a link of type IV.

NB: in this case, it is possible to have $\mathcal{C}_i = \mathcal{C}_j$ (see example below).

- If $\mathcal{W}_i \cap \mathcal{W}_j$, $\mathcal{W}_i$ and $\mathcal{W}_j$ all belong to the same chamber. Again X_i, X_j have the same Picard number. This is a link of type II.
- If $\mathcal{W}_i \cap \mathcal{W}_j$ and $\mathcal{W}_i$ are in a same chamber, and $\mathcal{W}_j$ in another. Then the Picard number of X_j is one more than the Picard number of X_i. This is a link of type I. In the obvious symmetric situation we get a link of type III.

4.5. **Example.** $\mathbb{P}^3$ blown-up at two distinct points

For those who know some toric geometry (if not, the book [CLS] is a great place to start with), I give below the fan (in $\mathbb{Z}^3$, with canonical basis (v_1, v_2, v_3)) of the toric variety X obtained from $\mathbb{P}^3$ by blowing-up the two points $[0:1:0:0]$ and $[0:0:1:0]$:



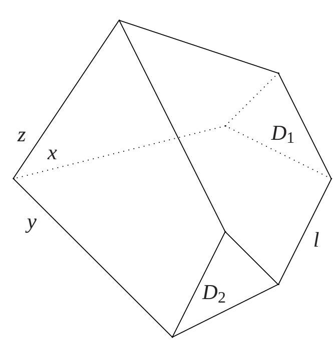
The dotted line corresponds to a flopping curve.

Now I describe divisors on X (without using toric geometry).

Note D_x (resp. D_y, D_z, D_t the strict transform on X of the plane $x = 0$ in $\mathbb{P}^3$ (resp. $y = 0$, etc...). Note D_1, D_2 the exceptional divisors obtained by blowing-up $[1:0:0:0]$ and $[0:1:0:0]$ respectively.

Note l_1, l_2 lines in D_1, D_2 and l the transform of the line $z = t = 0$ through the two blown-up points.

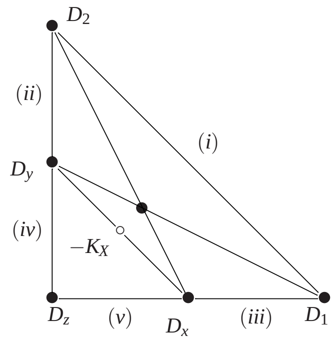
The divisors D_z, D_1, D_2 generate the cone $\mathrm{Eff}(X)$ of effective divisors, and the curves l, l_1, l_2 generate the cone of curves.



From the following intersection table, we see that $D_x \sim D_z + D_1, D_y \sim D_z + D_2$ and $D_z + D_1 + D_2$ generate the nef cone.

	D_z	D_x	D_y	D_t	D_1	D_2	$D_z+D_1+D_2$
l_1	$+1$	0	$+1$	$+1$	-1	0	0
l_2	$+1$	$+1$	0	$+1$	0	-1	0
l	-1	0	0	-1	$+1$	$+1$	$+1$

We have the following chamber decomposition in the cone $NE(X)$:



The five external faces of this cone correspond to all Mori fiber spaces that we can obtain from X by running a MMP:

- (i): The trivial fibration $\mathbb{P}^3 \to pt$;
- (ii) and (iii): The two $\mathbb{P}^1$-fibrations to $\mathbb{P}^2$ obtained by blowing-up $\mathbb{P}^3$ at one of the two points;
- (iv) and (v): The two $\mathbb{P}^1$-fibrations to the Hirzebruch surface $\mathbb{F}_1$ that exist on Y (which is obtained from X by the flop of l).

Two adjacent windows correspond to Sarkisov links. Note that the anticanonical divisor $-K_X$ is equivalent to $D_x + D_y$, and is nef, movable but not ample. Thus Mori fiber spaces $(i),(ii),(iii)$ can be obtained by K-MMP, whereas (iv) and (v) cannot.

5. Sarkisov program after Hacon-McKernan: overview

As it is often the case, the global features that we observed on projective toric varieties partially generalize to more general projective varieties.

We start with a classical example that shows the difficulty; and then we review the construction by Hacon and McKernan that leads to a proof of existence of a Sarkisov program in any dimension.

5.1. **A surface with infinitely many -1-curves.** See [Fri98, p. 131], or [McK10].

Proposition 13. *Consider S the blow-up of $\mathbb{P}^2$ at 9 general points. Then there is a bijection between the set of exceptional curves on S and $\mathbb{Z}^8$.*

Proof. First step: consider the 9 base points of a pencil of elliptic curves on $\mathbb{P}^2$. Note that these points are in special position, because (by a dimension argument), by 9 general points passes only one cubic. The surface S obtained by blowing-up these 9 points is an elliptic fibration over $\mathbb{P}^1$, and the 9 exceptional divisors are sections. Taking one of the exceptional divisor to define a neutral element on each elliptic curve of the fibration, the 8 other define automorphisms of S. We obtain a subgroup $\mathbb{Z}^8 \subset \mathrm{Aut}(S)$ (if the pencil is sufficiently general). The orbit under this group of any exceptional divisor gives infinitely many -1-curves.

Second step: consider perturbation $p_{i,t}$ of the 9 points in the first step, and the surface S_t obtained by blowing-up these points. The exceptional curves on $S = S_0$ persists on the nearby fibers, because they are characterized topologically by their intersection numbers : $K_S \cdot E = E^2 = -1$.

An alternative proof would be to show that, fixing E_0 an exceptional curve of a general S, and denoting by $\mathcal{E}$ the set of exceptional curves on S, the map

$$\begin{aligned} \mathcal{E} &\to (K_S)^{\perp}/K_S \\ E &\to E - E_0 \end{aligned}$$

is a bijection. Then since $(K_S)^{\perp}/K_S = \mathbb{Z}^8$ and any class e satisfying $e^2 = K_S \cdot e = -1$ can be represented by an effective curve (Riemann-Roch), we obtain the result. □

5.2. **Minimal and canonical models.** [KM98, §3.8]

A birational map $\phi \colon X \dashrightarrow Y$ is called a **birational contraction** if ϕ^{-1} does not contract any divisor.

Let (X,Δ) be a log canonical pair and $\phi \colon (X,\Delta) \dashrightarrow (Y,\phi_*\Delta)$ a birational contraction.

We call $(Y,\phi_*\Delta)$ a **weak canonical model** of (X,Δ) if

(1) $K_Y + \phi_*\Delta$ is nef;
(2) $a(E,X,\Delta) \le a(E,Y,\phi_*\Delta)$ for every ϕ-exceptional divisor $E \subset X$.

There are two ways in which this definition can be strengthened.

If in (1) we put "ample" instead of "nef", we say that $(Y,\phi_*\Delta)$ is a **canonical model**. If a canonical model Y of (X,Δ) exists then it is unique and is given by

$$Y = \operatorname{Proj} \bigoplus_{m\ge 0} H^0(X, \lfloor m(K_X+\Delta)\rfloor).$$

If in (2) we ask for a strict inequality then we say that $(Y,\phi_*\Delta)$ is a **minimal model**. If $K_X+\Delta$ is big and we run a log MMP starting from X, the end product will be a minimal model.

Note the following result from [BCHM10, Corollary 1.1.2]:

Corollary 14. *Let (X,Δ) be a projective Kawamata log terminal pair, where $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Then the ring*

$$\bigoplus_{m\ge 0} H^0(X, \lfloor m(K_X+\Delta)\rfloor)$$

is finitely generated.

In particular taking the Proj we obtain the canonical model (we also say an **ample model**, in particular when the dimension of the model is strictly less than the dimension of Z) of $K_X+\Delta$.

5.3. **Shokurov polytopes.** Let Z be a smooth projective variety, $V \subset \mathrm{WDiv}_{\mathbb{R}}(Z)$ a finite dimensional vector subspace which generates $N^1(Z)$, $A \geq 0$ an ample $\mathbb{Q}$-divisor.

We define

$$\mathcal{E}_A(V) = \{D = \lambda(K_Z + A + B); \\ \lambda \geq 0, B \geq 0 \in V, (Z, A+B) \text{ log canonical}, D \text{ pseudo-effective}\}.$$

A key result from [BCHM10, Theorem E] is

Theorem 15 (Finiteness of models)**.** *There are finitely many birational contractions $\psi_j \colon Z \dashrightarrow X_j$ such that if $\psi \colon Z \dashrightarrow X$ is a weak canonical model for $K_Z + \Delta$, with $K_Z + \Delta \in \mathcal{E}_A(V)$, then $\psi = \psi_j$ up to an isomorphism $X \simeq X_j$.*

Using Corollary 14 it makes sense to consider the equivalence relation on $\mathcal{E}_A(V)$ given by Definition 11.

We note $\mathcal{A}_i$ the equivalence classes, $f_i \colon Z \dashrightarrow X_i$ the associated ample models, and $\mathcal{C}_i$ the closure of $\mathcal{A}_i$.

Proposition 16 ([HM09, Theorem 3.3])**.** *The $\mathcal{A}_i$ are in finite number, with the following properties:*

(1) $\{\mathcal{A}_i; 1 \leq i \leq m\}$ is a partition of $\mathcal{E}_A(V)$. Each $\mathcal{C}_i$ is a rational polyhedral cone, and is the disjoint union of some of the $\mathcal{A}_j$.
(2) If i, j are two indices such that $\mathcal{C}_i \cap \mathcal{A}_j \neq \emptyset$, there exists $f_{i,j} \colon X_i \to X_j$ a morphism such that $f_j = f_{i,j} \circ f_i$.
(3) f_i is birational and X_i is $\mathbb{Q}$-factorial iff $\mathcal{C}_i$ is of maximal dimension.
(4) The relative Picard number of $f_{i,j} \colon X_i \to X_j$ is equal to the difference in the dimensions of $\mathcal{C}_i$ and $\mathcal{C}_i \cap \mathcal{C}_j$.

Proof. (1) is essentially a consequence of Theorem 15, but I skip the details.

(2) Pick $\theta_0 \in \mathcal{C}_i \cap \mathcal{A}_j$, and $\theta_1 \in \mathcal{A}_i$, such that $\theta_t = t\theta_1 + (1-t)\theta_0 \in \mathcal{A}_i$ for all $t \in]0,1]$.

We may assume that there exists a birational contraction $f \colon Z \dashrightarrow X$ which is a weak canonical model for $t \in]0,1]$.

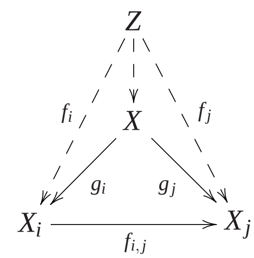
Set $\Delta_t = f_*\theta_t$. The Base Point Free Theorem 17 implies that $K_X + \Delta_t$ is semiample, and so by unicity of the ample model we have a morphism $g_i \colon X \to X_i$ and ample divisors $H_{1/2}$ and H_1 such that

$$K_X + \Delta_{1/2} = g_i^* H_{1/2} \text{ and } K_X + \Delta_1 = g_i^* H_1.$$

One can verify that $K_X + \Delta_t = g_i^* H_t$ for all $t \in [0,1]$.

Since $K_X + \Delta_0$ is semiample we also have a morphism $g_j \colon X \to X_j$.

The curves contracted par g_i are the curves C such that $(K_X + \Delta_t) \cdot C = 0$ for $t \in]0,1]$, and so also for $t = 0$. Hence $\mathrm{NE}(g_i) \subset \mathrm{NE}(g_j)$, and the rigidity Lemma 18 gives a morphism $f_{i,j}\colon X_i \to X_j$ such that the following diagram commutes:



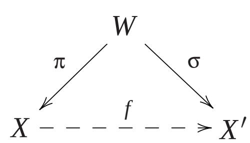
(3) I skip details.

(4) Let us prove this when $\mathcal{C}_i = \mathrm{Nef}(Z)$ (closure of $\mathcal{A}_i = \mathrm{Amp}(Z)$). In fact the statement is equivalent to a dual statement (Cone Theorem), which says that to any $K + A$ negative face in the cone of curve $\overline{\mathrm{NE}}(Z)$ corresponds a contraction morphism with relative Picard number the dimension of the face. □

Theorem 17 (Base Point Free, see [Deb01, Theorem 7.32])**.** *Let X be a smooth projective variety, and let D be a nef divisor on X such that $aD - K_X$ is nef and big for some positive a. Then the linear system $|mD|$ is base point free for all m sufficiently large.*

Lemma 18 (Rigidity, see [Deb01, Proposition 1.14])**.** *Let X, Y, Y' be projective varieties and let $\pi\colon X \to Y$, $\pi'\colon X \to Y'$ be morphisms with connected fibers. If* $\mathrm{NE}(\pi) \subset \mathrm{NE}(\pi')$*, there is a unique morphism $f\colon Y \to Y'$ such that $\pi' = f \circ \pi$.*

5.4. **Sarkisov program again.** We start with a birational map $f\colon X/S \dashrightarrow X'/S'$ between (smooth) Mori fiber spaces. For instance $X = X' = \mathbb{P}^3$, in which case $S = S'$ is a point. Consider a smooth resolution of f:



Now we consider the chamber decomposition of $\mathcal{E}_A(V)$, for a small ample and effective divisor A on W, and $V \subset \mathrm{WDiv}(W)$ a finite dimensional vector subspace which generate $N^1(W)$ and contains all effective divisors we need in what follows.

Consider H a divisor on X such that $K_X + H$ is ample. If H is chosen to be a sum of divisor with small enough coefficients and simply normal crossing support, then (W, π^*H) is klt and $K_W + \pi^*H \in \mathcal{E}_A(W)$. So we see that X corresponds to an external chamber $\mathcal{C}_X$, and X can be reached from W by running a K-MMP; i.e. there exists B_X an ample divisor on W such that the segment $[B_X, K_W]$ pass through the chamber $\mathcal{C}_X$. The same is true for X', by the same argument (so there exists $B_{X'}$ ample on W, etc...). There is a sequence of adjacent external chambers from $\mathcal{C}_X$ to $\mathcal{C}_{X'}$ (consider all the K-MMP starting from $B \in [B_X, B_{X'}]$). This gives a finite sequence of Sarkisov links, by the same argument as in §4.4.

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Exercises

Exercises you can try to do after LECTURE I.

Exercise 1. Find the canonical divisor of $\mathbb{P}^1 \times \mathbb{P}^1$ by computing explicitely the zeros and poles of a 2-form (for instance, you can work with the form $dx \wedge dy$ in a local affine chart $\mathbb{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$).

Do the same computation again starting with another 2-form (say $\frac{dx}{x} \wedge \frac{dy}{y}$). Check that you find a divisor linearly equivalent to the previous one.

Exercise 2. Prove (or take for granted) that a smooth quadric surface S in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Compute the canonical divisor of S using the adjunction formula. Compare with the previous exercise.

Exercise 3. Consider a smooth plane curve $C \subset \mathbb{P}^2$ of degree d: the equation of C is $F(X,Y,Z) = 0$ where F is homogeneous of degree d.

(1) In the affine chart $x = X/Z, y = Y/Z$, consider the 1-form $\omega = \frac{dx}{\partial_y f}$, where $f(x,y) = F(x,y,1)$.

Prove that ω is regular on this chart (hint: note that $df \equiv 0$ on C).

(2) Consider now the chart $u = X/Y, v = Z/Y$, and denote by $g(u,v) = F(u,1,v)$ the equation of C in this chart.

Prove that (ω) admits a zero of order $d-3$ along the hyperplane section $v = 0$.

(3) Is this result coherent with the adjunction formula ?

Exercise 4. Prove that if an exceptional divisor E over a (normal, $\mathbb{Q}$-factorial) variety X satisfies $a(E,X) < -1$, then one can find a sequence of exceptional divisors E_n over X such that $a(E_n;X) \to -\infty$.

Hint: suppose first that X is a surface, and that E is a divisor with $a(E,X) < -1$. Blow-up a point on E to produce an exceptional curve E_1. Blow-up again the point $E_1 \cap E$. Compute discrepancies...

Exercise 5. If C is a projective curve, recall the relation between the genus of C and the degree of the canonical divisor: $2g(C) - 2 = \deg K_C$. Use the adjunction formula to compute the genus of a smooth curve of degree d in $\mathbb{P}^2$. Same question for a smooth curve of bidegree (m,n) in $\mathbb{P}^1 \times \mathbb{P}^1$. Same question for a smooth curve which is the complete intersection of two smooth surfaces of degrees p and q. Conclude that some curves in $\mathbb{P}^3$ are not complete intersections.

Exercise 6. If $C \subset X$ is a smooth curve in a smooth 3-fold, we want to establish

$$\deg(N_{C/X}) = 2g(C) - 2 - K_X \cdot C$$

where $N_{C/X} = T_X|_C / T_C$ is the normal bundle of C in X (vector bundle of rank 2).

(1) If $C = S_1 \cap S_2$ is a complete intersection (both surfaces smooth), use adjunction formula.

(2) In general, recall that if we have an exact sequence of vector bundles on C

$$0 \to V \to V' \to V'' \to 0$$

then the first Chern class is additive: $\deg c_1(V') = \deg c_1(V) + \deg c_1(V'')$. Apply this to the sequence

$$0 \to \mathcal{N}_{C/X} \to T_X|C \to T_C \to 0$$

and conclude.

Exercise 7. Let $Y \to X$ be the blow-up of a smooth curve C in a smooth projective 3-fold X. Let E be the exceptional divisor and $L \subset E$ a contracted curve. Show that

$$K_Y \cdot L = E \cdot L = -1.$$

Exercise 8. Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Show that Q is the compactification of $\mathbb{A}^n$ by an irreducible divisor as soon as $n \geq 3$ (hint: consider a tangent hyperplane section).

Exercise 9. Consider the blow-up of a cuspidal cubic in $\mathbb{P}^3$ (say $z = y^2 - x^3 = 0$ in a local affine chart $\mathbb{A}^3 \subset \mathbb{P}^3$).

(1) Check that the resulting 3-fold has exactly one isolated singular point.
(2) Consider the blow-up of this singular point. What is the exceptional divisor ? Compute the discrepancy.

Exercises you can try to do after LECTURE III.

Exercise 10. On $\mathbb{P}^1 \times \mathbb{P}^1$, consider $F_1 = \{x = 0\}$, $F_2 = \{y = 0\}$. Compute ϕ_D for $D = F_1$, $D = F_1 + F_2$ and $D = F_1 + 2F_2$.

Exercise 11. Consider a smooth cubic surface $S \subset \mathbb{P}^3$.

(1) What is the dimension of the real vector space $N^1(S)$?
(2) How many extremal rays are there in $\overline{\mathrm{NE}}(S)$?
(3) Running the Minimal Model Program from S, how many different ways are there to reach a Mori fiber space?

Exercise 12. What is the result of the Sarkisov links starting from $\mathbb{P}^3$ with the blow-up of (i) a point or (ii) a line?

Exercise 13. (1) What is the result of the Sarkisov link starting from $\mathbb{P}^3$ with the blow-up of a smooth conic?
(2) Show that with two such links we can construct a Cremona map of bidegree (2,2) (i.e. a birational map $f\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ of degree 2 such that f^{-1} has also degree 2).
(3) Can you produce an example of a quadratic birational map $f\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that f^{-1} has degree 3? 4? 5?
(4) Back to the first question: what happen if we start with the blow-up of a plane curve of higher degree?

Exercise 14. Apply the Sarkisov program to the following two quadratic birational maps of $\mathbb{P}^2$:

(1) $\sigma\colon [x:y:z] \dashrightarrow [yz:xz:xy]$.
(2) $f\colon [x:y:z] \dashrightarrow [xz+y^2:yz:z^2]$.

Exercise 15. Show that a smooth cubic 3-fold X is unirational, i.e. there exists a dominant rational map $\mathbb{P}^3 \dashrightarrow X$:

(1) Show that there exists a line L_0 in X.
(2) Consider the variety B of lines which are tangent to X in a point $x \in L_0$. Show that B is rational (i.e., birational to $\mathbb{P}^3$).
(3) Show that there exist a dominant map $B \dashrightarrow X$ and conclude (hint : if L is one of the lines parametrized by B, consider $L \cap X$).

Exercise 16. What are the Sarkisov links starting from a smooth cubic 3-fold with the blow-up of a point; a line; a conic; a plane smooth cubic?

Exercise 17. Prove that the exceptional locus of an extremal divisorial contraction is irreducible.

Hint: you'll probably need a version of the negativity lemma which says: if $E = \sum a_i E_i$ is a π-nef exceptional divisor of a morphism $\pi\colon Y \to X$, then $a_i \leq 0$ for all i.

NB: for a small contraction, the exceptional locus might be reducible (see next exercise)!

Exercise 18. Consider $S \subset \mathbb{P}^3$ a smooth cubic surface, and a morphism $f\colon S \to \mathbb{P}^2$ (given by blowing-up 6 points).

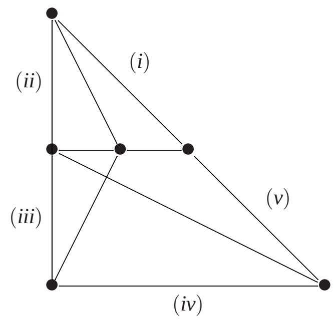
(1) If $C \subset S$ is the pull-back by f of a general conic, what is the degree of C (as a curve in $\mathbb{P}^3$)?
(2) Prove that any quadrisecant of C (line intersecting C 4 times) lies inside S. How many such quadrisecants are there?
(3) Consider the blow-up of $\mathbb{P}^3$ along C. What are the two extremal rays of the resulting nef cone?
(4) Prove that if we perform an Atiyah flop on one of the (strict transform of the) quadrisecants, we end up with an algebraic but non projective 3-fold. Compare with the cover illustration of Shafarevic, vol 2...

Exercise you can try to do after LECTURE IV.

Exercise 19. Consider the following two situations:

(1) $X = \mathbb{P}^3$ blown-up at a point and an infinitely near line;
(2) $X = \mathbb{P}^2$ blown-up at a point and an infinitely near point.

In each case compute the chamber decomposition of $\mathrm{Eff}(X)$: you should find something like the following picture.



Identify the Mori fiber spaces corresponding to the windows *(i-v)*. Where is the anticanonical divisor $-K_X$? Which chambers correspond to varieties with terminal singularities ? Verify that these can be reached via a K_X-MMP.