Algebraic Geometry 11 (1) (2024) 111–162
doi:10.14231/AG-2024-004

Generating the plane Cremona groups by involutions

Stéphane Lamy and Julia Schneider

Abstract

We prove that over any perfect field, the plane Cremona group is generated by involutions.

1. Introduction

The plane Cremona group over a field $\mathbf{k}$ is the group $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ of birational transformations of the projective plane. In concrete terms, a map $g \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ can be written in homogeneous coordinates as

$$g\colon [x : y : z] \dashrightarrow [P_0 : P_1 : P_2]\,,$$

where the $P_i \in \mathbf{k}[x, y, z]$ are homogeneous polynomials of the same degree and without non-constant common factor, and such that g admits an inverse of the same form. In a more geometric way, over an algebraically closed field, any birational map between surfaces can be understood as a sequence of blow-ups and inverses of blow-ups. Similarly, in the context of surfaces defined over a perfect field, the elementary operations to factorize birational maps are blow-ups of Galois orbits. It is remarkable that a single class of elementary transformations allows one to reconstruct any birational map. However, a blow-up is a transformation between two non-isomorphic surfaces, so it leaves open the question of finding a natural generating set for the group $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$. Over an algebraically closed field, Noether's theorem gives a neat answer: the Cremona group is generated by $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right) = \mathrm{PGL}_3(\mathbf{k})$ and by a single extra generator, the standard quadratic involution $\sigma\colon [x : y : z] \dashrightarrow [yz : xz : xy]$. Since in this case $\mathrm{PGL}_3(\mathbf{k})$ is generated by involutions, we obtain in particular that the Cremona group is generated by involutions.

The main result of the present paper is a generalization of this last statement to the case of an arbitrary perfect field $\mathbf{k}$.

Theorem 1.1. *Let $\mathbf{k}$ be a perfect field. The Cremona group $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ is generated by involutions.*

One motivation for this result was to understand the abelianization of the Cremona group, or in other words the possible surjective homomorphisms from the Cremona group to an abelian group. Over many perfect fields including all number fields, finite fields, or the real numbers, we know [Zim18, LZ20, Sch22] that the abelianization of the Cremona group contains an infinite direct sum of groups of order 2. An immediate corollary of Theorem 1.1 is that there is no

Received 15 June 2021, accepted in final form 10 October 2022.
2020 Mathematics Subject Classification 14E07.
Keywords: involution, Sarkisov link, Jonquières map, orthogonal group.

J. S. was supported by the Swiss National Science Foundation project P2BSP2_200209 and hosted by the Institut de Mathématiques de Toulouse.

surjective homomorphism from $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ to $\mathbf{Z}$, or to $\mathbf{Z}/n\mathbf{Z}$ for $n \geqslant 3$. We can rephrase this remark as follows.

COROLLARY 1.2. *Let* $\mathbf{k}$ *be a perfect field. The abelianization of the Cremona group* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *is a group of exponent* 2 (*any non-trivial element has order* 2).

In contrast, in higher dimension it was recently proved by E. Shinder and H.-Y. Lin [LS22] that $\mathrm{Bir}_{\mathbf{Q}}(\mathbb{P}^3)$ and $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^4)$ are *not* generated by involutions, as a consequence of the fact that these groups admit homomorphisms to $\mathbf{Z}$. In another direction, C. Shramov has proved that some Severi–Brauer surfaces (forms of $\mathbb{P}^2$ without any rational point) admit an infinite group of birational maps without any involution [Shr20, Theorem 1.2]. So it was not clear a priori whether the groups $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ should always be generated by involutions. In retrospect the result seems to be an accident of low dimension, and this might justify that the proof cannot be completely conceptual: even if we use the Sarkisov program as a general framework, at some points some combinatorial miracles have to occur in order for the result to hold true.

We mention that over the field of real numbers, the generation of $\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)$ by involutions was proved by S. Zimmermann in her Ph.D. thesis [Zim16, Corollary II.4.12]; she has also given a complete description of the abelianization of $\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)$ [Zim18]. Over the field with two elements, the generation of $\mathrm{Bir}_{\mathbf{F}_2}(\mathbb{P}^2)$ by involutions was established by the second author [Sch21], with a strategy similar to that in the present paper, but with some cases relying on an exhaustive search assisted by computer.

We now explain our strategy of proof for Theorem 1.1. To obtain a set of generators for $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, our first step is to apply the Sarkisov program. As a byproduct of the Sarkisov factorization, we get a natural invariant associated with each element $f \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, namely the minimal number $\mathrm{sl}(f) \geqslant 0$ of Sarkisov links necessary to factorize f. Then we can define a notion of irreducible element with respect to this Sarkisov length, and we obtain in Proposition 2.3 an abstract factorization result in terms of irreducible elements.

Then we look more closely at the geometry of the involved links. There is a known list of possible Sarkisov links between rational surfaces, due to V. Iskovskikh [Isk91]; see also [Cor95] and [Isk96]. Here we face the same problem as with blow-ups: a Sarkisov link is (in general) not a birational map between two isomorphic surfaces, so to deduce a set of generators for the Cremona group, we have to explain how to concatenate Sarkisov links. This was also done by V. Iskovskikh but resulted in very long lists that are not easy to use (see for instance [Isk96, Theorem 2.6], whose statement runs over eight pages). In Theorem 2.4 and Proposition 2.5, we propose a compact way to express these results; we also provide in Appendix A a mostly self-contained proof of the classification of Sarkisov links between rational surfaces.

We can distinguish two kinds of irreducible elements in $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$: those involving links between conic bundles, and so preserving a rational fibration, and other sporadic cases with no such fibration. In Section 3.1, we describe the possible fibrations, which turn out to be of one of the following types: pencil of lines through a rational point, or pencil of conics through a Galois orbit of size 4, or two Galois orbits of size 2. Then in Proposition 3.7, we get a more geometric set of generators. We can see this as an analog of another classical result, Castelnuovo's theorem, which asserts that over an algebraically closed field, the Cremona group is generated by linear automorphisms and Jonquières maps.

The last step towards the proof of Theorem 1.1 is to factorize each of these generators into a product of involutions. For some generators, we identify some matrix groups to which they belong, such as a projective linear group or an orthogonal group. Then we use classical results about the generation of these groups by involutions, such as the Cartan–Dieudonné theorem. For the remaining sporadic generators, in Section 4, we manage to write them as products of quadratic, Geiser, or Bertini involutions. To find these involutions we use elementary relations between Sarkisov links, which we can visualize as polygonal pieces encoded by Del Pezzo surfaces of Picard rank 3. In Appendix B, we give an exhaustive list of all such pieces, even if in the present paper we only use a few of them.

2. Sarkisov links

In this section, we recall the notion of Sarkisov link and state the classification of Sarkisov links for rational surfaces over a perfect field. We also introduce the notion of irreducible element in the Cremona group, and in Proposition 2.5, we get a first description of such irreducible elements.

2.1 Generation by irreducible elements

Let $\mathbf{k}$ be a perfect field and $\mathbf{k}^a$ an algebraic closure of $\mathbf{k}$. Let X be an algebraic variety defined over $\mathbf{k}$, and denote by $X(\mathbf{k})$ the set of $\mathbf{k}$-rational points on X. The absolute Galois group $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ acts on $X \times_{\mathrm{Spec}\,\mathbf{k}} \mathrm{Spec}\,\mathbf{k}^a$ through the second factor, and in particular it acts on $X(\mathbf{k}^a)$. We call *d-point* on X an orbit of size d in $X(\mathbf{k}^a)$ under the action of $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$. When $d = 1$ we keep the terminology *rational point* instead of 1-point. If $p = \{p_1, \dots, p_d\}$ is a d-point on a surface X, we say that each $p_i \in X(\mathbf{k}^a)$ is a *geometric component* of p (or simply a component). Given $g \in \mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$, we denote by p_i^g the image of the component p_i under the action of g.

Following [LZ20], we now recall the definition of a rank r fibration, which unifies the concepts of Del Pezzo surfaces and conic bundles, of Sarkisov links between such surfaces, and of elementary relations between such links.

By a surface we always mean a smooth projective surface defined over a perfect field $\mathbf{k}$. We say that a surface X is rational if it is birational to $\mathbb{P}^2$ over $\mathbf{k}$. Let X be a rational surface and $r \geqslant 1$ an integer. We say that X is a *rank r fibration* if there exists a surjective morphism $X \longrightarrow B$ with connected fibers, relatively ample anticanonical divisor, and relative Picard rank equal to r, where B is a point or a smooth curve. Since we assume X rational, there are only two possibilities for B: either $B = \mathrm{pt}$ is a point and X is a Del Pezzo surface of Picard rank r over $\mathbf{k}$, or $B = \mathbb{P}^1$ and $X \longrightarrow \mathbb{P}^1$ is a conic bundle with $r-1$ orbits of singular fibers. In particular, a rank 1 fibration is the same as a rational surface with a structure of Mori fiber space. Since in this paper we are only interested in rational surfaces, we put the rationality condition in the definition in order to avoid repeating “rational” rank r fibration everywhere.

A *marked* rank r fibration is a rank r fibration X/B together with a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^2$. Let $(X/B, \varphi)$ and $(X'/B', \varphi')$ be two marked fibrations, of respective ranks r and r', and consider the birational map $\varphi'^{-1} \circ \varphi\colon X \dashrightarrow X'$ induced by the markings. We say that X/B *factorizes through* X'/B', or that X'/B' is *dominated* by X/B, if this induced map is a morphism and if there exists a morphism $B' \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\varphi'^{-1}\circ\varphi"'] & & B \\ & X' \arrow[r] & B' \arrow[u] \end{tikzcd}$$

This implies $r \geqslant r'$. If $B' \longrightarrow B$ and $\varphi'^{-1} \circ \varphi\colon X \longrightarrow X'$ are both isomorphism, we say that the two fibrations are *equivalent*. The Cremona group acts on equivalence classes of marked fibration, via post-composition:

$$f \cdot (X/B, \varphi) = (X/B, f \circ \varphi)\,.$$

From here on, all rank r fibrations are supposed to be marked, but we usually keep the marking implicit.

We say that a d-point on a Del Pezzo surface (respectively, on a conic bundle) is *general* if the blow-up of the orbit is still a Del Pezzo surface (respectively, a conic bundle over the same base curve).

Lemma 2.1. *Assume that X/B is a rank $r+1$ fibration that factorizes through a rank r fibration X'/B'. Then one of the following holds:*

(1) *either $B \simeq B'$, and there exists a general d-point on X' such that $X \longrightarrow X'$ is the blow-up of p;*

(2) *or $B = \mathbb{P}^1$, $B' = \mathrm{pt}$, and $X \longrightarrow X'$ is an isomorphism.*

Proof. By the additivity of the relative Picard rank, one of the morphisms $X \longrightarrow X'$ or $B' \longrightarrow B$ is an isomorphism, and the other one has relative Picard rank 1. This gives the two cases of the statement. $\square$

The *piece* of a rank r fibration X/B is the $(r-1)$-dimensional combinatorial polytope constructed as follows: Each rank d fibration dominated by X/B is a $(d-1)$-dimensional face, and for each pair of faces X_i/B_i, $i = 1, 2$, the face X_2/B_2 lies in X_1/B_1 if and only if X_1/B_1 dominates X_2/B_2. We write $(r-1)$-piece when we want to emphasize the dimension of the piece (associated with a rank r fibration). We now consider more closely the case of 1-pieces, which turn out to encode Sarkisov links.

Let Y/B_Y be a rank 2 fibration. As a consequence of the two-rays game, there are exactly two rank 1 fibrations X/B and X'/B' dominated by Y/B_Y. We say that the induced birational map $X \dashrightarrow X'$ is a *Sarkisov link*. Using Lemma 2.1, we can distinguish between four types of Sarkisov links, depending on whether the domination of X/B (respectively, X'/B') by Y/B_Y is a blow-up or a change of base. In Table 2.1, we describe these four types. The first column is the usual numbering in the literature, where for type II we moreover distinguish between the cases where the common base of the fibrations is $\mathbb{P}^1$ or a point. Observe here that, by definition, a link of type II over $\mathbb{P}^1$ sends a general fiber of $X/\mathbb{P}^1$ to a fiber of $X'/\mathbb{P}^1$. The second column shows the classical diagram, which is often taken as a definition for the Sarkisov links. Here an arrow $\xrightarrow{d}$ refers to the blow-up of a general d-point. The third column shows the 1-piece with the rank 2 fibration in the center of the edge, dominating the two rank 1 fibrations X/B and X'/B' on the left and right vertices. Finally, the last column shows the shorthand that we shall use in the text.

In [LZ20, Proposition 2.6], it was shown that a 2-piece (or rather its geometric realization) is homeomorphic to a disk, and we will draw it as a regular polygon. The boundary of the polygon corresponds to a sequence of Sarkisov links whose product is an automorphism: we say that the piece encodes an *elementary relation* between Sarkisov links. See Appendix B for a list of all 2-dimensional pieces given by a rank 3 fibration X/pt, where X is a Del Pezzo surface. It is in fact true that for any d, a d-piece is homeomorphic to a convex polytope of dimension d, with vertices correspond to Mori fiber spaces, edges to Sarkisov links, and 2-dimensional faces to elementary relations. We do not give details since in this paper we shall only use d-pieces with $d = 1$ or 2.

Table 2.1. Sarkisov links

Type	Diagram	Piece	Short notation
I	$\begin{tikzcd} & X' \arrow[ld, "d"'] \arrow[d] \\ X \arrow[d] & \mathbb{P}^1 \arrow[ld] \\ \mathrm{pt} & \end{tikzcd}$	$X/\mathrm{pt} \overset{d}{\textbf{—}} X'/\mathrm{pt} \textbf{ — } X'/\mathbb{P}^1$	$X \overset{d}{\dashrightarrow} X'$
II/pt	$\begin{tikzcd} & Y \arrow[ld, "d"'] \arrow[rd, "d'"] & \\ X \arrow[rd] & & X' \arrow[ld] \\ & \mathrm{pt} & \end{tikzcd}$	$X/\mathrm{pt} \overset{d}{\textbf{—}} Y/\mathrm{pt} \overset{d'}{\textbf{—}} X'/\mathrm{pt}$	$X \overset{d\ d'}{\dashrightarrow} X'$
II/$\mathbb{P}^1$	$\begin{tikzcd} & Y \arrow[ld, "d"'] \arrow[rd, "d"] & \\ X \arrow[rd] & & X' \arrow[ld] \\ & \mathbb{P}^1 & \end{tikzcd}$	$X/\mathbb{P}^1 \overset{d}{\textbf{—}} Y/\mathbb{P}^1 \overset{d}{\textbf{—}} X'/\mathbb{P}^1$	$X \overset{d\ d}{\dashrightarrow} X'$
III	$\begin{tikzcd} X \arrow[d] \arrow[rd, "d"] & \\ \mathbb{P}^1 \arrow[rd] & X' \arrow[d] \\ & \mathrm{pt} \end{tikzcd}$	$X/\mathbb{P}^1 \textbf{ — } X/\mathrm{pt} \overset{d}{\textbf{—}} X'/\mathrm{pt}$	$X \xrightarrow{d} X'$
IV	$\begin{tikzcd} & X \arrow[ld] \arrow[rd] & \\ \mathbb{P}^1 \arrow[rd] & & \mathbb{P}^1 \arrow[ld] \\ & \mathrm{pt} & \end{tikzcd}$	$X/\mathbb{P}^1 \textbf{ — } X/\mathrm{pt} \textbf{ — } X/\mathbb{P}^1$	$X \xrightarrow{\mathrm{IV}} X$

We define $\mathrm{BirMori}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ as the groupoid of birational maps between rank 1 fibrations (or equivalently, between rational Mori fiber spaces, hence the name). The Sarkisov program can be phrased as follows (see [LZ20, Proposition 3.14]).

Proposition 2.2. *The groupoid* $\mathrm{BirMori}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ *is generated by Sarkisov links and automorphisms.*

Given $g \in \mathrm{BirMori}_{\mathbf{k}}\left(\mathbb{P}^2\right)$, we call *Sarkisov length* of g, denoted by $\mathrm{sl}(g)$, the minimal number of Sarkisov links necessary to factorize g. Now assume $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$. We say that f is *reducible* if we can write $f = f_2 \circ f_1$ with $f_i \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ and $\mathrm{sl}(f_i) < \mathrm{sl}(f)$ for $i = 1, 2$. Otherwise, we say that f is *irreducible*. In particular, $\mathrm{sl}(f) = 0$ if and only if $f \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right)$, and such elements are trivially irreducible. As an immediate consequence of Proposition 2.2 and of the definition of irreducible elements, we have the following.

Proposition 2.3. *The Cremona group* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ *is generated by its irreducible elements.*

2.2 The graph of Sarkisov links

Let $\mathcal{D}$ be the set of isomorphy classes of $\mathbf{k}$-rational Del Pezzo surfaces of Picard rank 1 and $\mathcal{C}$ the set of isomorphy classes of $\mathbf{k}$-rational conic bundles of relative Picard rank 1. We now define several subsets $\mathcal{D}_i \subset \mathcal{D}$ and $\mathcal{C}_j \subset \mathcal{C}$. The index always refers to the degree of the corresponding surfaces, by which we mean the self-intersection of the canonical divisor. Starting from $\mathbb{P}^2 \in \mathcal{D}$, each definition is in terms of explicit Sarkisov links, from an already defined class of surfaces.

- Let $\mathcal{D}_8 \subset \mathcal{D}$ be the set of surfaces obtained by blowing up a 2-point on $\mathbb{P}^2$ and then blowing down the transform of the line through this point. Observe that any $X \in \mathcal{D}_8$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbf{k}^a$, and so we can speak of the bidegree of a divisor on X. We call *diagonal* any curve on X of bidegree $(1,1)$. Similarly, we call *vertical ruling* (respectively, *horizontal ruling*) any curve of bidegree $(1,0)$ (respectively, $(0,1)$), necessarily defined over $\mathbf{k}^a$ but not over $\mathbf{k}$.
- Let $\mathcal{D}_5 \subset \mathcal{D}$ be the set of surfaces obtained by blowing up a general 5-point on $\mathbb{P}^2$ and then blowing down the transform of the smooth conic through this point.
- Let $\mathcal{D}_6 \subset \mathcal{D}$ be the set of surfaces obtained by blowing up a general 3-point on a surface $X \in \mathcal{D}_8$ and then blowing down the transform of the smooth diagonal through this point.
- Let $\mathcal{C}_5 \subset \mathcal{C}$ be the set of conic bundles obtained by blowing up a general 4-point on $\mathbb{P}^2$ and taking the transform of conics through this point.
- Let $\mathcal{C}_6 \subset \mathcal{C}$ be the set of conic bundles obtained by blowing up a general 2-point on $X \in \mathcal{D}_8$, and taking the transform of diagonals through this point. Observe that when we think of X as coming from the blow-up of a 2-point on $\mathbb{P}^2$ followed by the contraction of a line, the conic bundle on a surface in the set $\mathcal{C}_6$ corresponds to the transform of conics in $\mathbb{P}^2$ passing through two 2-points.
- Let $\mathcal{C}_8 = \{\mathbb{F}_n \mid n \geqslant 0\} \subset \mathcal{C}$ be the set of Hirzebruch surfaces, with their structure of conic bundle (precisely, of $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, with a section of self-intersection $-n$).

We say that a surface $X \in \mathcal{D}_8$ is of *type* $\mathcal{D}_8$, and similarly with the other subsets. In Appendix A, we give a proof of the following theorem, which can also be extracted from [Isk96]. The content of the theorem is summed up in Figure 2.1, adapted from [Sch22]. The label on each edge indicates the type of Sarkisov link (according to the numbering in Table 2.1) and the size of the blown-up Galois orbits.

Theorem 2.4. *Let $\mathbf{k}$ be a perfect field.*

(1) *Given any (marked) rank 1 fibration X/B, the surface X lies in one of the following seven pairwise disjoint sets, which form the vertices of the graph in Figure 2.1:*

$$\{\mathbb{P}^2\},\ \mathcal{D}_5,\ \mathcal{D}_6,\ \mathcal{D}_8,\ \mathcal{C}_5,\ \mathcal{C}_6,\ \mathcal{C}_8\,.$$

In particular, $\mathcal{D} = \mathcal{D}_5 \cup \mathcal{D}_6 \cup \mathcal{D}_8 \cup \{\mathbb{P}^2\}$ and $\mathcal{C} = \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_8$.

(2) *Let $X \dashrightarrow X'$ be a Sarkisov link of type I or II between rank 1 fibrations. Then the type of X' is uniquely defined by the type of X and the size of the blown-up Galois orbit, as indicated by the edges on the graph in Figure 2.1.*

(3) *Let $X \dashrightarrow X'$ be a Sarkisov link of type IV between rank 1 fibrations. Then $X = Y = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and the link is the change of ruling.*

From Theorem 2.4, any factorization into Sarkisov links of a birational map between rank 1 fibrations corresponds to a path in the graph of Figure 2.1. In particular, any factorization of an

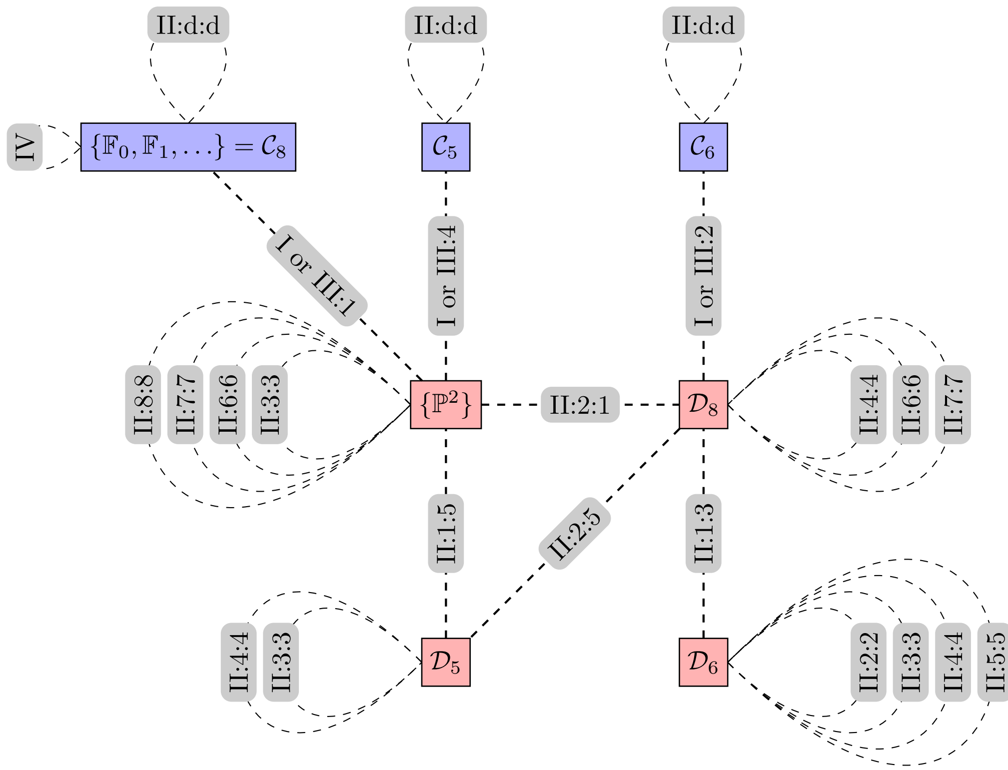

Figure 2.1. Sarkisov links between rational surfaces over a perfect field

element of $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ corresponds to a closed path based at the vertex $\left\{\mathbb{P}^2\right\}$. A natural question is to ask about the converse: is any path in the graph realized by at least one composition of Sarkisov links? We shall use the following partial positive answer to this question, which follows from the definition of the sets $\mathcal{D}_i$ and $\mathcal{C}_i$:

- Given $X \in \mathcal{D}_8 \cup \mathcal{D}_5 \cup \mathcal{C}_5$, there exists a Sarkisov link from X to $\mathbb{P}^2$.
- Given $X \in \mathcal{D}_6 \cup \mathcal{C}_6$, there exists a composition of two Sarkisov links from X to $\mathbb{P}^2$, via an intermediate surface $X' \in \mathcal{D}_8$;
- Given $X \in \mathcal{C}_8$, there exists a Sarkisov link from X to $\mathbb{P}^2$ if and only if $X = \mathbb{F}_1$.

Let $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ be an irreducible element. We say that f is of *Del Pezzo type* if it admits a minimal factorization using only links of type II over pt, and f is of *fibering type* if it admits a minimal factorization containing a link of type I (and so also a link of type III). In other words, in terms of the closed path associated with a minimal factorization, f is of Del Pezzo type if the path visits at most the vertices $\left\{\mathbb{P}^2\right\}$, $\mathcal{D}_8$, $\mathcal{D}_6$, $\mathcal{D}_5$, and is of fibering type if the path visits at least one of the vertices $\mathcal{C}_8$, $\mathcal{C}_6$, or $\mathcal{C}_5$. Observe that these two classes of irreducible elements have no reason to be disjoint since a minimal factorization into Sarkisov links is not unique. (From Figure B.11 the interested reader can cook up an example of a map $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ with $\mathrm{sl}(f) = 3$ and admitting two minimal factorizations of distinct types.)

Proposition 2.5. *Let $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ be an irreducible element.*

(1) *A minimal factorization of f never contains a link of type IV.*

Table 2.2. Irreducible generators of Del Pezzo type in Proposition 2.5(4) (the last column corresponds to the study in Section 4)

$\mathrm{sl}(f)$	Factorization type	Note
0	–	Automorphism
1	$\mathbb{P}^2 \overset{3\,3}{\dashrightarrow} \mathbb{P}^2$	Quadratic simplification
	$\mathbb{P}^2 \overset{6\,6}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.12
	$\mathbb{P}^2 \overset{7\,7}{\dashrightarrow} \mathbb{P}^2$	Geiser simplification
	$\mathbb{P}^2 \overset{8\,8}{\dashrightarrow} \mathbb{P}^2$	Bertini simplification
2	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	Quadratic simplification
	$\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.6
3	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{4\,4}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.8
	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{6\,6}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	Geiser simplification
	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{7\,7}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	Bertini simplification
	$\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{3\,3}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$	Geiser simplification
	$\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{4\,4}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$	Bertini simplification
	$\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{2\,5}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.11
	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{5\,2}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$	inverse of the previous one
4	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.7
5	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{2\,2}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.10
	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{3\,3}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	See Lemma 4.9
	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{4\,4}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	Geiser simplification
	$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{5\,5}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$	Bertini simplification

(2) *A minimal factorization of f never contains a link of type III immediately following a link of type I.*

(3) *If f is of fibering type, then f admits a minimal factorization of one of the following forms:*

   (i) $\mathbb{P}^2 \overset{1}{\dashrightarrow} \mathbb{F}_1 \in \mathcal{C}_8$, *followed by a certain number $r \geqslant 1$ of links of type II between Hirzebruch surfaces, and a final link $\mathbb{F}_1 \overset{1}{\longrightarrow} \mathbb{P}^2$, so in particular $\mathrm{sl}(f) = 2 + r \geqslant 3$;*

   (ii) $\mathbb{P}^2 \overset{4}{\dashrightarrow} \mathcal{C}_5 \overset{d\,d}{\dashrightarrow} \mathcal{C}_5 \overset{4}{\longrightarrow} \mathbb{P}^2$, *so $\mathrm{sl}(f) = 3$;*

   (iii) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{2}{\dashrightarrow} \mathcal{C}_6 \overset{d\,d}{\dashrightarrow} \mathcal{C}_6 \overset{2}{\longrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$, *so $\mathrm{sl}(f) = 5$.*

(4) *If f is of Del Pezzo type, then $\mathrm{sl}(f) \leqslant 5$, and f admits a minimal factorization of one the forms listed in Table 2.2.*

Proof. (1) By Theorem 2.4(3), we know that a link of type IV is the change of ruling on $\mathbb{F}_0$. Assume that f admits a minimal factorization $f = \alpha^{-1} \circ \chi \circ \beta$, where $\chi\colon \mathbb{F}_0 \longrightarrow \mathbb{F}_0$ is a link of type IV and $\alpha, \beta\colon \mathbb{P}^2 \longrightarrow \mathbb{F}_0$ are compositions of Sarkisov links. Observe that $\mathbb{P}^2$ and $\mathbb{F}_0$ cannot be connected by a single link, so $\mathrm{sl}(\alpha) \geqslant 2$ and $\mathrm{sl}(\beta) \geqslant 2$. Let $\gamma^{-1}\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$ be the composition of two Sarkisov links, corresponding to the blow-up of a rational point $x \in \mathbb{F}_0$ followed by the contraction of the two rules through x. Then we can write

$$f = \left(\alpha^{-1} \circ \chi \circ \gamma\right) \circ \left(\gamma^{-1} \circ \beta\right).$$

We have $\operatorname{sl}\left(\alpha^{-1} \circ \chi \circ \gamma\right) \leqslant \operatorname{sl}(f)$ and $\operatorname{sl}\left(\gamma^{-1} \circ \beta\right) < \operatorname{sl}(f)$, so by the irreducibility of f, the first inequality must be an equality, and this gives $\operatorname{sl}(\beta) = \operatorname{sl}(\gamma) = 2$. Similarly, $\operatorname{sl}(\alpha) = 2$, and both α and β are composition of two links $\mathbb{P}^2 \dashrightarrow \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$. But then $\alpha^{-1}\chi\alpha \in \operatorname{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right)$, and writing

$$f = \left(\alpha^{-1} \circ \chi \circ \alpha\right) \circ \left(\alpha^{-1} \circ \beta\right),$$

we contradict our assumption that f is irreducible.

(2) If $X \in \mathcal{C}_i$ with $i \in \{5, 6, 8\}$, the Picard rank of X is 2, and the two extremal rays correspond respectively to the fibration to $\mathbb{P}^1$ and to a link of type III. In particular, if $X_1 \overset{d}{\dashrightarrow} X \xrightarrow{d} X_2$ is a composition of a link of type I followed by a link of type III, then $X_1 \longrightarrow X_2$ is an isomorphism, and this contradicts the fact that such a composition of two such links can be part of a minimal factorization.

(3) and (4). Let $m = \operatorname{sl}(f)$, and consider the successive vertices

$$v_0 = \left\{\mathbb{P}^2\right\},\ v_1,\ \ldots,\ v_{m-1},\ v_m = \left\{\mathbb{P}^2\right\}$$

visited by the closed path associated with a minimal factorization of f. So each pair of vertices v_i, v_{i+1} are joined by an edge (which can be a loop) in the graph of Figure 2.1. Since f is irreducible, any intermediate vertex v_i, $1 \leqslant i \leqslant m-1$, is distinct from $\left\{\mathbb{P}^2\right\}$. If one of the vertices is in $\mathcal{C}_8$, then since we know by assertion (1) that we cannot use any link of type IV, by inspection of the graph, the factorization is as described in item (i). Since the factorization of f is minimal, if the path does not visit the vertex $\mathcal{C}_8$, then by the remark after Theorem 2.4, the associated path is unimodal in the following sense:

- For each index i such that $1 \leqslant i \leqslant \frac{1}{2}m$, we have $d(v_i, v_0) = d(v_{i-1}, v_0) + 1$, where $d(\cdot,\cdot)$ is the distance in the graph.
- For each index i such that $\frac{1}{2}m + 1 \leqslant i \leqslant m$, we have$d(v_i, v_0) = d(v_{i-1}, v_0) - 1$.
- If $m = 2n+1$ is odd, then $d(v_n, v_0) = d(v_{n+1}, v_0)$.

Then the possible factorization types in each case (fibering or Del Pezzo type) are obtained by inspection of Figure 2.1, together with assertion (2). □

Remark 2.6. We do not claim that any factorization of the forms given in Table 2.2 is automatically minimal, or even that there exists an irreducible element corresponding to a given line of the table (in particular, the existence can depend on the field $\mathbf{k}$ we are working with). See Figure B.10 for an example of a map with a factorization of type $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$ which is not minimal because it also admits a factorization of type $\mathbb{P}^2 \overset{3\,3}{\dashrightarrow} \mathbb{P}^2$.

3. Subgroups of matrix type

Over an algebraically closed field, it is a classical result that any element in $\operatorname{Bir}\left(\mathbb{P}^2\right)$ preserving a pencil of rational curves is conjugate to a Jonquières map, that is, a map preserving a pencil of lines. Over a perfect field, we have to introduce two more types of normal forms, which we call Jonquières type $2+2$ and Jonquières type 4, and we call Jonquières maps of type 1 the classical ones. In Proposition 3.7, we obtain a set of generators: first, automorphisms and Jonquières maps of type 1, which are easily seen to be compositions of involutions; second, Jonquières maps of type $2+2$ or type 4, which are studied in the rest of this section; and finally, irreducible elements of Del Pezzo type, which will be studied in Section 4. The proof that type 4 and type $2+2$ Jonquières maps are compositions of involutions is the most technical part of this paper. The argument naturally splits into two parts: Jonquières maps that preserve the fibration fiberwise correspond to an orthogonal group, and by explicit computations we obtain Jonquières involutions whose compositions can realize all possible permutations between fibers.

3.1 Three types of fibrations

We call any dominant rational map $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ a *rational fibration* on $\mathbb{P}^2$. We say that $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ *preserves the fibration* if there exists an $\alpha \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^1\right)$ such that $\pi \circ f = \alpha \circ \pi$, and we say that it *fixes the fibration* if $\alpha = \mathrm{id}_{\mathbb{P}^1}$:

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"'] & \mathbb{P}^2 \arrow[d, dashed, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\alpha"] & \mathbb{P}^1 . \end{tikzcd}$$

We write $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi\right)$ for the group that preserves the fibration, $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right)$ for the one that fixes the fibration, and $\mathrm{Aut}^{\pi}_{\mathbf{k}}\left(\mathbb{P}^1\right)$ for the image in $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^1\right)$ of the application $f \longmapsto \alpha$. So by definition we have an exact sequence

$$1 \longrightarrow \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right) \longrightarrow \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi\right) \longrightarrow \mathrm{Aut}^{\pi}_{\mathbf{k}}\left(\mathbb{P}^1\right) \longrightarrow 1 . \tag{Seq}$$

We say that $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ is of

- *Jonquières type* 1 if f preserves the fibration given by the pencil of lines through a rational point,
- *Jonquières type* $2+2$ with residue fields L and L' if f preserves the fibration given by the pencil of conics through two general 2-points with respective residue fields L and L',
- *Jonquières type* 4 with residue field F if f preserves the fibration given by the pencil of conics through a general 4-point with residue field F.

Note that when we speak about "the" residue field we always mean up to $\mathbf{k}$-isomorphism. For Jonquières type 1, the sequence (Seq) splits and gives a description as a semidirect product $\mathrm{PGL}_2(\mathbf{k}) \ltimes \mathrm{PGL}_2(\mathbf{k}(x))$. For Jonquières type $2+2$ and 4, it is not clear whether the sequence always splits.

Remark 3.1. A field extension $F/\mathbf{k}$ is the residue field of a general 4-point if and only if it is the residue field of an irreducible polynomial of degree 4:

Choosing a conic C defined over $\mathbf{k}$ going through the general 4-point p, and applying a $\mathbf{k}$-coordinate change, we can assume that C is given by $y^2 - xz = 0$. Hence, the components of p are of the form $[a_i^2 : a_i : 1] \in \mathbb{P}^2(\mathbf{k}^a)$, and its residue field is $\mathbf{k}(a_1) \simeq \mathbf{k}(a_2) \simeq \mathbf{k}(a_3) \simeq \mathbf{k}(a_4)$, which equals the residue field of the irreducible polynomial $f = (x - a_1) \cdots (x - a_4) \in \mathbf{k}$ of degree 4.

For the converse direction, Example 3.5 shows how one can construct a general 4-point given an irreducible polynomial of degree 4.

Lemma 3.2. *Let $f \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ be an irreducible element of fibering type. Then there exists an $\alpha \in \mathrm{PGL}_3(\mathbf{k})$ such that $\alpha \circ f$ is of Jonquières type* 1, $2+2$, *or* 4.

Proof. We consider each of the three cases given by Proposition 2.5(3).

An irreducible element going through $\mathcal{C}_8$ is a map that sends the pencil of lines through one point onto the pencil of lines through possibly some other point. Hence, there exists an

$\alpha \in \mathrm{PGL}_3(\mathbf{k})$ such that $\alpha \circ f$ preserves the pencil of lines through a point, and so f is of Jonquières type 1.

If f has factorization type $\mathbb{P}^2 \overset{4}{\dashrightarrow} X \overset{d\ d}{\dashrightarrow} X' \overset{4}{\longrightarrow} \mathbb{P}^2$ with $X, X' \in \mathcal{C}_5$, then f maps the pencil of conics through the 4-point associated with X onto the one through the 4-point associated with X'. By Lemma A.10, the surfaces X and X' are isomorphic, and the associated 4-points are equivalent under the action of $\mathrm{PGL}_3(\mathbf{k})$. This gives the existence of α.

Finally, if f has factorization type $\mathbb{P}^2 \overset{2\ 1}{\dashrightarrow} Y \overset{2}{\dashrightarrow} X \overset{d\ d}{\dashrightarrow} X' \overset{2}{\longrightarrow} Y' \overset{1\ 2}{\dashrightarrow} \mathbb{P}^2$ with $X, X' \in \mathcal{C}_6$ and $Y, Y' \in \mathcal{D}_8$, then f send the pencil of conics through the two 2-points associated with X and Y onto the one associated with X' and Y'. By Lemma A.11, we know that the two surfaces $X, X' \in \mathcal{C}_6$ of the middle link are dominated by isomorphic surfaces Z, Z', each of which is the blow-up of $\mathbb{P}^2$ at two 2-points, and the two sets of four geometric points are equivalent under the action of $\mathrm{PGL}_3(\mathbf{k})$. □

Let f, g be two elements of Jonquières type 1, $2+2$, or 4, preserving fibrations π_f, π_g. We say that f and g are *equivalent* if there exists an $\alpha \in \mathrm{PGL}_3(\mathbf{k})$ such that $\alpha^{-1} \circ g \circ \alpha$ preserves π_f.

In order to classify Jonquières maps up to equivalence, we first need a reinforcement of [Sch22, Lemma 6.11], which states that any Galois equivariant bijection between two Galois-invariant sets of four points can be realized by an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$.

Lemma 3.3. *Let $\mathbf{k}$ be a perfect field, and let $p, q \in \mathbb{P}^2$ be two general 4-points. Then the following are equivalent:*

(1) *There exists an $\alpha \in \mathrm{PGL}_3(\mathbf{k})$ such that $\alpha(p_i) = q_i$ for $i = 1, \dots, 4$.*
(2) *For every $g \in \mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$, there exists a $\sigma \in S_4$ such that $g(p_i) = p_{\sigma(i)}$ and $g(q_i) = q_{\sigma(i)}$ for $i = 1, \dots, 4$.*
(3) *The residue fields of p and q are $\mathbf{k}$-isomorphic.*

Proof. The fact that assertion (2) implies assertion (1) is [Sch22, Lemma 6.11]. As elements of $\mathrm{PGL}_3(\mathbf{k})$ do not change the residue field, assertion (1) implies assertion (3). It remains to see that assertion (3) implies assertion (2). As in Remark 3.1, there exist $\alpha, \beta \in \mathrm{PGL}_3(k)$ such that $\alpha(p_i) = [a_i : a_i^2 : 1]$ and $\beta(q_i) = [b_i : b_i^2 : 1]$, where $a_1, \dots, a_4$ and $b_1, \dots, b_4$ are the roots of two irreducible polynomials $f, f' \in \mathbf{k}[x]$ of degree 4. In particular, the residue field of p_i is $F_i = \mathbf{k}(a_i)$, and that of q_i is $F'_i = \mathbf{k}(b_i)$. The residue field F of f is $\mathbf{k}$-isomorphic to F_i, and the residue field F' is $\mathbf{k}$-isomorphic to F'_i.

Let $L/\mathbf{k}$ be a finite Galois extension containing $a_1, \dots, a_4, b_1, \dots, b_4$. Having that F_i and F'_i are $\mathbf{k}$-isomorphic means that there exists an $h \in \mathrm{Gal}(L/\mathbf{k})$ (with corresponding $\tau \in S_4$ such that $h(b_i) = b_{\tau(i)}$) such that $F'_{\tau(i)} = \mathbf{k}(b_{\tau(i)}) = \mathbf{k}(a_i) = F_i$. Therefore, after exchanging b_i with $b_{\tau(i)}$, we find $F_i = F'_i$. By the fundamental theorem of Galois theory, this is equivalent to $\mathrm{Gal}(L/F_i) = \mathrm{Gal}(L/F'_i)$. In other words, for every $g \in \mathrm{Gal}(L/\mathbf{k})$, there exists a $\sigma \in S_4$ such that $g(a_i) = a_{\sigma(i)}$ and $g(b_i) = b_{\sigma(i)}$ for $i = 1, \dots, 4$.

Therefore, we also have $\alpha(g(p_i)) = g(\alpha(p_i)) = g([a_i : a_i^2 : 1]) = [a_{\sigma(i)} : a_{\sigma(i)}^2 : 1] = \alpha(p_{\sigma(i)})$, and so by applying α^{-1}, we find $g(p_i) = p_{\sigma(i)}$. Similarly, we find $\beta(g(q_i)) = \beta(q_{\sigma(i)})$ and so $g(q_i) = q_{\sigma(i)}$. □

Lemma 3.4. *The classification up to equivalence of elements of Jonquières type is as follows:*

(1) *All elements of Jonquières type 1 are equivalent.*

(2) *Two elements of Jonquières type* $2+2$ *are equivalent if and only if the pairs of associated residue fields are* $\mathbf{k}$*-isomorphic (and in fact equal).*

(3) *Two elements of Jonquières type* 4 *are equivalent if and only if the associated residue fields are* $\mathbf{k}$*-isomorphic.*

Proof. Assertion (1) is clear, Assertion (2) follows directly from [Sch22, Lemma 6.11], and assertion (3) follows from Lemma 3.3. □

With Lemma 3.4, for each fibering type and each (pair of) residue field(s), we choose one fibration $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. We will denote it by $\pi_\times$ (Jonquières type 1), $\pi_{L,L'}$ (Jonquières type $2+2$ with residue fields L, L'), π_F (Jonquières type 4 with residue field F). Such a choice can be made explicit, as follows.

Example 3.5.

Jonquières type 1. Set

$$\pi_\times\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1\,, \quad [x:y:z] \longmapsto [x:z]\,,$$

which is the fibration corresponding to the pencil of lines through $[0:1:0]$.

Jonquières type $2+2$. Let $L=\mathbf{k}(a_1)$ and $L'=\mathbf{k}(a_1')$ be two quadratic extensions over $\mathbf{k}$ with respective minimal polynomials $(t-a_1)(t-a_2)$, $(t-a_1')(t-a_2')\in\mathbf{k}[t]$. Let $p=\{[a_i:1:0]\}_{i=1,2}$ and $p'=\{[a_i':0:1]\}_{i=1,2}$, and set

$$\pi_{L,L'}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1\,, \quad [x:y:z] \longmapsto \left[(x-a_1y)(x-a_2y)+(x-a_1'z)(x-a_2'z)-x^2:yz\right],$$

which is a fibration corresponding to the pencil of conics through p, p'.

Jonquières type 4. Let $L=\mathbf{k}(a_1,a_2,a_3,a_4)$ be a splitting field of an irreducible polynomial of degree 4 over $\mathbf{k}$ with minimal polynomial $(t-a_1)(t-a_2)(t-a_3)(t-a_4)=t^4+at^3+bt^2+ct+d\in\mathbf{k}[t]$. Then $p=\left\{\left[a_i^2:a_i:1\right]\right\}_{i=1,\dots,4}$ is a general 4-point, and we set

$$\pi_F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1\,, \quad [x:y:z] \longmapsto \left[x^2+axy+by^2+cyz+dz^2:y^2-xz\right],$$

which is a fibration corresponding to the pencil of conics through p. The associated residue field is $F=\mathbf{k}(a_1)\simeq\mathbf{k}(a_2)\simeq\mathbf{k}(a_3)\simeq\mathbf{k}(a_4)$, all of which are $\mathbf{k}$-isomorphic.

Remark 3.6. Note that two d-points with $d\leqslant 3$ have $\mathbf{k}$-isomorphic residue fields if and only if they have isomorphic splitting fields. For $d=4$ this is not true: Let $\mathbf{k}=\mathbf{Q}$ and consider the 4-points given by $p_1=\left[\sqrt[4]{2}:\sqrt{4}^2:1\right]\in\mathbf{Q}\left[\sqrt[4]{2}\right]$, respectively $q_1=\left[\sqrt[4]{2}(1-i):\left(\sqrt[4]{2}(1-i)\right)^2:1\right]\in\mathbf{Q}\left[\sqrt[4]{2}(1-i)\right]$. They both have the splitting field $\mathbf{Q}\left[\sqrt[4]{2},i\right]$, but their residue fields are not $\mathbf{k}$-isomorphic. However, one can show that this can happen only if the Galois group over the splitting field is isomorphic to the dihedral group D_8.

3.2 A geometric generating set

We can now describe a generating set for the Cremona group, valid over any perfect field.

Proposition 3.7. *Let* $\mathbf{k}$ *be a perfect field. The Cremona group* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ *is generated by the following elements:*

- $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right)\simeq\mathrm{PGL}_3(\mathbf{k})$,
- *irreducible elements* f *of Del Pezzo type with* $1\leqslant\mathrm{sl}(f)\leqslant 5$,
- *the Jonquières group* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2,\pi_\times\right)$ *of type* 1,

- *the Jonquières groups* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_{L,L'}\right)$ *of type* $2+2$, *for each pair of quadratic extensions* $L/\mathbf{k}$, $L'/\mathbf{k}$,
- *the Jonquières groups* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_F\right)$ *of type* 4, *for each residue field* $F/\mathbf{k}$ *of an irreducible polynomial of degree* 4.

Proof. By Proposition 2.3, the Cremona group $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ is generated by the irreducible elements, which are of Del Pezzo type and/or of fibering type. An irreducible element of Del Pezzo type has Sarkisov length at most 5 by Proposition 2.5(4), and the case $\mathrm{sl}(f)=0$ corresponds to the case $f \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right)$. If an irreducible element is of fibering type, then by Lemma 3.2, up to an automorphism, it is equal to an element of Jonquières type 1, 2+2, or 4. Then Lemma 3.4 implies the statement, where we use Remark 3.1. □

Our aim in the rest of the paper is to study each type of generator in Proposition 3.7 and show that it can be written as a product of involutions.

We shall use the basic remark that in any group, the subgroup generated by involutions is a normal subgroup. This implies that if G is a simple group containing an involution, then G is generated by involutions. For instance, for any field $\mathbf{k}$ and any $n \geqslant 2$ (except for $n=2$ and $|\mathbf{k}|=2$ or 3), the group $\mathrm{PSL}_n(\mathbf{k})$ is simple and contains the involution $(x_1, x_2, \ldots, x_n) \longmapsto (-x_1, -x_2, x_3, \ldots, x_n)$ (in characteristic different from 2) or $(x_1, x_2, \ldots, x_n) \longmapsto (x_2, x_1, x_3, \ldots, x_n)$ (in characteristic 2), so it is generated by involutions. Observe that $\mathrm{PSL}_2(\mathbf{F}_2) \simeq S_3$ also is generated by involutions even if it is not simple, but $\mathrm{PSL}_2(\mathbf{F}_3) \simeq A_4$ is not generated by involutions. On the other hand, it is known that $\mathrm{PGL}_n(\mathbf{k})$ is not always generated by involutions: for instance, a necessary and sufficient condition for $\mathrm{PGL}_3(\mathbf{k})$ to be generated by involutions is that all elements in $\mathbf{k}$ are cubes. However, using the ambient birational group, we have the following.

Lemma 3.8. *Let* $\mathbf{k}$ *be any field. The group* $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right) = \mathrm{PGL}_3(\mathbf{k})$ *and the Jonquières group* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_\times\right) = \mathrm{PGL}_2(\mathbf{k}) \ltimes \mathrm{PGL}_2(\mathbf{k}(x))$ *are contained in the subgroup of* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ *generated by involutions.*

Proof. The group $\mathrm{PGL}_3(\mathbf{k})$ is generated by the involutions in $\mathrm{PSL}_3(\mathbf{k})$ and by the subgroup of dilatations of the form $[x:y:z] \longmapsto [ax:y:z]$, with $a \in \mathbf{k}^*$. But in affine coordinates, we have

$$(ax, y) = \left(\frac{1}{x}, y\right) \circ \left(\frac{1}{ax}, y\right),$$

so we see that any dilatation is a composition of two birational involutions.

Similarly, over any field $\mathbf{k}$ with $|\mathbf{k}| \neq 3$, the Jonquières group is generated by the involutions in $\mathrm{PSL}_2(\mathbf{k}) \ltimes \mathrm{PSL}_2(\mathbf{k}(x))$ and by dilatations of the form $(x,y) \longmapsto (ax, y)$ and $(x,y) \dashrightarrow (x, a(x)y)$, which we can write as a product of two birational involutions as above.

Finally, we can handle the case $\mathbf{k} = \mathbf{F}_3$ by noticing that $\mathrm{PSL}_2(\mathbf{F}_3) \subset \mathrm{PSL}_2(\mathbf{F}_3(y)) \subset \mathrm{Bir}_{\mathbf{F}_3}\left(\mathbb{P}^2\right)$ and using that $\mathrm{PSL}_2(\mathbf{F}_3(y))$ is generated by involutions. □

3.3 Study of $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right)$ via quadratic forms

In this subsection, we show that in the context of Jonquières groups of type 4 or 2+2, $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right)$ is isomorphic to a special orthogonal group $\mathrm{SO}\left(\mathbf{k}(t)^3, q\right)$, where q is a quadratic form on $\mathbf{k}(t)^3$ corresponding to the conic fibration.

3.3.1 *Quadratic forms and the Cartan–Dieudonné theorem.* Here, after recalling basic facts about quadratic forms in arbitrary characteristic, we give a short proof of the Cartan–Dieudonné theorem in the special case of anisotropic quadratic spaces.

Let K be an arbitrary field, and consider the vector space $E = K^n$ for $n \geqslant 1$. Given a symmetric (or skew-symmetric) bilinear form $b\colon E \times E \longrightarrow K$, we set $E^\perp = \{x \in E \mid b(x, y) = 0 \ \forall y \in E\}$. We say that b is *non-degenerate* if $E^\perp = \{0\}$.

A quadratic form on E is a homogeneous polynomial $q \in K[x_1, \dots, x_n]_2$ of degree 2. To treat the cases of $\operatorname{char} K \neq 2$ and $\operatorname{char} K = 2$ simultaneously while keeping the classical notation in each case, we set

$$\delta = \begin{cases} 2 & \text{if } \operatorname{char} K \neq 2\,, \\ 1 & \text{if } \operatorname{char} K = 2\,. \end{cases}$$

We say that the symmetric bilinear form b given by

$$b(x, y) = \frac{1}{\delta}(q(x + y) - q(x) - q(y))$$

is the *polar form* of the quadratic form q. Note that

$$b(x, x) = \frac{1}{\delta}(q(2x) - 2q(x)) = \frac{1}{\delta}(2q(x)) = \begin{cases} q(x) & \text{if } \operatorname{char} K \neq 2\,, \\ 0 & \text{if } \operatorname{char} K = 2\,. \end{cases}$$

We call (E, q) a *quadratic space* and say that it is *non-degenerate* if b is. A non-zero vector $x \in E$ is *isotropic* if $q(x) = 0$ and *anisotropic* otherwise. Similarly, a quadratic space (E, q) is *isotropic* if E contains an isotropic vector, and *anisotropic* otherwise. We say that q has *defect* $d \geqslant 1$ if $d = \dim E^\perp$ and $E^\perp$ is anisotropic. Defect $d \geqslant 1$ occurs only in characteristic 2 and can be thought of as a weaker form of non-degeneracy.

The quadratic spaces we are interested in will turn out to be non-degenerate or with defect 1, and anisotropic (see Lemma 3.16).

Example 3.9. Consider $q(x) = x_1^2 + x_0x_2$ as a quadratic form on $E = K^3$, where K is an arbitrary field. The polar form of q is given by $b(x, y) = 2x_1y_1 - \frac{1}{\delta}(x_0y_2 + x_2y_0)$. If $\operatorname{char} K \neq 2$, q is non-degenerate. If $\operatorname{char} K = 2$, we have $E^\perp = \{(0, a, 0) \mid a \in K\}$. Since $q((0, a, 0)) \neq 0$ for $a \in K^*$, the quadratic space (E, q) has defect 1.

The *orthogonal group* $\mathrm{O}(E, q)$ of the quadratic space (E, q) is the group consisting of maps $\varphi \in \mathrm{GL}(E)$ such that $q(\varphi(x)) = q(x)$ for all $x \in E$. For an anisotropic vector $a \in E$, we define the map $\tau_a\colon E \longrightarrow E$ by

$$\tau_a(x) = x - \delta\frac{b(x, a)}{q(a)}a\,.$$

Note that $\tau_a(a) = -a$, and $\tau_a(x) = x$ if and only if $x \in a^\perp$. When (E, q) is *not totally degenerate*, that is, $b \neq 0$, the kernel $a^\perp$ of the linear map $b(a, \cdot)$ has codimension 1 for $a \notin E^\perp$. In this case, we use the following terminology for τ_a:

- If $\operatorname{char} K \neq 2$, the linear map τ_a is the *orthogonal reflection* along a. In particular, $\tau_a \in \mathrm{O}(E, q)$ is an involution with determinant -1; see [Gro01, Chapter 5].
- If $\operatorname{char} K = 2$, assume $a \notin E^\perp$. Then τ_a is the *transvection* with fixed hyperplane $a^\perp$. In particular, $\tau_a \in \mathrm{GL}(E)$ is an involution with determinant 1; see [Gro01, Chapter 1]. Observe that $\tau_a \in \mathrm{O}(E, q)$. Indeed, setting $\lambda = b(x, a)/q(a)$, we have $q(\tau_a(x)) = q(x + \lambda a) = b(x, \lambda a) + \lambda^2 q(a) + q(x) = \lambda(b(x, a) + \lambda q(a)) + q(x) = q(x)$ because $\lambda q(a) = b(x, a)$. We will call τ_a an *orthogonal transvection*.

Also note that a quadratic space (E,q) of dimension n is not totally degenerate as soon as q is non-degenerate or with defect less than n.

Lemma 3.10. *Let x, y be two anisotropic elements in the quadratic space (E,q) such that $q(x)=q(y)$ and $x-y$ is anisotropic. Then $\tau_{x-y}(x)=y$.*

Proof. Using $q(x)=q(y)=q(-y)$ and $\delta b(x,x)=2q(x)$, in any characteristic we have

$$q(x-y)=q(x)+q(-y)+\delta b(x,-y)=\delta b(x,x)-\delta b(x,y)=\delta b(x,x-y)\,.$$

Hence,

$$\tau_{x-y}(x)=x-\frac{\delta b(x,x-y)}{q(x-y)}(x-y)=x-(x-y)=y\,.\qquad\square$$

The Cartan–Dieudonné theorem states that $\mathrm{O}(E,q)$ is generated by reflections, except when the underlying field has two elements, the dimension of E is 4, and q is hyperbolic (see [Che97, Theorem I.5.1]). When (E,q) is anisotropic, there is an elementary proof.

Lemma 3.11 (Anisotropic Cartan–Dieudonné theorem). *Let (E,q) be an anisotropic quadratic space that is not totally degenerate. Then every element φ of $\mathrm{O}(E,q)$ can be expressed as a product of $\operatorname{codim}(F)$ orthogonal reflections (or transvections), where $F=F_\varphi\subset E$ is the space of fixed points of φ.*

Proof. First note that any product φ of k orthogonal reflections (or transvections) fixes a space of dimension at least $n-k$, namely the intersection of all the fixed hyperplanes from the involution. Hence, $\operatorname{codim}(F)\leqslant k$.

Let $\varphi\in\mathrm{O}(E,q)$. We proceed by induction on $\operatorname{codim}(F)$. If $\operatorname{codim}(F)=0$, then φ is the identity. Assume that F has codimension $k\geqslant 1$ in E, and let $v\in E\setminus F$, so that $v-\varphi(v)\neq 0$. Since by assumption E is anisotropic, applying Lemma 3.10, we find that $\tau_{v-\varphi(v)}(v)=\varphi(v)$. In particular, $\tau_{v-\varphi(v)}\neq\mathrm{id}$, so $v-\varphi(v)\notin E^\perp$ and so the map is indeed an orthogonal reflection (or transvection) since we assume q to be not totally degenerate.

Let $x\in F$; that is, $\varphi(x)=x$. We compute

$$b(x,v-\varphi(v))=b(x,v)-b(x,\varphi(v))=b(x,v)-b(\varphi(x),\varphi(v))=0\,.$$

Therefore, $\tau_{v-\varphi(v)}(x)=x$, and so $\tau_{v-\varphi(v)}$ fixes F.

This implies that $\varphi'=\varphi^{-1}\circ\tau_{v-\varphi(v)}$ fixes both F and v, hence it fixes a subspace containing F and v, which has codimension at most $k-1$. By the remark at the beginning of the proof, the fixed space of φ' has codimension exactly $k-1$, and hence by induction φ' is the composition of $k-1$ orthogonal reflections (or transvections), and so φ is the composition of $k=\operatorname{codim}(F)$ such involutions. $\square$

3.3.2 *Similitudes and the special orthogonal group.* Let (E,q) be a quadratic space over a field K. A *similitude* of (E,q) is a map $f\in\mathrm{GL}(E)$ such that there exists a $\lambda\in K^*$ with $q(f(x))=\lambda q(x)$ for all $x\in E$. The constant λ is called the *multiplier* of f. We denote the group of similitudes by $\mathrm{GO}(E,q)\subset\mathrm{GL}(E)$. The map $\mathrm{GO}(E,q)\longrightarrow K^*$ given by $f\longmapsto\lambda$ is a group homomorphism with kernel $\mathrm{O}(E,q)$. We define $\mathrm{PGO}(E,q)=\mathrm{GO}(E,q)/K^*$ to be the image of $\mathrm{GL}(E)\longrightarrow\mathrm{PGL}(E)$ restricted to $\mathrm{GO}(E,q)$, so we can write

$$\begin{aligned}\mathrm{PGO}(E,q)&=\{[A]\in\mathrm{PGL}_n(K)\mid A\in\mathrm{GO}(E,q)\}\\&=\{[A]\in\mathrm{PGL}_n(K)\mid\exists\lambda\in K^*\colon q\circ A=\lambda q\}\,,\end{aligned}$$

where n is the dimension of E.

Lemma 3.12. *The following hold:*

(1) *Let b be a (skew-)symmetric non-degenerate bilinear form on E, and let $M \in \mathrm{GL}(E)$ be such that $b(Mx, My) = b(x, y)$ for all $x, y \in E$. Then $\det M = \pm 1$.*

(2) *Let (E, q) be a quadratic space that is non-degenerate or with defect 1. Then $\det(M) = \pm 1$ for all $M \in \mathrm{O}(E, q)$.*

Proof. In part (1), since b is a non-degenerate bilinear form. there exists an $A \in \mathrm{GL}(E)$ such that $b(x, y) = x^t A y$ for all $x, y \in E$. The assumption on M implies that $x^t A y = x^t M^t A M y$ for all $x, y \in E$. Therefore, $A = M^t A M$. As A is invertible, this implies that $1 = \det(M)^2$.

For part (2), observe that if $M \in \mathrm{O}(E, q)$, then $b(Mx, My) = b(x, y)$ for all $x, y \in E$, where b is the polar form of q. So if q is non-degenerate, part (1) implies the statement. Now assume that q is with defect 1; that is, $E^\perp$ is 1-dimensional and anisotropic. In particular, we are in characteristic 2, and so we will show that $\det(M) = 1$.

First of all, if M fixes a 1-dimensional subspace L on which q does not vanish outside the origin, then it fixes L pointwise: Let $x \in L$, $x \neq 0$, and let $\lambda \in K$ be such that $M(x) = \lambda x$. Since $M \in \mathrm{O}(E, q)$, we have $q(x) = q(M(x)) = q(\lambda x) = \lambda^2 q(x)$, and as $q(x) \neq 0$, this implies that $\lambda = \pm 1$. Being in characteristic 2 implies that M fixes the point.

By extension of the basis, we can write $E = E^\perp \oplus W$ for some $(n-1)$-dimensional subspace W. Writing M in this basis, we get a matrix $\left(\begin{smallmatrix} a & B \\ 0 & M' \end{smallmatrix}\right)$with $a \in K$, $M' \in \mathrm{GL}_{n-1}(K)$. Note that M fixes $E^\perp$. By the remark above, this implies that M fixes $E^\perp$ pointwise; therefore, $a = 1$. Next, we show that $\det(M') = 1$. For this we consider M' as an endomorphism on the quotient $\bar{E} = E/E^\perp$. Note that the polar bilinear form b on E induces a bilinear form $\bar{b}$ on $\bar{E}$ by $\bar{b}(\bar{x}, \bar{y}) = b(x, y)$ for $\bar{x}, \bar{y} \in \bar{E}$. Now, $\bar{b}$ is symmetric and non-degenerate, and for all $\bar{x}, \bar{y}$, we have $\bar{b}(M'\bar{x}, M'\bar{y}) = \bar{b}(\bar{x}, \bar{y})$. By part (1), the determinant of M' is 1. Therefore, $\det(M) = a \det(M') = 1$. □

We set

$$\mathrm{SO}(E, q) = \{f \in \mathrm{O}(E, q) \mid \det f = 1\}\,.$$

In odd dimensions, there is the following relationship between similitudes and the orthogonal group.

Proposition 3.13 ([KMRT98, Propositions 12.4 and 12.6]). *Let (E, q) be a quadratic space in odd dimension over an arbitrary field K. Assume that q is non-degenerate or with defect 1. Then*

$$\mathrm{GO}(E, q) = \mathrm{SO}(E, q) \cdot K^* \simeq \mathrm{SO}(E, q) \times K^*\,,$$
$$\mathrm{PGO}(E, q) \simeq \mathrm{SO}(E, q)\,.$$

Lemma 3.14. *Let (E, q) be an anisotropic quadratic space of odd dimension n over an arbitrary field K. Assume that q is not totally degenerate. Then each element of $\mathrm{SO}(E, q)$ can be written as the composition of at most n involutions (and in fact at most $n - 1$ when $\operatorname{char} K \neq 2$).*

Proof. If $\operatorname{char} K = 2$, then $\mathrm{SO}(E, q) = \mathrm{O}(E, q)$ and the result is Lemma 3.11.

So now assume $\operatorname{char} K \neq 2$, and let $A \in \mathrm{SO}(E, q) \subset \mathrm{O}(E, q)$. Lemma 3.11 implies that A is a composition of at most n orthogonal reflections in $\mathrm{O}(E, q)$, all of which have determinant -1. Hence, there exist $B_1, \ldots, B_k \in \mathrm{O}(E, q)$ with determinant -1 and $k \leqslant n$ such that $A = B_1 \cdots B_k$. Taking the determinant on both sides gives that k has to be even, so $k \leqslant n - 1$. Since the dimension is odd, $\det(-A) = -\det(A)$ for $A \in \mathrm{GL}_n(K)$. Use the surjective group homomorphism $\mathrm{O}(E,q) \longrightarrow \mathrm{SO}(E,q)$ given by

$$\eta \colon A \longmapsto \begin{cases} -A & \text{if } \det(A) = -1\,, \\ \ \ A & \text{if } \det(A) = 1\,, \end{cases}$$

which sends B_i onto involutions $\eta(B_i) \in \mathrm{SO}(E,q)$ whose composition is $\eta(A) = A$. □

3.3.3 *Conic fibrations and quadratic forms.* The following lemma shows in particular that defect 1 is a useful notion in characteristic 2, tightly related to the notion of "strange curves" [Har77, Section IV.3].

Lemma 3.15. *Let $q \in K[x,y,z]$ be a homogeneous polynomial of degree 2 over an arbitrary field K (not necessarily perfect). Assume that q is irreducible over K^a, and consider the conic $C = V_{\mathbb{P}^2_K}(q)$. Then the quadratic space (K^3, q) is non-degenerate if* $\operatorname{char} K \neq 2$ *and has defect* 1 *if* $\operatorname{char} K = 2$.

Moreover, if $\operatorname{char} K = 2$, *the* 1*-dimensional kernel $E^\perp$ corresponds to the point $p \in \mathbb{P}^2(K)$ where all tangent lines to C meet.*

Proof. Write $E = K^3$. Note that C contains a K-point if and only if there exists a non-zero element $x \in K^3$ such that $q(x) = 0$, that is, if (E,q) is isotropic.

If (E,q) is isotropic, then C contains a rational point and hence there is a linear K-automorphism of E sending C onto the conic $V\big(x_1^2 - x_0x_2\big)$ from Example 3.9. Hence, q is non-degenerate if $\operatorname{char} K \neq 2$ and with defect 1 if $\operatorname{char} K = 2$.

Now consider the anisotropic case. If $\operatorname{char} K \neq 2$, having $q(x) = b(x,x) \neq 0$ for all non-zero $x \in K^3$ directly implies that q is non-degenerate. Now assume $\operatorname{char} K = 2$. Let $B = (b_{ij})_{1 \leqslant i,j \leqslant 3}$ be the matrix with $b_{ij} = b(e_i, e_j)$, which has the form $B = \left(\begin{smallmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{smallmatrix}\right) \in \mathrm{GL}_3(K)$. Since we are in the anisotropic case, it is enough to show that $\dim E^\perp = 1$. If B is the zero matrix, then q is a sum of square monomials, hence q is reducible over K^a, giving a contradiction. So the kernel $E^\perp = \{(cx, bx, ax) \in E \mid x \in K\}$ is 1-dimensional.

Finally, note that in characteristic 2, all $V(q)$ with the same polar form b have the same tangent lines. With coordinates as above, we can assume up to a permutation that $a \neq 0$, and so the tangent lines are given by $\mu x + \lambda y + \frac{1}{a}(\mu c + \lambda b)z = 0$ for $[\lambda : \mu] \in \mathbb{P}^1(K^a)$. All of them go through the point $[c : b : a]$. □

The following lemma shows that any quadratic space $\big(\mathbf{k}(t)^3, q\big)$ corresponding to a Jonquières group $\mathrm{Bir}_{\mathbf{k}}\big(\mathbb{P}^2/\pi\big)$ of type $2+2$ or 4 is anisotropic and either is non-degenerate or has defect 1.

Lemma 3.16. *Let $\{p_1, \dots, p_4\} \subset \mathbb{P}^2(\mathbf{k}^a)$ be a Galois-invariant set of four points such that no three are collinear, and let $q_1, q_2 \in \mathbf{k}[x,y,z]$ be two homogeneous polynomials of degree 2 such that $C_1 = V(q_1)$, $C_2 = V(q_2)$ are two distinct conics intersecting at the four points. Let $q = q_1 + tq_2 \in \mathbf{k}[t,x,y,z] \subset \mathbf{k}(t)[x,y,z]$ and $C = V_{\mathbb{P}^2_{\mathbf{k}(t)}}(q)$. Then the quadratic space $\big(\mathbf{k}(t)^3, q\big)$ is non-degenerate* ($\operatorname{char} \mathbf{k} \neq 2$) *or has defect* 1 ($\operatorname{char} \mathbf{k} = 2$), *and the following are equivalent:*

(1) *The intersection $C_1 \cap C_2$ contains a $\mathbf{k}$-point.*

(2) *The subspace $C \subset \mathbb{P}^2_{\mathbf{k}(t)}$ contains a $\mathbf{k}(t)$-point.*

(3) *The quadratic space $\big(\mathbf{k}(t)^3, q\big)$ is isotropic.*

Proof. First, note that since no three of the four points are collinear, q is irreducible over $\mathbf{k}(t)^a$. By Lemma 3.15, the polynomial q is non-degenerate if $\operatorname{char} K \neq 2$ and has defect 1 if $\operatorname{char} K = 2$.

The equivalence (2) $\iff$ (3) is direct. Since $\mathbf{k} \subset \mathbf{k}(t)$, part (1) implies part (2). We now show the converse direction. Assume that C contains a $\mathbf{k}(t)$-point. Hence, C is isomorphic to $\mathbb{P}^1_{\mathbf{k}(t)}$ over $\mathbf{k}(t)$. In particular,

$$X = \{([x:y:z],t) \mid q(t,x,y,z) = 0\} \subset \mathbb{P}^2 \times \mathbb{A}^1$$

contains sections that are defined over $\mathbf{k}$. Note that the projection $\rho\colon X \longrightarrow \mathbb{P}^2$ is the blow-up at the four points $p_1, \dots, p_4$. We now assume that $C_1 \cap C_2$ contains no $\mathbf{k}$-point, and we will show that X contains no sections defined over $\mathbf{k}$, giving a contradiction. As none of the four points $p_1, \dots, p_4 \in \mathbb{P}^2(\mathbf{k}^a)$ is a $\mathbf{k}$-point, they form either one 4-point or two 2-points (say $\{p_1, p_2\}$ and $\{p_3, p_4\}$). Let $D \subset X$ be a curve, defined over $\mathbf{k}$, and let $D' = \rho(D) \subset \mathbb{P}^2$. The essential remark is that all geometric components of a point have the same multiplicity on D'. Write d for the degree of D' and m for the multiplicity of D' at p_1, p_2 and m' for that at p_3, p_4. After resolving the singularities $x_1, \dots, x_r$ of D and writing L for the strict transform of a general line in $\mathbb{P}^2$, and E_i for the exceptional divisor of p_i, and E_{x_i} for that of x_i, we can write the intersection of D with a general fiber f of X as

$$\begin{aligned} D \cdot f &= (dL - m(E_1 + E_2) - m'(E_3 + E_4) - \sum m_{x_i}(D) E_{x_i})(2L - (E_1 + \cdots + E_4)) \\ &= 2d - 2m - 2m' . \end{aligned}$$

As $D \cdot f$ is even, D is not a section. $\square$

Remark 3.17. For $\operatorname{char} \mathbf{k} = 2$, we have seen in Lemma 3.15 that the defect corresponds to the point where all the tangent lines to $C \subset \mathbb{P}^2_{\mathbf{k}(t)}$ meet. This in turn corresponds to the line in $\mathbb{P}^2_{\mathbf{k}}$ that is tangent to all conics that go through $p_1, \dots, p_4$. This line plays a crucial role in [Sch21], where involutive generators of the groups $\operatorname{Bir}_{\mathbb{F}_2}\left(\mathbb{P}^2/\pi_{\mathbb{F}_4,\mathbb{F}_4}\right)$ and $\operatorname{Bir}_{\mathbb{F}_2}\left(\mathbb{P}^2/\pi_{\mathbb{F}_{16}}\right)$ are described explicitly.

Lemma 3.18. *Let $\{p_1, \dots, p_4\} \subset \mathbb{P}^2(\mathbf{k}^a)$ be a Galois-invariant set of four points such that no three are collinear, and let $q_1, q_2 \in \mathbf{k}[x,y,z]$ be two homogeneous polynomials of degree 2 such that the pencil of conics through $p_1, \dots, p_4$ is given by $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $\pi([x:y:z]) = [q_1 : q_2]$. Let $q = q_1 + tq_2 \in \mathbf{k}[t][x,y,z] \subset \mathbf{k}(t)[x,y,z]$, which we see as a quadratic form of the 3-dimensional vector space $E = \mathbf{k}(t)^3$ over $\mathbf{k}(t)$. Then*

$$\operatorname{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right) \simeq \operatorname{PGO}\left(\mathbf{k}(t)^3, q\right).$$

Proof. In Lemma 3.16, we have observed that the quadratic space $\left(\mathbf{k}(t)^3, q\right)$ is non-degenerate or with defect 1. By construction, the generic fiber C of π is given by the zero set of q in $\mathbb{P}^2_{\mathbf{k}(t)}$. Note that the splitting field L of the four points $p_1, \dots, p_4$ is a Galois extension of $\mathbf{k}$ over which the generic fiber C is non-empty. So [Sch21, Lemma 3.29] yields

$$\operatorname{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right) \simeq \operatorname{Bir}_{\mathbf{k}(\mathbb{P}^1)}(C) = \left\{f \in \operatorname{Aut}_{\mathbf{k}(\mathbb{P}^1)}\left(\mathbb{P}^2\right) \mid f(C) = C\right\}.$$

We have $f(C) = C$ if and only if $V(q \circ f) = f^{-1}(C) = C = V(q)$, so the above group is equal to

$$\left\{f \in \operatorname{Aut}_{\mathbf{k}(\mathbb{P}^1)}\left(\mathbb{P}^2\right) \mid \exists \lambda \in \mathbf{k}\left(\mathbb{P}^1\right)^* \colon q \circ f = \lambda q\right\} = \operatorname{PGO}\left(\mathbf{k}\left(\mathbb{P}^1\right)^3, q\right).$$

Since $\mathbf{k}\left(\mathbb{P}^1\right) \simeq \mathbf{k}(t)$, the statement follows. $\square$

PROPOSITION 3.19. *The following hold:*

(1) *Let $F/\mathbf{k}$ be a residue field of an irreducible polynomial of degree* 4*. Then* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi_F\right) \simeq \mathrm{SO}\left(\mathbf{k}(t)^3, q_F\right)$.

(2) *Let L, L' be two field extensions of degree* 2 *over* $\mathbf{k}$*. Then* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi_{L,L'}\right) \simeq \mathrm{SO}\left(\mathbf{k}(t)^3, q_{L,L'}\right)$.

In particular, each element of these groups is a composition of at most three (*two if* $\mathrm{char}\,\mathbf{k} \neq 2$) *involutions.*

Proof. None of the four geometric points where π_F, respectively $\pi_{L,L'}$, is not defined is $\mathbf{k}$-rational. Therefore, Lemma 3.16 implies that in both cases, the associated quadratic space $\left(\mathbf{k}(t)^3, q\right)$ is anisotropic, and so by Lemma 3.14 each element in the group $\mathrm{SO}\left(\mathbf{k}(t)^3, q\right)$ is the product of at most three (two if $\mathrm{char}\,\mathbf{k} \neq 2$) involutions. With Lemma 3.18 and Proposition 3.13, we get

$$\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2/\pi\right) \simeq \mathrm{PGO}\left(\mathbf{k}(t)^3, q\right) \simeq \mathrm{SO}\left(\mathbf{k}(t)^3, q\right). \qquad \square$$

3.4 The image for Jonquières type 4

Here we study the image $\mathrm{Aut}_{\mathbf{k}}^{\pi_F}\left(\mathbb{P}^1\right)$ from the exact sequence (Seq) from page 120, for fibering type 4. Denote by $\{p_1, \ldots, p_4\} \in \mathbb{P}^2(\mathbf{k}^a)$ the base points of π_F and by L_{ij} the line through p_i and p_j. In the following lemma, we show that the permutation of the three singular fibers by a map in $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_F\right)$ is always achieved by an automorphism.

LEMMA 3.20. *Let $F/\mathbf{k}$ be a residue field of an irreducible polynomial of degree* 4*. Let* $\varphi \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_F\right)$*. Then there exists an* $\alpha \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right) \cap \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_F\right)$ *such that* $\pi_F \circ \alpha = \pi_F \circ \varphi$.

Proof. The blow-up of the 4-point gives a link $\mathbb{P}^2 \overset{4}{\dashrightarrow} X_5$ with $X_5 \in \mathcal{C}_5$. Hence, $\varphi \in \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_F\right)$ factors through a birational map $X_5 \dashrightarrow X_5$ that preserves the fibration $X_5 \longrightarrow \mathbb{P}^1$. By [Sch22, Corollary 3.2], the map φ sends the set of three singular fibers onto itself. As any element of $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^1\right)$ is uniquely determined by its value at three points, the image of $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi\right)$ in $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^1\right)$ is isomorphic to a subgroup of the symmetric group S_3 determined by the action on the projection of the three singular fibers $L_{ij} \cup L_{kl}$ for $\{i,j,k,l\} = \{1,2,3,4\}$. Hence, it is enough to find an $\alpha \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right) \cap \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2, \pi_F)$ such that $\alpha(L_{ij} \cup L_{kl}) = \varphi(L_{ij} \cup L_{kl})$ for $\{i,j,k,l\} = \{1,2,3,4\}$. We show that there is an α that satisfies $\alpha(L_{ij}) = \varphi(L_{ij})$.

We follow the proof of [Sch22, Proposition 6.12]. Write $F_{ij} = \varphi(L_{ij})$, and note that two configurations are possible (see [Sch22, Lemma 6.5(4)]):

(1) for each i, the three lines F_{ij} for $j \neq i$ intersect in one point, or

(2) for each i, the three lines F_{jk} for $i \notin \{j,k\}$ intersect in one point.

In both cases, write q_i for the intersection point of the three lines, and note that $q_i \in \{p_1, \ldots, p_4\}$ since $\varphi(L_{ij}) = L_{kl}$ for some k, l. Let L be the Galois closure of $F/\mathbf{k}$. For $g \in \mathrm{Gal}(L/\mathbf{k})$, let $\tau \in S_4$ be such that $p_i^g = p_{\tau(i)}$. In case (1), q_i^g is the intersection of the lines $\varphi(L_{ij})^g = \varphi(L_{\tau(i),\tau(j)})$ for $j \neq i$, which is $q_{\tau(i)}$. In case (2), q_i^g is the intersection of the three lines $\varphi(L_{jk})^g = \varphi(L_{\tau(j),\tau(k)})$ for $i \notin \{j,k\}$, which is again $q_{\tau(i)}$. Hence, in both cases there exists an $\alpha \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ such that $\alpha(p_i) = q_i$; see [Sch22, Lemma 6.11]. In particular, α preserves the pencil of conics through the p_i, which concludes the proof. $\square$

COROLLARY 3.21. *Let* $\pi_F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ *be a rational fibration of type* 4*, associated with a residue field $F/\mathbf{k}$ of an irreducible polynomial of degree* 4 (*see Example* 3.5)*. Then* $\mathrm{Bir}\left(\mathbb{P}^2, \pi_F\right)$ *is contained in the subgroup of* $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ *generated by involutions.*

Proof. By the exact sequence (Seq) and Lemma 3.20, the group $\mathrm{Bir}\left(\mathbb{P}^2, \pi_F\right)$ is generated by $\mathrm{Bir}\left(\mathbb{P}^2/\pi_F\right)$ and by $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right) \cap \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_F\right)$. By Proposition 3.19, any element in $\mathrm{Bir}\left(\mathbb{P}^2/\pi_F\right) \simeq \mathrm{SO}_3\left(\mathbf{k}(t)^3, q_F\right)$ is a product of at most three (two if $\operatorname{char} \mathbf{k} \neq 2$) involutions. On the other hand, by Lemma 3.8, the full automorphism group $\mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right) \simeq \mathrm{PGL}_3(\mathbf{k})$ is contained in the subgroup of $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ generated by involutions, which gives the result. □

3.5 The image for Jonquières type 2+2

Let $L/\mathbf{k}$ and $L'/\mathbf{k}$ be two quadratic extensions, and write $K = LL'$ for the composite field. The main result of this section is Proposition 3.25, which gives a family of involutions in $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_{L,L'}\right)$ that surject onto the image $\mathrm{Aut}_{\mathbf{k}}^{\pi_{L,L'}}\left(\mathbb{P}^1\right)$.

3.5.1 *The statement.* Denote by $\pi_1, \pi_2 \colon \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ the projection onto the first, respectively, second $\mathbb{P}^1$. We call the fibers of π_1 *vertical* and the fibers of π_2 *horizontal* curves. First, we will describe geometrically a birational map $\varepsilon \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ that is defined over $K = LL'$. We will see that the fibration $\pi_{L,L'}$ corresponding to the pencil of conics through the two points of degree 2 is sent onto the fibration that is given by $\pi_1 \colon \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ (see Lemma 3.22). Next, we will describe the map ε explicitly on affine charts and keep track of the induced Galois action $\varepsilon \operatorname{Gal}(K/\mathbf{k})\varepsilon^{-1}$.

Lemma 3.22. *Let $\{p_1, \ldots, p_4\} \subset \mathbb{P}^2(\mathbf{k}^a)$ be four points in general position and $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the fibration corresponding to the pencil of conics through the four points. Denote by K the splitting field of $p_1, \ldots, p_4$. Let $\varepsilon \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the composition $\gamma \circ \beta \circ \alpha$ of the following maps, all defined over K:*

- *$\alpha \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up of p_1, p_2 followed by the contraction of the strict transform of the line through p_1, p_2;*
- *$\beta \colon \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$ is an automorphism preserving the two rulings;*
- *$\gamma \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up of $\beta \circ \alpha(p_3)$, $\beta \circ \alpha(p_4)$ followed by the contraction of the strict transforms of the horizontal curves through $\beta \circ \alpha(p_3)$, respectively $\beta \circ \alpha(p_4)$.*

Then, ε preserves the fibrations $\mathbb{P}^2/\pi$ and $\mathbb{P}^1 \times \mathbb{P}^1/\pi_1$; that is, there exists a $\varphi \in \mathrm{Aut}_{\mathbf{k}^a}\left(\mathbb{P}^1\right)$ such that $\pi \circ \varepsilon^{-1} = \varphi \circ \pi_1$.

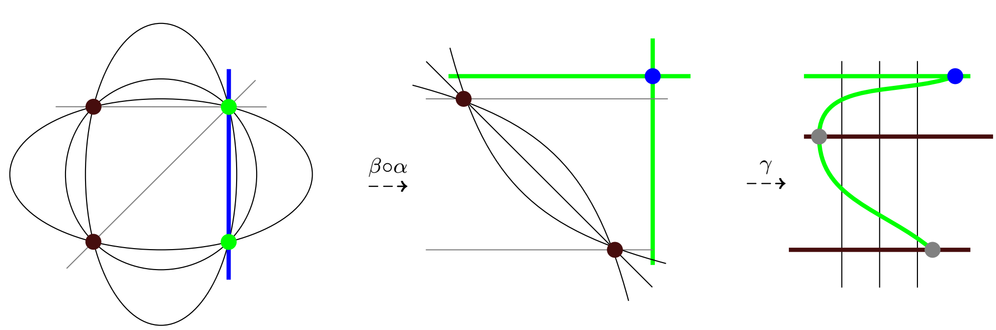

Figure 3.1. The birational map $\varepsilon \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ from Lemma 3.22

Proof. The birational map $\beta \circ \alpha$ sends the pencil of conics through $p_1, \ldots, p_4$ onto the pencil of diagonal curves going through $\beta \circ \alpha(p_3)$, $\beta \circ \alpha(p_4)$, which is sent by γ onto the pencil of vertical curves (see Figure 3.1). □

COROLLARY 3.23. *Take the notation from Lemma 3.22. Then*

$$\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2,\pi\right)\simeq\left(\mathrm{PGL}_2(K)\ltimes\mathrm{PGL}_2(K(x))\right)^{\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}}.$$

Proof. Lemma 3.22 implies that $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2,\pi\right)\simeq\mathrm{Bir}_K\left(\mathbb{P}^1\times\mathbb{P}^1,\pi_1\right)^{\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}}$. The statement follows since

$$\mathrm{Bir}_K\left(\mathbb{P}^1\times\mathbb{P}^1,\pi_1\right)\simeq\mathrm{PGL}_2(K)\ltimes\mathrm{PGL}_2(K(x)),$$

using that the elements $j\in\mathrm{Bir}_K\left(\mathbb{P}^1\times\mathbb{P}^1,\pi_1\right)$ are of the form $j(x,y)=\left(\frac{ax+b}{cx+d},\frac{A(x)y+B(x)}{C(x)y+D(x)}\right)$ with $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{PGL}_2(K)$ and $M'=\left(\begin{smallmatrix}A(x)&B(x)\\C(x)&D(x)\end{smallmatrix}\right)\in\mathrm{PGL}_2(K(x))$. □

SETUP 3.24. Let $L/\mathbf{k}$, $L'/\mathbf{k}$ be two quadratic extensions with $L=\mathbf{k}(\theta)$, $L'=\mathbf{k}(\theta')$, and write $K=LL'$ for the composite field.

(1) If $L=L'$, let g be the generator of $\mathrm{Gal}(K/\mathbf{k})=\mathrm{Gal}(L/\mathbf{k})\simeq\mathbf{Z}/2\mathbf{Z}$.

(2) If $L\neq L'$, let h, h' be the generators of the Galois groups $\mathrm{Gal}(L/\mathbf{k})$, $\mathrm{Gal}(L'/\mathbf{k})$, and set $g=hh'$. So g, h generate $\mathrm{Gal}(K/\mathbf{k})=\mathrm{Gal}(L/\mathbf{k})\times\mathrm{Gal}(L'/\mathbf{k})\simeq\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$.

We write x^g for the action of g on $x\in\mathbf{k}^a$ and A^g for the action of g on the coefficients of $A\in\mathbf{k}^a(x)$, and similarly for h, h'.

Note that $\theta^g=\theta^h$ and $\theta'^g=\theta'^{h'}$ for $L\neq L'$. As in Example 3.5, we take $\pi_{L,L'}$ to be the fibration given by the pencil of conics through the 2-points $p'=\{p_1,p_2\}=\{[\theta':1:0],[\theta'^g:1:0]\}$ and $p=\{p_3,p_4\}=\{[\theta:0:1],[\theta^g:0:1]\}$. Hence, $g\in\mathrm{Gal}(K/\mathbf{k})$ acts on $\{p_1,\ldots,p_4\}$ as the permutation $(12)(34)$. In the case $L\neq L'$, the elements h and h' act as, respectively, the transpositions (34) and (12).

We shall use Lemma 3.22 with the following maps α, β, γ, which we express in the affine charts

$$\begin{array}{rcl}\mathbb{A}^2&\longrightarrow&\mathbb{P}^2\\(x,y)&\longmapsto&[x:y:1]\end{array}\qquad\text{and}\qquad\begin{array}{rcl}\mathbb{A}^2&\longrightarrow&\mathbb{P}^1\times\mathbb{P}^1,\\(x,y)&\longmapsto&([x:1],[y:1]).\end{array}$$

We set

$$\begin{aligned}\alpha(x,y)&=(x-\theta'y,x-\theta'^gy), & \alpha^{-1}(x,y)&=\left(\frac{\theta'^gx-\theta'y}{\theta'^g-\theta'},\frac{x-y}{\theta'^g-\theta'}\right),\\ \beta(x,y)&=\left(\frac{x-\theta}{-x+\theta^g},\frac{y-\theta^g}{-y+\theta}\right), & \beta^{-1}(x,y)&=\left(\frac{\theta^gx+\theta}{x+1},\frac{\theta y+\theta^g}{y+1}\right),\\ \gamma(x,y)&=(xy,y), & \gamma^{-1}(x,y)&=\left(\frac{x}{y},y\right).\end{aligned}$$

The map $\alpha\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^1\times\mathbb{P}^1$ blows up the two components p_1, p_2 of p' and contracts the line at infinity in $\mathbb{P}^2$. Moreover, α sends the components $p_3=(\theta,0)$, $p_4=(\theta^g,0)$ of p onto the points (θ,θ), (θ^g,θ^g), which are then sent by $\beta\in\mathrm{Aut}\left(\mathbb{P}^1\times\mathbb{P}^1\right)$ onto $([0:1],[1:0]),([1:0],[0:1])$. Observe that $\varepsilon=\gamma\circ\beta\circ\alpha$ is defined over K.

In the computations we shall use the notation of the following diagram for the action of g, and similarly for h, h':

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "\alpha"] \arrow[d, "g"] \arrow[rrr, bend left, "\varepsilon"] & \mathbb{A}^2 \arrow[r, "\beta"] \arrow[d, "g_\alpha"] & \mathbb{A}^2 \arrow[r, "\gamma"] \arrow[d, "g_\beta"] & \mathbb{A}^2 \arrow[d, "g_\gamma"] \\ \mathbb{A}^2 \arrow[r, "\alpha"] & \mathbb{A}^2 \arrow[r, "\beta"] & \mathbb{A}^2 \arrow[r, "\gamma"] & \mathbb{A}^2 . \end{tikzcd} \tag{Dia}$$

In the rest of this section, we use the notation of Setup 3.24, and we prove the following.

PROPOSITION 3.25. *Denote by $H_{L,L'} \subset \mathrm{Bir}_K\left(\mathbb{A}^2\right)$ the group generated by involutions of the form*

$$(x,y) \dashrightarrow \left(\frac{1}{\mu x}, \frac{1}{\lambda y}\right),$$

where $\mu = \lambda\lambda^g$ and

(1) $\lambda \in K^* = L^*$ *if* $L = L'$,

(2) $\lambda \in K^*$ *such that* $\lambda\lambda^h = 1$ *if* $L \neq L'$.

Then the group $\varepsilon^{-1} H_{L,L'} \varepsilon \subset \mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_{L,L'}\right)$ surjects onto $\mathrm{Aut}_{\mathbf{k}}^{\pi_{L,L'}}\left(\mathbb{P}^1\right)$.

3.5.2 *The induced Galois action*

LEMMA 3.26. *The induced Galois action $\varepsilon \circ g \circ \varepsilon^{-1}$ is given by*

$$g_\gamma(x,y) = \left(x^g, \frac{x^g}{y^g}\right).$$

Proof. We compute step by step the maps on diagram (Dia). First,

$$g_\alpha(x,y) = \alpha g \alpha^{-1}(x,y) = \alpha\left(\frac{\theta' x^g - \theta'^g y^g}{\theta' - \theta'^g}, \frac{x^g - y^g}{\theta' - \theta'^g}\right) = \left(y^g, x^g\right).$$

We observe that (y^g, x^g) commutes with β, so $g_\beta(x,y) = \beta g_\alpha \beta^{-1}(x,y) = (y^g, x^g)$. Finally,

$$g_\gamma(x,y) = \gamma g_\beta \gamma^{-1}(x,y) = \gamma g_\beta\left(\frac{x}{y}, y\right) = \gamma\left(y^g, \frac{x^g}{y^g}\right) = \left(x^g, \frac{x^g}{y^g}\right). \qquad \square$$

LEMMA 3.27. *Assume $L \neq L'$. Then the induced Galois action $\varepsilon \circ h \circ \varepsilon^{-1}$ is given by*

$$h_\gamma(x,y) = \left(\frac{1}{x^h}, \frac{1}{y^h}\right).$$

Proof. We compute step by step the maps on diagram (Dia): Since $\theta'^{gh} = \theta'^g$ and $\theta'^h = \theta'$, we have

$$h_\alpha(x,y) = \alpha h \alpha^{-1}(x,y) = \left(x^h, y^h\right).$$

On the other hand, $\theta^{gh} = \theta$ and $\theta^h = \theta^g$, so we get

$$\begin{aligned} h_\beta(x,y) = \beta h_\alpha \beta^{-1}(x,y) &= \beta h_\alpha\left(\frac{\theta^g x + \theta}{x+1}, \frac{\theta y + \theta^g}{y+1}\right) \\ &= \beta\left(\frac{\theta x^h + \theta^g}{x^h + 1}, \frac{\theta^g y^h + \theta}{y^h + 1}\right) = \left(\frac{1}{x^h}, \frac{1}{y^h}\right). \end{aligned}$$

Finally, since $\bigl(1/x^h, 1/y^h\bigr)$ commutes with $\gamma = (xy, y)$, we get

$$h_\gamma(x,y) = \gamma h_\beta \gamma^{-1} = \left(\frac{1}{x^h}, \frac{1}{y^h}\right). \qquad \square$$

Lemma 3.28. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(K)$ and $M' = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{PGL}_2(K(x))$. Then $j = (M, M') \in \mathrm{PGL}_2(K) \ltimes \mathrm{PGL}_2(K(x))$ is g_γ-invariant if and only if $M = M^g$ in $\mathrm{PGL}_2(K)$, and the equality*

$$\begin{pmatrix} A(x)x & B(x) \\ C(x)x & D(x) \end{pmatrix} = \begin{pmatrix} D^g(x)(a^g x + b^g) & C^g(x)(a^g x + b^g) \\ B^g(x)(c^g x + d^g) & A^g(x)(c^g x + d^g) \end{pmatrix} \tag{3.1}$$

holds in $\mathrm{PGL}_2(K(x))$.

Proof. Using the formula $g_\gamma(x,y) = \bigl(x^g, x^g/y^g\bigr)$ from Lemma 3.26, we compute

$$\begin{aligned} g_\gamma\bigl(j\bigl(x^g, y^g\bigr)\bigr) &= g_\gamma\left(\frac{ax^g + b}{cx^g + d}, \frac{A(x^g)y^g + B(x^g)}{C(x^g)y^g + D(x^g)}\right) \\ &= \left(\frac{a^g x + b^g}{c^g x + d^g}, \frac{(a^g x + b^g)(C^g(x)y + D^g(x))}{(c^g x + d^g)(A^g(x)y + B^g(x))}\right), \\ j\bigl(g_\gamma\bigl(x^g, y^g\bigr)\bigr) &= j\left(x^g, \frac{x^g}{y^g}\right) = \left(\frac{ax + b}{cx + d}, \frac{A(x)\frac{x}{y} + B(x)}{C(x)\frac{x}{y} + D(x)}\right) \\ &= \left(\frac{ax + b}{cx + d}, \frac{A(x)x + B(x)y}{C(x)x + D(x)y}\right). \end{aligned}$$

Therefore, $j = (M, M')$ is g_γ-invariant if and only if equation (3.1) holds, as well as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a^g & b^g \\ c^g & d^g \end{smallmatrix}\right)$ in $\mathrm{PGL}_2(K)$. $\square$

Remark 3.29. The condition $M^g = M$ is equivalent to $M \in \mathrm{PGL}_3(\mathbf{k})$ in the case $L = L'$, and to $M \in \mathrm{PGL}_3(L'')$, where $\mathbf{k} \subset L'' \subset K$ is the fixed field of $g = hh'$, in the case $L \neq L'$.

Lemma 3.30. *Assume $L \neq L'$. Then $(M, M') = \bigl(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\bigr) \in \mathrm{PGL}_2(K) \ltimes \mathrm{PGL}_2(K(x))$ is h_γ-invariant if and only if the following equalities in $\mathrm{PGL}_2(K)$, respectively $\mathrm{PGL}_2(K(x))$, are satisfied:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^h & c^h \\ b^h & a^h \end{pmatrix}, \tag{3.2}$$

$$\begin{pmatrix} A\bigl(\frac{1}{x}\bigr) & B\bigl(\frac{1}{x}\bigr) \\ C\bigl(\frac{1}{x}\bigr) & D\bigl(\frac{1}{x}\bigr) \end{pmatrix} = \begin{pmatrix} D^h(x) & C^h(x) \\ B^h(x) & A^h(x) \end{pmatrix}. \tag{3.3}$$

Proof. Using the formula $h_\gamma(x,y) = \bigl(1/x^h, 1/y^h\bigr)$ from Lemma 3.27, we study the action of h_γ:

$$\begin{aligned} h_\gamma\bigl(j\bigl(x^h, y^h\bigr)\bigr) &= \left(\frac{c^h x + d^h}{a^h x + b^h}, \frac{C^h(x)y + D^h(x)}{A^h(x)y + B^h(x)}\right), \\ j\bigl(h_\gamma\bigl(x^h, y^h\bigr)\bigr) &= \left(\frac{a\frac{1}{x} + b}{c\frac{1}{x} + d}, \frac{A\bigl(\frac{1}{x}\bigr)\frac{1}{y} + B\bigl(\frac{1}{x}\bigr)}{C\bigl(\frac{1}{x}\bigr)\frac{1}{y} + D\bigl(\frac{1}{x}\bigr)}\right) = \left(\frac{bx + a}{dx + c}, \frac{B\bigl(\frac{1}{x}\bigr)y + A\bigl(\frac{1}{x}\bigr)}{D\bigl(\frac{1}{x}\bigr)y + C\bigl(\frac{1}{x}\bigr)}\right). \end{aligned}$$

Hence, $h_\gamma \circ (M, M') = (M, M') \circ h_\gamma$ if and only if the matrix equalities (3.2) and (3.3) hold in $\mathrm{PGL}_2(K)$, respectively $\mathrm{PGL}_2(K(x))$. $\square$

Seeing PGL_2 as a subvariety of $\mathbb{P}^4$, two matrices $\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$ are equal in PGL_2 if and only if $a_i b_j = a_j b_i$ for all $i, j = 1, \ldots, 4$. If $a_i \neq 0$ for a fixed i, then this is equivalent to $a_i b_j = a_j b_i$ for $j = 1, \ldots, 4$.

Corollary 3.31. *Let $a \in K$ and $P(x) \in K(x)$. Then*

$$\left(\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ P(x) & 0 \end{pmatrix}\right) \in \mathrm{PGL}_2(K) \ltimes \mathrm{PGL}_2(K(x))$$

is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant if and only if

(1) *$a^g = a$, $P(x)P^g(x) = a$, and*

(2) *$aa^h = 1$ and $P(1/x)P^h(x) = 1$ if $L \neq L'$.*

Moreover, under these conditions, $\left(\left(\begin{smallmatrix} 0 & 1 \\ a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ P(\mu) & 0 \end{smallmatrix}\right)\right)$ is also $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant for all $\mu \in \mathbf{k}^*$.

Proof. Note that if $L \neq L'$, then $g = hh'$ and h generate $\mathrm{Gal}(K/\mathbf{k})$. The equations come from Lemmas 3.28 and 3.30.

For the second part, write $P(x) = A(x)/B(x)$ with two polynomials $A, B \in K[x]$ without common factor. Condition (1) implies that $B^g(x)$ is a multiple of $A(x)$. If $\mu \in \mathbf{k}$ is a root of B, then it is also a root of B^g (since $B^g(\mu) = B^g(\mu^g) = (B(\mu))^g = 0$), hence of A, giving a contradiction to A, B being without common factors. Therefore, evaluating $P(x)$ at $\mu \in \mathbf{k}^*$, we get a constant function $P(\mu)$ again satisfying conditions (1) and (2). □

Remark 3.32. Over $\mathbf{k} = \mathbf{Q}$ and $L = L' = \mathbf{Q}(i)$, taking $a = 1$ and $P(x) = (x+i)/(x-i)$ is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant. More generally, if $L = L' = \mathbf{k}(\theta)$, one can take $a = 1$ and $P(x) = (x+\theta)/(x+\theta^g)$.

In the case $L = \mathbf{k}(\theta) \neq \mathbf{k}(\theta') = L'$, one can also construct a non-constant $P \in K(x)$ with $a = 1$: consider $Q(x) = (x+\theta)/(x+\theta^g)$, and choose $P(x) = Q(x)Q(1/x)$.

3.5.3 *Proof of Proposition* 3.25

Lemma 3.33. *For $A, B, C, D \in K[x]$ with $AD - BC \neq 0$ and $a, b, c, d \in K$ with $ad - bc \neq 0$, the following hold:*

(1) *If $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\right)$ is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant, then if $A \neq 0$, the map $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)\right)$ also is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant, and if $B \neq 0$, then $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)\right)$ also is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant.*

(2) *The map $\left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)\right)$ is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant if and only if $\left(\left(\begin{smallmatrix} 0 & a \\ d & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & A \\ D & 0 \end{smallmatrix}\right)\right)$ is.*

(3) *The map $\left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)\right)$ is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant if and only if $\left(\left(\begin{smallmatrix} 0 & d \\ a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} C & 0 \\ 0 & B \end{smallmatrix}\right)\right)$ is. Moreover, this is equivalent to $\left(\left(\begin{smallmatrix} 0 & d \\ a & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & xC(x) \\ B(x) & 0 \end{smallmatrix}\right)\right)$ being $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant.*

(4) *The map $\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$ is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant.*

(5) *If $A = D = 0$ or $B = C = 0$, then either $a = d = 0$ or $b = c = 0$.*

Proof. We use the conditions of $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariance from Lemma 3.28 (for the action of g_γ) and Lemma 3.30 (for the action of h_γ if $L \neq L'$). Note that (3.1) implies that $A \neq 0$ if and only if $D \neq 0$, and $B \neq 0$ if and only if $C \neq 0$. Assertions (1) and (4) are then immediate. For assertion (2), and similarly for assertion (3), one checks that the conditions given by (3.1) on the two pairs of matrices in the statement are the same (the other equations are clear).

To prove assertion (5), we write $\delta(a) = 0$ if $a = 0$, and $\delta(a) = 1$ if $a \neq 0$, and similarly for b, c, d. Since $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is invertible, we cannot have $A = 0$ and $B = 0$ simultaneously. Assume $A \neq 0$, $B = C = 0$ (the other case $B \neq 0$, $A = D = 0$ is similar and left to the reader). Equation (3.1) gives

$$A^g(x)xA^g(x)(c^gx+d^g)=D(x)D^g(x)(a^gx+b^g)\,.$$

Taking the degree in x, we get

$$2\deg(A)+\delta(c)+1=2\deg(D)+\delta(a)\,.$$

Reducing this last equality modulo 2 gives two possibilities: either $\delta(c)=0$ and $\delta(a)=1$, hence $b=c=0$ and $a,d\neq 0$; or $\delta(a)=0$ and $\delta(c)=1$, hence $a=d=0$ and $b,c\neq 0$. □

Lemma 3.33 has the following corollary, which says that the diagonal and antidiagonal elements in $(\mathrm{PGL}_2(K)\ltimes\mathrm{PGL}_2(K(x)))^{\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}}$ surject onto $\mathrm{Aut}_{\mathbf{k}}^{\pi_{L,L'}}(\mathbb{P}^1)$, which is the image of the projection

$$(\mathrm{PGL}_2(K)\ltimes\mathrm{PGL}_2(K(x)))^{\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}}\longrightarrow\mathrm{PGL}_2(K)\,.$$

Corollary 3.34. *Let* $\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\right)\in\mathrm{PGL}_2(K)\ltimes\mathrm{PGL}_2(K(x))$ *be* $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$*-invariant. Then one of the following holds:*

1. *We have* $b=c=0$*, and there exist* $A',D'\in K(x)$ *such that the map* $\left(\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right),\left(\begin{smallmatrix}A'&0\\0&D'\end{smallmatrix}\right)\right)$ *is* $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$*-invariant.*
2. *We have* $a=d=0$*, and there exist* $B',C'\in K(x)$ *such that the map* $\left(\left(\begin{smallmatrix}0&b\\c&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&B'\\C'&0\end{smallmatrix}\right)\right)$ *is* $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$*-invariant.*

Proof. Up to multiplying by a common denominator, we can assume that A, B, C, D are polynomials. If $A\neq 0$, we can assume $B=C=0$ by Lemma 3.33(1). By Lemma 3.33(5), either $b=c=0$ and we are in the diagonal case, or $a=d=0$. In the latter case, Lemma 3.33(3) implies that $\left(\left(\begin{smallmatrix}0&b\\c&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&xA\\D&0\end{smallmatrix}\right)\right)$ is invariant, and we are in the antidiagonal case.

If $B\neq 0$, we proceed similarly: We can assume $A=D=0$ and find that either $a=d=0$ and we are in the antidiagonal case, or $b=c=0$. In the latter case, the matrices are of the form $\left(\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right),\left(\begin{smallmatrix}0&B\\C&0\end{smallmatrix}\right)\right)$. Lemma 3.33(3) implies that the antidiagonal $\left(\left(\begin{smallmatrix}0&d\\a&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&xC\\B&0\end{smallmatrix}\right)\right)$ is invariant, but then $\left(\left(\begin{smallmatrix}d&0\\0&a\end{smallmatrix}\right),\left(\begin{smallmatrix}xC&0\\0&B\end{smallmatrix}\right)\right)$ also is, by Lemma 3.33(2). Note that this corresponds to the birational map $f\colon(x,y)\longmapsto(dx/a,xC(x)y/B)$. Finally, conjugating this with the Galois-invariant $(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$ (Lemma 3.33(4)), that is, the map $\iota\colon(x,y)\longmapsto(1/x,1/y)$, we find

$$\iota\circ f\circ\iota(x,y)=\iota(f(1/x,1/y))=\iota\left(\frac{d}{ax},\frac{C(1/x)}{xB(1/x)y}\right)=\left(\frac{ax}{d},\frac{xB(1/x)y}{C(1/x)}\right).$$

So $\left(\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right),\left(\begin{smallmatrix}B(1/x)x&0\\0&C(1/x)\end{smallmatrix}\right)\right)$ is $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant, and we are in the diagonal case again. □

We are now ready for the proof of Proposition 3.25.

Proof of Proposition 3.25. First of all, observe that maps of the form $f(x,y)=(1/\mu x,1/\lambda x)$ are indeed involutions and that the conditions on λ,μ assert that f is g_γ-invariant (see Lemma 3.26), and also h_γ-invariant if $L\neq L'$ (see Lemma 3.27). So indeed $\varepsilon H_{L,L'}\varepsilon^{-1}\subset\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2,\pi_{L,L'})$.

By Corollary 3.23, we have

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2,\pi)\simeq(\mathrm{PGL}_2(K)\ltimes\mathrm{PGL}_2(K(x)))^{\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}}\,,$$

and $\mathrm{Aut}_{\mathbf{k}}^{\pi_{L,L'}}(\mathbb{P}^1)$ corresponds to the image of the projection onto $\mathrm{PGL}_2(K)$. Corollary 3.34 gives that diagonal and antidiagonal elements surject onto the image of the projection. The diagonal elements are generated by antidiagonal ones, namely

$$\left(\left(\begin{smallmatrix}\mu & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}P(x) & 0\\ 0 & 1\end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\right) \circ \left(\left(\begin{smallmatrix}0 & 1\\ \mu & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ P(x) & 0\end{smallmatrix}\right)\right)$$

is a composition of two $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant antidiagonal elements; see Lemma 3.33(2) and (4). Corollary 3.31 describes the conditions on the entries of the antidiagonal matrices $\left(\left(\begin{smallmatrix}0 & 1\\ a & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ P(x) & 0\end{smallmatrix}\right)\right)$ to be $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant. In particular, it says that evaluating P at any $a \in \mathbf{k}^*$ gives again an $\varepsilon\,\mathrm{Gal}(K/\mathbf{k})\varepsilon^{-1}$-invariant antidiagonal matrix $\left(\left(\begin{smallmatrix}0 & 1\\ a & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ P(a) & 0\end{smallmatrix}\right)\right)$. Setting $P(a) = \lambda \in K$ gives the conditions $\mu^g = \mu$, $\lambda\lambda^g = \mu$, and if $L \neq L'$ also $\mu\mu^h = 1 = \lambda\lambda^h$. Note that $\mu = \lambda\lambda^g$ implies $\mu^g = \mu$, and so $\lambda\lambda^h = 1$ implies $\mu\mu^h = 1$. Hence, $H_{L,L'}$ surjects onto the image of the projection to $\mathrm{PGL}_2(K)$, and the statement follows. □

4. Irreducible elements of Del Pezzo type

In this last section, we study irreducible elements of Del Pezzo type, whose possible factorization types in terms of Sarkisov links were given in Table 2.2. First, we identify some "easy cases" where we can lower the Sarkisov length by composing with a quadratic, Geiser, or Bertini involution. Then, for the remaining "hard cases," we produce factorizations into involutions by using elementary relations between Sarkisov links. Most of these factorizations rely on an assumption about the existence of points in general position, and in Section 4.3 we show that these conditions are satisfied. Finally, in Section 4.4, we put everything together and finish the proof of Theorem 1.1.

4.1 Easy simplifications

First, we discuss quadratic involutions, using the following result.

Lemma 4.1. *Let $f \in \mathrm{Bir}_{\mathbf{k}^a}(\mathbb{P}^2)$ be a quadratic map with three proper base points p_1, p_2, p_3. For each $\{i,j,k\} = \{1,2,3\}$, let $q_k = f(L_{ij})$, where $L_{ij} \subset \mathbb{P}^2$ is the line through p_i, p_j. Then:*

1. *There exists an $\alpha \in \mathrm{Aut}_{\mathbf{k}^a}(\mathbb{P}^2)$ that sends q_i to p_i for each $i = 1,2,3$.*
2. *For any such automorphism α, the quadratic map $\alpha \circ f$ is an involution.*
3. *If the set $\{p_1, p_2, p_3\}$ is invariant under the Galois action, then in part (1) we can choose α in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$.*

Proof. Assertion (1) is just the classical fact that $\mathrm{PGL}_3(\mathbf{k}^a)$ acts transitively on triple of non-collinear points. For assertion (3), see [Sch22, Lemma 6.11].

To prove assertion (2), up to conjugation we can assume $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the blow-up of p_1, p_2, p_3. Since by construction $\alpha \circ f$ has the same base points as its inverse, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, dashed, "\alpha\circ f"] & \mathbb{P}^2 \end{tikzcd}$$

with $g \in \mathrm{Aut}_{\mathbf{k}^a}(X)$. The automorphism group of the Del Pezzo surface X of degree 6 is well known [Dol12, Theorem 8.4.2]:

$$\mathrm{Aut}_{\mathbf{k}^a}(X) = (\mathbb{G}_m)^2 \rtimes (S_3 \times \mathbf{Z}/2)\,,$$

where $(\mathbb{G}_m)^2$ is the standard toric action $[ax : by : z]$ with $a, b \in \mathbf{k}^{a*}$, the symmetric group S_3 permutes the homogeneous coordinates and the generator of $\mathbf{Z}/2$ is the standard quadratic involution $\sigma = [yz : xz : xy]$. In particular, the action of g on the six (-1)-curves of X uniquely determines g up to composition by an element of $(\mathbb{G}_m)^2$. Finally, since $p_k = \alpha \circ f(L_{ij})$, we observe that g has the same action as σ and that for any $(a, b) \in (\mathbb{G}_m)^2$, the map $[ayz : bxz : xy]$ is an involution, which ends the proof. □

Corollary 4.2. *Let $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a map of minimal factorization type*

(Q1) $\mathbb{P}^2 \overset{3\ 3}{\dashrightarrow} \mathbb{P}^2$ *or*

(Q2) $\mathbb{P}^2 \overset{2\ 1}{\dashrightarrow} \mathcal{D}_8 \overset{1\ 2}{\dashrightarrow} \mathbb{P}^2$.

Then there exists an $\alpha \in \operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ such that $\alpha \circ f$ is a quadratic involution.

Proof. In the case (Q1), we can directly apply Lemma 4.1. In the case (Q2), we take $p = \{p_1, p_2\}$ to be the 2-point blown up by the first link, and p_3 to be the rational point on $\mathbb{P}^2$ corresponding to the point blown up by the second link. Note that they are not collinear. Again, we conclude by Lemma 4.1. □

Now we discuss Geiser and Bertini involutions; see [Dol12, Sections 8.7.2 and 8.8.2] for details. Let X be a Del Pezzo surface of degree 1 (respectively, of degree 2). Then the linear system $|-2K_X|$ (respectively, $|-K_X|$) corresponds to a 2-to-1 morphism $X \longrightarrow \mathbb{P}^2$, and the deck transformation $\sigma\colon X \longrightarrow X$ is an automorphism of X called a Bertini involution (respectively, a Geiser involution). The involution σ acts on the Néron–Severi space $\operatorname{Pic}(X) \otimes \mathbf{R}$ preserving the class of K_X, and it acts as minus the identity on $(K_X)^\perp$.

Lemma 4.3. *Let X be a Del Pezzo surface of degree 1 (respectively, of degree 2), and let $\sigma \in \operatorname{Aut}(X)$ be the Bertini involution (respectively, the Geiser involution). Then for any collection $\{C_i\}_{i \in I}$ of rational curves with $C_i^2 \in \{-1, 0\}$ and $C_i \cdot C_j = 0$ for $i \neq j$, we have $\sigma(C_i) \neq C_j$ for all $i \in I$.*

Proof. As mentioned above, the class of a curve $C \in \operatorname{Pic}(X)$ is fixed by σ if and only if C is a multiple of $-K_X$. In particular, any curve that is fixed by σ must have positive self-intersection. So the assumption $C_i^2 \leqslant 0$ implies directly that $\sigma(C_i) \neq C_i$ for $i = 1, \ldots, r$. Now to obtain a contradiction, assume $\sigma(C_i) = C_j$ for some indexes $i \neq j$. Then $C_i + C_j$ is fixed by the involution σ, and since $C_i \cdot C_j = 0$ by assumption, we get $(C_i + C_j)^2 = C_i^2 + C_j^2 \leqslant 0$, hence $C_i + C_j$ is not fixed by σ, giving a contradiction. □

Corollary 4.4. *Let $\mathcal{P}$ be the piece associated with a Del Pezzo surface of degree 1 (respectively, of degree 2). Then the Bertini involution (respectively, the Geiser involution) acts on the piece without fixing any proper face. If the piece is 1- or 2-dimensional, this means that the involution acts on the piece as $-\mathrm{id}$ ("central symmetry").*

Proof. Let X be a Del Pezzo surface of degree 1 or 2, of Picard rank $n \geqslant 2$. So X/pt is a rank n fibration giving rise to an $(n-1)$-dimensional piece. Let $r \leqslant n$, and let Y/B be a rank r fibration that is dominated by X, giving rise to an $(r-1)$-dimensional face. For $r = n$, there is nothing to prove. Now assume $r < n$. Consider all the rank $n-1$ fibrations Z_j/B_j that dominate Y and are dominated by X, with dominating maps $f_j\colon X \longrightarrow Z_j$. Consider the set $\{C_{j,i}\}_{i \in I_j}$ of curves on X that are contracted by f_j if $B_j = \mathrm{pt}$, or by $\pi_j \circ f_j$, where $\pi_j\colon Z_j \longrightarrow B_j$, if $B_j = \mathbb{P}^1$. So the set of $C_{j,i}$ is either a Galois orbit of pairwise disjoint (-1)-curves or the transform of fibers of $Z_j \longrightarrow B_j$. Now take the union $\bigcup_j \{C_{j,i}\}_{i \in I_j}$. The curves satisfy $C_{ji} \cdot C_{j'i'} = 0$ for $C_{ji} \neq C_{j'i'}$. So the curves $\{C_{ji}\}$ are as in Lemma 4.3. Hence, the Bertini, respectively Geiser, involution does not fix Y. This proves that no proper face is fixed by the involution.

If the piece has dimension $n = 1$ or 2, the combinatorial piece can be embedded into $\mathbf{R}^n$ as an interval or a regular polygon centered at the origin, so that any combinatorial bijection comes from the restriction of a Euclidean isometry of $\mathbf{R}^n$. Since $-\mathrm{id}$ is the only Euclidean involution preserving the polygon that does not fix any proper face, we get the result. □

We say that $\iota \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is conjugate to a Bertini involution (respectively, to a Geiser involution) if there exists a birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow X$, where X is a Del Pezzo surface of degree 1 (respectively, of degree 2) such that $\iota = \varphi^{-1}\sigma\varphi$, where $\sigma\colon X \longrightarrow X$ is the Bertini involution (respectively, the Geiser involution) on X. The following result explains the meaning of "Bertini simplification" or "Geiser simplification" in the last column of Table 2.2.

PROPOSITION 4.5. *Let $f \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ be an irreducible map of minimal factorization type one of the following:*

(G1) $\mathbb{P}^2 \overset{7\,7}{\dashrightarrow} \mathbb{P}^2$,

(G2) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{6\,6}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(G3) $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{3\,3}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$,

(G4) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{4\,4}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(B1) $\mathbb{P}^2 \overset{8\,8}{\dashrightarrow} \mathbb{P}^2$,

(B2) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{7\,7}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(B3) $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{4\,4}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$,

(B4) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{5\,5}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$.

Then we can write $f = g \circ \iota$, where ι is conjugate to a Geiser or Bertini involution and $\mathrm{sl}(g) < \mathrm{sl}(f)$.

Proof. The diagram below explains the proof for the case (G3). By the minimality of the factorization, we have $\mathrm{sl}(f) = 3$.

$$\begin{tikzcd}
 & X_2 \arrow[r, "\sigma"] \arrow[d, "3"'] & X_2 \arrow[r, dashed, "\varphi^{-1}"] \arrow[d, "3"] & \mathbb{P}^2 \arrow[dd, dashed, bend left, "g \text{ with } \mathrm{sl}(g)\leqslant 2"] \\
\mathbb{P}^2 \arrow[ur, dashed, "\varphi"] \arrow[r, dashed, "5\,1"] \arrow[urrr, dashed, bend left, "\iota = \varphi^{-1}\sigma\varphi"] \arrow[drrr, dashed, bend right, "f"'] & X_5 \arrow[r, dashed, "3\,3"] & X_5 \arrow[ur, dashed, "1\,5"'] \arrow[dr, dashed, "1\,5"] & \\
 & & & \mathbb{P}^2
\end{tikzcd}$$

Here $\sigma\colon X_2 \longrightarrow X_2$ is the Geiser involution on the fibration of rank 2 dominating the link $X_5 \overset{3\,3}{\dashrightarrow} X_5$; see Corollary 4.4. The cases (G2), (G4), (B2), (B3), (B4) are similar.

For the case (G1), we have to slightly adapt the argument since here g will be an automorphism that we usually keep hidden in the definition of Sarkisov link "up to isomorphism." We use the diagram

$$\begin{tikzcd} X_2 \arrow[r, "\sigma"] \arrow[d, "7"'] & X_2 \arrow[d, "7"] & \\ \mathbb{P}^2 \arrow[r, dashed, "7\,7", "\iota"'] \arrow[rr, dashed, bend right, "f"'] & \mathbb{P}^2 \arrow[r, dashed, "g\in\mathrm{Aut}\left(\mathbb{P}^2\right)"] & \mathbb{P}^2 , \end{tikzcd}$$

where again σ is the Geiser involution on the Del Pezzo surface X_2. The argument for the case (B1) is similar. □

4.2 Remaining cases

For the remaining cases of Table 2.2, we show that any map f with factorization type in the following list is generated by involutions, provided that certain generality conditions are satisfied:

(i) $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$,

(ii) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(iii) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{4\,4}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(iv) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{3\,3}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(v) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{2\,2}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(vi) $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{2\,5}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$,

(vii) $\mathbb{P}^2 \overset{6\,6}{\dashrightarrow} \mathbb{P}^2$.

If there exist an a-point and a b-point on $X \in \mathcal{D}$ such that the two points are in general position, we denote by $\mathcal{P}\big(X; a, b\big)$ the piece of the corresponding relation involving the two links starting at X associated with the blow-up of the a-point, respectively the b-point. (This is coherent with the notation used in Appendix B.)

Lemma 4.6 (Case (i)). *Let f be a map of minimal factorization type $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$. The following holds:*

(1) *Assume that the two rational points on X_5 are general. Then the map f is a composition of two maps of Jonquières type* 1 (*and an automorphism of $\mathbb{P}^2$ at the end*).

(2) *Assume that there is a 2-point on X_5 that is general with the two rational points. Then the map f is a composition of two Geiser involutions from Del Pezzo surfaces with Picard rank* 3 *and a quadratic map* (Q2).

Proof. Since the factorization is minimal, the two rational points on X_5 are distinct. The assumption in part (1) implies that the map f lies inside the piece $\mathcal{P}\big(X_5; 1, 1\big) = \mathcal{P}\big(\mathbb{P}^2; 1, 5\big)$. It can be glued with the piece $\mathcal{P}\big(\mathbb{P}^2; 1, 1\big)$ along the edge corresponding to the exchange of the rulings of $\mathbb{F}_0$. This gives a decomposition of f into two maps of Jonquières type 1, up to an automorphism; see Figure 4.1.

By the assumption in part (2), the map f fits into two pieces $\mathcal{P}\big(X_5; 1, 2\big) = \mathcal{P}\big(\mathbb{P}^2; 2, 5\big)$ glued along the edge $\mathcal{D}_5 \overset{2\,5}{\dashrightarrow} \mathcal{D}_8$ that meets the factorization of f. By Corollary 4.4, the central symmetry of $\mathcal{P}\big(\mathbb{P}^2; 2, 5\big)$ corresponds to a Geiser involution. Hence, the map f is given by the Geiser involution of the first piece, followed by a quadratic map (Q2) followed by the Geiser involution of the second piece. □

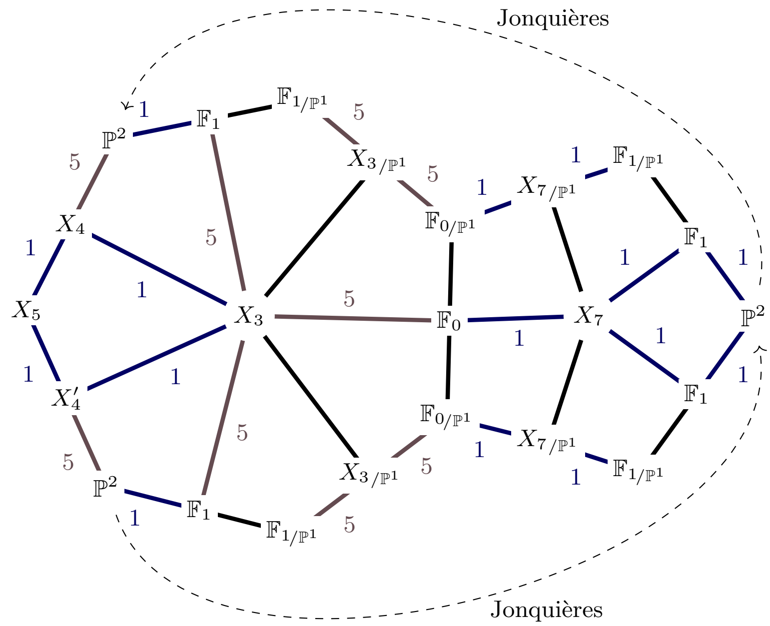

Figure 4.1. The two pieces of Lemma 4.6(1)

Lemma 4.7 (Case (ii)). *Let f be a map of minimal factorization type $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$. Assume that the two rational points on X_6 are general. Then f is the composition of two quadratic maps (Q2) and one quadratic map (Q1).*

Proof. By the minimality of the factorization, the two rational points on X_6 are distinct, hence general by the assumption. This implies that the middle part of f, namely $\mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8$, is inside the piece $\mathcal{P}(X_6; 1, 1) = \mathcal{P}(\mathbb{P}^2; 2, 3)$. Hence, the map f is given by a quadratic map (Q2), followed by a quadratic map (Q1), followed by another quadratic map (Q2). □

Lemma 4.8 (Case (iii)). *Let f be a map of minimal factorization type $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{4\,4}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$.*

(1) *Assume that there exists a rational point on X_8 that is general with the 4-point. Then the map f is the product of (at most) two quadratic maps (Q2) and a map of fibering type in $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2, \pi_F)$, where $F/\mathbf{k}$ is the residue field of the 4-point.*

(2) *Assume that there is a 2-point that is general with the 4-point on X_8. Then the map f is the product of a Geiser involution from a Del Pezzo surface of Picard rank 3 and a map of Jonquières type $2 + 2$ in $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_{L,L'}\right)$ (and an automorphism of $\mathbb{P}^2$), where $L/\mathbf{k}$ is the residue field of the 2-point on $\mathbb{P}^2$ and $L'/\mathbf{k}$ is the residue field of the 2-point on X_8.*

(3) *Assume that there exists a 3-point on X_8 that is general with the 4-point. Then the map f is the product of a Bertini involution from a Del Pezzo surface of Picard rank 3 and a Geiser simplification (G4).*

Proof. For assertion (1), observe that the assumption implies that the middle part of f, namely $\mathcal{D}_8 \overset{4\,4}{\dashrightarrow} \mathcal{D}_8$, fits into a piece $\mathcal{P}(X_8; 1, 4) = \mathcal{P}(\mathbb{P}^2; 2, 4)$. Two opposite vertices of this piece are $\mathbb{P}^2$, and the map between these two $\mathbb{P}^2$ is of fibering type 4 with residue field F. The beginning and the end of f, that is, $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8$ and $\mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$, are not necessarily contained in the piece. However, composing it with a quadratic map (Q2) at the beginning and/or the end gives the statement.

By the assumption in assertion (2), the middle part of f is part of a piece $\mathcal{P}\big(X_8; 2, 4\big)$, whose central symmetry is a Geiser involution (Corollary 4.4). So f can be written as a Geiser involution followed by a map $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} X_8 \overset{2}{\dashrightarrow} \mathcal{D}_6 \overset{4\,4}{\dashrightarrow} \overset{2}{\longrightarrow} X_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$, which is a map of Jonquières type $2+2$ in $\mathrm{Bir}_{\mathbf{k}}\left(\mathbb{P}^2, \pi_{L,L'}\right)$ (after composing with an automorphism of $\mathbb{P}^2$).

By the assumption in assertion (3), the middle part of f is part of a piece $\mathcal{P}\big(X_8; 3, 4\big)$. Observe that by Corollary 4.4, its central symmetry is a Bertini involution. So f can be decomposed into this Bertini involution and a Geiser simplification (G4). □

Lemma 4.9 (Case (iv)). *Let f be a map of minimal factorization type*

$$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{3\,3}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2\,.$$

Assume that there exists a rational point on X_6 that is general with the 3-point on X_6. Then the map f is the composition of two maps of the form (ii) and a Geiser involution.

Proof. The assumption implies that the middle part of f, namely $\mathcal{D}_6 \overset{3\,3}{\dashrightarrow} \mathcal{D}_6$, is part of a piece $\mathcal{P}\big(X_6; 1, 3\big)$. By Corollary 4.4, the central symmetry of the piece is a Geiser involution. Choosing a rational point on a $\mathcal{D}_8$ in the piece gives an edge $\mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$ going out of the piece, and the central Geiser involution gives an opposite edge to $\mathbb{P}^2$. Therefore, the map f is the product of a Geiser involution and maps of the form (ii) at the beginning and/or at the end. □

Lemma 4.10 (Case (v)). *Let f be a map of minimal factorization type*

$$\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{2\,2}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2\,.$$

(1) *Assume that there is a rational point on X_6 that is general with the 2-point on X_6. Then the map f is the composition of (at most) two maps of the form (ii) and a map of fibering type, more precisely a map in $\mathrm{Bir}_{\mathbf{k}}\big(\mathbb{P}^2, \pi_{L,L'}\big)$, where $L/\mathbf{k}$ is the residue field of the 2-point in X_6 and $L'/\mathbf{k}$ is the degree 2 extension coming from the link $\mathbb{P}^2 \dashrightarrow X_8$.*

(2) *Assume that there is a 3-point on X_6 that is general with the 2-point on X_6. Then the map f is the composition of a Bertini involution of a Del Pezzo surface of Picard rank 3 and (if needed) a map of the form (iv).*

Proof. The assumption in assertion (1) implies that the middle part of f, namely $\mathcal{D}_6 \overset{2\,2}{\dashrightarrow} \mathcal{D}_6$, is part of a piece $\mathcal{P}\big(X_6; 1, 2\big) = \mathcal{P}\big(X_8; 2, 3\big)$. We choose a rational point on one of the $\mathcal{D}_8$ in the piece, obtaining an edge to $\mathbb{P}^2$ going out of the piece. On the opposite side of the piece, we can choose an edge to $\mathbb{P}^2$ such that the induced map of the piece between the opposite copies of $\mathbb{P}^2$ is of fibering type. Hence, by composing this map, if needed, with a map of the form (ii) at the beginning and/or the end, statement (1) follows.

The assumption in assertion (2) implies that the middle part of f is part of a piece $\mathcal{P}\big(X_6; 2, 3\big)$. By Corollary 4.4, the central symmetry of the piece is a Bertini involution. Hence, the map f is the composition of the central Bertini involution, followed by a map of the form (iv). □

Lemma 4.11 (Case (vi)). *Let f be a map of factorization type $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{2\,5}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$. Assume that there is a rational point on X_5 that is general with the 2-point on X_5. Then f is the product of a Geiser involution of a Del Pezzo surface of Picard rank 3 and (if needed) maps of the form (Q2) and (i).*

Proof. The assumption implies that the middle part of f, namely $\mathcal{D}_5 \overset{2\ 5}{\dashrightarrow} \mathcal{D}_8$, is part of a piece $\mathcal{P}(X_5;1,2) = \mathcal{P}(\mathbb{P}^2;2,5)$. By Corollary 4.4, the central symmetry of the piece is a Geiser involution. Hence, f can be written as the product of a Geiser involution and, if needed, a map of the form (i) at the beginning and a quadratic map (Q1) at the end. □

Lemma 4.12 (Case (vii)). *Let f be a link $\mathbb{P}^2 \overset{6\ 6}{\dashrightarrow} \mathbb{P}^2$.*

(1) *Assume that there exists a rational point on $\mathbb{P}^2$ that is general with the 6-point on $\mathbb{P}^2$. Then f is the composition of a Geiser involution, a map of Jonquières type 1, and an automorphism of $\mathbb{P}^2$.*

(2) *Assume that there exists a 2-point on $\mathbb{P}^2$ that is general with the 6-point on $\mathbb{P}^2$. Then, the map f is the composition of a Bertini involution of a Del Pezzo surface of Picard rank 3 and a Geiser simplification (G2).*

Proof. The assumption in assertion (1) implies that f is part of a piece $\mathcal{P}(\mathbb{P}^2;1,6)$ whose central symmetry is a Geiser involution. Hence, f can be decomposed into a Geiser involution and a map of Jonquières type 1 (and possibly an automorphism of $\mathbb{P}^2$).

The assumption in assertion (2) implies that f is part of a piece $\mathcal{P}(\mathbb{P}^2;2,6)$. Note that the central symmetry is a Bertini involution. Hence, doing first the central Bertini involution and then the Geiser simplification (G2) along the edge $\mathcal{D}_8 \overset{6\ 6}{\dashrightarrow} \mathcal{D}_8$ gives the statement. □

4.3 General position

In this section, we show that the assumptions of general position in the lemmas in Section 4.2 are satisfied, at least for the first part of the respective lemmas. As always, the ground field $\mathbf{k}$ is assumed to be an arbitrary perfect field.

Lemma 4.13 (Cases (i) and (ii)). *Let $X \in \mathcal{D}_5 \cup \mathcal{D}_6$. Then, any two rational points on X are general.*

Proof. See Appendix A and Lemmas A.6 and A.8. □

Lemma 4.14 (Case (iii), Lemma 4.8(1)). *Let $p \in X_8$ be a general 4-point, where $X_8 \in \mathcal{D}_8$. Then, any rational point $q \in X_8$ is general with p.*

Proof. Recall that five points on $\mathbb{P}^1 \times \mathbb{P}^1$ are general if and only if no two lie on the same vertical or horizontal curve and no four lie on a diagonal (see for example [Sch21, Lemma 4.3]). Denote by s the horizontal curve through q and by f the vertical curve through q. Observe that s and f form a Galois orbit. If p_1 lies on $s \cup f$, then all four p_i lie on $s \cup f$, hence (at least) two of the p_i lie on the same vertical or horizontal line, contradicting the generality of p.

If q lies on the diagonal through three of the p_i, then q lies on all the diagonals through three of the four p_i. Since the diagonal through three points is unique, this means that all these diagonals are equal and so that p lies on a diagonal, again contradicting the generality of p. □

Lemma 4.15 (Case (iv), Lemma 4.9). *Assume that the ground field $\mathbf{k}$ is not the field with two elements, and let $X_6 \in \mathcal{D}_6$. For any general 3-point $p \in X_6$, there exists a rational point q on X_6 that is general with p.*

Proof. Consider the blow-up $X_3 \longrightarrow X_6$ of X_6 at p. It is enough to see that there is a rational point that does not lie on any of the 27 (-1)-curves of the cubic surface X_3. Since each (-1)-curve contains at most one rational point (if it contains two, it is defined over $\mathbf{k}$ since two lines intersect at most once), this is clear if $\mathbf{k}$ is infinite.

Now assume that $\mathbf{k}$ is finite. We have (see Remark A.2)

$$|X_3(\mathbf{k})| = |X_6(\mathbf{k})| = |\mathbf{k}|^2 - |\mathbf{k}| + 1\,.$$

In the following, we give a bound on the number of rational points that lie on the 27 (-1)-curves of X_3 and see that this number is smaller than $|X_3(\mathbf{k})|$. Assume that $p \in X_3(\mathbf{k})$ lies on a (-1)-curve that is defined over $\mathbf{k}^a$, and let $E_1, \ldots, E_d$ be its Galois orbit. Since p is a rational point, it lies on each E_i. The 27 (-1)-curves on X_3 are such that at most three lines intersect in one point. Hence, $d \leqslant 3$. Since the link given by p is of the form $X_6 \overset{3\,3}{\dashrightarrow} X_6'$, the only Galois orbits of pairwise disjoint (-1)-curves on X_3 are two Galois orbits of size 3. In particular, there is no (-1)-curve defined over $\mathbf{k}$, and so $d \geqslant 2$. The exceptional divisors of the two points of degree 3 that come from the link give six (-1)-curves that do not contain a rational point. The hexagon of the six (-1)-curves on X_6 is one Galois orbit (in [SZ21, Figure 1, Section 4.1] all configurations of the hexagon give rise to a birational morphism except when the hexagon forms one Galois orbit). Similarly, the hexagon of the six (-1)-curves on X_6' also forms one Galois orbit, and its strict transform on X_3 is different from the hexagon on X_6.

Hence, at most $27 - 18 = 9$ (-1)-curves contain a rational point. Since each (-1)-curve contains at most one rational point and each rational point lies on at least two (-1)-curves, the total number of rational points lying on the set of (-1)-curves is at most $9/2$, so it is at most 4. We find

$$|\mathbf{k}|^2 - |\mathbf{k}| + 1 - 4 \geqslant 3\,. \qquad \square$$

Remark 4.16. Lemma 4.15 does not hold for $\mathbf{k} = \mathbb{F}_2$: The blow-up of the two general 3-points on X_6 found in [Sch21, Lemma 4.33(2)] gives a cubic surface such that all three rational points of X_6 lie on (-1)-curves. Hence, Lemma 4.9 is not applicable. Moreover, there is also no 2-point that is general with the 3-point, and so also Lemma 4.10(2) is not applicable, which would have allowed us to decompose a link of the form $\mathcal{D}_6 \overset{3\,3}{\dashrightarrow} \mathcal{D}_6$ into a Bertini involution and a link $\mathcal{D}_6 \overset{2\,2}{\dashrightarrow} \mathcal{D}_6$. Therefore, this approach fails to give a decomposition of maps of the form (iv) into involutions. Luckily, it was already proved in [Sch21, Corollary 1.3] that over $\mathbf{F}_2$, these maps are in fact involutions.

LEMMA 4.17. *Let $p \in X_8$ be a general 2-point, and let $q \in X_8$ be a general 3-point. Assume that the diagonal through q does not contain any of the components of p. Then q is general with p.*

Proof. The Galois orbit of the horizontal and vertical curves containing p has four components. Therefore, none of the vertical or horizontal curves through one of the p_i contains any of the q_i since the latter is a component of a 3-point.

Assume that two points of q, say q_2, q_3, lie on one diagonal with p; call it D. Note that since $\mathrm{Gal}(L/\mathbf{k})$ is a transitive subgroup of S_3, it contains the cyclic group, where $L/\mathbf{k}$ is the splitting field of q. Let σ be the generator of the cyclic group in $\mathrm{Gal}(L/\mathbf{k})$. So $\sigma(D)$ contains p as well as q_1, q_3, and therefore $D = \sigma(D)$ is the diagonal through q, giving a contradiction. $\square$

LEMMA 4.18 (Case (v), Lemma 4.10(1)). *Let p be a general 2-point on $X_6 \in \mathcal{D}_6$. Then any rational point on X_6 is general with p.*

Proof. Let $q \in X_6$ be a rational point, and denote the image of p under the link $X_6 \overset{1\,3}{\dashrightarrow} X_8$ again by p. Write $q = \{q_1, q_2, q_3\} \subset X_8$ for the contracted 3-point. In particular, q as well as p are general points in X_8. Note that the diagonal through q is the exceptional divisor of the link. Since the diagonal through the three geometric points of q is contracted by the link, it does not contain any of the p_i. Lemma 4.17 implies the statement. $\square$

Lemma 4.19 (Case (vi), Lemma 4.11). *Let $p \in X_5$ be a general 2-point, where $X_5 \in \mathcal{D}_5$. Then any rational point $q \in X_5$ is general with p.*

Proof. Consider the link $X_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$ given by the rational point q, denote by r the 5-point in $\mathbb{P}^2$ (this is a general point), and denote the image of p in $\mathbb{P}^2$ again by p. Note that p on X_5 is general with q if and only if p on $\mathbb{P}^2$ is general with r because the surface obtained by blowing up p and q on X_5 is isomorphic to the one obtained by blowing up p and r on $\mathbb{P}^2$. Note that since the exceptional divisor of q is sent onto the conic through the five geometric components of r, the point p does not lie on this conic. Lemma A.5 implies the statement. □

Lemma 4.20 (Case (vii), Lemma 4.12(1)). *Let $p \in \mathbb{P}^2$ be a general 6-point. Then, all rational points except finitely many are general with p. Moreover, if the ground field $\mathbf{k}$ is finite, then all rational points except at most one are general with p.*

Proof. Let r be a rational point. We first show that it does not lie on any conic through five components of p. Assume that r lies on the conic through all components of p except p_i, and call that conic C_i. Let $\sigma \in \mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ be such that $\sigma(p_i) = p_j \neq p_i$. So $\sigma(C_i) = C_j$ contains q and all components of p except p_j. So C_i and C_j have the five points q and $p \setminus \{p_i, p_j\}$ in common and are therefore equal. This contradicts the generality of p.

The lines L_{ij} through two components p_i, p_j of p contain at most one rational point (if a line contains two, then it is defined over $\mathbf{k}$, which is not possible). Therefore, there are only finitely many rational points that are collinear with two components of p. (In fact, $\binom{6}{2} = 15$ gives an upper bound.)

Now assume that $\mathbf{k}$ is finite. We order the components of p cyclically and denote by σ the generator of $\mathrm{Gal}(L/\mathbf{k})$, where L is the extension of $\mathbf{k}$ of degree 6. The lines L_{ij} form three orbits; namely, those of L_{12} and L_{13} are both of size 6, and that of L_{14} is of size 3. If $q \in L_{12}$, then $q \in \sigma(L_{12}) = L_{23}$, so $L_{12} = L_{23}$ and so p_1, p_2 and p_3 are collinear, contradicting the generality of p. If $q \in L_{13}$, then $q \in \sigma^2(L_{13}) = L_{35}$ and so p_1, p_3, and p_5 are collinear, again giving a contradiction. If $q \in L_{14}$, then $q \in L_{25} \cap L_{36}$, and so q is the unique intersection point of these three lines. □

4.4 Proof of Theorem 1.1

We start from Proposition 3.7, and we want to show that each of the generators is contained in the subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ generated by involutions.

This is Lemma 3.8 for $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ and $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2, \pi_\times)$, and it is Corollary 3.21 for a Jonquières group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2, \pi_F)$ of type 4. For a Jonquières group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2, \pi_{L,L'})$ of type $2+2$, this follows from Propositions 3.19 and 3.25.

We are left with the case of an irreducible elements f of Del Pezzo type with $1 \leqslant \mathrm{sl}(f) \leqslant 5$. Assume for a contradiction that there exists such an element that is not a composition of involutions, and assume that $\mathrm{sl}(f)$ is minimal for this property. Then the factorization type of a minimal factorization of f is not one of the 10 cases covered by Corollary 4.2 and Proposition 4.5, and we are left with the seven cases studied in Sections 4.2 and 4.3:

(i) $\mathbb{P}^2 \overset{5\,1}{\dashrightarrow} \mathcal{D}_5 \overset{1\,5}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.6(1) and 4.13),

(ii) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.7 and 4.13),

(iii) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{4\,4}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.8(1) and 4.14),

(iv) $\mathbb{P}^2 \overset{2\,1}{\dashrightarrow} \mathcal{D}_8 \overset{3\,1}{\dashrightarrow} \mathcal{D}_6 \overset{3\,3}{\dashrightarrow} \mathcal{D}_6 \overset{1\,3}{\dashrightarrow} \mathcal{D}_8 \overset{1\,2}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.9 and 4.15 if $|\mathbf{k}| \geqslant 3$; for $\mathbf{k} = \mathbb{F}_2$, see [Sch21, Corollary 1.3]),

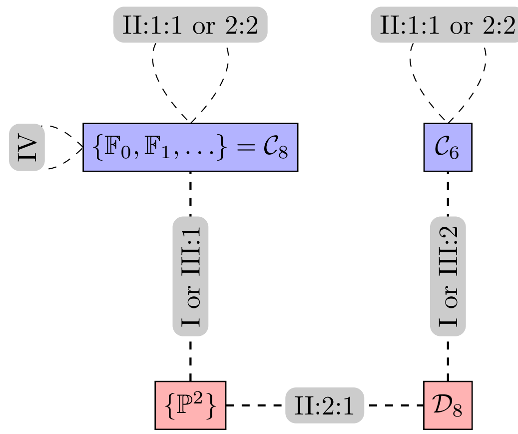

Figure 4.2. Sarkisov links between rational surfaces over $\mathbf{R}$

(v) $\mathbb{P}^2 \overset{2\ 1}{\dashrightarrow} \mathcal{D}_8 \overset{3\ 1}{\dashrightarrow} \mathcal{D}_6 \overset{2\ 2}{\dashrightarrow} \mathcal{D}_6 \overset{1\ 3}{\dashrightarrow} \mathcal{D}_8 \overset{1\ 2}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.10(1) and 4.18),

(vi) $\mathbb{P}^2 \overset{5\ 1}{\dashrightarrow} \mathcal{D}_5 \overset{2\ 5}{\dashrightarrow} \mathcal{D}_8 \overset{1\ 2}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.11 and 4.19),

(vii) $\mathbb{P}^2 \overset{6\ 6}{\dashrightarrow} \mathbb{P}^2$ (see Lemmas 4.12(1) and 4.20).

In each case, we obtain a factorization into involutions (using automorphisms, elements of Jonquières type, quadratic, Geiser, and Bertini involutions), which gives the expected contradiction and finishes the proof of the theorem. □

We can be slightly more precise and give the following involutive generating set.

Proposition 4.21. *The group* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *is generated by linear involutions, involutions of Jonquières type* 1, $2+2$, *and* 4, *quadratic involutions, and maps conjugate to a Geiser or Bertini involution on a Del Pezzo surface of Picard rank* 2 *or* 3.

Proof. In the proof of Theorem 1.1, it is enough to give the composition into involutions of the irreducible maps of Del Pezzo type, which the mentioned lemmas provide except in the case of (G1)–(B4).

For the Geiser and Bertini simplifications $f = g \circ \iota$ in Proposition 4.5, the map $\iota \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is conjugate to a Geiser (or Bertini) involution on a Del Pezzo surface of degree 2 or 1 that has Picard rank 2, and g has factorization type (Q2), (i), or (ii). It is enough to observe that g either is an automorphism or has a minimal factorization (Q1), (Q2), (i), or (ii), as follows.

First, assume that g has $\mathrm{sl}(g) \neq 0$ and factorization type (Q2) or (i). If it is not minimal, then $\mathrm{sl}(g) = 1$, and hence g has a minimal factorization $\mathbb{P}^2 \overset{d\ d}{\dashrightarrow} \mathbb{P}^2$ with a d-point as base point. By Figure 2.1, we have $d \in \{3, 6, 7, 8\}$, but g does not have any such d-point as base point, giving a contradiction. Moreover, if g with $\mathrm{sl}(g) \neq 0$ has factorization type (ii), then g has as base points at most one 2-point, one 3-point, and two rational points. Considering Figure 2.1, either the factorization is minimal, or g has a minimal factorization as either (Q1) or (Q2). □

Remark 4.22. For $\mathbf{k} = \mathbf{R}$, or more generally for any field $\mathbf{k}$ with $[\mathbf{k}^a : \mathbf{k}] = 2$, the graph of Sarkisov links reduces to Figure 4.2, and Proposition 3.7 becomes the following (compare with [Zim18, Corollary 4.2]): the Cremona group is generated by $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, by the Jonquières group of type 1, and by the Jonquières group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2, \pi_{L,L})$ of type $2+2$, with $L = \mathbf{k}^a$. So to prove that $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$

is generated by involutions, there is no need to consider sporadic cases of Del Pezzo type, and then our proof reduces essentially to the one given in [Zim16, Corollary II.4.12].

Remark 4.23. Over $\mathbf{k} = \mathbf{F}_2$, some miracles occur with the sporadic cases. For instance, it is shown in [Sch21] that over $\mathbf{F}_2$, a link $\mathbb{P}^2 \overset{6\ 6}{\dashrightarrow} \mathbb{P}^2$ is always an involution (up to PGL_3), but this is certainly not the case for general fields. In fact, it can be observed from [DD19] that if the link is an involution, then the cubic surface dominating the link must admit an Eckardt point (three (-1)-curve meeting at a same point), and it turns out that over $\mathbf{F}_2$ all rational cubic surfaces of Picard rank 2 admit Eckardt points.

Appendix A. Sarkisov links between rational surfaces over a perfect field

In this appendix, we provide a mostly self-contained proof of Theorem 2.4. Precisely, we use some references such as [Sch22, SZ21], but we avoid any reference to the paper [Isk96].

One should keep in mind that the theorem does not say anything about the existence of the links. In particular,

- over an algebraically closed field, the graph of Figure 2.1 reduces to only two vertices $\{\mathbb{P}^2\}$, $\mathcal{C}_8$;
- over the field $\mathbf{R}$, or more generally a field such that $[\mathbf{k}^a : \mathbf{k}] = 2$, the graph of Figure 2.1 reduces to only four vertices $\{\mathbb{P}^2\}$, $\mathcal{C}_8$, $\mathcal{D}_8$, $\mathcal{C}_6$ (see Figure 4.2);
- over the field $\mathbb{F}_2$, it was shown in [Sch21] that all vertices appear, and all edges appear except the loop $\mathcal{D}_8 \overset{4\ 4}{\dashrightarrow} \mathcal{D}_8$.

The following result holds in greater generality, with a proof almost as short; see [RY00, Proposition A.6]. However, our argument allows a precise counting in the case of a finite field.

Lemma A.1. *Let $X \dashrightarrow Y$ be a birational map between surfaces defined over $\mathbf{k}$. Then X admits at least one rational point if and only if the same holds for Y.*

Proof. Since any birational map is a composition of blow-ups, it is sufficient to consider the case of the blow-up of a d-point $X \xrightarrow{d} Y$. If $d \geqslant 2$, the rational points of X and Y are in bijection, and if $d = 1$, the exceptional divisor is isomorphic to $\mathbb{P}^1$ over $\mathbf{k}$ and so contains rational points. □

Remark A.2. Over a finite field $\mathbf{k} = \mathbf{F}_q$, the above argument shows that the blow-up of a rational point produces an exceptional divisor with $q + 1$ rational points. In particular, given a link $X \overset{d\ 1}{\dashrightarrow} X'$ of type II with $d \geqslant 2$, we have $|X'(\mathbf{k})| = |X(\mathbf{k})| - q$. This shows

$$\begin{aligned} \left|\mathbb{P}^2(\mathbf{k})\right| &= q^2 + q + 1\,, \quad & |X_8(\mathbf{k})| &= q^2 + 1\,, \\ |X_6(\mathbf{k})| &= q^2 - q + 1\,, \quad & |X_5(\mathbf{k})| &= q^2 + 1 \end{aligned}$$

for any $X_i \in \mathcal{D}_i$. Also observe that all these numbers are greater than or equal to 3.

In the next proposition, we gather some characterizations of the rank 1 fibrations involved in our links (except for the class $\mathcal{D}_5$, where we could not find a reference).

Proposition A.3. *Let X be a rational Del Pezzo surface defined over a perfect field $\mathbf{k}$. Assume that X has degree d and Picard rank 1.*

1. *If $d = 9$, then $X = \mathbb{P}^2$.*
2. *If $d = 8$, then $X \in \mathcal{D}_8$.*

(3) *If $d = 6$, then $X \in \mathcal{D}_6$.*

Let $X/\mathbb{P}^1$ be a rational conic bundle defined over a perfect field $\mathbf{k}$. Assume $K_X^2 = d$ and that the relative Picard rank of $X/\mathbb{P}^1$ is 1.

(4) *If $d = 5$, then $X \in \mathcal{C}_5$.*

(5) *If $d = 6$, then $X \in \mathcal{C}_6$.*

(6) *If $d = 8$, then $X \in \mathcal{C}_8$.*

Proof. (1) This is a result of Châtelet; see for example [GS06, Theorem 5.1.3] or [Kol16, Corollary 13].

(2) See [SZ21, Lemma 3.2].

(3) See [SZ21, Section 4].

(4) We follow [Sch22, Lemma 6.13]. The surface X admits $8 - K_X^2 = 3$ singular fibers. Since the Picard rank of $X/\mathbb{P}^1$ is 1, for each singular fiber $C + C'$, there is an element $g \in \mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ that sends C onto C', and since the intersection form is preserved, the same g sends C' onto C. Since X is rational, by Lemma A.1 it contains a rational point, and so we can apply [Sch22, Lemma 6.5] and conclude that X is the blow-up of four points in $\mathbb{P}^2$, invariant under $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$. Moreover, since the Picard rank of X is 2, the four points form a single Galois orbit.

(5) The surface X admits $8 - K_X^2 = 2$ singular fibers. A singular fiber contains at most one rational point (the meeting point of the two components), so by Remark A.2 we can pick a rational point p outside the singular fibers. Let $Y \longrightarrow X$ be the blow-up of p. Then as in the previous case, we can apply [Sch22, Lemma 6.5] (with [Sch22, Observation 6.9]) to Y and conclude that Y is the blow-up of $\mathbb{P}^2$ at four points. Since the Picard rank of Y is 3, the four points are divided into two Galois orbits. The singular fibers correspond on $\mathbb{P}^2$ to pairs of lines going through all four points. If one of the points is rational, none of these lines is defined over $\mathbf{k}$. This is not possible since one of them corresponds to the exceptional curve E of p, which is rational. Therefore, $Y \longrightarrow \mathbb{P}^2$ is the blow-up at two 2-points. Let $F = \{F_1, F_2\}$ and $F' = \{F_1', F_2'\}$ be the orbits of (-1)-curves on Y that can be contracted to $\mathbb{P}^2$. By [Sch22, Lemma 6.5(3)], exactly one of them is pairwise disjoint with E, say F. We obtain a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d, "E"'] & X_7 \arrow[r, "F'"] \arrow[d, "E"] & \mathbb{P}^2 \\ X \arrow[r, "F"] & X_8 . \end{tikzcd}$$

In particular, $X \in \mathcal{C}_6$.

(6) Over $\mathbf{k}^a$, the surface X is isomorphic to a Hirzebruch surface $\mathbb{F}_n$ for some $n \geqslant 0$. If $n = 1$, then $X = \mathbb{F}_1$ because the (-1)-curve is rational and hence can be contracted to a Del Pezzo surface of degree 9 with a rational point, which is $\mathbb{P}^2$. For $n \geqslant 2$, we proceed by induction. Since $X \longrightarrow \mathbb{P}^1$ is defined over $\mathbf{k}$, there exist fibers defined over $\mathbf{k}$. Pick a rational point on such a fiber that is not the intersection point with the $(-n)$-curve. Blowing it up and contracting the transform of the fiber gives a birational map $X \dashrightarrow X'$ such that X' is isomorphic to $\mathbb{F}_{n-1}$ over $\mathbf{k}^a$, and with a morphism $X' \longrightarrow \mathbb{P}^1$ over $\mathbf{k}$. By induction, we arrive at $\mathbb{F}_1$ after $n-1$ steps. Therefore, each step was a map $\mathbb{F}_i \dashrightarrow \mathbb{F}_{i-1}$ and so $X = \mathbb{F}_n$. Similarly, in the case $n = 0$, by blowing up a rational point on a fiber defined over $\mathbf{k}$ and contracting the transform of the fiber, we obtain $\mathbb{F}_1$, and we conclude as above. $\square$

We say that a Sarkisov link of type II $X \overset{d\ d'}{\dashrightarrow} Y$ is *auto-similar* if X and Y are of the same type. Comparing the Picard numbers of X and Y over $\mathbf{k}^a$, we see that in this case $d = d'$. On the graph of Figure 2.1, a loop from one vertex to itself should be understood as the assertion that any corresponding Sarkisov link is auto-similar. For instance, the loop $\mathbb{P}^2 \overset{8\ 8}{\dashrightarrow} \mathbb{P}^2$ means that blowing up a general 8-point on $\mathbb{P}^2$, we should always come back to $\mathbb{P}^2$ after contracting an orbit of eight disjoint (-1)-curves. In the following lemma, we check this assertion and other similar ones.

Lemma A.4. *Let $X \in \mathcal{D}$, let $d \geqslant 1$, and let $p \in X$ be a general d-point. Assume that X and d are such that the corresponding edge on the graph of Figure 2.1 is a loop. Then the Sarkisov link starting with the blow-up of p is indeed an auto-similar link.*

Proof. Let $Z \longrightarrow X$ be the blow-up of the d-point p. If Z is a Del Pezzo surface of degree 1 or 2, then by Lemma 4.3, up to an automorphism, the link is induced by the Bertini or Geiser involution on Z, and so the conclusion is clear (and in fact, we get a link $X \dashrightarrow X$ between two isomorphic surfaces).

The Bertini case corresponds to the links

$$\mathbb{P}^2 \overset{8\ 8}{\dashrightarrow} \mathbb{P}^2\,, \quad \mathcal{D}_8 \overset{7\ 7}{\dashrightarrow} \mathcal{D}_8\,, \quad \mathcal{D}_5 \overset{4\ 4}{\dashrightarrow} \mathcal{D}_5\,, \quad \mathcal{D}_6 \overset{5\ 5}{\dashrightarrow} \mathcal{D}_6\,.$$

The Geiser case corresponds to the links

$$\mathbb{P}^2 \overset{7\ 7}{\dashrightarrow} \mathbb{P}^2\,, \quad \mathcal{D}_8 \overset{6\ 6}{\dashrightarrow} \mathcal{D}_8\,, \quad \mathcal{D}_5 \overset{3\ 3}{\dashrightarrow} \mathcal{D}_5\,, \quad \mathcal{D}_6 \overset{4\ 4}{\dashrightarrow} \mathcal{D}_6\,.$$

We now do a case-by-case analysis for the five remaining loops:

- $\mathbb{P}^2 \overset{3\ 3}{\dashrightarrow} \mathbb{P}^2$. We blow down the orbit of three lines through two of the p_i; by Proposition A.3(1), the resulting surface is $\mathbb{P}^2$.
- $\mathbb{P}^2 \overset{6\ 6}{\dashrightarrow} \mathbb{P}^2$. We blow down the orbit of six conics through five of the p_i; again by Proposition A.3(1), the resulting surface is $\mathbb{P}^2$.
- $\mathcal{D}_8 \overset{4\ 4}{\dashrightarrow} \mathcal{D}_8$. We blow down the orbit of four diagonals through three of the p_i; by Proposition A.3(2), the resulting surface is in $\mathcal{D}_8$.
- $\mathcal{D}_6 \overset{2\ 2}{\dashrightarrow} \mathcal{D}_6$. On a Del Pezzo surface X of degree 4, given two disjoint (-1)-curves E_1, E_2, there exists a unique pair of two other disjoint (-1)-curves E_3, E_4 such that $\sum_{i=1}^{4} E_i = -K_X$. We apply this remark to the exceptional divisor of the blow-up of the 2-point, and we find an orbit of two curves that we can contract. By Proposition A.3(3), the resulting surface is in $\mathcal{D}_6$.
- $\mathcal{D}_6 \overset{3\ 3}{\dashrightarrow} \mathcal{D}_6$. On a Del Pezzo surface X of degree 3, given three pairwise disjoint (-1)-curves E_1, E_2, E_3, there exists a unique triple of other pairwise disjoint (-1)-curves E_4, E_5, E_6 such that $\sum_{i=1}^{6} E_i = -2K_X$: intersect the cubic surface X with the unique quadric surface containing E_1, E_2, E_3. We apply this remark to the exceptional divisor of the blow-up of the 3-point, and we find an orbit of three curves that we can contract. Again, by Proposition A.3(3), the resulting surface is in $\mathcal{D}_6$. □

We now consider the remaining edges between the Del Pezzo type vertices.

Lemma A.5. *Let $r \in \mathbb{P}^2$ be a general 5-point, and let $p \in \mathbb{P}^2$ be either a rational point or a 2-point. Assume that p does not lie on the conic through r. Then p is general with r.*

Proof. In [Sch21, Lemma 4.9], it was proved that given any general 5-point on $\mathbb{P}^2$ and any 2-point on $\mathbb{P}^2$, either all seven components are in general position, or they all lie on the same conic. This implies the statement when p is a 2-point.

We now show that a rational point on $\mathbb{P}^2$ is never collinear with two geometric components of r, which then implies the other part of the statement. The lines L_{ij} through r_i, r_j do not contain any rational point: There exist a $\sigma \in \mathrm{Gal}(\mathbf{k}'/\mathbf{k})$ such that $\sigma(r_i) = r_j$ and $\sigma(r_j) \neq r_i$, where $\mathbf{k}'/\mathbf{k}$ is the splitting field of r. Hence, if a rational point p lies on L_{ij}, then $p = \sigma(p) \in L_{j,\sigma(j)}$ and so $L_{ij} = L_{j,\sigma(j)}$ contains three components of r, contradicting the generality of r. $\square$

Lemma A.6. *Let $X \in \mathcal{D}_5$, let $p \in X$ be a rational point such that the blow-up of p corresponds to a link $X \overset{1\ 5}{\dashrightarrow} \mathbb{P}^2$, and let $q \in X$ be an arbitrary rational point, distinct from p. Then:*

(1) *The points p and q are in general position.*

(2) *The Sarkisov link associated with the blow-up of q also is of type $\mathcal{D}_5 \overset{1\ 5}{\dashrightarrow} \mathbb{P}^2$.*

In particular, any two rational points on X are in general position.

Proof. Let $r \in \mathbb{P}^2$ be the general 5-point that is contracted by the Sarkisov link $X \overset{1\ 5}{\dashrightarrow} \mathbb{P}^2$ associated with p. The point $q \in X$ corresponds to a rational point $q' \in \mathbb{P}^2$, which does not lie on the conic through r. By Lemma A.5, the blow-up of r and q' is a Del Pezzo surface Z, and since this is the same as the blow-up of X at p and q, this gives assertion (1).

The piece associated with the Del Pezzo surface Z is the piece $\mathcal{P}(\mathcal{D}_5; 1, 1) = \mathcal{P}(\mathbb{P}^2; 1, 5)$, which is described in Lemma B.1. In particular, we get that the blow-up of q also gives a link $X \overset{1\ 5}{\dashrightarrow} \mathbb{P}^2$.

The last assertion is immediate. $\square$

Lemma A.7. *Let $X \in \mathcal{D}_5$, and let $p \in X$ be a rational point and $q \in X$ a 2-point. Then:*

(1) *The points p and q are in general position.*

(2) *The Sarkisov link associated with the blow-up of q is of type $\mathcal{D}_5 \overset{2\ 5}{\dashrightarrow} \mathcal{D}_8$.*

Proof. By Lemma A.6, the blow-up of p yields a link $X \overset{1\ 5}{\dashrightarrow} \mathbb{P}^2$. Let $r \in \mathbb{P}^2$ be the general 5-point that is blown up by the inverse of this link. The 2-point $q \in X$ corresponds to a 2-point $q' \in \mathbb{P}^2$, which does not lie on the conic through r. By Lemma A.5, the blow-up of r and q' is a Del Pezzo surface Z, and since this is the same as the blow-up of X at p and q, this gives assertion (1). The piece associated with the Del Pezzo surface Z is the piece $\mathcal{P}(\mathcal{D}_5; 1, 2) = \mathcal{P}(\mathbb{P}^2; 2, 5)$, which is described in Lemma B.3. In particular, we get that the blow-up of q gives a link $X \overset{2\ 5}{\dashrightarrow} X' \in \mathcal{D}_8$. $\square$

Lemma A.8. *Let $X \in \mathcal{D}_6$, and let $p \in X$ a rational point such that the blow-up of p corresponds to a link $X \overset{1\ 3}{\dashrightarrow} X'$ with $X' \in \mathcal{D}_8$ and $q \in X$ an arbitrary rational point, distinct from p. Then:*

(1) *The points p and q are in general position.*

(2) *The Sarkisov link associated with the blow-up of q also is of type $\mathcal{D}_6 \overset{1\ 3}{\dashrightarrow} \mathcal{D}_8$.*

In particular, any two rational points on X are in general position.

Proof. The proof is similar to that of Lemma A.6, now using the piece $\mathcal{P}(\mathcal{D}_6; 1, 1) = \mathcal{P}(\mathcal{D}_8; 1, 3)$, which is described in Lemma B.2. $\square$

Lemma A.9. *Let $X \in \mathcal{D}$, let $d \geqslant 1$, and let $p \in X$ be a general d-point. Assume that the corresponding edge on the graph of Figure 2.1 is of type II and is not a loop. Then the Sarkisov link starting with the blow-up of p is as prescribed by Figure 2.1.*

Proof. For the edges $\mathbb{P}^2 \overset{2\ 1}{\dashrightarrow} \mathcal{D}_8$, $\mathbb{P}^2 \overset{5\ 1}{\dashrightarrow} \mathcal{D}_5$, and $\mathcal{D}_8 \overset{3\ 1}{\dashrightarrow} \mathcal{D}_6$, this follows from the definition of the classes $\mathcal{D}_i$. Now we consider the inverses of these links.

If $X \in \mathcal{D}_8$ and $p \in X$ is a rational point, after the blow-up of X at p, we can contract the transform of the horizontal and vertical rulings through p. By Proposition A.3(1), the resulting surface is $\mathbb{P}^2$, so we get a link $\mathcal{D}_8 \overset{1\ 2}{\dashrightarrow} \mathbb{P}^2$.

If $X \in \mathcal{D}_5$ and $p \in X$ is a rational point, the existence of a link $\mathcal{D}_5 \overset{1\ 5}{\dashrightarrow} \mathbb{P}^2$ starting with the blow-up of p is given by Lemma A.6. If $X \in \mathcal{D}_6$ and $p \in X$ is a rational point, the existence of a link $\mathcal{D}_6 \overset{1\ 3}{\dashrightarrow} \mathcal{D}_8$ is given by Lemma A.8. Similarly, if $X \in \mathcal{D}_5$ and $p \in X$ is a general 2-point, the existence of a link $\mathcal{D}_5 \overset{2\ 5}{\dashrightarrow} \mathcal{D}_8$ is given by Lemma A.7.

Finally, let $X \in \mathcal{D}_8$, and let $p \in X$ be a general 5-point. We claim that any rational point $r \in X$ is general with p. Indeed, let C be the curve of bidegree $(2,2)$ passing through p, r, and with a double point at r. Since p is general, C is irreducible. Consider the link $X \overset{1\ 2}{\dashrightarrow} \mathbb{P}^2$ starting with the blow-up of r, and let $q \in \mathbb{P}^2$ be the 2-point image of the horizontal and vertical rulings through r. Then the image of C is a conic which does not contain q. We know by Lemma A.5 that q and the image of p are general, so q and r are also general, as claimed. We conclude the existence of a link $\mathcal{D}_8 \overset{5\ 2}{\dashrightarrow} \mathcal{D}_5$ by using the piece $\mathcal{P}\big(\mathcal{D}_8; 1, 5\big) = \mathcal{P}\big(\mathbb{P}^2; 2, 5\big)$, which is described in Lemma B.3. $\square$

We now turn to links involving the classes $\mathcal{C}_i$. First, recall [Sch22, Lemma 6.12].

Lemma A.10. *Let $\mathcal{P}, \mathcal{Q} \subset \mathbb{P}^2(\mathbf{k}^a)$ be two sets of four points and $X_{\mathcal{P}} \longrightarrow \mathbb{P}^2$, $X_{\mathcal{Q}} \longrightarrow \mathbb{P}^2$ be the corresponding blow-ups. Assume that $\mathcal{P}$ is either a general 4-point or the union of two 2-points that are general. If there exists a birational map $X_{\mathcal{P}} \dashrightarrow X_{\mathcal{Q}}$ preserving the fibrations associated with the pencils of conics through $\mathcal{P}$ and $\mathcal{Q}$, respectively, then there exists an automorphism $\alpha \in \mathrm{Aut}_{\mathbf{k}}\left(\mathbb{P}^2\right)$ such that $\alpha(\mathcal{P}) = \mathcal{Q}$, and in particular $X_{\mathcal{P}}$ and $X_{\mathcal{Q}}$ are isomorphic.*

Let p, p' be two distinct 2-points in general position in $\mathbb{P}^2$, and let $Y \longrightarrow \mathbb{P}^2$ be the blow-up of p and p'. Then the transform of the pencil of conics through p, p' gives the structure of a rank 2 fibration $Y/\mathbb{P}^1$. Contracting the transform of the line either through p or through p', we obtain two distinct surfaces $X, X' \in \mathcal{C}_6$. We say that X and X' are *twin* elements in $\mathcal{C}_6$ and that Y is their *parent*. Observe that X and X' are not necessarily isomorphic and that they are uniquely defined by Y.

Lemma A.11. *Let $X \in \mathcal{C}_5 \cup \mathcal{C}_6$, and let $p \in X$ be a general d-point (here d can be arbitrary large). Let $\chi\colon X \overset{d\ d}{\dashrightarrow} X'$ be the Sarkisov link of type II over $\mathbb{P}^1$ constructed from the blow-up of p.*

(1) *If $X \in \mathcal{C}_5$, then X' is isomorphic to X.*

(2) *If $X \in \mathcal{C}_6$, then X' is isomorphic to X or to the twin of X.*

Proof. We know from Proposition A.3 that X and X' belong to the same class $\mathcal{C}_i$, $i = 5$ or 6. Then Lemma A.10 gives the result when $X \in \mathcal{C}_5$, and when $X \in \mathcal{C}_6$, it gives that X, X' have the same parent, hence the result. $\square$

Lemma A.12. *Let $\mathbb{F}_n \in \mathcal{C}_8$ be a Hirzebruch surface. Then any Sarkisov link from $\mathbb{F}_n$ is one of the following:*

- *a link $\mathbb{F}_n \dashrightarrow \mathbb{F}_m$ of type II over $\mathbb{P}^1$,*
- *when $n = 1$, a link $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ of type III,*
- *when $n = 0$, a link $\mathbb{F}_0 \longrightarrow \mathbb{F}_0$ of type IV.*

Proof. If $\mathbb{F}_n \overset{d\,d}{\dashrightarrow} X$ is a link of type II over $\mathbb{P}^1$, the surface $X/\mathbb{P}^1$ is again a Hirzebruch surface by Proposition A.3(6). Since $\mathbb{F}_n$ has Picard rank 2, it admits exactly two extremal rays, with one corresponding to the given structure of rank 1 fibration $\mathbb{F}_n/\mathbb{P}^1$. If $n \geqslant 2$, the second extremal ray is generated by the exceptional section, which cannot be contracted to a smooth point. So the only possibilities for a link of type III or IV are the cases $n \in \{0, 1\}$, as expected. □

Proof of Theorem 2.4. (1) Let $(X/B, \varphi)$ be a marked rank 1 fibration. By Proposition 2.2, we know that the birational map $\varphi\colon X \dashrightarrow \mathbb{P}^2$ admits a factorization into Sarkisov links. In the above lemmas, we systematically explored all possible sequences of Sarkisov links starting from $\mathbb{P}^2$, and we showed that we never leave the seven classes $\{\mathbb{P}^2\}$, $\mathcal{D}_5$, $\mathcal{D}_6$, $\mathcal{D}_8$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_8$.

Similarly, statements (2) and (3) follow from the exhaustive description of links. A link of type IV must occur on a rank 1 fibration $X/\mathbb{P}^1$ such that the second extremal ray also corresponds to a fibration, and $\mathbb{F}_0$ is the only candidate. □

Appendix B. Elementary relations

In this appendix, we describe all elementary relations associated with a rank 3 fibrations X_d/pt, where X_d is a Del Pezzo surface of Picard rank 3 and degree d. In some sense, this gives a modern version of the results in [IKT93]. However, the point of view is slightly different since [IKT93] describes a set of relations for the Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, whereas by [LZ20, Theorem 3.1] our elementary relations constitute a set of relations for the groupoid $\mathrm{BirMori}_{\mathbf{k}}(\mathbb{P}^2)$, with respect to the Sarkisov links as generators.

Recall from Section 2.1, page 114, that with any rank 3 fibration we can associate a 2-piece, which we represent as a polygon with vertices corresponding to some Mori fiber spaces and edges to some Sarkisov links. There are two possibilities for the surface X_d:

- Either X_d is the blow-up of general a-point and b-point on $X_i \in \mathcal{D}_i \subset \mathcal{D}$, with $a+b+d=i$; we write $\mathcal{P}(\mathcal{D}_i; a, b)$ for the corresponding 2-piece;
- or X_d is the blow-up of a general a-point on $\mathbb{F}_0$, with $a+d=8$; we write $\mathcal{P}(\mathbb{F}_0; a)$ for the corresponding 2-piece.

In total there are 27 distinct such 2-pieces; see Table B.1 and the figures below. We do not include here the elementary relations associated with a rank 3 fibration $X_d/\mathbb{P}^1$ since these are always of the same form: the corresponding piece is a square with all four vertices corresponding to conic bundles in the same subset $\mathcal{C}_i \subset \mathcal{C}$, $i = 5, 6$, or 8.

In the pictures, we use the following convention. An edge labeled with d is the blow-up of a general d-point, with one color associated with each d (from $d = 1$ to $d = 7$). A black edge without label is a change of base, as in Lemma 2.1(2). A surface X_i is a Del Pezzo surface of degree i. In particular, when a surface X_i corresponds to a vertex, it has Picard rank 1 and so belongs to the class $\mathcal{D}_i$, with its structure of fibration to the point. Similarly, a surface $X_i/\mathbb{P}^1$ at a vertex denotes a surface in $\mathcal{C}_i$, with its structure of conic bundle. We put some prime such as X_i' when several surfaces with the same degree appear in the diagram and we see no good reason why they should be isomorphic (a typical good reason is that they are related by a Geiser or Bertini involution or that we can apply Lemma A.11(1)).

The proof that each piece has the form given in the pictures below can be done as follows. Over $\mathbf{k}^a$, we know the number of (-1)-curves and of rational fibrations on X_d. Then we can study the Galois action on these curves and find which orbits correspond to pairwise disjoint (-1)-curves and so correspond to a blow-down defined over $\mathbf{k}$. We now give detailed statements for the pieces $\mathcal{P}(\mathbb{P}^2;1,5)$, $\mathcal{P}(\mathbb{P}^2;2,3)$, $\mathcal{P}(\mathbb{P}^2;2,5)$ since these were used in our proof of Theorem 2.4 given in Appendix A.

Lemma B.1 ($\mathcal{P}(\mathbb{P}^2;1,5)$). *Let $p \in \mathbb{P}^2$ be a rational point and $q \in \mathbb{P}^2$ a 5-point such that p, q are in general position. Let $X_3 \longrightarrow \mathbb{P}^2$ be the blow-up of p and q. Then X_3 is a Del Pezzo surface of degree 3 and Picard rank 3 and admits exactly seven extremal rays described as follows (we order them such that the intersection product of two consecutive rays is zero):*

- *the exceptional divisor E_1 from p,*
- *the exceptional divisor E_5 from q,*
- *the pencil of lines through p, giving a structure of rank 2 fibration $X_3/\mathbb{P}^1$,*
- *the transform L_5 of the five lines through p and one of the q_i,*
- *the pencil of cubics through p, q and with a double point at p, corresponding to a second structure of rank 2 fibration $X_3/\mathbb{P}^1$,*
- *the transform O_5 of the five conics through p and four of the q_i,*
- *the transform O_1 of the conic through q.*

In consequence, the piece $\mathcal{P}(\mathbb{P}^2;1,5)$ does not depend on the particular choice of the blown-up points and coincides with the piece $\mathcal{P}(\mathcal{D}_5;1,1)$.

Proof. It is routine to check that the listed curves become smooth rational curves on X_3:

- either of self-intersection 0, and so moving in a pencil,
- or a disjoint union of (-1)-curves, forming a single Galois orbit.

At this point, we know that the piece is a heptagon and that the surfaces corresponding to vertices have respective degree 9, 5, 9, 8, 8, 8, 8. By Proposition A.3, the surfaces of degree 9 are $\mathbb{P}^2$. Then since the surface of degree 5 is obtained by a Sarkisov link from $\mathbb{P}^2$, it belongs to $\mathcal{D}_5$. For the four vertices corresponding to surfaces in $\mathcal{C}_8$, two of them come from the blow-up of a rational point on $\mathbb{P}^2$ and so are $\mathbb{F}_1$, and the two other are related by a link of type IV and so are $\mathbb{F}_0$. □

The proofs for the next two lemmas are similar, so we omit them.

Lemma B.2 ($\mathcal{P}(\mathbb{P}^2;2,3) = \mathcal{P}(\mathcal{D}_8;1,3) = \mathcal{P}(\mathcal{D}_6;1,1)$). *Let $X_8 \in \mathcal{D}_8$, and let $q \in X_8$ be a general 3-point.*

(1) *For any rational point p not on the diagonal passing through q, the points p, q are in general position.*

(2) *Let $X_4 \longrightarrow X_8$ be the blow-up of such points p, q. Then X_4 is a Del Pezzo surface of degree 4 and Picard rank 3 and admits exactly five extremal rays described as follows (we order them such that the intersection product of two consecutive rays is zero):*
   - *the exceptional divisor E_3 from q,*
   - *the exceptional divisor E_1 from p,*
   - *the transform D_1 of the diagonal through q,*
   - *the transform D_3 of the three diagonals through p and two of the q_i,*
   - *the transform R_2 of the horizontal and vertical rulings through p.*

As a consequence, the piece $\mathcal{P}(\mathcal{D}_8;1,3)$ does not depend on the particular choice of the blown-up points and coincides with the pieces $\mathcal{P}(\mathbb{P}^2;2,3)$ and $\mathcal{P}(\mathcal{D}_6;1,1)$.

Lemma B.3 $\left(\mathcal{P}\left(\mathbb{P}^2; 2, 5\right)\right)$. *Let $q \in \mathbb{P}^2$ be a 2-point. Let $p \in \mathbb{P}^2$ be a 5-point in general position with q, and let $X_2 \longrightarrow \mathbb{P}^2$ be the blow-up of p, q. Then X_2 is a Del Pezzo surface of degree 2 and Picard rank 3 and admits exactly six extremal rays described as follows:*

- *the exceptional divisor E_5 from p,*
- *the line L_1 through q,*
- *the cubics C_2 passing through p, q and singular at one of the q_i,*
- *the cubics C_5 passing through p, q and singular at one of the p_i,*
- *the conic O_1 through p,*
- *the exceptional divisor E_2.*

Let $\mathbb{P}^2 \overset{2\ 1}{\dashrightarrow} X_8$ be the Sarkisov link starting with the blow-up of q, let $r \in X_8$ be the rational point image of the line through q, and still denote by p the image of the 5-point on X_8. Then we can see the surface X_2 as the blow-up of r and p on X_8, and in term of transform of curves coming from X_8, the above list of extremal rays becomes the following:

- *the exceptional divisor E_5 from p,*
- *the exceptional divisor E_1 from r,*
- *the curves C_2 of bidegree $(1, 2)$ and $(2, 1)$ through p,*
- *the curves C_5 of bidegree $(2, 2)$ through p and r, and singular at one of the p_i,*
- *the curves O_1 of bidegree $(2, 2)$ through p and r, and singular at r,*
- *the vertical and horizontal rulings through r.*

As a consequence, the piece $\mathcal{P}\left(\mathbb{P}^2; 2, 5\right)$ does not depend on the particular choice of the blown-up points and coincides with the pieces $\mathcal{P}\left(\mathcal{D}_8; 2, 5\right)$.

Remark B.4. Over the field $\mathbf{k} = \mathbf{R}$, a similar list of 2-pieces was used by Zimmermann in [Zim22], where she calls "disc of type 1 to 6" our pieces involving only rational points or 2-points. The correspondence is as follows:

- Disc of type 1: $\mathcal{P}\left(\mathbb{P}^2; 1, 2\right)$,
- Disc of type 2: square relation between conic bundles in $\mathcal{C}_6$,
- Disc of type 3: $\mathcal{P}\left(\mathbb{P}^2; 2, 2\right)$,
- Disc of type 4: $\mathcal{P}\left(\mathcal{D}_8; 2, 2\right)$,
- Disc of type 5: $\mathcal{P}\left(\mathbb{P}^2; 1, 1\right)$,
- Disc of type 6: square relation between Hirzebruch surfaces.

Table B.1. The 27 elementary relations over a point

Piece	Figure
$\mathcal{P}(\mathbb{P}^2; 1, 1)$	Fig. B.1
$\mathcal{P}(\mathbb{P}^2; 1, 2)$	Fig. B.2
$\mathcal{P}(\mathbb{P}^2; 1, 3)$	Fig. B.3
$\mathcal{P}(\mathbb{P}^2; 1, 4)$	Fig. B.4
$\mathcal{P}(\mathbb{P}^2; 1, 5)$	Fig. B.5
$\mathcal{P}(\mathbb{P}^2; 1, 6)$	Fig. B.6
$\mathcal{P}(\mathbb{P}^2; 1, 7)$	Fig. B.8
$\mathcal{P}(\mathbb{P}^2; 2, 2)$	Fig. B.9
$\mathcal{P}(\mathbb{P}^2; 2, 3)$	Fig. B.10
$\mathcal{P}(\mathbb{P}^2; 2, 4)$	Fig. B.11
$\mathcal{P}(\mathbb{P}^2; 2, 5)$	Fig. B.12
$\mathcal{P}(\mathbb{P}^2; 2, 6)$	Fig. B.7
$\mathcal{P}(\mathbb{P}^2; 3, 3)$	Fig. B.13
$\mathcal{P}(\mathbb{P}^2; 3, 4)$	Fig. B.15
$\mathcal{P}(\mathbb{P}^2; 3, 5)$	Fig. B.16
$\mathcal{P}(\mathbb{P}^2; 4, 4)$	Fig. B.17

Piece	Figure
$\mathcal{P}(\mathcal{D}_8; 1, 1)$	Fig. B.2
$\mathcal{P}(\mathcal{D}_8; 1, 2)$	Fig. B.9
$\mathcal{P}(\mathcal{D}_8; 1, 3)$	Fig. B.10
$\mathcal{P}(\mathcal{D}_8; 1, 4)$	Fig. B.11
$\mathcal{P}(\mathcal{D}_8; 1, 5)$	Fig. B.12
$\mathcal{P}(\mathcal{D}_8; 1, 6)$	Fig. B.7
$\mathcal{P}(\mathcal{D}_8; 2, 2)$	Fig. B.18
$\mathcal{P}(\mathcal{D}_8; 2, 3)$	Fig. B.19
$\mathcal{P}(\mathcal{D}_8; 2, 4)$	Fig. B.22
$\mathcal{P}(\mathcal{D}_8; 2, 5)$	Fig. B.24
$\mathcal{P}(\mathcal{D}_8; 3, 3)$	Fig. B.14
$\mathcal{P}(\mathcal{D}_8; 3, 4)$	Fig. B.23
$\mathcal{P}(\mathcal{D}_5; 1, 1)$	Fig. B.5
$\mathcal{P}(\mathcal{D}_5; 1, 2)$	Fig. B.12
$\mathcal{P}(\mathcal{D}_5; 1, 3)$	Fig. B.16
$\mathcal{P}(\mathcal{D}_5; 2, 2)$	Fig. B.24

Piece	Figure
$\mathcal{P}(\mathbb{F}_0; 1)$	Fig. B.2
$\mathcal{P}(\mathbb{F}_0; 2)$	Fig. B.25
$\mathcal{P}(\mathbb{F}_0; 3)$	Fig. B.3
$\mathcal{P}(\mathbb{F}_0; 4)$	Fig. B.26
$\mathcal{P}(\mathbb{F}_0; 5)$	Fig. B.5
$\mathcal{P}(\mathbb{F}_0; 6)$	Fig. B.27
$\mathcal{P}(\mathbb{F}_0; 7)$	Fig. B.8
$\mathcal{P}(\mathcal{D}_6; 1, 1)$	Fig. B.10
$\mathcal{P}(\mathcal{D}_6; 1, 2)$	Fig. B.19
$\mathcal{P}(\mathcal{D}_6; 1, 3)$	Fig. B.14
$\mathcal{P}(\mathcal{D}_6; 1, 4)$	Fig. B.23
$\mathcal{P}(\mathcal{D}_6; 2, 2)$	Fig. B.20
$\mathcal{P}(\mathcal{D}_6; 2, 3)$	Fig. B.21

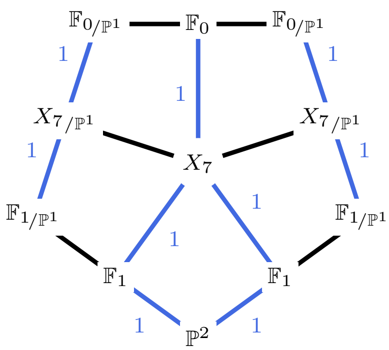

Figure B.1. $\mathcal{P}\big(\mathbb{P}^2;1,1\big)$, $\mathcal{P}(\mathbb{F}_0;1)$

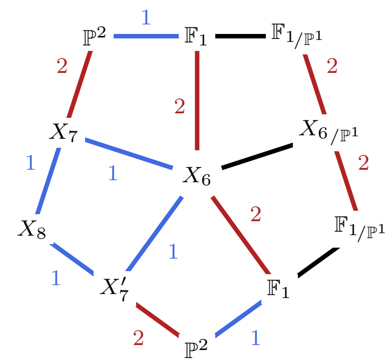

Figure B.2. $\mathcal{P}\big(\mathbb{P}^2;1,2\big)$, $\mathcal{P}\big(\mathcal{D}_8;1,1\big)$

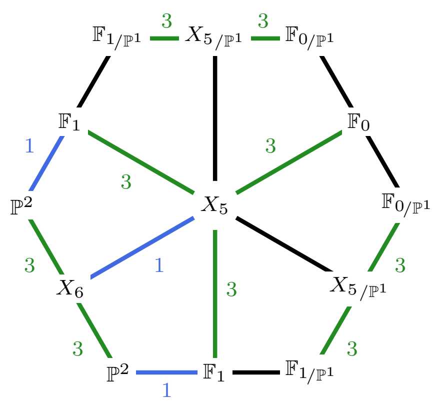

Figure B.3. $\mathcal{P}\big(\mathbb{P}^2;1,3\big)$, $\mathcal{P}(\mathbb{F}_0;3)$

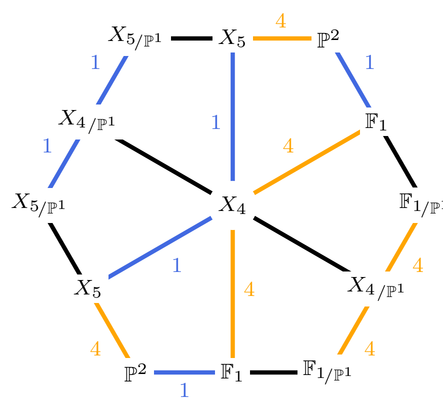

Figure B.4. $\mathcal{P}\big(\mathbb{P}^2;1,4\big)$

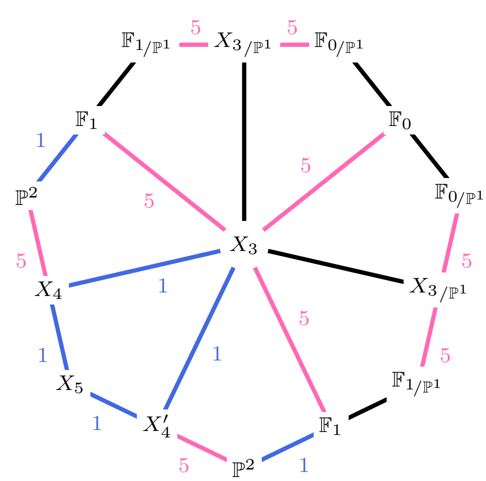

Figure B.5. $\mathcal{P}\big(\mathbb{P}^2;1,5\big)$, $\mathcal{P}\big(\mathcal{D}_5;1,1\big)$, $\mathcal{P}(\mathbb{F}_0;5)$

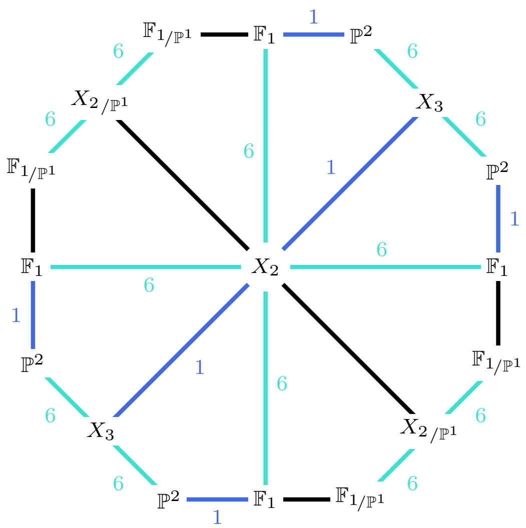

Figure B.6. $\mathcal{P}\big(\mathbb{P}^2; 1, 6\big)$

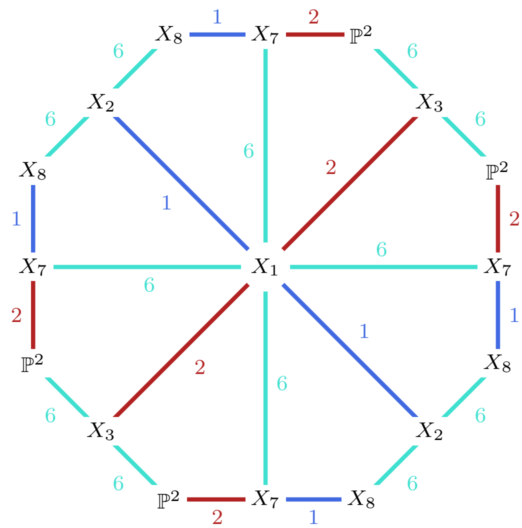

Figure B.7. $\mathcal{P}\big(\mathbb{P}^2; 2, 6\big)$, $\mathcal{P}\big(\mathcal{D}_8; 1, 6\big)$

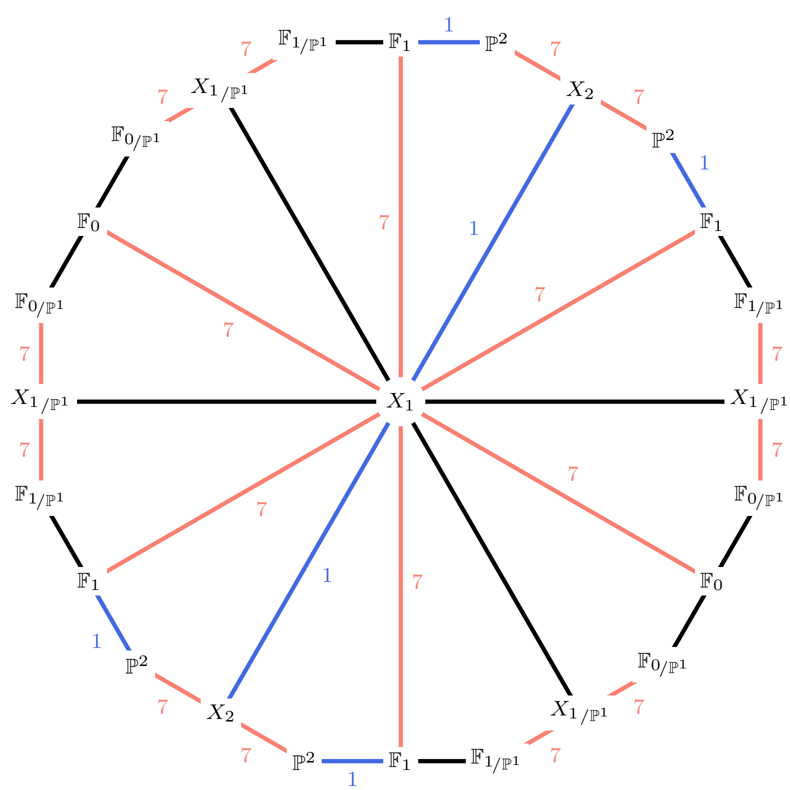

Figure B.8. $\mathcal{P}\big(\mathbb{P}^2; 1, 7\big)$, $\mathcal{P}(\mathbb{F}_0; 7)$

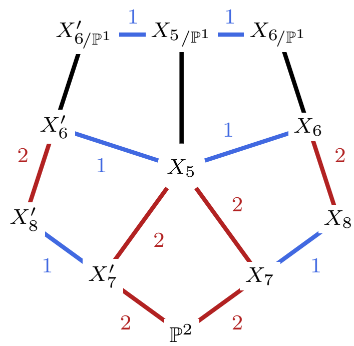

Figure B.9. $\mathcal{P}\big(\mathbb{P}^2;2,2\big)$, $\mathcal{P}\big(\mathcal{D}_8;1,2\big)$

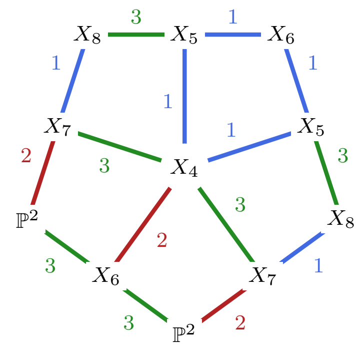

Figure B.10. $\mathcal{P}\big(\mathbb{P}^2;2,3\big)$, $\mathcal{P}\big(\mathcal{D}_8;1,3\big)$, $\mathcal{P}\big(\mathcal{D}_6;1,1\big)$

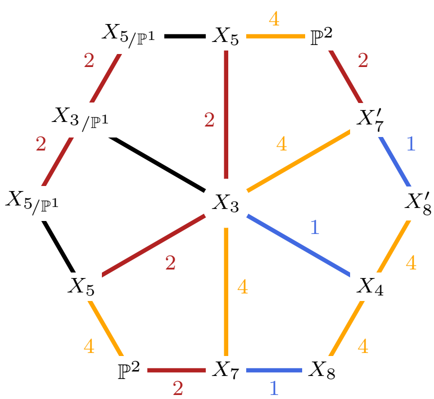

Figure B.11. $\mathcal{P}\big(\mathbb{P}^2;2,4\big)$, $\mathcal{P}\big(\mathcal{D}_8;1,4\big)$

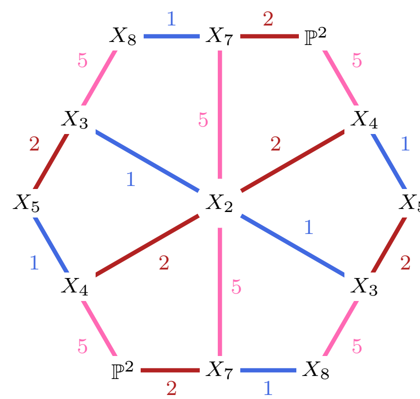

Figure B.12. $\mathcal{P}\big(\mathbb{P}^2;2,5\big)$, $\mathcal{P}\big(\mathcal{D}_8;1,5\big)$, $\mathcal{P}\big(\mathcal{D}_5;1,2\big)$

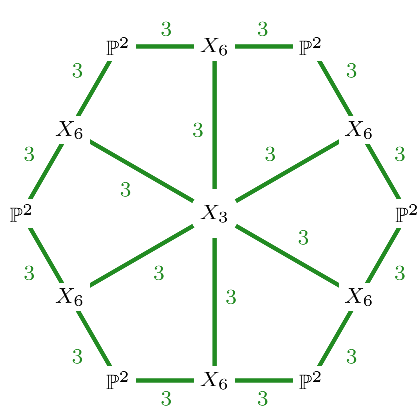

Figure B.13. $\mathcal{P}\big(\mathbb{P}^2;3,3\big)$

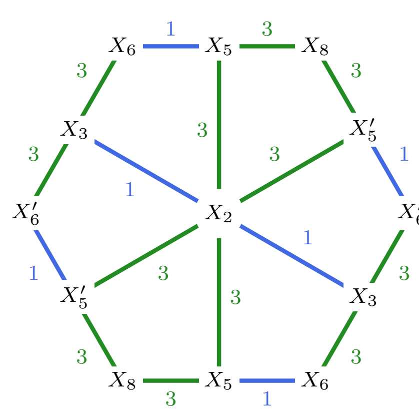

Figure B.14. $\mathcal{P}\big(\mathcal{D}_8;3,3\big)$, $\mathcal{P}\big(\mathcal{D}_6;1,3\big)$

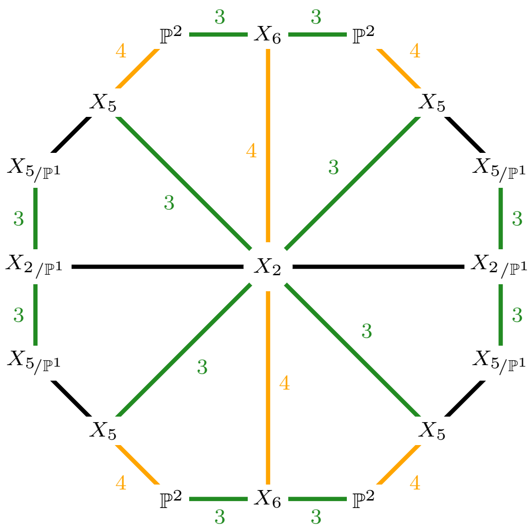

Figure B.15. $\mathcal{P}(\mathbb{P}^2; 3, 4)$

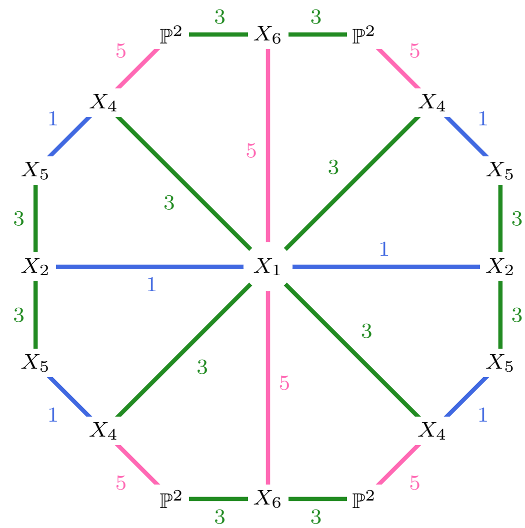

Figure B.16. $\mathcal{P}(\mathbb{P}^2; 3, 5)$, $\mathcal{P}(\mathcal{D}_5; 1, 3)$

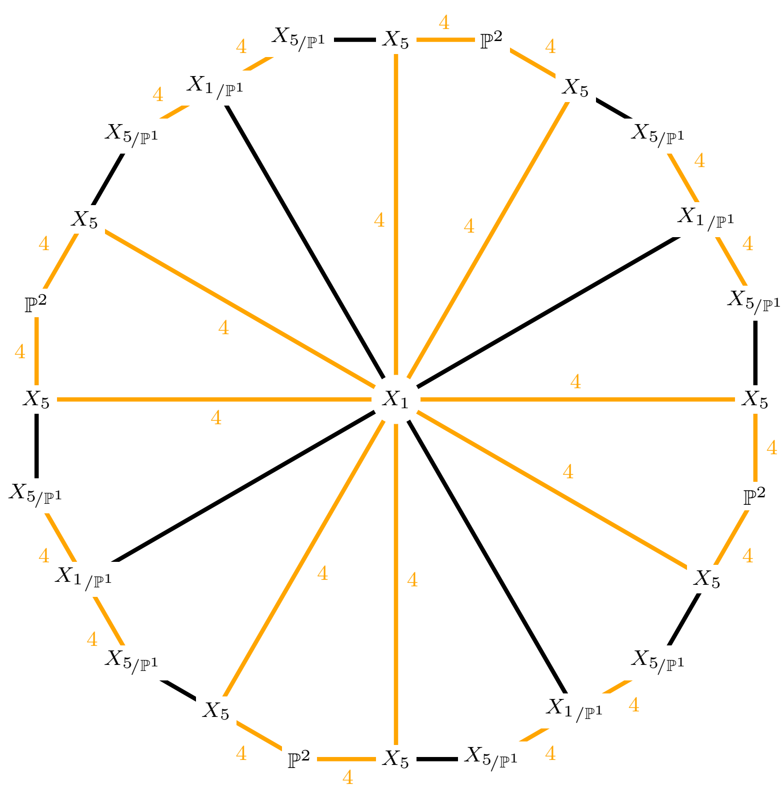

Figure B.17. $\mathcal{P}(\mathbb{P}^2; 4, 4)$

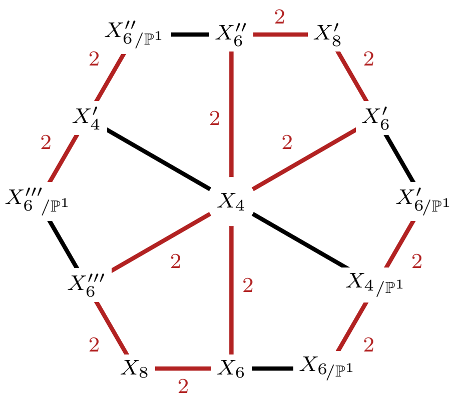

Figure B.18. $\mathcal{P}(\mathcal{D}_8; 2, 2)$

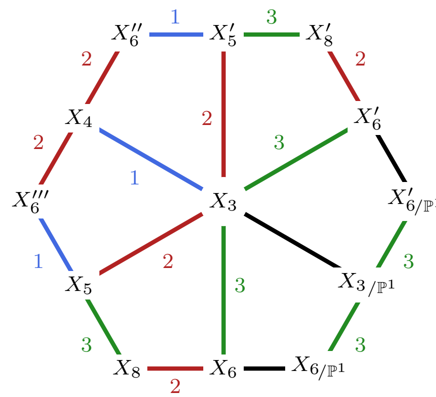

Figure B.19. $\mathcal{P}(\mathcal{D}_8; 2, 3)$, $\mathcal{P}(\mathcal{D}_6; 1, 2)$

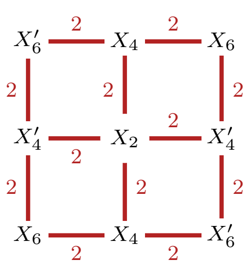

Figure B.20. $\mathcal{P}(\mathcal{D}_6; 2, 2)$

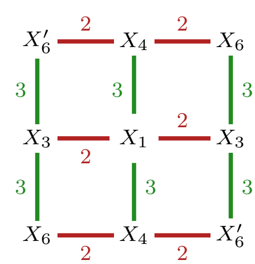

Figure B.21. $\mathcal{P}(\mathcal{D}_6; 2, 3)$

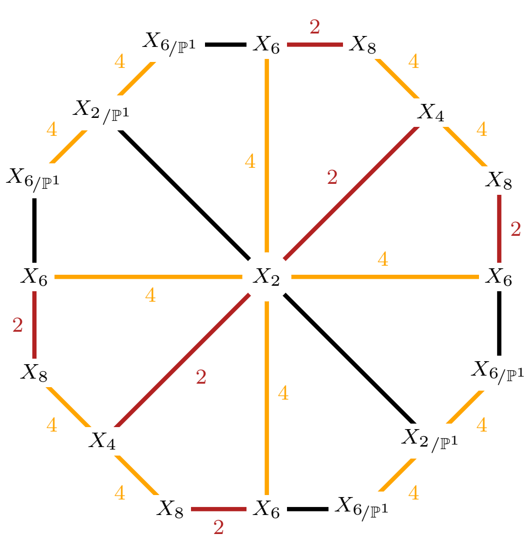

Figure B.22. $\mathcal{P}(\mathcal{D}_8; 2, 4)$

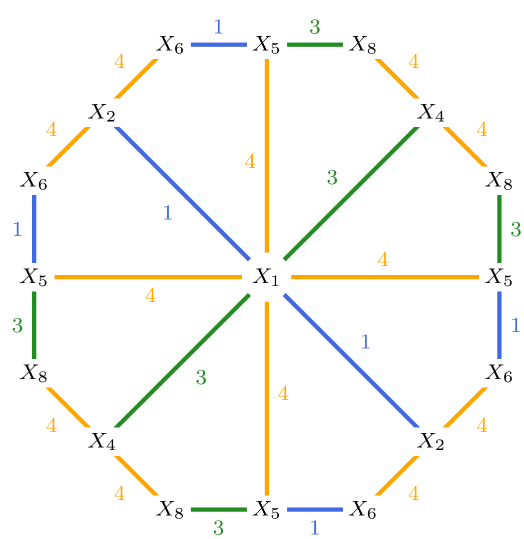

Figure B.23. $\mathcal{P}(\mathcal{D}_8; 3, 4)$, $\mathcal{P}(\mathcal{D}_6; 1, 4)$

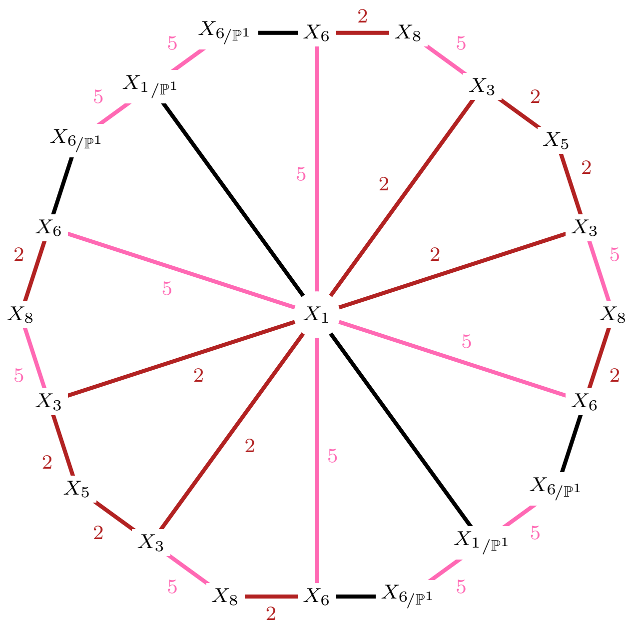

Figure B.24. $\mathcal{P}\big(\mathcal{D}_8; 2, 5\big)$, $\mathcal{P}\big(\mathcal{D}_5; 2, 2\big)$

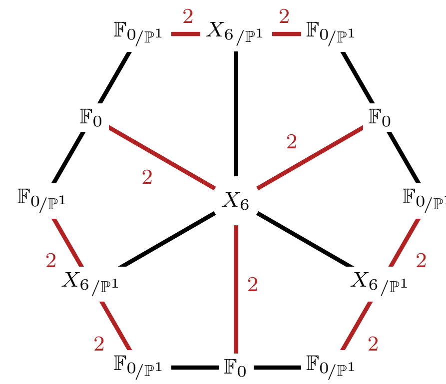

Figure B.25. $\mathcal{P}(\mathbb{F}_0; 2)$

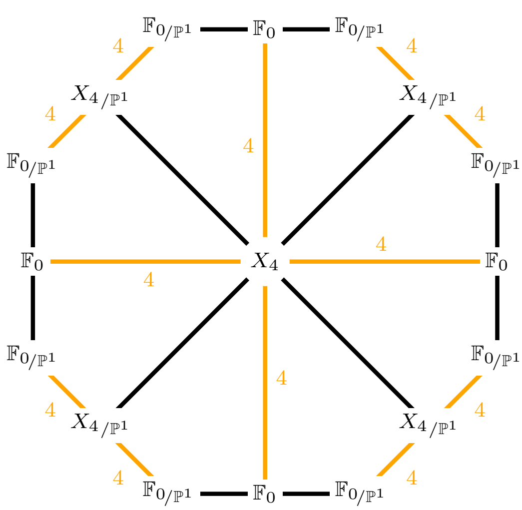

Figure B.26. $\mathcal{P}(\mathbb{F}_0; 4)$

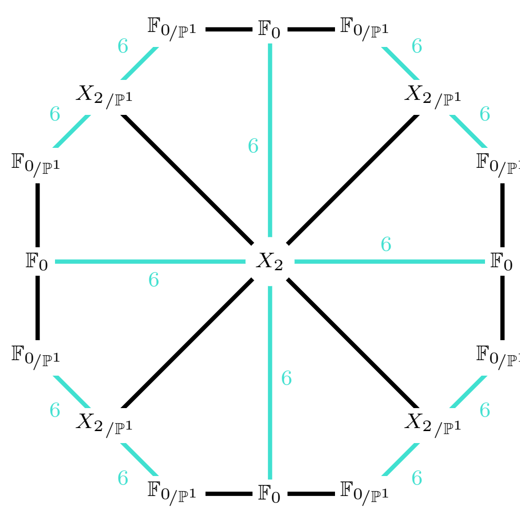

Figure B.27. $\mathcal{P}(\mathbb{F}_0; 6)$

References

Che97 C. Chevalley, *The algebraic theory of spinors and Clifford algebras* in *Collected works of Claude Chevalley. Vol. 2* (edited and with a foreword by P. Cartier and C. Chevalley, with a postface by J.-P. Bourguignon) (Springer-Verlag, Berlin, 1997).

Cor95 A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebr. Geom. **4** (1995), no. 2, 223–254.

Dol12 I. Dolgachev, *Classical algebraic geometry: A modern view*, (Cambridge Univ. Press, Cambridge, 2012); doi:10.1017/CBO9781139084437.

DD19 I. Dolgachev and A. Duncan, *Automorphisms of cubic surfaces in positive characteristic*, Izv. Ross. Akad. Nauk. Ser. Mat. **83** (2019), no. 3, 15–92, doi:10.4213/im8803 (Russian); Izv. Math. **83** (2019), no. 3, 424–500 (English); doi:10.1070/IM8803.

GS06 P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Stud. Adv. Math., vol. 101 (Cambridge Univ. Press, Cambridge, 2006); doi:10.1017/CBO9780511607219.

Gro01 L. C. Grove, *Classical groups and geometric algebra*, Grad. Stud. Math., vol. 39 (Amer. Math. Soc., Providence, RI, 2001); doi:10.1090/gsm/039.

Har77 R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52 (Springer, New York, 1977); doi:10.1007/978-1-4757-3849-0.

Isk91 V. A. Iskovskikh, *Generators in the two-dimensional Cremona group over a nonclosed field*, in *Number theory, algebra, mathematical analysis and their applications*, Trudy Mat. Inst. Steklov. 200 (1991), 157-–170 (Russian); Proc. Steklov Inst. Math. (1993), no. 2, 173-–188 (English).

Isk96 ———, *Factorization of birational maps of rational surfaces from the point of viewpoint of Mori theory*, Russ. Math. Surv., **51** (1996), no. 4, 585–652; doi:10.1070/RM1996v051n04ABEH002962.

IKT93 V. A. Iskovskikh, F. K. Kabdykairov, and S. L. Tregub, *Relations in the two-dimensional Cremona group over a perfect field*, Russ. Acad. Sci. Izv. Math. **42** (1993), no. 3, 3–69.

KMRT98 M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Amer. Math. Soc. Colloq. Publ., vol. 44 (Amer. Math. Soc., Providence, RI, 1998); doi:10.1090/coll/044.

Kol16 J. Kollár, *Severi–Brauer varieties; a geometric treatment*, 2016, arXiv:1606.04368.

LS22 H.-Y. Lin and E. Shinder, *Motivic invariants of birational maps*, 2022, arXiv:2207.07389.

LZ20 S. Lamy and S. Zimmermann, *Signature morphisms from the Cremona group over a non-closed field*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 10, 3133–3173; doi:10.4171/jems/983.

RY00 Z. Reichstein and B. Youssin, *Essential dimensions of algebraic groups and a resolution theorem for G-varieties* (with an appendix by J. Kollàr and E. Szabó), Canad. J. Math. **52** (2000), no. 5, 1018–1056; doi:10.4153/CJM-2000-043-5.

Sch21 J. Schneider, *Generators of the plane Cremona group over the field with two elements*, 2021, arXiv:2008.07991v3.

Sch22 ———, *Relations in the Cremona group over a perfect field*, Ann. Inst. Fourier (Grenoble) **72** (2022), no. 1, 1–42; doi:10.5802/aif.3463.

SZ21 J. Schneider and S. Zimmermann, *Algebraic subgroups of the plane Cremona group over a perfect field*, Épijournal Géom. Algébr., EPIGA **5** (2021), article no. 14; doi:10.46298/epiga.2021.6715.

Shr20 C. A. Shramov, *Birational automorphisms of Severi–Brauer surfaces*, Sb. Math., **211** (2020), no. 3, 466–480; doi:10.1070/SM9304.

Zim16 S. Zimmermann, *Compositions and relations in the Cremona groups*, Ph.D. thesis, University Basel, 2016, available at https://edoc.unibas.ch/44298.

Zim18 ———, *The Abelianization of the real Cremona group*, Duke Math. J. **167** (2018), no. 2, 211–267; doi:10.1215/00127094-2017-0028.

Zim22 ______, *The real plane Cremona group is an amalgamated product*, Ann. Inst. Fourier (Grenoble) **71** (2022), no. 3, 1023–1045; doi:10.5802/aif.3415.

Stéphane Lamy slamy@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse UMR 5219, Université de Toulouse, UPS F-31062 Toulouse Cedex 9, France

Julia Schneider julia.schneider@math.ch
École Polytechnique Fédérale de Lausanne, Chair of Algebraic Geometry (Bâtiment MA), Station 8, CH-1015 Lausanne, Switzerland
Current address: Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland