THE TITS ALTERNATIVE FOR AUT[$\mathbb{C}^2$]

STÉPHANE LAMY

This is a translation of the paper *L'alternative de Tits pour* Aut[$\mathbb{C}^2$], Journal of Algebra, 239, p. 413-437, 2001.

ABSTRACT. Letting Aut[$\mathbb{C}^2$] act on a tree, we classify the subgroups of Aut[$\mathbb{C}^2$], and show that the Tits alternative is true. Further we get another formulation for the notion of the Green function for a Hénon type automorphism.

1. INTRODUCTION.

This article contains the detail of the results announced in the note [7].

It is well known that the group Aut[$\mathbb{C}^2$] of polynomial automorphisms of the complex plane can be described as an amalgamated product. More precisely following [5] we set:

$$\begin{aligned} E &= \{(x,y) \to (\alpha x + P(y), \beta y + \gamma);\ \alpha,\beta,\gamma \in \mathbb{C}, \alpha\beta \neq 0, P \in \mathbb{C}[X]\}; \\ A &= \{(x,y) \to (a_1x + b_1y + c_1, a_2x + b_2y + c_2);\ a_i, b_i, c_i \in \mathbb{C}, a_1b_2 - a_2b_1 \neq 0\}; \\ S &= A \cap E. \end{aligned}$$

We call E the group of elementary automorphisms; of course A is the group of affine automorphisms. Then Aut[$\mathbb{C}^2$]= $A *_S E$. In the sequel by abuse of notation we always note $f = (f_1(x,y), f_2(x,y))$ to refer to an element $f \in$ Aut[$\mathbb{C}^2$] (instead of $f : (x,y) \to (f_1(x,y), f_2(x,y))$).

Following a strategy already used by Wright [12] (in the context of the study of abelian subgroups), we want to study subgroups in Aut[$\mathbb{C}^2$] by means of the action of Aut[$\mathbb{C}^2$] on a tree, which is provided by the theory of Bass and Serre [9]. The idea is that questions such as "Do f and g commute?" or "Do f and g generate a free group?" will be easier to tackle by considering the actions of f and g on the tree.

Friedland and Milnor [5] classified the elements of Aut[$\mathbb{C}^2$] up to conjugacy. For $f \in$ Aut[$\mathbb{C}^2$] we have two possibilities:

(1) f is conjugate to an element in E;
(2) f is conjugate to a composition of generalized Hénon maps *i.e.*

$$\varphi f \varphi^{-1} = g_m \circ \cdots \circ g_1$$

where $\varphi \in \mathrm{Aut}[\mathbb{C}^2]$, $g_i = (y, P_i(y) - \delta_i x)$ with $\delta_i \in \mathbb{C}^*$, and $P_i \in \mathbb{C}[X]$ has degree ≥ 2.

We say respectively that f is of elementary or Hénon type.

This alternative may be rephrased as follows: for $g \in \mathrm{Aut}[\mathbb{C}^2]$ let us define the dynamical degree $d(g) = \lim_{n \to +\infty} (d^\circ g^n)^{1/n}$ where $d^\circ g^n$ is the ordinary degree of g^n. A good feature of the dynamical degree is its invariance under conjugacy, and we have:

$$d(g) = 1 \quad \Leftrightarrow \quad g \text{ is conjugate to an element in } E;$$
$$d(g) \geq 2 \quad \Leftrightarrow \quad g \text{ is of Hénon type.}$$

The article is organized as follows.

According to Bass-Serre theory [9], we can canonically associate a tree to any amalgamated product. We recall this construction in Section 2, and we rephrase once again the above alternative, this time in terms of fixed subtrees. We also introduce some normal forms which will be useful for computations. Then, we state our main theorem, from which the Tits alternative follows immediately.

In Section 3 we consider automorphisms of (dynamical) degree 1. We first show that, except for some very special rotations, the fixed tree under the action of an automorphism f with degree 1 is bounded; in particular this gives an obstruction to a relation $f \circ g = g \circ f$ with $d(g) \geq 2$. Similarly if we consider a group all elements of which have degree 1, we show that except for some special cases of the same nature as above, this group is conjugate to a subgroup of E or A.

In Section 4 we consider the case of groups which contain elements of Hénon type. We characterize some pairs f, g of such automorphisms which generate a free group, and we show that the centralizer of an automorphism with degree ≥ 2 is a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$. This terminates the proof of the main theorem.

Finally in Section 5 we establish a relation with the point of view developed for instance by Bedford, Smillie and Sibony ([1, 2], [10]): we characterize the automorphisms with the same Green function, as well as the automorphisms which preserve an attracting basin. The resolution of these questions with a dynamics flavor was the main motivation to start this work.

2. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREM.

In the beginning of this section we explain how the general construction in [9] works in our particular case.

We construct a simplicial tree $\mathcal{T}$ as follows: we take the disjoint union of $\mathrm{Aut}[\mathbb{C}^2]\,/A$ and $\mathrm{Aut}[\mathbb{C}^2]/E$ as the set of vertices, and $\mathrm{Aut}[\mathbb{C}^2]\,/S$ as the set of edges. All these quotients must be understood as being left cosets; the cosets of $g \in \mathrm{Aut}[\mathbb{C}^2]$ are noted respectively gA, gE and gS. By definition, the edge hS links the vertices fA and gE if $hS \subset fA$ and $hS \subset gE$ (and so $fA = hA$ and $gE = hE$).

In this way we obtain a graph $\mathcal{T}$; the fact that A and E are amalgamated along S is equivalent to the fact that $\mathcal{T}$ is a tree (see [9]).

This tree is uniquely characterized (up to isomorphism) by the following property: there exists an action of Aut[$\mathbb{C}^2$] on $\mathcal{T}$, such that the fundamental domain of this action is a segment, *i.e.* an edge and two vertices, with E and A equal to the stabilizers of the vertices of this segment (and so S is the stabilizer of the entire segment). This action is simply the left translation: $g(hS) = (g \circ h)S$.

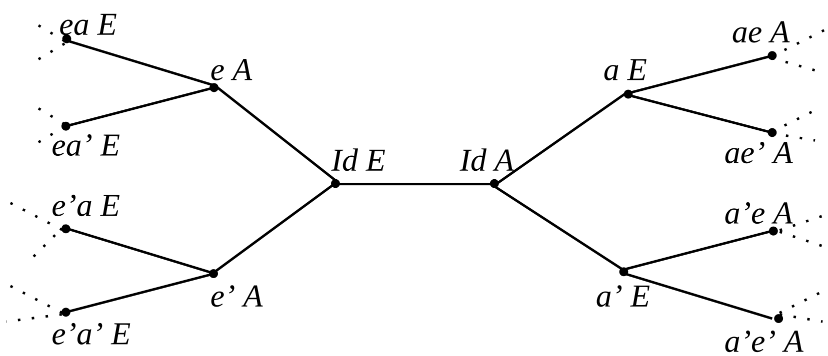

FIGURE 1. A few vertices in the tree $\mathcal{T}$ $(a, a' \in A \setminus E; e, e' \in E \setminus A)$.

There exists a natural metric on the set of vertices of $\mathcal{T}$: if p, q are two vertices, $\mathrm{dist}(p,q) \in \mathbb{N}$ is the number of edges of the shorter (that is, without going back and forth) path from p to q. We will see in Section 3 that the left translation induces a *faithful* representation of Aut[$\mathbb{C}^2$] into the isometry group of $\mathcal{T}$. Thus in the sequel we will identify an element f in Aut[$\mathbb{C}^2$] with the corresponding isometry on $\mathcal{T}$.

It is easy to see that if the action of f admits two fixed points, then all points on the path that links these two points are also fixed by f; so it makes sense to define the subtree Fix(f) fixed by f (do not confuse this set with the set of fixed points of f as an automorphism of $\mathbb{C}^2$...). Note that, by construction, E is the stabilizer of IdE. This immediately implies that for any $g \in$ Aut[$\mathbb{C}^2$], gEg^{-1} is the stabilizer of gE (*idem* with A and S).

Take now f with Fix(f) = $\emptyset$. Consider the set of vertices which realize the infimum $\inf_p \mathrm{dist}(p, fp)$; these vertices define an infinite geodesic on which f acts as a translation (see [9], p. 88). We note Geo(f) this geodesic, and lg(f) ('length of f') the number $\inf_p \mathrm{dist}(p, fp)$. Note that the action of f naturally induces an orientation on Geo(f); however we will not use this notion of an oriented geodesic before Section 5.

Terminology. Given an element f in Aut[$\mathbb{C}^2$] it is therefore equivalent to say that f has elementary type, (*i.e.* is conjugate to an elementary automorphism), or that $d(f) = 1$, or that Fix(f) is non empty. Similarly, the following three properties are equivalent: f is of Hénon type, $d(f) \geq 2$ and Fix(f) is empty. In the text we use these three points of view.

Remark that an affine automorphism is always of elementary type (by the triangulation of matrices).

Finally, we say that a subgroup of Aut[$\mathbb{C}^2$] has degree 1 if all its elements have elementary type.

We gather some straightforward properties in the following proposition:

Proposition 2.1. (1) *If g has Hénon type then lg(g) is always even, because a vertex of type* φ*E (resp.* φ*A) is always sent on a vertex of same type by g;*
(2) *If* $d(g) \geq 2$*, and* $n \in \mathbb{Z}$*, then Geo(*g^n*) = Geo(g) and lg(*g^n*) =* $|n|$*.lg(g).*
(3) *If* $d(f) = 1$ *and* $\varphi \in$ Aut[$\mathbb{C}^2$]*, then Fix(*$\varphi f \varphi^{-1}$*) =* φ*.Fix(f).*
(4) *Similarly, if* $d(g) \geq 2$*, Geo(*$\varphi g \varphi^{-1}$*) =* φ*.Geo(g).*

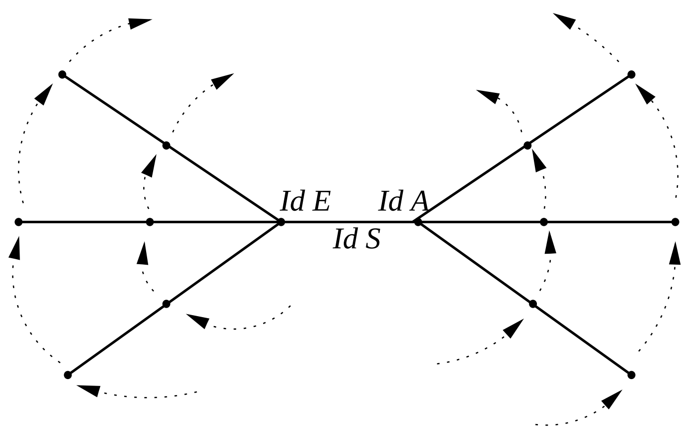

FIGURE 2. Action of an element f with Fix(f) = $\{IdS\}$.

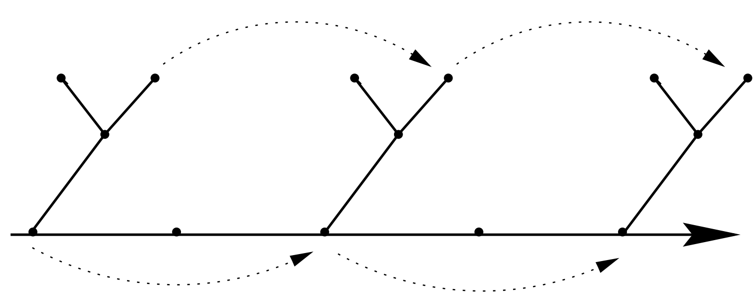

FIGURE 3. Action of an element f with Fix(f) = $\emptyset$ and lg(f) = 2.

We now come back to the notion of an amalgamated product. The fact that Aut[$\mathbb{C}^2$] = $A *_S E$ means that for each f in Aut[$\mathbb{C}^2$] we have a decomposition of the form $f = a_n \circ e_n \circ \cdots \circ a_1 \circ e_1$ where $a_i \in A \setminus S$, $e_i \in E \setminus S$ (possibly f starts with an e_i or ends with an a_i), and this decomposition is unique up to change of the type

$(a_i \circ s^{-1}) \circ (s \circ e_i)$ instead of $a_i \circ e_i$ (where $s \in S$). In particular the "size" of f, *i.e.* the number of e_i and a_i necessary to write f, is well defined (for instance here this is $2n$). We say that f is cyclically reduced if f has minimal size in its conjugacy class. So we see that a difficulty in working with the notion of an amalgamated product is that we do not immediately have a normal form for each element in Aut$[\mathbb{C}^2]$ (as it would be the case for a free product). However if we choose $(a_i)_{i\in I}$ and $(e_j)_{j\in J}$ some representatives of the left cosets A/S and E/S, we obtain a normal form for each element in Aut$[\mathbb{C}^2]$, and for each vertex and edge of $\mathcal{T}$ as well.

Given such system of representatives, consider the set M of all (reduced) words obtained by juxtaposition of a finite number of a_i and e_j:

$$M = \{a_{i_o} e_{j_1} ... e_{j_n} a_{i_n};\ \text{where all } a_{i_k}, e_{j_k} \text{ are non trivial except possibly } a_{i_0} \text{ and } a_{i_n}\}.$$

Then we have a bijection ([9], p. 9):

$$\begin{array}{rcl} M \times S & \to & Aut[\mathbb{C}^2] \\ (a_{i_o} e_{j_1} \cdots a_{i_n}, s) & \to & a_{i_o} \circ e_{j_1} \circ \cdots \circ a_{i_n} \circ s. \end{array}$$

hence the bijections:
$M \to$ edges of $\mathcal{T}$;
$M^e \to$ vertices of type φA in $\mathcal{T}$;
$M^a \to$ vertices of type φE in $\mathcal{T}$.
where by M^e (resp. M^a) we denote the subset of words in M the last (non trivial) element of which is an e_j (resp. an a_i).

All this will be particularly useful since, following Wright [12], one can produce very simple systems of representatives (a_i) and (e_j) in our setting. For all $\lambda \in \mathbb{C}$, and for all $P \in Y^2\mathbb{C}[Y] \setminus \{0\}$ (*i.e.* P is a non zero polynomial such that $P(0) = P'(0) = 0$), we define:

$$a(\lambda) = (\lambda x + y, x);$$

$$e(P) = (x + P(y), y).$$

Then the $(a(\lambda))_{\lambda\in\mathbb{C}}$ (resp. the $(e(P))_{P\in Y^2\mathbb{C}[Y]\setminus\{0\}}$) are systems of representatives of the left cosets A/S (resp. E/S). Therefore an automorphism $\varphi \in$ Aut$[\mathbb{C}^2]$ admits a unique factorization as a composition of $a(\lambda)$ and $e(P)$ (corrected on the right by an automorphism $s \in S$): we shall say that this is the normal form of φ. Similarly we will speak of a normal form for a vertex or an edge in $\mathcal{T}$.

Example 2.2. Consider the Hénon map $g = (y, y^2 + \delta x)$. We have

$$g = a(0) \circ e(y^2) \circ (\delta x, y).$$

The automorphism g corresponds to an edge gS and to two vertices gE, gA which admit respectively $a(0)e(y^2)S$, $a(0)E$ and $a(0)e(y^2)A$ as normal forms.

Remark 2.3. (1) Given g of Hénon type, it is equivalent to say that g is cyclically reduced or that Geo(g) contains the edge IdS. Indeed it is clear that dist(IdE, gE) = dist(IdA, gA) if and only if $IdS \subset$ Geo(g), and in this situation this distance coincide with the size of g.

(2) All the $a(\lambda)$ and $e(P)$ fix the origin in $\mathbb{C}^2$, in consequence $f \in \mathrm{Aut}[\mathbb{C}^2]$ fixes 0 if and only if its normal form is $f = a_{i_0} \circ e_{j_1} \circ \cdots \circ a_{i_n} \circ s$ with $s(0) = 0$, *i.e.* $s = (a_1 x + b_1 y, b_2 y)$.

We now state our main theorem. The proof will be the subject of the next two sections:

Theorem 2.4. *Let G be a subgroup of* $\mathrm{Aut}[\mathbb{C}^2]$. *One, and only one, of the following possibility occurs:*

(1) *G is a group of degree 1 conjugate to a subgroup of E or A.*
(2) *G is group of degree 1 but is not conjugate to a subgroup of E or A. Then G is abelian.*
(3) *G contains elements of Hénon type, and all such automorphisms in G share the same geodesic. Then G is solvable.*
(4) *G contains two elements of Hénon type with distinct geodesics. Then G contains a free subgroup over two generators.*

From this we easily deduce the

Corollary 2.5. *The group* $\mathrm{Aut}[\mathbb{C}^2]$ *satisfies the Tits alternative: If G is a subgroup in* $\mathrm{Aut}[\mathbb{C}^2]$ *then one of the following possibilities occurs:*

(1) *G contains a solvable subgroup with finite index;*
(2) *G contains a non abelian free group.*

Proof. In cases 2, 3 or 4 of the theorem the result is clear. We are left with the case 1, that is with a subgroup of E or A. It is easy to compute the derived subgroups of E:

$$\begin{aligned} E^{(1)} &= [E,E] = \{(x+P(y), y+\gamma); P \in \mathbb{C}[X], \gamma \in \mathbb{C}\}; \\ E^{(2)} &= \{(x+P(y), y); P \in \mathbb{C}[X]\}; \\ E^{(3)} &= \{Id\}. \end{aligned}$$

Thus E is solvable. On the other hand A is clearly a subgroup of $GL_3(\mathbb{C})$ via the injective morphism:

$$\begin{aligned} A &\rightarrow GL_3(\mathbb{C}) \\ (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2) &\rightarrow \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

We conclude using the classical Tits alternative in the context of linear groups, see [6] for a presentation of this difficult result. □

Note that there exist groups which act faithfully on a tree but do not satisfy the Tits alternative. We refer to [11] for such an example: it is an infinite group, of finite type, all elements of which are finite (thanks to E. Ghys for this reference).

3. AUTOMORPHISMS OF ELEMENTARY TYPE.

In this section we focus first on automorphisms with (dynamical) degree 1. In particular we characterize the automorphisms f which admit a non bounded fixed tree $\mathrm{Fix}(f)$. Such automorphisms appear naturally when we consider the commutator of two Hénon type automorphisms which share the same geodesic (case 3 of the main theorem), or when we consider automorphisms which fix an end of the tree (this will occur in the case 2).

Then we study the subgroups of degree 1, which correspond to cases 1 and 2 of Theorem 2.4.

Note that it is not *a priori* completely clear that there exist some automorphisms (except the identity map) which fix a non bounded subtree of $\mathcal{T}$. The following lemma allows us to produce many such examples.

Lemma 3.1. *Let* $f, g \in \mathrm{Aut}[\mathbb{C}^2]$, *with* $d(f) = 1$ *and* $d(g) \geq 2$. *Suppose that* $f \circ g = g \circ f$. *Then Geo(g)* $\subset$ *Fix(f).*

Proof. Let p be a vertex in $\mathrm{Fix}(f)$. Then for any $n \in \mathbb{Z}$

$$f(g^n(p)) = g^n(f(p)) = g^n(p)$$

i.e. $g^n(p) \in \mathrm{Fix}(f)$. Thus for all n the subtree $\mathrm{Fix}(f)$ contains the path from $g^{n-1}(p)$ to $g^n(p)$, and each of these paths contains $\lg(g)$ edges of $\mathrm{Geo}(g)$. The result follows (Fig. 4). □

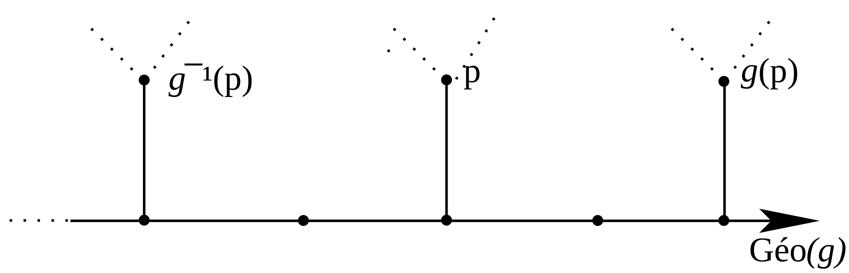

FIGURE 4. $\mathrm{Fix}(f)$ contains the $g^n(p) \Rightarrow \mathrm{Fix}(f)$ contains $\mathrm{Geo}(g)$.

We remark now that for some very simple automorphisms we can apply the previous lemma:

Example 3.2. If $f = (\alpha x, \beta y)$ with α, β roots of the unity of the same order, then it is easy to construct examples of g with $d(g) \geq 2$ such that $f \circ g = g \circ f$. Lemma 3.1 applies: $\mathrm{Fix}(f)$ is a subtree of $\mathcal{T}$ with infinite diameter, because it contains $\mathrm{Geo}(g)$.

- If $\alpha = \beta$ and $\alpha^n = 1$, we can take $g = (y, y^{n+1} + x)$.
- If $\alpha \neq \beta$ then there exists p, $q \geq 2$ such that $\alpha^p = \beta$, $\beta^q = \alpha$. Set $g_1 = (y, y^p + x)$, $g_2 = (y, y^q + x)$, then we can take $g = g_1 \circ g_2$.

The aim of the following proposition is to show that the examples listed in 3.2 are the only one (up to conjugacy). This is a crucial result for the proof of Theorem

2.4; we would like to emphasize that the result is not a mere consequence of Bass-Serre theory but is very particular to the group $\mathrm{Aut}[\mathbb{C}^2]$. In fact we show that if $\mathrm{Fix}(f)$ is bounded then $\mathrm{Fix}(f)$ is small (diameter at most 6), which allows us to do a proof by brute force:

Proposition 3.3. *Let f in $\mathrm{Aut}[\mathbb{C}^2]$ with degree 1. Then the diameter of Fix(f) is infinite if and only if f is conjugate to a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order.*

Proof. Since $d(f) = 1$, up to conjugacy we can assume $f \in E$. Conjugating inside E we can assume that f has one of the following form (see [5]):

(1) $(\alpha x, \beta y)$ with $\alpha, \beta \in \mathbb{C}^*$;
(2) $(x+1, \beta y)$ or $(\beta x, y+1)$ with $\beta \in \mathbb{C}^*$;
(3) $(\beta^d x + \beta^d y^d, \beta y)$ with $d \geq 1$, $\beta \in \mathbb{C}^*$;
(4) $(\beta^d x + \beta^d y^d q(y^r), \beta y)$ with $d \geq 1$, q non constant with higher coefficient equal to +1, β r^{th} root of the unity.

We study now each of these cases, from the less to the more complicated $\mathrm{Fix}(f)$.

In case 4, and in case 3 with $d \geq 2$, even if we allow conjugacy in $\mathrm{Aut}[\mathbb{C}^2]$ we cannot decrease the degree of f (Lemma 6–7 in [5]). In particular f is not conjugate to an element in S, so $\mathrm{Fix}(f)$ is reduced to a unique vertex (of type φE).

In all the remaining cases we have $f \in S$, thus $\mathrm{Fix}(f)$ contains the edge IdS. Recall that $\mathrm{Fix}(f)$ is a tree, so if f fixes another edge it fixes also the whole path from this edge to IdS. The idea is now to use the normal forms to obtain some equations that must be satisfied by f in order for a neighbor edge to be also fixed. For instance we have:

$$f \text{ fixes the edge } a(\lambda)S \Leftrightarrow f \in a(\lambda)Sa(\lambda)^{-1} \Leftrightarrow a(\lambda)^{-1} f a(\lambda) \in S.$$

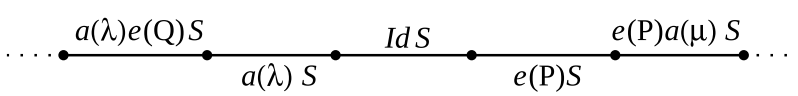

FIGURE 5. Normal forms of the edges next to IdS.

Consider Case 3 with $d = 1$, *i.e.* $f = (\beta x + \beta y, \beta y)$. We have:

$$\begin{aligned}
a(\lambda)^{-1} f a(\lambda) &= (y, x - \lambda y) \circ (\beta x + \beta y, \beta y) \circ (\lambda x + y, x) \\
&= (\beta x, \beta y + \beta x); \\
e(P)^{-1} f e(P) &= (x - P(y), y) \circ (\beta x + \beta y, \beta y) \circ (x + P(y), y) \\
&= (\beta x + \beta y + \beta P(y) - P(\beta y), \beta y).
\end{aligned}$$

Thus f does not fix any edge of the form $a(\lambda)S$, because $(\beta x, \beta y + \beta x) \notin S$. Moreover f fixes an edge $e(P)S$ only if $P(\beta y) = \beta P(y)$. In such a case β is a root of the unity, and we note that $e(P)^{-1} f e(P) = f$. So by the previous computation $a(\lambda)^{-1} e(P)^{-1} f e(P) a(\lambda) \notin S$, that is f does not fix any edge of the form $e(P)a(\lambda)S$. Finally $\mathrm{Fix}(f)$ contains only the edge IdS if β is not a root of the unity, and contains IdS plus some edges of the form $e(P)S$ if β is a root. Thus $\mathrm{Fix}(f)$ has diameter at most 2.

Consider now Case 2, *i.e.* $f=(x+1,\beta y)$ (the case $f=(\beta x,y+1)$ is similar, up to conjugacy by (y,x)). We have:

$$a(\lambda)^{-1}fa(\lambda)=(\beta x,\lambda(1-\beta)x+y+1).$$

So f fixes $a(\lambda)S$ if $\lambda=0$ or if $\beta=1$, and in both cases we have $a(\lambda)^{-1}fa(\lambda)=(\beta x,y+1)$. Conjugating by $e(Q)^{-1}$ we obtain:

$$e(Q)^{-1}\circ(\beta x,y+1)\circ e(Q)=(\beta x+\beta Q(y)-Q(y+1),y+1).$$

This automorphism is in S if and only if $\beta=1$ and $Q(y)=ay^2$. In this case we have $e(Q)^{-1}\circ(x,y+1)\circ e(Q)=(x-2ay-a,y+1)$. A third conjugacy yields:

$$a(\mu)^{-1}\circ(x-2ay-a,y+1)\circ a(\mu)=(x+1,y-2ax-a-\mu)$$

and this automorphism cannot be in S, regardless of the choice for μ. So f does not fix any edge of the form $a(\lambda)e(Q)a(\mu)S$. Consider now the edges $e(P)S$:

$$e(P)^{-1}fe(P)=(x+1+P(y)-P(\beta y),\beta y).$$

We see that f fixes $e(P)S$ as soon as $P(y)=P(\beta y)$ and in this case we have $e(P)^{-1}fe(P)=f$; that is we are reduced to the previous computation We conclude again that $\mathrm{Fix}(f)$ has finite diameter. The case $\beta=1$ gives the maximal possible diameter; the computations above show that in this case $\mathrm{Fix}(f)$ contains edges of the form $a(\lambda)e(Q)S, a(\lambda)S, IdS, e(P)S, e(P)a(\mu)S$ and $e(P)a(\mu)e(Q)S$: we see that $\mathrm{Fix}(f)$ has diameter 6.

Finally let us study Case 1, *i.e.* $f=(\alpha x,\beta y)$. We compute as before:

$$a(\lambda)^{-1}fa(\lambda)=(\beta x,\lambda(\alpha-\beta)x+\alpha y);$$

$$e(P)^{-1}fe(P)=(\alpha x+\alpha P(y)-P(\beta y),\beta y).$$

Thus f fixes $a(\lambda)S$ if $\alpha=\beta$ or if $\lambda=0$, and in both cases we have $a(\lambda)^{-1}fa(\lambda)=(\beta x,\alpha y)$. Moreover f fixes $e(P)S$ if $P(\beta y)=\alpha P(y)$, which implies $\alpha=\beta^n$, where n is the degree of P. Note that in this case $e(P)^{-1}fe(P)=f$.

So $e(Q)^{-1}a(\lambda)^{-1}fa(\lambda)e(Q)\in S$ implies $\beta=\alpha^m$ where m is the degree of Q. It is clear that the existence of $n,m\geq 2$ such that $\alpha^m=\beta$ and $\alpha=\beta^n$ implies that α and β are roots of the unity of the same order. We are in the setting of the examples 3.2, and we have seen that all such examples admit an unbounded fixed subtree. □

Remark 3.4. The computations made in the above proof (in Case 1) allow us to make precise which edges belong to $\mathrm{Fix}(f)$ when this subtree is unbounded. Let $f=(\alpha x,\beta y)$ with α, β roots of the unity of the same order, and let $\varphi S=a(\lambda_1)e(P_1)\cdots a(\lambda_n)e(P_n)S$ be an edge. We distinguish two cases:

(1) $\alpha=\beta$: $\varphi S\in\mathrm{Fix}(f)$ if and only if all P_j satisfy $P_j(\alpha x)=\alpha P_j(x)$ (the λ_i can be arbitrarily chosen). In other words f commutes with each $a(\lambda_i)$ and $e(P_j)$.

(2) $\alpha \neq \beta$: $\varphi S \in \mathrm{Fix}(f)$ if and only if all λ_i are null and the P_j satisfy $P_{2k+1}(\alpha y) = \beta P_{2k+1}(y)$ and $P_{2k}(\beta y) = \alpha P_{2k}(y)$. In other words $e(P_{2k+1})$ (resp. $e(P_{2k})$) commutes with $(\beta x, \alpha y)$ (resp. $(\alpha x, \beta y)$).

We have similar results when the factorization of φ begins with an $e(P)$ or ends with an $a(\lambda)$.

Remark 3.5. It is now clear that the action of $\mathrm{Aut}[\mathbb{C}^2]$ on the tree $\mathcal{T}$ is faithful. Indeed if $f \in \mathrm{Aut}[\mathbb{C}^2]$ acts trivially on the tree, by Proposition 3.3 f is conjugate to a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order, and by the remark above such a rotation fixes the whole tree if and only if it is the identity map. Note that there exist amalgamated products where the induced action is not faithful: for instance in $SL(2,\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ the matrix $-Id$ acts trivially on the Bass-Serre tree.

Corollary 3.6. *Take $f = (\alpha x, \beta y)$ with α, β roots of the unity of the same order, and take $g \in \mathrm{Aut}[\mathbb{C}^2]$ with degree ≥ 2. Assume that Geo(g) $\subset$ Fix(f). Then there exists $\varphi \in \mathrm{Aut}[\mathbb{C}^2]$ such that*

- *$\varphi g \varphi^{-1}$ is cyclically reduced;*
- *$\varphi f \varphi^{-1} = (\alpha x, \beta y)$ or $(\beta x, \alpha y)$.*

Proof. One can take φ such that $\varphi^{-1} S \in \mathrm{Geo}(g)$. Indeed $\varphi^{-1}.\mathrm{Geo}(\varphi g \varphi^{-1}) = \mathrm{Geo}(g)$, so $IdS \in \mathrm{Geo}(\varphi g \varphi^{-1})$ and we apply Remark 2.3. Moreover $\varphi f \varphi^{-1}$ is still diagonal by Remark 3.4. □

We will need the following elementary lemma:

Lemma 3.7. *Let f_1, $f_2 \in \mathrm{Aut}[\mathbb{C}^2]$, with Fix($f_1$) $\cap$ Fix(f_2) unbounded. Then f_1 and f_2 commute; moreover they both admit the same unique fixed point (in $\mathbb{C}^2$).*

Proof. Up to conjugacy we can assume that f_1 and f_2 are in E (since they both fix a vertex of type φE, in fact they even fix infinitely many such vertices). Then the commutator $h := f_1 f_2 f_1^{-1} f_2^{-1}$ has the form $(x + P(y), y + \gamma)$; since $\mathrm{Fix}(h)$ is unbounded by Proposition 3.3 h has finite order. We conclude that $h = Id$.

Since f_1 and f_2 have unbounded fixed subtree, they are both conjugate to a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order. In particular each one has a unique fixed point in $\mathbb{C}^2$, and since they commute their fixed points must coincide. □

We are now in position to describe the subgroups of degree 1 in $\mathrm{Aut}[\mathbb{C}^2]$. A first idea could be that such a group must be conjugate to a subgroup of A or E. We will see that this is not the case in general (examples of Wright), but only true under additional assumptions. We will need the following three lemmas:

Lemma 3.8. *Let G be a subgroup of degree 1, and let f, $g \in G$. Then Fix(g) $\cap$ Fix(f) $\neq \emptyset$.*

Proof. The idea is to consider $p \in \mathrm{Fix}(g \circ f)$, and q the middle point of the path from p to $f(p)$. Then $q \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ (see [9], Proposition 26 p. 89). □

Lemma 3.9. *Let X be a tree, and $X_1,\dots,X_n$ some subtrees with non empty pairwise intersection. Then $\bigcap_i X_i \neq \emptyset$.*

Proof. See [9], Lemma 10 p. 91.

Lemma 3.10. *Let X be a tree, and $(X_i)_{i\in I}$ a family of subtrees of X, such that:*

(1) $X_i \cap X_j \neq \emptyset \quad \forall i,j \in I$;
(2) *There exists Y a bounded subtree of X, such that $\forall i \in I, X_i \subset Y$.*

Then $\bigcap_i X_i \neq \emptyset$.

Proof. We proceed by induction on the diameter n of Y. If $n = 0$ (*i.e.* if Y is reduced to a single vertex), then for all i we have $X_i = Y$ and so $\bigcap_i X_i = Y$. If $n \geq 1$, then either there exists a terminal vertex of Y (*i.e.* a vertex which belongs to only one edge) contained in all X_i, and we are done, or there is no such vertex and we can do the same reasoning on $Y' := Y \setminus \{$ terminal vertices and edges of Y $\}$, $X_i' := X_i \cap Y'$. □

We can now state

Proposition 3.11. *Let G be a subgroup of degree 1 of* $\text{Aut}[\mathbb{C}^2]$. *Assume that one of the following assumptions is satisfied:*

(1) *G is finitely generated;*
(2) *G contains an element f with Fix(f) bounded.*

Then G is conjugate to subgroup of A or E.

Proof. Suppose first that G is finitely generated, and write $G = \langle g_1,\cdots,g_n\rangle$, $g_i \in \text{Aut}[\mathbb{C}^2]$. We set $X_i = \text{Fix}(g_i)$. Lemma 3.8 says that the pairwise intersection of the X_i are non empty, and so by Lemma 3.9 their global intersection contains at last one vertex P. This vertex P is equal to φA or φE (with $\varphi \in \text{Aut}[\mathbb{C}^2]$), and G is contained in the stabilizer of P, *i.e.* $G \subset \varphi A \varphi^{-1}$ or $\varphi E \varphi^{-1}$.

Consider now Case 2, *i.e.* there exists $f \in G$ with $\text{Fix}(f)$ bounded. For $g \in G$ we set $X_g = \text{Fix}(g) \cap \text{Fix}(f)$. If g_1, $g_2 \in G$, Lemma 3.8 applid three times (to the pairs (g_1, f), (g_2, f) and (g_1,g_2)) implies that $X_{g_1} \cap X_{g_2} \neq \emptyset$. Since each element in the family $(X_g)_{g\in G}$ is contained in the bounded tree $\text{Fix}(f)$, we are exactly in the setting of Lemma 3.10. Thus G is again contained in the stabilizer of a vertex P (where $P \in \bigcap_g X_g$). □

With respect to the existence of subgroups of degree 1 which are not conjugate to a subgroup of E or A, we refer to [12] where an explicit example is given. In the following proposition we characterize such subgroups:

Proposition 3.12. *Let G be a subgroup of degree 1 which is not conjugate to a subgroup of A or E. Then:*

(1) *G is abelian;*
(2) *G is equal to the union of an increasing sequence of groups H_i , $i \in \mathbb{N}$, where each H_i is conjugate to a finite cyclic group generated by a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order;*

(3) *Each element of G admits a unique fixed point (as an automorphism of $\mathbb{C}^2$) and this point is the same for all elements of G;*
(4) *The action of G fixes an end of the tree $\mathcal{T}$.*

Proof. Since G is not conjugate to a subgroup of A or E, it does not fix any vertex By Proposition 3.11 we obtain that each element of G admits an unbounded fixed subtree, *i.e.* is conjugate to a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order. Moreover, if $f, g \in G$ then $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is unbounded, indeed otherwise we could apply Lemma 3.10 with $Y = \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$, $X_{g_i} = \mathrm{Fix}(g_i)$ $\cap Y$, and this would contradict the fact that G is not contained in the stabilizer of any vertex. By Lemma 3.7, we get (1) and (3).

Let us now show the assertion (2). Again by Proposition 3.11 the group G is not finitely generated. Moreover if $f, g \in G$ have the same order then there exists $n \in \mathbb{Z}$ such that $f^n = g$. Indeed otherwise one could assume (up to conjugacy) that $f, g \in E$ and we could find $m \in \mathbb{N}$ such that

$$f^m \circ g = (\alpha x + P(y), \beta y + \gamma) \neq Id$$

with $\alpha = 1$ or $\beta = 1$. But this is impossible because on one hand α and β are roots of the same order, so $\alpha = \beta = 1$, and $f^m \circ g$ must be of finite order. Thus there exists a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that n_i is the order of an element in G. Define H_i as the subgroup of G generated by all elements with order less or equal to n_i. This group H_i is finite and contains at least one element f_i with maximal order. Let us show that f_i generate H_i. Let $g \in H_i$; there exists $n \in \mathbb{N}$ such that f_i^n and g have the same order. By the previous reasoning we have $f_i^{n'} = g$ for some $n' \in \mathbb{N}$.

Finally it is clear that $\mathrm{Fix}(f_{i+1}) \subset \mathrm{Fix}(f_i)$, moreover $\bigcap_i \mathrm{Fix}(f_i)$ is empty, and we obtain (4) (see [9, p. 92–93]). □

4. AUTOMORPHISMS OF HÉNON TYPE.

Given $g \in \mathrm{Aut}[\mathbb{C}^2]$ of degree ≥ 2 we want to characterize all f which commute with g, or on the contrary all f such that there is no relation between f and g.

To make precise the relative positions of two geodesics (associated with automorphisms) we will use the following

Proposition 4.1. *Let $f, g \in \mathrm{Aut}[\mathbb{C}^2]$ with $d(g) \geq 2$ and $d(f) = 1$. Suppose Geo(g) $\cap$ Fix(f) is unbounded. By Proposition 3.3 we can write $\varphi f \varphi^{-1} = (\alpha x, \beta y)$ with α, β roots of the unity of the same order, and then:*

(1) *If $\alpha \neq \beta$, $\alpha^n = \beta$, $\beta^n = \alpha$ and lg(g) = 2 mod 4, then $g f g^{-1} = \varphi^{-1}(\beta x, \alpha y)\varphi = f^n$ (and so f and g^2 commute);*
(2) *In all other cases f and g commute.*

In particular $Geo(g) \subset Fix(f)$.

Proof. Conjugating by φ we can assume $f = (\alpha x, \beta y)$, and so $f(0) = 0$. Recall that $\text{Fix}(gfg^{-1}) = g.\text{Fix}(f)$, and so:

$$\text{Geo}(g) \cap \text{Fix}(f) \text{ unbounded } \Rightarrow \text{Fix}(gfg^{-1}) \cap \text{Fix}(f) \text{ unbounded}.$$

By Lemma 3.7 we deduce that gfg^{-1} fixes 0, and so g also fixes 0.

So we can write $g = m \circ s$, where $m = a(\lambda_n) \circ e(P_n) \circ \cdots \circ a(\lambda_1) \circ e(p_1)$ and $s = (a_1 x + b_1 y, b_2 y)$ (precisely we can assume m of this form up to conjugacy). Up to conjugacy again, the edges mS and $msa(\lambda_n)S$ are in Fix(f), and we have $msa(\lambda_n)S = ma(\mu_n)S$ where μ_n is such that $a(\mu_n)^{-1}sa(\lambda_n) \in S$.

If $\alpha = \beta$ then f commutes with s (easy) and with m (Remark 3.4), and so also with g.

If $\alpha \neq \beta$ then in the expression above we have $\mu_n = \lambda_n = 0$ (Remark 3.4), hence $s = (a_1 x, b_2 y)$. Thus f still commutes with s. Moreover, again by Remark 3.4, we have $g \circ (\alpha x, \beta y) \circ g^{-1} = (\alpha x, \beta y)$ (resp. $(\beta x, \alpha y)$) when lg(g) = 0 mod 4 (resp. 2 mod 4). Finally, in the second case, one checks that there exists n such that $\alpha^n = \beta$ and $\beta^n = \alpha$. □

Corollary 4.2. *Let f and g be two automorphisms of Hénon type. Then either Geo(f) = Geo(g), or Geo(f) ∩ Geo(g) is bounded (possibly empty).*

Proof. Assume that Geo(f) ∩ Geo(g) is unbounded. Then by taking powers of f and g (this does not change the respective geodesics), one can assume that lg(f) = lg(g), lg(f) = 0 mod 4 and that f and g induce the same orientation on Geo(f) ∩ Geo(g).

Then the automorphism fg^{-1} fixes an infinite number of vertices in Geo(g), so by Proposition 4.1 we get that g and fg^{-1} commute, and so

$$g(fg^{-1})f^{-1} = (fg^{-1})gf^{-1} = Id$$

In other words f and g commute, and by looking at the action on the tree we check easily that Geo(f) = Geo(g) (figure 6). □

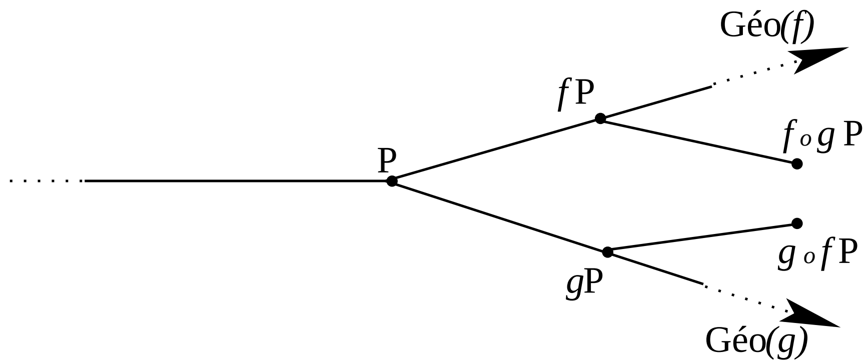

FIGURE 6. Geo(f) $\neq$ Geo(g) and Geo(f) ∩ Geo(g) unbounded $\Rightarrow f \circ g \neq g \circ f$.

If we have $f, g \in \mathrm{Aut}[\mathbb{C}^2]$ of Hénon type with Geo(f) = Geo(g), we deduce from Proposition 3.3 that there exists a relation between f and g. Indeed $fgf^{-1}g^{-1}$ fixes Geo(g), so there is a n such that $(fgf^{-1}g^{-1})^n = Id$. We now study the case Geo(f) $\neq$ Geo(g):

Proposition 4.3. *Let $f,\ g \in \mathrm{Aut}[\mathbb{C}^2]$ with degree ≥ 2, with Geo(f) $\neq$ Geo(g). Assume lg(f) $> N$ and lg(g) $> N$ where N is the diameter of Geo(f) $\cap$ Geo(g).*

Then f and g generate a free group, moreover all elements (except the identity) in $\langle f, g\rangle$ have degree ≥ 2.

Proof. We must check that for any h defined by

$$h = f^{n_p} \circ g^{m_p} \circ \cdots f^{n_1} \circ g^{m_1}$$

where the n_i, m_i are in $\mathbb{Z} \setminus \{0\}$, we have $d(h) \geq 2$ (and so in particular $h \neq Id$). This is equivalent to check that such an h does not fix any vertex in $\mathcal{T}$.

Let Q be a vertex in $\mathcal{T}$. We introduce the following notations: $\mathrm{dist}_f(Q)$ is the distance from Q to Geo(f) (and similarly for g), and $\mathcal{T}_f$ (resp. $\mathcal{T}_g$) is the subtree of $\mathcal{T}$ containing all vertices P such that $\mathrm{dist}_f(P) \leq \mathrm{dist}_g(P)$ (resp. $\mathrm{dist}_f(P) \geq \mathrm{dist}_g(P)$). We deduce the result, by induction on p, from the two following assertions. Take $n,\ m \in \mathbb{Z} \setminus \{0\}$ and set $Q' = (f^n \circ g^m)Q$.

(1) If $Q \in \mathcal{T}_f$, then $Q' \in \mathcal{T}_f$ and $\mathrm{dist}_f(Q') > \mathrm{dist}_f(Q)$.

Indeed the assumption lg(g) $> N$ implies $g^m(Q) \in \mathcal{T}_g$ along with the strict inequality:

$$\mathrm{dist}_g(g^m(Q)) < \mathrm{dist}_f(g^m(Q)).$$

The assumption lg(f) $> N$ then implies $Q' = f^n \circ g^m(Q) \in \mathcal{T}_f$. Moreover we have

$$\mathrm{dist}_g(g^m(Q)) = \mathrm{dist}_g(Q) \geq \mathrm{dist}_f(Q);$$

The equality is clear, the inequality comes from $Q \in \mathcal{T}_f$. Thus

$$\mathrm{dist}_f(g^m(Q)) > \mathrm{dist}_f(Q) \Rightarrow \mathrm{dist}_f(f^n \circ g^m(Q)) > \mathrm{dist}_f(f^n(Q)) = \mathrm{dist}_f(Q).$$

(2) If $Q \in \mathcal{T}_g$ and $Q' \in \mathcal{T}_g$, then $\mathrm{dist}_g(Q') < \mathrm{dist}_g(Q)$.

It is clear that $g^m(Q) \notin \mathcal{T}_g$ (otherwise $f^n \circ g^m(Q) \in \mathcal{T}_f$); in other words

$$\mathrm{dist}_g(g^m(Q)) > \mathrm{dist}_f(g^m(Q))$$

and the result comes from the relations

$$\mathrm{dist}_g(Q) = \mathrm{dist}_g(g^m(Q));$$

$$\mathrm{dist}_f(g^m(Q)) = \mathrm{dist}_f(f^n \circ g^m(Q)) \geq \mathrm{dist}_g(f^n \circ g^m(Q)). \quad \square$$

Remark 4.4. This proof is essentially a "ping-pong" argument, which a classical technical tool in this kind of problem (see [6]). However it seemed interesting to us to make a few more computations to obtain that *all* elements of G have Hénon type. In particular in Section 5 we use the fact that $d(fgf^{-1}g^{-1}) \geq 2$.

We obtain the two corollaries:

Corollary 4.5. *Let f, g be of Hénon type. If Geo(f) $\neq$ Geo(g), then $\langle g, f\rangle$ contains a non abelian free group.*

Proof. Let N be the diameter of $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$. If we take n such that $\mathrm{lg}(f^n) > N$ and $\mathrm{lg}(g^n) > N$, then we have $\langle f^n, g^n \rangle = \mathbb{Z} * \mathbb{Z}$. □

Corollary 4.6. *Let f, g be of Hénon type. If $g \circ f = f \circ g$, then Geo(f) = Geo(g).*

Our next aim is to compute the centralizer

$$\mathrm{Cent}(g) = \{f \in \mathrm{Aut}[\mathbb{C}^2];\ f \circ g = g \circ f\}$$

of an automorphism g of Hénon type.

Lemma 4.7. *Take $g \in \mathrm{Aut}[\mathbb{C}^2]$ with degree ≥ 2. We set*

$$H = \{f \in Cent(g);\ d(f) = 1\}.$$

Then H is conjugate to a group generated by a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order.

Proof. Note first that for each $f \in H$ the tree $\mathrm{Fix}(f)$ is unbounded, in fact by Lemma 3.1)it contains $\mathrm{Geo}(g)$. Thus by Proposition 3.3 f has finite order. Moreover the order of f is bounded by the (dynamical) degree of g. Indeed f induces a permutation on the set of fixed points (in $\mathbb{C}^2$) of g, which has cardinal equal to the degree of g if we count multiplicities (see [5]). Let $f_0 \in H$ be an element of maximal order. We now use similar arguments as in the proof of Proposition 3.12. If $h \in H$ then the order of f_0 is a multiple of the order of h, moreover if h_1 and h_2 are two elements in H with the same order, then there exist n and $m \in \mathbb{N}^*$ such that $h_1^n = h_2$ and $h_2^m = h_1$. Finally $H = \langle f_0 \rangle$ which gives the expected result. □

Proposition 4.8. *Take $g \in \mathrm{Aut}[\mathbb{C}^2]$ of degree ≥ 2. Then Cent(g) is generated by two elements h and f satisfying:*

1. *$d(h) \geq 2$ and Geo(g) = Geo(h);*
2. *f is conjugate to a rotation $(\alpha x, \beta y)$ with α, β roots of the unity of the same order;*
3. *There exists n such that $f \circ h = h \circ f^n$.*

In particular $Cent(g)$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$, where p is the order of f.

Proof. The automorphism f is given by Lemma 4.7, and among all automorphisms of degree ≥ 2 in $\mathrm{Cent}(g)$ (which all share the same geodesic by Corollary 4.6) we choose h which minimizes $\mathrm{lg}(h)$.

If $\varphi \in \mathrm{Cent}(g)$ has degree ≥ 2, then $\mathrm{lg}(h)$ divides $\mathrm{lg}(\varphi)$ (use an euclidean division), so there exists $q \in \mathbb{Z}$ such that $\varphi \circ h^q$ have degree 1, *i.e.* $\varphi \circ h^q \in \langle f \rangle$ and so we have $\varphi \in \langle h, f \rangle$.

The integer n comes from Proposition 4.1, so n is equal either to 1 or to $(p+1)/2$. □

Remark 4.9. (1) We should compare this situation with the case of an automorphism of degree 1 which always admits an uncountable centralizer.

(2) Most of the time the integer n of the proposition is equal to 1, *i.e.* the group $\mathrm{Cent}(g)$ is often isomorphic to $\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ (or even to $\mathbb{Z}$).
(3) Be warned that the converse of Corollary 4.6 is false. For instance, if we take $g=(y,y^2+x)$ and $f=(jx,j^2y)$ where j is a cubic root of the unity, then g and $g\circ f$ are both automorphisms of Hénon type which share the same geodesic but do not commute.

To finish the proof of Theorem 2.4 it only remains to consider the case of a subgroup all elements of which have Hénon type with the same geodesic. Such subgroups are always solvable as the following proposition shows.

Proposition 4.10. *Let Γ be an infinite geodesic, such that $\Gamma = Geo(g)$ for some $g\in \mathrm{Aut}[\mathbb{C}^2]$. Then there exists a unique subgroup G of $\mathrm{Aut}[\mathbb{C}^2]$, maximal for the property: "All elements of Hénon type in G admit Γ as a geodesic".*

Moreover G is solvable, and contains a subgroup of finite index isomorphic to $\mathbb{Z}$.

Proof. If G exists it must contain all iterates of g. Note first that any ψ[1] in such a group G must globally preserve Γ. Indeed assume $\psi(\Gamma)\neq\Gamma$, then we would have $d(\psi\circ g\circ\psi^{-1})\geq 2$ with $\mathrm{Geo}(\psi\circ g\circ\psi^{-1}) = \psi\Gamma\neq\Gamma$, which contradicts the assumption on G.

So we are lead to set G equal to the group of all automorphisms which globally preserve Γ. We want to show that G is solvable. For this, let us find some generators for G. We distinguish three subsets in G which form a partition:

(1) The set T_Γ of automorphisms that act on Γ by a (non trivial) translation;
(2) The group F_Γ of automorphisms that fix Γ;
(3) The set S_Γ of automorphisms that act on Γ by symmetry with respect to a vertex.

Note that F_Γ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, indeed $F_\Gamma\subset\mathrm{Cent}(g^2)$ (Proposition 4.1) and we can apply Lemma 4.7.

Choose $h\in T_\Gamma$ which minimizes $\lg(h)$, $f\in F_\Gamma$ a generator of F_Γ and $\varphi\in S_\Gamma$ arbitrary ($\varphi = Id$ if $S_\Gamma=\emptyset$). We claim that $G=\langle h,f,\varphi\rangle$. Indeed if $\psi\in T_\Gamma$, then $\lg(h)$ must divide $\lg(\psi)$, so $\psi = h^n\circ f^q$. On the other hand if $\psi\in S_\Gamma$, either ψ and φ have the same center and $\psi\circ\varphi\in\langle f\rangle$, or ψ and φ do not have the same center and $\psi\circ\varphi\in T_\Gamma$. In both cases $\psi\in<h,f,\varphi>$.

Let us show that G is solvable. We have:

$$[T_\Gamma,F_\Gamma]\subset F_\Gamma,$$
$$[T_\Gamma,S_\Gamma]\subset T_\Gamma,$$
$$[F_\Gamma,S_\Gamma]\subset F_\Gamma,$$

where $[T_\Gamma,F_\Gamma]$ is the set of commutators of the form $h_0f_0h_0^{-1}f_0^{-1}$ with $h_0\in T_\Gamma$ and $f_0\in F_\Gamma$.

Thus $G^{(1)} = [G,G]\subset<T_\Gamma,F_\Gamma>$, and $G^{(2)}\subset F_\Gamma$ is abelian.

[1] In the original paper this is a ϕ. I was young and nobody told me that we are not supposed to use both \phi and \varphi in the same paper...

Finally the quotient of G by the finite group $\langle f, \varphi\rangle$ is isomorphic to $\mathbb{Z}$ (the generator is the class of h in $G/\langle f, \varphi\rangle$). $\square$

Remark 4.11. If G is a group where all elements of degree ≥ 2 share the same geodesic Γ, if there exists $\varphi \in G$ that acts by symmetry on Γ, let us show that all elements of G have a Jacobian determinant of module 1.

According to the proof of the proposition above $G = \langle h, f, \varphi\rangle$ with $\mathrm{Geo}(h) = \Gamma$, $\Gamma \subset \mathrm{Fix}(f)$. Note that $|\det D\varphi| = 1$, indeed $\varphi^2 \in \langle f\rangle$ and $|\det Df| = 1$. But we can also write $G = \langle f, \varphi, \varphi \circ h\rangle$, and $\varphi \circ h$ acts by symmetry on Γ. Thus all three generators admit a Jacobian determinant of module 1.

A summary of the proof of Theorem 2.4. Cases 1 and 2 have been taken care of in Section 3. Precisely Case 2 corresponds to the "groups of Wright" which are described in Proposition 3.12.

Case 3 is given by Proposition 4.10, we see that these groups are solvable groups of a very particular form.

Finally Case 4 corresponds to Corollary 4.5; note that this result is a mere corollary of Bass-Serre theory, on the contrary the more precise statements 4.2 and 4.3 are particular to $\mathrm{Aut}[\mathbb{C}^2]$.

In the way through the proof we obtained to auxiliary results interesting in their own rights:

(1) Description of all $f \in \mathrm{Aut}[\mathbb{C}^2]$ with $\mathrm{Fix}(f)$ unbounded (Proposition 3.3);
(2) Description of the centralizer of an automorphism of Hénon type (Proposition 4.8).

5. Green functions and Fatou-Bieberbach domains

We first review a few facts and definitions about the dynamics of automorphisms of degree ≥ 2; for more details we refer to [1, 2] and [10].

Consider $f \in \mathrm{Aut}[\mathbb{C}^2]$ of Hénon type; we have seen that we can write $f = \varphi g \varphi^{-1}$ where $\varphi \in \mathrm{Aut}[\mathbb{C}^2]$ and g is a composition of generalized Hénon maps. We set:

$$K_g^+ = \{(x,y) \in \mathbb{C}^2 \text{ such that } \{g^n(x,y)\}_{n\in\mathbb{N}} \text{ is bounded}\};$$
$$J_g^+ = \partial K_g^+ \ (\partial \text{ is the topological boundary}).$$

The set $J_g^+ \subset \mathbb{C}^2$ is the (positive) Julia set of g. We introduce also the Green function associated with g:

$$G_g^+ = \lim_{n\to+\infty} \frac{1}{d(g)^n} \log^+ \|g^n\|.$$

The map G_g^+ is continuous, positive and plurisubharmonic, and satisfies the properties:

- $K_g^+ = \{G_g^+ = 0\}$;
- $G_g^+ \circ g = d(g).G_g^+$;

- G_g^+ is pluriharmonic on $\mathbb{C}^2 \setminus K_g^+$.

The Green current associated with g is the positive closed $(1,1)$-current with potential G_g^+:

$$\mu_g^+ = \frac{i}{\pi}\partial\overline{\partial}G_g^+ = \sum \frac{\partial^2 G_g^+}{\partial z_i \partial \overline{z_j}} dz_i \wedge d\overline{z_j}.$$

ts support is exactly the Julia set J_g^+. Note that $g^*\mu_g^+ = d(g).\mu_g^+$.

We obtain similar notions by considering the backward iterations: we define in this way K_g^-, J_g^-, G_g^- and μ_g^-. It is interesting to consider the extension of g to $\mathbb{CP}^2$; there is two particular points on the line at infinity. One the one hand we have $p_- = [1:0:0]$ which is an indeterminacy point, on the other hand we have $p_+ = [0:1:0]$ which is a superattracting fixed point with basin equal to $\{G_g^+ > 0\}$ in $\mathbb{C}^2$.

We define the mass of a current T on $\mathbb{CP}^2$ by mean of the standard Kähler form ω:

$$||T|| = \int_{\mathbb{CP}^2} T \wedge \omega.$$

The current μ_g^+ (or more precisely its trivial extension to $\mathbb{CP}^2$) has mass 1.

One can extend the definitions and properties above to f by taking

$$G_f^+ = k.G_g^+ \circ \varphi^{-1} = k.\varphi^{-1^*}G_g^+;$$

$$\mu_f^+ = k.\varphi^{-1^*}\mu_g^+ = k.\frac{i}{\pi}\partial\overline{\partial}(\varphi^{-1^*}G_g^+).$$

where $k > 0$ is chosen in order to get $||\mu_f^+|| = 1$.

We have the following remarkable result (see [10]):

Theorem 5.1 (Sibony)**.** *Let T be a positive closed $(1,1)$-current in $\mathbb{CP}^2$ with mass 1 and with support contained in $\overline{K_g^+}$. Then $T = \mu_g^+$.*

From this we are able to deduce the following result:

Proposition 5.2. *Let f be of Hénon type, and $h \in \mathrm{Aut}[\mathbb{C}^2]$. Assume that $G_f^+ \circ h = G_f^+$. Then $d(h) = 1$.*

Proof. Suppose that $d(h) \geq 2$. We know (see [1]) that h admits a periodic saddle point p and that J_h^+ (resp. J_h^-) is the closure of the stable variety (resp. instable) associated with p. We deduce from the relation $G_f^+ \circ h = G_f^+$ that J_h^+ and J_h^- belong to a same level $\{G_f^+ = c\}$.

If $c = 0$ then the theorem of Sibony implies $\mu_h^+ = \mu_f^+$, hence $h^*\mu_f^+ = d(h).\mu_f^+$. But the assumption $G_f^+ \circ h = G_f^+$ implies $h^*\mu_f^+ = \mu_f^+$: a contradiction.

If $c > 0$ we also want to get a contradiction. First the set P of periodic saddle points of h does not contain any isolated point, indeed the closure of P is the support of the equilibrium measure $\mu_h^+ \wedge \mu_h^-$ (see [3]), and the support of this measure is nowhere locally polar ([1]). Now G_f^+ is pluriharmonic in a neighborhood of p, so locally G_f^+ is the real part of an holomorphic function φ. Up to a small perturbation of p one can assume that φ is a submersion or the n power of a a submersion in a neighborhood of p. Thus locally $\{G_f^+ = c\}$ is diffeomorphic to a real hyperplane or to n real hyperplanes meeting at p, and each one of these hyperplanes contains only one complex direction. In particular the stable and unstable varieties associated with p cannot be contained in the level $\{G_f^+ = c\}$. □

Remark 5.3. It might be well known that all levels of the Green function are smooth (except of course the level 0): this would simplify he end of the proof above. However I do not know any simple demonstration of this fact.

We can now state the main theorem of this section: we establish an equivalence between the notions of (oriented) geodesics and of Green functions associated with an automorphism of Hénon type.

Theorem 5.4. *Let f, $g \in \mathrm{Aut}[\mathbb{C}^2]$ of Hénon type. The following assertions are equivalent:*

(1) *Geo(f) = Geo(g) with the same orientations;*
(2) *There exist $n,m \in \mathbb{N}^*$ such that $f^n = g^m$;*
(3) $G_g^+ = G_f^+$.

Proof. Assume that $\mathrm{Geo}(g) = \mathrm{Geo}(f) = \Gamma$. In vienw of the proof of Proposition 4.10 we know that there exist two automorphisms h and e such that:

- h is of Hénon type and $\mathrm{Geo}(h) = \Gamma$;
- e has order r and fixes Γ;
- $f = h^{n_1} e^{p_1}$, $g = h^{n_2} e^{p_2}$, with $n_1, n_2, p_1, p_2 \in \mathbb{N}$.

By taking the square of f and g we can assume that n_1 and n_2 are even, this by Proposition 4.1 e commutes with h^{n_1} and h^{n_2}. Taking $n = rn_2$ and $m = rn_1$, we obtain:

$$f^n = g^m = h^{rn_1n_2}.$$

We just proved $(1) \Rightarrow (2)$; moreover $(2) \Rightarrow (3)$ is immediate. Let us show $(3) \Rightarrow (1)$. Assume $G^+ := G_g^+ = G_f^+$. If $\mathrm{Geo}(f) \neq \mathrm{Geo}(g)$ then up to taking powers of f and g (which does not change G^+), one can assume that f and g generate a free group all elements of which (except Id) have degree ≥ 2 (Proposition 4.3) In particular $fgf^{-1}g^{-1}$ has degree ≥ 2. The two relations

$$G^+ \circ f = d(f).G^+$$
$$G^+ \circ g = d(g).G^+$$

imply $G^+ \circ fgf^{-1}g^{-1} = G^+$. But then Proposition 5.2 implies $d(fgf^{-1}g^{-1}) = 1$, hence a contradiction. □

Remark 5.5. In the same spirit it is possible to show that two automorphisms Hénon type share the same geodesic if and only if they have the same Julia sets, or if and only if they admit the same invariant measure; however for the proof we need to study more closely the notions of Green functions and Green current, in particular in the case where g and g^{-1} have the same indeterminacy points (see [8], Theorem 2.24).

Consider now $g \in \mathrm{Aut}[\mathbb{C}^2]$ of degree ≥ 2; assume that g admits an attracting periodic point p. Up to taking a power, and up to conjugacy by a translation, we can assume that $p = 0$ is an attracting fixed point. Define Σ the attracting basin of 0: this is Fatou-Bieberbach domain, *i.e.* a domain biholomorphic to $\mathbb{C}^2$ strictly included in $\mathbb{C}^2$. In the following proposition, which makes precise a result in [4], we compute the group $\mathrm{Aut}[\Sigma]$ of all $f \in \mathrm{Aut}[\mathbb{C}^2]$ preserving Σ:

Proposition 5.6. *With the notations above, Aut*$[\Sigma]$ *contained exactly all automorphisms which fix 0 (in* $\mathbb{C}^2$*) and which preserve globally Geo(g). Moreover Aut*$[\Sigma]$ *does not contain any automorphism acting by symmetry on Geo*(g)*. Finally Aut*$[\Sigma]$ *is isomorphic to a semi-direct product* $\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z}$ *pour some* $n \in \mathbb{N}^*$.

Proof. To show that all elements in $\mathrm{Aut}[\Sigma]$ preserve $\mathrm{Geo}(g)$ it is sufficient to show that for any $f \in \mathrm{Aut}[\Sigma]$ of Hénon type we have $\mathrm{Geo}(f) = \mathrm{Geo}(g)$ (see poof Proposition 4.10). Up to taking powers of f and g it is sufficient to show that $d(fgf^{-1}g^{-1}) = 1$. Note that fgf^{-1} admits an attracting fixed point $p = f(0)$ with basin Σ, where $J_g^+ = J^+_{fgf^{-1}} = \partial\Sigma$ (see [2]). Then by the theorem of Sibony $\mu_g^+ = k.\mu^+_{fgf^{-1}}$ (with $k > 0$), hence $G_g^+ = k.G^+_{fgf^{-1}}$ (indeed $\partial\overline{\partial}(G_g^+ - k.G^+_{fgf^{-1}}) = 0$, so we have a pluriharmonic map null over Σ, hence everywhere null). Finally we obtain the relation $G_g^+ \circ fgf^{-1}g^{-1} = G_g^+$, hnce $d(fgf^{-1}g^{-1}) = 1$ by Proposition 5.2.

Note that f cannot act by symmetry on $\mathrm{Geo}(g)$; otherwise by Remark 4.11 g would have a Jacobian determinant of module 1 and this would contradict the existence of an attracting fixed point.

Let us show now that any $f \in \mathrm{Aut}[\Sigma]$ fixes 0. We just showed that f preserves $\mathrm{Geo}(g)$, which implies that f commutes with g^m for some m (if $\mathrm{d}(f) = 1$ then we can take $m = 2$ by Proposition 4.1; and if $\mathrm{d}(f) \geq 2$ we know that $f^n = g^m$ for some n and m). Consider $w \in \Sigma$, we have $f(w) \in \Sigma$, and

$$\lim_{k\to+\infty} g^k(f(w)) = 0 \Rightarrow \lim_{k\to+\infty} g^{km}(f(w)) = 0 \Rightarrow \lim_{k\to+\infty} f(g^{km}(w)) = 0 \Rightarrow f(0) = 0.$$

Conversely assume that $f(0) = 0$, and that f globally preserves $\mathrm{Geo}(g)$. There exists $m \in \mathbb{N}$ such that f commutes with g^m, so f preserves K_g^+, in particular f acts on the connected components of the interior of K_g^+. Since $f(0) = 0$, we deduce that the connected component of K_g^+ containing 0 is fixed by f, in other words Σ is fixed by f.

The group of automorphisms that preserve $\mathrm{Geo}(g)$ without reversing the orientation on $\mathrm{Geo}(g)$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ for some $p \in \mathbb{N}^*$ (see proof of Proposition 4.10). The group $\mathrm{Aut}[\Sigma]$ is a subgroup of this group, and so is also a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z}$ (in fact it is easy to show that $n = 1$ or p). $\square$

During the redaction of this article I benefited from numerous discussions: I thank M. Nicolau, W. Dicks, N. Sibony and more particularly D. Cerveau who proposed me the problems studied in Section 5. The idea to use the Bass-Serre theory to study the subgroups of $\mathrm{Aut}[\mathbb{C}^2]$ was suggested by E. Ghys.

References

1. E. Bedford and J. Smillie, *Polynomial diffeomorphisms of* $\mathbb{C}^2$ *: currents, equilibrium measure and hyperbolicity.*, Inven. Math. **103** (1991), 69–99.
2. ______, *Polynomial diffeomorphisms of* $\mathbb{C}^2$ *II : stable manifolds and recurrence.*, J. Amer. Soc. **4** (1991), 657–679.
3. E. Bedford, J. Smillie, and M. Lyubish, *Polynomial diffeomorphisms of* $\mathbb{C}^2$ *IV : the measure of maximal entropy and laminar currents.*, Inven. Math. **112** (1993), 77–125.
4. D. Cerveau, *Sur la linéarisation de certains sous-groupes de difféomorphismes polynomiaux du plan et les domaines de Fatou-Bieberbach.*, Prépublication IRMAR (1997).
5. S. Friedland and J. Milnor, *Dynamical properties of plane polynomial automorphisms.*, Erg. Th. and Dyn. Sys. **9** (1989), 67–99.
6. P. de la Harpe, *Free groups in linear groups.*, Ens. Math. **29** (1983), 129–144.
7. S. Lamy, *L'alternative de Tits pour* $\mathrm{Aut}[\mathbb{C}^2]$., CRAS **327** (1998), 537–540.
8. ______, *Automorphismes polynomiaux du plan complexe : étude algébrique et dynamique*, Ph.D. thesis, Toulouse, 2000.
9. J.-P. Serre, *Arbres, amalgames,* sl_2., Astérisque, vol. 46, SMF, 1977.
10. N. Sibony, *Dynamique des applications rationnelles de* $\mathbb{P}^k$., Dynamique et géoétrie complexe, Panoramas et synthèses 8., 1999, pp. 97–195.
11. S. Sidki, *On a 2-generated infinite 3-group : the presentation problem.*, J. of Algebra **110** (1987), 13–23.
12. D. Wright, *Abelian subgroups of* $Aut_k(k[x,y])$ *and applications to actions on the affine plane.*, Illin. J. Math. **23** (1979), 579–635.