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# NORMAL SUBGROUP GENERATED BY A PLANE POLYNOMIAL AUTOMORPHISM

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**Abstract.** We study the normal subgroup $\langle f \rangle_N$ generated by an element $f \neq \mathrm{id}$ in the group $G$ of complex plane polynomial automorphisms having Jacobian determinant 1. On the one hand, if $f$ has length at most 8 relative to the classical amalgamated product structure of $G$, we prove that $\langle f \rangle_N = G$. On the other hand, if $f$ is a sufficiently generic element of even length at least 14, we prove that $\langle f \rangle_N \neq G$.

## Introduction

Let $\mathrm{Aut}[\mathbb{C}^2]$ denote the group of complex plane polynomial automorphisms and let $G$ be the subgroup of automorphisms having Jacobian determinant 1. In this paper we deal with normal subgroups of $G$ generated by a single element.

It is easy to check that $G$ is equal to the commutator subgroup of $\mathrm{Aut}[\mathbb{C}^2]$ and to its own commutator subgroup as well (see Proposition 10). It is more difficult to decide whether $G$ is a simple group or not. There does not seem to exist any natural morphism whose kernel is a proper normal subgroup of $G$. However, in a short note published in 1974 that seems to have been quite forgotten, Danilov [Dan74] proves that $G$ is not a simple group. He uses results from Schupp [Sch71], namely the so-called small cancellation theory in the context of an amalgamated product. To be precise, he shows that the normal subgroup generated by the automorphism $(ea)^{13}$ where $a = (y, -x)$ and $e = (x, y + 3x^5 - 5x^4)$ is a strict subgroup of $G$. In fact, he writes $(ea)^{12}$ because he uses a slightly erroneous definition of the condition $C'(\frac{1}{6})$ (see Subsection 3.1).

We now introduce the algebraic length of an automorphism in order to state our main result. The theorem of Jung, Van der Kulk and Nagata asserts that $\mathrm{Aut}[\mathbb{C}^2]$ is the amalgamated product over their intersection of the groups $A$ and $E$ of affine and elementary automorphisms (see Subsection 1.1). Let $f$ be an element of $\mathrm{Aut}[\mathbb{C}^2]$. If $f$ is not in the amalgamated part $A \cap E$, its algebraic length $|f|$ is defined as the least integer $m$ such that $f$ can be expressed as a composition

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$f = g_1 \dots g_m$, where each $g_i$ is in some factor ($A$ or $E$) of $\mathrm{Aut}[\mathbb{C}^2]$. If $f$ is in the amalgamated part, by convention we set $|f| = 0$ (see [Ser77, §1.3]).

The normal subgroup generated by an element $f$ of $G$ will be denoted by $\langle f \rangle_N$. Of course, $\langle f \rangle_N$ remains unchanged when replacing $f$ by one of its conjugates in $G$. So, one can assume $f$ of minimal algebraic length in its conjugacy class (see Subsection 1.4). If $|f| \neq 1$, this amounts to saying that $|f|$ is even (indeed, if $|f|$ is even, it is clear that $f$ is strictly cyclically reduced in the sense of Subsection 3.1 below). This is for example the case for the previous automorphism $(ea)^{13}$ which has length 26.

Here are the two main results of our paper.

**Theorem 1.** *If $f \in G$ satisfies $|f| \leqslant 8$ and $f \neq \mathrm{id}$, then $\langle f \rangle_N = G$.*

**Theorem 2.** *If $f \in G$ is a generic element of even length $|f| \geqslant 14$, then the normal subgroup generated by $f$ in $\mathrm{Aut}[\mathbb{C}^2]$ (or a fortiori in $G$) is different from $G$.*

Here the genericness means that if we write $f^{\pm 1} = a_1 e_1 \dots a_l e_l$, where $l \geqslant 7$, $a_1, \dots, a_l \in A \setminus E$ and each $e_i = (x + P_i(y), y)$, then there exists an integer $D$ such that for any sequence $d_1, \dots, d_l$ of integers $\geqslant D$, $(P_1, \dots, P_l)$ can be chosen generically (in the sense of algebraic geometry, i.e., outside a Zariski-closed hypersurface) in the affine space $\prod_{1 \leqslant i \leqslant l} \mathbb{C}[y]_{\leqslant d_i}$, where we have set $\mathbb{C}[y]_{\leqslant d} = \{P \in \mathbb{C}[y]; \deg P \leqslant d\}$.

Theorems 1 and 2 correspond to Theorems 32 and 45 in the text below. Note that in the latter statements we use a geometric notion of length coming from Bass–Serre theory (see Subsection 1.2). This geometric length allows us to obtain more natural statements. In fact, Theorem 45 deals with automorphisms satisfying the special condition (C2) (see Definition 27). The proof that this condition is indeed generic is postponed to the annex. To convince the reader that such a condition is necessary, we now give examples of automorphisms of arbitrary even length and generating normal subgroups equal to $G$.

**Example 3.** Consider the three automorphisms

$$a = (y, -x), \qquad e_1 = (x + P(y), y), \qquad e_2 = (x + Q(y), y),$$

where $P$ (resp. $Q$) is an even (resp. odd) polynomial of degree $\geqslant 2$, and set $f = ae_1(ae_2)^n$ where $n \geqslant 1$ is an integer. If $u = -\mathrm{id}$, we get $au = ua$, $e_2 u = ue_2$ and $e_1 u = ue_1^{-1}$, so that the commutator $[f, u] = fuf^{-1}u^{-1}$ is equal to

$$[f, u] = ae_1(ae_2)^n u(ae_2)^{-n}(ae_1)^{-1}u^{-1} = ae_1 ue_1^{-1}a^{-1}u^{-1} = ae_1^2 a^{-1}.$$

Since $[f, u] \in \langle f \rangle_N$, we get $e_1^2 \in \langle f \rangle_N$, so that $\langle f \rangle_N = G$ by Theorem 1 (or by Lemma 30 below).

One motivation for this work is the still open question of the simplicity of the Cremona group $\mathrm{Cr}_2$, i.e., the group of birational transformations of $\mathbb{C}^2$. For instance, in [Giz94] the question is explicitly stated and Gizatullin gives several criteria that would prove that $\mathrm{Cr}_2$ is simple. Recently Blanc [Bla10] proved that $\mathrm{Cr}_2$ is simple as an infinite-dimensional algebraic group. In this respect we should mention that Shafarevich claimed that the group $\mathrm{Aut}_1[\mathbb{C}^n]$ of automorphisms of the affine space $\mathbb{C}^n$ having Jacobian determinant 1 is simple as an infinite-dimensional algebraic group for any $n \geqslant 2$ (see [Sha66, Theorem 5] and [Sha81, Theorem 5]). However, it is known that these two papers contain some inaccuracies (see [Kam96], [Kam03]), so the status of this question is not clear to us.

After studying the polynomial case, our opinion is that $\mathrm{Cr}_2$, view as an abstract group, could be not simple as well. Indeed, it is known since Iskovskikh [Isk85] that $\mathrm{Cr}_2$ admits a presentation as the quotient of an amalgamated product by the normal subgroup generated by a single element. Take $H_1 = (\mathrm{PGL}(2) \times \mathrm{PGL}(2)) \rtimes \mathbb{Z}/2\mathbb{Z}$, the group of birational transformations that extend as automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$, and take $H_2$ as the group of transformations that preserve the pencil of vertical lines $x = cte$. Take $\tau = (y, x) \in H_1 \setminus H_2$ and $e = (1/x, y/x) \in H_2 \setminus H_1$; then $\mathrm{Cr}_2$ is equal to the quotient

$$(H_1 *_{H_1 \cap H_2} H_2) \,/\langle f \rangle_N$$

where $f = (\tau e)^3$. To prove that $\mathrm{Cr}_2$ is not simple it would be sufficient to find an element $g$ in the amalgamated product of $H_1$ and $H_2$ (that should correspond to a sufficiently general birational transformation) such that the normal subgroup $\langle f, g \rangle_N$ is proper. This is similar to the results we obtain in this paper; but the problem seems harder in the birational setting.

As a final remark on these matters, we would like to mention a nice reinterpretation of Iskovskikh's result by Wright (see [Wri92, Theorem 3.13]). Let $H_3 = \mathrm{PGL}(3)$ be the group of birational transformations that extend as automorphisms of $\mathbb{P}^2$. Then Wright proves that the group $\mathrm{Cr}_2$ is the free product of $H_1, H_2$, and $H_3$ amalgamated along their pairwise intersection in $\mathrm{Cr}_2$.

In this paper we chose to work over the field $\mathbb{C}$ of complex numbers, even if most of the results could be adapted to any base field. Note that in the case of a finite field the nonsimplicity result is almost immediate. Let $\mathbb{F}_q$ denote the finite field of $q = p^n$ elements, where $p$ is prime and $n \geqslant 1$. Let $\mathrm{Aut}[\mathbb{F}_q^2]$ be the group of automorphisms of the affine plane $\mathbb{A}^2_{\mathbb{F}_q} = \mathbb{F}_q^2$ and let $\mathrm{Aut}_1[\mathbb{F}_q^2]$ be the normal subgroup of automorphisms having Jacobian determinant 1. If $X$ is a finite set, let $\mathfrak{Per}(X)$ (resp. $\mathfrak{Per}^+(X)$) be the group of permutations (resp. even permutations) of $X$. Since the natural morphism $\phi : \mathrm{Aut}[\mathbb{F}_q^2] \to \mathfrak{Per}(\mathbb{F}_q^2)$ induces a nonconstant morphism $\mathrm{Aut}_1[\mathbb{F}_q^2] \to \mathfrak{Per}(\mathbb{F}_q^2)$ (consider the translations!), it is clear that $\mathrm{Aut}_1[\mathbb{F}_q^2]$ is not simple.

*Remark 4.* If $q$ is odd (i.e., the characteristic $p$ of $\mathbb{F}_q$ is odd), one can easily check that $\phi(\mathrm{Aut}_1[\mathbb{F}_q^2]) = \mathfrak{Per}^+(\mathbb{F}_q^2)$. Indeed, $\phi$ is surjective (see [Mau01]), so that $\phi(\mathrm{Aut}_1[\mathbb{F}_q^2])$ is a normal subgroup of $\mathfrak{Per}(\mathbb{F}_q^2)$. However, if the cardinal of $X$ is different from 4, it is well known that $\mathfrak{Per}^+(X)$ is the only nontrivial normal subgroup of $\mathfrak{Per}(X)$ (see, e.g., [Rot95, Ex. 3.21, p. 51]). Therefore, it is enough to show that $\phi(\mathrm{Aut}_1[\mathbb{F}_q^2]) \subseteq \mathfrak{Per}^+(\mathbb{F}_q^2)$. But on one hand, $\mathrm{Aut}_1[\mathbb{F}_q^2]$ is generated by the elementary automorphisms $(x + P(y), y)$ and $(x, y + Q(x))$ where $P \in \mathbb{C}[y]$, $Q \in \mathbb{C}[x]$ are any polynomials. On the other hand, it is straightforward to check that such automorphisms induce even permutations of $\mathbb{F}_q^2$.

As a final remark we would like to stress the importance of translations in getting our results. Let $\mathrm{Aut}^0[\mathbb{C}^2]$ be the group of automorphisms fixing the origin and let $J_n$ be the natural group-morphism associating to an element of $\mathrm{Aut}^0[\mathbb{C}^2]$ its $n$-jet at the origin (for $n \geqslant 1$). For $n \geqslant 2$, the kernel of $J_n$ is a nontrivial normal subgroup of $G^0 = G \cap \mathrm{Aut}^0[\mathbb{C}^2]$, so that this latter group is not simple. Of course, for $\mathrm{Aut}[\mathbb{C}^2]$ the morphism $J_n$ does not exist. This explains the fact that our paper strongly relies on translations (see Lemmas 7 and 16).

*Remark 5.* It results from [Ani83] that the image of $J_n$ is exactly the group of $n$-jets of polynomial endomorphisms fixing the origin and whose Jacobian determinant is a nonzero constant. The precise statement can be found in [Fur07, Prop. 3.2].

The paper is organized as follows.

In Section 1 we gather the results from the Bass–Serre theory that we need: this includes some basic definitions and facts but also some quite intricate computations, such as in the characterization of tripods (Subsection 1.7). This is also the place where we define precisely the condition (C2) that we need in Theorem 45.

Section 2 is devoted to the proof of Theorem 1. This is the most elementary part of the paper. We only use Lemma 7 from Section 1.

In Section 3 we deal with R-diagrams. This field of combinatorial group theory has been introduced by Lyndon and Schupp in relation with condition $C'(\frac{1}{6})$ from small cancellation theory (see Subsection 3.1). A noteworthy feature of our work is that we use R-diagrams in a completely opposite setting (positive curvature).

In Section 4 we are able to give a proof of Theorem 2 using the full force of both Bass–Serre and Lyndon–Schupp theories.

We briefly discuss in Section 5 the cases not covered by Theorems 1 and 2, that is to say when the automorphism has length 10 or 12.

Finally, in the Annex, we prove that condition (C2) is generic and we also give explicit examples of automorphisms satisfying this condition.

## 1. The Bass–Serre tree

### 1.1. Generalities

The classical theorem of Jung, van der Kulk and Nagata states that the group $\mathrm{Aut}[\mathbb{C}^2]$ is the amalgamated product of the *affine* group

$$A = \{(\alpha x + \beta y + \gamma, \delta x + \epsilon y + \zeta); \alpha, \ldots, \epsilon \in \mathbb{C}, \alpha\epsilon - \beta\delta \neq 0\}$$

and the *elementary* group

$$E = \{(\alpha x + P(y), \beta y + \gamma); \alpha, \beta, \gamma \in \mathbb{C}, \alpha\beta \neq 0, P \in \mathbb{C}[y]\}$$

over their intersection (see [Jun42], [vdK53], [Nag72]). This is usually written in the following way.

**Theorem 6.** $\mathrm{Aut}[\mathbb{C}^2] = A *_{A\cap E} E$.

A geometric proof of this theorem and many references may be found in [Lam02]. Let us also recall that elements of $E$ are often called *triangular* automorphisms.

The Bass–Serre theory [Ser77] associates a simplicial tree to any amalgamated product. In our context, let us denote this tree by $\mathcal{T}$. By definition, the set of vertices of $\mathcal{T}$ is the disjoint union of the left cosets modulo $A$ (vertices of *type* $A$) and modulo $E$ (vertices of *type* $E$). The edges of $\mathcal{T}$ are the left cosets modulo $(A \cap E)$. Finally, if $\phi \in \mathrm{Aut}[\mathbb{C}^2]$, the edge $\phi(A \cap E)$ links the vertices $\phi A$ and $\phi E$. Since $\mathrm{Aut}[\mathbb{C}^2]$ is generated by $A$ and $E$, $\mathcal{T}$ is connected. Thanks to the amalgamated structure, $\mathcal{T}$ contains no loop, so that it is indeed a tree.

The group $\mathrm{Aut}[\mathbb{C}^2]$ acts naturally on $\mathcal{T}$ by left multiplication: for any $g, \phi \in \mathrm{Aut}[\mathbb{C}^2]$, we set $g.\phi A = (g\phi)A$, $g.\phi E = (g\phi)E$ and $g.\phi(A \cap E) = (g\phi)(A \cap E)$. It turns out that this action gives an embedding of $\mathrm{Aut}[\mathbb{C}^2]$ into the group of simplicial isometries of $\mathcal{T}$ (see [Lam01, Remark 3.5]). This action is transitive on the set of edges, on the set of vertices of type $A$ and on the set of vertices of type $E$. The stabilizer of a vertex $\phi A$ (resp. of a vertex $\phi E$, resp. of an edge $\phi(A \cap E)$) is the group $\phi A \phi^{-1}$ (resp. $\phi E \phi^{-1}$, resp. $\phi(A \cap E)\phi^{-1}$).

Following [Wri79], [Lam01], one can define systems of representatives of the nontrivial left cosets $A/A \cap E$ and $E/A \cap E$ by taking

$$\begin{aligned} a(\lambda) &= (\lambda x + y, -x), \qquad && \lambda \in \mathbb{C}, \\ e(Q) &= (x + Q(y), y), \qquad && Q(y) \in y^2\mathbb{C}[y] \setminus \{0\}. \end{aligned}$$

Note that the minus sign in the expression of $a(\lambda)$ did not appear in [Wri79], [Lam01]. We have to introduce it in the present paper in order to get automorphisms with Jacobian determinant 1 (see Subsection 1.4).

Then any element $g \in \mathrm{Aut}[\mathbb{C}^2]$ may be uniquely written $g = ws$ where $w$ is a product of factors of the form $a(\lambda)$ or $e(Q)$, successive factors being of different forms, and $s \in A \cap E$ (see, e.g., [Ser77, Chap. I, 1.2, Theorem 1]). Similarly, any edge (resp. vertex of type $A$, resp. vertex of type $E$) may be uniquely written $w(A \cap E)$ (resp. $wA$, resp. $wE$) where $w$ is as above.

We call a (directed) *path* a sequence of consecutive edges in $\mathcal{T}$. To denote a path we enumerate its vertices separated by $-$. For instance, the path $\mathcal{P}$ of two edges containing the vertices $\mathrm{id}A$, $\mathrm{id}E$, $eA$, where $e \in E \setminus A$ will be denoted $\mathcal{P} = \mathrm{id}A - \mathrm{id}E - eA$. If we are only interested in the type of the vertices we say, for example, that $\mathcal{P}$ is of type $A - E - A$.

If two vertices of $\mathcal{T}$ are fixed by an automorphism of $\mathrm{Aut}[\mathbb{C}^2]$, then the path relating them is also fixed. Therefore, the subset of $\mathcal{T}$ fixed by an automorphism is either empty or a subtree. Up to conjugation, this subset has been computed for any automorphism in [Lam01, Proof of Prop. 3.3]. In particular, it has been computed for the translation $(x+1, y)$. The following easy and technical lemma is a slight variation of this computation. As in the latter paper, this analogous statement turns out to be very useful. The proof is given for the sake of completeness.

**Lemma 7.** *The subtree of $\mathcal{T}$ fixed by the translation $(x + c, y)$, $c \in \mathbb{C}^*$, is exactly the union of the paths*

$$\mathrm{id}E - e(P)A - e(P)a(\lambda)E - e(P)a(\lambda)e(Q)A,$$

*where $P \in y^2\mathbb{C}[y]$, $\lambda \in \mathbb{C}$, and $Q(y) = \alpha y^2$, $\alpha \in \mathbb{C}^*$.*

*Note that we (exceptionally) allow $P$ to be zero. In that case, the path should rather be written*

$$\mathrm{id}\,E - \mathrm{id}\,A - a(\lambda)E - a(\lambda)e(Q)A.$$

*In particular, the fixed subtree does not depend on $c$, has diameter 6 and contains the closed ball of radius 2 centered at $\mathrm{id}\,E$, i.e., the union of the paths*

$$\mathrm{id}\,E - e(P)A - e(P)a(\lambda)E, \qquad P \in y^2\mathbb{C}[y], \quad \lambda \in \mathbb{C}.$$

*Proof.* If $P, Q \in y^2\mathbb{C}[y]$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}(x+c,y)\circ e(P) &= e(P)\circ(x+c,y),\\ (x+c,y)\circ a(\lambda) &= a(\lambda)\circ(x,y+c),\\ (x,y+c)\circ e(Q) &= e(Q)\circ f,\end{aligned}$$

where $f = (x + Q(y) - Q(y+c), y+c)$, so that

$$(x+c,y)e(P)a(\lambda)e(Q) = e(P)a(\lambda)e(Q)f.$$

Therefore, the vertex $e(P)a(\lambda)e(Q)A$ is fixed by $(x+c,y)$ if and only if $f \in A$, i.e., $\deg(Q(y) - Q(y+c)) \leqslant 1$, i.e., $\deg(Q) \leqslant 2$. If $Q = \alpha y^2$, this vertex is fixed. Since the vertex $\mathrm{id}\,E$ is also (obviously) fixed, this shows that the following path is fixed:

$$\mathrm{id}\,E - e(P)A - e(P)a(\lambda)E - e(P)a(\lambda)e(Q)A.$$

If $Q = \alpha y^2$, where $\alpha \neq 0$ and $\mu \in \mathbb{C}$, it remains to show that the vertex $e(P)a(\lambda)e(Q)a(\mu)E$ is not fixed. Indeed, an easy computation shows that

$$(x+c,y)e(P)a(\lambda)e(Q)a(\mu) = e(P)a(\lambda)e(Q)a(\mu)g,$$

where $g = (x - c, 2\alpha c x + y + \mu c - \alpha c^2) \notin E$. $\square$

### 1.2. Algebraic and geometric lengths

We will use two notions of *length* on $\mathrm{Aut}[\mathbb{C}^2]$.

The *algebraic length* has been defined in the Introduction: if $g \in \mathrm{Aut}[\mathbb{C}^2]$ is not in the amalgamated part, $|g|$ is defined as the least integer $m$ such that $g$ can be expressed as a composition $g = g_1 \dots g_m$ where each $g_i$ is in some factor of the amalgam. If $g$ is in the amalgamated part, we set $|g| = 0$.

The *geometric length* is defined by $\lg(g) = \inf_{v\in\mathcal{V}} \mathrm{dist}(g.v, v)$, where $\mathcal{V}$ is the set of vertices of $\mathcal{T}$ and $\mathrm{dist}(\cdot\,,\cdot)$ is the simplicial distance on $\mathcal{T}$.

By Lemma 8 we almost always have $\lg(g) = \min\{|\phi g \phi^{-1}|; \phi \in \mathrm{Aut}[\mathbb{C}^2]\}$, the only exception being when $g$ is conjugate to an elementary automorphism which is not conjugate to an element in the amalgamated part.

### 1.3. Elliptic and hyperbolic elements

Elements $g$ of $\mathrm{Aut}[\mathbb{C}^2]$ may be sorted into two classes according to their action on $\mathcal{T}$.

If $\lg(g) = 0$ (i.e., $g$ has at least one fixed point on $\mathcal{T}$), we say that $g$ is *elliptic.* This corresponds to the case where $g$ is conjugate to an element belonging to some factor ($A$ or $E$) of $\mathrm{Aut}[\mathbb{C}^2]$. Since any element of $A$ is conjugate to some element of $E$, this amounts to saying that $g$ is triangularizable (i.e., conjugate to some triangular automorphism).

If $\lg(g) > 0$, we say that $g$ is *hyperbolic.* This corresponds to the case where $g$ is conjugate to a composition of generalized Hénon transformations $h_1 \dots h_l$ (see [FM89]). We recall that a generalized Hénon transformation is a map of the form

$$h = (y, ax + P(y)) = (y, x) \circ (ax + P(y), y),$$

where $a \in \mathbb{C}^*$ and $P(y)$ is a polynomial of degree at least 2. Equivalently, $g$ is conjugate to an automorphism of the form

$$f = a_1 e_1 \dots a_l e_l,$$

where each $a_i \in A \setminus E$ and each $e_i \in E \setminus A$.

The set of points $v \in \mathcal{T}$ satisfying $\mathrm{dist}(g.v, v) = \lg(g)$ defines an infinite geodesic of $\mathcal{T}$ denoted by $\mathrm{Geo}(g)$. Furthermore, $g$ acts on $\mathrm{Geo}(g)$ by translation of length $\lg(g)$. It is not difficult to check that $\lg(g) = \lg(f) = |f| = 2l$ and that the geodesic of $f$ is composed of the path $\mathrm{id}A - a_1E - a_1e_1A - \cdots - a_1e_1 \dots a_le_lA$ and its translations by the $f^k$'s ($k \in \mathbb{Z}$). If $g = \phi f \phi^{-1}$ with $\phi \in \mathrm{Aut}[\mathbb{C}^2]$, we have, of course, $\mathrm{Geo}(g) = \phi(\mathrm{Geo}(f))$.

The proof of the following easy result is left to the reader. Note that these two sets of equivalent conditions correspond to the notions of strictly and weakly cyclically reduced elements given in Subsection 3.1.

**Lemma 8.** *Let $g \in \mathrm{Aut}[\mathbb{C}^2]$ be a hyperbolic element.*

(1) *The following assertions are equivalent:*
   (i) $|g| = \lg(g)$*;*
   (ii) $\mathrm{Geo}(g)$ *contains the vertices* $\mathrm{id}A$ *and* $\mathrm{id}E$*.*

(2) *The following assertions are equivalent:*
   (iii) $|g| \leqslant \lg(g) + 1$*;*
   (iv) $\mathrm{Geo}(g)$ *contains the vertex* $\mathrm{id}A$ *or* $\mathrm{id}E$*.*

### 1.4. The group $G$

In this subsection we prove two basic facts about $G$. Let us set $A_1 = A \cap G$ and $E_1 = E \cap G$. Theorem 6 easily implies the following result.

**Proposition 9.** $G = A_1 *_{A_1 \cap E_1} E_1$.

*Proof.* By [Ser77, Chap. I, no. 1.1, Prop. 3] it is sufficient to prove that any $g \in G$ is a composition of affine and triangular automorphisms with Jacobian determinant 1. We know that we can write $g$ as a composition of $a(\lambda)$ and $e(Q)$, with a correcting term $s \in A \cap E$. Note that the $a(\lambda)$ and $e(Q)$ are automorphisms with

Jacobian determinant 1, so $s$ is also of Jacobian determinant 1 and we are done. □

As a consequence of this proposition the whole discussion of the previous subsection still applies to $G$. In particular, we can make the same choice of representatives $a(\lambda)$ and $e(Q)$ to write edges and vertices, so that there exists a natural bijection between the trees associated to $\mathrm{Aut}[\mathbb{C}^2]$ and to $G$.

**Proposition 10.** *The group $G$ is the commutator subgroup of the group $\mathrm{Aut}[\mathbb{C}^2]$, and is also equal to its own commutator subgroup.*

*Proof.* Using Proposition 9 it is sufficient to check that the commutator subgroup of $G$ contains $\mathrm{SL}(2,\mathbb{C})$ and all triangular automorphisms of the form $(x+P(y),y)$. But, on the one hand, it is well known that $\mathrm{SL}(2,\mathbb{C})$ is equal to its own commutator subgroup; on the other hand, any triangular automorphism $(x+\lambda y^n, y)$, with $n \geqslant 2$ and $\lambda \in \mathbb{C}$, is the commutator of $(x+\lambda(1-b)^{-1}y^n, y)$ and $\big(bx, b^{-1}y\big)$ where $b \neq 1$ is a $n$th root of the unity. Finally, any translation $(x+c,y)$ is the commutator of $(-x,-y)$ and $(x-c/2,y)$. □

## 1.5. The color

We now introduce the *color* of a path of type $A-E-A$. This notion will be used to make precise the genericness assumptions we need. Note that any path of type $A-E-A$ can be written $\mathcal{P} = \psi e_1 A - \psi E - \psi e_2 A$ where $\psi \in \mathrm{Aut}[\mathbb{C}^2]$ and $e_1, e_2 \in E$.

**Definition 11.** The *color* of $\mathcal{P}$ is the double coset $(A\cap E)e_1^{-1}e_2(A\cap E)$.

One verifies easily that this definition does not depend on the choice of $e_1, e_2$. The color is clearly invariant under the action of $\mathrm{Aut}[\mathbb{C}^2]$. In fact, given two paths of type $A-E-A$ one could even show that one can send one to the other (by an element of $\mathrm{Aut}[\mathbb{C}^2]$) if and only if they have the same color. However, we will not use this result. For an illustration of the notion of color we can note that the color of the path $e(P)A - e(P)a(\lambda)E - e(P)a(\lambda)e(Q)A$ appearing in Lemma 7 has color $(A\cap E)e(Q)(A\cap E)$.

If $P \in \mathbb{C}[y]$ is such that the color of $\mathcal{P}$ is equal to the double coset $(A \cap E)e(P)(A\cap E)$, we say that $P$ represents the color of $\mathcal{P}$. The following lemma implies that this notion does not depend on the orientation of the path. Its proof is easy and is left to the reader.

**Lemma 12.** *Let $P, Q \in \mathbb{C}[y]$ be polynomials of degree $\geqslant 2$. Then $P$ and $Q$ represent the same color if and only if there exist $\alpha, \dots, \epsilon$ with $\alpha\beta \neq 0$ such that $Q(y) = \alpha P(\beta y+\gamma) + \delta y + \epsilon$.*

*Remark 13.* Note that any path of type $A-E-A$ can be sent by an automorphism to a path of the form $\mathrm{id}A - \mathrm{id}E - e(P)A$. It is easy to check that the vertices $e(P)A$ and $e(Q)A$ are equal if and only if there exists $\alpha, \beta \in \mathbb{C}$ such that $Q(y) = P(y) + \alpha y + \beta$.

**Fundamental Example 14.** Let $g$ be a hyperbolic automorphism of geometric length $\lg(g) = 2l$. We know that $g$ is conjugate to an automorphism of the form $f = a_1 e_1 \dots a_l e_l$ where each $a_i \in A \setminus E$ and each $e_i \in E \setminus A$. Then the geodesic of $g$ (and $f$) carries the $l$ colors $(A \cap E)e_i(A \cap E)$ $(1 \leqslant i \leqslant l)$ which are repeated periodically.

### 1.6. General color

**Definition 15.** A polynomial $P \in \mathbb{C}[y]$ of degree $d \geqslant 5$ is said to be *general* if it satisfies:

$$\forall \alpha, \beta, \gamma \in \mathbb{C}, \quad \deg(P(y) - \alpha P(\beta y + \gamma)) \leqslant d - 4 \quad \Longrightarrow \quad \alpha = \beta = 1 \text{ and } \gamma = 0.$$

The color $(A \cap E)e(P)(A \cap E)$ is said to be *general* if $P$ is general. Lemma 12 implies that this notion does not depend on the choice of a representative $P$.

**Lemma 16.** *Let $Q \in y^2\mathbb{C}[y]$ be general. The stabilizer of the path $\mathcal{P} = e(Q)A - \mathrm{id}E - \mathrm{id}A$ is equal to $\{(x + \beta y + \gamma, y); \beta, \gamma \in \mathbb{C}\}$. Furthermore, if $\beta \neq 0$, the automorphism $(x + \beta y + \gamma, y)$ does not fix any path strictly containing $\mathcal{P}$.*

*Proof.* We know that $f \in \mathrm{Aut}[\mathbb{C}^2]$ fixes the path $\mathrm{id}E - \mathrm{id}A$ if and only if $f \in A \cap E$. In this case there exist constants $\alpha, \dots, \zeta$, with $\alpha\varepsilon \neq 0$ such that $f = (\alpha x + \beta y + \gamma, \varepsilon y + \zeta)$. Since $fe(Q) = e(Q)g$, where $g = (\alpha x + \beta y + \alpha Q(y) - Q(\varepsilon y + \zeta), \varepsilon y + \zeta)$, the vertex $e(Q)A$ is fixed by $f$ if and only if $g \in A$, i.e., $\deg(\alpha Q(y) - Q(\varepsilon y + \zeta)) \leqslant 1$. The polynomial $Q$ being general, this is equivalent to $\alpha = \varepsilon = 1$ and $\zeta = 0$.

The second assertion comes from the following simple observation:

$$(x + \beta y + \gamma, y)a(\lambda)E = a(\lambda - \beta)E.$$

Indeed, since $(x + \beta y + \gamma, y)e(Q) = e(Q)(x + \beta y + \gamma, y)$, we also have

$$(x + \beta y + \gamma, y)e(Q)a(\lambda)E = e(Q)a(\lambda - \beta)E.$$

Therefore, the vertices $a(\lambda)E$ and $e(Q)a(\lambda)E$ are fixed by $(x + \beta y + \gamma, y)$ if and only if $\beta = 0$. □

*Remark 17.* Lemma 16 is a kind of converse to Lemma 7. Precisely, we obtain that if $\phi$ fixes a general path of four edges centered on $\mathrm{id}E$, then $\phi = (x + c, y)$ (here by general we mean that the color supported by the two central edges of the path is general; see Definition 11 and below).

Note also that since $(x, y + c) = a(0) \circ (x - c, y) \circ a(0)^{-1}$, the subset of $\mathcal{T}$ fixed by $(x, y + c)$ is the image by $a(0)$ of the subset fixed by $(x - c, y)$. In particular, it contains the closed ball of radius 2 centered at $a(0)E$. Furthermore, if $\phi$ fixes a general path of four edges centered at $a(0)E$, it can be written as $\phi = (x, y + c)$.

We now apply the notion of a general color to prove a technical result that we need to prove Theorem 45. We consider a hyperbolic automorphism $f$ and $g = \varphi f \varphi^{-1} \neq f$ a conjugate of $f$. We want to show that if $f$ is sufficiently general then $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$ is a path of length at most 4. More precisely, we also describe all possible types of such paths.

**Definition 18.** We say that a hyperbolic automorphism of geometric length $2l$ satisfies condition (C1) if the $l$ colors supported by its geodesic (see Example 14) are general and distinct.

In the Annex we show that this condition is generic in a natural sense.

**Proposition 19.** *Let $f$ and $g = \phi f \phi^{-1}$ be two distinct conjugate automorphisms satisfying condition* (C1). *If the intersection* $\operatorname{Geo}(f) \cap \operatorname{Geo}(g)$ *contains at least one edge then this path is of type*

$$A - E, \quad E - A - E, \quad A - E - A \quad \text{or} \quad E - A - E - A - E.$$

*Proof.* There is no restriction to assume that $\mathcal{P}' = \operatorname{Geo}(f) \cap \operatorname{Geo}(g) = \operatorname{Geo}(f) \cap \phi(\operatorname{Geo}(f))$ contains a path of type $A - E - A$, because otherwise $\mathcal{P}'$ is at most a path of type $E - A - E$.

Let us call $v$ the central vertex of type $E$ of this subpath of $\mathcal{P}'$. Since $\phi^{-1}(v) \in \operatorname{Geo}(f)$, there exists an integer $k$ such that $\operatorname{dist}(f^k(v), \phi^{-1}(v)) = \operatorname{dist}((\phi f^k)(v), v) < \lg(f) = 2l$. Replacing $\phi$ by $\phi f^k$, we do not change $g$, but we now have $\operatorname{dist}(\phi(v), v) < 2l$. By condition (C1), the geodesic of $f$ carries $l$ distinct colors which are repeated periodically. Therefore, $\operatorname{dist}(\phi(v), v) \in 2l\mathbb{Z}$ and finally we get $\phi(v) = v$, so that $\phi$ is elliptic.

Let us set $\mathcal{P} = \phi^{-1}(\mathcal{P}') = \operatorname{Geo}(f) \cap \phi^{-1}(\operatorname{Geo}(f))$. Equivalently, one may define $\mathcal{P}$ as the maximal path such that $\mathcal{P} \subseteq \operatorname{Geo}(f)$ and $\phi(\mathcal{P}) \subseteq \operatorname{Geo}(f)$.

The path $\mathcal{P}$ contains a path of type $A - E - A$ whose central vertex is $v$. Without loss of generality, one can now conjugate and assume that this subpath is of the form $e(Q)A - \operatorname{id}E - \operatorname{id}A$. In particular, $v = \operatorname{id}E$.

There are two subcases, depending on whether $\phi \colon \mathcal{P} \to \phi(\mathcal{P})$ preserves the orientation induced by $\operatorname{Geo}(f)$.

If $\phi$ preserves this orientation, then $\phi$ fixes $\mathcal{P}$ point by point. We may assume that $\mathcal{P}$ is strictly greater than $e(Q)A - \operatorname{id}E - \operatorname{id}A$ because, otherwise, there is nothing to show. Then, by Lemmas 7 and 16, we get $\phi = (x + \gamma, y)$. Since the colors of $\operatorname{Geo}(f)$ are general, Lemma 7 shows us that $\mathcal{P}$ is of the form $e(Q)a(\lambda)E - e(Q)A - \operatorname{id}E - \operatorname{id}A - a(\mu)E$, so that it is of type $E - A - E - A - E$.

If $\phi$ does not preserve this orientation, then $\phi$ fixes only the vertex $v$ of $\operatorname{Geo}(f)$. One can show that $\phi$ has to be an involution (see Lemma 20 below). This implies that $\mathcal{P}$ contains an even number of edges and is centered on $v$. Since the $l$ colors supported by $\operatorname{Geo}(f)$ are distinct, $\mathcal{P}$ contains only one color, so that it is of type $A - E - A$ or $E - A - E - A - E$. □

**Lemma 20.** *Let $\mathcal{P}$ be a path of type $A - E - A$ carrying a general color. If $\phi \in \operatorname{Aut}[\mathbb{C}^2]$ exchanges the two ends of $\mathcal{P}$ then $\phi^2 = \operatorname{id}$.*

*Proof.* Without loss of generality, one can conjugate and assume that the path $\mathcal{P}$ is of the form $e(Q)A - \operatorname{id}E - \operatorname{id}A$ (see Remark 13). Note that $\phi_1 = e(Q) \circ (-x, y)$ is an involution that exchanges the two vertices $e(Q)A$ and $\operatorname{id}A$. Thus $\phi_1 \phi$ fixes the path $\mathcal{P}$ point by point, and since $Q$ is general by Lemma 16 we get $\phi = \phi_1 \circ (x + \beta y + \gamma, y)$. Remark that $\phi_1 \circ (x + \beta y + \gamma, y) = (x + \beta y + \gamma, y)^{-1} \circ \phi_1$, hence

$$\phi^2 = \phi_1 \circ (x + \beta y + \gamma, y) \circ (x + \beta y + \gamma, y)^{-1} \circ \phi_1 = \operatorname{id}. \quad \square$$

**Example 21.** Here we show that all cases allowed by Proposition 19 can be realized. In the following examples we suppose that $\operatorname{Geo}(f)$ contains the path $a(0)E - \operatorname{id}A - \operatorname{id}E - e(Q)A - e(Q)a(\mu)E$ where $Q$ is a general polynomial and we choose $\phi$ such that the path $\mathcal{P}$ has various forms.

(1) Examples with $\phi$ fixing at least one edge:

- $\phi = (x + P(y), y)$ with $\deg P \geqslant 2$, $\mathcal{P} = \mathrm{id}A - \mathrm{id}E$;
- $\phi = (\alpha x, \beta y)$ with $\alpha\beta \neq 0$ and $(\alpha, \beta) \neq (1, 1)$, $\mathcal{P} = a(0)E - \mathrm{id}A - \mathrm{id}E$;
- $\phi = (x + by, y)$ with $b \neq 0$, $\mathcal{P} = \mathrm{id}A - \mathrm{id}E - e(Q)A$;
- $\phi = (x+c, y)$ with $c \neq 0$, $\mathcal{P} = a(0)E - \mathrm{id}A - \mathrm{id}E - e(Q)A - e(Q)a(\mu)E$.

(2) Examples with $\phi$ reversing the orientation:

- $\phi = (y, x)$ exchanges $a(0)E$ and $\mathrm{id}E$, $\mathcal{P} = a(0)E - \mathrm{id}A - \mathrm{id}E$;
- $\phi = (-x + Q(y), y)$ exchanges $\mathrm{id}A$ and $e(Q)A$, $\mathcal{P}$ is of length 4 or 2 depending if $\mu = 0$ or not.

(3) Example with $\phi$ hyperbolic:

- $\phi = e(Q)a(\mu)u$ with $u = (-x, -y)$ sends $\mathcal{P} = a(0)E - \mathrm{id}A - \mathrm{id}E$ to $\phi(\mathcal{P}) = \mathrm{id}E - e(Q)A - e(Q)a(\mu)E$ (the reader should verify that $a(\mu)ua(0) = (x - \mu y, y) \in A \cap E$).

### 1.7. Independent colors and tripods

**Definition 22.** A family of polynomials $P_i \in \mathbb{C}[y]$ $(1 \leqslant i \leqslant l)$ is said to be *independent* if, given any $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$ with $\alpha_k\beta_k \neq 0$ and $i_k \in \{1, \ldots, l\}$ for $1 \leqslant k \leqslant 3$, we have

$$\deg \sum_{1\leqslant k\leqslant 3} \alpha_k P_{i_k}(\beta_k y + \gamma_k) \leqslant 1 \quad \Longrightarrow \quad i_1 = i_2 = i_3.$$

The family of colors $(A\cap E)e(P_i)(A\cap E)$ $(1 \leqslant i \leqslant l)$ is said to be *independent* if the family $P_i$ $(1 \leqslant i \leqslant l)$ is independent. Lemma 12 implies that this notion does not depend on the choice of the representatives $P_i$.

**Definition 23.** Three paths $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ of the tree $\mathcal{T}$ define a *tripod* if

- For each $i \neq j$, $\mathcal{P}_i \cap \mathcal{P}_j$ contains at least one edge;
- The intersection $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3$ consists of exactly one vertex $v$.

The three paths $\mathcal{P}_i \cap \mathcal{P}_j$ are called the *branches* of the tripod. The vertex $v$ is called the *center* of the tripod.

If we have a center of type $E$, we can consider the three colors associated with the three paths of type $A-E-A$ containing the center and included in the tripod. In this situation we say that any one of these colors is a *mixture* of the two other colors.

**Lemma 24.** *Let $P_1, P_2, P_3 \in \mathbb{C}[y]$ be polynomials of degree $\geqslant 2$. The following assertions are equivalent:*

(1) *$(A\cap E)e(P_3)(A\cap E)$ is a mixture of the $(A\cap E)e(P_i)(A\cap E)$'s $(1 \leqslant i \leqslant 2)$;*
(2) *there exist $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \delta, \epsilon \in \mathbb{C}$ with $\alpha_1\beta_1\alpha_2\beta_2 \neq 0$ such that*

$$P_3(y) = \alpha_1 P_1(\beta_1 y + \gamma_1) + \alpha_2 P_2(\beta_2 y + \gamma_2) + \delta y + \epsilon.$$

*Proof.* (1) $\Rightarrow$ (2) Assume that there exists a tripod admitting the three colors $(A \cap E)e(P_i)(A \cap E)$ $(1 \leqslant i \leqslant 3)$.

We may assume that the center of this tripod is $\mathrm{id}E$ and that one of its branches is $\mathrm{id}E - \mathrm{id}A$. Let $\widetilde{P}_1, \widetilde{P}_2 \in \mathbb{C}[y]$ be such that the other two branches are $\mathrm{id}E -$

$e(\widetilde{P}_1)A$ and $\mathrm{id}E - e(\widetilde{P}_2)A$ and such that $(E\cap A)e(P_1)(A\cap E) = (E\cap A)e(\widetilde{P}_1)(A\cap E)$ and $(E\cap A)e(P_2)(A\cap E) = (E\cap A)e(\widetilde{P}_2)(A\cap E)$. By Lemma 12, for $1 \leqslant i \leqslant 2$, there exists $\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i$ with $\alpha_i\beta_i \neq 0$ such that $\widetilde{P}_i = \alpha_i P_i(\beta_i y + \gamma_i) + \delta_i y + \epsilon_i$.

We then have $(E\cap A)e(P_3)(A\cap E) = (E\cap A)e(\widetilde{P}_3)(A\cap E)$, where $\widetilde{P}_3 = \widetilde{P}_1 - \widetilde{P}_2$, so (still by Lemma 12) this shows that $P_3$ has the desired form.

(2) $\Rightarrow$ (1) Set $\widetilde{P}_1 = \alpha_1 P_1(\beta_1 y + \gamma_1)$, $\widetilde{P}_2 = -\alpha_2 P_2(\beta_2 y + \gamma_2)$ and $\widetilde{P}_3 = \widetilde{P}_1 - \widetilde{P}_2 = \alpha_1 P_1(\beta_1 y + \gamma_1) + \alpha_2 P_2(\beta_2 y + \gamma_2)$. By Lemma 12 we have $(E\cap A)e(\widetilde{P}_i)(A\cap E) = (E\cap A)e(P_i)(A\cap E)$ for $1 \leqslant i \leqslant 3$. Since $e(\widetilde{P}_2)^{-1}e(\widetilde{P}_1) = e(\widetilde{P}_3) \notin A$, the vertices $e(\widetilde{P}_1)A$ and $e(\widetilde{P}_2)A$ are distinct. Consider the tripod with center $\mathrm{id}E$ and branches $\mathrm{id}E - \mathrm{id}A$, $\mathrm{id}E - e(\widetilde{P}_1)A$ and $\mathrm{id}E - e(\widetilde{P}_2)A$. Its three colors are $(E\cap A)e(\widetilde{P}_i)(A\cap E)$ for $1 \leqslant i \leqslant 3$. This shows that $(A\cap E)e(P_3)(A\cap E)$ is a mixture of $(A\cap E)e(P_1)(A\cap E)$ and $(A\cap E)e(P_2)(A\cap E)$. □

*Remark 25.* The second condition of Lemma 24 may be written under the following symmetric form.

For $1 \leqslant k \leqslant 3$, there exists $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$ with $\alpha_k\beta_k \neq 0$ such that

$$\deg \sum_{1\leqslant k\leqslant 3} \alpha_k P_k(\beta_k y + \gamma_k) \leqslant 1.$$

Therefore, the following lemma is an easy consequence of the previous one.

**Lemma 26.** *Consider three colors represented by $P_1, P_2, P_3 \in \mathbb{C}[y]$ which are polynomials of degree $\geqslant 2$. The following assertions are equivalent:*

(1) *The three colors $(A\cap E)e(P_i)(A\cap E)$ $(i = 1, 2, 3)$ are independent.*
(2) *For any $i_1, i_2, i_3 \in \{1, 2, 3\}$, if $(A\cap E)e(P_{i_3})(A\cap E)$ is a mixture of $(A\cap E)e(P_{i_1})(A\cap E)$ and $(A\cap E)e(P_{i_2})(A\cap E)$, then $i_1 = i_2 = i_3$.*

**Definition 27.** We say that a hyperbolic automorphism of geometric length $2l$ satisfies condition (C2) if the $l$ colors supported by its geodesic (see Example 14) are general and independent.

In the Annex we show that this condition is generic in a natural sense.

*Remark 28.* One could easily check that independent colors are necessarily distinct. Therefore, condition (C2) is stronger than condition (C1).

By misuse of language, we will say that three hyperbolic automorphisms $g_1$, $g_2$, $g_3$ define a *tripod* if their geodesics $\mathrm{Geo}(g_1), \mathrm{Geo}(g_2), \mathrm{Geo}(g_3)$ define a tripod.

**Lemma 29.** *A tripod associated with three conjugates of a hyperbolic automorphism $f$ satisfying condition* (C2) *admits branches of length at most* 2.

*Proof.* If the center of the tripod is of type $A$, by Proposition 19 there is nothing to do. Assume now that the center of the tripod is of type $E$. Without loss of generality, one can conjugate and assume that the center is $\mathrm{id}E$, and that $\mathrm{Geo}(f)$ contains the vertices $\mathrm{id}A$ and $a(0)E$. We denote $g = ufu^{-1}$ and $h = vfv^{-1}$, the two conjugates of $f$ involved in the tripod.

By condition (C2) the three colors centered on $\mathrm{id}E$ in the tripod must be equal. Indeed, if $(A\cap E)e(P_i)(A\cap E)$, $1 \leqslant i \leqslant l$, are the $l$ colors supported by $\mathrm{Geo}(f)$, then there exist $i_1, i_2, i_3 \in \{1, \dots, l\}$ such that these three colors are $(A\cap E)e(P_{i_k})(A\cap E)$, $1 \leqslant k \leqslant 3$. By Definition 22 and Lemma 24 (see also Remark 25) we get $i_1 = i_2 = i_3$, so that the three colors are equal.

Let us prove that $u$ can be chosen fixing the center $\alpha = \mathrm{id}E$ of the tripod. Since $\alpha \in \mathrm{Geo}(f)\cap\mathrm{Geo}(g) = \mathrm{Geo}(f)\cap u(\mathrm{Geo}(f))$ we get $u^{-1}(\alpha) \in \mathrm{Geo}(f)$, so that there exists an integer $k$ such that $\mathrm{dist}(f^k(\alpha), u^{-1}(\alpha)) < \lg(f) = 2l$. Replacing $u$ by $uf^k$, we do not change $g$, but we now have $\mathrm{dist}(u(\alpha), \alpha) < 2l$. By condition (C1) (cf. Remark 28), the geodesic of $g$ carries $l$ distinct colors which are repeated periodically. Therefore, $\mathrm{dist}(u(\alpha), \alpha) \in 2l\mathbb{Z}$ and finally we get $u(\alpha) = \alpha$. We would prove in the same way that $v$ can be chosen fixing $\alpha = \mathrm{id}E$. In other words, we have $u, v \in E$.

Let us now assume that there exists a branch, say $\mathrm{Geo}(f) \cap \mathrm{Geo}(h)$, of length strictly greater than 2. Then, by Proposition 19, this branch has length 4, with middle point $a(0)E$ (see Figure 1). Since $v$ fixes point by point the general path $\mathrm{Geo}(f) \cap \mathrm{Geo}(h)$, by Remark 17 it can be written as $v = (x, y + c)$.

Let $e = e(P) = (x+P(y), y) \in E$ be such that the vertex $eA \in \mathrm{Geo}(f)\cap\mathrm{Geo}(g)$. Since $\mathrm{Geo}(h) = v(\mathrm{Geo}(f))$, the vertex $veA \in \mathrm{Geo}(h)$ and, finally, $veA \in \mathrm{Geo}(g) \cap \mathrm{Geo}(h)$.

We assume that the orientation induced by $g$ on $\mathrm{id}E - eA$ is opposite to the one of $f$, the other case being symmetric.

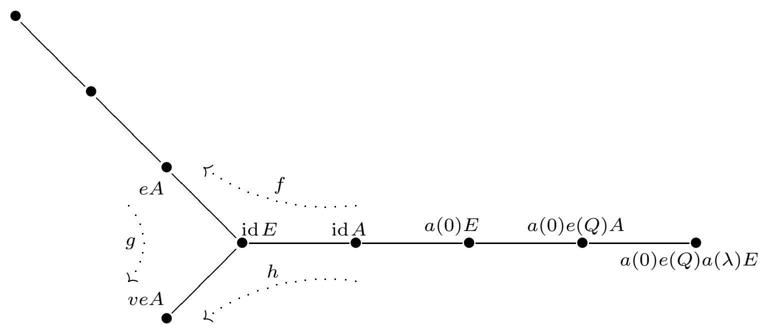


FIGURE 1.

Since $\mathrm{Geo}(g) = u(\mathrm{Geo}(f))$, $u$ sends the path $\mathrm{id}A - \mathrm{id}E - eA$ to the path $eA - \mathrm{id}E - veA$.

On the one hand, $u$ sends $\mathrm{id}A$ to $eA$, i.e., $uA = eA$, i.e., $e^{-1}u \in A$, i.e., $e^{-1}u \in A\cap E$. Since $e^{-1}u \in A\cap E$ it can be written as $s_1s_2$, where $s_1 = (a_1x, b_1y+c_1)$, $s_2 = (x + \beta y + \gamma, y) \in A \cap E$ and we have $u = es_1s_2$.

On the other hand, $u$ sends $eA$ to $veA$, i.e., $ueA = veA$, i.e., $es_1s_2eA = veA$. Since $s_2e = es_2$, we have $es_1s_2eA = es_1eA$, so that $es_1eA = veA$. This last equality is still equivalent to $e^{-1}v^{-1}es_1e \in A$. We compute

$$e^{-1}v^{-1}es_1e = (a_1x + a_1P(y) + P(b_1y + c_1) - P(b_1y + c_1 - c), b_1y + c_1 - c).$$

We should have $\deg(a_1P(y) + P(b_1y + c_1) - P(b_1y + c_1 - c)) \leqslant 1$. Since $a_1 \neq 0$ and $\deg(P(b_1y + c_1) - P(b_1y + c_1 - c)) < \deg P$, this is impossible. $\square$

## 2. Proof of Theorem 1

We start by looking at the case of an automorphism of algebraic length $\leqslant 1$, i.e., a triangular or affine automorphism. Note that similar results in the context of birational transformations are proved in [Giz94] and [CD08].

**Lemma 30.** *If $f \in G$ satisfies $|f| \leqslant 1$ and $f \neq \mathrm{id}$, then $\langle f \rangle_N = G$.*

*Proof.* Let $g, h \in G$. Note that if $g$ or $h$ belongs to $\langle f \rangle_N$, then so does the commutator $[g,h] = ghg^{-1}h^{-1}$. We show that $G = \langle f \rangle_N$ by making the following observations:

• If $f \in \mathrm{SL}(2,\mathbb{C})$ and $f \neq \pm\mathrm{id}$, we obtain $\mathrm{SL}(2,\mathbb{C}) \subseteq \langle f \rangle_N$. We used the fact that $\{\pm\mathrm{id}\}$ is the unique nontrivial normal subgroup of $\mathrm{SL}(2,\mathbb{C})$. Indeed, if $H$ is a normal subgroup of $\mathrm{SL}(2,\mathbb{C})$ not included in $\{\pm\mathrm{id}\}$, we get $\mathrm{SL}(2,\mathbb{C}) = H \cup (-H)$ by simplicity of $\mathrm{PSL}(2,\mathbb{C})$. Therefore, if $g = (y, -x)$, we get $g \in H$ or $-g \in H$, so that $-\mathrm{id} = g^2 = (-g)^2 \in H$ and, finally, $H = \mathrm{SL}(2,\mathbb{C})$.

Now, if $\alpha, \beta \in \mathbb{C}$, we get

$$[(x+\alpha, y+\beta), (-x,-y)] = (x+2\alpha, y+2\beta)$$

so that $A \subseteq \langle f \rangle_N$.

If $b \neq 1$ is an $n$th root of the unity ($n \geqslant 2$) and $\lambda \in \mathbb{C}$, we get

$$[(x+\lambda(1-b)^{-1}y^n, y), (bx, b^{-1}y)] = (x+\lambda y^n, y)$$

and we are done.

• If $f$ is a translation, then, conjugating by $\mathrm{SL}(2,\mathbb{C})$, we see that $\langle f \rangle_N$ contains all translations. So it contains the commutator

$$[(x, y+1), (x+y^2, y)] = (x-2y+1, y)$$

and also the linear automorphism $(x-2y, y)$. We conclude by the previous case.

• If $f$ is an affine automorphism which is not a translation, then there exists a translation $g$ which does not commute with $f$. Therefore, the commutator $[f,g]$ is a nontrivial translation belonging to $\langle f \rangle_N$ and we conclude by the previous case.

• Finally, if $f = (ax+P(y), a^{-1}y+c)$ is a triangular nonaffine automorphism, then, up to replacing $f$ by $[f,g]$, where $g$ is a triangular automorphism noncommuting with $f$, we may assume that $a=1$. Still replacing $f$ by $[f,g]$, where $g$ is a triangular automorphism noncommuting with $f$, we may even assume that $c=0$. Therefore, $f$ is of the form $(x+P(y), y)$. Remark then that the commutator

$$[(x, y+1), (x+P(y), y)]$$

is a triangular automorphism of the form $(x+R(y), y)$ with $\deg R = \deg P - 1$. By induction on the degree we obtain the existence of a nontrivial translation $(x+c, y)$ in $\langle f \rangle_N$. This case has already been done. $\square$

**Corollary 31.** *If $f \in G$ is elliptic (i.e., triangularizable) and $f \neq \mathrm{id}$, then $\langle f \rangle_N = G$.*

We are now ready to prove Theorem 1. In fact, we will prove the following stronger and more geometric version.

**Theorem 32.** *If $f \in G$ satisfies $\mathrm{lg}(f) \leqslant 8$ and $f \neq \mathrm{id}$, then $\langle f \rangle_N = G$.*

*Proof.* The crucial fact we use here is the knowledge of the subtree fixed by translations $(x+c, y)$. We know that this subtree is of diameter 6, centered in $\mathrm{id}E$, and that the closed ball of radius 2 and center $\mathrm{id}E$ is contained in this subtree (see Lemma 7). In consequence, given an arbitrary path of type $E - A - E - A - E$, there exists a conjugate $\psi$ of $(x+1, y)$ fixing this path point by point.

Let us choose such a path contained in the geodesic of $f$ and let us set $g = \psi f \psi^{-1}$. Then if $\mathrm{lg}(f) = 2$ or 4, it is clear that $f \circ g^{-1}$ is elliptic, so we can conclude by Corollary 31. If $\mathrm{lg}(f) = 6$, then $\mathrm{lg}(f \circ g^{-1}) \leqslant 4$ so we are done by the previous case.

The case where $\mathrm{lg}(f) = 8$ is more subtle and we have to refine the above argument. Replacing $f$ by one of its conjugates, we may assume $|f| = \mathrm{lg}(f) = 8$. We can then assume (maybe replacing $f$ by $f^{-1}$) that

$$f = e_1 a_1 e_2 a_2 e_3 a_3 e_4 a_4$$

where $a_i \in A \setminus E$, $e_j \in E \setminus A$. Without loss of generality, we can further assume that each $e_j$ is of the form $e_j = e(P_j) = (x + P_j(y), y)$ and that $\deg(e_1) \leqslant \deg(e_j)$ for $j = 2, 3, 4$.

We know that any translation $(x+c, y)$ fixes the closed ball of radius 2 and center $\mathrm{id}E$. Note also that for any $s \in A \cap E$, $s(x+1, y)s^{-1}$ is still a translation of the form $(x+c, y)$. In consequence, if we write $e_1 a_1$ under the form $e_1 a_1 = e(P)a(\lambda)s$ with $s \in A \cap E$, the automorphism

$$\begin{aligned}
\tilde{e}_1 &= e_1 a_1 (x+1, y) a_1^{-1} e_1^{-1} \\
&= (x + P(y), y) \circ (\lambda x + y, -x) \circ (x + c, y) \circ (-y, \lambda y + x) \circ (x - P(y), y) \\
&= (x + \lambda c + P(y - c) - P(y), y - c)
\end{aligned}$$

fixes the closed ball of radius 2 and center $e_1 a_1 E$. Note that $\deg \tilde{e}_1 = \deg e_1 - 1$. Consider

$$g = \tilde{e}_1 f \tilde{e}_1^{-1} \qquad \text{and} \qquad h = g^{-1} f.$$

By construction the geodesics $\mathrm{Geo}(g)$ and $\mathrm{Geo}(f)$ have at least four edges in common. By Lemma 7 we also know that they have at most six edges in common. Then we can check (see Figure 2) that $h$ sends the vertex $v = a_4^{-1} e_4^{-1} a_3^{-1} E$ to a vertex at distance at most 8 (and at least 6) of $v$. Explicitly, one can compute

$$h = \tilde{e}_1 a_4^{-1} e_4^{-1} a_3^{-1} \tilde{e}_3 a_3 e_4 a_4,$$

where $\tilde{e}_3 = e_3^{-1} a_2^{-1} (x - 1, y) a_2 e_3$ is a triangular automorphism with $\deg(\tilde{e}_3) = \deg(e_3) - 1$.

If $\deg(\tilde{e}_1) = \deg(\tilde{e}_3) = 1$, then $\mathrm{lg}(h) = 4$. This corresponds to the case when $\mathrm{Geo}(g)$ and $\mathrm{Geo}(f)$ share six edges. Note that $a_3^{-1} \tilde{e}_3 a_3$ and $a_4 \tilde{e}_1 a_4^{-1}$ are indeed nontriangular affine automorphisms.

If $\deg(\tilde{e}_1) = 1$ and $\deg(\tilde{e}_3) \geqslant 2$, then $\lg(h) = 6$. In this case, $\mathrm{Geo}(g)$ and $\mathrm{Geo}(f)$ share five edges: the vertices $\mathrm{id}\,A$ and $\tilde{e}_1 A$ coincide.

In the two cases above we are done by the first part of the proof.

Finally, if $\deg(\tilde{e}_1) \geqslant 2$, then $h$ admits a factorization similar to the one of the $f$ we started with except that the first triangular automorphism has a strictly smaller degree. By induction we can produce an element of length 8 in $\langle f \rangle_N$ with the first triangular automorphism of degree 2, and we are done by the previous argument. □

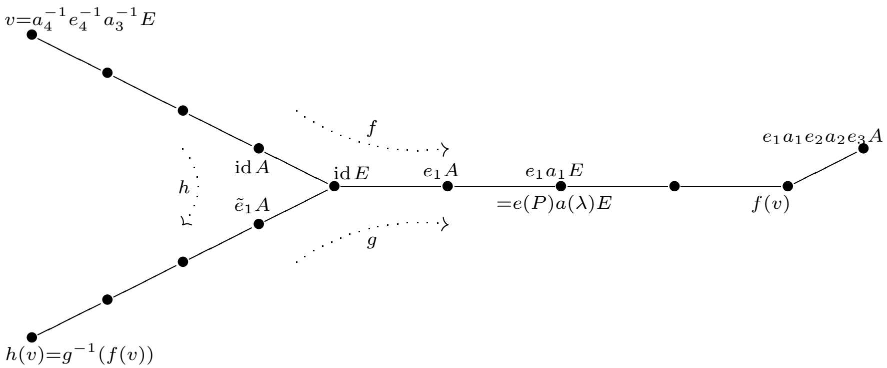


FIGURE 2. Proof of Theorem 32

## 3. R-diagrams

### 3.1. Generalities on small cancellation theory

In this subsection we consider $H = H_1 *_{H_1 \cap H_2} H_2$ a general amalgamated product of two factors. Of course our motivation is to apply the theory to the group $\mathrm{Aut}[\mathbb{C}^2]$ of plane automorphisms.

The following definitions are taken from [LS01, Chap. V, §11, p. 285]. If $u$ is an element of $H$, not in the amalgamated part $H_1 \cap H_2$, a *normal form* of $u$ is any sequence $x_1 \cdots x_m$ such that $u = x_1 \cdots x_m$, each $x_i$ is in a factor of $H$, successive $x_i$ come from different factors of $H$, and no $x_i$ is in the amalgamated part. The *length* of $u$ is defined by $|u| = m$. This definition does not depend on the chosen normal form, but only on $u$. If $u$ is in the amalgamated part of $H$, by convention we set $|u| = 0$.

We call a *word* an element $u \in H$ given with a factorization $u = u_1 \cdots u_k$, where $u_i \in H$ for $i = 1, \dots, k$. A word $u = u_1 \cdots u_k$ is said to have *reduced form* if $|u_1 \cdots u_k| = |u_1| + \cdots + |u_k|$.

Suppose $u$ and $v$ are elements of $H$ with the normal forms $u = x_1 \cdots x_m$ and $v = y_1 \cdots y_n$. If $x_m y_1$ is in the amalgamated part, we say that there is *cancellation* between $u$ and $v$ in forming the product $uv$. Equivalently, this means that $|uv| \leqslant |u| + |v| - 2$. If $x_m$ and $y_1$ are in the same factor of $H$ and $x_m y_1$ is not in the amalgamated part, we say that $x_m$ and $y_1$ are *consolidated* in forming a normal form of $uv$. Equivalently, this means that $|uv| = |u| + |v| - 1$.

A word is said to have *semireduced form* $u_1 \cdots u_k$ if there is no cancellation in this product. Consolidation is expressly allowed.

A word $u = x_1 \cdots x_m$ in normal form is *strictly* (resp. *weakly*) *cyclically reduced* if $m \leqslant 1$ or if $x_m$ and $x_1$ are in different factors of $H$ (resp. the product $x_m x_1$ is not in the amalgamated part). These two notions correspond to the two sets of equivalent conditions given in Lemma 8

A subset $R$ of $H$ is *symmetrized* if all elements of $R$ are weakly cyclically reduced and for each $r \in R$, all weakly cyclically reduced conjugates of both $r$ and $r^{-1}$ belong to $R$.

If $f$ is strictly cyclically reduced, $R(f)$ denotes the symmetrized set generated by $f$, i.e., the smallest symmetrized set containing $f$. It is clear that $R(f)$ is equal to the set of conjugates of $f^{\pm 1}$ of length $\leqslant |f| + 1$.

We now discuss briefly the condition $C'(\lambda)$ (mostly used with $\lambda = \frac{1}{6}$). We do not need this notion in our construction, but this was the original setting where the notion of an R-diagram (see next subsection) was introduced. Let $R$ be a symmetrized subset of $H$. A word $b$ is said to be a *piece* (relative to $R$) if there exists distinct elements $r_1, r_2$ of $R$ such that $r_1 = bc_1$ and $r_2 = bc_2$ in semireduced form.

**Lemma 33.** *If $0 < \lambda < 1$ and $\forall\, r \in R$, $|r| > 1/\lambda$, the following assertions are equivalent:*

(1) *If $r \in R$ admits a semireduced form $r = bc$, where $b$ is a piece of $R$, then $|b| < \lambda |r|$;*
(2) *for all $r_1, r_2 \in R$ such that $r_1 r_2 \neq 1$, $|r_1 r_2| > |r_1| + |r_2| - 2\lambda \min\{|r_1|, |r_2|\} + 1$.*

*Proof.* The equivalence is easily obtained from the following claim.

Let $r_1 = bc_1$ and $r_2 = bc_2$ be semireduced expressions with $b \neq 1$ and $r_1 \neq r_2$.

*Claim.* There exists $b', c_1', c_2'$ such that:

(a) the equalities $r_1 = b'c_1'$ and $r_2 = b'c_2'$ hold;
(b) these expressions are semireduced;
(c) exactly one of these expressions is reduced;
(d) the expression $(c_1')^{-1} c_2'$ is reduced; and
(e) $|b'| \geqslant |b|$. □

**Definition 34.** When the equivalent assertions of Lemma 33 are satisfied, we say that $R$ satisfies condition $C'(\lambda)$.

The first assertion is the one used by Lyndon and Schupp. The second one is used by Danilov, except that he forgets the $+1$ in the formula. This leads to the slight error in his statement that we mentioned in the Introduction. Let us finish this subsection by recalling one of the main theorems of small cancellation theory (see [LS01, Theorem 11.2, p. 288]).

**Theorem 35.** *Let $R$ be a symmetrized subset of the amalgamated group $H$. Suppose that $R$ satisfies condition $C'(\lambda)$ with $\lambda \leqslant \frac{1}{6}$, then the normal subgroup generated by $R$ in $H$ is different from $H$.*

### 3.2. Construction of an R-diagram

The idea of associating diagrams in the Euclidean plane to some products in amalgamated groups appears in [VK33].

In 1966 Lyndon independently arrived at the same idea and Weinbaum rediscovered van Kampen's paper (see [Lyn66], [Wei66] and [LS01, p. 236]). For the basic definition of a *diagram*, we refer to [LS01, Chap. V, §1, p. 235]. Here follows a quick review of this notion.

A diagram is a plane graph (or, more generally, a graph on an orientable surface, we will consider spherical diagrams in Lemma 42). Vertices are divided into two types, *primary* and *secondary*. Any edge joining two vertices gives rise to two directed edges (according to the chosen directions) which we call *half-segments*. If $e$ denotes one of these half-segments, $e^{-1}$ will refer to the other one (obtained by reversing the direction of $e$). The notation "edge" will be used later on to refer to some special unions of half-segments (see the remark on terminology below). A half-segment will always join vertices of different types. By definition, *segments* will denote some special successions of two half-segments that we now describe. If $e_1, \dots, e_r$ are the half-segments arriving at some secondary vertex $v$ and taken counterclockwise, then, by definition, the segments passing through $v$ are the successive half-segments $e_i, e_{i+1}^{-1}$ and their inverses $e_{i+1}, e_i^{-1}$ for $1 \leqslant i \leqslant r$, where $i$ and $i+1$ are taken modulo $r$. If two successive half-segments $e, e'$ define a segment, the latter will be noted $ee'$. Note that the initial and terminal vertices of a segment have to be primary. By convention, each segment (resp. half-segment) has length 1 (resp. $\frac{1}{2}$). Each oriented half-segment $e$ will be labeled by an element $\phi(e)$ belonging to a factor of $\operatorname{Aut}[\mathbb{C}^2]$, with the labels on successive half-segments at a secondary vertex belonging to the same factor. The identity $\phi(e^{-1}) = \phi(e)^{-1}$ is required. This labeling gives a labeling on segments, by taking $\phi(ee') = \phi(e)\phi(e')$. The label on an individual half-segment may be in the amalgamated part, but if $e, e'$ are the two half-segments of a segment, we will usually insist that $\phi(ee')$ is not in the amalgamated part (in fact, there will be only one exception to this rule, see Step 4 in the proof of Theorem 36). We call a *region* a bounded connected component of the complement of the graph in the surface. A *boundary cycle* of a region $D$ is a collection of half-segments that run along the entire boundary of $D$ (say counterclockwise in the case of the plane, or in a way compatible with the orientation in general) with initial vertex of primary type. Similarly, a boundary cycle of the diagram is a collection of half-segments that run along the boundary of the diagram. Let us note that a segment necessarily belongs to the boundary of some region and/or to the boundary of the diagram.

Now let $f$ be an element of $\operatorname{Aut}[\mathbb{C}^2]$ and consider $R(f)$ the associated symmetrized set. We say that a diagram is an *$R(f)$-diagram* if for any region $D$ and any boundary cycle $e_1 \dots e_s$ of $D$, we have $\phi(e_1) \dots \phi(e_s) \in R(f)$.

**Terminology.** Note that we use two kinds of graph in this paper: the Bass–Serre tree and the diagrams of Lyndon and Schupp. In the context of the Bass–Serre tree we have already used the term *edge*, and we have called a *path* the union of several edges. In the context of the Lyndon–Schupp diagrams we have *segments* and *half-segments*. We call *edge* in this context a connected component of the intersection of the boundary of two regions, which is a collection of half-segments.

The following result will be the key ingredient for the proof of Theorem 2. Its proof will occupy the rest of this subsection.

**Theorem 36.** *Let $f \in G$ be a strictly cyclically reduced element of $G$ of (even) algebraic length $|f| \geqslant 2$. Assume that the normal subgroup generated by $f$ in* $\mathrm{Aut}[\mathbb{C}^2]$ *is equal to $G$. Then there exists a planar $R(f)$-diagram $M$ such that:*

1. *$M$ is connected and simply connected;*
2. *The boundary of $M$ has length $\frac{1}{2}$ or $1$;*
3. *If $e_1e_1' \dots e_te_t'$ is a boundary cycle of some region of $M$, then $t = |f|$ and $\phi(e_1e_1') \dots \phi(e_te_t')$ is a reduced form of a strictly cyclically reduced conjugate of $f$.*

*Proof.* We start by choosing an element $g \neq \mathrm{id}$ with $\lg(g) = 0$. By assumption we can write

$$g = (\phi_1 f^{\pm 1} \phi_1^{-1}) \cdots (\phi_n f^{\pm 1} \phi_n^{-1}),$$

with $\phi_i \in \mathrm{Aut}[\mathbb{C}^2]$.

We assume that we have chosen $g$ such that $n$ is minimal. By Lemma 37 we may assume that each $\phi_i f^{\pm 1} \phi_i^{-1}$ is expressed under reduced form $\psi_i r_i \psi_i^{-1}$ (i.e., $|\psi_i r_i \psi_i^{-1}| = |\psi_i| + |r_i| + |\psi_i^{-1}|$) where $r_i \in R(f)$. There is no restriction to assume that $|\psi_i| = 0$ if and only if $\psi_i = \mathrm{id}$. Note also that the four following assertions are equivalent:

(a) $r_i$ is strictly cyclically reduced;
(b) $|r_i| = |f|$;
(c) $|r_i|$ is even;
(d) $|\psi_i r_i \psi_i^{-1}|$ is even.

If any one of these assertions is satisfied, we necessarily have $\psi_i = \mathrm{id}$ (since the expression $\psi_i r_i \psi_i^{-1}$ is reduced).

Let us now explain the construction of $M$, that we perform in several steps.

*Step* 1. We associate a diagram to each $\psi_i r_i \psi_i^{-1}$.

Our construction will involve a base point $O$ which will be considered as a primary vertex. Let $r_i = x_1 \dots x_m$ be a normal form of $r_i$.

• Assume that $r_i$ is strictly cyclically reduced, i.e., $m = |f|$.

The diagram for $\psi_i r_i \psi_i^{-1} = r_i$ is the loop at the base point $O$ consisting of $2m$ half-segments $d_1, d_1', \dots, d_m, d_m'$ such that $\phi(d_j d_j') = x_j$ for each $j$.

• Assume that $r_i$ is not strictly cyclically reduced, i.e., $m = |f| + 1$. Note that in this case $(x_m x_1) x_2 \cdots x_{m-1}$ is strictly cyclically reduced.

The diagram for $\psi_i r_i \psi_i^{-1}$ is a loop at a vertex $v$ joined to the base point $O$ by a path.

Let $\psi_i = z_1 \dots z_k$ be a normal form of $\psi_i$.

The path $Ov$ consists of $2k$ half-segments $e_1, e_1', \dots, e_k, e_k'$ such that $\phi(e_j e_j') = z_j$ for each $j$ and an additional final half-segment $e$.

The loop at $v$ consists of $2m - 2$ half-segments $b, d_2, d_2', \dots, d_{m-1}, d_{m-1}', c$ such that $\phi(d_j d_j') = x_j$ for each $j$.

The three half-segments $e, b, c$ which meet at the secondary vertex $v$ are labeled to satisfy the necessary (and compatible) conditions $\phi(eb) = x_1$, $\phi(ce^{-1}) = x_m$ and $\phi(cb) = x_m x_1$. For instance, we can take $\phi(b) = x_1$, $\phi(c) = x_m$ and $\phi(e) = \mathrm{id}$.

*Step* 2. The initial diagram for the composition

$$g = (\psi_1 r_1 \psi_1^{-1}) \cdots (\psi_n r_n \psi_n^{-1})$$

consists of the initial diagrams for each $\psi_i r_i \psi_i^{-1}$ arranged, in counterclockwise order, around the base point $O$. This initial diagram has the desired properties (1) and (3).

*Step* 3. We will now proceed to the identification of some half-segments of $M$ until the boundary length of $M$ is $\leqslant 2$.

Note that in these identifications:

- We shall always identify primary vertices with primary vertices and secondary vertices with secondary vertices, preserving this distinction.
- The label of a segment will never be in the amalgamated part.
- The number $n$ of regions of $M$ will not change and (1) and (3) will be satisfied at each stage.
- If $\alpha$ is a boundary cycle of $M$, then $\phi(\alpha)$ is conjugate to $g$.

For grounds of brevity, the tiresome and easy verification of the second point (on label of segments) has been omitted in the two cases below.

If the boundary length of $M$ is $\geqslant 3$, there necessarily exist successive segments $ee'$ and $ff'$ in $\partial M$ such that the labels $\phi(ee')$ and $\phi(ff')$ are in the same factor of $\mathrm{Aut}[\mathbb{C}^2]$. Indeed, otherwise, any boundary cycle $\alpha = e_1 e_1' \ldots e_i e_i'$ of $M$ would have even length $i \geqslant 4$ and its label $\phi(\alpha) = \phi(e_1 e_1') \ldots \phi(e_i e_i')$ would be a strictly cyclically reduced conjugate of $g$, a contradiction.

So we consider the element $s = \phi(ee')\phi(ff')$ which lies in a factor of $\mathrm{Aut}[\mathbb{C}^2]$.

*Case* 1. Assume that $s$ is not in the amalgamated part.

Change the label on the half-segment $e'$ to 1, readjusting the labels on the other half-segments at the secondary vertex separating $e$ and $e'$. In other words, this amounts, for each half-segment $g$ ending at this secondary vertex, to replacing its label $\phi(g)$ by $\phi(ge')$.

In the same way, change the label on the half-segment $f$ to 1, readjusting the labels on the other half-segments at the secondary vertex separating $f$ and $f'$.

Then we identify the (oriented) half-segments $e'$ and $f^{-1}$ (which now have the same labels) (see Figure 3 where the $\bullet$ are primary vertices and the $\circ$ are secondary vertices).

*Case* 2. Assume that $s$ is in the amalgamated part.

Note first that the diagram has no loop of length $\leqslant 2$ with total label in one of the factors of $\mathrm{Aut}[\mathbb{C}^2]$. Indeed, such a loop $\alpha$ would be a boundary cycle of some strictly smaller subdomain so that, by Lemma 38 below, $\phi(\alpha)$ would be the product of strictly less that $n$ conjugates of $f$. This would contradict the minimality of $n$.

Therefore, if $u$ is the initial vertex of $ee'$, $v$ its terminal vertex (as well as the initial vertex of $ff'$) and $w$ the terminal vertex of $ff'$, then the vertices $u, v, w$ are distinct.

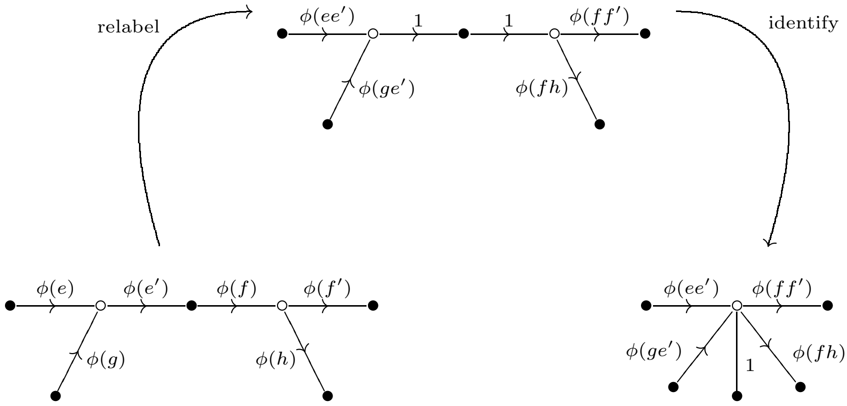


FIGURE 3. Relabelings and identifications in Case 1.

Recall that $\phi(f)\phi(f') = \phi(e')^{-1}\phi(e)^{-1}s$. We change the labels in the following way (see Figure 4):

- we change the label of $f$ to $\phi(e')^{-1}$, readjusting the labels on the other half-segments at the secondary vertex separating $f$ and $f'$;
- we change the label of $f'$ to $\phi(e)^{-1}$;
- for each half-segment $g$ having $w$ as initial vertex, we replace its label $\phi(g)$ by $s\phi(g)$.

Then we identify the (oriented) segments $e, e'$ and $f'^{-1}, f^{-1}$ (which now have the same labels).

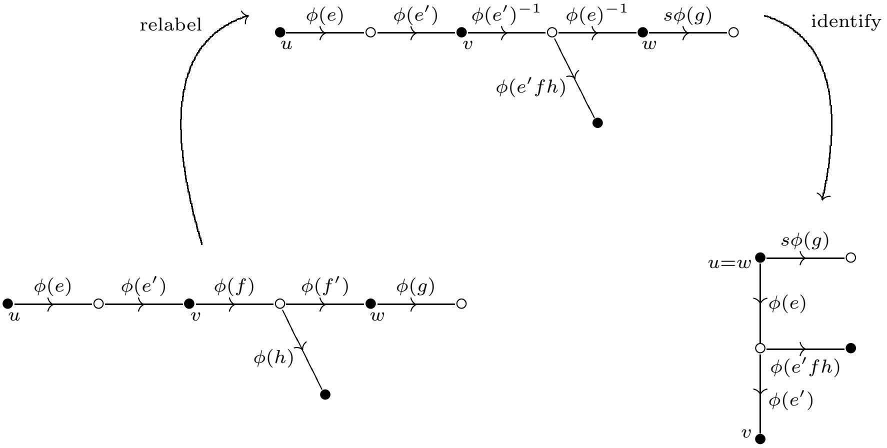


FIGURE 4. Relabelings and identifications in Case 2.

Note that after performing the identification in Case 1 (resp. in Case 2) the boundary length drops by 1 (resp. by 2). Note also that if two regions $D_1$ and $D_2$ share at least one half-segment, and if $r_1, r_2$ are two boundary cycles of these regions with respect to a common starting point, then we cannot have $r_1 = r_2^{-1}$. Indeed, if this was the case, removing the two regions from the diagram and applying Lemma 38 we would obtain a new element in $R(f)$ that would contradict the minimality of $n$. In fact, by Lemma 39, two regions in the diagram never share an edge of length greater than 4.

*Step* 4. By induction, the previous step gave us a diagram with a boundary length $\leqslant 2$. We now perform one last identification to obtain that the boundary length of $M$ is at most 1. If the last identification falls under Case 1 there is no particular problem. However, if we are in Case 2, then we can no longer assume that the vertices $u$ and $w$ are disjoint. So we slightly modify the procedure: we keep the label of $f'$ to be $\phi(e)^{-1}s$ and we only identify the half-segments $e'$ and $f$. It may happen that after this identification the label of the segment $ef'$ on the boundary of $M$ is in the amalgamated part: apart from being slightly non aesthetic, this will not be a problem in the proof of Theorem 45. □

**Lemma 37.** *Any conjugate of $f$ (notation as in Theorem 36) can be written under reduced form $\psi r \psi^{-1}$, where $r$ is a weakly cyclically reduced conjugate of $f$.*

*Proof.* Recall that a hyperbolic element of $\mathrm{Aut}[\mathbb{C}^2]$ is strictly (resp. weakly) cyclically reduced if and only if its geodesic contains (resp. intersects) the edge $e = \mathrm{id}(A \cap E)$ in the Bass–Serre tree (see Lemma 8). Let now $g$ be a conjugate of $f$. If the geodesic of $g$ intersects $e$, we can just set $\psi = \mathrm{id}$, $r = g$. Therefore, let us assume that this geodesic does not intersect $e$.

Let $d$ be the natural distance on the Bass–Serre tree and let $I$ be the middle of the edge $e$. For any element $h$ of $G$, we have $|h| = d(I, h(I))$.

Let $p \in \mathrm{Geo}(g)$ be the unique vertex such that $d(\mathrm{Geo}(g), e) = d(p, e)$. Since $d(p, e) \geqslant 1$, there exists a unique point $I'$ on the geodesic $[p, I]$ such that $d(p, I') = \frac{1}{2}$. The group $G$ acting transitively on the middle of the edges of the Bass–Serre tree, there exists an element $\psi$ of $G$ such that $\psi(I) = I'$. Let us set $r = \psi^{-1} g \psi$. We have $\mathrm{Geo}(r) = \psi^{-1}(\mathrm{Geo}(g))$ and $d(\mathrm{Geo}(g), I') = \frac{1}{2}$, so that $d(\mathrm{Geo}(r), I) = \frac{1}{2}$ and $\mathrm{Geo}(r)$ meets $e$, i.e., $r$ is weakly cyclically reduced. Finally, we have $|g| = d(I, g(I)) = \lg(g) + 2d(I, \mathrm{Geo}(g)) = |f| + 2d(I, I') + 1$, $|\psi| = d(I, I')$ and $|r| = |f| + 1$, so that $|g| = |\psi| + |r| + |\psi^{-1}|$. □

The following result can be proven similarly as in [LS01, Chap. V, §1, Lemma 1.2, p. 239] (i.e., by induction on the number $m$ of regions).

**Lemma 38.** *Let $M$ be an oriented connected and simply connected diagram with $m$ regions $D_1, \ldots, D_m$. Let $\alpha$ be a boundary cycle of $M$ (beginning at some vertex of $\partial M$) and let $\beta_i$ be a boundary cycle of $D_i$ (beginning at some vertex of $\partial D_i$), for $1 \leqslant i \leqslant m$. Then $\phi(\alpha)$ belongs to the normal subgroup generated by the $\phi(\beta_i)$, $1 \leqslant i \leqslant m$. More precisely, there exist $u_1, \ldots, u_m$ in $\mathrm{Aut}[\mathbb{C}^2]$ such that*

$$\phi(\alpha) = (u_1\, \phi(\beta_1)\, u_1^{-1}) \ldots (u_m\, \phi(\beta_m)\, u_m^{-1}).$$

### 3.3. A dictionary between the Bass–Serre and Lyndon–Schupp theories

Let $\alpha$ be a boundary cycle of some region of $M$ (as in Theorem 36) beginning at some vertex $v$. If $v$ is primary (resp. secondary), $\phi(\alpha)$ is a reduced form of a strictly cyclically reduced (resp. nonstrictly cyclically reduced) element of $R(f)$.

**Lemma 39.** *If $D_1, D_2$ are two distinct regions of a diagram $M$ having a common edge, there exists a primary vertex $v$ of $\partial D_1 \cap \partial D_2$ such that the labels $g_1, g_2$ of the boundary cycles of $D_1, D_2$ beginning at $v$ satisfy*

$$|\mathrm{Geo}(g_1) \cap \mathrm{Geo}(g_2)| \geqslant |\partial D_1 \cap \partial D_2|.$$

*Proof.* If $k$ is the largest integer such that $k < |\partial D_1 \cap \partial D_2|$, there exists a path of $k$ segments $s_1, \dots, s_k$ included into $\partial D_1 \cap \partial D_2$. We can just take for $v$ the initial or terminal vertex of this path (if $k = 0$, these two vertices coincide). Indeed, we may assume that $g_1$ has the normal form $g_1 = \phi(s_1) \dots \phi(s_k) x_1 \dots x_m$ (where each $x_i$ is in some factor of $G$). Therefore, $g_2^{-1}$ has the normal form $g_2^{-1} = \phi(s_1) \dots \phi(s_k) y_1 \dots y_m$ (where each $y_i$ is in some factor of $G$).

The geodesics $\mathrm{Geo}(g_1)$ and $\mathrm{Geo}(g_2^{-1}) = \mathrm{Geo}(g_2)$ contain the $k+1$ consecutive edges,

$$\mathrm{id}\,(A \cap E),\ \phi(s_1)\,(A \cap E), \dots, \phi(s_1) \dots \phi(s_k)\,(A \cap E). \quad \square$$

**Example 40.** Assume that $M$ contains the two regions depicted in Figure 5 (the $\bullet$ are primary vertices, the secondary vertices are denoted by $\circ$ only when they have valence $\geqslant 3$).

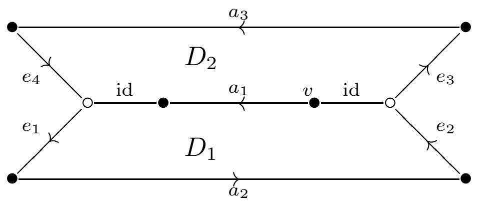


FIGURE 5.

We get $g_1 = a_1 e_1 a_2 e_2$, $g_2 = e_3 a_3 e_4 a_1^{-1}$ and Figure 6 gives the picture in the Bass–Serre tree. Note that here for simplicity we took $D_1$ and $D_2$ with boundary length 4, but in the context of Theorem 36 any region has boundary length at least 10.

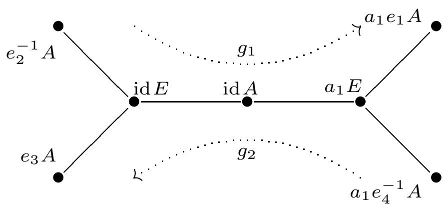


FIGURE 6.

**Lemma 41.** *If $v$ is a vertex of valence 3 of $M$ with regions $D_1, D_2, D_3$ meeting at $v$ and if $g_1, g_2, g_3$ are the labels of the boundary cycles of these regions beginning at $v$, then the geodesics of the $g_i$'s form a tripod in the Bass–Serre tree and for all $i, j$'s,*

$$|\mathrm{Geo}(g_i) \cap \mathrm{Geo}(g_j))| \geqslant |\partial D_i \cap \partial D_j|.$$

*Proof.* The vertex $v$ is necessarily secondary. Let $e_1$ (resp. $e_2$, resp. $e_3$) be the (oriented) half-segment having $v$ as initial vertex and included into $\partial D_2 \cap \partial D_3$ (resp. $\partial D_1 \cap \partial D_3$, resp. $\partial D_1 \cap \partial D_2$). The $\phi(e_i)$'s are in the same factor of $G$ and if $i \neq j$, $\phi(e_i)\phi(e_j)^{-1}$ is not in the amalgamated part. As in Lemma 39, let $k$ be the largest integer such that $k < |\partial D_1 \cap \partial D_2|$ and let $s_1, \ldots, s_k$ be segments such that the path $e_3, s_1, \ldots, s_k$ is included in $\partial D_1 \cap \partial D_2$. We may assume that $g_1$ has the normal form

$$g_1 = \phi(e_3)\phi(s_1)\ldots\phi(s_k)x_1\ldots x_m\phi(e_2)^{-1},$$

where each $x_i$ is in some factor of $G$. Therefore, $g_1' = \phi(e_3)^{-1}g_1\phi(e_3)$ is strictly cyclically reduced and has the normal form

$$g_1' = \phi(s_1)\ldots\phi(s_k)x_1\ldots x_{m+1},$$

where $x_{m+1} = \phi(e_2)^{-1}\phi(e_3)$. Since the geodesic of $g_1'$ contains the consecutive edges

$$\mathrm{id}(A \cap E),\ \phi(s_1)(A \cap E), \ldots, \phi(s_1)\ldots\phi(s_k)(A \cap E),$$

it is clear that the geodesic of $g_1$ contains the consecutive edges

$$\phi(e_3)(A \cap E),\ \phi(e_3)\phi(s_1)(A \cap E), \ldots, \phi(e_3)\phi(s_1)\ldots\phi(s_k)(A \cap E).$$

One would show in the same way that these edges are also contained in the geodesic of $g_2$, so that we get $|\mathrm{Geo}(g_1) \cap \mathrm{Geo}(g_2)| \geqslant |\partial D_1 \cap \partial D_2|$. The other inequalities are proven similarly. We finish the proof by noting that $\mathrm{Geo}(g_1) \cap \mathrm{Geo}(g_3)$ contains the edge $\phi(e_2)(A \cap E)$ and that $\mathrm{Geo}(g_2) \cap \mathrm{Geo}(g_3)$ contains the edge $\phi(e_1)(A \cap E)$. If the $\phi(e_i)$'s are in the factor $A$ (resp. $E$), it is clear that the three edges $\phi(e_i)(A \cap E)$ intersect at the vertex $\mathrm{id}A$ (resp. $\mathrm{id}E$). □

## 4. Proof of Theorem 2

### 4.1. A result about curvature

Let us recall some notations from [LS01]. If $v$ is a vertex of a diagram $M$, the degree $d(v)$ (or valence) of $v$ will denote the number of oriented edges having $v$ as initial vertex (thus, if an edge has both endpoints at $v$, we count it twice). If $D$ is a region, the degree $d(D)$ of $D$ will denote the number of edges of $D$. The following formula defines a curvature contribution for each region:

$$\delta(D) = 2 - d(D) + \sum_{v \in D} \frac{2}{d(v)}.$$

**Lemma 42.** *For any diagram on the* 2*-sphere we have*

$$4 = \sum_{D} \delta(D).$$

*Proof.* Let $V$, $E$, and $F$ be the numbers of vertices, edges and faces of the diagram. The formula is a direct consequence of Euler's formula on the sphere $2 = V - E + F$ and of the obvious relations $2E = \sum_{(v,D)} 1$, $V = \sum_{(v,D)} 1/d(v)$ and $F = \sum_{(v,D)} 1/d(D)$:

$$4 = 2V + 2F - 2E = \sum_{(v,D)} \left( \frac{2}{d(v)} + \frac{2}{d(D)} - 1 \right) = \sum_{D} \delta(D),$$

where the first sum runs over the couples $(v, D)$ with $v$ a vertex and $D$ a face such that $v \in D$. □

**Corollary 43.** *For any planar diagram homeomorphic to the disk we have*

$$2 \leqslant \sum_{D} \delta(D).$$

*Proof.* Let $K$ be this diagram. Let $L$ be the spherical diagram obtained by sticking along their boundaries two copies $K_1$ and $K_2$ of $K$. Since $L$ is homeomorphic to the sphere, we have $4 = \sum_{D \in L} \delta(D)$, i.e.,

$$4 = \sum_{D \in K_1} \delta(D) + \sum_{D \in K_2} \delta(D) = 2 \sum_{D \in K_1} \delta(D) \leqslant 2 \sum_{D \in K} \delta(D).$$

The last inequality comes from the fact that for each boundary region $D$ in $K$ the contribution curvature $\delta(D)$ computed in the disk diagram is bigger than the contribution computed in the spherical diagram. □

*Remark 44.* Here is a (noncomplete) list of faces $D$ having negative or zero curvature:

- $D$ with $d(D) \geqslant 6$;
- $D$ with $d(D) = 5$ and at most three vertices of $D$ are tripods;
- $D$ with $d(D) = 4$ and each vertex of $D$ has valence at least 4;
- $D$ with $d(D) = 4$ and $D$ admits a tripod, two vertices of valence at least 4 and a fourth vertex of valence at least 6;
- $D$ with $d(D) = 3$ and each vertex of $D$ has valence at least 6.

### 4.2. End of the proof

We are now in a position to prove Theorem 2. As in Theorem 1, we will prove a stronger and more geometric version.

**Theorem 45.** *If* $f \in G$ *is a hyperbolic element of geometric length* $\lg(f) \geqslant 14$ *satisfying conditions* (C2)*, then the normal subgroup generated by* $f$ *in* $\mathrm{Aut}[\mathbb{C}^2]$ *is different from* $G$*.*

*Proof.* We can assume that $f$ is a strictly cyclically reduced element of length $\lg(f) = |f| = 2l \geqslant 14$. If the normal subgroup generated by $f$ in $\mathrm{Aut}[\mathbb{C}^2]$ was equal to $G$ then, by Theorem 36, there would exist an $\mathrm{Aut}[\mathbb{C}^2]$-labeled oriented diagram $M$ such that:

(1) $M$ is connected and simply connected;
(2) the perimeter of $M$ is $\leqslant 1$; and
(3) if $e_1e_1' \ldots e_te_t'$ is a boundary cycle of some region of $M$, then $t = |f|$ and $\phi(e_1e_1') \ldots \phi(e_te_t')$ is a reduced form of a strictly cyclically reduced conjugate of $f$.

Let $D_1, D_2$ be two distinct regions of $M$ having a common edge. By Proposition 19 and Lemma 39 we have $|\partial D_1 \cap \partial D_2| \leqslant 4$. Since $|\partial D_1| \geqslant 14$, we conclude that any interior region has at least four edges.

Furthermore, if $D_1, D_2, D_3$ are three distinct regions of $M$ having a common vertex of valence 3, by Lemmas 29 and 41, we know that each edge $\partial D_i \cap \partial D_j$ is at most of length 2. In consequence, if an interior region has at least one interior vertex of valence 3, then this region has at least five edges. Similarly, if an interior region has at least three interior vertices of valence 3, then this region has at least six edges.

By the previous observations, and using Remark 44, we conclude that the curvature contribution $\delta(D)$ of any interior region $D$ is nonpositive. Let us examine now the contribution of the boundary regions. Since the perimeter is at most 1 (i.e., at most two half-segments), there are at most two boundary regions.

Suppose there are exactly two boundary regions. Since the boundary edge of such a region $D$ is a half-segment, it is easy to check that $D$ has at least five edges, and that if at least one interior vertex is of valence 3 then $D$ has at least six edges. Thus $\delta(D) \leqslant 0$.

Assume now that there is only one boundary region $D$. Then the only boundary vertex of $D$ (which has to be counted twice) has valence at least 4. So $D$ has at least five edges and if $D$ has exactly five edges, then the three interior vertices cannot be of valence 3, and again we obtain $\delta(D) \leqslant 0$.

In conclusion we have $\sum \delta(D) \leqslant 0$, which is contradictory to Lemma 43. We conclude that the normal subgroup generated by $f$ in $\mathrm{Aut}[\mathbb{C}^2]$ cannot be equal to $G$. $\square$

## 5. The remaining cases: Lengths 10 and 12

In this section we present some of the problems that await the reader who would like to extend our results to the case of an automorphism of length 10 or 12, along with two striking examples of configuration in the Bass–Serre tree.

### 5.1. Length 12

The main problem in adapting our strategy to the case of $f$ with $\lg(f) = 12$ is that we have to deal with regions in an $R(f)$-diagram that are triangles with three edges of length 4. Then we would have to study not only tripods coming from three conjugates of $f$, but their generalization, which we call $n$-pods, coming from $n$ conjugates $f_i$ $(0 \leqslant i \leqslant n-1)$ of $f$. This is the case where the geodesics $\mathrm{Geo}(f_i)$

have a common vertex and where each pair, $\text{Geo}(f_i), \text{Geo}(f_{i+1})$, has at least one edge in common (where $i = 0, ..., n-1$ and the indices are taken modulo $n$). To be sure precisely that the curvature of such a triangle is nonpositive it would be sufficient to have the following

**Lemma/Conjecture 46.** *If $n$ conjugates of $f$ form an $n$-pod in the Bass–Serre tree, with two consecutive branches of length 4, then $n \geqslant 6$.*

We believe that this result is true, but the verification seems to have to involve a very long list of cases: that is why we do not think it reasonable to try to present a proof. However, it is interesting to note that there exist 6-pods with branches of length 4.

**Example 47** (6-pod with all branches of length 4)**.** Let us consider the following automorphism $f_0$ of length $2l \geqslant 8$,

$$f_0 = e_1 a e_2 a \cdots e_l a$$

where $a = a(0) = (y, -x)$. We suppose that $e_1 = (x + P(y), y)$ and we set $e = e_1$. We are going to construct $f_1, \ldots, f_5$ five conjugates of $f_0$ such that their geodesics form a 6-pod (see Figure 7).

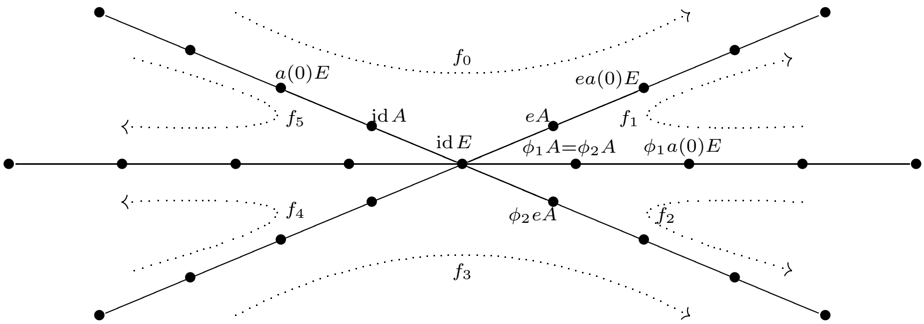


FIGURE 7. A 6-pod with all branches of length 4 (example 47)

For $i = 1, \ldots, 5$ we choose constants $c_i \neq 0$ and we set $t_i = (x, y + c_i)$. We take $f_i = \phi_i f_0 \phi_i^{-1}$ where

$$\begin{aligned}
\phi_1 &= e t_1 e^{-1}, \\
\phi_2 &= e t_1 e^{-1} t_2, \\
\phi_3 &= e t_1 e^{-1} t_2 e t_3 e^{-1}, \\
\phi_4 &= e t_1 e^{-1} t_2 e t_3 e^{-1} t_4, \\
\phi_5 &= e t_1 e^{-1} t_2 e t_3 e^{-1} t_4 e t_5 e^{-1},
\end{aligned}$$

are all elements of $E$.

We claim that for each $i = 0, \cdots, 4$, the geodesics of $f_i$ and $f_{i+1}$ share a path of four edges with $\mathrm{id}E$ as an extremity.

Consider the case $i = 0$. We have $\mathrm{Geo}(f_1) = \phi_1(\mathrm{Geo}(f_0))$. Recall that $t_1$ fixes the ball of radius 2 centered on $a(0)E$ (Remark 17), so $\phi_1$ fixes the ball of radius 2 centered on $ea(0)E$, hence the claim.

Now take $i = 1$. Note that $f_2 = \phi_1 t_2 f_0 t_2^{-1} \phi_1^{-1} = \phi_1 t_2 \phi_1^{-1} f_1 \phi_1 t_2^{-1} \phi_1^{-1}$ and $\phi_1 t_2 \phi_1^{-1}$ fixes the ball of radius 2 centered at $\phi_1 a(0)E$. Thus the geodesic of $f_1$ and $f_2$ share four edges. We can make a similar computation for $i = 2, 3, 4$.

Suppose now that the constants $c_i$ satisfy

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ c_2 + c_3 + c_4 &= 0, \\ c_3 + c_4 + c_5 &= 0. \end{aligned}$$

For instance, one can take $(c_1, c_2, c_3, c_4, c_5) = (1, 1, -2, 1, 1)$.

A straightforward computation shows that

$$\begin{aligned} \phi_5 &= e t_1 e^{-1} t_2 e t_3 e^{-1} t_4 e t_5 e^{-1} \\ &= \big(x + P(y + c_1 + c_2 + c_3 + c_4 + c_5) - P(y + c_2 + c_3 + c_4 + c_5) \\ &\quad + P(y + c_3 + c_4 + c_5) - P(y + c_4 + c_5) + P(y + c_5) - P(y), \\ &\quad y + c_1 + c_2 + c_3 + c_4 + c_5\big) = (x, y - c_3). \end{aligned}$$

Since $(x, y - c_3)$ fixes the ball of radius 2 centered at $a(0)E$, this implies that the geodesics of $f_0$ and $f_5$ share four edges, as shown in Figure 7.

### 5.2. Length 10

The case of $f$ of length 10 seems even more doubtful. For instance, one could have pentagonal regions with all edges of length 2 and all vertices of valence 3. It is probably easy to rule out this case, but there are some harder ones. One could have triangular regions with edges of length $4, 4, 2$. Example 47 allows us to glue six such triangles along their edge of length 4 to obtain an $R(f)$-diagram with boundary length 12. One can wonder if it is possible to glue two such diagrams to obtain an $R(f)$-diagram on a sphere (in this case our strategy would fail). One would need to have 4-pods with branches $4, 2, 4, 2$. We do not know if this is possible, but the following example shows again that we would have to rely on very careful computations to exclude this case (note also that the assumption "consecutive" was crucial in the statement of Lemma 46).

**Example 48 (**4-pod with branches of length 4, 1, 4, 1**).** Similarly to the previous example, we take $f_i = \phi_i f_0 \phi_i^{-1}$ where

$$\begin{aligned} \phi_1 &= e t_1 e^{-1}, \\ \phi_2 &= e t_1 e^{-1} t_2, \\ \phi_3 &= e t_1 e^{-1} t_2 e t_3 e^{-1}, \end{aligned}$$

with $t_1 = t_3 = (x + c, y)$ and $t_2 = (-x, y - c)$. Then one can verify that $\phi_3 = (-x, y + c)$ and the geodesics of the $f_i$ form a 4-pod as in Figure 8.

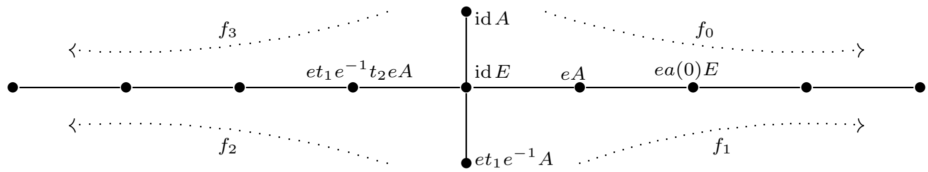


FIGURE 8. A 4-pod with branches 4, 1, 4, 1 (Example 48).

## Annex: Genericness of condition (C2)

We begin with a reformulation of Theorem 45.

**Theorem 49.** *Let $l \geqslant 7$ be an integer. Assume that the polynomials $P_1, \ldots, P_l \in \mathbb{C}[y]$ are general and independent. If the element $f$ of $G$ can be written $f = a_1e_1 \ldots a_le_l$ where $e_i = e(P_i)$ and $a_i \in A \setminus E$ for each $i$, then the normal subgroup generated by $f$ in* $\mathrm{Aut}[\mathbb{C}^2]$ *is different from $G$.*

In this Annex we will show that if $P_1, \ldots, P_l$ are generic (in some sense), then they are general and independent. We will also finish by giving explicit examples.

### A. Genericness of condition (C1)

The aim of this subsection is to show that condition (C1) is generic (see Corollary 57 and Remark 58). For technical purposes we introduce a variation of the notion of a general polynomial (see Definition 15).

**Lemma 50.** *Let $Q \in \mathbb{C}[y]$ be a polynomial. The following assertions are equivalent:*

(1) *for all $\alpha, \beta, \gamma \in \mathbb{C}$, $Q(y) = \alpha Q(\beta y + \gamma) \Rightarrow \alpha = \beta = 1$ and $\gamma = 0$;*
(2) *for all $\alpha, \beta, \gamma \in \mathbb{C}$, $Q(y) = \alpha Q(\beta y + \gamma) \Rightarrow \beta = 1$.*

*Proof.* (1)⇒(2) is obvious. Let us prove (2)⇒(1). If $Q$ satisfies (2), note that $Q$ cannot be constant. If $Q(y) = \alpha Q(y + \gamma)$, it is enough to show that $\gamma = 0$. Let $\zeta$ be a root of $Q$. Since $\zeta + n\gamma$ is also a root of $Q$ for any integer $n$ we must have $\gamma = 0$. □

**Definition 51.** We say that $Q$ is *weakly general* if it satisfies the equivalent assertions of Lemma 50.

*Remark 52.* Clearly, if $Q'$ is weakly general, then $Q$ is also weakly general. Furthermore, $Q^{(k)}$ is weakly general if and only if the following equivalent assertions are satisfied:

(1) for all $\alpha,\beta,\gamma \in \mathbb{C}$, $\deg(Q(y)-\alpha Q(\beta y+\gamma))<k \Rightarrow \alpha=\beta=1$ and $\gamma=0$;
(2) for all $\alpha,\beta,\gamma \in \mathbb{C}$, $\deg(Q(y)-\alpha Q(\beta y+\gamma))<k \Rightarrow \beta=1$.

In other words, a polynomial $Q$ of degree $d \geqslant 5$ is general if and only if $Q^{(d-3)}$ is weakly general.

**Lemma 53.** *The following assertions are equivalent:*

(1) *$Q$ is not weakly general;*
(2) *there exists $c \in \mathbb{C}$, $R \in \mathbb{C}[y]$, $k \geqslant 0, n \geqslant 2$, such that $Q(y+c)=y^k R(y^n)$.*

*Proof.* (1)⇒(2) If $Q$ is not weakly general, there exists $\alpha,\beta,\gamma$ with $\beta \neq 1$ such that $Q(y)=\alpha Q(\beta y+\gamma)$. If we set $c=\gamma/(1-\beta)$, then the polynomial $P(y)=Q(y+c)$ satisfies $P(y)=\alpha P(\beta y)$. Writing $P=\sum_i p_i y^i$, the last equation is equivalent to, for all $i, (1-\alpha\beta^i)p_i=0$. If $\beta$ is not a root of unity, this implies that there exists $k \geqslant 0$ such that $P=p_k y^k$. Assume now that $\beta$ is a primitive $n$th root of unity. If $P \neq 0$, there exists $k \geqslant 0$ such that $p_k \neq 0$ and so $\alpha=\beta^{-k}$. Since $p_i \neq 0$ implies $i \equiv k \pmod n$, we get $P=y^k R(y^n)$ where $R(y)=\sum_i p_{k+ni} y^i$.

(2)⇒(1) This is a consequence of the previous computation. □

**Proposition 54.**

(1) *If $d \geqslant 3$, the generic element of $\mathbb{C}[y]_{\leqslant d}$ is weakly general.*
(2) *If $d \geqslant 5$, the generic element of $\mathbb{C}[y]_{\leqslant d}$ is general.*

*Proof.* If $u \in \mathbb{R}$, we denote its integer part by $[u]$.

(1) If $Q \in \mathbb{C}[y]_{\leqslant d}$ is not weakly general, by Lemma 53 we can write

$$Q(y)=(y-c)^k R\left((y-c)^n\right),$$

where $0 \leqslant k \leqslant d$, $2 \leqslant n \leqslant d$, $c \in \mathbb{C}$, $e=[d/n]$ and $R \in \mathbb{C}[y]_{\leqslant e}$. Therefore, $Q$ belongs to the image of the following morphism:

$$\varphi_{k,n}:\mathbb{C}\times\mathbb{C}[y]_{\leqslant e}\to\mathbb{C}[y], \qquad (c,R(y))\mapsto (y-c)^k R((y-c)^n).$$

However,

$$\dim \mathrm{Im}\varphi_{k,n} \leqslant \dim(\mathbb{C}\times\mathbb{C}[y]_{\leqslant e})=e+2\leqslant \frac{d}{n}+2\leqslant\frac{d}{2}+2<d+1=\dim\mathbb{C}[y]_{\leqslant d}.$$

Part (2) is a direct consequence of (1), by considering the map $Q \mapsto Q^{(d-3)}$, and using Remark 52. □

**Proposition 55.** *If $d_1, d_2 \geqslant 5$ and $(P_1,P_2)$ is a generic element of $\mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2}$, then $P_1$, $P_2$ represent different colors.*

*Proof.* By Lemma 12, if $P_1, P_2$ represent the same color, then $(P_1,P_2)$ belongs to the image of the following morphism:

$$\varphi:\mathbb{C}[y]_{\leqslant d_1}\times\mathbb{C}^5\to\mathbb{C}[y]\times\mathbb{C}[y], \qquad (P_1,(\alpha,\beta,\gamma,\delta,\epsilon))\mapsto(P_1,\alpha P_1(\beta y+\gamma)+\delta y+\epsilon).$$

However,

$$\dim\mathrm{Im}\varphi\leqslant d_1+6<\dim\mathbb{C}[y]_{\leqslant d_1}\times\mathbb{C}[y]_{\leqslant d_2}. \qquad \square$$

*Remark 56.* If $d_1 \neq d_2$, Proposition 55 is still more obvious. Indeed, the generic element $P_i$ of $\mathbb{C}[y]_{\leqslant d_i}$ has degree $d_i$. Therefore, if $(P_1, P_2)$ is a generic element of $\mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2}$, then $\deg P_1 \neq \deg P_2$, which clearly implies that $P_1$, $P_2$ represent different colors.

Propositions 54 and 55 give us the following result.

**Corollary 57.** *Fix a sequence of integers $d_1, \ldots, d_l \geqslant 5$. If $(P_1, \cdots, P_l)$ is a generic element of $\prod_{1 \leqslant i \leqslant l} \mathbb{C}[y]_{\leqslant d_i}$, then the polynomials $P_i$ are general and represent distinct colors.*

*Remark 58.* In other words, if $a_i \in A \setminus E$ and $e_i = e(P_i)$ for $1 \leqslant i \leqslant l$, then the automorphism $a_1 e_1 \ldots a_l e_l$ satisfies condition (C1).

### B. Genericness of condition (C2)

The aim of this subsection is to show that condition (C2) is generic (see Corollary 61 and Remark 62).

**Proposition 59.** *If $d_1, d_2, d_3 \geqslant 8$ and $(P_1, P_2, P_3)$ is generic in $\prod_{1 \leqslant i \leqslant 3} \mathbb{C}[y]_{\leqslant d_i}$, then the polynomials $P_1, P_2, P_3$ are independent.*

*Proof.* By permutations, it is enough to show the following two points:

(1) If $(P_1, P_2)$ is generic in $\mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2}$, then $(A \cap E)e(P_2)(A \cap E)$ is not a mixture of $(A \cap E)e(P_1)(A \cap E)$ and $(A \cap E)e(P_1)(A \cap E)$.

(2) If $(P_1, P_2, P_3)$ is generic in $\mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2} \times \mathbb{C}[y]_{\leqslant d_3}$, then $(A \cap E)e(P_3)(A \cap E)$ is not a mixture of $(A \cap E)e(P_1)(A \cap E)$ and $(A \cap E)e(P_2)(A \cap E)$.

*Proof of* (1). Define $\phi : \mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}^8 \to \mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]$,

$$(P_1, (\alpha, \ldots, \theta)) \mapsto (P_1, \alpha P_1(\beta y + \gamma) + \delta P_1(\epsilon y + \zeta) + \eta y + \theta).$$

We have $\dim \operatorname{Im} \phi \leqslant d_1 + 1 + 8 < \dim \mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2}$. If $(P_1, P_2) \in (\mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2}) \setminus \operatorname{Im} \phi$, it is clear that $(A \cap E)e(P_2)(A \cap E)$ is not a mixture of $(A \cap E)e(P_1)(A \cap E)$ and $(A \cap E)e(P_1)(A \cap E)$.

*Proof of* (2). Define $\psi : \mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2} \times \mathbb{C}^8 \to \mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2} \times \mathbb{C}[y]$,

$$(P_1, P_2, (\alpha, \ldots, \theta)) \mapsto (P_1, P_2, \alpha P_1(\beta y + \gamma) + \delta P_2(\epsilon y + \zeta) + \eta y + \theta).$$

We have $\dim \operatorname{Im} \phi \leqslant (d_1 + 1) + (d_2 + 1) + 8 < \dim \mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2} \times \mathbb{C}[y]_{\leqslant d_3}$. If $(P_1, P_2, P_3) \in (\mathbb{C}[y]_{\leqslant d_1} \times \mathbb{C}[y]_{\leqslant d_2} \times \mathbb{C}[y]_{\leqslant d_3}) \setminus \operatorname{Im} \psi$, it is clear that $(A \cap E)e(P_3)(A \cap E)$ is not a mixture of $(A \cap E)e(P_1)(A \cap E)$ and $(A \cap E)e(P_2)(A \cap E)$. □

**Corollary 60.** *Fix a sequence of integers $d_1, \ldots, d_l \geqslant 8$. The generic element $(P_1, \ldots, P_l)$ of $\prod_{1 \leqslant i \leqslant l} \mathbb{C}[y]_{\leqslant d_i}$ is an independent sequence.*

Combining Corollaries 57 and 60 we get

**Corollary 61.** *Fix a sequence of integers $d_1, \ldots, d_l \geqslant 8$. The generic element $(P_1, \ldots, P_l)$ of $\prod_{1 \leqslant i \leqslant l} \mathbb{C}[y]_{\leqslant d_i}$ defines a sequence of general and independent polynomials.*

*Remark 62.* In other words, if $a_i \in A \setminus E$ and $e_i = e(P_i)$ for $1 \leqslant i \leqslant l$, then the automorphism $a_1 e_1 \dots a_l e_l$ satisfies condition (C2).

### C. Explicit examples

Lemmas 63 and 66 below will allow us to give explicit examples of polynomials $P_1, \dots, P_l \in \mathbb{C}[y]$ which are general and independent (see Example 67).

**Lemma 63.** *Let $P \in \mathbb{C}[y]$ be a polynomial of degree $d \geqslant 3$ and let $M = -p_{d-1}/dp_d$ be the arithmetic mean of its roots. If there exist two consecutive integers $k \geqslant 0$ such that $P^{(k)}(M) \neq 0$, then $P$ is weakly general.*

*Proof.* If $P(y) = \alpha P(\beta y + \gamma)$, then the automorphism $f$ of the affine line given by $f(y) = \beta y + \gamma$ permutes the roots of $P$. Since $f$ is affine, we must have $f(M) = M$. By substituting $M$ for $y$ in the equality $P^{(k)}(y) = \alpha \beta^k P^{(k)}(f(y))$, we get $(1 - \alpha\beta^k) P^{(k)}(M) = 0$, whence the result. □

*Remark 64.* We always have $P^{(d-1)}(M) = 0$. Therefore, if $P$ has degree 2, it is not possible to find two consecutive integers $k$ such that $P^{(k)}(M) \neq 0$. As a consequence, it is not possible to show that $P$ is weakly general by using an analogous version of Lemma 63. In fact, it is easy to check that no polynomial of degree 2 is weakly general!

**Example 65.** Let $P = \sum_i p_i y^i$ be a polynomial of degree $d \geqslant 5$.

(1) If $p_{d-1} = 0$ and $p_{d-2} p_{d-3} \neq 0$, then $P$ is general.
(2) If $p_{d-1} \neq 0$ and $p_{d-2} = p_{d-3} = 0$, then $P$ is general.

**Lemma 66.** *A family $(P_i)_i$ of general polynomials satisfying $|\deg P_i - \deg P_j| > 3$ for any $i \neq j$ is independent.*

*Proof.* Let us assume (by contradiction) that $\deg \sum_{1 \leqslant k \leqslant 3} \alpha_k P_{i_k}(\beta_k y + \gamma_k) \leqslant 1$ and that we do not have $i_1 = i_2 = i_3$.

*First case.* $i_1, i_2, i_3$ are distinct.

By the assumption, $\deg P_{i_1}, \deg P_{i_2}, \deg P_{i_3}$ are distinct, this is impossible.

*Second case.* $i_1, i_2, i_3$ are not distinct.

We may assume that $i_1 = i_2 \neq i_3$.

Since $P_{i_1}$ is general, for any $\alpha, \beta, \gamma$, the polynomial $P_{i_1}(y) - \alpha P_{i_1}(\beta y + \gamma)$ either has degree $\geqslant$ deg $P_{i_1} - 3$ or is null. More generally, the same result holds for $Q(y) = \sum_{1 \leqslant k \leqslant 2} \alpha_k P_{i_1}(\beta_k y + \gamma_k)$. But $|\deg P_{i_3} - \deg P_{i_1}| > 3$ by the assumption, so that $\deg Q \neq \deg P_{i_3}$. Therefore, we cannot have $\deg(Q + \alpha_3 P_{i_3}(\beta_3 y + \gamma_3)) \leqslant 1$. □

**Example 67.** By Example 65 the polynomial $y^d + y^{d-1}$ is general for $d \geqslant 5$. Therefore, if we set $P_d = y^{4d+1} + y^{4d}$, the polynomials $P_1, \dots, P_l$ are general and independent (for any $l$). As a consequence, if $a_i \in A \setminus E$ and $e_i = e(P_i)$ for $1 \leqslant i \leqslant l$, then $f = a_1 e_1 \dots a_l e_l$ satisfies condition (C2). If we assume, furthermore, that $f \in G$ and $l \geqslant 7$, then $\langle f \rangle_N \neq G$ by Theorem 45.

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