TRANSFORMATION GROUPS ACTING ON HYPERBOLIC SPACES
FIVE LECTURES AT ENNEY, SWITZERLAND
JANUARY 26-30, 2015

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Abstract. The general theme of these lectures is the study of some transformation groups via their action on some (often not immediately obvious) hyperbolic spaces. Each lecture focus on a particular group, and several applications are discussed along the way, such as the Tits alternative, the WPD property and the non-simplicity, the centralizer of a hyperbolic isometry, dynamical degree and abelian subgroups...

Each lecture ends with a few exercises. I'm not sure that I know how to solve all of them...

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Date: January 2015.

Partially supported by ANR Grant "BirPol" ANR-11-JS01-004-01.

LECTURE 1. $\mathrm{SL}_2(\mathbb{Z})$

In this lecture we explore two ways in which $\mathrm{SL}_2(\mathbb{Z})$ acts on hyperbolic spaces. This group, or its closely related siblings $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z})$, will serve as a toy model for all the results presented in the other lectures. In particular we will be interested in groups of birational transformations, and $\mathrm{GL}_2(\mathbb{Z})$ can be seen as the group of birational monomial maps of the projective (or affine, this makes no difference) plane, by identifying $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to the map:

$$(x, y) \in \mathbb{A}^2 \mapsto (x^a y^b, x^c y^d) \in \mathbb{A}^2.$$

1.1. **Amalgamated product and Bass-Serre tree.**

We take as a starting point the following presentation of $\mathrm{SL}_2(\mathbb{Z})$:

Proposition 1.1. $\mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ *where the respective generators for* $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ *and* $\mathbb{Z}/6\mathbb{Z}$ *are the matrices*

$$-\mathrm{id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \textit{and} \quad R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Equivalently we could have written $\mathrm{SL}_2(\mathbb{Z}) \simeq \langle a, b;\ a^4 = b^6 = 1, a^2 = b^3 \rangle$. We chose instead to use the notion of amalgamated product, so let us recall what it means more generally. Let G be a group, A, B two subgroups, and denote by $i_A \colon A \cap B \to A$, $i_B \colon A \cap B \to B$ the two inclusion morphisms. We say that G is the amalgamated product of its subgroups A, B along their intersection, denoted $G \simeq A *_{A\cap B} B$, if G is isomorphic to $(A * B)/N$, where N is the normal subgroup in the free product $A * B$ generated by all products $i_A(h) i_B(h)^{-1}$, with $h \in A \cap B$. In particular, given choices of representatives (a_i), (b_j) of the non-trivial left cosets $A/(A \cap B)$, $B/(A \cap B)$, any element $g \in G$ admits a unique factorization of the form $g = wc$, where w (for *word*) is a finite alternate composition of some a_i and b_j, and $c \in A \cap B$.

Given such a structure we can construct an abstract simplicial tree $\mathcal{T}$ on which the group G acts: this is the simplest instance of what is known as Bass-Serre theory. The construction is as follows. The vertices of $\mathcal{T}$ are of two types, namely the left cosets G/A and G/B, and the edges are the left cosets $G/(A \cap B)$, with the following gluing rule: for each $f \in G$, the vertices fA and fE are linked by the edge $f(A \cap E)$. This defines a graph, and the fact that G is the amalgamated product of A and B translates in the property that this graph has no loop, that is, this is a tree. The group G acts naturally by isometry on this tree. If $f, g \in G$, we have

$$g \cdot fA = (g \circ f)A, \quad g \cdot fB = (g \circ f)B, \quad \text{and } g \cdot f(A \cap B) = (g \circ f)(A \cap B).$$

Moreover the action is without inversion: if an edge is invariant it is point-wise fixed. This follows from the fact that the two types of vertices are preserved by the action.

Coming back to the case of $\mathrm{SL}_2(\mathbb{Z})$, the associated tree has vertices of valence 2 or 3.

1.2. **Action on the hyperbolic half-space.**

The natural geometric playground for $\mathrm{SL}_2(\mathbb{Z})$ is the hyperbolic half-space $\mathbb{H}^2$, since $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$. Proposition 1.1 is equivalent to the following assertion, that we will prove in this paragraph:

$$\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

Recall the following criterion (see [dlH00, §II.B]) for finding free products.

Lemma 1.2 (Ping-Pong). *Let G be a group acting on a set X, let Γ_1, Γ_2 be two subgroups of G and let $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ be the subgroup of G generated by Γ_1 and Γ_2. Assume there exists X_1, X_2 two disjoint non empty subsets of X such that*

$$\begin{array}{lll} \gamma(X_2) \subseteq X_1 & \text{for all} & \gamma \in \Gamma_1, \gamma \neq 1 \\ \gamma(X_1) \subseteq X_2 & \text{for all} & \gamma \in \Gamma_2, \gamma \neq 1 \end{array}$$

Assume moreover that one of the Γ_i has order at least 3. Then Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$.

Now we apply the Ping-Pong Lemma to the action by homography of $\mathrm{PSL}_2(\mathbb{Z})$ on the real projective line $\mathbb{R} \cup \{\infty\}$ (boundary of the half-plane $\mathbb{H}^2$), and to the subgroups $\Gamma_1 = \langle S \rangle$, $\Gamma_2 = \langle R \rangle$. We have

$$Sx = -\frac{1}{x} \quad \text{and} \quad Rx = 1 - \frac{1}{x}.$$

Let $X_1 = \mathbb{R}_{\leq 0}$ and $X_2 = \mathbb{R}_{>0} \cup \{\infty\}$. One checks that $S(X_2) \subseteq X_1$ and $R(X_1) \subseteq X_2$. This gives $\langle S, R \rangle \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

Now we sketch (see [Ser70, §VII.1]) the proof of the equality $\langle S, R \rangle = \mathrm{PSL}_2(\mathbb{Z})$, which is equivalent to $\langle S, T \rangle = \mathrm{PSL}_2(\mathbb{Z})$, where we put $T = RS$. Observe that this element is the translation $Tz = z + 1$. Now consider

$$D = \left\{ z \in \mathbb{H}^2; |z| \geq 1, \, |\mathrm{Re}\, z| \leq \frac{1}{2} \right\} \subset \mathbb{H}^2.$$

Fact 1.3. *For any $z \in \mathbb{H}^2$, there exists a matrix in $\langle S, T \rangle$ that sends z to a point in D.*

Indeed one checks that if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ we have $\mathrm{Im}\, Mz = \frac{\mathrm{Im}\, z}{|cz+d|^2}$. So there exists $M \in \langle S, T \rangle$ such that $\mathrm{Im}\, Mz$ is maximum. By composing with a power of T we can assume $|\mathrm{Re}\, Mz| \leq \frac{1}{2}$. Then we have $|Mz| \geq 1$, otherwise we would get the contradiction $\mathrm{Im}\, SMz > \mathrm{Im}\, Mz$.

Fact 1.4. *Let $M \in \mathrm{PSL}_2(\mathbb{Z})$, and assume there exists $z \neq z' \in D$ such that $Mz = z'$. Then z, z' both belong to the frontier ∂D, and $z' = Tz, T^{-1}z$ or Sz.*

Fact 1.5. *For any $z \in \mathring{D}$, and any $M \in \mathrm{PSL}_2(\mathbb{Z})$, $Mz = z$ implies $M = \mathrm{id}$.*

Now let $M \in \mathrm{PSL}_2(\mathbb{Z})$, and pick any $z \in \mathring{D}$. By Fact 1.3 there exists $N \in \langle S, T \rangle$ such that $NMz \in D$. Since $z \notin \partial D$, by Fact 1.4 we have $NMz = z$. Then by Fact 1.5 we get $NM = \mathrm{id}$, that is, $M = N^{-1} \in \langle S, T \rangle$.

In passing we also get the following

Proposition 1.6. *The set D is a fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}^2$.*

As a nice corollary we get a more concrete geometric realization of the Bass-Serre tree of $\mathrm{SL}_2(\mathbb{Z})$.

Corollary 1.7. *The orbit under the action of $\mathrm{PSL}_2(\mathbb{Z})$ of the geodesic segment between $e^{\frac{i\pi}{2}}$ and $e^{\frac{i\pi}{3}}$ is a realization of the Bass-Serre tree of $\mathrm{SL}_2(\mathbb{Z})$.*

1.3. **More geometry.** Let X be a geodesic metric space. We define now two notions of non-positive curvature, both in terms of triangles (Gromov hyperbolicity, $\mathrm{CAT}(\kappa)$ spaces). This allows to answer the question: What is the common feature between $\mathbb{H}^2$ and a tree?

Let $\delta \geq 0$, and let $\Delta \subset X$ be a triangle, that is Δ is the data of 3 geodesic segments $[x, y]$, $[y, z]$, $[x, z]$. We say that Δ is δ-thin is each segment is contained in the δ-neighborhood of the union of the other two.

The space X is called hyperbolic, if there exists a uniform δ such that any triangle in X is δ-thin.

Given a triangle Δ in X, let $\bar{\Delta} \in \mathbb{H}^2$ be a triangle with the same lengths. We call $\bar{\Delta}$ a comparison triangle for Δ, and denote by $\bar{x}, \bar{y}, \bar{z}$ its vertices. We say that Δ is thinner than $\bar{\Delta}$ if for any $t, u \in \Delta$, we have $d_X(t, u) \leq d_{\mathbb{H}^2}(\bar{t}, \bar{u})$ where $\bar{t}$, $\bar{u}$ are the corresponding comparison points in $\bar{\Delta}$.

The space X is called $\mathrm{CAT}(-1)$ is any triangle Δ in X is thinner than its comparison triangle $\bar{\Delta} \subset \mathbb{H}^2$. By considering a comparison space of constant curvature $\kappa \in \mathbb{R}$ we can similarly define a notion of $\mathrm{CAT}(\kappa)$ space. In particular a $\mathrm{CAT}(0)$ space is a space where all triangles are thinner than Euclidean ones.

Observe that a tree is both 0-hyperbolic and $\mathrm{CAT}(\kappa)$ for any $\kappa \in \mathbb{R}$ (one could say that a tree is $\mathrm{CAT}(-\infty)$).

In Lecture 3, I shall point out analogies between the Cremona group and the Mapping Class Group. Observe that $\mathrm{SL}_2(\mathbb{Z})$ is one of the most simple example of mapping class group, namely, the one associated with the torus T^2.

Proposition 1.8. *The morphism* $\sigma\colon \mathrm{Mod}(T^2) \to \mathrm{SL}_2(\mathbb{Z})$ *given by the action of* $H_1(T^2, \mathbb{Z}) \simeq \mathbb{Z}^2$ *is an isomorphism.*

Idea of proof, after [FM12, Theorem 2.5]. Let α, β be generators of $H_1(T^2, \mathbb{Z})$. The symmetry that fixes point-wise α induces the map $(x, y) \mapsto (x, -y)$, hence corresponds to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. It follows that maps preserving orientation have their image in $\mathrm{SL}_2(\mathbb{Z})$. It is easy to see that the map in surjective: let $A \in \mathrm{SL}_2(\mathbb{Z})$, then the linear action of A on $\mathbb{R}^2$ descends to the quotient $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$, and acts as the matrix A on homology. The injectivity is a little more delicate. $\square$

In fact there is a direct relation between $\mathrm{SL}_2(\mathbb{Z})$ seen as a group of monomial maps over $\mathbb{C}$ or as the mapping class group of the torus: consider the restriction of the map $(x, y) \mapsto (x^a y^b, x^c y^d)$ to the torus $|x| = |y| = 1$ (see [CF03, Example 1.3]).

1.4. **Exercises.**

Exercise 1.1. Find the factorization of $\left(\begin{smallmatrix} 3 & -2 \\ -4 & 3 \end{smallmatrix}\right)$ in the amalgamated product of Proposition 1.1. Is the associated isometry on the Bass-Serre tree elliptic or hyperbolic? Same question for $\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)$.

Exercise 1.2. What is the trouble if we allow both groups Γ_1 and Γ_2 to be of cardinal 2 in the Ping-Pong Lemma 1.2?

Exercise 1.3. Is the embedding of the Bass-Serre tree into $\mathbb{H}^2$ given in Corollary 1.7 a quasi-isometry?

Exercise 1.4. Can you find examples of geodesic metric spaces that are

- $\mathrm{CAT}(0)$ but not δ-hyperbolic?

- δ-hyperbolic but not CAT(0)?
- CAT(0) and δ-hyperbolic but not CAT($-\varepsilon$) for any $\varepsilon > 0$?

Lecture 2. $\mathrm{Aut}(\mathbb{A}^2)$

Let $\mathbf{k}$ be any field, and let $\mathbb{A}^2$ be the affine plane over $\mathbf{k}$. In this lecture we explore the properties of the group $\mathrm{Aut}(\mathbb{A}^2)$, that is, the group of polynomial automorphisms of the affine plane. In particular we show that this group satisfies the Tits alternative.

2.1. **Amalgamated product.** In $\mathrm{Aut}(\mathbb{A}^2)$ we have the following two natural subgroups: first the affine group $A = \mathrm{GL}_2(\mathbf{k}) \ltimes \mathbf{k}^2$, and second the group of elementary automorphisms

$$E = \{(x, y) \mapsto (ax + P(y), by + c);\ a, b, c \in \mathbf{k}, ab \neq 0, P \in \mathbf{k}[y]\}.$$

A classical theorem gives a presentation by "generators and relations" for this group.

Theorem 2.1 (Jung-van der Kulk-Nagata, see for instance [Lam02])**.** *The group* $\mathrm{Aut}(\mathbb{A}^2)$ *is the amalgamated product of its subgroups A and E along their intersection:*

$$\mathrm{Aut}(\mathbb{A}^2) = A *_{A \cap E} E.$$

The group $\mathrm{Aut}(\mathbb{A}^2)$ acts on its Bass-Serre tree $\mathcal{T}$. Recall from Lecture 1 that the vertices of $\mathcal{T}$ are of two types, namely the left cosets $\mathrm{Aut}(\mathbb{A}^2)/A$ and $\mathrm{Aut}(\mathbb{A}^2)/E$, and the edges are the left cosets $\mathrm{Aut}(\mathbb{A}^2)/(A \cap E)$, with the following gluing rule: for each $f \in \mathrm{Aut}(\mathbb{A}^2)$, the vertices fA and fE are joined by the edge $f(A \cap E)$. In contrast with the case of $\mathrm{SL}_2(\mathbb{Z})$, observe that this tree is not locally finite (even if $\mathbf{k}$ is the field with two elements!), and if $\mathbf{k}$ is uncountable then all vertices have uncountable valence.

Let f be an isometry of a tree $\mathcal{T}$ (possibly not simplicial). We denote by $\mathrm{Min}(f)$ the subtree of points minimizing the distance $d(x, f(x))$, and $\ell(f)$ this minimum (translation length). The set $\mathrm{Min}(f)$ is never empty by the following classical lemma.

Lemma 2.2. *Let g be an isometry of a tree $\mathcal{T}$.*

(1) *If gx is the middle point of the segment $[x, g^2x]$ then the segment $[x, g^2x]$ is contained in* $\mathrm{Min}(g)$.
(2) *For any point $x \in \mathcal{T}$, the middle point m of the segment $[x, gx]$ is contained in* $\mathrm{Min}(g)$.

Proof. (1) If $gx = x$ there is nothing to show. Otherwise, the segments $[g^ix, g^{i+1}x]$ form an infinite geodesic Γ on which g acts as a translation of length $\ell(g) = d(x, gx)$. If y is any other point, and z is the projection of y on Γ, then $d(y, gy) = \ell(g) + 2d(y, z)$. So $\Gamma = \mathrm{Min}(g)$, and in particular it contains the segment $[x, g^2x]$.

(2) Let $[x, p]$ be the maximal subsegment in $[x, m]$ such that $[gp, gx] \subset [x, gx]$. Then gp is the middle point of $[p, g^2p]$ (which also contains m) and we can apply the previous assertion. □

From the proof of the lemma we also recover that $\mathrm{Min}(f)$ corresponds either to $\mathrm{Fix}(f)$ or $\mathrm{Ax}(f)$, according whether $\ell(f) = 0$ (f is elliptic) or $\ell(f) > 0$ (f is hyperbolic). In particular on a tree there is no parabolic isometry.

As a first application of the action on the tree, one can easily show that $\mathrm{Aut}(\mathbb{A}^2)$ contains huge free subgroups:

Lemma 2.3. *Let* $e = (x, y) \mapsto (x + y^2, y)$, $a_\lambda \colon (x, y) \mapsto (\lambda x + y, x)$ *where* $\lambda \in \mathbf{k}$, $a_\infty = \mathrm{id}$, *and set* $g_\lambda = a_\lambda e a_\lambda^{-1}$. *Then the subgroup generated by the* g_λ *is the free product of the* $\langle g_\lambda \rangle$ *(parametrized by* $\mathbb{P}^1_{\mathbf{k}} = \mathbf{k} \cup \{\infty\}$*).*

Observe that $\langle g_\lambda \rangle \simeq \mathbb{Z}$ if $\operatorname{char} \mathbf{k} = 0$, but $\langle g_\lambda \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ if $\operatorname{char} \mathbf{k} = p$.

Proof. By construction g_λ fixes the vertex $a_\lambda E$ but not the vertex $\mathrm{id}A$. Denote by U_λ the subtree of $\mathcal{T}$ defined as the set of vertices whose projection on the segment $[\mathrm{id}A, a_\lambda E]$ is equal to $a_\lambda E$. Then for any $\lambda' \neq \lambda$ we have $g_\lambda U_{\lambda'} \subset U_\lambda$. We conclude by a variant of the Ping-Pong Lemma 1.2. □

2.2. **Tits alternative.** A group G satisfies the Tits alternative if any subgroup $H \subset G$ either contains a solvable group of finite index, or contains a free group over two generators. A classical theorem of Tits [Tit72] says that if $\mathbf{k}$ is a field of characteristic 0, then $\mathrm{GL}_n(\mathbf{k})$ satisfies the Tits alternative. On the other hand, if $\mathbf{k}$ is an infinite field of characteristic p (for instance the algebraic closure of the finite field $\mathbb{F}_p$), then $\mathrm{SL}_n(\mathbf{k})$ does not contains non abelian free groups (since all elements have finite order), and is not virtually solvable either (if H is a solvable subgroup of finite index, by taking the intersection of the gHg^{-1} one can reduce to the case where H is also normal, then H is a group of scalar matrices, contradiction with finite index). The usual way of bypassing this problem in positive characteristic is to restrict to subgroups H that are finitely generated: one obtains the weak Tits alternative.

When we have a group G acting on a tree $\mathcal{T}$, we have a geometric Tits alternative (see [PV91]), as follows: any subgroup $H \subset G$

(1) either fixes a vertex in $\mathcal{T}$ (elliptic),
(2) or fixes a point at infinity (parabolic),
(3) or preserves two points at infinity (elementary hyperbolic),
(4) or contains a free group over two generators (general hyperbolic).

In the case $G = \mathrm{Aut}(\mathbb{A}^2)$ acting on its Bass-Serre tree, in case (1) we have H conjugate to a subgroup of A (linear group) or E (solvable group). Hence to prove the Tits alternative, it is sufficient to understand cases (2) and (3). We conclude this section by sketching a proof of:

Proposition 2.4 (see [Lam01])**.** *Let* $H \subset \mathrm{Aut}(\mathbb{C}^2)$ *be a subgroup of the automorphism group of the complex affine plane.*

(1) *If* H *is parabolic, that is,* H *fixes exactly one point at infinity, then* H *is an injective limit of finite cyclic groups.*
(2) *If* H *is elementary hyperbolic, that is,* H *contains a hyperbolic isometry* g *and preserves the two ends of* $\mathrm{Ax}(g)$*, then* H *is virtually cyclic, and more precisely isomorphic to* $\mathbb{Z} \ltimes F$ *with* F *either finite cyclic or containing a finite cyclic subgroup of index 2.*

Fact 2.5. *Let* $f \in \mathrm{Aut}(\mathbb{C}^2)$. *Then* $\mathrm{Fix}(f) \subset \mathcal{T}$ *has infinite diameter if and only if* f *is conjugate to* $(\alpha x, \beta y)$ *with* α, β *primitive roots of the unity of the same order.*

From this it follows that if $H \subset \mathrm{Aut}(\mathbb{C}^2)$ is a parabolic subgroup, all elements of H are conjugate to some $(\alpha x, \beta y)$ as above, and they commute (so they all share a common fixed point in $\mathbb{C}^2$). Then by setting $H_i \subset H$ the subgroup of H of elements of order at most n_i, where n_i is any sequence of integers such that, for all i, $n_i | n_{i+1}$

and n_i is the order of an element of H, we get that H is the injective limit of the H_i.

Fact 2.6. *Let $f, g \in \mathrm{Aut}(\mathbb{C}^2)$, with f elliptic and g hyperbolic. Assume $\mathrm{Ax}(g) \cap \mathrm{Fix}(f)$ unbounded. Then f and g^2 commute.*

Fact 2.7. *Let $g \in \mathrm{Aut}(\mathbb{C}^2)$ be a hyperbolic element. Then the group*

$$H = \{f \in \mathrm{Cent}(g);\ f \text{ fixes pointwise } \mathrm{Ax}(g)\}$$

is a finite cyclic group.

Indeed if $f \in H$, then f is conjugate to $(\alpha x, \beta y)$ and induces a permutation on the fixed points of g distinct from the origin. Up to replacing g by a power there are such fixed points (g admits countably many periodic saddle points), and their number is bounded by the degree of g. Finally f has order bounded by the degree of g, so we can assume that f has maximal order in $\mathrm{Cent}(g)$, and then it is easy to show that f generates the subgroup of elliptic elements in $\mathrm{Cent}(g)$.

2.3. **Examples.**

Example 2.8 (Elementary hyperbolic groups)**.** We give a collection of examples for the second case in Proposition 2.4. Let $n \geq 3$, and let α be a primitive nth root of the unity. Set $g = (y, y^{n-1} - x)$, $f = (\alpha x, \alpha^{-1} y)$, and $a = (y, x)$. One check that f commutes with g^2 (but not with g), and that the involution a conjugates g to its inverse g^{-1}. Thus the group $\langle g, f, a\rangle$ is an elementary hyperbolic group isomorphic to $(\mathbb{Z} \ltimes \mathbb{Z}/n) \rtimes \mathbb{Z}/2$.

Example 2.9 (Parabolic groups)**.** Examples of parabolic subgroups in $\mathrm{Aut}(\mathbb{A}^2)$ where first found by Wright [Wri79]. Up to conjugacy, it turns out that any parabolic subgroup has the following form.

Let $(p_k)_{k \geq 1}$ be a sequence of integers ≥ 2, and let $(\alpha_k, \beta_k)_{k \geq 0}$ be pairs of primitive root of the unity of the same order satisfying $(\alpha_0, \beta_0) = (1, 1)$ and $\alpha_k^{p_k} = \alpha_{k-1}$, $\beta_k^{p_k} = \beta_{k-1}$ for all $k \geq 1$.

Set $f_k = (\alpha_k x, \beta_k y)$. Now we claim that there exists a sequence (g_k) of elements in $\mathrm{Aut}(\mathbb{C}^2)$ such that

- The factorization of each g_k in the amalgamated product $A *_{A \cap E} E$ has even length and start (on the left) by an element in A;
- The union of the cyclic groups H_k generated by the
$$h_k = (g_1 \cdots g_k) f_k (g_k^{-1} \cdots g_1^{-1})$$
is a parabolic subgroup.

The condition to put on the g_k is easier to express using the action on the Bass-Serre tree. A sequence (g_k) defines a sequences of (non contiguous) edges $\mathrm{id}S$, $g_1 S$, $g_1 g_2 S, \ldots, g_1 g_2 \cdots g_k S, \ldots$ and also a decreasing sequences of geodesic rays Γ_k: we define Γ_k as the convex hull of the edges in the previous sequence starting from $g_1 g_2 \cdots g_k S$. We want to choose the g_k such that for any k we have

$$\Gamma_k \subset \mathrm{Fix}(h_k) \text{ but } \Gamma_{k-1} \not\subset \mathrm{Fix}(h_k).$$

This will be the case if g_j commutes with f_k for any $j > k$, but not with f_k. For instance, setting $d = \mathrm{order}(\alpha_{k-1})$, if $\alpha_{k-1} = \beta_{k-1}$ one can take $g_k = (y, y^{d+1} - x)$; and if $\alpha_{k-1}^p = \beta_{k-1}$, $\beta_{k-1}^q = \alpha_{k-1}$ with $2 \leq p, q < d$, one can take $g_k = (y, y^p - x) \circ (y, y^q - x)$.

2.4. **Exercises.**

Exercise 2.1. Can you find $f \in \mathrm{Aut}(\mathbb{C}^2)$ such that the action of f on the Bass-Serre tree is elliptic, with $\mathrm{Fix}(f)$ of diameter 2? of diameter 4? Of diameter 6? Same question with the base field $\mathbb{F}_2$.

Exercise 2.2. Find an example of $g \in \mathrm{Aut}(\mathbb{C}^2)$ such that the action of g on the Bass-Serre tree is hyperbolic, g is not conjugate to g^{-1}, but there exists $n > 1$ such that g^n is conjugate to g^{-n}.

Exercise 2.3. Find an example of an embedding of the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Aut}(\mathbb{A}^2)$ such that any non-trivial element acts on the Bass-Serre tree as a hyperbolic isometry.

Exercise 2.4. Let $\mathbf{k}$ be a infinite field of characteristic p. Find an example of a subgroup in $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ that fixes two points at infinity in the Bass-Serre tree, but which is not virtually cyclic (compare with Proposition 2.4).

Lecture 3. $\mathrm{Mod}(\Sigma_g)$

In this lecture we leave for a moment the world of birational geometers, and introduce the Mapping Class Group, which is a favourite among topologists. This group presents many striking similarities with birational groups of transformations. In particular, we shall see that it acts on a non-locally finite hyperbolic graph.

3.1. **The curve complex.** Let Σ_g be the compact surface of genus g (unique from the diffeomorphic viewpoint). The modular group $\mathrm{Mod}(\Sigma_g)$ of the surface Σ_g, or mapping class group, is defined as the quotient of the group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(\Sigma_g)$ by the normal subgroup of diffeomorphisms isotopic to identity (working with homeomorphisms gives the same group, and is in fact sometimes more convenient).

The simplest elements of infinite order in $\mathrm{Mod}(\Sigma_g)$ are the Dehn twists: "cut along a simple closed curve, perform a full turn, and glue again". The Dehn twists are natural generators for $\mathrm{Mod}(\Sigma_g)$: they play a similar role as elementary matrices for linear groups, or as Jonquières maps for the Cremona group. By a result of Thurston, any element in $\mathrm{Mod}(\Sigma_g)$ admits a representative f that falls in one of the three following types:

(1) Periodic: $f^n = \mathrm{id}$;
(2) Reducible: f preserves a finite collection of pairwise disjoint simple closed curves (Dehn twists are examples);
(3) Pseudo-Anosov: there exists λ, called the stretch factor of f, such that, away from finitely many points, f locally looks like $(\lambda x, \frac{1}{\lambda} y)$ (precisely f preserves two measured foliations, dilating or contracting the measure by a factor λ).

The group $\mathrm{Mod}(\Sigma_g)$ is a finitely presented group [FM12, Theorems 4.1 & 5.3], but its Cayley graph is not hyperbolic, because there are some $\mathbb{Z}^2$ embedded in $\mathrm{Mod}(\Sigma_g)$. On the other hand $\mathrm{Mod}(\Sigma_g)$ acts on the fundamental group $\pi_1(\Sigma_g)$, but not on the Cayley graph of $\pi_1(\Sigma_g)$(the action does not send an edge on an edge), so again we fail to get an action on a hyperbolic space.

The curve complex is a simplicial complex on which $\mathrm{Mod}(\Sigma_g)$ acts. We recall the construction. The vertices of the complex are isotopy classes of simple closed curves in Σ_g. We put an edge between two vertices if they admit disjoint representatives. In this way we obtain the 1-skeleton of the curve complex, the curve graph, that we denote by $\mathcal{C}$. Similarly we can construct a simplicial complex by associating a n-simplex to each collection of $n+1$ vertices with pairwise disjoint representatives.

Here are some natural questions about $\mathcal{C}$: Is $\mathcal{C}$ unbounded? Is $\mathcal{C}$ hyperbolic?

3.2. **Laminations and infinite diameter.** To answer the first question we need the concept of lamination. A lamination on Σ_g is a closed subset that is a union of embedded geodesics, called the leaves. There is a natural order on the set of laminations given by inclusion, hence a notion of minimal or maximal lamination. Observe that a lamination is minimal if and only if any leaf is dense. A trivial example of minimal (but not maximal) lamination is provided by taking a single closed geodesic. We say that a lamination L is full if the components of the complement of L are isometric to ideal polygons of $\mathbb{H}^2$. A measured lamination is a lamination endowed with a transversal measure. We denote by $\mathcal{ML}$ the set of measured laminations, and by $\mathcal{PML}$ the set of projective measured laminations (that is, where the transverse measure is only defined up to a multiplicative constant).

Facts:

(1) In $\mathcal{ML}$, we have the implication L full and minimal $\Rightarrow$ L maximal (because the only topological possibilities would be to add an isolated leaf spiraling towards L, but then we would have to put an atomic measure on this extra leaf, in contradiction with the locally finite mass of the initial L).

(2) The set of multi-curves (with atomistic weight) is dense in $\mathcal{ML}$, and the natural intersection product on this set extends as a continuous map $i\colon \mathcal{ML}\times\mathcal{ML}\to [0,+\infty[$. In particular if $L\in\mathcal{ML}$ is full and γ is any geodesic, we have $i(L,\gamma)>0$.

(3) The set $\mathcal{PML}$ is compact.

(4) There exist sequences of simple closed curves (normalized by their length) that converge in $\mathcal{ML}$ to a full and minimal lamination L (consider the iterates of any closed curve under a pseudo-Anosov diffeomorphism).

Proposition 3.1. *The curve graph is unbounded.*

Idea of proof, after [MM99, p. 124]. Let α_n be a sequence of normalized simple closed curves that converge in $\mathcal{ML}$ to a full and minimal lamination L. Let a_n be the corresponding sequence of vertices in $\mathcal{C}$. Assume that the sequence $d(a_0,a_n)$ is bounded by $N>0$. Taking a subsequence, we can assume that $d(a_0,b_n)=N$ for all $n\geq 1$. Then there exist $(b_n)_{n\geq 1}$ such that $d(a_0,b_n)=N-1$ and $d(b_n,a_n)=1$ for all $n\geq 1$. By compactness $\mathcal{PML}$, the sequence b_n corresponds to a normalized sequence β_n that converges to a lamination $L'\in\mathcal{ML}$. By continuity of the intersection product, since $i(a_n,b_n)=0$ for all n we have $i(L,L')=0$. This implies $L'=L$ (any leaf of L' intersects trivially L, thus is in L by maximality, but L is also minimal). We can continue this reduction process until $N=0$, that is, $\alpha_0=L$. But α_0 is not full: contradiction. □

3.3. **Hyperbolicity.** The proof of the hyperbolicity of the curve graph was first proved by Masur and Minsky [MM99], but since then other simplified proofs were found, in particular by Bowditch [Bow06, Bow14]. Recently Sisto [Sis13] proposed a particularly short proof, which I now sketch. See [Lon14] for a detailed report.

The basic tool is the following criterion for hyperbolicity by Bowditch.

Proposition 3.2. *Let X be a graph, and $h \geq 0$. Assume that to each pair of vertices $x, y \in X$ we associate a connected subgraph $\mathcal{L}(x, y)$ containing x and y such that*

(1) *If $d(x, y) \leq 1$, then the diameter of $\mathcal{L}(x, y)$ is at most h;*
(2) *For any $x, y, z \in X$, $\mathcal{L}(x, z)$ is contained in the h-tubular neighborhood of $\mathcal{L}(x, y) \cap \mathcal{L}(y, z)$.*

Then X is δ-hyperbolic, for some δ depending only on h.

The result of Bowditch is even more precise, since he also prove that any path from x to y in $\mathcal{L}(x, y)$ is a quasi-geodesic for some uniform constant. It is possible to adapt the statement to any geodesic metric space.

Proposition 3.3. *The curve graph is hyperbolic.*

Idea of proof, after Sisto. For convenience, work in the augmented curve graph, where we put an edge between two vertices if representatives intersect at most once. This graph is quasi-isometric to the initial one, so from the perspective of hyperbolicity it doesn't change anything.

Then, to each pair of vertices a, b in the curve graph we define $\mathcal{L}(a, b)$ as the collection of vertices c which are given by the union of an arc of a and one arc of b, and with edges when two such vertices are at distance 1 in the augmented graph. It is easy to see that $\mathcal{L}(a, b)$ is connected, and if we allow empty arcs in the definition it contains a and b.

Now let a, b, c be three vertices, let $d \in \mathcal{L}(a, b)$ be any vertex. Consider 3 consecutive intersection points of c with d (if $i(c, d) \leq 2$, c is not far from d). We can assume that 2 of the point, p and q, are on a. Then by taking the a-arc and the c-arc between p and q we construct $e \in \mathcal{L}(a, c)$ that intersect at most once d. □

3.4. **WPD property.** First, in relation with Lecture 2, let us mention that the Tits alternative is known to hold for mapping class groups [McC85]. The proof uses the action on the space of projective lamination, which can be identified with a compactification of the curve complex. (As far as I know it is an important open question whether mapping class groups are linear or not).

As another application to the previous material, we now introduce the WPD condition of Bestvina-Fujiwara [BF02], that allows for instance to construct normal subgroups with all elements pseudo-Anosov (as a corollary of Theorem 3.5 below).

Let G a group acting by isometry on a metric space X. Let $A \subseteq X$ be a subset, and let $\eta \geq 0$. We define the η-stabilizer of A in G as

$$\operatorname{Fix}_\eta A = \{g \in G;\ d(a, ga) \leq \eta \text{ for all } a \in A\}.$$

Observe that $\operatorname{Fix}_\eta A$ is stable under taking the inverse, but in general not under composition, so a priori this is not a subgroup of G!

Now let $g \in G$. We say that g satisfies the WPD property if

$$\exists x \in X, \forall \eta \geq 0, \exists M \in \mathbb{N} \text{ such that } \operatorname{Fix}_\eta\{x, g^M x\} \text{ is finite.}$$

Proposition 3.4 ([BF02, Proposition 11])**.** *Any pseudo-Anosov class g in the mapping class group satisfies the WPD property for the action of* $\operatorname{Mod}(\Sigma_g)$ *on the curve graph.*

Sketch of proof. We pick $x \in \operatorname{Ax}(g)$, and $\eta \geq 0$. We can assume that η is an integer. Let L^+ and L^- be the two laminations corresponding to the ends of $\operatorname{Ax}(g)$. One knows that $i(L^\pm, L) = 0$ implies $L = L^\pm$. By continuity of $i(\cdot,\cdot)$, we can construct a tower of closed neighborhood of $L^\pm$ in $\mathcal{PML}$

$$U_\eta^\pm \supseteq U_{\eta-1}^\pm \supseteq \cdots \supseteq U_0^\pm$$

such that, for any $0 \le j < \eta$ and any $L_j \in U_j^\pm$, $i(L_j, L) = 0$ implies $L \in U_{j+1}^\pm$. Moreover we can assume that for any $L \in U_\eta^+$ and $L' \in U_\eta^-$, we have $i(L, L') \neq 0$. Now there exist $p > q$ in $\mathbb{Z}$ such that $g^p x \in U_0^+$ and $g^q x \in U_0^-$. Since $\mathrm{Fix}_\eta\{x, g^{p-q}x\} = \mathrm{Fix}_\eta\{g^p x, g^q x\}$, by setting $M = p - q$, it is sufficient to show that for any curves $a \in U_0^+$, $b \in U_0^-$, the set $\mathrm{Fix}_\eta\{a, b\}$ is finite.

By contradiction, assume (f_n) is an infinite sequence of pairwise distinct elements in $\mathrm{Fix}_\eta\{a, b\}$. For each $n \in \mathbb{N}$, there exist vertices $c_0 = a$, c_1, ..., $c_\eta = f_n(a)$ such that $d(c_j, c_{j+1}) \le 1$. In particular by induction $c_j \in U_j^+$ for all $0 \le j \le \eta$, and $f_n(a) \in U_\eta^+$. By taking subsequences, we can assume $f_n(a)$ (resp. $f_n(b)$) converges to a lamination $A \in U_\eta^+$ (resp. $B \in U_\eta^-$) in $\mathcal{PML}$. Normalizing by the length r_n of $f_n(a)$, we obtain a convergent sequence $\frac{f_n(a)}{r_n}$ in $\mathcal{ML}$. If $r_n \to \infty$, then we would have $i(A, B) = 0$, contradiction. Hence the length of the curves $f_n(a)$ (and similarly $f_n(b)$) are bounded. But there are only finitely many classes of a given bounded length, so passing again to a subsequence we can assume that $f_n(a)$ and $f_n(b)$ are constant. Finaly (f_n) is a sequence in the stabilizer $S(a, b)$ of both curves a and b, which is finite: contradiction. □

A motivation for looking for WPD elements is that by taking power one can construct small cancellation families, and then produce many normal subgroups. We don't say more on small cancellation theory (but see Exercise 3.4 for a first glimpse of what it is about), and we content ourselves by extracting the following result from [DGO11]:

Theorem 3.5. *Let X be a δ-hyperbolic metric space, and let G be a non-virtually cyclic group acting by isometry on X. Assume $g \in G$ satisfies the WPD property. Then there exists $n > 0$ such that all non-trivial elements of the normal subgroup $\ll g^n \gg$ are hyperbolic with translation length at least $\ell(g^n)$.*

The WPD property is relevant mostly when X is a non-locally compact hyperbolic metric space: This the case for the curve graph, and it was also the case for the Bass-Serre tree of $\mathrm{Aut}(\mathbb{A}^2)$. The WPD property for $\mathrm{Aut}(\mathbb{A}^2)$ was recently established (over any base field) in [MO13]; this extends previous work by Danilov [Dan74], see also [FL10].

3.5. **Exercises.**

Exercise 3.1. Let Σ be a surface of genus 2. Can you draw two simple closed curves on Σ that correspond to vertices at distance 2 in the curve graph? At distance 3? At distance 4?

Exercise 3.2. Show that the WPD property is equivalent to the following apparently more restrictive version:

$$\forall x \in X, \forall \eta \ge 0, \exists M \in \mathbb{N} \text{ such that } \mathrm{Fix}_\eta\{x, g^M x\} \text{ is finite.}$$

Exercise 3.3. Consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$ by homography, and let $A \in \mathrm{SL}_2(\mathbb{Z})$ be any matrix with $|\operatorname{tr} A| > 2$. Show that A satisfies the WPD condition.

Exercise 3.4. Draw a picture of the axes of hyperbolic isometries $h_1, \ldots, h_n$ on a tree $\mathcal{T}$ such that, for some $L > 0$:

(1) All h_i have the same translation length $3L$;
(2) for any $i \neq j$, $\mathrm{Ax}(h_i) \cap \mathrm{Ax}(h_j)$ has diameter at most L;
(3) $h_1 \circ h_2 \circ \cdots \circ h_n$ is elliptic.

Can you do the same if now in (1) we ask for a translation length of $4L$? $5L$? $6L$?

Lecture 4. $\mathrm{Bir}(\mathbb{P}^2)$

4.1. **Dynamical degree and monomial maps.** Let $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a dominant rational map. One defines the dynamical degree of f as the limit

$$\lambda(f) = \lim_{n\to\infty} (\deg f^n)^{1/n}.$$

More generally, if $f\colon X \dashrightarrow X$ is a dominant rational map on any projective variety X, $\|\cdot\|$ is any norm on the Néron-Severi real vector space $N^1(X)$, and f^* is the induced action by f on $N^1(X)$, we can define

$$\lambda(f) = \lim_{n\to\infty} \|(f^n)^*\|^{1/n}.$$

This quantity is a birational invariant, that is, if $\varphi\colon X \dashrightarrow Y$ is a birational map, and $g = \varphi f \varphi^{-1}$, then $\lambda(f) = \lambda(g)$.

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\varphi"'] & X \arrow[d, dashed, "\varphi"] \\ Y \arrow[r, dashed, "g"'] & Y \end{tikzcd}$$

Proposition 4.1. *Let $f = f_A\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a monomial map, associated with the matrix $A \in \mathcal{M}_n(\mathbb{Z})$. Then $\lambda(f)$ is equal to the spectral radius ρ of A.*

First proof, after [HP07, §6]. The degree of f_A is given by the formula, where $\mathrm{Max} = \max(0, \cdot)$:

$$\deg f_A = \sum_{j=1}^{n} \mathrm{Max}\{-a_{ij};\ 1 \le i \le n\} + \mathrm{Max}\left\{\sum_{j=1}^{n} a_{ij};\ 1 \le i \le n\right\}.$$

Let v be a unit eigenvector associated with an eigenvalue realizing the spectral radius ρ. Then $\|A^N v\| = \rho^N$. In particular, if $0 < c < \rho$, and N is large enough, then A^N admits at least one coefficient with module greater than c^N. From the above formula we deduce that $\deg f_{A^N} > c^N$, and so $\lambda(f) \ge c$. Since this true for any $c < \rho$, we have $\lambda(f) \ge \rho$. The converse inequality is easy. □

Second proof, after [Lin, §7.2]. We view f_A as an element of $\mathrm{Bir}((\mathbb{P}^1)^n)$ instead of $\mathrm{Bir}(\mathbb{P}^n)$. Then one can check that f_A^* acting on $N^1((\mathbb{P}^1)^n) = \mathbb{R}^n$ corresponds to the matrix $(|a_{ji}|)$. This immediately gives $\lim_{n\to\infty} \|(f_A^*)^n\|^{1/n} = \rho$. □

If $f \in \mathrm{Bir}(\mathbb{P}^2)$ is a map with $\lambda(f) > 1$, then the dynamical degree $\lambda(f)$ is an algebraic integer with all Galois conjugates in the unit disk. More precisely, we distinguish between the two following situations: either all Galois conjugates are in the open unit disk ($\lambda(f) \in \mathbf{Pis}$ is a Pisot number), or at least one Galois conjugate has modulus 1 ($\lambda(f) \in \mathbf{Sal}$ is a Salem number).

It is known that $\mathbf{Pis} = \overline{\mathbf{Pis}} \subset \overline{\mathbf{Sal}}$. The smallest Pisot number is the plastic number $\lambda_P \simeq 1.324718$, which is a root of $X^3 - X - 1$. The smallest *known* Salem number is the Lehmer number $\lambda_L \simeq 1.176280$, which is a root of $X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$.

Theorem 4.2 (Gap property, [BC13, Corollary 2.7])**.** *Let $f \in \mathrm{Bir}(\mathbb{P}^2)$. Then*

$$\lambda(f) \notin]1, \lambda_L[.$$

Two main ingredients of the proof are as follows.

If $\lambda(f) > 1$ is a Salem number for some $f \in \mathrm{Bir}(\mathbb{P}^2)$, then f is birationally conjugate to an automorphism of a rational surface S ([BC13, Theorem A]).

If $f\colon S \to S$ is an automorphism of a (rational) surface with $\lambda(f) > 1$, then $\lambda(f) \geq \lambda_L$ ([McM07]).

4.2. **Action on the Picard-Manin space.** Different models for the hyperbolic 2-space: disk model, half-plane model, Klein model, hyperboloid model. The latter one generalizes to give a model of the hyperbolic n-space.

Let S be a smooth projective surface with Picard number ρ, $N^1(S) \simeq \mathbb{R}^\rho$ the Néron-Severi space (divisors with real coefficients, up to numerical equivalence). The intersection form gives a symmetric bilinear form on $N^1(S)$ with signature $(1, \rho - 1)$. In general this follows from the Hodge Index Theorem, which is a consequence of Riemann-Roch Formula, see [Har77, p.364]. Observe however that when S is the blow-up of $\mathbb{P}^2$ along several points, this is an easy direct observation. By considering divisors $D \in N^1(S)$ satisfying $D^2 = 1$, we obtain a 2-sheeted hyperboloid, and by adding the condition $D \cdot H > 0$ for some choice of ample divisor H we select one of the sheets (the one containing nef divisors) and get a model $\mathbb{H}(S)$ of the hyperbolic space $\mathbb{H}^n$. Recall that the distance on $\mathbb{H}^n$ is given by

$$d(D_1, D_2) = \cosh^{-1}(D_1 \cdot D_2),$$

and that geodesic are obtained by intersecting $\mathbb{H}(S)$ with a hyperplane in $N^1(S)$.

Now consider $\pi\colon S' \to S$ a birational morphism (for instance the blow-up of a point). The pull-back map defines a map from $N^1(S)$ to $N^1(S')$ that preserves intersection forms:

$$\pi^* D_1 \cdot \pi^* D_2 = D_1 \cdot D_2.$$

In particular we get an isometric embedding of $\mathbb{H}(S)$ into $\mathbb{H}(S')$.

Now if $S_1 \to S$ and $S_2 \to S$ are two birational morphisms, there exist a third surface S_3 and morphisms $S_3 \to S_1$ and $S_3 \to S_2$ such that the following diagrams commute:

$$\begin{tikzcd} & S_3 \arrow[dl] \arrow[dr] & \\ S_1 \arrow[dr] & & S_2 \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{H}(S_3) \subset N^1(S_3) & \\ \mathbb{H}(S_1) \subset N^1(S_1) \arrow[ur] & & \mathbb{H}(S_2) \subset N^1(S_2) \arrow[ul] \\ & \mathbb{H}(S) \subset N^1(S) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We are ready to consider simultaneously all possible surfaces S' dominating S, and to take an injective limit

$$\mathbb{H}^\infty_C(S) = \lim_{\leftarrow} \mathbb{H}(S') \subset \lim_{\leftarrow} N^1(S') = \mathcal{Z}_C(S).$$

The C in index is for "Cartier", and $\mathcal{Z}_C(S)$ is sometimes called the space of Cartier b-divisors. One could similarly consider a projective limit, and obtain $\mathcal{Z}_W(S)$ which is called the space of Weil b-divisors. The space that will really be useful to us is an intermediate space $\mathbb{H}^\infty(S)$, which is the L^2 completion of $\mathbb{H}^\infty_C(S)$. In concrete terms we have

$$\mathcal{Z}_C(S) = \{D = D_0 + \sum_p a_p E_p;\ D_0 \in N^1(S), a_p = 0 \text{ except finitely many}\};$$

$$\mathcal{Z}_W(S) = \{D = D_0 + \sum_p a_p E_p;\ D_0 \in N^1(S)\};$$

$$\mathcal{Z}(S) = \{D = D_0 + \sum_p a_p E_p;\ D_0 \in N^1(S), \sum a_p^2 < \infty\};$$

Observe finally that if $\pi\colon S_2 \to S_1$ is a birational morphism, then we get a canonical isometry π^* (and not simply an embedding!) between $\mathbb{H}^\infty(S_1)$ and $\mathbb{H}^\infty(S_2)$. This allows to define an action of $\mathrm{Bir}(\mathbb{P}^2)$ on $\mathbb{H}^\infty = \mathbb{H}^\infty(\mathbb{P}^2)$. If $f \in \mathrm{Bir}(\mathbb{P}^2)$ and

$$\begin{tikzcd} & S \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"'] & & \mathbb{P}^2 \end{tikzcd}$$

is a resolution of f, then we set $f_* := (\sigma^*)^{-1} \circ \pi^*$.

Now let $f \in \mathrm{Bir}(\mathbb{P}^2)$ be a map with $\lambda(f) > 1$, and let $D \in \mathbb{H}^\infty$ be an arbitrary point. The sequence $(f^n)_*(D)/\lambda^n$ converges to a point $\omega \in \partial\mathbb{H}^\infty$, which corresponds to a class in the isotropic cone of the intersection form: $\omega \cdot \omega = 0$. Similarly $(f^{-n})_*(D)/\lambda^n$ converges to $\alpha \in \partial\mathbb{H}^\infty$. We can normalize such that $\alpha \cdot \omega = \frac{1}{2}$, and then the set

$$\{u\alpha + v\omega;\ uv = 1\} \in \mathbb{H}^\infty$$

is a geodesic line Γ invariant by f. Moreover, since $f_*\alpha = \frac{1}{\lambda}\alpha$ and $f_*\omega = \lambda\omega$, we can compute the distance L between $P = \alpha + \omega \in \Gamma$ and $fP = \frac{\alpha}{\lambda} + \lambda\alpha$:

$$e^L + \frac{1}{e^L} = 2\cosh L = 2\cosh d(P, fP) = 2P \cdot fP = \lambda + \frac{1}{\lambda}.$$

We see that the logarithm of the dynamical degree (sometimes called algebraic entropy) is the translation length for the action on $\mathbb{H}^\infty$.

4.3. **Centralizer of hyperbolic elements.** We can use the gap property and the action on the Picard-Manin space to get a description of the centralizer of a hyperbolic element.

Proposition 4.3 ([BC13, Corollary 4.7])**.** *Let $h \in \mathrm{Bir}(\mathbb{P}^2)$ be an element of hyperbolic type (that is, $\lambda(h) > 1$). Then the cyclic group $\langle h \rangle$ has finite index in the centralizer group of h in $\mathrm{Bir}(\mathbb{P}^2)$. In particular $\mathrm{Cent}(h) \simeq \mathbb{Z} \rtimes F$ with F finite, and if $g, h \in \mathrm{Bir}(\mathbb{P}^2)$ are two hyperbolic maps that commute, there exist $n, m > 0$ such that $g^m = h^n$.*

Proof. We have a morphism $\mathrm{Cent}(h) \to \mathbb{R}^*$ that sends f to $\lambda(f)$.

By the gap property, the image of this morphisms is a discrete subgroup of $\mathbb{R}^*$.

We are reduced to prove that the kernel F is finite. Any element in the kernel fixes the axis of h point-wise. In particular, if d is the distance from the class ℓ

of a line to $\mathrm{Ax}(h)$, then $d(\ell, f\ell) \leq 2d$, and this implies that $\deg(f)$ is uniformly bounded.

By a result of Blanc and Furter [BF02], the Zariski closure of F is an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (so in fact F is equal to its Zariski closure).

If F is not finite, it contains a 1-dimensional algebraic subgroup A whose orbits are preserved by h: this contradicts $\lambda(h) > 1$. Indeed if L is an orbit of A we have $h^*L = L$, but when $\lambda(h) > 1$ there are exactly two eigenclasses in the Picard-Manin space, which are multiplied by $\lambda(h)^{\pm 1}$ under the action of h^*. □

Using the action on the Picard-Manin space, Cantat [Can11] obtains the (weak, that is, for finitely generated subgroups) Tits alternative for $\mathrm{Bir}(\mathbb{P}^2)$. A first step is to establish a geometric Tits alternative, similarly as in page 6.

Proposition 4.4. *Let $G \subset \mathrm{Bir}(\mathbb{P}^2)$ be a subgroup, and consider the action of G on $\mathbb{H}^\infty$. Then*

(1) *Either G fixes a point in $\mathbb{H}^\infty$,*
(2) *or G fixes exactly one point in $\partial\mathbb{H}^\infty$,*
(3) *or G fixes, or exchanges, exactly two points in $\partial\mathbb{H}^\infty$,*
(4) *or G contains a free group over two generators.*

Idea of proof. The difficult case is when any $h \in G$ is elliptic. If G admits a bounded orbit, the circumcenter of the orbit (well defined in any $CAT(0)$ space) is fixed, so we are in Case (1). If G admits an orbit with limit set a single point in $\partial\mathbb{H}$, this point is fixed and we are in Case (2). Now assume that G admits an orbit with limit set containing two distinct points $a, b \in \partial\mathbb{H}$. So there exists $x \in \mathbb{H}$, $f_n, g_n \in G$ such that $f_n x \to a$ and $g_n x \to b$. Then one can prove that for n large $f_n g_n$ is hyperbolic, hence a contradiction: This is easy in a tree, and one can treat the general case by an approximation argument reducing to the case of a tree (see [GdlH90, Lemme 8.35]). □

4.4. **Exercises.**

Exercise 4.1. Let f be the standard quadratic involution:

$$f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[x : y : z] \mapsto [yz : xz : xy]$$

Let ℓ denote the class of a line in $\mathcal{Z}(\mathbb{P}^2)$, and let e_p denote the class corresponding to the exceptional divisor of the blow-up of a point $p \in \mathbb{P}^2$ (p might be a base point for f). Compute $f_*\ell$ and f_*e_p, where f_* is the induced action on the Picard-Manin space $\mathcal{Z}(\mathbb{P}^2)$. Check that

$$f_*\ell \cdot f_*\ell = 1, \quad f_*e_p \cdot f_*e_p = -1, \quad f_*\ell \cdot f_*e_p = 0.$$

Exercise 4.2. Can you find an example of $g \in \mathrm{Bir}(\mathbb{P}^2)$, such that the action of g on $\mathbb{H}$ is elliptic, and the set $\mathrm{Fix}(g) \subseteq \mathbb{H}$ is bounded?

Exercise 4.3. Can you find an example of $g \in \mathrm{Bir}(\mathbb{P}^2)$ such that $\lambda(g) > 1$ and g is not conjugate to g^{-1}?

Lecture 5. Tame($\mathbb{C}^3$)

5.1. **The tame group.**

Let $\mathbf{k}$ be a field, and let $\mathbb{A}^n = \mathbb{A}^n_{\mathbf{k}}$ be the affine space over $\mathbf{k}$. We are interested in the group $\mathrm{Aut}(\mathbb{A}^n)$ of algebraic automorphisms of the affine space. Concretely, we choose once and for all an origin and a coordinate system $(x_1, \ldots, x_n)$ for $\mathbb{A}^n$. Then any element $f \in \mathrm{Aut}(\mathbb{A}^n)$ is a map of the form

$$f\colon (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),$$

where the f_i are polynomials in n variables, such that there exists a map g of the same form satisfying $f \circ g = \mathrm{id}$. We shall abbreviate this situation by writing $f = (f_1, \ldots, f_n)$, and $g = f^{-1}$.

The group $\mathrm{Aut}(\mathbb{A}^n)$ contains the following natural subgroups. First we have the affine group $A_n = \mathrm{GL}_n(\mathbf{k}) \ltimes \mathbf{k}^n$. Secondly we have the group E_n of elementary automorphisms, which have the form

$$f\colon (x_1, \ldots, x_n) \mapsto (x_1 + P(x_2, \ldots, x_n), x_2, \ldots, x_n),$$

for any choice of polynomial P in $n-1$ variables. The group $\mathrm{Tame}(\mathbb{A}^n) = \langle A_n, E_n \rangle$ is called the subgroup of **tame automorphisms**.

A natural question is whether the inclusion $\mathrm{Tame}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is strict or not. As we saw in Lecture 2, it is a well-known result that the answer is *yes* for $n = 2$ (over any base field). On the other hand, it is a result by Shestakov and Umirbaev ([SU04], see also [Kur10]) that the answer is *no* for $n = 3$, at least when $\mathbf{k}$ is a field of characteristic zero.

5.2. **Abelian subgroups with dynamics.** We recall a well-known construction (see e.g. [DS04, Example 4.5]):

Proposition 5.1. *There exist $\langle f, g \rangle \in \mathrm{Bir}(\mathbb{P}^3)$ a free abelian subgroup of rank 2, such that any non trivial element in $\langle f, g \rangle$ has dynamical degree > 1.*

Proof. We construct f, g in the subgroup of monomial birational maps, which is isomorphic to $\mathrm{GL}_3(\mathbb{Z})$, using some basic arithmetic theory. Let $\mathbf{k} = \mathbb{Q}(\delta)$ be a totally real cubic extension of $\mathbb{Q}$, that is $\delta \in \mathbb{R}$ is a root of $P \in \mathbb{Z}[X]$ with P irreducible of degree 3 and with 3 real roots. Denote by $\mathcal{O}_{\mathbf{k}}$ the ring of integers in $\mathbf{k}$. This is a classical fact (see [Sam67, p. 48]) that $\mathcal{O}_{\mathbf{k}}$ is a free $\mathbb{Z}$-module of rank 3. Then the Theorem of Units of Dirichlet (see [BS66, p. 112] or [Sam67, p. 72]), says that the multiplicative group U of positive units in $\mathcal{O}_{\mathbf{k}}$ is isomorphic to $\mathbb{Z}^2$. Denote by α, β two generators of U. Then the multiplication by α, β on $\mathcal{O}_{\mathbf{k}} \simeq \mathbb{Z}^3$ gives two commuting matrices $A, B \in \mathrm{GL}_3(\mathbb{Z})$, thus two commuting elements $f, g \in \mathrm{Bir}(\mathbb{P}^3)$. The assertion about dynamical degrees is equivalent to the assertion that any non-trivial matrix C in $\langle A, B \rangle$ has spectral radius > 1. Let $(i, j) \in \mathbb{Z}^2 \smallsetminus \{(0, 0)\}$ such that $C = A^i B^j$, and set $\gamma = \alpha^i \beta^j \in U \smallsetminus \{1\}$. Observe that $\gamma \notin \mathbb{Q}$, hence $(1, \gamma, \gamma^2)$ is a basis of a rank 3 submodule $\mathbb{Z}[\gamma] \subseteq \mathcal{O}_{\mathbf{k}}$, in particular this is also a basis of $\mathbf{k}$ as a $\mathbb{Q}$-vector space. Let C' be the matrix of the multiplication by γ in this basis : C' is the companion matrix of the minimal polynomial μ_γ of γ, hence the spectral radius $\rho_{C'}$ (which is equal to ρ_C) is the maximum of the absolute value of the three real roots of μ_γ, whose product is 1. Since $|\gamma| \neq 1$, we get $\rho_C > 1$. $\square$

Example 5.2. (1) Extracted from [Lou12]:

Consider $P(X) = X^3 - (m+3)X^2 + mX + 1 \in \mathbb{Z}[X]$, $m \geq -1$. Let α be the root of P satisfying $\alpha > 1$. Then

$$\alpha \quad \text{and} \quad \beta = 1 - \frac{1}{\alpha} = \alpha^2 - (m+3)\alpha + m + 1$$

generate the positive units in $\mathbb{Z}[\alpha] = \{a + b\alpha + c\alpha^2;\ a, b, c \in \mathbb{Z}\}$, and considering multiplication by α and β on $\mathbb{Z}[\alpha]$, we obtain the following commuting matrices in $\mathrm{GL}_3(\mathbb{Z})$:

$$f = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -m \\ 0 & 1 & m+3 \end{pmatrix} \text{ and } g = \begin{pmatrix} m+1 & -1 & 0 \\ -(m+3) & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(2) Extracted from [Tho79, Proposition 3.6]:

Consider $P(X) = X^3 - (n-1)X^2 + nX - 1 \in \mathbb{Z}[X]$. We check that P is irreducible by observing that it has no roots mod 2.

Let α be the root of P satisfying $\alpha > 1$. Then (under the condition $n \geq 7$)

$$\alpha \quad \text{and} \quad \beta = \alpha - 1$$

generate the positive units in $\mathbb{Z}[\alpha] = \{a + b\alpha + c\alpha^2;\ a, b, c \in \mathbb{Z}\}$, and considering multiplication by α and β on $\mathbb{Z}[\alpha]$, we obtain the following commuting matrices in $\mathrm{GL}_3(\mathbb{Z})$:

$$f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & n-1 \end{pmatrix} \text{ and } g = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & -n \\ 0 & 1 & n-2 \end{pmatrix}.$$

5.3. **Action on a simplicial complex.** We now describe a natural simplicial complex on which the group $\mathrm{Tame}(\mathbb{A}^n)$ acts. In contrast with the previous paragraph, we hope that it could help us prove the following conjecture (among many other wonderful stuff about $\mathrm{Tame}(\mathbb{C}^3)$).

Conjecture 5.3. The tame group $\mathrm{Tame}(\mathbb{C}^3)$ does not contain a subgroup isomorphic to $\mathbb{Z}^2$ such that any non-trivial element has dynamical degree > 1.

Here is the construction. For any $1 \leq r \leq n$, we call r-tuple of components a morphism

$$f \colon \mathbb{A}^n \to \mathbb{A}^r$$
$$x = (x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_r(x))$$

that can be extended as a tame automorphism $f = (f_1, \dots, f_n)$ of $\mathbb{A}^n$. One defines n distinct types of vertices, by considering r-tuple of components modulo composition by an affine automorphism on the range, $r = 1, \dots, n$:

$$[f_1, \dots, f_r] := A_r(f_1, \dots, f_r) = \{a \circ (f_1, \dots, f_r); a \in A_r\}$$

where $A_r = \mathrm{GL}_r(\mathbf{k}) \ltimes \mathbf{k}^r$ is the r-dimensional affine group.

Now for any tame automorphism $(f_1, \dots, f_n) \in \mathrm{Tame}(\mathbb{A}^n)$ we attach a $(n-1)$-simplex on the vertices $[f_1]$, $[f_1, f_2]$, ..., $[f_1, \dots, f_n]$. In particular for any $1 \leq i < j \leq n$, there is an edge between $[f_1, \dots, f_i]$ and $[f_1, \dots, f_j]$. This definition is independent of a choice of representatives and produces a $(n-1)$-dimensional simplicial complex $\mathcal{C}_n$ (see Figure 1 for the case $n = 3$) on which the tame group acts by isometries, by the formulas

$$g \cdot [f_1, \dots, f_r] := [f_1 \circ g^{-1}, \dots, f_r \circ g^{-1}].$$

Lemma 5.4. *The group* $\mathrm{Tame}(\mathbb{A}^n)$ *acts on* $\mathcal{C}_n$ *with fundamental domain the simplex*

$$[x_1], [x_1, x_2], \dots, [x_1, \dots, x_n].$$

In particular the action is transitive on vertices of a given type, and on edges of a given type.

Proof. Let $v_1, \dots, v_n$ be the vertices of a simplex (with index corresponding to the type). By definition there exists $f = (f_1, \dots f_n) \in \mathrm{Tame}(\mathbb{A}^n)$ such that $v_i = [f_1, \dots, f_i]$ for each i. Then

$$[x_1, \dots, x_i] = [(f_1, \dots, f_i) \circ f^{-1}] = f \cdot v_i. \qquad \square$$

Remark 5.5. One could make a similar construction by working with the full automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ instead of the tame group. The complex $\mathcal{C}_n$ we consider is the gallery connected component of the vertex [id] in this bigger complex. See [BFL14, §6.2.1] for more details.

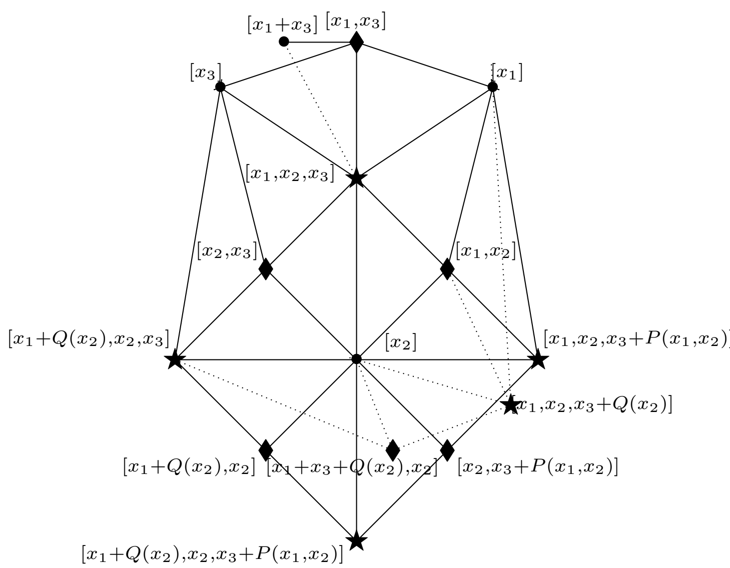

FIGURE 1. A few simplexes of the complex $\mathcal{C}_3$

When $n = 2$, the previous construction yields a graph $\mathcal{C}_2$. We now show (following [BFL14, §2.5.2]) that $\mathcal{C}_2$ is isomorphic to the Bass-Serre tree of $\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Tame}(\mathbb{A}^2)$ constructed in Lecture 2. We use the affine groups:

$$A_1 = \{t \mapsto at + b;\ a \in \mathbf{k}^*, b \in \mathbf{k}\};$$

$$A_2 = \left\{(t_1, t_2) \mapsto (at_1 + bt_2 + c, a't_1 + b't_2 + c');\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2, c, c' \in \mathbf{k}\right\}.$$

The vertices of our graph $\mathcal{C}_2$ are of two types: classes $A_1 f_1$ where $f_1 \colon \mathbf{k}^2 \to \mathbf{k}$ is a component of an automorphism, and classes $A_2(f_1, f_2)$ where $(f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2)$. For each automorphism $(f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2)$, we attach an edge between $A_1 f_1$ and $A_2(f_1, f_2)$. Note that $A_2(f_1, f_2) = A_2(f_2, f_1)$, so there is also an edge between the vertices $A_2(f_1, f_2)$ and $A_1 f_2$.

Recall that $\mathrm{Aut}(\mathbb{A}^2)$ is the amalgamated product of A_2 and E_2 along their intersection, where E_2 is the (extended) elementary group defined as:

$$E_2 = \{(x, y) \mapsto (ax + P(y), by + c);\ a, b \in \mathbf{k}^*, c \in \mathbf{k}\}.$$

The Bass-Serre tree associated with this structure consists in taking cosets $A_2(f_1, f_2)$, $E_2(f_1, f_2)$ as vertices, and cosets $(A_2 \cap E_2)(f_1, f_2)$ as edges (here we use right cosets instead of left cosets for consistency with the convention for $\mathcal{C}_2$).

Proposition 5.6. *The graph $\mathcal{C}_2$ is isomorphic to the Bass-Serre tree associated with the structure of amalgamated product of* $\mathrm{Aut}(\mathbb{A}^2)$.

Proof. We define a map φ from the set of vertices of the Bass-Serre tree to the graph $\mathcal{C}_2$ by taking

$$A_2(f_1, f_2) \mapsto A_2(f_1, f_2),$$
$$E_2(f_1, f_2) \mapsto A_1 f_2.$$

Clearly φ is a local isometry. Moreover φ is bijective, since we can define $\varphi^{-1}(A_1 f_2)$ to be $E_2(f_1, f_2)$ where (f_1, f_2) is an automorphism. Indeed any other way to extend f_2 is of the form $(af_1 + P(f_2), f_2)$, and so the class $E_2(f_1, f_2)$ does not depend on the extension we choose. □

Let us conclude with a few remarks and open questions in the case $n = 3$. The fact that the 2-dimensional simplicial complex $\mathcal{C}_3$ is simply connected has been proved by Wright [Wri13], under the following algebraic reformulation: $\mathrm{Tame}(\mathbb{C}^3)$ is the amalgamated product along their pairwise intersection of the stabilizers of the 3 vertices of a givien simplex. It is not known whether the complex $\mathcal{C}_3$ is contractible. It is not even clear whether $\mathcal{C}_3$ has infinite diameter: recall that the similar question for the curve complex has a pretty technical answer (see Proposition 3.1). If this is indeed the case, the next natural question would be whether $\mathcal{C}_3$ is δ-hyperbolic. In view of the previous work [BFL14], that deals with a variation of these questions ($\mathbb{C}^3$ being replaced by an affine quadric 3-fold), I risk the following conjecture:

Conjecture 5.7. The complex $\mathcal{C}_3$ is contractible (but not CAT(0), see Exercise 5.2), unbounded and hyperbolic.

5.4. **Exercises.**

Exercise 5.1. Can you find an element $g \in \mathrm{Tame}(\mathbb{C}^3)$ such that

- The dynamical degree $\lambda(g)$ is not an integer?
- The dynamical degrees $\lambda(g)$ and $\lambda(g^{-1})$ are distinct?
- $\lambda(g) > 1$ and the centralizer of g is uncountable?

Exercise 5.2. By Lemma 5.4, for any choice of Euclidean structure on a triangle of $\mathcal{C}$ one gets a uniquely defined $\mathrm{Tame}(\mathbb{C}^3)$ -invariant metric on $\mathcal{C}$. Show that such a metric is never CAT(0) (Hint: use Figure 1).

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