INTRODUCTION TO THE CREMONA GROUP FOUR LECTURES AT CUERNAVACA, MEXICO JANUARY 6-10, 2020

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Abstract.
- Lecture 1: Birational maps, base points, resolution by blow-ups, examples, intersection form.
- Lecture 2: Generators for the Cremona group: Noether and Castelnuovo's theorems with a Sarkisov flavor.
- Lecture 3: The Picard Manin space and the infinite dimensional hyperboloid, dynamical degree and translation length.
- Lecture 4: Tits alternative, Normal subgroup theorem, and more...

Lecture 1. The Cremona group

1.1. **Affine vs homogeneous coordinates.** We work over the field $\mathbb{C}$ of complex number. The *Cremona group* is the group $\mathrm{Bir}(\mathbb{P}^2)$ of birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. A rational map can be given in homogeneous coordinates as

$$f\colon [x:y:z] \mapsto [P_0(x,y,z) : P_1(x,y,z) : P_2(x,y,z)]$$

with the P_i homogeneous of the same degree d, and without common factor. One can also describe a map $f \in \mathrm{Bir}(\mathbb{P}^2)$ by working in a affine chart, say $z = 1$:

$$f(x,y) = (f_0(x,y), f_1(x,y))$$

with f_0, f_1 rational fractions. By definition such a rational map is *birational* if it admits an inverse of the same form.

Here follows a few examples of natural subgroups in $\mathrm{Bir}(\mathbb{P}^2)$:

(1) The group $\mathrm{PGL}_3(\mathbb{C})$ corresponds to the biregular subgroup $\mathrm{Aut}(\mathbb{P}^2)$ in $\mathrm{Bir}(\mathbb{P}^2)$.

(2) The group $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is the component of the identity of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. Given a choice of birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ can be identified to the following subgroup of $\mathrm{Bir}(\mathbb{P}^2)$:

$$(x,y) \mapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{ay+b}{cy+d} \right).$$

(3) The group $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2$ can also be seen as a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$:

$$(x,y) \mapsto \left(\frac{\alpha(y) x + \beta(y)}{\gamma(y) x + \delta(y)}, \frac{ay+b}{cy+d} \right).$$

These transformations are called *Jonquières*, geometrically these are the transformations preserving the pencil of lines $y = $ constant.

(4) The group $\mathrm{GL}_2(\mathbb{Z})$ can be seen as the subgroup of monomial transformations:

$$(x,y) \mapsto (x^a y^b, x^c y^d).$$

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(5) The group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms, such as the Hénon map $(x, y) \mapsto (y, y^2 - x)$.

1.2. **Base locus, exceptional set, linear system.** A birational map between two surfaces $S_1 \dashrightarrow S_2$ is defined as an isomorphism between two Zariski open subsets $U_1 \subset S_1 \to U_2 \subset S_2$. The complement of the maximal such subset U_1 is called the *exceptional set*. The set where the map is not defined is called the *indeterminacy set*, or the *base locus* when equipped with the natural scheme structure. This is a finite number of points.

For simplicity assume now that $S_2 = \mathbb{P}^2$. The preimage of lines is a two-dimensional family of curves on S_1, called the *linear system* associated with the map $S_1 \dashrightarrow \mathbb{P}^2$.

Consider the example of the standard quadratic involution $f\colon (x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$, or in homogeneous coordinates

$$f\colon [x : y : z] \mapsto [yz : xz : xy].$$

The base locus is defined by the vanishing of yz, xz and xy, and so is equal to the set $\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. The exceptional set corresponds to the complement of $(\mathbb{C}^*)^2$ inside $\mathbb{P}^2$, and so is equal to the union of three lines. The linear system is the system of conics passing through the three base points (including the three singular ones).

Exercise 1.1. Viewing $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ as compactifications of the same $\mathbb{C}^2$, we get a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. Determine the base locus, exceptional locus and linear system of this map.

1.3. **Blow-ups.** The map from $\mathbb{C}^2$ to $\mathbb{C}^2$ given by $(x, y) \mapsto (x, xy)$ is injective outside the line $x = 0$, and maps the line $x = 0$ to the point $(0, 0)$. This simple quadratic map plays an important role in the theory of birational maps on surfaces, and is called a *blow-up*.

Formally :

Proposition 1.2. *Let S be a smooth surface, and $p \in S$ a point. Then there exists a (unique) birational morphism $\pi\colon S' \to S$ from a smooth surface S', such that $\pi^{-1}(p) = E$ is a smooth rational curve, and π is an isomorphism from $S' \smallsetminus E$ to $S' \smallsetminus \{p\}$.*

One way to construct blow-ups is by taking graphs of rational functions, the following exercise gives the main idea:

Exercise 1.3. Let $g\colon \mathbb{C}^2 \to \mathbb{P}^1$ be the rational function defined by $g(x, y) = [x : y]$. Let $S' \subset \mathbb{C}^2 \times \mathbb{P}^1$ be the (closure of the) graph of g, and let $\pi\colon S' \to \mathbb{C}^2$ be the natural projection on the first factor. Find some affine charts on S' such that the expression of π is the simple quadratic map given above.

The importance of blow-ups:

Theorem 1.4. *Let $f\colon S_1 \dashrightarrow S_2$ be a birational map between smooth projective surfaces. Then there exist a smooth surface S and compositions of blow-ups $\pi_1\colon S \to S_1$, $\pi_2\colon S \to S_2$ such that the following diagram commutes:*

$$\begin{tikzcd} & S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S_1 \arrow[rr, dashed, "f"] & & S_2 \end{tikzcd}$$

The proof of this result relies on the notion of intersection between divisors.

1.4. **Intersection form on surfaces.** We work on S a smooth projective surface. A *divisor* D on S is a finite collection of irreducible curves C_i, with multiplicity $a_i \in \mathbb{Z}$. We write

$$D = \sum a_i C_i.$$

Equivalently, D is defined by zeros and poles of local equations f_α, where (U_α) is a (Zariski) open covering of S, and f_α is a rational function on U_α (with obvious compatibility assumption on each $U_\alpha \cap U_\beta$). We say that D is *effective* if all $a_i \geqslant 0$, or equivalently if all f_α are regular.

Example 1.5.
- Given a rational function $f\colon S \dashrightarrow \mathbb{P}^1$, the divisor $\operatorname{div} f := (f)_0 - (f)_\infty$ given by zeros and poles of f is called a *principal divisor*.
- Given a rational 2-form ω written locally $\omega = f_\alpha dz_1 \wedge dz_2$, the divisor defined by the zeros and poles of the f_α is called a *canonical divisor* on S. (Fact: the difference of any two canonical diviors is a principal divisor).
- Given a hyperplane $H \subset \mathbb{P}^N \supset S$, the hyperplane section $H \cap S$ is called a *very ample divisor*.

Two divisors D_1, D_2 are *linearly equivalent* if there exists a rational function f such that $D_1 - D_2 = \operatorname{div} f$.

Let D_1, D_2 be two effective divisors on S with non common component, $x \in S$, and $f_1, f_2 \in \mathcal{O}_x$ some local equations of D_1, D_2. We define the *local intersection* of D_1 and D_2 at x by

$$(D_1 \cdot D_2)_x := \dim_{\mathbb{C}} \mathcal{O}_x/(f_1, f_2).$$

We define the *global intersection* of D_1 and D_2 by

$$D_1 \cdot D_2 := \sum_{x \in \operatorname{Supp} D_1 \cap \operatorname{Supp} D_2} (D_1 \cdot D_2)_x.$$

We can extend these definition to non effective divisors by linearity. This is an exercise to check that this definition corresponds to the intuitive notion of "intersection number with multiplicity". In particular:

Exercise 1.6. Prove that
- $(D_1 \cdot D_2)_x = 0 \iff x$ does not belong to the support of D_1 or D_2.
- $(D_1 \cdot D_2)_x = 1 \iff f_1, f_2$ are local parameters at x.

Lemma 1.7. *Let D_1, D_2, D_2' be three divisors on S. Assume $D_2 \sim D_2'$ and D_2, D_2' have non commom component with D_1. Then*

$$D_1 \cdot D_2 = D_1 \cdot D_2'.$$

Lemma 1.8. *Let D_1, D_2 be two divisors on S. Then there exists a divisor D_2' linearly equivalent to D_2 such that D_1 and D_2' have no common component in their supports.*

Idea of proof. One shows that given finitely points $x_1, \dots, x_n \in S$, there exists $D_2' \sim D_2$ such that $x_i \notin \operatorname{Supp} D_2'$ for each i. □

With these two lemmas we can define the intersection number of two arbitrary divisors on S, in particular we can define the self-intersection $D^2 = D \cdot D$ of any divisor.

Exercise 1.9 (Bezout theorem)**.** Prove as a corollary of the above discussion that if $C_1, C_2 \subset \mathbb{P}^2$ are two plane curves of respective degrees D_1, D_2, with no common component, then $C_1 \cdot C_2 = d_1.d_2$.

An important property of the intersection number is:

Proposition 1.10. *Let $f\colon S' \to S$ be a birational morphism, and D_1, D_2 two divisors on S. Then*

$$D_1 \cdot D_2 = f^*D_1 \cdot f^*D_2.$$

Proof. First if $D \sim D'$ on S, then $f^*D \sim f^*D'$ on S': $D - D' = \operatorname{div} g \implies f^*D - f^*D' = \operatorname{div} g \circ f$. Then we can move D_1 and D_2 using linear equivalence to avoid the finitely many points in the base locus of f^{-1}. □

Another important tool is the behaviour of intersection under blow-up:

Proposition 1.11. *Let $\pi\colon S' \to S$ be the blow-up of a point $x \in S$, with exceptional divisor E.*

(1) *For any divisor D on S, we have $E \cdot \pi^*D = 0$.*
(2) *If $C \subset S$ is a curve with multiplicity m at x, then $\pi^*C = C' + mE$, where $C' := \overline{\pi^{-1}(C \smallsetminus \{x\}}$ is called the strict transform of C.*
(3) *$E^2 = -1$.*
(4) *If $C \subset S$ is a curve with multiplicity m at x, $E \cdot \bar{C} = m$.*
(5) *If $C_1, C_2 \subset S$ are curves with multiplicity m_1, m_2 at x, and strict transforms C_1', C_2' on S', then*
$$C_1' \cdot C_2' = C_1 \cdot C_2 - m_1 \cdot m_2.$$
(6) *The canonical divisors on S and S' are related by the formula*
$$K_{S'} = \pi^*K_S + E.$$

Finally we have the Castelnuovo contraction criterion:

Lemma 1.12. *Let S' be a smooth projective surface, and $E \subset S'$ a smooth curve isomorphic to $\mathbb{P}^1$, with $E \cdot E = -1$. Then there exists a morphism $\pi\colon S' \to S$ to another smooth projective surface S such that π is the blow-up of a point $x \in S$, with exceptional divisor E.*

Lecture 2. Castelnuovo and Noether's Theorems (with Sarkisov flavor)

2.1. **Hirzebruch surfaces.** We call *Hirzebruch surface* any surface S with a morphism $S \to \mathbb{P}^1$ such that all fibers are isomorphic to $\mathbb{P}^1$. We denote $\mathbb{F}_n$, $n \geqslant 1$, a Hirzebruch surface admitting a section s with self-intersection $-n$. We say that s is the exceptional section of $\mathbb{F}_n$, and we denote by f the class of a fiber (up to linear equivalence).

We denote $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, where the notation $\mathbb{F}_0$ also implies a choice between the two rulings, which plays the role of the fiber f, whereas any rule of the other ruling plays the role of the section s. In any case we have

$$f \cdot f = 0, \ s \cdot s = -n \text{ and } f \cdot s = 1.$$

Exercise 2.1. We admit that any divisor on $\mathbb{F}_n$ is linearly equivalent to $af + bs$ for some $a, b \in \mathbb{Z}$.

(1) Show that any curve $C \subset \mathbb{F}_n$ is linearly equivalent to $af + bs$ with $a, b \in \mathbb{N}$.

$$
\begin{tikzcd}
 & & \mathbb{P}^1 & \\
 & \mathbb{F}_1 \arrow[ur] \arrow[dl] & & \\
\mathbb{P}^2 & & S \arrow[ul] \arrow[dl] \arrow[r] & \mathbb{F}_0 \arrow[uul] \arrow[ddl] \\
 & \mathbb{F}_1 \arrow[ul] \arrow[dr] & & \\
 & & \mathbb{P}^1 &
\end{tikzcd}
$$

FIGURE 1

(2) If $n \geqslant 1$, show that s is the only curve on $\mathbb{F}_n$ such that $s \cdot s < 0$.

Using blow-ups one can construct Hirzebruch surfaces of any index:

(1) The blow-up of a point on $\mathbb{P}^2$ produces $\mathbb{F}_1$.
(2) An elementary transformation from $\mathbb{F}_n$ (blow-up a point, contract a fiber) changes the index by 1: we get F_{n+1} if the blown-up is on the exceptional section of $\mathbb{F}_n$, and $\mathbb{F}_{n-1}$ otherwise.

It is a classification result that up to isomorphism there exists a unique surface $\mathbb{F}_n$ for each $n \geqslant 0$.

2.2. **Statements.** We want to prove:

Theorem 2.2 (see [KSC04, Theorem 2.24])**.** *Any birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be reduced to a linear automorphism by a sequence of the following 4 types of transformations:*

(I) *A blow-up $\mathbb{P}^2 \leftarrow \mathbb{F}_1$;*
(II) *An elementary transformation $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm 1}$;*
(III) *A contraction (inverse of blow-up) $\mathbb{F}_1 \to \mathbb{P}^2$;*
(IV) *The involution $\tau\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ exchanging the two rulings.*

This is a factorization theorem for selfmaps of $\mathbb{P}^2$ via transformations between other surfaces: this is a modern point of view that generalizes in any dimension as the *Sarkisov program*. We can deduce the following result of Castelnuovo (1901) :

Corollary 2.3 (Castelnuovo [Cas01])**.** *Any birational map of $\mathbb{P}^2$ is a composition of Jonquières maps and linear automorphisms.*

Proof. First, using the commutative diagram of Figure 1, an elementary transformation $\mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ followed by τ can be rewritten

$$\mathbb{F}_1 \dashrightarrow \mathbb{F}_0 \to \mathbb{F}_0 = \mathbb{F}_1 \to \mathbb{P}^2 \leftarrow \mathbb{F}_1 \dashrightarrow \mathbb{F}_0.$$

So in Theorem 2.2 we can use only the first three types of generators.

Now a map with a factorization of the form (no $\mathbb{P}^2$ in the middle)

$$\mathbb{P}^2 \leftarrow \mathbb{F}_1 \dashrightarrow \cdots \dashrightarrow \mathbb{F}_1 \to \mathbb{P}^2$$

is Jonquières (up to linear automorphisms on the left and right, to identify the fibrations on the $\mathbb{F}_n$ to the pencil $y =$ constant on $\mathbb{P}^2$). $\square$

In turn, the motivation for Castelnuovo to state this result was to fill a gap in a Theorem by M. Noether:

Theorem 2.4 (Noether)**.** *Any birational map of $\mathbb{P}^2$ is a composition of quadratic and linear maps* (*and one can even only use the quadratic map* $(x, y) \to (\frac{1}{x}, \frac{1}{y})$)*.*

I leave the reduction from Theorem 2.3 to 2.4 as an exercise (not so easy!) (for the solution on can look at [KSC04]). The difficulty is almost completely contained in the following special case:

Exercise 2.5 (**)**.** What is the minimal number of quadratic maps necessary to factorize the Jonquières map $(x, y) \mapsto (x + y^3, y)$?

2.3. **Degrees and multiplicities.** Let $\varphi\colon S \dashrightarrow \mathbb{P}^2$ be a birational map, where $S = \mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \geqslant 0$. The map φ corresponds to a linear system Γ_S on S (pull-back of the system Γ of lines on $\mathbb{P}^2$):

(1) If $S = \mathbb{P}^2$, $\Gamma_S \subset |d\ell|$, where ℓ is the class of a line and $d \geqslant 1$ is the ordinary degree of φ ($d = 1$ iff φ is an automorphism).
(2) If $S = \mathbb{F}_n$, $\Gamma_S \subset |af + ds|$, where $d \geqslant 1$ and $a \geqslant nd$ (take the intersection with s, which must be $\geqslant 0$).

With these notations, we define the ("Sarkisov") *degree* of φ:

(1) If $S = \mathbb{P}^2$, $\deg \varphi := d/3$;
(2) If $S = \mathbb{F}_n$, $\deg \varphi := d/2$.

Unified definition: "usual degree normalised by the canonical divisor":

(1) If $S = \mathbb{P}^2$, $\deg \varphi = \frac{\Gamma_S \cdot \ell}{-K_S \cdot \ell}$;
(2) If $S = \mathbb{F}_n$, $\deg \varphi = \frac{\Gamma_S \cdot f}{-K_S \cdot f}$.

Observe that the set of possible degrees is contained in $\frac{1}{6}\mathbb{N}$ and so is discrete, and the minimal one $\frac{1}{3}$ corresponds to the case of an automorphism on $\mathbb{P}^2$: we are in good shape to try a proof by induction.

For the proof of Theorem 2.2, we consider a resolution of φ by a sequence of blow-ups of points p_i, each p_i being a point of multiplicity $m_i \geqslant 1$ for the linear system Γ_S.

$$
\begin{tikzcd}
 & M \arrow[dl, "\pi"'] \arrow[dr] & \\
S \arrow[rr, dashed] & & \mathbb{P}^2
\end{tikzcd}
$$

Then we have integers $a_i, b_i \geqslant 1$ defined by the relations:

$$K_M = \pi^* K_S + \sum_i a_i E_i$$

$$\Gamma_M = \pi^* \Gamma_S - \sum_i b_i E_i$$

$$K_M + \frac{1}{\deg \varphi}\Gamma_M = \pi^* \left(K_S + \frac{1}{\deg \varphi}\Gamma_S \right) + \sum_i \lambda_i E_i$$

where $\lambda_i = a_i - \dfrac{b_i}{\deg \varphi}$. Observe that

$$a_i - \frac{b_i}{\deg \varphi} < 0 \iff \deg \varphi < \frac{b_i}{a_i}.$$

Lemma 2.6.

(1) *If p_i is a proper base point, then $a_i = 1$ and $b_i = m_i$, hence $\frac{b_i}{a_i}$ is the usual multiplicity of p_i.*
(2) *If p_j is infinitely near from p_i, then $\frac{b_j}{a_j} < \frac{b_i}{a_i}$.*

Proof. (1) follows from the definition, and (2) follows from Exercise 2.7. □

Exercise 2.7. Let $S = \mathbb{P}^2$ or $\mathbb{F}_n$, and $\varphi\colon S \dashrightarrow \mathbb{P}^2$ a birational map. Assume that p_1, p_2 are two base points of φ, with p_2 infinitely near to p_1 (meaning p_2 lies on the exceptional divisor E_1 produced by blowing-up p_1). Let m_1, m_2 be the respective multiplicities of p_1 and p_2. Show that $m_1 \geqslant m_2$. (Hint: $m_1 = H \cdot E_1$, where H is any member of the linear system associated with φ, and $(H \cdot E_1)_{p_2} \geqslant m_2$).

2.4. **Not nef case.** A divisor D on a surface is *nef* if $D \cdot C \geqslant 0$ for any curve C. One can extend this definition to $\mathbb{Q}$-divisors, i.e. formal sums of irreducible curves with coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$.

Lemma 2.8. *The divisor $K_S + \frac{1}{\deg\varphi}\Gamma_S$ is not nef iff we are in one of the following situations:*

(1) *$S = \mathbb{F}_0$, and $\Gamma_S \subset |af + ds|$ with $a < d$;*
(2) *$S = \mathbb{F}_1$, and $\Gamma_S \subset |af + ds|$ with $\frac{a}{3} < \frac{d}{2}$.*

Proof. First observe that if $S = \mathbb{P}^2$ then $K_S + \frac{1}{\deg\varphi}\Gamma_S$ is trivial hence nef: so we can assume $S = \mathbb{F}_n$. As before let f, s be a fiber and an exceptional section of $\mathbb{F}_n$. By adjunction

$$K_S \cdot s = -2 - s^2 = n - 2.$$

In particular if $n \geqslant 2$ we have

$$\left(K_S + \frac{1}{\deg\varphi}\Gamma_S\right) \cdot f = 0;$$
$$\left(K_S + \frac{1}{\deg\varphi}\Gamma_S\right) \cdot s \geqslant 0;$$

so that $K_S + \frac{1}{\deg\varphi}\Gamma_S$ is nef (because any curve on $\mathbb{F}_n$ is equivalent to $af + bs$ with $a, b \geqslant 0$). So the condition $K_S + \frac{1}{\deg\varphi}\Gamma_S$ not nef is equivalent to $n = 0$ or 1, with

$$\left(K_S + \frac{1}{\deg\varphi}\Gamma_S\right) \cdot s < 0,$$

which we rewrite

$$\Gamma_S \cdot s < -\frac{d}{2}K_S \cdot s.$$

If $n = 0$, we have $\Gamma_S \cdot s = (af + ds) \cdot s = a$ and $K_S \cdot s = -2$, we get $a < d$. If $n = 1$, we have $\Gamma_S \cdot s = (af + ds) \cdot s = a - d$ and $K_S \cdot s = -1$, we get $\frac{a}{3} < \frac{d}{2}$. □

Remark 2.9. In both cases of Lemma 2.8, it is easy to decrease the degree of φ using one of the operation of Theorem 2.2, namely:

- If $S = \mathbb{F}_0$ and $a < d$, we apply (IV): exchange of the rulings. The new degree is $\frac{a}{2} < \deg\varphi = \frac{d}{2}$.
- If $S = \mathbb{F}_1$ and $\frac{a}{3} < \frac{d}{2}$, we apply (III): contraction of the exceptional section. The new degree is $\frac{a}{3} < \deg\varphi = \frac{d}{2}$.

2.5. **Nef case.**

Lemma 2.10. *Let $S = \mathbb{P}^2$ or $\mathbb{F}_n$, and $\varphi\colon S \dashrightarrow \mathbb{P}^2$ a birational map, with linear system Γ_S. Assume φ is not an isomorphism, and $K_S + \frac{1}{\deg\varphi}\Gamma_S$ is nef. Then Γ_S admits at least one proper base point of multiplicity $> \deg\varphi$.*

Proof. Take a resolution

$$\begin{tikzcd} & M \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ S \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd}$$

Let ℓ be a general line on the target $\mathbb{P}^2$, we also denote ℓ its pull-back on M. Since φ is not an isomorphism, we have

$$\deg\varphi > \frac{1}{3} = \frac{\ell\cdot\ell}{-K_{\mathbb{P}^2}\cdot\ell},$$

which we rewrite as

$$0 > \left(K_{\mathbb{P}^2} + \frac{1}{\deg\varphi}\ell\right)\cdot\ell.$$

On the other hand, denoting E_i, E'_j the respective exceptional divisors of π, σ:

$$\begin{aligned} K_M + \frac{1}{\deg\varphi}\Gamma_M &= \pi^*\left(K_S + \frac{1}{\deg\varphi}\Gamma_S\right) + \sum_i\left(a_i - \frac{b_i}{\deg\varphi}\right)E_i \\ &= \sigma^*\left(K_{\mathbb{P}^2} + \frac{1}{\deg\varphi}H\right) + \sum_j \star E'_j. \end{aligned}$$

Since ℓ is general, $E_i\cdot\ell \geqslant 0$ for all i, and $E'_j\cdot\ell = 0$ for all j. Intersecting with ℓ on M we get

$$0 > \mathrm{nef}\cdot\ell + \sum_i\left(a_i - \frac{b_i}{\deg\varphi}\right)E_i\cdot\ell.$$

So there is at least one index i such that $\left(a_i - \frac{b_i}{\deg\varphi}\right) < 0$, or equivalently $\frac{b_i}{a_i} > \deg\varphi$. By Lemma 2.6 we can assume that this negative coefficient corresponds to a proper base point. This gives $a_i = 1$, $b_i = m_i$ and finally $m_i > \deg\varphi$. □

Lemma 2.11. *Let $\varphi\colon S \dashrightarrow \mathbb{P}^2$ be a birational map with linear system Γ_S. Assume that Γ_S admits a base point p with multiplicity $> \deg\varphi$.*

1. *If $S = \mathbb{P}^2$, let $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$ be the blow-up of p. Then $\deg(\varphi\circ\pi) < \deg\varphi$;*
2. *If $S = \mathbb{F}_n$, let $\alpha\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm1}$ be the elementary transformation associated with p. Then $\deg(\varphi\circ\alpha^{-1}) = \deg\varphi$, and the sum of multiplicities of the resolution of $\varphi\circ\alpha^{-1}$ is strictly less than the one of φ.*

Proof. If $S = \mathbb{P}^2$, the system Γ'_S on $\mathbb{F}_1$ corresponds to $\pi^*d\ell - mE$. The class of a fiber on $\mathbb{F}_1$ is given by $f = \pi^*\ell - E$, and so

$$\deg\Gamma'_S = \frac{\Gamma'_S\cdot f}{2} = \frac{d-m}{2} < \frac{3d-d}{6} = \frac{d}{3} = \deg\Gamma_S.$$

If $S = \mathbb{F}_n$ the elementary transformation α is an isomorphism in a neighborhood of a general fiber f, so

$$\deg\Gamma'_S = \frac{\Gamma'_S\cdot f}{2} = \frac{\Gamma_S\cdot f}{2} = \deg\Gamma_S.$$

The sum of multiplicities goes down because we replaced a point of multiplicity m by a point of multiplicity $d - m$, and $m > d/2$. □

2.6. **Proof of Theorem 2.2.** Let $\varphi\colon S \dashrightarrow \mathbb{P}^2$ be a birational map, with S equal to $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$. Assume φ is not an isomorphism. We explain how to make the degree smaller by applying one of the 4 operations given in the statement, so that we can conclude by induction.

If all base points have multiplicity $\leqslant \deg\varphi$, we are in one the special situations of Lemma 2.10, and we saw in Remark 2.9 that an operation of type III or IV decreases the degree:

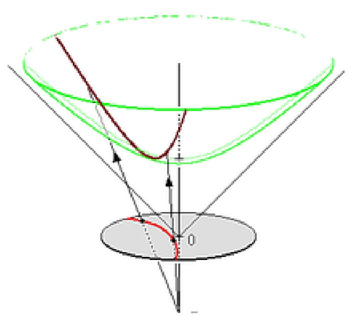
- If $S = \mathbb{F}_0$, then (IV) changing the projection we exchange the role of a and d, so the degree goes down since $a < d$.
- If $S = \mathbb{F}_1$ then (III) we contract the section s. The degree of the new map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is $a/3$, and $a/3 < d/2$.

Assume now that there is a base point p of multiplicity $m > \deg\varphi$. We apply Lemma 2.11.

- If $S = \mathbb{P}^2$, then (I) we consider $\mathbb{F}_1 \to \mathbb{P}^2$ the blow-up of p. The degree of the new map $\mathbb{F}_1 \dashrightarrow \mathbb{P}^2$ is $\frac{d-m}{2} < \frac{d}{3} = \deg\varphi$.
- If $S = \mathbb{F}_n$, then (II) we apply the elementary transformation associated with p. The degree stays the same, but the sum of multiplicities goes down. After finitely many such steps, we must be in one the three previous cases.

Lecture 3. Action on the Picard-Manin space

3.1. **Construction of the infinite dimensional hyperboloid.** Different models for the hyperbolic 2-space: disk model, half-plane model, Klein model, hyperboloid model. The latter one generalizes to give a model of the hyperbolic n-space.



Let S be a smooth projective surface with Picard number $\rho = n+1$, $N^1(S) \simeq \mathbb{R}^{n+1}$ the Néron-Severi space (divisors with real coefficients, up to numerical equivalence). The intersection form gives a symmetric bilinear form on $N^1(S)$ with signature $(1, n)$. In general this follows from the Hodge Index Theorem, which is a consequence of Riemann-Roch Formula, see [Har77, p.364]. Observe however that when S is the blow-up of $\mathbb{P}^2$ along n points, this is an easy direct observation. By considering divisors $D \in N^1(S)$ satisfying $D^2 = 1$, we obtain a 2-sheeted hyperboloid, and by adding the condition $D \cdot H > 0$ for any choice of ample divisor H we select one of the sheets (the one containing nef divisors) and get a model $\mathbb{H}(S)$

of the hyperbolic space $\mathbb{H}^n$. Recall that the distance on $\mathbb{H}^n$ is given by

$$d(D_1, D_2) = \operatorname{arcosh}(D_1 \cdot D_2),$$

and that geodesics are obtained by intersecting $\mathbb{H}(S)$ with a hyperplane in $N^1(S)$.

Now consider $\pi\colon S' \to S$ a birational morphism (for instance the blow-up of a point). The pull-back map defines a map from $N^1(S)$ to $N^1(S')$ that preserves intersection forms:

$$\pi^* D_1 \cdot \pi^* D_2 = D_1 \cdot D_2.$$

In particular we get an isometric embedding of $\mathbb{H}(S)$ into $\mathbb{H}(S')$.

Now if $S_1 \to S$ and $S_2 \to S$ are two birational morphisms, there exist a third surface S_3 and morphisms $S_3 \to S_1$ and $S_3 \to S_2$ such that the following diagrams commute:

$$\begin{tikzcd}
 & S_3 \arrow[dl] \arrow[dr] & \\
S_1 \arrow[dr] & & S_2 \arrow[dl] \\
 & S &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{H}(S_3) \subset N^1(S_3) & \\
\mathbb{H}(S_1) \subset N^1(S_1) \arrow[ur] & & \mathbb{H}(S_2) \subset N^1(S_2) \arrow[ul] \\
 & \mathbb{H}(S) \subset N^1(S) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We are ready to consider simultaneously all possible surfaces S' dominating S, and to take an injective limit

$$\mathbb{H}^\infty_C(S) = \lim_{\leftarrow} \mathbb{H}(S') \subset \lim_{\leftarrow} N^1(S') = \mathcal{Z}_C(S).$$

The C in index is for "Cartier", and $\mathcal{Z}_C(S)$ is sometimes called the space of Cartier b-divisors. One could similarly consider a projective limit, and obtain $\mathcal{Z}_W(S)$ which is called the space of Weil b-divisors. The space that will really be useful to us is an intermediate space $\mathbb{H}^\infty(S)$, which is the L^2 completion of $\mathbb{H}^\infty_C(S)$. In concrete terms we have

$$\begin{aligned}
\mathcal{Z}_C(S) &= \{D = D_0 + \sum_p a_p E_p;\ D_0 \in N^1(S), a_p = 0 \text{ except finitely many}\};\\
\mathcal{Z}_W(S) &= \{D = D_0 + \sum_p a_p E_p;\ D_0 \in N^1(S)\};\\
\mathcal{Z}(S) &= \{D = D_0 + \sum_p a_p E_p;\ D_0 \in N^1(S), \sum a_p^2 < \infty\};
\end{aligned}$$

$\mathbb{H}^\infty(S) \subset \mathcal{Z}(S) = \{D \in \mathcal{Z}(S);\ D^2 = 1\};$

3.2. **Action of the Cremona group.** Observe that if $\pi\colon S_2 \to S_1$ is a birational morphism, then we get a canonical isometry π^* (and not simply an embedding!) between $\mathbb{H}^\infty(S_1)$ and $\mathbb{H}^\infty(S_2)$. This allows to define an action by iometry of $\mathrm{Bir}(\mathbb{P}^2)$ on $\mathcal{Z}(\mathbb{P}^2)$, and so also on $\mathbb{H}^\infty = \mathbb{H}^\infty(\mathbb{P}^2)$. If $f \in \mathrm{Bir}(\mathbb{P}^2)$ and

$$\begin{tikzcd}
 & S \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2
\end{tikzcd}$$

is a resolution of f, then we set $f_* := (\sigma^*)^{-1} \circ \pi^*$.

Let ℓ denote the class of a line in $\mathcal{Z}(\mathbb{P}^2)$, and let e_p denotes the class corresponding to the exceptional divisor of the blow-up of a point $p \in \mathbb{P}^2$.

$$\mathcal{Z}(\mathbb{P}^2) = \{D = a_0\ell + \sum_p a_p e_p;\ a_0 \in N^1(S), \sum a_p^2 < \infty\}.$$

To be more concrete we now describe the action of a particular element.

Let f be a birational quadratic map, with three proper base points p_1, p_2, p_3, and q_1, q_2, q_3 the proper base points of f^{-1}. (Numerotation such that if L_{ij} is the line through p_i and p_j, then $f(L_{ij}) = q_k$ where $\{i, j, k\} = \{1, 2, 3\}$.)

Let L be a line not passing through any of the p_i. Then $C = f(L)$ is a smooth conic passing through each q_i. We have

$$\sigma^* C = \bar{C} + E_{q_1} + E_{q_2} + E_{q_3} = \pi^* L + E_{q_1} + E_{q_2} + E_{q_3},$$

so

$$f_*(\ell) = 2\ell - e_{q_1} - e_{q_2} - e_{q_3}.$$

On the other hand

$$f_*(\ell - e_{p_i} - e_{p_j}) = e_{q_k};$$
$$f_*(e_{p_k}) = \ell - e_{q_i} - e_{q_j}.$$

If p is not on any of the L_{ij}, then

$$f_*(e_p) = e_{f(p)}.$$

Exercise 3.1. Check that

$$f_*\ell \cdot f_*\ell = 1, \quad f_*e_p \cdot f_*e_p = -1, \quad f_*\ell \cdot f_*e_p = 0.$$

Exercise 3.2 (**)**.** Can you find an example of $g \in \mathrm{Bir}(\mathbb{P}^2)$, such that the action of g on $\mathbb{H}^\infty$ is elliptic, and the set $\mathrm{Fix}(g) \subseteq \mathbb{H}^\infty$ is bounded?

3.3. **Dynamical degree and translation length.** Let $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a dominant rational map. One defines the *dynamical degree* of f as the limit

$$\lambda(f) = \lim_{n\to\infty} (\deg f^n)^{1/n}.$$

More generally, if $f\colon X \dashrightarrow X$ is a dominant rational map on any projective surface X, $\|\cdot\|$ is any norm on the Néron-Severi real vector space $N^1(X)$, and f^* is the induced action by f on $N^1(X)$, we can define

$$\lambda(f) = \lim_{n\to\infty} \|(f^n)^*\|^{1/n}.$$

This quantity does not depend on the choice of the norm, and is a birational invariant, that is, if $\varphi\colon X \dashrightarrow Y$ is a birational map, and $g = \varphi f \varphi^{-1}$, then $\lambda(f) = \lambda(g)$:

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\varphi"'] & X \arrow[d, dashed, "\varphi"] \\ Y \arrow[r, dashed, "g"'] & Y \end{tikzcd}$$

Now let $f \in \mathrm{Bir}(\mathbb{P}^2)$ be a map with $\lambda(f) > 1$, and let $D \in \mathbb{H}^\infty$ be an arbitrary point. The sequence $(f^n)_*(D)/\lambda^n$ converges to a point $\omega \in \partial\mathbb{H}^\infty$, which corresponds to a class in the isotropic cone of the intersection form: $\omega \cdot \omega = 0$. Similarly

$(f^{-n})_*(D)/\lambda^n$ converges to $\alpha \in \partial\mathbb{H}^\infty$. We can normalize such that $\alpha \cdot \omega = \frac{1}{2}$, and then the set

$$\{u\alpha + v\omega;\ uv = 1\} \in \mathbb{H}^\infty$$

is a geodesic line Γ invariant by f. Moreover, since $f_*\alpha = \frac{1}{\lambda}\alpha$ and $f_*\omega = \lambda\omega$, we can compute the distance L between $P = \alpha + \omega \in \Gamma$ and $fP = \frac{1}{\lambda}\alpha + \lambda\omega$:

$$e^L + \frac{1}{e^L} = 2\cosh L = 2\cosh d(P, fP) = 2P \cdot fP = \lambda + \frac{1}{\lambda},$$

so that $L = \log\lambda$: the logarithm of the dynamical degree (sometimes called *algebraic entropy*) is equal to the translation length for the action on $\mathbb{H}^\infty$.

Exercise 3.3. Let $f = f_A\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a monomial map, associated with the matrix $A \in \mathcal{M}_2(\mathbb{Z})$. Show that $\lambda(f)$ is equal to the spectral radius ρ of A. (hint: view f_A as an element of $\mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$, and consider the action on $N^1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{R}^2$).

Classification of elements in $\mathrm{Bir}(\mathbb{P}^2)$ (see [Can11, §2.3]).

Definition 3.4. One says that $g \in \mathrm{Bir}(\mathbb{P}^2)$ is

- *Virtually isotopic to the identity* if there is a positive iterate g^n and a birational map $\varphi\colon S \dashrightarrow \mathbb{P}^2$ such that $\varphi g^n \varphi^{-1}$ is in the connected component of the identity of $\mathrm{Aut}(S)$.
- *A Jonquières twist* if g preserves a pencil of rational curves and is not virtually isotopic to the identity.
- *A Halphen twist* if g preserves a pencil of elliptic curves and is not virtually isotopic to the identity.

Definition 3.5. One says that $g \in \mathrm{Bir}(\mathbb{P}^2)$ is

- *elliptic* if g admits (at least) one fixed point in $\mathbb{H}^\infty$.
- *parabolic* if g is not elliptic and admits exactly one fixed point on $\partial\mathbb{H}^\infty$.
- *loxodromic* if g is not elliptic and admits exactly two fixed point on $\partial\mathbb{H}^\infty$. In this case the geodesic between these two fixed points is called the axis of g, and g acts by translation of length $L(g) > 0$ on its axis.

Theorem 3.6. *Any $g \in \mathrm{Bir}(\mathbb{P}^2)$ belongs to one of the following mutually exclusive four classes:*

(1) *g is virtually isotopic to the identity, or equivalently g is elliptic, or equivalently the sequence $\deg g^n$ is bounded.*
(2) *g is a Jonquières twist, which is the first kind of parabolic elements, equivalently the sequence $\deg g^n$ grows linearly.*
(3) *g is a Halphen twist, which is the second kind of parabolic elements, equivalently the sequence $\deg g^n$ grows quadratically.*
(4) *g is loxodromic, or equivalently $\lambda(g) > 1$, and we have $L(g) = \log\lambda(g)$.*

Lecture 4. Some results

4.1. Tits alternative.

Theorem 4.1 ([Can11], [Ure18])**.** *Let $G \subseteq \mathrm{Bir}(\mathbb{P}^2)$ be a subgroup of the Cremona group. Then*

(1) *either G contains a solvable subgroup of finite index;*
(2) *or G contains a free group $\mathbb{Z} * \mathbb{Z}$.*

A first step is to establish a geometric Tits alternative:

Proposition 4.2. *Let $G \subseteq \mathrm{Bir}(\mathbb{P}^2)$ be a subgroup, and consider the action of G on $\mathbb{H}^\infty$. Then*

(1) *Either G fixes at least one point in $\mathbb{H}^\infty$,*
(2) *or G fixes exactly one point in $\partial\mathbb{H}^\infty$,*
(3) *or G fixes, or exchanges, exactly two points in $\partial\mathbb{H}^\infty$,*
(4) *or G contains a free group over two generators.*

Idea of proof for (1), (2), (3). The difficult case is when any $h \in G$ is elliptic. If G admits a bounded orbit, the circumcenter of the orbit (well defined in any $CAT(0)$ space) is fixed, so we are in Case (1). If G admits an orbit with limit set a single point in $\partial\mathbb{H}$, this point is fixed and we are in Case (2). Now assume that G admits an orbit with limit set containing two distinct points $a, b \in \partial\mathbb{H}$. So there exists $x \in \mathbb{H}$, $f_n, g_n \in G$ such that $f_n x \to a$ and $g_n x \to b$. Then one can prove that for n large $f_n g_n$ is hyperbolic, hence a contradiction: This is easy in a tree, and one can treat the general case by an approximation argument reducing to the case of a tree (see [GdlH90, Lemme 8.35]). □

Now we explain how to produce a free group $\mathbb{Z} * \mathbb{Z}$ in case (4) of the theorem. Recall the following criterion (see [dlH00, §II.B]) for finding free products.

Lemma 4.3 (Ping-Pong). *Let G be a group acting on a set X, let Γ_1, Γ_2 be two subgroups of G and let $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ be the subgroup of G generated by Γ_1 and Γ_2. Assume there exists X_1, X_2 two disjoint non empty subsets of X such that*

$$\gamma(X_2) \subseteq X_1 \qquad \textit{for all} \qquad \gamma \in \Gamma_1, \gamma \neq 1$$
$$\gamma(X_1) \subseteq X_2 \qquad \textit{for all} \qquad \gamma \in \Gamma_2, \gamma \neq 1$$

Assume moreover that one of the Γ_i has order at least 3. Then Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$.

Exercise 4.4. Apply the Ping-Pong Lemma to the action by homography of $\mathrm{PSL}_2(\mathbb{Z})$ on the real projective line $\mathbb{R} \cup \{\infty\}$ to show that $\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. (hint: use the homographies $x \mapsto -\frac{1}{x}$ and $x \mapsto 1 - \frac{1}{x}$).

Exercise 4.5. What is the trouble if we allow both groups Γ_1 and Γ_2 to be of order 2 in the Ping-Pong Lemma 4.3?

Lemma 4.6. *Let $g_1, g_2 \in \mathrm{Bir}(\mathbb{P}^2)$ be two loxodromic elements with axis Γ_1, Γ_2. Assume that the endpoints of the Γ_i are distincts. Then there exists an integer $n \geqslant 1$ such that g_1^n, g_2^n generate a free subgroup $\mathbb{Z} \times \mathbb{Z} \subset \mathrm{Bir}(\mathbb{P}^2)$.*

Idea of proof. Let $O_1 \in \Gamma_1$ and $O_2 \in \Gamma_2$ be the points realizing the distance between Γ_1 and Γ_2. Let $B > 0$, and for $i = 1, 2$ define $X_i \subset \mathbb{H}^\infty$ as the set of points whose projection onto Γ_i is at distance $\geqslant B$ from O_i. Then the claim is that for B sufficiently large $X_1 \cap X_2 = \emptyset$, and by choosing n sufficiently large we can apply the Ping-Pong Lemma to $\Gamma_1 = \langle g_1^n \rangle$ and $\Gamma_2 = \langle g_2^n \rangle$.

Alternative way of setting-up a ping-pong: Γ_1 and Γ_2 are contained in a unique copy of $\mathbb{H}^3$ (or $\mathbb{H}^2$ is they are secant), one can consider the action on the boundary sphere of this $\mathbb{H}^3$ (or boundary circle of this $\mathbb{H}^2$). □

4.2. **Normal subgroups.** Let G a group acting by isometry on a metric space X. Let $A \subseteq X$ be a subset, and let $\eta \geqslant 0$. We define the η-stabilizer of A in G as

$$\mathrm{Fix}_\eta A = \{g \in G;\ d(a, ga) \leqslant \eta \text{ for all } a \in A\}.$$

Observe that $\mathrm{Fix}_\eta A$ is stable under taking the inverse, but in general not under composition, so a priori this is not a subgroup of G!

Now let $g \in G$. We say that g satisfies the WPD property if

$$\exists x \in X, \forall \eta \geqslant 0, \exists M \in \mathbb{N} \text{ such that } \mathrm{Fix}_\eta\{x, g^M x\} \text{ is finite.}$$

Proposition 4.7 ([Lon16])**.** *For any $n \geqslant 2$, the polynomial automorphism $(x, y) \mapsto (y, y^n - x)$ is an example of WPD element in $\mathrm{Bir}(\mathbb{P}^2)$ with respect to the action on $\mathbb{H}^\infty$.*

Theorem 4.8 ([CL13], [Lon16], [DGO17])**.** *Let $g \in \mathrm{Bir}(\mathbb{P}^2)$ be a WPD element. Then for any $C > 0$, there exists an integer $n \geqslant 1$ such that any element $h \neq \mathrm{id}$ in the normal subgroup $\langle\langle g^n \rangle\rangle$ is loxodromic with translation length $L(h) > C$.*

4.3. **Gap property.** If $f \in \mathrm{Bir}(\mathbb{P}^2)$ is a map with $\lambda(f) > 1$, then the dynamical degree $\lambda(f)$ is an algebraic integer with all Galois conjugates in the unit disk. More precisely, we distinguish between the two following situations:

- All Galois conjugates are in the open unit disk: $\lambda(f) \in \mathbf{Pis}$ is a *Pisot number*,
- At least one Galois conjugate has modulus 1: $\lambda(f) \in \mathbf{Sal}$ is a *Salem number.*

It is known that $\mathbf{Pis} = \overline{\mathbf{Pis}} \subset \overline{\mathbf{Sal}}$. The smallest Pisot number is the plastic number $\lambda_P \simeq 1.324718$, which is a root of $X^3 - X - 1$. The smallest *known* Salem number is the Lehmer number $\lambda_L \simeq 1.176280$, which is a root of $X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$.

Theorem 4.9 (Gap property, [BC16, Corollary 2.7])**.** *Let $f \in \mathrm{Bir}(\mathbb{P}^2)$. Then*

$$\lambda(f) \notin]1, \lambda_L[.$$

Two main ingredients of the proof are as follows.

If $\lambda(f) > 1$ is a Salem number for some $f \in \mathrm{Bir}(\mathbb{P}^2)$, then f is birationally conjugate to an automorphism of a rational surface S ([BC16, Theorem A]).

If $f\colon S \to S$ is an automorphism of a (rational) surface with $\lambda(f) > 1$, then $\lambda(f) \geqslant \lambda_L$ ([McM07]).

4.4. **Centralizer of hyperbolic elements.** We can use the gap property and the action on the Picard-Manin space to get a description of the centralizer of a hyperbolic element.

Proposition 4.10 ([BC16, Corollary 4.7])**.** *Let $h \in \mathrm{Bir}(\mathbb{P}^2)$ be an element of hyperbolic type (that is, $\lambda(h) > 1$). Then the cyclic group $\langle h \rangle$ has finite index in the centralizer group of h in $\mathrm{Bir}(\mathbb{P}^2)$. In particular $\mathrm{Cent}(h) \simeq \mathbb{Z} \rtimes F$ with F finite, and if $g, h \in \mathrm{Bir}(\mathbb{P}^2)$ are two hyperbolic maps that commute, there exist $n, m > 0$ such that $g^m = h^n$.*

Proof. We have a morphism $\mathrm{Cent}(h) \to \mathbb{R}^*$ that sends f to $\lambda(f)$.

By the gap property, the image of this morphisms is a discrete subgroup of $\mathbb{R}^*$.

We are reduced to prove that the kernel F is finite. Any element in the kernel fixes the axis of h point-wise. In particular, if d is the distance from the class ℓ of a line to $\mathrm{Ax}(h)$, then $d(\ell, f\ell) \leqslant 2d$, and this implies that $\deg(f)$ is uniformly bounded.

By a result of Blanc and Furter [BF02], the Zariski closure of F is an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ (so in fact F is equal to its Zariski closure).

If F is not finite, it contains a 1-dimensional algebraic subgroup A whose orbits are preserved by h: this contradicts $\lambda(h) > 1$. Indeed if L is an orbit of A we have

$h^*L = L$, but when $\lambda(h) > 1$ there are exactly two eigenclasses in the Picard-Manin space, which are multiplied by $\lambda(h)^{\pm 1}$ under the action of h^*. □

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