GROUPS OF POLYNOMIAL AUTOMORPHISMS OF THE PLANE

SERGE CANTAT AND STÉPHANE LAMY

This is a translation of our paper "Groupes d'automorphismes polynomiaux du plan", Geom. Dedicata 123 (2006), 201–221. (translation by S. Lamy)

ABSTRACT. We study the embeddings of lattices from simple LIE groups into the group of polynomial automorphisms of the affine plane and answer a question of Dekimpe concerning crystallographicc polynomial groups of the plane.

1. INTRODUCTION

Which are the finite type groups that act by polynomial transformations on the plane? Here we try to give a partial answer to questions of this kind. Of course we only consider some specific groups of finite type, namely lattices in simple connected LIE groups. This is motivated by some conjectures of ZIMMER (*see* [18] and [9]) and by a question by DEKIMPE about polynomial crystallographic groups (*see* [13]).

1.1. **Polynomial automorphisms.** If k is a field, the group $\mathrm{Aut}(k^2)$ of polynomial automorphisms of the affine plane k^2 contains two important subgroups: the affine group A and the group of automorphisms that preserve the foliation of k^2 by affine horizontal lines. These latter automorphisms are called *elementary* and the group they composed is the *elementary group* E. By the theorem of JUNG - VAN DER KULK, $\mathrm{Aut}(k^2)$ is the amalgamated product of A and E along their intersection S.

The BASS-SERRE theory allows us to associate a tree to this amalgamated product structure and to embed the group $\mathrm{Aut}(k^2)$ into the group of simplicial automorphisms of this tree. The stabilizers of vertices are conjugate in $\mathrm{Aut}(k^2)$ to the group A or to the group E. Thus, when a group G is embedded in $\mathrm{Aut}(k^2)$, either it acts on the tree without fixed vertex, or it can be embedded in A or E.

From this we can study embeddings of KAZHDAN group, and in particular lattices in real LIE groups of real rank at least 2. Using a recent result of SHALOM, we obtain the following theorem.

Theorem A. *Let k be a field. Let G be a real simple* LIE *group and Γ a lattice in G. If there exists an injective morphism $\rho : \Gamma \to Aut(k^2)$, the group G is isomorphic to* $\mathsf{PSO}(1,n)$ *or to* $\mathsf{PSU}(1,n)$ *for some integer n; furthermore if G is distinct from* $\mathsf{PSO}(1,2)$ *then the image of ρ is contained in a conjugate of the* *affine group.*

This result was recently used by DÉSERTI to prove a rigidity result for the group $\mathsf{SL}(3, \mathbf{Z})$ relatively to the group of birational transformations of the complex projective plane (*see* [15]).

Since the group $\mathsf{PSO}(1, 2)$ is isomorphic to $\mathsf{PSL}(2, \mathbf{R})$ it is easy to embed lattices of this group into the group $\mathrm{Aut}(\mathbf{R}^2)$. On the other hand it is more delicate to find embeddings whose image is not conjugate to a subgroup of the affine group. The existence of such embeddings will be the main theme of this text.

1.2. **Fundamental groups of surfaces.** Let Γ be a lattice of $\mathsf{PSL}(2, \mathbf{R})$. We want to know if there exists embeddings of Γ into $\mathrm{Aut}(\mathbf{C}^2)$ or $\mathrm{Aut}(\mathbf{R}^2)$ which are not conjugate to an embedding into the affine group.

If we replace Γ by one of its finite index subgroups, the lemma of SELBERG allows us to assume that Γ is a lattice without torsion. The quotient of the POINCARE disc by Γ is then an orientable closed surface minus a finite number of points. When this surface is compact, Γ is isomorphic to the fundamental group

$$\Gamma_g = \langle a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{i=g} [a_i, b_i] = 1 \rangle$$

where g is the genus of the surface and $[a_i, b_i]$ is the commutator $a_i b_i a_i^{-1} b_i^{-1}$ of a_i and b_i. When the surface is obtained by removing p points, with $p \geq 1$, Γ is isomorphic to the free group F_k over k generators, with k equal to $2g + p - 1$.

Before describing the embeddings of Γ_g or F_k in the group $\mathrm{Aut}(\mathbf{R}^2)$, let us make precise some points of vocabulary (*see* §2.1). A generalized HÉNON transformation is an automorphism that reads

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ P(y) - ax \end{pmatrix}$$

where P is a polynomial in one variable which degree is at least 2. It turns out that any element h of $\mathrm{Aut}(\mathbf{C}^2)$ is conjugate either to an element of E or to a composition of generalized HÉNON transformations (*see* [17]); we will say that h is of elementary or HÉNON type. In a similar way, we say that h is of affine type if it is conjugate to an element of the affine group. By [32], it is equivalent to say that h is of HÉNON type or that the topological entropy of h, view as a continuous transformation of $\mathbf{C}^2$, is strictly positive. On the other hand, an element of $\mathrm{Aut}(\mathbf{R}^2)$ may be of HÉNON type but nevertheless conjugate to a translation of $\mathbf{R}^2$; such elements present a very rich dynamical behavior on $\mathbf{C}^2$ and a very poor one on $\mathbf{R}^2$.

Proposition. *For every integer $k \geq 1$ there exists a subgroup Γ in Aut($\mathbf{R}^2$) isomorphic to the free group F_k such that:*

- *Any non trivial element in Γ is an* HÉNON *type automorphism;*
- *any element in Γ is analytically conjugate to a translation;*

- Γ *acts properly discontinuously on the plane* $\mathbf{R}^2$.

This proposition is proved in part 4. The case of the fundamental groups Γ_g of compact orientable surfaces of genus g is more delicate. The following theorem shows that there exists some embeddings of Γ_g into $\mathrm{Aut}(\mathbf{R}^2)$ which are not conjugate to an affine embedding, but there exists some constraints on the possible embeddings. We give the proof in two steps in paragraphs 5.2 and 5.3, theorems 5.1 and 5.3.

Theorem B.

- *For any integer* $g \geq 2$ *there exists subgroups of* $Aut(\mathbf{R}^2)$ *isomorphic to* Γ_g *that contain* HÉNON *type automorphisms.*
- *Any subgroup of* $Aut(\mathbf{R}^2)$ *isomorphic to* Γ_g *(with* $g \geq 2$*) contains an element distinct from the identity which possess a fixed point in* $\mathbf{R}^2$.

1.3. **Crystallographic groups.** A subgroup of $\mathsf{Diff}^\infty(\mathbf{R}^n)$ is a crystallographic group if its action on $\mathbf{R}^n$ is discrete and cocompact. Recently, DEKIMPE and IGODT proved that for any polycyclic group Γ there exists an integer n such that Γ is isomorphic to a crystallographic group of polynomial diffeomorphisms of $\mathbf{R}^n$ (see [14]). On the other hand, we can ask for a classification of the polynomial crystallographic groups when n is small. The results above lead to the following theorem that answers a question by DEKIMPE (see [13] or [6]). This result is proved in paragraph 5.2.

Theorem C.

- *It is not possible to find a model of the universal covering of the compact orientable surface of genus* $g \geq 2$ *such that the group of automorphisms would act by polynomial transformations of the plane.*
- *Any polynomial crystallographic group of the plane admits a finite index subgroup which is polynomially conjugate to the group of integral translations.*

1.4. **Acknowledgements.** This text originates from a question by François BEGUIN about BROUWER homeomorphisms (*see* paragraph 4.1); we would like to thank François for his beautiful lecture on this subject. Thanks also to Dominique CERVEAU for his advice and to Étienne GHYS for his encouragements.

2. ACTIONS OF GROUPS ON TREES

2.1. BASS-SERRE **tree.** Let k be a field. By the theorem of JUNG - VAN DER KULK (a proof of which we can find in [25]), $\mathrm{Aut}(k^2)$ is the product of the affine group

$$A = \{(x,y) \mapsto (a_1x + a_2y + a_3, b_1x + b_2y + b_3); a_i, b_i \in k, a_1b_2 - a_2b_1 \neq 0\}$$

and of the elementary group

$$E = \{(x,y) \mapsto (\alpha x + P(y), \beta y + \gamma); \alpha, \beta, \gamma \in k, \alpha\beta \neq 0, P \in k[X]\}$$

amalgamated along their intersection S. In particular, any polynomial automorphism of the affine plane is a composition of affine and elementary transformations.

Remark 2.1.

1.- The groups E and S are solvable of length 3 and 2 respectively.
2.- When k is algebraically closed, any element of A is conjugate, in A, to an element of S.

Since $\mathrm{Aut}(k^2)$ is an amalgamated product, we can construct its BASS-SERRE tree ([30]). The vertices of this tree are in bijection with right cosets modulo A (*type* A vertices) and modulo E (*type* E vertices). Every element ϕ in $\mathrm{Aut}(k^2)$ thus gives to distinct vertices, ϕA and ϕE. The edges are in bijection with cosets modulo S: an edge ϕS joins the vertices ϕA and ϕE. The CW-complex so constructed is a tree. This is the BASS-SERRE tree of $\mathrm{Aut}(k^2)$, that we will denote by $\mathcal{T}$.

The group $\mathrm{Aut}(k^2)$ acts on $\mathcal{T}$ by left translation; for instance, the image of the vertex ϕA by the translation associated with ψ is the vertex $(\psi \circ \phi)A$. In this way we embed $\mathrm{Aut}(k^2)$ into the group of simplicial isometries of $\mathcal{T}$. This action is transitive on the set of edges and on the set of type A vertices (resp. type E vertices). The stabilizer of the vertex ϕA (*resp.* of the vertex ϕE, *resp.* of the edge ϕS) is the group $\phi A \phi^{-1}$ (*resp.* $\phi E \phi^{-1}$, *resp.* $\phi S \phi^{-1}$).

The elements g of $\mathrm{Aut}(k^2)$ may be classified into two types according to their action on $\mathcal{T}$. If g acts on the BASS-SERRE tree with at least one fixed point, then g is conjugate to an affine or elementary automorphism. When k is algebraically closed, g is then conjugate to an elementary automorphism; we will say that an element of $\mathrm{Aut}(k^2)$ is of *elementary type* (even if k is not algebraically closed) if it admits a fixed point on the BASS-SERRE tree. We will say that it is of affine type if it is conjugate to an element in the affine group.

Let $\mathrm{long}(g)$ be the translation length of g, defined as the minimum of the distances $\mathrm{dist}(g(s), s)$ where s runs on the set of all vertices of $\mathcal{T}$ and $\mathrm{dist}(.,.)$ is the simplicial distance on $\mathcal{T}$. Thus, an automorphism g is of elementary type if and only if its translation length is zero. When the length is strictly positive, we say that g is of HÉNON *type*. This is the case for the usual HÉNON automorphism

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + y^2 + 0,35 \end{pmatrix}. \tag{2.1}$$

The justification of the terminology comes from the dynamics. Indeed, when $k = \mathbf{C}$, an automorphism is either of elementary type or of HÉNON type, and the latter case corresponds exactly to the automorphisms with a non elementary dynamics on $\mathbf{C}^2$ (infinity of hyperbolic periodic points, etc...). If g is an HÉNON type element, the set of vertices s in the tree $\mathcal{T}$ that satisfy $\mathrm{dist}(g(s), s) = \mathrm{long}(g)$ form a geodesic in $\mathcal{T}$ called *geodesic of* g and noted $\mathrm{Geo}(g)$.

2.2. **The property (FA).** Let X be a tree. We say that the action of a group Γ on X is *without inversion* if there does not exist a couple of adjacent vertices which are exchanged by an element of Γ. A group Γ has the property (FA) if, for any action without inversion of Γ on a tree X, there exists a vertex of X which is invariant for all the elements in Γ. A countable group Γ has the property (FA) if and only if it satisfies the following three properties (*see* [30]):(i) Γ is not a trivial amalgamated product, (ii) the abelianized group of Γ is finite and (iii) the group Γ is of finite type.

Let Γ be a group with the property (FA). Note $\rho : \Gamma \to \mathrm{Aut}(k^2)$ a morphism from Γ to the group of plane automorphisms. We thus obtain an action without inversion of Γ on the tree $\mathcal{T}$ and by the property (FA) there exists a vertex invariant for Γ. In other words there exists an element f of $\mathrm{Aut}(k^2)$ such that $f\rho(\Gamma)f^{-1}$ is contained in the affine or elementary group. We will use this remark in the next section in order to classify the lattices in simple Lie groups that can be imbedded into $\mathrm{Aut}(k^2)$.

2.3. **Graph of groups.** (*see* [30, 29]) A graph of groups $(\mathcal{G}, G)$ is a graph $\mathcal{G}$, with a group G_s labelling every vertex s of $\mathcal{G}$, and a group G_a labelling every (non oriented) edge a of $\mathcal{G}$, and with two injective morphisms $\rho_{a_0} : G_a \to G_{a_0}$ and $\rho_{a_1} : G_a \to G_{a_1}$ for every edge a with vertices a_0 and a_1 (possibly equal).

Consider a vertex s_0 in $\mathcal{G}$. For every vertex s (*resp.* every edge a) we choose a pointed topological space X_s (*resp.* X_a) which is a $K(G_s, 1)$ (*resp.* a $K(G_a, 1)$). The morphisms ρ_{a_i} are then realized by continuous applications between pointed topological spaces $f_{a_i} : X_a \to X_{a_i}$, unique up to homotopy. Let $X(\mathcal{G}, G)$ be the topological space obtained by gluing the spaces X_s with the spaces $X_a \times [0, 1]$ by means of the applications $f_{a_i} : X_a \times \{i\} \to X_{a_i}$. The fundamental group of this topological space is then uniquely determined by the graph of groups $(\mathcal{G}, G)$ and the choice of s_0. When the graph $\mathcal{G}$ is connected, this group is unique up to isomorphism and is noted $\pi_1(\mathcal{G}, G)$.

The two main examples of graphs of groups correspond respectively to the notion of amalgamated product, when $\mathcal{G}$ is a segment, and to HNN-extension (for Higman, Neumann, Neumann), when $\mathcal{G}$ is a loop; the fundamental group of any graph of groups can be decomposed as a sequence of amalgamated products and HNN-extensions.

2.4. **Bass-Serre theory.** There exists a bijective correspondence between groups acting without inversion on trees and fundamental groups of graphs of groups. If Γ acts without inversion on a tree $\mathcal{A}$, we construct the associated graph of groups $(\mathcal{G}, G)$ as the quotient graph $\mathcal{G} = \Gamma\backslash\mathcal{A}$, labeled in the following way. We choose a maximal subtree $\mathcal{M}$ in $\mathcal{G}$ that we lift as a tree $\tilde{\mathcal{M}}$ in $\mathcal{A}$; if s (*resp.* a) is a vertex (*resp.* an edge) of $\mathcal{M}$ we note $\tilde{s}$ (*resp.* $\tilde{a}$) its lifting in $\tilde{\mathcal{M}}$. For every vertex s of $\mathcal{G}$, s is in $\mathcal{M}$ and by definition the group G_s is the stabilizer of the associated vertex $\tilde{s}$. The construction extends to the edges of $\mathcal{M}$ and the morphisms $G_a \to G_{a_i}$ are inclusions. If a is an edge of $\mathcal{G}$ which is not in $\mathcal{M}$, we first lift its vertices a_0 and a_1, which are in $\mathcal{M}$, to two vertices $\tilde{a_0}$ and $\tilde{a_1}$ of $\tilde{\mathcal{M}}$. Then we consider the edge a' of $\mathcal{A}$ starting from $\tilde{a_0}$ that lifts a and we define G_a as the stabilizer of this edge, the morphism from G_a in G_{a_0} being the inclusion. Let γ be an element in Γ that sends $\tilde{a_1}$ on the end of a' which is not in $\tilde{\mathcal{M}}$; then we define the morphism $\rho_{a_1} : G_a \to G_{a_1}$ by $\rho_{a_1}(\alpha) = \gamma^{-1}\alpha\gamma$. This construction done, it turns out that the morphism from $\pi_1(\mathcal{G}, G)$ to Γ induced by the inclusions of the G_s and G_a into Γ is an isomorphism. This is the content of the BASS-SERRE theory.

Thus BASS-SERRE theory shows that groups that act on a tree without global fixed point may be decomposed as a sequence of amalgamated product and HNN-extension. In order to embed a non solvable group into $\mathrm{Aut}(k^2)$ but not into the affine group, a necessary condition is that the group may be non trivially decomposed as an amalgamated product or an HNN-extension.

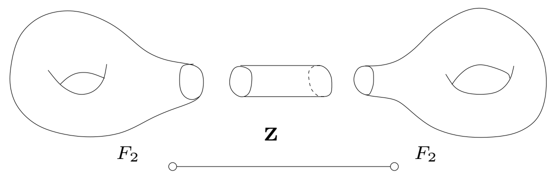

FIGURE 1. The fundamental group of the compact orientable surface of genus 2, $\Gamma_2 = \langle a_1, b_1, a_2, b_2 | [a_1, b_1] = [a_2, b_2] \rangle$, is isomorphic to the fundamental group of the graph of groups pictured above. In particular, the group Γ_2 acts without inversion and without fixed point on some trees.

2.5. **Two examples.** In this paragraph, we show that two classical examples of amalgamated product and HNN-extension can not be embedded into $\mathrm{Aut}(\mathbf{C}^2)$. They are the braid group B_n and the BAUMSLAG-SOLITAR group $\mathrm{BS}(2,3)$. This will illustrate the technics used further in the article.

For any couple of strictly positive integers (p, q), the BAUMSLAG-SOLITAR group $\mathrm{BS}(p, q)$ (non solvable except for $p = 1$ or $q = 1$) may be defined by the following presentation

$$\mathrm{BS}(p, q) = \langle s, t |\, st^p s^{-1} = t^q \rangle. \tag{2.2}$$

When $p = 2$ and $q = 3$, this finite type group is not residually finite and thus can not be embedded into any $\mathsf{GL}(n, \mathbf{C})$ (*see* [2], [3]).

Proposition 2.2. *Let p and q be two distinct strictly positive integers. Then any morphism from $BS(p, q)$ to $Aut(\mathbf{C}^2)$ admits a solvable image. In consequence the group $Aut(\mathbf{C}^2)$ does not contain any subgroup isomorphic to $BS(2, 3)$.*

Remark 2.3. By the main theorem of [1], a finite type subgroup of the polynomial automorphisms of $\mathbf{C}^n$ is residually finite, and this is not the case for $\mathrm{BS}(2, 3)$. So part of this proposition is a corollary of [1].

Proof. Let $\rho : \mathrm{BS}(p, q) \to \mathrm{Aut}(\mathbf{C}^2)$ be a group morphism. Note a and b the images of t and s by this morphism. Since a^p is conjugate to a^q, the length of translation of a is zero as soon as p is distinct from q.

If a is of infinite order, proposition 3.3 from [24] and its proof show that the fixed points of the tree $\mathcal{T}$ for the action of a^n are a bounded subtree of diameter at most 6. This implies that the set of points of the tree $\mathcal{T}$ which are periodic for a is a tree F of diameter at most 6. If we conjugate the morphism ρ by an element of $\mathrm{Aut}(\mathbf{C}^2)$, we can assume that F has its center on the vertex E, the vertex A, or the edge S. Since ba^pb^{-1} is equal to a^q, the tree F and its center are invariants under the action of b (*see* [30, p. 32]). Thus the image of ρ is contained in E, in A or in S.

From this it is easy to deduce that the image of ρ is solvable. It is immediate if it is contained in E. If it is contained in A the relation $ba^pb^{-1} = a^q$ implies that the linear parts of a and b or b^2 share a common eigendirection; and so the image of ρ is solvable.

When the order of a is finite, the image of ρ is an extension of $\mathbf{Z}$ by a finite cyclic group and so is solvable.

Since $\mathrm{BS}(2,3)$ is not solvable, it can not be embedded into $\mathrm{Aut}(\mathbf{C}^2)$. □

The braid group B_3 admits the following presentation

$$B_3 = \langle u, v \,|\, u^2 = v^3 \rangle. \tag{2.3}$$

So this group is the product of two copies of $\mathbf{Z}$ amalgamated along $\mathbf{Z}$ and as such it can be embedded into the automorphism group of a simplicial tree.

Proposition 2.4.

- *If ρ is an injective morphism from B_3 to $Aut(\mathbf{C}^2)$, its image is contained in some conjugate of the affine group.*
- *It does not exist any embedding of the braid group B_n into $Aut(\mathbf{C}^2)$ when $n \geq 4$.*

Remark 2.5. The BURAU representation gives a representation of B_n in $\mathsf{GL}(n, \mathbf{Q}[t^{\pm 1}])$. This representation is reducible, it splits in a representation of dimension 1 and a representation of dimension $n-1$. For $n = 3$, this representation of dimension 2 is faithful; by replacing t with a transcendant complex number we obtain an injective irreducible representation of B_3 in $\mathsf{GL}(2, \mathbf{C})$ (*see* [8] and references therein).

Remark 2.6. The groups B_3 and B_4 are the only braid groups which admit a non trivial decomposition into an amalgamated product (*see* [23]), but we will not need this property.

Proof. Let ρ be a morphism from B_3 to $\mathrm{Aut}(\mathbf{C}^2)$. The polynomial automorphism

$$h = \rho(u)^2 = \rho(v)^3 \tag{2.4}$$

commutes with $\rho(u)$ and with $\rho(v)$. If h is of HÉNON type, the image of ρ is then contained in a solvable group and ρ can not be injective (*see* [24], thm. 2.4 and prop.4.8).

If h is an automorphism of elementary type, $\rho(u)$ and $\rho(v)$ are also elementary and so we can assume that the image of ρ is contained in A or E (apply a similar argument as the one above). Since B_3 is not solvable, the only morphisms that can be injective are the ones with value in A.

Now we show the second point. It is sufficient to consider the case $n = 4$ because B_4 is contained in B_n for all n greater than 4. The group B_4 admits the following presentation:

$$B_4 = \langle a, b, c |\, ac = ca,\ aba = bab,\ bcb = cbc \rangle \tag{2.5}$$

If we take $u = aba$ and $v = ab$, or $u = bcb$ and $v = bc$, we see that B_4 contains two copies of B_3, one generated by a and b, and the other by b and c.

Suppose there exists an injective morphism ρ from B_4 to $\mathrm{Aut}(\mathbf{C}^2)$. The study of morphisms from B_3 to $\mathrm{Aut}(\mathbf{C}^2)$ shows that, up to conjugacy, the image of $h = (ab)^3$ by ρ is an element of the affine group which fixed points (on $\mathcal{T}$) are a subtree of finite diameter and with center on A. This property remains true for elements that commute with h, that is a and b, and then also for elements that commute with a, for instance c. Thus, the image of ρ is contained in the affine group.

It is easy to see that no representation of B_4 in the affine complex group can be injective (*see* [16]). So we obtain a contradiction that ends the proof. □

2.6. **Modular groups.** Let Γ be the fundamental group of an compact orientable surface with boundary. The group $\mathrm{Out}(\Gamma)$ can be embedded in the automorphism group $\mathrm{Aut}(V)$ of a complex algebraic variety V (*see* [1]), and it would be interesting to know what are the smallest varieties which $\mathrm{Out}(\Gamma)$ acts polynomially and faithfully on. Suppose that Γ is a free group F_n with $n \geq 3$ or that S is closed with genus at least 2. Then $\mathrm{Out}(\Gamma)$ has property (FA) (*see* [10]), is not solvable and can not be embedded into the group $\mathrm{Aff}(\mathbf{C}^2)$. The arguments of the previous paragraphs then show that $\mathrm{Out}(\Gamma)$ can not be embedded into $\mathrm{Aut}(\mathbf{C}^2)$. We can also prove that $\mathrm{Out}(\Gamma)$ can not be embedded into $\mathrm{Aut}(X)$ if X is a complex projective surface. So it seems that algebraic faithful actions of these groups do not exist in dimension 2.

3. Lattices in simple Lie groups

In this section we prove the following theorem.

Theorem 3.1. *Let k be a field. Let G be a real simple* Lie *group and Γ be a lattice in G. If there exists an injective morphism $\rho : \Gamma \to Aut(k^2)$, the group G is isomorphic to $\mathsf{PSO}(1, n)$ or to $\mathsf{PSU}(1, n)$ for some integer n; furthermore if G is distinct from $\mathsf{PSO}(1, 2)$ then the image of ρ is contained in a conjugate of the affine group.*

3.1. **The property (T).** We are going to apply the remarks of the previous section to groups satisfying the property (T) of Kazhdan. We do not define this property here, the interested reader should consult [12], chapter I, or [26], chapter III. Let us simply recall some consequences of property (T). Let G be a locally compact topological group with countable basis and Γ be a lattice in G.

(a) G has property (T) if and only if Γ has property (T).
(b) If G has property (T), then any continuous morphism from G to a solvable group has a relatively compact image.

(c) If G has property (T), then G is generated by a compact neighborhood of the identity element.
(d) If Γ has property (T) then it has property (FA).

Thus, when G is a locally compact topological group with countable basis that has property (T), any lattice in G is of finite type (apply (c)) and has property (FA). In particular, when Γ is a lattice in G and $\rho : \Gamma \to \mathrm{Aut}(k^2)$ is a group morphism, conjugating ρ by an element of $\mathrm{Aut}(k^2)$ we can assume that the image of ρ is contained in the affine group or in the group E of elementary automorphisms. In the latter case, the image of ρ is finite because E is solvable. If the image of ρ is contained in the affine group, the linear parts of elements in $\rho(\gamma)$ give a representation $\rho' : \Gamma \to \mathsf{GL}(2,k)$ and, if we replace Γ by finite index subgroup, we can assume that the image of ρ' is in $\mathsf{SL}(2,k)$. Using [19] we deduce that the image of ρ' (and of ρ) is finite.

Proposition 3.2. *Let k be a field. Let G be a locally compact topological group satisfying property (T), and let Γ be a lattice in G. Any morphism from Γ to Aut(k^2) has a finite image.*

By this proposition we can manage all countable groups satisfying property (T) and lattices in real simple LIE groups which are not locally isomorphic to $\mathsf{SO}(1,n)$ or $\mathsf{SU}(1,n)$ (*see* [12]). These two groups do not have property (T), and indeed some of their lattices act without inversion and without global fixed point on trees.

3.2. **A result by** SHALOM**, [31].** In order to conclude, we are going to apply a result by SHALOM about the actions on trees of lattices in $\mathsf{SO}(1,n)$ or $\mathsf{SU}(1,n)$. First, let us recall that these two LIE groups naturally act on the real or complex hyperbolic space of dimension n. Note $\mathbb{H}$ this hyperbolic space, and fix a base point o in $\mathbb{H}$. We note $d(.,.)$ the hyperbolic distance. For any discrete group Γ of isometries of $\mathbb{H}$, the critical exponent of Γ is the positive real number

$$\delta(\Gamma) = \inf\left\{ s \in \mathbf{R} \,:\, \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma(o))} < +\infty \right\}.$$

The critical exponent of a lattice is equal to $n-1$ in the case of the real hyperbolic space and to $2n$ for the complex hyperbolic space; the exponent of a solvable group is zero.

Theorem 3.3 (SHALOM)**.** *Let X be a simplicial tree. Let Γ be a lattice in $\mathsf{SO}(1,n)$ or in $\mathsf{SU}(1,n)$, with $n \in \mathbf{N}^*$. Let $\Gamma \times X \to X$ be an action without inversion nor fixed point. Then there exists an edge α in X which stabilizer C in Γ satisfies*

$$\delta(C) \geq \delta(\Gamma) - 1.$$

Thus, as soon as $\delta(\Gamma)$ is strictly bigger than 1, the critical exponent of C is strictly positive and C can not be a solvable group. Since stabilizers of edges of the tree T associated with $\mathrm{Aut}(k^2)$ are solvable groups, we deduce that any representation of Γ in $\mathrm{Aut}(k^2)$ stabilizes a vertex. So, any faithful representation of Γ in $\mathrm{Aut}(k^2)$ has value in the affine group.

3.3. **Conclusion.** The demonstration of theorem 3.1 is complete: we dealt with lattices in LIE groups satisfying property (T) in paragraph 3.1, and with the remaining case of lattices in $\mathsf{SO}(1,n)$ and $\mathsf{SU}(1,n)$ in the paragraph above.

Remark 3.4. It could be that no lattice of $\mathsf{SO}(1,n)$ can be embedded in the affine group of $\mathbf{C}^2$ when $n \geq 4$. This would precise theorem 3.1. Unfortunately, we do not know how to handle this problem.

4. FREE GROUPS

In this section and the next one we describe some embeddings of lattices in $\mathsf{PSL}(2,\mathbf{R})$ into the group $\mathrm{Aut}(\mathbf{R}^2)$ which image contains HÉNON type elements. As already mentioned in the introduction, there are two classes of lattices to consider, the uniform (or cocompacts) lattices and the others. Any non uniform lattice contains a finite index subgroup which is isomorphic to a free group F_k over a finite number of generators.

Proposition 4.1. *For any integer $k \geq 1$ there exists a subgroup Γ of $Aut(\mathbf{R}^2)$ isomorphic to the free group F_k such that:*

- *any non trivial element in Γ is an HÉNON type automorphism;*
- *any element in Γ is analytically conjugate to a translation;*
- *Γ acts properly discontinuously on the plane $\mathbf{R}^2$.*

Remark 4.2. Since the free group F_2 contains a copy of every F_k, $k \geq 1$, we will only consider the case k equal to 2.

Remark 4.3. The elements in the group we are about to construct preserve the orientation and so are BROUWER homeomorphisms, that is to say they are homeomorphisms without fixed point that preserve the orientation. By a theorem of BROUWER, these elements do not have any periodic point. A priori, they could nevertheless have some interesting dynamical behavior (*see* the introduction of [5]): for instance, there exists a BROUWER homeomorphism which does not act properly discontinuously on any non empty invariant closed subset of $\mathbf{R}^2$ (*see* [11]). However, we show in the first paragraph of this section that any automorphism of the plane which is a BROUWER homeomorphism is analytically conjugate to a translation. This is the result that allows us to prove the second point in the proposition. This result contrasts with the existence of analytic BROUWER diffeomorphisms which have interesting dynamics.

Remark 4.4. The proposition above say nothing about the set of non-uniform lattices since we put aside lattices with elements of finite order. Here are two examples:

• there exists an embedding of the group $\mathsf{PSL}(2,\mathbf{Z}) \simeq \mathbf{Z}/2\star\mathbf{Z}/3$ into $\mathrm{Aut}(\mathbf{R}^2)$ which image contains HÉNON type automorphisms. For instance consider the group generated by the automorphisms

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -x-\sqrt{3}y \\ \sqrt{3}x - y \end{pmatrix} \quad \text{et} \quad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+y^2 \\ y \end{pmatrix} \tag{4.1}$$

and apply BASS-SERRE theory (*see* §5.3 for other reasonings of this kind).

- any morphism of the triangular group

$$T_{l,m,n} = \langle a, b | a^m = b^m = (ab)^l = 1 \rangle$$

to the group $\mathrm{Aut}(k^2)$ is conjugate to a morphism with value in the affine group when l, m and n are positive integers (this group is a lattice in $\mathsf{PSL}(2, \mathbf{R})$ when $1/l + 1/m + 1/n < 1$), this comes from the fact that these groups have the property (FA) (*see* [34] or [30]).

4.1. BROUWER **automorphisms.** In this paragraph we want to show the following proposition.

Proposition 4.5. *Any polynomial* BROUWER *homeomorphism is analytically conjugate to a translation.*

Proof. In order to classify polynomial automorphisms which are not BROUWER homeomorphisms, we consider three cases according if an automorphism, say g, is of elementary, affine or HÉNON type (*see* paragraph 2.1 for the definitions). In these three cases, the strategy is the same: the point is to show that the group generated by g acts properly discontinuously on $\mathbf{R}^2$. Indeed, $\Sigma(g) = \mathbf{R}^2/\langle g \rangle$ is then an orientable analytic surface homeomorphic to a cylinder. So this surface is isomorphic, as a real analytic surface, to the standard cylinder obtained as the quotient of the plane $\mathbf{R}^2$ by the horizontal unitary translation (*see* [21], p. 65 §II.5). Thus there exists an analytic covering $\Phi : \mathbf{R}^2 \to \Sigma(g)$ which automorphism group is the group of integral translations $\tau_n : (x, y) \mapsto (x+n, y)$, $n \in \mathbf{Z}$. We can lift this covering to an analytic map $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ which conjugates g to the unitary translation:

$$\phi^{-1} \circ g \circ \phi = \tau_1. \tag{4.2}$$

This is the conjugacy stated in proposition 4.5.

If g is an elementary type automorphism that preserves the orientation and without fixed point, then g is conjugate to a transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x + P(y) \\ \beta y + \gamma \end{pmatrix} \tag{4.3}$$

where P is a polynomial in one variable and the quadruplet $(\alpha, \beta, \gamma, P)$ satisfies to one of the three following properties:

(1) $\beta = 1$ and $\gamma \neq 0$;
(2) $\alpha = \beta = 1$, $\gamma = 0$ and P does not have any real root.
(3) $\beta \neq 1$, $\alpha = 1$ and $P(\gamma/(1-\beta))$ is not zero. In this case if we conjugate by a translation we can assume $\gamma = 0$.

In each case it is easy to verify that $\langle g \rangle$ acts properly discontinuously on $\mathbf{R}^2$.

If g is an affine type automorphism that preserves the orientation and without fixed point, g is conjugate to an affine automorphism which linear part admits 1 as an eigenvalue. By conjugacy in $\mathsf{GL}(2, \mathbf{R})$ we can obtain a triangular affine automorphism, and so we are in the previous case.

Suppose now that g is an HÉNON type automorphism which is also a BROUWER homeomorphism. Note G^+ the (positive) Green function of g, restricted to the real affine plane $\mathbf{R}^2$. It is defined on $\mathbf{C}^2$ by the following formula

$$G^+(z) = \lim_{n\to+\infty} \left\{ \frac{1}{d^n} \log^+ (\|g^n(z)\|) \right\}, \tag{4.4}$$

where $d \geq 2$ is the (dynamical) degree of g (*see* [22]). in particular, G^+ is positive and $G^+ \circ g = dG^+$. Let us recall that G^+ is smooth on the open set $G^+ > 0$, and that the set $G^+ = 0$ is exactly the locus of points in $\mathbf{C}^2$ with bounded positive orbits. Since g is a BROUWER homeomorphism, all its orbits go to infinity and so G^+ is smooth and everywhere strictly positive on $\mathbf{R}^2$.

Let m be a point in $\mathbf{R}^2$; we can choose $\alpha > 0$ such that m is in a fundamental domain:

$$m \in \{p \in \mathbf{R}^2; \alpha < G^+(p) < \alpha.d\}.$$

let $\mathcal{U}$ be a neighborhood of m included in this fundamental domain. Then we remark that for any $k \in \mathbf{Z}$ we have $g^k(\mathcal{U}) \subset \{p \in \mathbf{R}^2; \alpha d^k < G^+(p) < \alpha d^{k+1}\}$, and we conclude that $\langle g \rangle$ acts properly discontinuously on $\mathbf{R}^2$.

This ends the proof of proposition 4.5 □

4.2. **An embedding of F_2.** Now we give the proof of the proposition 4.1. If f is a non affine automorphism of the plane $\mathbf{R}^2$, its birational extension to the real projective plane admits a unique indeterminacy point $i(f)$. We can choose f such that the points $i(f)$ and $i(f^{-1})$ are distinct. We can further assume that f is a BROUWER homeomorphism. For instance, if $P \in \mathbf{R}[X]$ is any polynomial without real root, we can take

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ P(y) + 2y - x \end{pmatrix} \tag{4.5}$$

Then if we replace f by one of its iterates, we can choose a neighborhood $V^+(f)$ of $i(f^{-1})$ and a neighborhood $V^-(f)$ of $i(f)$ in the projective plane with disjoint adherence and satisfying the two following properties

$$f(\mathbf{R}^2 \setminus V^-(f)) \subset V^+(f) \quad \text{and} \quad \|f(p)\| > 2\|p\| \quad \forall p \in \mathbf{R}^2 \setminus V^-(f) \tag{4.6}$$
$$f^{-1}(\mathbf{R}^2 \setminus V^+(f)) \subset V^-(f) \quad \text{and} \quad \|f^{-1}(p)\| > 2\|p\| \quad \forall p \in \mathbf{R}^2 \setminus V^+(f) \tag{4.7}$$

We can also assume that $(0,0)$ is not in any of the $V^\pm(f)$.

For instance for the automorphism f above, which indeterminacy points are $i(f) = [1:0:0]$ and $i(f^{-1}) = [0:1:0]$, we can define $V^\pm(f)$ (or more precisely their trace on $\mathbf{R}^2$) by

$$V^+(f) \;=\; \{(x,y) \in \mathbf{R}^2; |y| > N, |x/y| < \varepsilon\}; \tag{4.8}$$
$$V^-(f) \;=\; \{(x,y) \in \mathbf{R}^2; |x| > N, |y/x| < \varepsilon\}; \tag{4.9}$$

where $\varepsilon > 0$ is arbitrarily small and $N > 0$ arbitrarily big (the conditions (4.6) and (4.7) will be fullfilled by an iterate f^k of f with k depending on ε and N).

Let g be another HÉNON type automorphism satisfying the same properties. It is possible to choose g such that

(i) $\{i(f),\, i(f^{-1})\}$ and $\{i(g),\, i(g^{-1})\}$ are all disjoint;

(ii) $\mathrm{Geo}(f)$ and $\mathrm{Geo}(g)$ are distinct.

For instance we can take g equal to the conjugate of f by a linear transformation that does not fix the horizontal nor the vertical axes. The constraints on g allow us to choose some neighborhoods $V^{\pm}(g)$ of $i(g)$ and $i(g^{-1})$ satisfying properties similar to (4.6) and (4.7), and disjoint from $V^+(f) \cup V^-(f)$.

Let N be an integer strictly positive such that the lengths of translation of f^N and of g^N are strictly greater than the diameter of the segment $\mathrm{Geo}(f) \cap \mathrm{Geo}(g)$. Replace f and g by their N powers (without changing notations). The group generated by f and g is then a free group all of which elements are Hénon type automorphisms (*see* [24]). This a consequence of the ping-pong lemma for the action of this group on the tree $\mathcal{T}$. We note Γ this group.

Let q be a point in the plane; we want to construct a neighborhood $\mathcal{U}$ of q such that $h(\mathcal{U}) \cap \mathcal{U} = \emptyset$ for all $h \in \Gamma \setminus \{Id\}$. Without loss of generality we can replace q by any point in the orbit of q under the action of Γ. If q is in the complement of the sets $V^{\pm}(f)$ and $V^{\pm}(g)$, we can choose $\mathcal{U}$ equal to a ball centered in q and without intersection with the sets $V^{\pm}(f), V^{\pm}(g)$.

Suppose now that the orbit of q under Γ is entirely contained in the union of the $V^{\pm}(f), V^{\pm}(g)$ (in particular it can not accumulate on $(0,0)$). Note $m = \inf\{||h(q)||; h \in \Gamma\}$, we can assume $0 < m \leq ||q|| < 2m$ (we can not a priori assume $||q|| = m$ because it is not clear if this infimum is realized). Suppose now that q is in $V^-(f)$ (the three other cases are similar). then $f(q) \in V^+(f)$ because otherwise we would have

$$2m > ||q|| = ||f^{-1}(f(q))|| > 2||f(q)|| \text{ and so } m > ||f(q)||$$

and this would contradicts the definition of m. If $\mathcal{U}$ is small enough ball centered on q, we can assume that $f(\mathcal{U}) \subset V^+(f)$ and that $\mathcal{U} \subset \{p \in \mathbf{R}^2; ||p|| < 2m\}$. Let h be an element of Γ distinct from the identity; h can be written in a unique way as a reduced word of length $l \geq 1$ in the letters f, f^{-1}, g and g^{-1}.

- if $h = f$, by construction we have $h(\mathcal{U}) \cap \mathcal{U} = \emptyset$;
- if the decomposition of h begins (on the right) by f and $l \geq 2$, then for any $p \in \mathcal{U}$

$$||h(p)|| \geq 2^{l-1}||f(p)|| \geq 2m$$

  and so $h(\mathcal{U}) \cap \mathcal{U} = \emptyset$;
- otherwise, for all $p \in \mathcal{U}$ we have $||h(p)|| \geq 2^l||p|| \geq 2m$ and again $h(\mathcal{U}) \cap \mathcal{U} = \emptyset$.

Thus we have proved that the group Γ is a free group, isomorphic to F_2, all of which elements distinct from the identity are of Hénon type, and that Γ acts discontinuously on the plane $\mathbf{R}^2$. In particular its elements distinct from the identity are Brouwer automorphisms and so are analytically conjugate to translations.

5. Fundamental groups of surfaces

In this section we want to study embeddings of the group

$$\Gamma_g = \langle a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle \tag{5.1}$$

in the group $\mathrm{Aut}(k^2)$ where k is a field. This group is the fundamental group of a compact orientable surface of genus g. So we can embed Γ_g in $\mathsf{SL}(2, \mathbf{R})$, and in $\mathrm{Aut}(\mathbf{R}^2)$. The embeddings we are interested in are those which are not conjugate to an embedding into the affine group. We will see that these are exactly the embeddings which image contains an Hénon type automorphism.

5.1. **A theorem of Zieschang.** Let $\rho : \Gamma_g \to \mathrm{Aut}(k^2)$ be an injective morphism. Then the group Γ_g acts on the Bass-Serre tree $\mathcal{T}$ associated with $\mathrm{Aut}(k^2)$. Let $\mathcal{T}_0$ be the smallest sub-tree of $\mathcal{T}$ containing the orbit of the edge IdS under the action of Γ_g. Let $(\mathcal{G}, G)$ be the graph of groups obtained as the quotient of $\mathcal{T}_0$ by the action of Γ_g. If a is an edge of $\mathcal{G}$, the group G_a associated with this edge is a solvable subgroup of Γ_g (because the stabilizer groups of edges are solvable): this group is then trivial or isomorphic to $\mathbf{Z}$. By a theorem of Zieschang (*see* [35] and [20]), a decomposition of Γ_g as the fundamental group of a graph of groups whose edges are labeled with a trivial group or a group isomorphic to $\mathbf{Z}$ is always given by a cut-out of the surface of genus g along disjoint simple closed curves. The groups attached to the vertices of this graph are the fundamental groups of each of the connected components of the cut-out surface. The reader might want to consult [27], pages 465 to 467, where this theorem is proved and partially attributed to Stallings, [33].

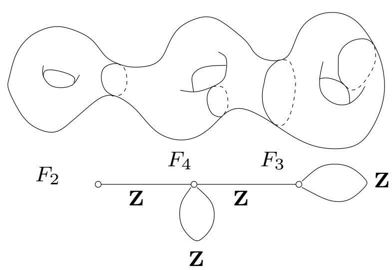

Figure 2. This figure presents a decomposition of Γ_3 as the fundamental group of a graph of groups: we have pictured the graph and the simple closed curves that can be chosen to realize this decomposition of Γ_3.

Let s be a vertex of $\mathcal{G}$ coming from a type E vertex of the tree $\mathcal{T}_0$. The group G_s corresponds to a solvable subgroup of Γ_g and so is isomorphic to $\mathbf{Z}$ or to the trivial group. If the groups associated with the edges joining s to the other vertices of $\mathcal{G}$ are trivial, the group G_s is also trivial, because otherwise $\Gamma_g = \pi_1(\mathcal{G}, G)$ would be the free product of $\mathbf{Z}$ and another group, and this is not the case (to see this, one can for example apply again the theorem of Zieschang cited above). So there exists at least one edge a linked to s for which the group G_a is isomorphic to $\mathbf{Z}$. The morphism from G_a to G_s, must be an isomorphism; so we can simplify the graph by erasing all the type E vertices.

Geometrically, this operation is associated to the following situation. The graph of groups $\mathcal{G}$ corresponds to a cut-out of the surface of genus g and the type E vertex corresponds to a component which is homeomorphic to a cylinder. This cylinder is produced by the cut-out of the surface along two boundary curves. To simplify the graph by removing the type E vertex is equivalent to simplify the cut-out by forgetting one of these two curves.

5.2. **Actions without fixed points.** We are now in position to prove the following theorem.

Theorem 5.1. *Any subgroup of Aut($\mathbf{R}^2$) isomorphic to Γ_g (with $g \geq 2$) contains an element distinct from identity which admits a fixed point in $\mathbf{R}^2$.*

As a corollary, it does not exist a properly discontinuous polynomial action of Γ_g on the plane $\mathbf{R}^2$. It is unfortunate: it would have been interesting to find a model of the universal covering of the surface of genus g ($g > 1$) for which the covering automorphism group acts by polynomial automorphisms of the plane.

Proof. Let F be a subgroup of the real affine group all of which elements act without fixed point on the plane. The linear parts of the elements of F satisfy the equation

$$\det(M - Id) = 0 \tag{5.2}$$

and so the group F is not Zariski dense. In consequence, F is a solvable subgroup of the affine group and in particular, F does not contain a free group on two generators.

Let Γ be a subgroup of $\mathrm{Aut}(\mathbf{R}^2)$ all of which non trivial elements are Brouwer homeomorphisms. The group Γ acts on the Bass-Serre tree $\mathcal{T}$ and so all the stabilizer groups in Γ of the vertices of $\mathcal{T}$ are solvable groups. Thus the groups associated to vertices of the graph of groups $\mathcal{T}\backslash\Gamma$ are solvable.

By the paragraph above, such a group can not be isomorphic to the group Γ_g. $\square$

This theorem allows us to obtain the following corollary, which answers a question by Dekimpe (see [6, question 5.2]).

Corollary 5.2. *Any polynomial crystallographic group of the plane admits a finite index subgroup conjugated by a polynomial automorphism to the group of integral translations.*

Proof. Let $G \subset \mathrm{Aut}(\mathbf{R}^2)$ be such a crystallographic group. There exists H of finite index in G such that H acts freely on $\mathbf{R}^2$ and such that the quotient $\mathbf{R}^2/H$ is compact and orientable. The theorem 5.1 forbids this quotient to be a surface of genus ≥ 2, so this is a torus and $H = \mathbf{Z}^2$. Thus it is impossible that H contains an Hénon type element (because all of those have a centralizer group in $\mathrm{Aut}(\mathbf{C}^2)$ isomorphic to $\mathbf{Z} \ltimes \mathbf{Z}/p\mathbf{Z}$: see [24]), so up to conjugacy we can assume that $H \subset A$ or $H \subset E$. We conclude by applying the main theorem of [7] which gives us the result as soon as we know that H is of bounded degree, and here this is an easy remark: if $H = \langle f, g\rangle \subset E$ then the degree of any element in H is at most $\max(\deg f, \deg g)$. □

5.3. **Examples of embeddings.** We are now going to prove the following

Theorem 5.3. *For any integer $g \geq 2$ there exists subgroups of $Aut(\mathbf{R}^2)$ isomorphic to Γ_g containing* HÉNON *type elements.*

In fact, we will construct two distinct embeddings. The graph of groups associated with the first one will correspond to a decomposition of Γ_g as an amalgamated product. The graph of groups associated with the second one will correspond to a decomposition of type HNN-extension. Here we didn't seek a general result, however these two examples lead us to think that any decomposition of a surface of genus g could be realized by an embedding in $\mathrm{Aut}(\mathbf{R}^2)$.

5.3.1. *Embeddings associated with an amalgamated product.* Let us consider a decomposition of Γ_g as an amalgamated product $\Gamma_g = F_{2k} \star_{\mathbf{Z}} F_{2l}$ where F_n is the free group over n generators and $k + l$ is equal to g ; The presentation of Γ_g associated with this decomposition is

$$\Gamma_g = \langle a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{i=k}[a_i, b_i] = \prod_{j=1}^{j=l}[a_{k+j}, b_{k+j}]\,\rangle$$

We want to construct an embedding ρ of Γ_g in $\mathrm{Aut}(\mathbf{R}^2)$ that "realizes" this decomposition: the orbit of the edge IdS under the action of $\rho(\Gamma_g)$ on the BASS-SERRE tree will be a tree $\mathcal{T}_0$ and $\mathcal{T}_0/\rho(\Gamma_g)$ will be the graph of groups pictured on figure 3. The labels A and E on this figure correspond to the type of the vertices. By the paragraph 5.1 the type E vertex is necessary.

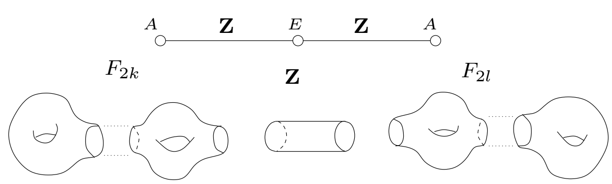

FIGURE 3. Graph of groups associated with the constructed embedding and decomposition of the related surface.

Lemma 5.4. *For all $n \geq 1$ there exists some matrices $A_1, B_1, \cdots, A_n, B_n \in$ $\mathsf{SL}(2, \mathbf{R})$ such that:*

1. *$\langle A_1, B_1, \cdots, A_n, B_n\rangle$ is a free group over $2n$ generators;*
2. *$h = \prod_{i=1}^{i=n}[A_i, B_i] = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$;*
3. *the only triangular matrices in $\langle A_1, B_1, \cdots, A_n, B_n\rangle$ are the powers of h.*

Proof. Such a choice of the A_i, B_i is possible because the Riemann surface Σ_n minus a point can be endowed with an hyperbolic metric of finite area. Explicitly it is sufficient to choose the A_i, B_i such that the fundamental domain for the action of the group on the Poincare disc is a polygon with $4n$ vertices, all of which are located on the circle at infinity. The figure 4 illustrates the case $n = 2$. As elements of $\mathsf{SL}(2, \mathbf{R})$ the matrices A_i and B_i are defined up to multiplication

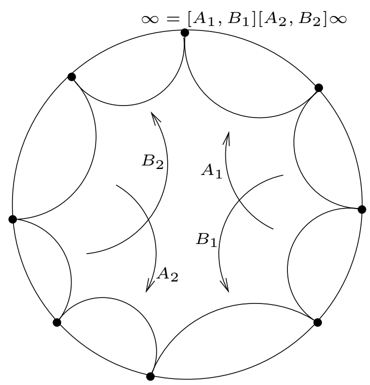

FIGURE 4. Case $n = 2$ in lemma 5.4.

by $-Id$, but their commutators $[A_i, B_i]$ are well defined. In particular, their traces are well defined. Since the matrice $h = \prod_{i=1}^{i=n}[A_i, B_i]$ fix the point ∞, it is triangular and its trace is ± 2. It turns out that the trace of h is equal to -2 (*see* for example [28], ex. 6.1, page 193 and [4] for results of this kind). Thus by conjugacy we can have $h(x, y) = (-x + y, -y)$. □

Let $u_1, v_1, ..., u_k, v_k$ be the elements of $\mathsf{SL}(2, \mathbf{R})$ given by the lemma with $n = k$, and $u'_{k+1}, v'_{k+1}, ..., u'_{k+l}, v'_{k+l}$ another set of elements given by the lemma with $n = l$. Choose ϕ an elementary automorphism which is not affine and which commutes with the linear automorphism h. For instance, we can take $\phi(x, y) = (x + y^3, y)$. For all integer j between 1 and l,, note u_{k+j} (*resp.* v_{k+j}) the automorphisms $\phi \circ u'_{k+j} \circ \phi^{-1}$ (*resp.* $\phi \circ v'_{k+j} \circ \phi^{-1}$).

Proposition 5.5. *If we define σ by $\sigma(a_i) = u_i$ and $\sigma(b_i) = v_i$ for all $i = 1, \cdots, k + l$, then we get an isomorphism between Γ_g and $\langle u_i, v_i\rangle$.*

Proof. By construction $G_{2k} = \langle u_1, v_1, \cdots, u_k, v_k\rangle \subset \mathsf{SL}(2, \mathbf{R})$ is a free group over $2k$ generators that fixes the vertex IdA in the BASS-SERRE tree. On the other hand, by condition (3) of lemma 5.4, the vertex IdE is not fixed by any $f \in G_{2k} \setminus \{Id\}$. In the same way, $G_{2l} = \langle u_{k+1}, v_{k+1}, \cdots, u_{k+l}, v_{k+l}\rangle \subset \phi\mathsf{SL}(2, \mathbf{R})\phi^{-1}$ is a free group over $2l$ generators that fixes the vertex ϕA, and any $g \in G_{2l} \setminus \{Id\}$ does not fix the vertex IdE (which is in the middle of the path of two edges joining the vertices IdA and ϕA).

By construction σ is a surjective group morphism. Take $m \in \Gamma_g \setminus \{1\}$. Up to conjugacy in Γ_g we can assume that m can be written

$$m = f_n g_n \cdots f_1 g_1 \text{ with } f_i \in F_{2k} \setminus \{1\}, g_i \in F_{2l} \setminus \{1\}, n \geq 1.$$

Then it is easy to check that the vertex $\sigma(m)IdE$ is at distance $2n$ from IdE, in particular $\sigma(m) \neq Id$ and σ is injective. $\square$

5.3.2. *Embeddings associated with an HNN-extension.* Now we want to describe an embedding of Γ_g into $\mathrm{Aut}(\mathbf{R}^2)$ based on the presentation of Γ_g as an HNN-extension (here a_g plays the role of the stable letter):

$$\Gamma_g = \langle a_1, b_1, \cdots, a_g, b_g \mid \left(\prod_{i=1}^{i=g-1} [a_i, b_i] \right) b_g = a_g b_g a_g^{-1} \rangle$$

Lemma 5.6. *For all $n \geq 1$ there exists matrices $A_1, B_1, \cdots, A_n, B_n \in \mathsf{SL}(2, \mathbf{R})$ such that, with the notation $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:*

(1) *$\langle A_1, B_1, \cdots, A_n, B_n, P\rangle$ generates a free group over $2n+1$ generators;*
(2) *$Q = \left(\prod_{i=1}^{i=n}[A_i, B_i]\right) P$ is a matrice of trace 2 (in particular Q is conjugate to P);*
(3) *the only triangular matrices in $\langle A_1, B_1, \cdots, A_n, B_n, P\rangle$ are the powers of P.*

Proof. The proof is similar to the one of lemma 5.4. The figure 5 illustrates the case $n = 1$. $\square$

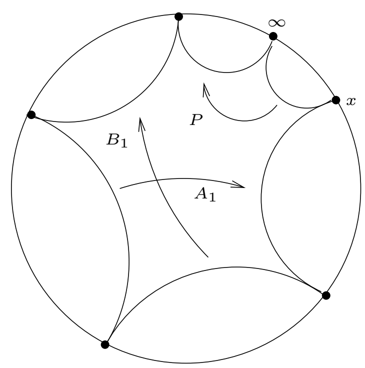

FIGURE 5. The vertex x satisfies $x = [A_1, B_1]P(x)$.

Let us apply this lemma with $n = g - 1$, and note M the matrice (non triangular, because it does not fix the point ∞) such that

$$MPM^{-1} = Q = \left(\prod_{i=1}^{g-1} [A_i, B_i] \right) P.$$

Chose e an element of $E \setminus A$ commuting with P: for instance we can take $e(x, y) = (x + y^2, y)$.

Proposition 5.7. *If we define σ by $\sigma(a_i) = A_i$, $\sigma(b_i) = B_i$ for $i = 1, \cdots, g-1$, $\sigma(b_g) = P$, $\sigma(a_g) = M \circ e$, we get a morphism from Γ_g to $Aut(\mathbf{R}^2)$ that realizes an isomorphism from Γ_g to its image.*

Proof. Since e commutes with P, σ is well defined and is a group morphism. The point is to prove that σ is injective.

Let h be an element of $\Gamma_g \setminus \{1\}$, we want to show that $\sigma(h)$ is distinct from Id. Note $F \subset \Gamma_g$ the free subgroup over $2g-1$ generators generated by $a_1, b_1, \cdots, a_{g-1}, b_{g-1}, b_g$. In restriction to F the morphism σ is injective; so we can assume that h is not in F. Up to conjugacy in Γ_g we can also assume that h admits a decomposition as follows:

$$h = f_n a_g^{l_n} \cdots f_2 a_g^{l_2} f_1 a_g^{l_1} \text{ with } f_i \in F \setminus \{1\}, l_i \in \mathbf{Z} \setminus \{0\} \text{ and } n \geq 1.$$

Finally, we can assume that this is a reduced decomposition, that is:

- If $l_i < 0$ and $l_{i+1} > 0$ then f_i is not a power of b_g;
- If $l_i > 0$ and $l_{i+1} < 0$ then f_i is not a power of $\left(\prod_{i=1}^{i=g-1}[a_i, b_i]\right) b_g$.

Indeed if the decomposition is not reduced we can modify it and observe that the sum of the $|l_i|$ drops. We are going to show that $\sigma(h)$ does not fix the vertex IdA, and this will prove $\sigma(h) \neq Id$. Note $x_0 = IdA$ and for all $i = 1, \cdots, n$

$$x_i = \sigma(f_i)(M \circ e)^{l_i} \cdots \sigma(f_1)(M \circ e)^{l_1}(IdA).$$

The automorphism $M \circ e = \sigma(a_g)$ is an Hénon type automorphism. The associated geodesic in the Bass-Serre tree contains the vertices $e^{-1}A, IdE, IdA, ME, MeA$ in this order, and $M \circ e$ acts on this geodesic as a translation of length 2. Note y_i the vertex in $\mathrm{Geo}(M \circ e)$ that is the closest from x_i, and $d_i \in \mathbf{N}$ the distance between y_i and x_i (*see* figure 6). In particular $y_0 = IdA$ et $d_0 = 0$. We want to show by induction that

- y_{i+1} is always one of the three vertices IdA, MeA or $e^{-1}A$;
- y_{i+1} is the vertex $e^{-1}A$ if and only if l_{i+1} is negative and $\sigma(f_{i+1})$ is in the group $\langle P \rangle$ generated by P;
- y_{i+1} is the vertex MeA if and only if l_{i+1} is positive and $\sigma(f_{i+1})$ is in the group $\langle Q \rangle$;
- $d_{i+1} \geq d_i$.

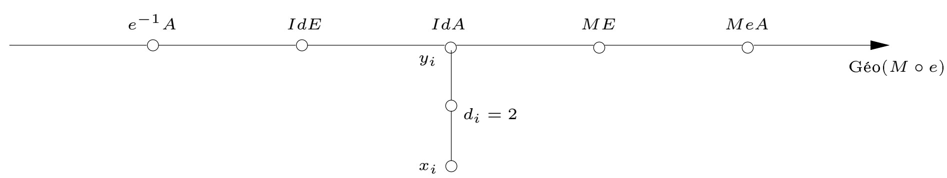

Figure 6. The geodesic of $M \circ e$.

Remark that the powers of P fix exactly three vertices of $\mathrm{Geo}(M \circ e)$: IdA, IdE and $e^{-1}A$ (these three vertices are fixed for P, and the proof of the proposition 3.3 of [24] shows that the tree fixed by P is of diameter 2). In the same way the powers of Q fix the vertices IdA, ME and MeA. Furthermore, by condition (3) of lemma 5.6, any other element of $\sigma(F)$ fixes only one vertex of $\mathrm{Geo}(M \circ e)$: IdA. The observations below follows easily (with the help of figure 6) and prove the properties stated above:

- If $y_i = IdA$ and
  - if $l_{i+1} < 0$ and $\sigma(f_{i+1}) \notin \langle P \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = IdA$.
  - if $l_{i+1} < -1$ and $\sigma(f_{i+1}) \in \langle P \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = e^{-1}A$.
  - if $l_{i+1} = -1$ and $\sigma(f_{i+1}) \in \langle P \rangle$ then $d_{i+1} = d_i$ and $y_{i+1} = e^{-1}A$.
  - if $l_{i+1} > 0$ and $\sigma(f_{i+1}) \notin \langle Q \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = IdA$.
  - if $l_{i+1} > 1$ and $\sigma(f_{i+1}) \in \langle Q \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = MeA$.
  - if $l_{i+1} = 1$ and $\sigma(f_{i+1}) \in \langle Q \rangle$ then $d_{i+1} = d_i$ and $y_{i+1} = MeA$.
- If $y_i = e^{-1}A$ and
  - if $l_{i+1} < 0$ and $\sigma(f_{i+1}) \notin \langle P \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = IdA$.
  - if $l_{i+1} < 0$ and $\sigma(f_{i+1}) \in \langle P \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = e^{-1}A$.
  - if $l_{i+1} > 0$: this case is impossible because the decomposition of h is reduced and because of the induction hypotheses.
- If $y_i = MeA$ and
  - if $l_{i+1} > 0$ and $\sigma(f_{i+1}) \notin \langle Q \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = IdA$.
  - if $l_{i+1} > 0$ and $\sigma(f_{i+1}) \in \langle Q \rangle$ then $d_{i+1} > d_i$ and $y_{i+1} = MeA$.
  - if $l_{i+1} < 0$: this case also is impossible.

By induction we see that if $n > 1$ then $d_n > 0$, thus $x_n = \sigma(h)IdA$ is distinct from IdA and we obtain $\sigma(h) \neq Id$ (the case $n = 1$ is trivial). □

References

[1] Hyman Bass and Alexander Lubotzky. Automorphisms of groups and of schemes of finite type. *Israel J. Math.*, 44(1):1–22, 1983.

[2] Gilbert Baumslag. *Topics in combinatorial group theory*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[3] Gilbert Baumslag and Donald Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68:199–201, 1962.

[4] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.

[5] François Béguin and Frédéric Le Roux. Ensemble oscillant d'un homéomorphisme de Brouwer, homéomorphismes de Reeb. *Bull. Soc. Math. France*, 131(2):149–210, 2003.

[6] Yves Benoist. Pavages du plan. *Texte des Journï£¡s Mathï£¡atiques X-UPS*, pages 1–48, 2001.

[7] Yves Benoist and Karel Dekimpe. The uniqueness of polynomial crystallographic actions. *Math. Ann.*, 322(3):563–571, 2002.

[8] J. S. Birman. Review mr1399602 of the article : Braid groups are linear groups. *Adv. Math.*, 121(1):50–61, 1996.

[9] Serge Cantat. Version kählérienne d'une conjecture de Robert J. Zimmer. *Ann. Sci. École Norm. Sup. (4)*, 37(5):759–768, 2004.

[10] Marc Culler and Karen Vogtmann. A group-theoretic criterion for property FA. *Proc. Amer. Math. Soc.*, 124(3):677–683, 1996.

[11] Edward Warwick Daw. A maximally pathological Brouwer homeomorphism. *Trans. Amer. Math. Soc.*, 343(2):559–573, 1994.

[12] Pierre de la Harpe and Alain Valette. La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger). *Astérisque*, (175):158, 1989.

[13] Karel Dekimpe. Polynomial crystallographic actions on the plane. *Geom. Dedicata*, 93:47–56, 2002.

[14] Karel Dekimpe and Paul Igodt. Polycyclic-by-finite groups admit a bounded-degree polynomial structure. *Invent. Math.*, 129(1):121–140, 1997.

[15] Julie Déserti. Groupe de Cremona et dynamique complexe: une approche de la conjecture de Zimmer. *Int. Math. Res. Not.*, 2006.

[16] Edward Formanek. The irreducible complex representations of the braid group on n strings of degree $\leq n$. *J. Algebra Appl.*, 2(3):317–333, 2003.

[17] Shmuel Friedland and John Milnor. Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems*, 9(1):67–99, 1989.

[18] Étienne Ghys. Actions de réseaux sur le cercle. *Invent. Math.*, 137(1):199–231, 1999.

[19] Erik Guentner, Nigel Higson, and Shmuel Weinberger. The Novikov conjecture for linear groups. *Publ. Math. Inst. Hautes Études Sci.*, (101):243–268, 2005.

[20] Harrie Hendriks and Anant R. Shastri. A splitting theorem for surfaces. In *Topological structures, II (Proc. Sympos. Topology and Geom., Amsterdam, 1978), Part 1*, volume 115 of *Math. Centre Tracts*, pages 117–121. Math. Centrum, Amsterdam, 1979.

[21] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[22] John H. Hubbard and Peter Papadopol. Superattractive fixed points in $\mathbf{C}^n$. *Indiana Univ. Math. J.*, 43(1):321–365, 1994.

[23] Abe Karrass, Alfred Pietrowski, and Donald Solitar. Some remarks on braid groups. In *Contributions to group theory*, volume 33 of *Contemp. Math.*, pages 341–352. Amer. Math. Soc., Providence, RI, 1984.

[24] Stéphane Lamy. L'alternative de Tits pour Aut$[\mathbf{C}^2]$. *J. Algebra*, 239(2):413–437, 2001.

[25] Stéphane Lamy. Une preuve géométrique du théorème de Jung. *Enseign. Math.*, 48(3-4):291–315, 2002.

[26] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[27] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[28] David Mumford, Caroline Series, and David Wright. *Indra's pearls*. Cambridge University Press, New York, 2002. The vision of Felix Klein.

[29] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[30] Jean-Pierre Serre. *Arbres, amalgames,* SL_2. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[31] Yehuda Shalom. Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group. *Ann. of Math. (2)*, 152(1):113–182, 2000.

[32] John Smillie. The entropy of polynomial diffeomorphisms of $\mathbf{C}^2$. *Ergodic Theory Dynam. Systems*, 10(4):823–827, 1990.

[33] John R. Stallings. A topological proof of Grushko's theorem on free products. *Math. Z.*, 90:1–8, 1965.

[34] Yasuo Watatani. Property T of Kazhdan implies property FA of Serre. *Math. Japon.*, 27(1):97–103, 1982.

[35] Heiner Zieschang, Elmar Vogt, and Hans-Dieter Coldewey. *Surfaces and planar discontinuous groups*, volume 835 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Translated from the German by John Stillwell.

IRMAR, UMR 6625 du CNRS, Université de Rennes I, 35042 Rennes, France

ICJ, Université de Lyon 1, 21 av. Claude Bernard, 69622 Villeurbanne cedex, France

E-mail address: `cantat@univ-rennes1.fr`
E-mail address: `lamy@math.univ-lyon1.fr`