Mathematics of Machine Learning—Exercices

M2R Mathématiques Fondamentales et Applications
Aurélien Garivier and Sébastien Gerchinovitz

January 12, 2017

1 Minimax lower bounds

Exercise 1. Let $(E, \mathcal{B})$ be a measurable space.

1. Let P and Q be two probability measures on $(E, \mathcal{B})$. Show that if $P \ll Q$, then $P^{\otimes n} \ll Q^{\otimes n}$ and $\frac{\mathrm{d}\, P^{\otimes n}}{\mathrm{d}\, Q^{\otimes n}}(x_1, \ldots, x_n) = \frac{\mathrm{d}\, P}{\mathrm{d}\, Q}(x_1) \times \cdots \times \frac{\mathrm{d}\, P}{\mathrm{d}\, Q}(x_n)$.
2. Let P and Q be two probability measures on $(E, \mathcal{B})$. Show that, for all $n \in \mathbb{N}^*$,

$$\mathrm{KL}(P^{\otimes n}, Q^{\otimes n}) = n \, \mathrm{KL}(P, Q) \, .$$

Exercise 2. The goal of this exercise is to prove Varshamov-Gilbert's lemma: for all $d \geqslant 6$, there exists a subset $\Gamma \subseteq \{0, 1\}^d$ such that the two following conditions hold true:

$$\forall x \neq y \in \Gamma, \quad \sum_{j=1}^{d} \mathbb{1}_{\{x_j \neq y_j\}} > d/4 \tag{1}$$

$$|\Gamma| \geqslant e^{d/8} \tag{2}$$

In all the sequel, we consider any subset $\Gamma \subseteq \{0, 1\}^d$ such that (1) holds true and with maximal cardinality (i.e., all the other subsets Γ' satisfying (1) are such that $|\Gamma'| \leqslant |\Gamma|$).

1. Explain why such a subset Γ exists.
2. Denote by $\bar{B}(x, r) = \big\{ y \in \{0, 1\}^d : \sum_{j=1}^{d} \mathbb{1}_{\{y_j \neq x_j\}} \leqslant r \big\}$ the closed ball (in Hamming distance) centered at x and with radius $r \geqslant 0$. Prove that $\bigcup_{x \in \Gamma} \bar{B}(x, d/4) = \{0, 1\}^d$.
3. Use the above question to show that

$$1 \leqslant |\Gamma| \cdot 2^{-d} \sum_{k=0}^{\lfloor d/4 \rfloor} \binom{n}{k} \, .$$

4. Let $S \sim \mathcal{B}(d, 1/2)$. Prove that $\mathbb{P}(S \leqslant d/4) \leqslant e^{-d/8}$.
5. Conclude. What is the condition $d \geqslant 6$ for?

Exercise 3 (soon available)**.** Minimax lower bound of order $n^{-4/5}$ in the nonparametric density estimation model $\big\{ f : [0, 1] \to \mathbb{R}_+ : \int_0^1 f(x) \mathrm{d}x = 1 \text{ and } \|f''\|_\infty \leqslant a \big\}$.