Abstract

Let \((M, \partial M)\) be a compact 3-manifold with boundary which admits a complete, convex co-compact hyperbolic metric. For each hyperbolic metric \(g\) on \(M\) such that \(\partial M\) is smooth and strictly convex, the induced metric on \(\partial M\) has curvature \(K > -1\), and each such metric on \(\partial M\) is obtained for a unique choice of \(g\). A dual statement is that, for each \(g\) as above, the third fundamental form of \(\partial M\) has curvature \(K < 1\), and its closed geodesics which are contractible in \(M\) have length \(L > 2\pi\). Conversely, any such metric on \(\partial M\) is obtained for a unique choice of \(g\).

We are interested here in the similar situation where \(\partial M\) is not smooth, but rather looks locally like an ideal polyhedron in \(H^3\). We can give a fairly complete answer to the question on the third fundamental form — which in this case concerns the dihedral angles — and some partial results about the induced metric.

This has some by-products, like an affine piecewise flat structure on the Teichmüller space of a surface with some marked points, or an extension of the Koebe circle packing theorem to a setting where the sphere is replaced by the boundary of a 3-manifold.

Résumé

Soit \((M, \partial M)\) une variété compacte de dimension 3 à bord, qui admet une métrique complète convexe co-compacte. Pour chaque métrique hyperbolique \(g\) sur \(M\) telle que \(\partial M\) est régulier et strictement convexe, la métrique induite sur \(\partial M\) est à courbure \(K > -1\); réciproquement, chaque métrique à courbure \(K > -1\) sur \(\partial M\) est obtenue pour un unique choix de \(g\). Un énoncé dual est que, pour ces métriques \(g\) sur \(M\), la troisième forme fondamentale de \(\partial M\) est à courbure \(K < 1\), et ses géodésiques fermées qui sont contractiles dans \(M\) sont de longueur \(L > 2\pi\); réciproquement, chaque métrique de ce type est obtenue pour un unique choix de \(g\).

Nous nous intéressons au cas similaire où \(\partial M\) n’est pas régulière, mais ressemble au contraire localement à un polyèdre idéal dans \(H^3\). On donne un énoncé assez complet concernant la troisième forme fondamentale — qui dans ce cas se formule en termes d’angles dièdres — et un énoncé partiel pour la métrique induite sur le bord.

Ceci a comme conséquence l’existence d’une structure affine plate par morceaux sur l’espace de Teichmüller d’une surface munie de points marqués, ou une extension du théorème de Koebe sur les empilements de cercles à des situations où la sphère est remplacée par le bord d’une variété de dimension 3.

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Introduction

1.1 Hyperbolic manifolds with convex boundary

We will be motivated by the following results from [Sch02a], which show that the hyperbolic metrics on $M$ such that $\partial M$ is smooth and strictly convex can be well understood from quantities induced on the boundary.
Theorem 1.1. Let $g$ be a hyperbolic metric on $M$ such that $\partial M$ is $C^\infty$ and strictly convex. Then the induced metric $I$ on $\partial M$ has curvature $K > -1$. Each smooth metric on $\partial M$ with $K > -1$ is induced on $\partial M$ for a unique choice of $g$.

Theorem 1.2. Let $g$ be a hyperbolic metric on $M$ such that $\partial M$ is $C^\infty$ and strictly convex. Then the third fundamental form $\text{I}I\text{I}$ of $\partial M$ has curvature $K < 1$, and its closed geodesics which are contractible in $M$ have length $L > 2\pi$. Each such metric is obtained for a unique choice of $g$.

Theorem 1.1 was conjectured by Thurston, and its existence part was proved by Labourie [Lab92]. Note that the hypothesis of Theorem 1.2 is that the metric on $\partial M$, when lifted to the boundary of the universal cover of $M$, is globally CAT(1); this implies a local "curvature" condition, but also a global condition, namely that all closed geodesics have length larger than $2\pi$.

A striking feature of both those statements is that they appear to hold, at least in some cases, without any smoothness assumption (beyond the convexity). In the simple case where $M$ is a ball, both results can be stated in terms of convex surfaces in $H^3$:

Theorem 1.3 (Pogorelov [Pog73]). Let $h$ be a smooth metric with curvature $K > -1$ on $S^2$. Then $(S^2, h)$ has a unique isometric embedding into $H^3$.

Theorem 1.4 ([Sch96, Sch94]). Let $h$ be a smooth metric with curvature $K < 1$ on $S^2$. Then $(S^2, h)$ has an embedding in $H^3$ such that the third fundamental form of the image is $h$ if and only if all closed geodesics of $(S^2, h)$ have length $L > 2\pi$. This embedding is then unique.

Theorem 1.4 was conjectured in [Riv86, RH93]. Both those statements also hold without any smoothness assumption; it is a result of [Pog73] for Theorem 1.3, and has been announced recently by G. Moussong for Theorem 1.4 (see [DM99]). In particular, the polyhedral case corresponds to earlier results of Aleksandrov [Ale58] for Theorem 1.3, and to results of Andreev [And70], Rivin [Riv86] and Rivin and Hodgson [RH93] for Theorem 1.4.

It also appears that Theorems 1.1 and 1.2 hold, at least in some cases, for convex surfaces which are not complete. This is again most apparent when $M$ is a ball. In particular:

- direct analogs of Theorems 1.1 and 1.2 hold for smooth, non-compact convex surfaces in $H^3$ [Sch98a].
- the analog of Theorem 1.1 is true for ideal polyhedra, a result of Rivin [Riv92].
- the analog of Theorem 1.2 also holds for ideal polyhedra, a result of Andreev [And71] and Rivin [Riv96]. Thurston [Thu97] noted that those results are also strongly related to questions on circle packings, in particular the Koebe theorem.
- the analog of Theorem 1.1 holds for hyperideal polyhedra in $H^3$, this is a special case of a result of [Sch98a]1.
- Theorem 1.2 also holds for hyperideal polyhedra, see [Sch98a, BB02]

When $M$ is topologically more complicated than a ball, known results are limited to the "fuchsian" case, i.e. when $M$ is the product of a surface of genus at least 2 by an interval, with an isometric involution. In this case, a version of Theorem 1.2, concerning fuchsian manifolds whose boundary is locally like an ideal polyhedron, can be found hidden behind work of Thurston [Thu97, chapter 13], Colin de Verdière [CdV91] and more recently Bobenko and Springborn [BS04] on circle packings on hyperbolic surfaces; the relationship (which is well known) should be clear from section 3 below.

The main goal of this paper is to extend this result to the situation where $M$ is topologically "general" — but its boundary will always be supposed to be locally like an ideal polyhedron, in a sense which is precisely defined in section 3. In this case, we will call $M$ an ideal hyperbolic manifold.

We need the following definition. A cellulation of $\partial M$ is a decomposition of $\partial M$ into the images by diffeomorphisms of convex polygons in $\mathbb{R}^2$. We will call the images of the polygons the 2-cells of the cellulations, and the 1-cells will be the images of the edges of the polygons. We demand that the cellulations are well behaved in the sense that:

- the 2-cells have disjoint interiors.

1It was pointed out by Igor Rivin that he also proved this result, although he never wrote or published it.
two 1-cells which are distinct have disjoint interiors.

• the intersection of two 2-cells is a disjoint union of 1-cells and vertices.

• at least 3 images of polygons meet at the image of each vertex.

**Definition 1.5.** Let $\Gamma$ be the 1-skeleton of a cellulation of $\partial M$. A **circuit** in $\Gamma$ is a sequence $e_0, e_1, \ldots, e_n = e_0$ of edges of $\gamma$ such that the dual edges $e_0^*, e_1^*, \ldots, e_n^*$ are the successive edges of a closed path in the dual graph $\Gamma^*$, which is contractible in $M$. A circuit is **elementary** if the dual path bounds a face (i.e. containing no edge or vertex).

We can now describe the dihedral angles of ideal hyperbolic manifolds; it extends the results of [And71, Riv96] on ideal polyhedra, as well as results of Rivin [Riv03] and Bobenko and Springborn [BS04] in other contexts.

**Theorem 8.17.** Suppose that $M$ has incompressible boundary. Let $\Gamma$ be the 1-skeleton of a cellulation of $\partial M$. Let $w$ be a function from the edges of $\Gamma$ to $(0, \pi)$ such that:

1. for each elementary circuit in $\Gamma$, the sum of the values of $w$ is $2\pi$;

2. for each non-elementary circuit in $\Gamma$, the sum of the values of $w$ is strictly larger than $2\pi$.

Then there a unique hyperbolic metric $g$ on $M$, such that $(M, g)$ is an ideal hyperbolic manifold, with exterior dihedral angles given by $w$.

The hypothesis that $M$ has incompressible boundary is necessary for technical reasons (see section 8). Theorem 1.1 would seem to indicate that the result also holds without this hypothesis. Theorem 8.17 was actually proved before Theorem 1.2, and was an important motivation for the proof of Theorem 1.2, since it gave a strong indication that some analog of Theorem 1.1 (whose existence part was already known from [Lab92]) should hold with the metric replaced by the third fundamental form. The introduction of the current paper was then rewritten to mention Theorem 1.2, since it helps motivate Theorem 8.17. The proof used in [Sch02a], however, is completely different from the methods that we use here, and technically much more cumbersome.

Note that the hypothesis of this theorem on the function $w$, implies topological properties on $\Gamma$. In particular:

**Lemma 1.6.** Let $\sigma$ be a cellulation of $\partial M$, such that there is a function $w$ on the edges of $\sigma$, satisfying the hypothesis of theorem 8.17. Then:

1. the lift of each 1-cell to the universal cover of $\partial M$ is a disjoint union of segments.

2. the lift of each 2-cell to the universal cover of $\partial M$ is a disjoint union of disks.

3. the intersection of two 2-cells is either a vertex or an edge.

The proof is in section 3, along with some related properties.

### 1.2 Circle packings and circle patterns

Theorem 8.17 has some consequences in terms of circle packing and circle patterns on the boundary of 3-manifolds. Remember that a **circle packing** on the sphere $S^2$ is a set of closed disks with disjoint interiors in $S^2$. Given a circle packing, one can define its **incidence graph** as a graph on $S^2$ which has one vertex for each disk, and an edge between two vertices if and only if the corresponding disk are tangent.

The classical Koebe circle packing theorem states that, for each graph $\gamma$ in $S^2$ which is the 1-skeleton of a triangulation, there exists a unique circle packing whose incidence graph is $\Gamma$. It was proved by Koebe [Koe36] for triangulations, and extended by Thurston [Thu97] using the Andreev theorem on ideal polyhedra [And71]; in this more general case one should demand that, for any connected component of the complement of the disks, there is a circle which is orthogonal to all the neighboring disks.

Thurston also extended this theorem to circle packings on hyperbolic surfaces. Theorem 8.17 provides a further extension to the boundary of our 3-manifold $M$, when it is provided with the $CP^1$-structure on the boundary coming from a hyperbolic metric on $M$.

**Theorem 10.2.** Suppose that $M$ has incompressible boundary. Let $\Gamma$ be the 1-skeleton of a triangulation of $\partial M$. There is a unique couple $(g, c)$, where $g$ is a complete, convex co-compact hyperbolic metric on $M$, and $c$ is a circle packing on $\partial M$ (for the $CP^1$-structure defined on $\partial M$ by $g$) whose incidence graph is $\Gamma$. 
If one considers not only triangulations but more generally cellulations, the same result holds, with the additional condition that, for each connected component of the complement of the closed disks, there exists a circle which is orthogonal to all the adjacent circles — this condition is automatically satisfied for a triangulation.

The proof is given in section 10, with some details on more general statements on circle patterns — where the circles are not necessarily tangent but can intersect, with prescribed angles.

1.3 Induced metrics on the boundary

It would be interesting to know whether Theorem 1.1 extends to ideal hyperbolic manifolds. Unfortunately I do not know the answer to this question; but an infinitesimal version does hold:

Lemma 1.7. For each ideal hyperbolic manifold \( M \), each infinitesimal variation of the hyperbolic structure on \( M \) (among ideal hyperbolic manifolds) induces a non-trivial infinitesimal variation of the induced metric on \( \partial M \).

This assertion is worth mentioning because, in other questions related to Theorem 1.1, it is precisely this infinitesimal rigidity which is lacking (see section 1). It implies (by a simple dimension-counting argument) that, given an ideal hyperbolic manifold \( M \) and a small deformation of the induced metric on its boundary, there is a unique corresponding small deformation of \( M \).

However, in this case it is not sufficient to obtain a global result concerning the induced metrics on the boundary of ideal hyperbolic manifolds, like in Theorem 8.17 for the third fundamental form. The reason is that, to understand the induced metrics completely, one would have to go beyond the category of ideal hyperbolic manifolds, and obtain an infinitesimal rigidity result also for "bent" manifolds (as defined in section 3). We give more details on this in section 11.

We will also give below a more precise result in the special case of manifolds which we call "fuchsian"; they are ideal hyperbolic manifolds such that \( \partial M \) has two connected component, with an isometric involution exchanging the connected components of \( \partial M \).

1.4 Fuchsian polyhedra

In the course of the proof of Theorem 8.17, we are led to study fuchsian equivariant polyhedra; they are the objects arising as the universal covers of the boundaries of the fuchsian manifolds mentioned above. As a consequence of Theorem 8.17, we find a characterization of the dihedral angles of fuchsian ideal polyhedra. These results are strongly related to results of [Riv03] and to statements on circle patterns in hyperbolic surfaces, obtained independently by Bobenko and Springborn in a recent paper [BS04]. In addition, we will give in section 4 results concerning other fuchsian polyhedra, having some non-ideal vertices.

Definition 1.8. A fuchsian polyhedron is a triple \((S, \phi, \rho)\), where:

- \( S \) is a surface of genus \( g \geq 2 \), with \( N \) marked points \( x_1, x_2, \ldots, x_N \), \( N \geq 1 \).
- \( \phi \) is a polyhedral map from the universal cover \( \tilde{S} \) of \( S \) into \( H^3 \cup \partial_{\infty} H^3 \) such that the image is locally like a polyhedron in \( H^3 \), with vertices the images by \( \phi \) of the inverse images in \( \tilde{S} \) of the \( x_i \).
- \( \rho \) is a homomorphism from \( \pi_1 S \) into the subgroup of the isometry group of \( H^3 \) of isometries fixing a totally geodesic 2-plane \( P_0 \).
- For any \( x \in \tilde{S} \) and any \( \gamma \in \pi_1 S \), \( \phi(\gamma x) = \rho(\gamma)\phi(x) \).

Some special cases are of interest. Let \( I \) be the set of points in \( \tilde{S} \) which are sent by the canonical projection \( \tilde{S} \to S \) to a marked point. We say that \((S, \phi, \rho)\) is:

- ideal, if each point in \( I \) is sent to an ideal vertex.
- finite, if no point in \( I \) is sent to an ideal vertex.
- semi-ideal, if the points in \( I \) can be sent to either ideal or non-ideal vertices (this includes the finite and the ideal cases).
We obtain an existence result for the third fundamental forms of fuchsian polyhedra (compare with [Riv94, BS04]).

**Theorem 4.25.** Let $\Sigma$ be a surface of genus $g \geq 2$, and let $h$ be a spherical cone-metric on $\Sigma$, with negative singular curvature at the singular points. Suppose that all contractible closed geodesics of $(\Sigma, h)$ have length $L > 2\pi$, except when they bound a hemisphere. Then there is a unique fuchsian polyhedral embedding of $(\Sigma, h)$ into $H^3$ whose third fundamental form is $h$.

As a consequence of the analysis of the dihedral angles of ideal hyperbolic manifolds in the fuchsian case, we will obtain the following.

**Theorem 8.21.** For each $g \geq 2$ and each $N \geq 1$, there is a natural unimodular piecewise affine structure $A_{g,N}$ on the Teichmüller $T_{g,N}$ space of the genus $g$ surface with $N$ marked points.

This affine structure has some singularities, but also a number of interesting properties; for instance there is a natural function on $T_{g,N}$, which is defined by taking the volume of a fuchsian ideal hyperbolic manifold, and it is concave on the maximal dimension cells of $A_{g,N}$. There are also some questions which remain open concerning $A_{g,N}$.

Finally, in the case of fuchsian polyhedra, the infinitesimal rigidity result for the induced metrics (Lemma 1.7) happens to be sufficient to obtain a satisfactory global result on the induced metrics.

**Theorem 9.8.** Let $S$ be a surface of genus $g \geq 2$, and let $N \geq 1$. For each complete, finite area hyperbolic metric $h$ on $S$ with $N$ cusps, there is a unique ideal fuchsian hyperbolic manifold $M$ such that the induced metric on each component of the boundary is $h$.

The proof is in section 9.

### 1.5 Some hints on the techniques

The main idea behind the results obtained here is already a few years old: it is the use of the striking properties of the volume function on the space of hyperbolic structures. The first property is that the volume is a concave function when the space of hyperbolic structures is parametrized by the dihedral angles (this comes from Lemma 2.5 concerning the ideal simplex) and the second is the Schl"afli formula, which provides a fundamental link between the dihedral angles and the induced metric on ideal polyhedra, or on hyperbolic manifolds with ideal-like boundary.

Those ideas can be traced back to the treatment of the Andreev theorem in [Thur97]; Thurston gave a link between ideal polyhedra and circle packings and gave a "gradient-like" proof of circle packings results. The existence of a "generating function" for the problem was then pointed out in [CdV91], and it was discovered that this function was basically the hyperbolic volume in [Bea92] and independently in [Riv94].

The technique used here to prove the infinitesimal rigidity of ideal hyperbolic manifolds with respect to their boundary dihedral angles mostly follows the ideas of Rivin [Riv94]. Following this path for hyperbolic manifolds with ideal-like boundary, however, leads to some technical difficulties. The first is that it is not clear how one can find an ideal triangulation of the manifold; by the way, a similar difficulty appears in other situations where one wants to deform hyperbolic structures, see e.g. [PP00]. This is treated in section 5, where it is shown that some finite cover of the manifolds considered have an ideal triangulation. All the arguments can then be given in this finite cover, and an equivariance argument brings the result down to the manifolds we want to study.

Another technical point is to prove that the set of possible dihedral angle assignments (as it appears in Theorem 8.17) is connected; this is necessary because the proofs rely on a deformation argument. The solution chosen here uses the fact that the conditions are "independent" on each of the connected components of the boundary. Moreover, if $M$ has incompressible boundary, then the conditions on each connected component of $\partial M$ is the same as for the corresponding fuchsian ideal polyhedra; so that the connectedness in the general case follows from the understanding of the ideal fuchsian polyhedra. So those fuchsian polyhedra are studied in section 4. The methods used there are quite different from those of the other sections; an existence and uniqueness result is proved for the third fundamental forms of fuchsian manifolds whose boundary locally looks like a compact polyhedron in $H^3$, and from there one deduces an existence result for manifolds with ideal-like boundary by an approximation argument.

Although some other technically interesting details appear at different points in the paper, it does not seem necessary to describe them here — the reader will probably enjoy finding them as he reads along.
Even in the case of ideal polyhedra in $H^3$, I think that the approach used here is different, and I believe slightly more direct, than the one used in previous papers — although no new result is achieved. Moreover, I hope that the methods used — as mentioned above, mostly developed earlier to study ideal polyhedra — could hint at some approaches to the questions concerning the convex cores of complete, convex co-compact hyperbolic manifolds. Of course many difficulties remain in this direction, although recent works of Bonahon [Bon98a, Bon98b] might provide useful tools.

Note that some of the results presented here — notably Theorem 8.17 — have a non-empty intersection with some of the results obtained by Rivin [Riv94, Riv96, Riv03], by Leibon [Lei02a, Lei02b] and by Bobenko and Springborn [BS04]. The methods used here are similar to those of [Riv94]; their scope, however, are quite different, and the more general context considered here leads to a number of new technical difficulties.

1.6 Outline of the paper

Section 2 contains a reminder of the classical Schlöfli formula, as well as some remarks and interpretations in terms of symplectic geometry. Section 3 then gives some basic definitions and results on the geometry of ideal hyperbolic manifolds.

The special case of fuchsian manifolds is described in section 4, using methods that are much closer to those used in the classical theory of hyperbolic polyhedra (as developed in particular by Aleksandrov [Ale58]), and some methods from [LS00]. This special case is necessary for the proof of the general case, since a topological argument used in section 8 use it in an important way.

Section 5 deals with a technical issue on triangulations of ideal hyperbolic manifolds; the point is that, although I do not know how to construct a well-behaved triangulation of such a manifold, it is not too difficult to construct one on a finite cover, and this will be sufficient for this paper. This is done using ideas of Epstein and Penner [EP88] for complete hyperbolic manifolds of finite volume.

The geometrical constructions start in section 6, where more general hyperbolic structures on triangulated manifolds are investigated. The idea — which basically comes from earlier works, see [Thu07, CdV91, Bär92, Riv96] and in particular [Riv94] — is to use variational properties of the volume functional on a larger class of hyperbolic structures to obtain hyperbolic metrics. This is continued in section 7, where first-order properties of the volume are described. There a key infinitesimal rigidity statement is proved: given an ideal hyperbolic manifold, any first-order deformation of its dihedral angles is induced by a unique first-order deformation of the hyperbolic metric.

It is then possible, in the first part of section 8, to understand globally the space of ideal hyperbolic manifolds with a given boundary combinatorics: we prove first that the map sending such an ideal manifold to the dihedral angles on its edges is a covering. Then an argument based on the possible changes in the topology shows that, when one goes from a given combinatorics to an “adjacent” one — one obtained by deleting an edge and adding another — the number of inverse images of a point remains the same. But it follows from the results on the fuchsian situation — because $M$ is supposed to have incompressible boundary — that the space of possible dihedral angles is connected, so that the number of inverse images of a set of dihedral angles is actually constant. It is then possible to show that this constant is $1$, essentially by using some special examples which can be understood using the Mostow rigidity theorem.

We then move in section 9 to results on induced metrics on the boundary. Some applications to circles packings and circle patterns are mentioned in section 10, and other considerations stand in the last section.

2 The Schlöfli formula

We recall in this section the classical Schlöfli formula. It is the main tool used in the sequel, so we also describe some interesting interpretations of it and some related properties of the volume of simplices.

We first state the Schlöfli formula for compact polyhedra; for a proof, see e.g. [Mil94] or [Vin93].

**Lemma 2.1.** Let $(P_t)$ be a one-parameter family of convex polyhedra in $H^3$. Let $I$ be the set of its edges, $(L_i)_{i \in I}$ be its edge lengths, and $(\theta_i)_{i \in I}$ the corresponding (interior) dihedral angles. Then:

$$dV = -\frac{1}{2} \sum_i L_i d\theta_i.$$  

(1)
2.1 A symplectic viewpoint

There is an amusing symplectic interpretation of this formula. To explain it simply we choose an abstract polyhedron $P_b$, with $v$ vertices, $e$ edges and $f$ 2-faces, and consider only polyhedra with the same combinatorics. Call $\mathcal{L} := \mathbb{R}^e$ and $\Theta := (0, \pi)^v$ the sets containing the possible lengths and exterior dihedral angles, respectively, of hyperbolic polyhedra having the same combinatorial type as $P_b$. Now consider the symplectic vector space $(\mathcal{L} \times \Theta, \omega)$, with $\omega := \sum_i dL_i \wedge d\theta_i$. Let $\mathcal{P}$ be the subset corresponding to the edge lengths and dihedral angles of convex polyhedra of the same combinatorial type as $P_b$. It is well known, and not difficult to prove, that:

**Proposition 2.2.** $\mathcal{P}$ is a submanifold (with boundary) of $\mathcal{L} \times \Theta$ of dimension $e$.

We leave the proof to the reader, since this statement plays no role in the sequel; it uses the Euler formula and the fact that the number of constraints on the positions of the vertices is the number of non-triangular faces, counted with a multiplicity equal to the number of their vertices minus 3.

The Schlafli formula is essentially equivalent to the following:

**Corollary to the Schlafli formula.** $\mathcal{P}$ is Lagrangian in $(\mathcal{L} \times \Theta, \omega)$.

**Proof.** Define the 1-form:

$$\beta := \sum_i L_i d\theta_i .$$

Then $d\beta = \omega$. Since $\beta|_P = dV|_P$, $d\beta|_P = 0$ so that $\omega$ vanishes on $\mathcal{P}$. \hfill \Box

2.2 Ideal polyhedra

Now let $\mathcal{P}_\infty$ be the set of ideal polyhedra having the combinatorial type of $P_b$; they are obtained by letting the vertices go to infinity – this is possible for many combinatorial types. A basic remark is that those polyhedra still have finite volume. Their edge lengths, however, are not defined as such (they are infinite). To define an analog of the edge lengths for ideal polyhedra (following an idea of Milnor), we choose for each vertex $V$ a horosphere $H_V$ “centered” on $V$ that is, a level set for the Busemann function associated to $V$. We then define the length of the edge joining 2 vertices $V$ and $W$ as the distance, along the edge, between $H_V$ and $H_W$; we use the signed length, so that the length is negative if the horospheres overlap. The set of edge lengths of the elements of $\mathcal{P}_\infty$ is thus defined up to the addition of a constant for each vertex, and is contained in the set $\mathcal{L}_\infty := \mathbb{R}^e/\mathbb{R}^v$.

On the other hand, the dihedral angles of the ideal polyhedra are constrained. That is because the link of each vertex is a Euclidean polygon, so the sum of its exterior angles is $2\pi$, so that the sum of the exterior dihedral angles of the edges containing a given vertex of an ideal polyhedron is always $2\pi$. Therefore, the set of dihedral angles of the polyhedra in $\mathcal{P}_\infty$ stays in a $(e - v)$-dimensional space $\Theta_\infty$.

The amusing fact that we announced is then:

**Remark 2.3.**

1. $\mathcal{L}_\infty \times \Theta_\infty$ is obtained from $\mathcal{L} \times \Theta$ by symplectic reduction, and it therefore still carries a symplectic form $\omega_\infty$;

2. the Schlafli formula (1) still has a meaning, and still holds, for $\mathcal{P}_\infty$;

3. it still implies that $\mathcal{P}_\infty$ is a Lagrangian submanifold of $(\mathcal{L}_\infty \times \Theta_\infty, \omega_\infty)$.

**Proof.** The second point was originally proved by Milnor (following previous work going back to Lobachevsky), the proof can be found in [Riv94]. For the first point, consider the group $G := \mathbb{R}^v$, and its action $\phi$ on $\mathcal{L} \times \Theta$ by:

$$\phi: \quad G = \mathbb{R}^v \times (\mathcal{L} \times \Theta) \quad \rightarrow \quad \mathcal{L} \times \Theta$$

$$((\alpha_j)_{j=1,\ldots,v}, (L_i, \theta_i)_{i=1,\ldots,e}) \quad \mapsto \quad (L_i + \alpha_{i_+} + \alpha_{i_-}, \theta_i)_{i=1,\ldots,e}$$

where $i_+$ and $i_-$ are the two ends of the edge $i$.

It is quite easy to check that $\phi$ has a moment map $\mu$ defined by:

$$\mu: \quad \mathcal{L} \times \Theta \quad \rightarrow \quad \mathcal{G}^\ast = \mathbb{R}^v$$

$$(L_i, \theta_i) \quad \mapsto \quad \left( \sum_{k \in E_i} \theta_k \right)_{j=1,\ldots,v}$$
where $E_j$ is the set of edges containing a vertex $j$.

Then $L_\infty \times \Theta_\infty \simeq \mu^{-1}(2\pi, \ldots, 2\pi)/G$, where $G$ acts by $\phi$, so that $L_\infty \times \Theta_\infty$ is obtained by symplectic reduction from $L \times \Theta$ as announced, with a symplectic form $\omega_\infty$.

For the last point note that $\beta$ determines a well-defined 1-form $\beta_\infty$ on $L_\infty \times \Theta_\infty$, so again $\omega_\infty = d\beta_\infty$. But the Schl"afli formula for ideal polyhedra shows that $\beta_\infty$ vanishes on $P_\infty$, which is thus Lagrangian. 

\[ \square \]

2.3 The volume function

We will also need some elementary and well-known properties of the volume of hyperbolic simplices, which we recall here for the reader’s convenience. More details can be found for instance in [Thu97], chapter 7.

**Definition 2.4.** The Lobachevsky function is defined as:

$$\Lambda(\theta) := -\int_0^\theta \log |2 \sin u| du.$$ 

Now recall that there is a 2-parameter family of ideal simplices in $H^3$ (up to global isometries), which can be parametrized for instance by the complex cross-product of the four vertices in $\partial_\infty H^3 \simeq \mathbb{C}P^1$. For each ideal simplex, the dihedral angles of two opposite edges are equal, and the sum of the exterior dihedral angles of the edges containing a given vertex is $2\pi$. An ideal simplex is completely determined – again up to global isometry – by its three interior dihedral angles $\alpha, \beta$ and $\gamma$, under the condition that their sum is $\pi$.

The volume of an ideal simplex is given by a simple formula, discovered by Milnor (see [Thu97], chapter 7):

**Lemma 2.5.** The volume of the ideal simplex with dihedral angles $\alpha, \beta$ and $\gamma$ is $\Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$.

As a consequence, one finds (see [Riv94]):

**Corollary 2.6.** $V$ is a concave function of the dihedral angles $\alpha, \beta, \gamma$, which vanishes when one of the angles goes to 0.

This is proved by an elementary computation of the Hessian of $V$.

Note that the Schl"afli formula is not restricted to hyperbolic polyhedra; it is also valid in the other Riemannian or pseudo-Riemannian space-forms (see [SP00]) and also has an interesting extension to the setting of Einstein manifolds with boundary, see [RS00, RS99, Ber03]. I do not know whether any analog of the methods described in this paper – where the volume is used as a "generating function" to prove geometric results – can be found in this more general context.

There is another possible proof of Corollary 2.6, which uses the Schl"afli formula instead of explicit computations of the volume in terms of the Lobachevsky function. It is also more general. The key point is that, according to the Schl"afli formula, the infinitesimal deformations of an ideal simplex which do not change the lengths of the edges (i.e. the isometric deformations) are exactly the deformations which are in the kernel of the Hessian of the volume, seen as a function of the dihedral angles. Now it is easy to see — and well-known — that ideal simplices are rigid, i.e. they have no non-trivial infinitesimal isometric deformation. Therefore, the Hessian of the volume has constant signature. But an elementary argument (with the Schl"afli formula) shows that the regular simplex has maximal volume, so that $V$ is a concave function.

This line of reasoning can of course be used in different contexts; for instance, it shows that the signature of the Hessian of the volume, as a function of the dihedral angles, has constant signature on the space of compact hyperbolic simplices, since they are also known to be rigid.

3 Convex hulls of ideal points

The goal of this paper is to understand hyperbolic manifolds whose boundaries look locally like ideal polyhedra. We will introduce a class of manifolds with boundary obtained by taking the convex hull of a finite number of ideal points in a complete, convex co-compact hyperbolic manifold. The main point is that the class of those manifolds with boundary separates into two sub-class, according to whether their "convex core" intersects their boundary or not. We will then indicate why both cases actually exist in a non-trivial way, since it is not completely obvious at first sight.
3.1 Some definitions

We first define a notion of convexity — it is classical but other definitions are sometimes used.

**Definition 3.1.** Let $M$ be a hyperbolic manifold. We say that a subset $C \subset M$ is convex if any geodesic segment in $M$ with endpoints in $C$ is contained in $C$. For any subset $E$ of $M \cup \partial_{\infty} M$, the convex hull of $E$ is the smallest non-empty convex set containing $E$.

**Definition 3.2.** $M$ is a hyperbolic manifold with horns if there is a complete convex co-compact hyperbolic manifold $N$ and a finite family $x_1, \ldots, x_p$ of points in $\partial_{\infty} N$ such that $M$ is isometric to the convex hull in $N$ of $\{x_1, \ldots, x_p\}$.

The interior of an ideal polyhedron in $H^3$ satisfies this definition, except that $H^3$ is not convex co-compact. The convex core of a convex co-compact hyperbolic manifold is a hyperbolic manifold with horns, in this case $p = 0$.

**Remark 3.3.** Let $M$ be a compact hyperbolic manifold with convex boundary, such that the induced metric on the boundary is a complete hyperbolic metric of finite area on the boundary minus a finite number of points. Then $M$ is a hyperbolic manifold with horns.

*Proof.* Since $M$ is compact with convex boundary it is isometric to a subset of a complete convex co-compact hyperbolic manifold $N$. Let $\tilde{M}$ be the universal cover of $M$, then $\partial \tilde{M}$ is a convex surface in $H^3$ and its induced metric is hyperbolic; it is therefore the convex hull in $H^3$ of its boundary points in $\partial_{\infty} H^3$. Taking the quotient by $\pi_1 M$, we see that $\partial M$ is the convex hull of its boundary points in $\partial_{\infty} M$. Those points clearly correspond to the cusps of $\partial \tilde{M}$, so $M$ is the convex hull in $H^3/\pi_1 M$ of a finite number of ideal points. □

Consider a hyperbolic metric on $M$ for which $\partial M$ is convex. There is then a unique convex co-compact hyperbolic 3-manifold $N$ in which $M$ admits an isometric embedding which is surjective on the $\pi_1$; we call it the extension of $M$, and denote it by $E(M)$. The convex core of $E(M)$ can be defined as the smallest convex subset of $E(M)$, so it is contained in $M$; we will denote it by $C(M)$.

**Definition 3.4.** Let $M$ be a hyperbolic manifold with horns. $M$ is an ideal hyperbolic manifold if $C(M) \cap \partial M = \emptyset$, otherwise $M$ is a bent hyperbolic manifold.

For instance, the interiors of ideal polyhedra in $H^3$ are of the ideal kind, while the convex cores of convex co-compact manifolds are of the bent type. Most of what follows concerns ideal hyperbolic manifolds only, while the bent case appears as a problem lurking in the background. The relationship between the two will be explored after the next subsection, using the tools that it contains.

3.2 Relation with circle packings

To understand the behavior of the boundary of a hyperbolic manifold with horns $M$, it is relevant to consider its universal cover $\tilde{M}$, which can naturally be identified with a convex subset of $H^3$. The vertices of $\tilde{M}$ correspond to ideal points in $H^3$, i.e. to points in $\partial_{\infty} H^3$. The combinatorics of the faces of $\partial \tilde{M}$ is then described in terms of Delaunay cells of the sphere.

**Definition 3.5.** Let $\sigma$ be a cellulation of $S^2$. $\sigma$ is Delaunay if, for each cell $c$ of $\sigma$:

1. the vertices of $c$ are co-cyclic, i.e. lie on a circle $C(c)$ in $S^2$;
2. one of the closed disks bounded by $C(c)$ contains no other vertex of $\sigma$.

Note that this definition involves only the vertices of $\sigma$, not its edges. It depends on the Möbius structure of $S^2$, but not on its metric structure.

**Lemma 3.6.** Let $E$ be a discrete subset of $S^2$. There is a unique maximal Delaunay cellulation of a subset of $S^2$ whose vertices are the elements of $E$. It combinatorics is that of the convex hull of $E$ in $\mathbb{R}^3$.

The proof is classical, see e.g. [GW93]; the circles appearing in the definition of a Delaunay cellulation are the boundaries at infinity of support planes of the convex hull of $E$.

Let $M$ be a hyperbolic manifold with horns. By definition, $M$ is the convex hull of a finite set of ideal points $S = \{x_1, \ldots, x_p\} \subset \partial_{\infty} E(M)$. Consider the universal cover $E(M) = H^3$ of $E(M)$; $S$ lifts to a set of points $\tilde{S}$.
which is invariant under the action of $\pi_1(M)$, so that the accumulation set of $\tilde{S}$ is the limit set $\Lambda \subset S^2$ of the action of $\pi_1M$ on $H^3$.

Consider a cellulation $\sigma$ of a subset $\Omega$ of $S^2$; we consider it as of a partly geometric and partly combinatorial nature, with fixed vertices in $S^2$ but edges defined up to isotopy.

The following statement is a consequence of Lemma 3.6.

**Proposition 3.7.** There is a unique Delaunay cellulation of $S^2 \setminus \Lambda$ with vertices the elements of $\tilde{S}$; it is obtained as the combinatorial structure of the convex hulls of $\tilde{S}$ in $H^3$.

The circles appearing in the definition of a Delaunay cellulation are simply the traces on $\partial_{\infty}H^3$ of the 2-planes which are the faces of the convex hull of $\tilde{S}$. Moreover, the dihedral angles between those faces are the angles between the corresponding circles in $S^2 = \partial_{\infty}H^3$. Thus, dihedral angle questions on ideal hyperbolic manifolds can be translated as questions on circles patterns in $S^2$ having given angles (and invariant under group actions). A precise relationship with circle packings can be obtained using an idea of Thurston; this is recalled in section 10.

We can now show that ideal hyperbolic manifolds have a simple description in terms of the properties of their boundary. Of course this description does not apply to bent hyperbolic manifolds.

**Property 3.8.** Let $M$ be an ideal hyperbolic manifold. Then its boundary $\partial M$ is the union of a finite number of 2-faces, which are ideal polygons with a finite number of edges in totally geodesic 2-planes, and which intersect along geodesics.

**Proof.** Since $E(M)$ is convex co-compact, its convex core $C(M)$ is compact. By definition, $\partial M$ does not intersect $C(M)$, so $d(\partial M, C(M)) > 0$. Therefore, each face of $\partial M$ remains at a positive distance from $C(M)$. Thus each face of $\partial M$ remains in a compact subset of the complement of the limit set $\Lambda$ of the action of $\pi_1M$ on $H^3$. Therefore, the circles in $S^2 = \partial_{\infty}H^3$ which are the boundary at infinity of those planes are not tangent to $\Lambda$.

Let $\Omega$ be a fundamental domain with a compact closure in $S^2 \setminus \Lambda$ for the action of $\pi_1M$. $\Omega$ is contained in a compact subset $K \subset S^2 \setminus \Lambda$, so it intersects a finite number of those circles; therefore, $\partial M$ has a finite number of faces. The proof of the property follows. \qed

### 3.3 Existence of the bent case

Let $N$ be a complete, convex co-compact hyperbolic 3-manifold. Let $x_1, \ldots, x_p$ be a family of points in $\partial_{\infty}N$. If one of the connected component $\partial_0N$ of $\partial_{\infty}N$ does not contain any of the $x_i$, it is not difficult to see that the smallest convex subset of $N$ containing the $x_i$ will be of the bent type; indeed, its boundary will contain the component of the boundary of the convex core of $N$ facing $\partial_0N$. We will sketch here an argument intended to convince the reader that there are other situations where bent manifolds appear.

A first remark is that a hyperbolic manifold with horns is bent if and only if, among the circles in $S^2 \setminus \Lambda$ which are the boundary at infinity of its faces, one has non-empty intersection with $\Lambda$. Indeed it was already proved in Property 3.8 that this does not happen for an ideal hyperbolic manifold, while the converse is clear since, for such a circle, there should be a face of $\partial M$ which is at distance 0 from $C(M)$.

Thus, to show the existence of a bent hyperbolic manifold, it is simplest to search for one among hyperbolic manifolds with horns with only one ideal point in each boundary component (those manifolds could legitimately be called "unicorn manifolds"). Moreover we can restrict our attention to e.g. the quasi-fuchsian case — we will see below that fuchsian hyperbolic manifolds can not be bent.

Fix one of the boundary components of $M$, say $\partial_1M$, and let $x_1 \in \partial_{\infty}M$ be the corresponding ideal point. One of the faces of $\partial_1M$ has non-empty intersection with $\Lambda$ if and only if there is a circle $C_0$ in $S^2$, whose interior is in $S^2 \setminus \Lambda$, which intersects $\Lambda$, and whose interior contains no point of the orbit $(\pi_1M).x_1$.

So, to prove the existence of a bent hyperbolic manifold, it is enough to prove the existence of a closed disk $C_0$ which:

1. has its interior in $S^2 \setminus \Lambda$.
2. intersects $\Lambda$.
3. contains no fundamental domain for the action of $\pi_1M$ on $S^2$.

If such a circle exists, there will be a point $x_1$ whose orbit $(\pi_1M).x_1$ does not intersect the interior of $C_0$.

To simplify a little the picture, suppose that $M$ has a convex core whose pleating locus contains a closed geodesic $\gamma$, with a non-zero pleating angle. In $H^3$, $\gamma$ lifts to a geodesic $\tilde{\gamma}$ with endpoints $p_1, p_2 \in \Lambda$; both $p_1$ and
p_2 \) then correspond to "spikes" of \( \Lambda \). Moreover, \( \gamma \) lies in the boundary of \( C(\hat{M}) \), so there is a support plane \( P \) of \( C(M) \) along \( \gamma \). Let \( C_0 \) be the corresponding disk in \( S^2 = \partial_\infty H^3 \). By construction, the interior of \( C_0 \) does not intersect \( \Lambda \) (while \( C_0 \cap \Lambda \supset \{p_1, p_2\} \)).

Call \( \Omega_1 \) the connected component of \( S^2 \setminus \Lambda \) which corresponds to \( \partial_1 M \). Let \( g_1 \) be the hyperbolic metric on \( \Omega_1 \), conformal to the canonical metric \( g_0 \) on \( S^2 \) and invariant under the action of \( \pi_1 M \). Then, by a classical result of conformal geometry (see e.g. \([Ah66]\)), the conformal factor between \( g_0 \) and \( g_1 \) is bounded between \( c/r \) and \( C/r \), where \( c \) and \( C \) are two positive constants and \( r \) is the distance to \( \Lambda \) in the metric \( g_0 \).

Therefore, a simple computation shows that the interior of \( C_0 \) is contained in a neighborhood of a geodesic in \( (\Omega_1, g_1) \), and also — by taking the quotient by \( \pi_1 M \) — in \( (\partial_1 M, g_1) \). Thus, after taking a finite cover of \( \partial M \), the interior of \( C_0 \) contains no fundamental domain for the action of \( \pi_1 M \). Taking the corresponding finite cover \( \hat{M} \) of \( M \), we see that \( \hat{M} \) is a bent hyperbolic manifold.

On the other hand, this argument directly shows that bending laminations can not occur in the case of fuchsian hyperbolic manifolds.

**Remark 3.9.** Fuchsian hyperbolic manifolds with horns are ideal hyperbolic manifolds.

*Proof.* When \( M \) is a fuchsian manifold with horns, \( \Lambda \) is a circle in \( S^2 = \partial_\infty H^3 \). Circles tangent to \( \Lambda \) have interiors which are isometric — for the hyperbolic metric on the interior of \( \Lambda \) — to horoballs in \( H^3 \). Therefore they always contain fundamental domains for any co-compact action on \( H^3 \). \( \square \)

### 3.4 Necessary conditions on convex surfaces

We recall here some well-known properties of the induced metric and third fundamental form of convex surfaces in \( H^3 \); they are classical except for the length condition on the geodesics of the third fundamental form, which was understood more recently.

To understand the third fundamental form of non-smooth convex surfaces, it is helpful to know how the duality between \( H^3 \) and \( S^3_1 \) works, so we will recall it rapidly here; see e.g. \([Thu97, Riv86, RH93, Sch98a]\) for more details. Although it is basically simply the polar duality on \( \mathbb{R}P^3 \) for a bilinear form of signature \((3,1)\), its geometric importance and some of its main geometric properties were pointed out by Rivin \([Riv86, RH93]\).

Both the hyperbolic and the de Sitter space can be seen as quadrics in the Minkowski 4-space \( \mathbb{R}_4^3 \), with the induced metric:

\[
H^3 = \{ x \in \mathbb{R}_4^3 \mid \langle x, x \rangle = -1 \wedge x_0 > 0 \},
\]

\[
S^3_1 = \{ x \in \mathbb{R}_4^3 \mid \langle x, x \rangle = 1 \}.
\]

For \( x \in H^3 \), let \( D \) be the line in \( \mathbb{R}_4^3 \) going through 0 and \( x \), and let \( D^\perp \) be its orthogonal for the Minkowski inner product, so that \( D^\perp \) is a space-like plane. Then define the dual \( x^* \) of \( x \) as the intersection of \( D^\perp \) with \( S^3_1 \), which is a totally geodesic space-like plane in \( S^3_1 \). Similarly, the dual of a point in \( S^3_1 \) is an oriented plane in \( H^3 \). The dual of a convex polyhedron \( P \) in \( H^3 \) is the polyhedron in \( S^3_1 \) whose vertices are the duals of the faces of \( P \), and whose faces are the duals of the vertices of \( P \) (note that there are other approaches of this duality, which might actually be more illuminating; see e.g. \([Sch98a]\)).

Given a locally convex surface \( S \) in \( H^3 \), we can define its dual \( S^* \) as the set of points in \( S^3_1 \) which are duals of an (oriented) support plane to \( S \). It happens to be another locally convex surface, which is not necessarily smooth. If \( S \) is smooth, then \( S^* \) is smooth when \( S \) is locally strictly convex.

**Property 3.10.** Let \( S \) be a smooth locally strictly convex surface in \( H^3 \); its third fundamental form is the induced metric on the dual surface.

It is tempting to speak of the "third fundamental form" of a non-smooth surface in the sense "the metric induced on its dual". Some care is needed, however. For instance, if \( S \) is a connected component of the boundary of the convex core of a convex co-compact manifold, its dual is a graph; more generally, when \( S \) is a convex, developable surface, like the boundary of a horned hyperbolic manifold, its dual is one-dimensional.

On the other hand, this notion of third fundamental form works perfectly well for compact polyhedra in \( H^3 \) — and thus also for objects which locally look like them. For ideal polyhedra it also works quite well; in some cases (see below in part 3.5) it is helpful to "glue" a hemisphere in each of the length \( 2\pi \) circles corresponding to the ideal vertices.

**Lemma 3.11.** Let \( S \) be a smooth (resp. polyhedral) locally convex surface in \( H^3 \), and let I and III be its induced metric and third fundamental forms respectively. Then:
1. If has curvature $K \geq -1$ (resp. has curvature $-1$, except at the vertices, where the singular curvature is positive);

2. III has curvature $K \leq 1$ (resp. has curvature $1$, except at the dual vertices, where the singular curvature is negative);

3. the closed geodesics of III have length $L \geq 2\pi$.

In statements (1) and (2), the equality is attained only in the degenerate cases, i.e. when the surface is not locally strictly convex. In statement (3) it also corresponds to a very degenerate case, as we will see below.

**Proof.** The first and second point are consequences of the Gauss formula in the smooth case and can be checked locally in the polyhedral cases. For the last point the reader is referred to [RH93, CD95, BB02, Sch98a, Sch01] for different approaches of the polyhedral case, and e.g. to [Sch96] for the smooth case.

We will also prove here Lemma 1.6. The proof is based on a number of propositions. In all this subsection we consider a cellulation $\sigma$ of $\partial M$, along with a function $\pi$ on the edges of $\sigma$ verifying the hypothesis of Theorem 8.17.

**Proposition 3.12.** No 2-cell of $\sigma$ can have two edges sent to the same segment in $\partial M$. The intersection of two distinct 2-cells in $\partial M$ can not contain more than one 1-cell.

**Proposition 3.13.** No 2-cell can have two vertices sent to the same point in $\partial M$.

**Proposition 3.14.** The intersection of two 2-cells in $\partial M$ can not contain both a vertex and an edge (which are disjoint).

**Proposition 3.15.** The intersection of two 2-cells in $\partial M$ can not contain two vertices.

The proof of Proposition 3.12 clearly follows from those propositions.

**Proof of Proposition 3.12.** Suppose that two edges of a 2-cell $C$ are sent to the same segment in $\partial M$. This segment would then constitute a circuit in the 1-skeleton of $\sigma$, on which the sum of the values of $\pi$ is strictly less than $\pi$; this would contradict the hypothesis of Theorem 8.17.

Similarly, if two 2-cells $C$ and $C'$ in $\partial M$ have two edges in common, then they constitute a circuit on which the sum of the values of $\pi$ is strictly less than $2\pi$. \qed

**Proof of Proposition 3.13.** Let $C$ be a 2-cell having two vertices which are sent to the same point $x_0$ in $\partial M$. Since the edges of $C$ are sent to distinct segments by the Proposition 3.12, $C$ separates $\partial M$ in two parts, one compact and the other non-compact, whose closure intersect at $x_0$.

Let $c$ be the elementary circuit made of the edges adjacent to $x_0$. Then $c$ is the union of two sequences of edges, one, say $c_1$, made of the edges contained in the closure of the compact domain of the complement of $C$, and the other, say $c_2$, made of the other edges. Both sequence constitute a non-elementary circuit, and the sum of the values of $\pi$ is less than $2\pi$ on both. This again contradicts the hypothesis of Theorem 8.17. \qed

**Proof of Proposition 3.14.** Suppose that two cells $C$ and $C'$ have in common both a vertex, $x_0$, and an edge $c$. Let $c$ be the elementary circuit made of the edges adjacent to $x_0$. Again, $c$ can be written as the union of $c_1$ and $c_2$, where $c_1$ is the sequence of edges contained in the bounded domain in the complement of $C \cup C'$. Adding $c$ to either $c$ or $c'$ leads to a non-elementary circuit. But, according to the hypothesis of Theorem 8.17 the sum of the values of $\pi$ on the edges of $c$ is $2\pi$, so that the sum of the values of $\pi$ has to be less than $2\pi$ either on the edges of $c_1$ and $c$, or on the edges of $c_2$ and $c$. \qed

**Proof of Proposition 3.15.** Suppose now that $C$ and $C'$ share two vertices $x_0$ and $x_1$. Let $c$ be the elementary circuit made of the edges adjacent to $x_0$ and let $c'$ be the elementary circuit made of the edges adjacent to $x_1$. Again, both $c$ and $c'$ decompose into two sequences of edges, those in the closure of the bounded domain in the complement of $C \cup C'$ (call them $c_1$ and $c'_1$, respectively) and the others (let them be $c_2$ and $c'_2$, respectively).

Then $c_1 \cup c'_1$, $c_2 \cup c'_2$, $c_1 \cup c'_2$ and $c_2 \cup c'_1$ each is a non-elementary circuit, while the sum of the values of $\pi$ on at least one of them is clearly at most $2\pi$, again contradicting the hypothesis of Theorem 8.17. \qed
3.5  Third fundamental form versus dihedral angles

One of the points which should be clear from the introduction is that we want to consider the manifolds with polyhedral boundary together with those having smooth boundary. It is in that respect necessary to understand what the relationship between the third fundamental form and the dihedral angles is. This should clear up how question concerning the third fundamental form — as in Theorem 1.2 — are related to questions on the dihedral angles — as in Theorem 8.17. We will work out the relationship here; it is mostly well-known, the most interesting case is that of ideal polyhedra.

First note that for manifolds with a boundary that looks locally like a compact hyperbolic polyhedron one can consider the universal cover of $M$, and then the surface $\tilde{S}$ in $S^3$ which is dual to the universal cover of a connected component of $\partial M$. $\tilde{S}$ is a polyhedral surface — locally like a convex space-like polyhedron in $S^3$, so that it carries a spherical cone-metric with negative singular curvature at the singularities. The dihedral angles are then just the lengths of the dual edges.

This metric — which we will still call the third fundamental form of the boundary — lifts to a CAT(1) metric on the boundary of the universal cover of $M$. Indeed, this splits into a local curvature condition — which is satisfied by the local convexity, because the singular curvature is negative at each vertex — and a global condition on the length of the closed geodesics, which is also true here because of Lemma 3.11.

A rather important point is that, while knowing the dual metric (and the combinatorics of the polyhedral surface) determines the dihedral angles, the converse is not true — the dihedral angles determine the length of the edges of the dual surface, but it does not determine the shape of the dual faces with more than 4 edges. This is already the source of interesting questions for convex polyhedral in $H^3$, for instance, it is still an open problem to know whether a convex hyperbolic polyhedron can be infinitesimally deformed without changing its dihedral angles, see e.g. [Sch00], or [Sto68] for an analogous (and also open) problem in the Euclidean case.

For ideal polyhedra, there are two related ways of defining the third fundamental form. If one considers the dual surface to the universal cover of one of the components of the boundary, one obtains the induced metric on the dual of a pleated surface, which is a tree. The third fundamental form therefore reduces to the lengths of the edges of a graph. The length of the edges are the (exterior) dihedral angles of the corresponding edges of the ideal polyhedron.

To each ideal vertex corresponds a face of the dual graph, with the sum of the edge lengths equal to $2\pi$ (since the sum of the exterior dihedral angles at an ideal vertex is $2\pi$). One can glue in each of those faces a hemisphere (with its canonical metric). The result is a metric space on $\partial M$, which obviously has negative singular curvature at its singular points, because the singular points correspond to the vertices of the graph, and the total angle around those points is $\pi$ times the number of faces. We call $\tilde{\Omega}$ the corresponding metric on $\partial M$. Note that $\tilde{\Omega}$ is the “natural” third fundamental form of $\partial M$ for instance in a limit sense, as follows:

**Property 3.16.** Let $(\Omega_n)_{n \in \mathbb{N}}$ be an increasing sequence of open subsets of $M$ with smooth, convex boundary, such that $\bigcup_n \Omega_n = M$. Then the third fundamental forms of $\partial \Omega_n$ converge to $\tilde{\Omega}$.

We leave the proof to the reader.

The third fundamental form defined in this way has the important properties below. The second strongly contrasts with the situation for compact polyhedra.

**Property 3.17.** 1. $\tilde{\Omega}$ lifts to a CAT(1) metric on each boundary component of the universal cover of $M$.

2. There is a simple way to recover the cellulation of $\partial M$ from $\tilde{\Omega}$, and therefore also the dihedral angles.

**Proof.** The first point is again a consequence of Lemma 3.11.

For the second point, note that the dihedral angles of ideal polyhedra are in $(0, \pi)$, and so are the edge lengths of the dual surfaces. Now any geodesic segment that enters a hemisphere can exit it only after a path of length $\pi$. Since the dihedral angles are less than $\pi$, the edges of the dual cellulation can not enter the hemispheres. On the other hand all the segments in the boundary of the hemispheres must be edges, and this recovers the dual cellulation, and thus also the cellulation of $\partial M$. \qed

4  The fuchsian case

We will investigate in this section some properties of manifolds with polyhedral boundary in the fuchsian case; that is, we consider a metric $g$ on $M$ such that $\partial M$ is polyhedral, with an isometric involution $s$ of $(M, g)$ which fixes a compact surface. We also suppose that $\partial M$ have two connected components, which are exchanged by
4. We refer the reader to [Lei02a, Lei02b, BS04] for some recent results on those manifolds, mostly based on proofs which are different from those used in this section.

Another way to consider such manifolds is to take the universal cover $S$ of one of the two connected components of $\partial M$; it is a convex surface in $H^3$, which moreover is equivariant under the action of a surface group fixing a totally geodesic 2-plane.

We will first prove an existence and uniqueness result for the third fundamental forms of such surfaces, in the case where they locally look like a compact (rather than ideal) polyhedron in $H^3$. This is done using a deformation argument as in the case of hyperbolic polyhedra (see [Ale58]), and is also similar to what can be done in the smooth case [LS00].

This first result will then be used to prove a similar existence result for the dihedral angles of surfaces which locally look like an ideal polyhedron; the main idea is to approximate this case by the previous one, using a compactness result to prevent degeneracies from occurring. The results obtained here for those manifolds are strongly related to those obtained recently by Bobenko and Springborn [BS04]. The proof used here will also give a result concerning the manifolds with a polyhedral boundary with some ideal and some "non-ideal" vertices.

The first result which is obtained in this way — or at least its ideal part — might look like a weak and partial version of some of the results stated in the introduction. It is proved in a very different way, however, and turns out to be necessary for the more general cases, because it implies a technical statement — on the connectedness of some spaces of metrics on the boundary — which I do not know how to obtain directly.

Mathias Rousset [Rou04] has recently achieved another related result, concerning the dihedral angles of fuchsian hyperideal polyhedra, which includes the case of ideal fuchsian polyhedra. He reduces the study of hyperideal polyhedra to that of finite polyhedra.

In all this section, we fix a surface $S$ of genus $g \geq 2$; we will be interested in equivariant embeddings of $S$, or, in other terms, in fuchsian hyperbolic metrics on $S \times [-1, 1]$.

4.1 Infinitesimal rigidity of finite fuchsian polyhedra

We first consider equivariant polyhedra which look locally like compact polyhedra. First recall the definition of a polyhedral embedding.

**Definition 4.1.** A polyhedral embedding of a surface $S$ into $H^3$ (resp. $\mathbb{R}^3$, $S^3_1$) is a couple $(\sigma, \phi)$, where $\sigma$ is a cellulation of $S$ and $\phi$ is a map from $S$ to $H^3$ (resp. $\mathbb{R}^3$, $S^3_1$) which:

- is injective.
- sends each edge of $\sigma$ to a segment in $H^3$ (resp. $\mathbb{R}^3$).
- sends each 2-face of $\sigma$ to the interior of a compact, convex polygon in a totally geodesic 2-plane in $H^3$ (resp. $\mathbb{R}^3$, $S^3_1$).
- is locally convex at each vertex.

The last condition is not always necessary; it is included here since all the polyhedral objects that we will consider are convex.

**Definition 4.2.** A finite equivariant polyhedron is a couple $(\phi, \rho)$, where $\phi$ is a polyhedral embedding of the universal cover of a surface $S$ into $H^3$ and $\rho$ is a homomorphism from $\pi_1 S$ into $\text{Isom}(H^3)$, such that:

$$\forall x \in \hat{S}, \forall \gamma \in \pi_1 S, \phi(\gamma x) = \rho(\gamma)\phi(x).$$

We are specially interested in the equivariant polyhedra which are the boundary of the universal covers of the fuchsian hyperbolic manifolds mentioned above.

**Definition 4.3.** A finite equivariant polyhedron $(\phi, \rho)$ in $H^3$ is fuchsian if $\rho(\pi_1 S)$ is contained in the subgroup of elements which leave invariant a given plane $P \subset H^3$.

The definition of ideal equivariant polyhedra, and of semi-ideal equivariant polyhedra, is similar. We first define the notion of polyhedral map.

\[1\]I was not aware of the results of [BS04] when I wrote the first version of the current paper.
Definition 4.4. An **ideal polyhedral embedding** of a surface S into H^3 is a couple (σ, φ), where σ is a cellulation of S and φ is a map from S to H^3 ∪ ∂_∞ H^3 which:

- is injective.
- sends each vertex of σ to an ideal point (in ∂_∞ H^3).
- sends each edge of σ to a geodesic in H^3, which connects the ideal points corresponding to the vertices.
- sends each 2-face of σ to the interior of an ideal convex polygon in a totally geodesic 2-plane in H^3.
- is locally convex.

A **semi-ideal polyhedral embedding** is defined likewise, except that the vertices can be sent either to ideal points or to "usual" points of H^3.

Definition 4.5. An **ideal equivariant polyhedron** is a couple (φ, ρ) where φ is an ideal polyhedral embedding of the universal cover of a surface S into H^3 and ρ is a homomorphism from π_1 S into Isom(H^3), such that:

\[ ∀x ∈ \tilde{S}, ∀γ ∈ π_1 S, φ(γx) = ρ(γ)φ(x) \]

A **semi-ideal equivariant polyhedron** is defined in the same way, but with φ a semi-ideal polyhedral embedding.

Note that the induced metric on equivariant (semi-)ideal polyhedra is defined not only on the universal cover of the underlying surface, but also, by equivariance, on the quotient surface. For ideal equivariant polyhedra it is a hyperbolic metric with cusps corresponding to the vertices (i.e. a hyperbolic metric of finite area on the quotient surface S). For semi-ideal polyhedra, the induced metric is a cone-metric, with singular points corresponding to the vertices which are not ideal. It also has cusps corresponding to the ideal vertices.

Definition 4.6. An ideal (resp. semi-ideal) equivariant polyhedron (φ, ρ) is **fuchsian** if ρ(π_1 S) is contained in the subgroup of elements which leaves invariant a given plane P ⊂ H^3.

There is an analogous notion of fuchsian polyhedra in R^3_1 and in S^3_1; the de Sitter notion is dual to the hyperbolic notion, in the sense that the dual in de Sitter space of a convex, equivariant, fuchsian hyperbolic polyhedron is a convex, space-like, equivariant, fuchsian polyhedron in S^3_1. Indeed, it is not difficult to check (see [LS00]) that the dual in de Sitter space of a surface which is invariant under the action of a group fixing a plane is a surface which is invariant under the action of the same group, fixing a point — which is of course the de Sitter point dual to the hyperbolic point which is fixed.

Definition 4.7. An equivariant polyhedron (φ, ρ) in R^3_1 (resp. S^3_1) is **fuchsian** if ρ(π_1 S) is contained in the subgroup of elements which leaves invariant a given point x_0 ∈ R^3_1 (resp. x_0 ∈ S^3_1).

The first result we need to mention is an infinitesimal rigidity result for finite equivariant polyhedral embeddings in Minkowski 3-space.

Lemma 4.8. Let (φ, ρ) be a convex fuchsian space-like polyhedron in R^3_1, such that the representation ρ fixes the origin. There is no non-trivial infinitesimal deformation of (φ, ρ), among fuchsian polyhedra, which does not change the induced metric by φ to the first order.

This statement can be found as Theorem B in Igor Iskhakov’s thesis [Isk00], where it is proved by an extension of Cauchy’s ideas on the rigidity of polyhedra (see [Cau13, Sto68]) to surfaces of genus g ≥ 2.

I believe that an alternate proof could be given, along the approach given in [LS00] for smooth surfaces; the key point would be to replace the integration by part which works for the smooth case by a discrete version.

There is yet another way to prove this infinitesimal rigidity result, as well as the infinitesimal rigidity in [LS00], and more general results concerning for instance hyperideal fuchsian polyhedra. It is based on the Pogorelov map, which is also used for instance in [Sch02a], to bring the problem in R^3. There, the infinitesimal rigidity can be proved using the elementary fact that, if an infinitesimal deformation of a (smooth or polyhedral) convex surface is isometric, then the graph of its coordinates are saddle surfaces.

As a consequence, we find that the same result holds in the de Sitter space:

Lemma 4.9. Let (φ, ρ) be a convex fuchsian equivariant space-like polyhedron in S^3_1. There is no non-trivial infinitesimal deformation of (φ, ρ), among fuchsian equivariant polyhedra, which does not change the induced metric by φ to the first order.
The proof of Lemma 4.9 from Lemma 4.8 is related to a remarkable trick invented by Pogorelov [Pog73], which allows one to take an infinitesimal rigidity problem from a space-form to another. The crucial point is that this can be done in this setting, i.e. for equivariant objects when the representation fixes a point, as shown in [LS00] for smooth surfaces. Moreover, the polyhedral case works just like the smooth case, details on this stand in [Sch00, Sch98a, Sch01].

Note that the same could be done also in the anti-de Sitter space $H^3$, and this is indeed done in [LS00] for smooth surfaces. We leave this point to the reader, however, since it will not be necessary below.

Using the duality between $H^3$ and the de Sitter space $S^3_1$, we immediately find a translation of this lemma in terms of fuchsian surfaces in $H^3$:

**Corollary 4.10.** Let $(\phi, \rho)$ be a convex fuchsian equivariant polyhedron in $H^3$. There is no non-trivial infinitesimal deformation of $(\phi, \rho)$, among fuchsian equivariant polyhedra, which does not change the third fundamental form of the image of $\phi$ to the first order.

### 4.2 Compactness of fuchsian polyhedra

The main technical goal of this subsection is a compactness result, which is necessary to obtain the existence and uniqueness result for fuchsian polyhedra explained above, and stated below as Theorem 4.22. It is stated in a more general context, however, so as to be used also later in this section, to prove a result for ideal or semi-ideal fuchsian polyhedra.

We now fix two integers $N \geq 1$ and $g \geq 2$; $g$ will be the genus of the surface $S$ considered, and $N$ will be the number of vertices of the polyhedral surfaces and the number of singular points of the metrics considered.

**Definition 4.11.** We call:

- $\mathcal{P}^C$ the set of finite fuchsian polyhedra of genus $g$ with $N$ vertices in $H^3$.
- $\mathcal{P}^I$ the set of semi-ideal fuchsian polyhedra of genus $g$ with $N$ vertices in $H^3$.

**Definition 4.12.** We call:

- $\mathcal{M}^C$ the set of spherical cone-metrics on $S$ with $N$ cone-points where the singular curvature is negative, and such that all contractible closed geodesics have length $L > 2\pi$ (up to isotopy).
- $\mathcal{M}^I$ the set of spherical cone-metrics with $N$ singular points, where the singular curvature is negative, and such that contractible closed geodesics have length $L > 2\pi$, except when they bound a hemisphere (again up to isotopy).

It is clear (using Lemma 3.11) that the third fundamental forms of the elements of $\mathcal{P}^C$ are in $\mathcal{M}^C$. Similarly, the third fundamental forms of the elements of $\mathcal{P}^I$ are in $\mathcal{M}^I$, after one makes a simple surgery: gluing a hemisphere on each of the circles, of length $2\pi$, which correspond to the ideal vertices.

**Lemma 4.13.** Let $(\phi_n, \rho_n)$ be a sequence of finite fuchsian polyhedral embeddings in $S^3_1$, with representations $\rho_n$ fixing a point $x_0$. Let $(h_n) \in (\mathcal{M}^C)^N$ be the induced metrics. Suppose that $(h_n)$ converges, as $n \to \infty$, to a metric $h \in \mathcal{M}^I$. Then, after taking a subsequence and “renormalizing”, $(\phi_n, \rho_n)$ converges to a convex, fuchsian polyhedron $(\phi, \rho)$. Moreover $\rho$ fixes $x_0$.

In this statement, the “renormalization” is by composition on the left by a sequence of isometries.

Note that $(\phi, \rho)$ might have some faces which are tangent to the boundary at infinity of $H^3$ in the projective model of $H^3$ and $S^3_1$. More precisely, this happens exactly when $h$ has closed geodesics of length $2\pi$ bounding hemispheres, and the faces tangent to the boundary at infinity are precisely those hemispheres.

The proof of Lemma 4.13 depends on a series of propositions, it is done at the end of this sub-section.

**Proposition 4.14.** Let $x_0 \in S^3_1$. The function $u$ on the future or past cone of $x_0$, defined as the hyperbolic cosine of the distance to $x_0$ satisfies:

$$\text{Hess}(u) = -ug_0,$$

where $g_0$ is the metric of $S^3_1$. Moreover, the metric $g_0$ on the future cone of $x_0$ can be written as a warped product, as:

$$g_0 = (u^2 - 1)g_{H^2} - \frac{du^2}{u^2 - 1},$$

where $g_{H^2}$ is the metric on the hyperbolic plane.
The function $u$ will be used here — as in other similar problems, see e.g. [Sch96, LS00] — to control the lengths of closed geodesics of surfaces or the total extrinsic curvature of the surface along those geodesics.

**Proof.** The first point is elementary, it follows from the fact that $u$ is the restriction to $S^1_1$ — seen as a quadric in $\mathbb{R}^3$ — of one of the coordinate functions.

For the second point, for each $t > 0$, let $\Sigma_t$ be the set of points $x$ in the future cone of $x_0$ such that there is a time-like segment going from $x_0$ to $x$ of length $t$. Then a simple computation shows that $\Sigma_t$ has second fundamental form $H_t = \coth(t)I_t$, where $I_t$ is the induced metric on $\Sigma_t$. Moreover the surfaces $\Sigma_t$ are equidistant, so that an integration shows that the induced metrics are:

$$I_t = \sinh^2(t)g_{H^2},$$

and thus the metric on the future cone of $x_0$ can be written as:

$$\sinh^2(t)g_{H^2} - dt^2.$$

The proposition follows by setting $u = \cosh(t)$. \hfill $\Box$

We then need a simple proposition about the solution of an ordinary differential inequality; it will be applied below to the function $u$ restricted to geodesic segments in the metrics $h_n$.

**Proposition 4.15.** For any $L > 0$ and $u_0 > 1$, there exists $c > 0$ such that if $u : [0, L] \to \mathbb{R}_+$ is a function which:

- is smooth and satisfies $u'' = -u$ except at $N$ singular points $p_1, \cdots, p_N \in (0, L)$;
- has a positive jump in its derivative at the $p_i$;
- is bounded from below by $u_0$,

then:

$$\int_0^L \sqrt{\frac{1}{u^2 - 1} + \frac{u'^2}{(u^2 - 1)^2}} ds \leq c.$$

**Proof.** First note that:

$$\sqrt{\frac{1}{u^2 - 1} + \frac{u'^2}{(u^2 - 1)^2}} \leq \frac{1}{\sqrt{u_0^2 - 1}} + \frac{|u'|}{u^2 - 1} \leq \frac{1}{\sqrt{u_0^2 - 1}} + |(\coth(u))'|.$$

It is therefore enough to prove that the total variation over $[0, L]$ of $\coth(u)$ is bounded from above by a constant.

Let $0 \leq x_1 < \cdots < x_p \leq L$ be the sequence of local minima of $u$, and let $0 \leq y_1 < \cdots < y_q \leq L$ be its local maxima. The properties of $u$ clearly imply that there is indeed a finite number of minima and maxima. Then, if $y_j$ is a local maximum which immediately follows the local minimum $x_i$:

$$\int_{x_i}^{y_j} |(\coth(u))'| ds = \coth(u(x_i)) - \coth(u(y_j)) .$$

But, since $y_j$ is a local maximum, and since $u'$ has a positive jump at the singular points, $u'(y_j) = 0$, so that, for all $s \in [x_i, y_j]$:

$$u(y_j) \geq u(s) \geq u(y_j) \cos(y_j - s).$$

We now consider two cases:

1. $|y_j - x_i| \geq \pi/4$. Then:

$$|\coth(u(x_i)) - \coth(u(y_j))| \leq \coth(u_0) - 1,$$

because it is the difference between two numbers in $(1, \coth(u_0))$.
2. \(|y_j - x_i| \leq \pi/4\). Then:

\[
|\text{argcoth}(u(y_j) - \text{argcoth}(u(x_i)))| \leq \int_{u(x_i)}^{u(y_j)} \frac{dv}{v^2 - 1} \leq \frac{u(x_i)}{u(x_i)^2 - 1} \left( \frac{1}{\cos(y_j - x_i)} - 1 \right) \leq 4(y_j - x_i)^2 \frac{u(x_i)}{u(x_i)^2 - 1} \leq 4(y_j - x_i)^2 \frac{u_0}{u_0^2 - 1},
\]

the last inequality follows from the fact that \(u \rightarrow u/(u^2 - 1)\) is decreasing over \((1, \infty)\). An elementary symmetry argument shows that the same estimates apply when \(x_j\) is a local minimum which immediately follows a local maximum \(y_i\). As a consequence, we find that:

\[
\int_0^L \left| (\text{argcoth}(u(s)))' \right| ds \leq \frac{4L(\text{argcoth}(u_0) - 1)}{\pi} + \frac{4u_0L^2}{u_0^2 - 1},
\]

and the proposition follows.

We will also need the following more geometric estimate.

**Proposition 4.16.** For any \(r > 0\) and any integer \(N > 0\), there exists \(c > 0\) as follows. Let \(\sigma\) be a cellulation of the disk \(D\) with at most \(N\) vertices, and let \(\phi\) be a polyhedral space-like embedding of \(D\) in \(S^3_1\) with combinatorics given by \(\sigma\), such that:

- the boundary of \(\phi(D)\) is convex for the metric induced by \(\phi\).
- the boundary is at distance at least \(r\) from \(x_0\), the center of the disk \(D\), in the induced metric.
- \(\phi(D)\) is contained in the future cone \(C_+(x_1)\) of a point \(x_1\).

Then the absolute value of the distance between \(\phi(x_0)\) and \(x_1\) is at least \(c\).

**Proof.** Since \(\phi(D)\) is space-like with convex boundary, it remains outside the future cone of each of its points, in particular outside the future cone of \(x_0\). Since it also remains in \(C_+(x_1)\), it is clear that, if \(x_1\) were too close to \(x_0\), \(\phi(D)\) would have to remain in an arbitrarily small neighborhood of a light cone; this is not possible for a polyhedral surface having a fixed number of singular points.

We will also need below a basic result in the theory of hyperbolic surfaces; see e.g. [FLP91] for a proof. Its content is that, to prevent a sequence of hyperbolic metrics from degenerating, one only needs to bound from above the lengths of a finite set of closed geodesics.

**Lemma 4.17.** Let \(\Sigma\) be a surface of genus \(g \geq 2\). There exists a finite subset \(E\) of \(\pi_1\Sigma\) such that, for any \(\epsilon > 0\), the set of hyperbolic metrics on \(\Sigma\) such that the closed geodesics corresponding to the elements of \(E\) have length at most \(1/\epsilon\) is compact.

We can now state a proposition showing that, with the hypothesis of Lemma 4.13, the representations of the equivariant polyhedra \((\phi_n, \rho_n)\) do not diverge; the last part of the proof will be to show that, under the "length 2\(\pi^n\) condition, isolated vertices can not escape to infinity.

**Proposition 4.18.** In the setting of Lemma 4.13, the sequence of representations \((\rho_n)\) converges (after one takes a subsequence).

**Proof.** First we fix an integer \(n \in \mathbb{N}\), and consider an equivariant polyhedral embedding \(\phi_n : \tilde{S} \rightarrow S_1^3\) with representation fixing a point \(x_0\). As above we call \(u\) the function defined on the future cone of \(x_0\) as the hyperbolic cosine of the distance to \(x_0\). Let \(x_n\) be a point of \(\tilde{S}\) where the minimum of \(u\) is attained, and let \(u_n := u \circ \phi_n\). Proposition 4.16 shows that \(u_n(x_0)\) is bounded from below by a strictly positive constant. Let \(E \subset \Gamma = \pi_1\tilde{S}\) be a finite generating set, on which more details will be given below.

Let \(\gamma \in E\). There is a minimal geodesic segment \(c_{n,\gamma}\) going from \(x_n\) to \(\gamma x_n\) in \((\tilde{S}, h_n)\). Since \(E\) is finite, and since \(h_n \rightarrow h\) as \(n \rightarrow \infty\), the length \(L(c_{n,\gamma})\) of \(c_{n,\gamma}\) is bounded from above by a constant \(L_0\) for each \(\gamma \in E\) and each \(n \in \mathbb{N}\).

Since \(\phi_n(\tilde{S})\) is space-like and equivariant under the action of \(\Gamma\), which fixes \(x_0\), \(\phi_n(\tilde{S})\) remains in the future cone \(C_+(x_0)\) of \(x_0\). Calling \(u_n\) the restriction of \(u\) to \(c_{n,\gamma}\), Proposition 4.14 shows that \(u_n\) satisfies \(u_n'' = -u_n\), except when \(c_{n,\gamma}\) crosses an edge of \(\phi_n(\tilde{S})\), and then \(u_n''\) has a positive jump.
The second part of Proposition 4.14 implies that $C_+(x_0)$ has a natural submersion $\rho : C_+(x_0) \to H^2$, such that the restriction of $\rho$ to each surface $\{v = \text{const}\}$ in $C_+(x_0)$ is a homothety. Let $s$ be the length element induced by $\phi_n$ on $S$, and $t$ be the length element of the hyperbolic metric induced on $S$ by $\rho \circ \phi_n$. The second point of Proposition 4.14 then indicates that:

$$ds^2 = (u_n^2 - 1)dt^2 - \frac{du_n^2}{u_n^2 - 1},$$

$$dt^2 = \frac{ds^2}{u_n^2 - 1} + \frac{du_n^2}{(u_n^2 - 1)^2},$$

so that the length of the image of $c_{n,\gamma}$ by $\rho$ is:

$$L_t(c_{n,\gamma}) = \int_0^{L(c_{n,\gamma})} \sqrt{\frac{1}{u_n^2 - 1} + \frac{u_n^2}{(u_n^2 - 1)^2}} ds.$$

According to Propositions 4.16, $u_n$ is bounded from below by a constant $u_0 > 1$; Proposition 4.15 therefore shows that, as $n \to \infty$, the lengths for $t$ of the $c_{n,\gamma}$ remain bounded from above by a constant.

Now for each $n \in \mathbb{N}$, $\Gamma$ acts on $S^3$ fixing $x_0$, and therefore $\Gamma$ has an action on $C_+(x_0)$ which leaves globally invariant all the surfaces $\{u = \text{const}\}$. Thus it acts by isometries on $H^2$ through $\rho$. The previous argument shows that the translation distance of each element of $E$ remains bounded from above as $n \to \infty$. Lemma 4.17 then implies that the sequence $\rho_n$ remains in a compact subset of Teichmüller space, and therefore that one of its subsequences converges.

To prove that the sequence of equivariant polyhedra $(\phi_n, \rho_n)$ actually converges — and not only the representations — it is helpful to consider a projective model of the part of the de Sitter space $S^3_1$ which stands on one side of a totally geodesic space-like plane. One can be constructed as follows. Remember that $S^3_1$ is isometric to a quadric in Minkowski 4-space with the induced metric:

$$S^3_1 \simeq \{ x \in \mathbb{R}^4 | (x, x) = 1 \}.$$

Let $P_0$ be the affine hyperplane of equation $x_0 = 1$ in $\mathbb{R}^4$, and let:

$$S^3_{1,+} := \{ x \in S^3_1 | x_0 > 0 \}.$$

There is a natural map from $S^3_{1,+}$ to $P_0$ sending a point $x \in S^3_{1,+}$ to the intersection with $P_0$ of the line going through 0 and $x$. By construction it is projective, i.e. it sends geodesics to geodesics; indeed, an elementary argument using the action of $\text{SO}(3,1)$ shows that the geodesics of $S^3_{1,+}$ are the intersections with $S^3_{1,+}$ of the 2-planes of $\mathbb{R}^4$ containing 0. Note that, since the $\phi_n(S)$ remain in the future cone of a point, the projective model can be chosen so that a compact subset of $\mathbb{R}^3$ contains the image of the surfaces $\phi_n(S)$ for each $n$.

**Proposition 4.19.** After taking a subsequence and renormalizing, the sequence $(\phi_n(\tilde{S}))$ converges in the projective model described above to an equivariant polyhedron.

**Proof.** This is a direct consequence of the convergence (after taking a subsequence) of the representations, as stated in Proposition 4.18. Indeed, we can renormalize the sequence so that, for a given vertex $x \in \tilde{S}$, $\phi_n(x)$ is constant. Then an elementary compactness argument shows that, after taking a subsequence, the vertices adjacent to $x$ also have converging images. Going to the vertices adjacent to those and applying the same compactness argument shows that they also have converging images, and this can be done until all vertices in a fundamental domain of $S$ have converging image. Proposition 4.18 then implies the result.

We now have to exclude some cases, corresponding to a limit metric $h$ which has a closed geodesic of length $2\pi$. A similar assertion was used in [Riv96], and also — stated in a more general setting, including higher dimensions — in [Sch98a, Sch01].

**Proposition 4.20.** Suppose there is a finite set of vertices which converge to the same point $x_\infty$ in $\partial_\infty H^3$, while the other vertices do not. Then the limit metric $h$ has a closed geodesic of length $2\pi$ which does not bound a hemisphere containing no vertex.
**Proof.** We use Proposition 4.19 and suppose that, in the projective model described above, the sequence \((\phi_n(S))\) converges to a limit, \(P\). Then \(x_\infty \in P\), and, by convexity and the fact that the polyhedra \(\phi_n(S)\) remain in the exterior of the ball \(B^3\) corresponding to \(H^3\), \(P\) also contains a neighborhood of \(x_\infty\) in the plane \(\pi_0\) tangent to \(B^3\) at \(x_\infty\). More precisely, since \(P\) is a polyhedron, one of its faces is the interior of a convex polygon \(Q\) in \(\pi_0\).

We will show that the boundary polygon, \(\partial Q\), has length \(2\pi\) and is a geodesic for the limit induced metric \(h\). The fact that it has length \(2\pi\) is an elementary fact of Lorentz geometry, since it lies in a degenerate plane in \(S^1_1\) (see e.g. [Sch98a]). To show that it is a geodesic of \(h\), we have to show that both sides of \(P\) are concave for \(h\).

Consider first the interior of the polygon \(Q\). Since it carries a degenerate metric, it is not difficult to realize that the metrics induced on the corresponding faces of \(\phi_n(S)\) converge to the metric of a hemisphere. Therefore, this part of \(P\) is concave for \(h\).

For the other side, simply note that, for each \(n\), the metric \(h_n\) has a point of negative singular curvature at the vertices of \(\partial Q\), i.e. the limit total angle at those vertices is at least \(2\pi\). Since the limit metric is isometric to a hemisphere in the interior of \(Q\), it means that the limit total angle at each vertex of \(\partial Q\) of the complement of \(Q\) in \(P\) is at least \(\pi\), i.e. that the complement of \(Q\) in \(P\) is concave for \(h\), as needed.

**Proof of Lemma 4.13.** Proposition 4.18 shows that the sequence of representations \((\rho_n)\) converges (after taking a subsequence), while Proposition 4.20 indicates that no vertex can escape to infinity. Therefore \((\phi_n, \rho_n)\) converges. \(\square\)

### 4.3 Induced metrics on finite polyhedra

We can now consider the maps \(\Phi^C : \mathcal{P}^C \to \mathcal{M}^C\) and \(\Phi^I : \mathcal{P}^I \to \mathcal{M}^I\) sending a fuchsian polyhedron to its third fundamental form. We will prove that \(\Phi^C\) is a homeomorphism, and that \(\Phi^I\) is bijective. To show that for \(\Phi^C\), we will apply a deformation method, which I believe was invented by Aleksandrov [Ale58] to study the induced metrics on hyperbolic polyhedra, although it was later used in many very different fields.

First, it is clear that choosing an element of \(\mathcal{P}^C\) is equivalent to choosing a hyperbolic metric on \(S\) (or equivalently a fuchsian action of \(\Gamma := \pi_1(S)\) on \(H^3\)) along with \(N\) points \(x_1, \ldots, x_N\) in \(H^3/\Gamma\), under some conditions, i.e. that the \(x_i\) all lie on the boundary of their convex hull. Therefore, \(\mathcal{P}^C\) is a connected manifold with boundary, of dimension \(6g - 6 + 3N\).

It is also clear, using the results of Troyanov [Tro91], that choosing an element of \(\mathcal{M}^C\) is the same as choosing an element of the Teichmüller space of \(S\) with \(N\) marked points, along with the singular curvature at each of the marked points. Thus \(\mathcal{M}^C\) is also a manifolds with boundary of dimension \(6g - 6 + 3N\).

Moreover, Corollary 4.10 shows that \(\Phi^C\) is locally injective — and therefore a local diffeomorphism — between \(\mathcal{P}^C\) and \(\mathcal{M}^C\). To prove that it is a homeomorphism, we need to prove that \(\Phi^C\) is proper (this is a consequence of Lemma 4.13), that \(\mathcal{P}^C\) is connected, and that \(\mathcal{M}^C\) is simply connected.

The actual proof below is slightly more complicated than the outline here; since I do not know how to prove directly the connectedness of \(\mathcal{M}^C\), we will use a trick invented in [Riv86, RH93]: one uses the connectedness of a space of smooth metrics — which is easy to prove — to check that two metrics \(g_0\) and \(g_1\) with \(N\) singular points can be connected by a path of metrics with at most \(N'\) singularities, where \(N'\) is a (large) integer depending on \(g_0\) and \(g_1\).

**Lemma 4.21.** 1. Let \(g_0\) and \(g_1\) be elements of the space \(\mathcal{M}(S, N)\) of spherical cone-metrics on \(S\), with at most \(N\) singular points where the singular curvature is negative, and contractible closed geodesics of length \(L > 2\pi\). There exists an integer \(N'\) depending on \(g_0\) and \(g_1\) such that \(g_0\) and \(g_1\) can be connected in \(\mathcal{M}(S, N')\).

2. Let \(c : S^1 \to \mathcal{M}(S, N)\). There exists \(N' \geq N\) and a disk \(D \subset \mathcal{M}(S, N')\) with \(\partial D = c(S^1)\).

**Brief sketch of the proof.** We do not give a full proof, since it is almost the same as the one given by Rivin [Riv86] and Rivin and Hodgson [RH93]: the only difference is that the surfaces considered here have genus \(g \geq 2\), rather than 0, but this does not appear in the proof.

The main point is that the space of smooth metrics on \(S\) with curvature \(K < 1\) and contractible closed geodesics of length \(L > 2\pi\) is connected. Indeed, given two such metrics \(h_0\) and \(h_1\), one can take any path connecting them in the space of Riemannian metrics on \(S\), and then "scale up" the middle part, to make sure that the curvature remains small and the closed geodesics remain large (this is also explained in [Sch94, Sch96]).
To prove the first part of the lemma, one first shows that \( g_0 \) and \( g_1 \) can be approximated by smooth metrics \( h_0 \) and \( h_1 \) satisfying the curvature and geodesic length conditions. One thus obtains a path \( (h_t)_{t \in [0,1]} \) of metrics satisfying the same conditions. One then proves that there is an integer \( N' \) such that the metrics in \( (h_t)_{t \in [0,1]} \) can be approximated by polyhedral metrics in \( \mathcal{M}(S, N) \) in a continuous way.

The second point can be proved in an analogous way, using the fact that the space of smooth metrics on \( S \) with curvature \( K < 1 \) and contractible closed geodesics of length \( L > 2\pi \) is simply connected.

**Theorem 4.22.** Let \( S \) be a surface of genus \( g \geq 2 \), and let \( h \) be a spherical cone-metric on \( S \), with negative singular curvature at the singular points. Suppose that all contractible closed geodesics of \( (S, h) \) have length \( L > 2\pi \). Then there is a unique fuchsian polyhedral embedding of \( (S, h) \) into \( H^3 \) whose third fundamental form is \( h \).

Note that the uniqueness here is of course up to global isometries of \( H^3 \). Another remark is that the length condition is necessary by Lemma 3.11, because the curvature conditions at the vertices imply that the image is convex. By the way, it would be interesting to know whether a similar result also holds with \( S^2 \) replaced by the anti-de Sitter space.

**Proof.** As mentioned above, we already know that \( \Phi^C \) is a local diffeomorphism. Lemma 4.13 shows that \( \Phi^C \) is proper, so it is a covering of the connected components of \( \mathcal{M}^C \) which intersect its image. The first part of Lemma 4.21 shows that all of \( \mathcal{M}^C \) is in the image, while the second part, along with the fact that \( P^C \) is connected, indicates that each point of \( \mathcal{M} \) has a unique inverse image. \( \square \)

### 4.4 Fuchsian ideal manifolds

We have already mentioned above that the dihedral angles of ideal polyhedra — and of ideal fuchsian polyhedral embeddings, etc — is the analog of the third fundamental form of finite polyhedra (and of finite fuchsian polyhedral embeddings, etc). Theorem 4.22 should therefore have an analog for ideal fuchsian polyhedral embeddings in terms of dihedral angles. We will prove first the existence part of this statement; the more general result concerning ideal fuchsian polyhedra will be a consequence of other results proved here (specifically, of Theorem 8.17) but the existence result given here will be necessary to prove the more general statement. Moreover, we will consider here the case of semi-ideal polyhedral embeddings, which is not covered by Theorem 8.17. The first statement concerns ideal equivariant polyhedra, it is also contained in [BS04], where it is stated in terms of circle patterns.

**Lemma 4.23.** Let \( \Gamma \) be the 1-skeleton of a cellulation of a surface \( S \) of genus \( g \geq 2 \). Let \( w : \Gamma_1 \to (0, \pi) \) be a function on the set \( \Gamma_1 \) of edges of \( \Gamma \) such that:

1. for each elementary circuit in \( \Gamma \), the sum of the values of \( w \) is equal to \( 2\pi \);
2. for each non-elementary circuit, the sum of the values of \( w \) is strictly above \( 2\pi \).

Then there is a fuchsian ideal embedding of \( (S, h) \) into \( H^3 \) whose combinatorics is given by \( \Gamma \), with exterior dihedral angles given by \( w \).

This lemma is actually a consequence, using Property 3.17, of the more general statement below, so we don’t prove it separately.

**Lemma 4.24.** Let \( S \) be a surface of genus \( g \geq 2 \), and let \( h \) be a spherical cone-metric on \( S \), with negative singular curvature at the singular points. Suppose that all contractible closed geodesics of \( (S, h) \) have length \( L > 2\pi \), except when they bound a hemisphere. Then there is a fuchsian polyhedral embedding of \( (S, h) \) into \( H^3 \) whose third fundamental form is \( h \).

**Proof.** We choose a sequence of spherical cone-metrics \( (h_n)_{n \in \mathbb{N}} \) such that:

1. \( h_n \) converges to \( h \).
2. for each \( n \), \( h_n \) is a spherical cone-metric on \( S \), with negative singular curvature at the singularities.
3. for each \( n \), the contractible closed geodesics of \( (S, h_n) \) have length strictly above \( 2\pi \).
It is not difficult to find such an approximating sequence; one has to decrease slightly the length of some of the edges, and to replace the hemispheres by interiors of convex polygons in $S^2$.

Then apply Theorem 4.22 to obtain, for each $n$, a fuchsian finite polyhedral embedding of $S$ in $S^3$. Finally, Lemma 4.13 shows that, after renormalizing this sequence and taking a subsequence, it converges to a semi-ideal fuchsian polyhedral embedding inducing the metric $h$ on $S$. 

Mathias Rouset [Rou04] recently remarked that the uniqueness part of this statement can be obtained in a rather straightforward way (from Theorem 4.22), by using the infinitesimal Pogorelov map to show an infinitesimal rigidity result for semi-ideal fuchsian polyhedra; the uniqueness for semi-ideal polyhedra then follows from studying the map sending a fuchsian polyhedron to its third fundamental form in the neighborhood of semi-ideal polyhedra. He thus obtained that:

**Theorem 4.25 (Rouset [Rou04]).** Let $S$ be a surface of genus $g \geq 2$, and let $h$ be a spherical cone-metric on $S$, with negative singular curvature at the singular points. Suppose that all contractible closed geodesics of $(S, h)$ have length $L > 2\pi$, except when they bound a hemisphere. Then there is a unique fuchsian polyhedral embedding of $(S, h)$ into $H^3$ whose third fundamental form is $h$.

5 Triangulations

This section deals with questions concerning triangulations of a given ideal hyperbolic manifold. Although the existence of a triangulation inducing a given triangulation of the boundary might appear natural at first sight, it is not easy to prove — at least this is not proved here. This is similar to the situation concerning finite volume hyperbolic manifolds, where an ideal triangulation would be helpful but is not known to exist in general; see [PP00]. We will only prove that any ideal hyperbolic manifold $M$ is ‘virtually triangulable’ in the sense that it has a finite cover $\tilde{M}$ which does admit an ideal triangulation.

Note that alternative approaches could perhaps be followed. One is based on the fact that the main property of the ideal simplices which is used here, namely that the volume is a concave function, remains valid for more general ideal polyhedra (this is a result of [Riv94]). Another (which was pointed out by Francis Bonahon) uses a triangulation which might include some degenerate simplices. Lemma 2.5 does not apply for those simplices, since the volume function is concave but not strictly concave in those cases. It might however be possible to prove that the sum of the volumes of the simplices in a triangulation remains strictly concave, which is basically what one needs in the next sections.

Although those other approaches could presumably lead to shorter proofs, we chose the method described here because, once some technical points are proved, it gives rather simple picture of what goes on; and also because the method it contains might be useful in other settings.

5.1 Triangulations, cellulations

We first give more details about what we call a triangulation here.

**Definition 5.1.** Let $M$ be an ideal hyperbolic manifold. A cellulation $C$ of $M$ is a finite family $C_1, C_2, \ldots, C_n$ of non-degenerate, closed, convex, ideal polyhedra isometrically embedded in $M$, such that:

1. for $i \neq j$, the interiors of $C_i$ and $C_j$ are disjoint;
2. the union of the $C_i$ is all of $M$;
3. for $i \neq j$, if $C_i \cap C_j \neq \emptyset$, then it is a face of both $C_i$ and $C_j$.

The $C_i$ are the ‘cells’ of the cellulation $C$.

Here a polyhedron is ‘non-degenerate’ if it is not contained in a hyperbolic plane. The third condition excludes some ‘bad’ configuration, like the one depicted in figure 1, where two simplices have an intersection which is not a face in any of them.

**Definition 5.2.** A triangulation of $M$ is a cellulation whose cells are all simplices.

**Definition 5.3.** An ideal hyperbolic manifold $M$ is triangulable if it admits a triangulation. It is virtually triangulable if it has a finite cover which is triangulable.

The main goal of this section will be to prove the:
Lemma 5.4. Any ideal hyperbolic manifold is virtually triangulable.

The proof will proceed in several steps. The first point is the:

Proposition 5.5. Any ideal hyperbolic manifold admits a cellulation.

Proof. It is done along the ideas of Epstein and Penner [EP88]; the situation here is simpler since the action of $\pi_1 M$ on $S^2 \setminus \Lambda$ is discrete.

Let $M$ be an ideal hyperbolic manifold. Then $M$ is isometric to the convex hull of a set $\{x_1, \ldots, x_N\}$ of points in $\partial_{\infty} E(M)$, where $E(M)$ is the unique convex co-compact hyperbolic manifold in which $M$ admits an isometric embedding which is surjective on the $\pi_1$.

For each $i \in \{1, 2, \cdots, N\}$, choose a “small” horoball $b_i \subset E(M)$ with ideal point $x_i$; we suppose that the $b_i$ are small enough to be disjoint. Let $B_i$ be the lift of $b_i$ to $H^3$, which is $\pi_1 M$-invariant collection of disjoint horoballs in $H^3$.

Now we want to use this action of $\pi_1 M$ on the horoballs to produce a cellulation of $M$. This is done in [EP88] by considering the action of $\pi_1 M$ on the light cone in Minkowski 4-space, which contains $H^3$ as a quadric. We will use here a similar, slightly more complicated but maybe a little more geometric, approach. It is based more explicitly on the action of $\pi_1 M$ on the space of horospheres, with an explicit model from [Sch02b].

We consider the projective model of $H^4$ and half of $S^4_1$, the de Sitter space of dimension 4, corresponding to the 3-dimensional models already described above. It can be obtained as follows. $H^4$ and $S^4_1$ are both isometric to quadrics in Minkowski 5-space, with the induced metric:

$$H^4 \cong \{ x \in \mathbb{R}^4 | \langle x, x \rangle = -1 \land x_0 > 0 \} ,$$

$$S^4_1 \cong \{ x \in \mathbb{R}^4 | \langle x, x \rangle = 1 \} .$$

Let $P_0$ be the affine hyperplane of equation $x_0 = 1$ in $\mathbb{R}^4_1$; consider the map sending a point $x \in H^4$ (resp. $x \in S^4_1$ with $x_0 > 0$) to the intersection with $P_0$ of the line going through $x$ and $0$. It is not difficult to check that $\phi$ is projective; it maps $H^4$ to the interior of the radius one ball, and the part of $S^4_1$ standing on one side of a space-like hyperplane to its exterior. We now consider this model only, with $P_0$ identified with $\mathbb{R}^4$.

Using the classical Poincaré model of $H^3$, we can map conformally $H^3$ to the interior of a geodesic ball $B_0$ in $S^3$, e.g. to the hemisphere:

$$S^3_+ := \{ x \in \mathbb{R}^4 | \langle x, x \rangle = 1 \land x_1 \geq 0 \} .$$

Horospheres in $H^3$ are then mapped to spheres in $S^3$ which are interior to $S^3_+$ and tangent to its boundary. Those spheres are the boundaries at infinity of the totally geodesic 3-planes in $H^4$ which are asymptotic to a given 3-plane $H_1$, with $\partial_{\infty} H_1 = \partial S^3_+$.

Their dual points in $S^4_1$ (using the hyperbolic-de Sitter duality, see subsection 3.4) form the vertical cylinder $C_0^+$ which is tangent to $S^3$ along $\partial S^3_+$. In de Sitter terms, $C_0^+$ is the future (or past, depending of the orientation) light cone of a point $H_1^*$ which is at infinity in the projective model of $S^4_1$ which we use.

By construction, the action of $\pi_1 M$ on $H^3$ extends to a conformal action on $S^3$, and thus to a projective action on $\mathbb{R}^4$. This action leaves invariant $S^3_+$, and thus also $H_1$ and $H_1^*$, and therefore also $C_0^+$. For each $i \in \{1, \cdots, N\}$, the horoballs in $B_i$ corresponds to the points of an orbit $O_i$ of the action of $\pi_1 M$ on $C_0^+$. Since the horoballs in $B_i$ are disjoint, it is easy to see that $O_i$ is discrete, with no accumulation point outside $\partial S^3_+$.

To finish the proof, we proceed almost as in [EP88]; consider the convex hull (in $\mathbb{R}^4$) of $O_1 \cup \cdots \cup O_N \cup H_1$, which by the property just pointed out is locally finite outside $\partial S^3_+ = \partial H_1$, with faces which are polyhedra with
a finite number of edges and vertices. Then take the “projection” of this polyhedral surface on $H^3 \subset S^3$ in the vertical direction to obtain the required cellulation. By construction it is invariant under the action of $\pi_1 M$ on $H^3$.

The cellulation of $M$ obtained in this way has a finite number of cells. Otherwise, there would exist an edge $e$ meeting an infinite number of fundamental domains of the action of $\pi_1 M$ on $M$. So $e$ would connect a vertex $v$ of $M$ to the limit set $\Lambda$ of the action of $\pi_1 M$ on $H^3$. Now note that, by construction, $M$ is covered by a finite set of (non-disjoint) ideal simplices $T_1, \ldots, T_N$. So there would exist $i \in \{1, \ldots, N\}$ such that $e$ intersects $\gamma^{-1}T_i$ for an infinite set $S$ of elements $\gamma$ of $\pi_1 M$. Going back to the projective model used above, for each $\gamma \in S$, the segment $\gamma e$ should intersect $T_i$ — and it would also be in the boundary of the convex hull constructed above.

Now $\gamma e$ goes from $\gamma v$ to a point $\gamma v'$ of $\Lambda$; as $\gamma \to \infty$, the vertical coordinates of both $\gamma v$ and $\gamma v'$ go to zero, so that the whole segment $\gamma e$ goes to the horizontal hyperplane containing 0. Therefore, $\gamma e$ lies "below" $T_i$, so that the intersection $\gamma e \cap T_i$ can not be in the boundary of the convex hull. So such an $e$ can not exist, and the cellulation obtained has a finite number of cells.

5.2 From a cellulation to a triangulation

We now consider a cellulation $C$ of an ideal hyperbolic manifold $M$, and call $F_0, F_1, F_2$ and $F_3$ the sets of its faces of dimension 0, 1, 2 and 3 respectively. We also consider the universal cover $\tilde{M}$ of $M$; $C$ lifts to a cellulation $\tilde{C}$ of $\tilde{M}$, and we call $\tilde{F}_0, \tilde{F}_1, \tilde{F}_2$ and $\tilde{F}_3$ the sets of its faces of the 4 possible dimensions.

**Definition 5.6.** Let $C$ be a cellulation of an ideal hyperbolic manifold $M$. $C$ is **large** if, for any cell $\sigma \in \tilde{F}_3$, any vertex $v$ of $\sigma$ and any $\gamma \in \pi_1(\tilde{M})$, if $\gamma v$ is again a vertex of $\sigma$, then $\gamma = 1$.

**Proposition 5.7.** For any ideal hyperbolic manifold $M$ and any cellulation $C$ of $M$, $M$ has a finite cover $\overline{M}$ such that $C$ lifts to a large cellulation. 

**Definition 5.8.** Let $C$ be a cellulation of an ideal hyperbolic manifold $M$. A **polarization** of $C$ is a map $\rho : \tilde{F}_3 \to \tilde{F}_0$ which is equivariant under the action of $\pi_1 M$ and such that:

1. for any $\sigma \in \tilde{F}_3$, $\rho(\sigma)$ is a vertex of $\sigma$;
2. if $\sigma \in \tilde{F}_3$ and $\sigma' \in \tilde{F}_3$ are adjacent (i.e. $\sigma \cap \sigma' \in \tilde{F}_2$), and if $\rho(\sigma)$ and $\rho(\sigma')$ are both vertices of $\sigma \cap \sigma'$, then $\rho(\sigma) = \rho(\sigma')$.

Let’s pause to remark that, although a $(\pi_1 M)$-equivariant map from $\tilde{F}_3$ to $\tilde{F}_0$ induces a map from $F_3$ to $F_0$, the equivariant map contains much more information. Indeed this is already apparent in the simple case where one considers the manifold $S^1$, triangulated with only one edge and one vertex. Its universal cover is $\mathbb{R}$, triangulated with vertices at the integers. There are two $\mathbb{Z}$-equivariant maps sending a segment $[k, k+1]$ to one of its endpoints: $\rho_1 : [k, k+1] \mapsto k$, and $\rho_2 : [k, k+1] \mapsto k + 1$. There is however only one map from $F_3$ to $F_0$, since $F_0$ has only one element.

**Proposition 5.9.** Any large cellulation $C$ of an ideal hyperbolic manifold $M$ admits a polarization.
Proposition 5.10. Any polarized cellulation of an ideal hyperbolic manifold $M$ can be subdivided to obtain a triangulation. If the cellulation is large, so is the triangulation obtained.

The proof of Lemma 5.4 clearly follows from the three propositions above so it remains only to prove them.

Proof of Proposition 5.7. Let $\sigma \in \tilde{F}_3$. First note that, if $\gamma \in \pi_1 M$ fixes the vertices of $\sigma$, then $\gamma = e$; indeed, no non-identity, orientation-preserving element of $\text{Isom}(H^3)$ has more than 2 fixed points at infinity. Therefore, the elements of $\pi_1 M$ leaving $\sigma$ globally invariant are determined by their actions on its vertices, so that there is a finite subset $E_\sigma$ of $\pi_1 M$ of elements leaving $\sigma$ invariant.

$C$ has a finite number of cells, which we can call $\sigma_1, \ldots, \sigma_N$. For each $i$, let $\tau_i$ be a cell of $C$ whose projection on $M$ is $\sigma_i$. Then $E := \bigcup E_{\sigma_i}$ is finite, where $E$ is the set of elements of $\pi_1 M$ leaving one of the $\tau_i$ invariant. Since there is a finite number of cells $\tau_i$ sharing a vertex with some given cell $\tau_i$, the set $F_{\tau_i}$ of elements $\gamma$ of $\pi_1 M$ such that $\tau_i$ is invariant. Therefore, the set $F := \bigcup F_{\tau_i}$ is finite.

Since there is a finite number of cells $\tau_i$ sharing a vertex with some given cell $\tau_i$, the set $F_{\tau_i}$ of elements $\gamma$ of $\pi_1 M$ such that $\tau_i$ is invariant, and therefore residually finite (see e.g. [LS01], chapter III, 7.11). Thus there exists a normal subgroup $\Gamma$ of $\pi_1 M$ of finite index, such that $\Gamma \cap F = \{1\}$. The corresponding finite cover $\tilde{M}$ of $M$ has the required property. □

Proof of Proposition 5.9. We will construct the required polarization $\rho$ as the endpoint of a sequence of partially defined equivariant functions $\rho_i : \tilde{F}_3 \to \tilde{F}_0$ (that is, functions defined on a subset of $\tilde{F}_3$ only).

First choose $s_0 \in \tilde{F}_0$; since $C$ is large, for any $\sigma \in \tilde{F}_3$, at most one of the vertices of $\sigma$ is in $(\pi_1 M) s_0 \subset \tilde{F}_0$. Define $\rho_0$ on a cell $\sigma \in \tilde{F}_3$ as follows:

- if there exists $\gamma \in \pi_1 M$ such that $\gamma s_0$ is a vertex of $\sigma$, then set $\rho_0(\sigma) := \gamma s_0$;
- otherwise, leave $\rho_0$ undefined at $\sigma$.

It is clear that this partially defined map is equivariant.

Now choose $s_1 \in \tilde{F}_0$ such that some $\sigma \in \tilde{F}_3$ on which $\rho_0$ is not defined has $s_1$ as a vertex, and define $\rho_1$ as follows:

- if there exists $\gamma \in \pi_1 M$ such that $\gamma s_1$ is a vertex of $\sigma$, then $\rho_1(\sigma) := \gamma s_1$;
- otherwise, $\rho_1(\sigma) := \rho_0(\sigma)$.

The second case includes the possibility that $\rho_0$ is undefined at $\sigma$, then $\rho_1$ remains undefined at $\sigma$.

Then repeat this construction with $s_2$ to obtain a map $\rho_2$, etc. The number of cells of $\tilde{F}_3$ on which $\rho_i$ is not defined decreases by at least one unit at each step, and $C$ has a finite number of cells, so after a finite number of steps we obtain an equivariant map $\rho := \rho_N : \tilde{F}_3 \to \tilde{F}_0$ which is everywhere defined.

We now want to prove that $\rho$ is a polarization. It is clear by construction that, for any $\sigma \in \tilde{F}_3$, $\rho(\sigma)$ is a vertex of $\sigma$. Let $\sigma' \in \tilde{F}_3$ be another cell, such that $\sigma \cap \sigma' \in \tilde{F}_2$. Let:

$$i_0 := \text{max}\{i \in \{1, \ldots, N\} \mid \exists \gamma \in \pi_1 M, \gamma s_i \text{ is a vertex of } \sigma \},$$

$$j_0 := \text{max}\{j \in \{1, \ldots, N\} \mid \exists \gamma' \in \pi_1 M, \gamma' s_j \text{ is a vertex of } \sigma' \}.$$

We consider two cases:

1. $\gamma s_{i_0}$ is a vertex of $\sigma$ but not of $\sigma'$; then $\rho(\sigma) = \gamma s_{i_0}$ is not a vertex of $\sigma'$, and condition (2) of Definition 5.8 is satisfied. The same applies if $\gamma' s_{j_0}$ is not a vertex of $\sigma$.

2. $\gamma s_{i_0}$ and $\gamma' s_{j_0}$ are both vertices of both $\sigma$ and $\sigma'$. But then, by definition of $i_0$ and $j_0$, $i_0 = j_0$ and $\rho(\sigma) = \gamma s_{i_0} = \gamma' s_{j_0} = \rho(\sigma')$, so that condition (2) of Definition 5.8 again applies.

□

Proof of Proposition 5.10. Let $C$ be a cellulation of $M$, with a polarization $\rho$. We first built a $\pi_1 M$-invariant triangulation of $\tilde{F}_2$ as follows.

Let $f \in \tilde{F}_2$ be such that some vertex $s$ of $f$ is $\rho(\sigma)$, where $\sigma$ is one of the cells bounded by $f$. Then $s$ is the unique vertex of $f$ with this property, because of condition (2) of Definition 5.8. Define a triangulation of $f$ by adding the edges going from $s$ to all the other vertices of $f$. Repeat this for all the 2-faces with this property.

Then subdivide all the remaining non-triangular 2-faces of $\tilde{F}_2$, so as to obtain an equivariant triangulation of $\tilde{F}_2$.

Finally, define a triangulation of $M$ by subdividing each cell $\sigma$ of $\tilde{F}_3$ by adding triangles containing $\rho(\sigma)$ and any edge of $\sigma$ not containing $\rho(\sigma)$. It is clear that:
1. this defines an equivariant decomposition of $\tilde{M}$ into simplices, which is obtained by subdividing each cell into simplices, and thus a decomposition of $M$ into a finite number of simplices with disjoint interior;

2. the simplices are non-degenerate;

3. if $\sigma \in \tilde{F}_3$ and $\sigma' \in \tilde{F}_3$ are two adjacent simplices, then $\sigma \cap \sigma'$ is a face in both of them, because it has to be one of the triangles of the triangulation obtained above.

The definition then directly shows that the triangulation obtained in this manner from a large cellulation is itself large. □

5.3 Some elementary combinatorics

We now fix a triangulation $\Sigma$ of $M$, with which we will stick until section 8. We call $f$ the number of its 3-simplices, $t$ the number of its 2-faces, $e$ the number of its edges, $e_i$ and $e_b$ the number of interior and boundary edges respectively, and $v$ the number of vertices. We will need later on the following easy consequence of the Euler formula.

Lemma 5.11. $2f = 2e_i + e_b - v$.

Proof. Consider the closed triangulated manifold obtained by gluing two copies of $(M, \Sigma)$ along their boundary by the identity map. This triangulated manifold has $\tilde{f} := 2f$ simplices, $\tilde{t}$ 2-faces, $\tilde{e} := 2e_b + e_i$ edges, and $\tilde{v} := v$ vertices.

Since the Euler characteristic is 0 in odd dimensions:

$$\tilde{f} - \tilde{t} + \tilde{e} - \tilde{v} = 0.$$ 

Moreover, each 2-face bounds two simplices, and each simplex has 4 faces, so that:

$$\tilde{t} = 2\tilde{f}.$$ 

Therefore:

$$-\tilde{f} + \tilde{e} - \tilde{v} = 0,$$

and the result follows. □

6 Hyperbolic structures on triangulated manifolds

This section contains the definitions and basic properties concerning some simple notions of singular hyperbolic structures. The idea is to construct such structures by gluing ideal simplices, and then to show that the set of those structures with some constraints actually contains a smooth hyperbolic metric. This will be done in the next section using a variational argument, and will lead to an infinitesimal rigidity statement which plays an important role in the proof of the main theorem.

The ideas used here were mostly developed previously for ideal polyhedra in $H^3$. Their history is interesting. The first results were obtained by Andreev [And71], and then developed by Thurston [Thur97], who found the relationship between circle packings and ideal polyhedra. Colin de Verdière [CdV91] then noted (in the context of circle packings) that the results could be recovered using a variational approach, while Brägger [Brä92] (in the setting of circle packings) and independently Rivin [Riv94] (studying ideal polyhedra) understood that one could take as the functional the hyperbolic volume. The approach followed here is close to that of Rivin [Riv94], although we only use it here to obtain infinitesimal rigidity.

6.1 Sheared hyperbolic structures

We consider here a triangulation $\Sigma$ of $M$.

Definition 6.1. A sheared hyperbolic structure on $(M, \Sigma)$ is the choice, for each simplex $S$ in $\Sigma$, of a diffeomorphism from $S$ to an ideal simplex. We denote by $\mathcal{H}_{sh}$ the set of hyperbolic structures on $(M, \Sigma)$. 

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This vocabulary is justified by the elementary remark that, since all ideal triangles in $H^2$ are isometric, there is a unique way of gluing the hyperbolic simplices which are given by a sheared hyperbolic structure along their common faces. One then obtains a hyperbolic metric on the complement of the interior edges of $\Sigma$. But this metric does not extend over the edges; rather, the model of what happens along an edge is obtained by taking the quotient of the universal cover of $H^3$ minus a geodesic $g$ by the group generated by the composition of a rotation of angle $\theta$ around $g$ and a translation of length $\delta$ along $g$. We then call $\theta$ the angle around the edge, and $\delta$ the shear along the edge.

Those hyperbolic structures will be considered up to diffeomorphisms acting on the simplices; therefore, the choice of a sheared hyperbolic structure is equivalent to the choice of the three dihedral angles of each simplex, subject to the condition that their sum is $\pi$. So $H$ can be identified with the product of $f$-2-simplices, and it has a natural affine structure.

**Definition 6.2.** A sheared hyperbolic structure on $(M, \Sigma)$ is a **singular hyperbolic structure** if the shear of all the interior edges vanishes. It is **smooth** if, in addition, the total angle around all interior edges is $2\pi$.

The set of singular hyperbolic structures is denoted by $H_{si}^1$, the set of smooth hyperbolic structures by $H_{sm}$.

In other words, a singular hyperbolic structure on $(M, \Sigma)$ defines a hyperbolic cone-manifold structure on $M$, which is singular on the edges of $\Sigma$. A smooth hyperbolic structure defines a hyperbolic metric on $M$. In both cases the boundary is piecewise totally geodesic.

The shear at an interior edge of $\Sigma$ can be understood in the following elementary way.

**Definition 6.3.** Let $s$ be an ideal simplex, and let $e$ be an edge of $s$. Choose an orientation of $e$, and let $f_1$ and $f_2$ be the faces of $s$ containing $e$, in the order defined by the orientation of $e$. Let $x_1$ and $x_2$ be the orthogonal projections on $e$ of the vertices of $f_1$ and $f_2$ respectively which are not in $e$. The **shear** of $s$ at $e$ is the oriented distance between $x_1$ and $x_2$.

Note that the shear of a simplex at an edge is independent of the orientation chosen.

**Remark 6.4.** Let $h \in H_{sh}^1$. The shear of $h$ at an interior edge $e$ is the sum, over the simplices containing $e$, of their shears at $e$.

### 6.2 Angles

For each edge $e$ of an ideal triangulation of a sheared hyperbolic structure, we define the **angle** at $e$ to be the sum of the dihedral angles at that edge of the simplices containing it – this applies to interior as well as to boundary edges. For boundary edges this will also be called the **interior dihedral angle**, and the **exterior dihedral angle** is $\pi$ minus the interior dihedral angle. For interior edges, the **excess angle** is the angle minus $2\pi$, and the **singular curvature** around the corresponding edge is minus the excess angle.

Note that, for $h \in H_{sh}^1$, the sum of the exterior angles of the edges arriving at a vertex is equal to $2\pi$ plus the sum of the excess angles at the interior angles. This is checked by applying the Gauss-Bonnet formula to the link of the vertex, which is piecewise Euclidean manifold.

**Definition 6.5.** Let $\theta : \Sigma_1 \to \mathbb{R}_+$ be an assignment of “angles” to the edges of $\Sigma$. We will say that $\theta$ is “ideal” if:

- the angles assigned to boundary edges are in $(0, \pi)$;
- the angles assigned to interior edges are in $(0, 2\pi)$;
- at each vertex, the sum of the exterior dihedral angles of the boundary edges equals $2\pi$ plus the sum of the angle excess at the interior angles.

The set of ideal angle assignments is denoted by $\Theta$. For each $\theta \in \Theta$, we denote by $H_{sh}^2(\theta)$ the set of sheared hyperbolic structures on $\Sigma$ such that the angles associated to each edge is given by $\theta$; then $H_{si}^2(\theta) := H_{si}^1(\theta) \cap H_{si}^2$, and $H_{sm}(\theta) := H_{sh}^2(\theta) \cap H_{sm}$.

**Lemma 6.6.** If the triangulation $\Sigma$ is large, then $\Theta$ corresponds to the interior of a polytope of dimension $e-v$ in $\mathbb{R}^e$.

**Proof.** The only point is to prove that the constraints on the vertices are linearly independent. So let $C$ be a linear combination of those constraints which is zero. In other terms, $C$ is a function $C : \Sigma_0 \to \mathbb{R}$ such that, for each oriented edge $e$, $C(e_+) + C(e_-) = 0$. We supposed that $\Sigma$ is large, so it is quite obvious that the values of $C$ at the vertices of a triangular face of $\Sigma$ have to be 0, and therefore that $C = 0$. 

$\square$
Denition 6.7. We say that an ideal angle assignment is smooth if the angle assigned to each interior edge is $2\pi$. The set of those angle assignments is denoted by $\Theta_{sm}$.

7 First order variation of the volume

The main goal of this section is to use elementary properties of the volume — seen as a functional on the space $H_{sh}$ of sheared hyperbolic structures — to prove the existence of singular hyperbolic structures with given angles on the interior and boundary edges. In the next section, the results and some similar arguments will be used to understand the smooth hyperbolic structures with given dihedral angles at the boundary edges. The method used here is similar to that of [Riv94], while the use of sheared hyperbolic structures, for instance, can be traced back to Thurston [Thu97].

7.1 Definitions and first properties

We consider here again an ideal hyperbolic manifold, along with a triangulation $\Sigma$, and an angle assignment $\theta \in \Theta$. We first define the volume of a singular hyperbolic structure in the most obvious way.

Denition 7.1. For $h \in H_{sh}$, the volume of $h$, $V(h)$, is the sum of the hyperbolic volumes of the simplices of $\Sigma$.

As an immediate consequence of Lemma 2.5, we nd that:

Lemma 7.2. $V$ is a concave function on $H_{sh}$ (with the affine structure coming from the parametrization by the dihedral angles of the simplices).

We now need to understand how the volume varies when one deforms a sheared hyperbolic structure. Unfortunately the Schläfi formula (1) does not apply directly to sheared hyperbolic structures, and does not even make sense in this case; indeed there is no way to choose a horosphere centered at a given vertex, since the holonomy around an edge $e$ would act on it by translation along the edge, with a translation distance equal to the shear at $e$.

To understand this point better, we nd a sheared hyperbolic structure $h \in H_{sh}(\theta)$, and a rst order variation $h \in T_h H_{sh}(\theta)$. Suppose given an ideal triangulation of $h$. $h$ determines a rst order variation of the dihedral angles of the simplices, to which the Schläfi formula (1) can be applied, once a horosphere around each vertex is chosen for each simplex. To get a better understanding of the rst order variation of the volume, we can choose, for each vertex $v$, a horosphere centered on $v$ for each simplex containing $v$. We call this collection of horospheres a choice of horospheres at $v$.

Denition 7.3. Let $e$ be an edge of $\Sigma$, with vertices $e_-$ and $e_+$. Let $H_-$ and $H_+$ be choices of horosphere at $e_-$ and $e_+$ respectively. The volume variation at $e$ associated to $h$ is:

$$\sum_i L_i \alpha_i$$

where the sum is over the simplices containing $e$, $L_i$ is the oriented length of the part of $e$ which is between the horospheres in $H_-$ and $H_+$, and $\alpha_i$ is the rst-order variation of the dihedral angle at $e$.

7.2 Volume differential and shears

Of course the point is that, once horospheres are chosen, the volume variation at $e$ can be seen as the contribution coming from $e$ to the rst order variation of the volume; indeed, the Schläfi formula (1) indicates that:

Remark 7.4. If a horosphere choice is given for all vertices of $\Sigma$, the rst order variation of the volume under the deformation $h$ is the sum of the volume variations at the edges.

Denition 7.5. Let $T$ be a 2-face of the triangulation $\Sigma$, and let $v$ be a vertex contained in $T$. A choice of horospheres at $v$ is coherent at $T$ if the horospheres on both simplices containing $T$ have the same intersection with $T$.

The point is that, as explained above, if $h$ has a non-zero shear at an edge $e$ containing $v$, then it is not possible to nd a horosphere choice at $v$ which is coherent on all 2-faces containing $e$. 

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Proposition 7.6. Let $e$ be an oriented interior edge of $\Sigma$, with endpoints $e_-$ and $e_+$. Let $f_1, \ldots, f_N$ be the 2-faces containing $e$, oriented in cyclic order. Choose $i, j \in \{1, \ldots, N\}$. Let $H_-$ be a choice of horospheres at $e_-$ around $e$ which is coherent except at $f_i$, and let $H_+$ be a choice of horospheres at $e_+$ around $e$ which coherent except at $f_j$. Let $\hat{h}$ be an infinitesimal variation of $h$ which does not change the total angle around the interior edges. The variation of the volume at $e$ associated to $\hat{h}$ is:

$$V_e = s_e \sum_{k=j}^{i-1} \alpha_k,$$

where $s_e$ is the shear of $h$ at $e$, and $\alpha_k$ is the angle between $f_k$ and $f_{k+1}$.

Proof. The definition of $H_-$ and $H_+$ shows that there are two numbers $L, L' \in \mathbb{R}$ such that:

- for each $k \in \{i, i+1, \ldots, j-1\}$, the distance along $e$ between the horospheres of $H_-$ and $H_+$ in the simplex containing faces $f_k$ and $f_{k+1}$ is $L$.
- for each $k \in \{j, j+1, \ldots, i-1\}$, the same distance is $L'$.

Then $L' - L = s_e$, again by definition of $H_-, H_+$ and $s_e$. So, by Definition 7.3:

$$V_e = \sum_{k=j}^{i-1} L \cdot \alpha_k + \sum_{k=j}^{i-1} L' \cdot \alpha_k = \sum_{k=j}^{i-1} L \cdot \alpha_k + \sum_{k=j}^{i-1} (L + s_e) \cdot \alpha_k.$$

But the total angle around $e$ remains constant in the variation $\hat{h}$, which means that:

$$\sum_k \alpha_k = 0.$$

Subtracting $L$ times this equation to the previous one leads to the statement.

As a consequence we see that the first-order variation of the volume, under a deformation which does not change the total angle around the interior angles or the dihedral angles, has a remarkably simple form, more precisely it can be expressed only in terms of the shears at the interior edges.

Lemma 7.7. $dV(\hat{h})$, as a linear form on $\mathcal{H}_{sh}(\theta)$, depends only on the shear at the interior edges of $\Sigma$.

Proof. This is an immediate consequence of the previous proposition, and of the fact that the dihedral angle at the exterior edges is constant.

This means in particular that the volume — seen as a function on $\mathcal{H}_{sh}(\theta)$ — is critical when the shear vanishes at all interior edges. We will see below that the converse is true too. This will use a special type of deformations, defined as follows.

Definition 7.8. Let $h \in \mathcal{H}_{sl}$, and let $e \in \Sigma_1$ be an oriented interior edge of $\Sigma$. A local deformation $\hat{h}_0$ of $h$ at $e$ is determined as follows. Choose a face $f$ containing $e$. Then let $s_0, s_1, s_2, \ldots, s_N = s_0$ be the simplices of $\Sigma$ containing $e$, in cyclic order. Let $a_i^+$ and $b_i^+$ be the edges of $s_i$ containing $e_+$ (other than $e$), and $a_i^-$ and $b_i^-$ the edges of $s_i$ containing $e_-$ (again other than $e$), ordered so that $a_i^+ = b_{i+1}^+$ and that $a_i^- = b_{i+1}^-$. Then, in the deformation $\hat{h}_0$:

- the angles of the simplices $s_i$ at $e$ do not change;
- the angles of $s_i$ at $a_i^+$ and $b_i^+$ vary at speed 1;
- the angles of $s_i$ at $a_i^-$ and $b_i^-$ vary at speed $-1$.

It is a simple matter to check the following properties of local deformations:

Remark 7.9. The above definition indeed defines a deformation of $h$ in $\mathcal{H}_{sh}$. Moreover, it does not change the total angles at the edges, i.e. if $h \in \mathcal{H}_{sh}(\theta)$, then $\hat{h} \in T_h \mathcal{H}_{sh}(\theta)$. 30
The local deformations can then be used to prove that, among the sheared hyperbolic structures, those which have zero shear are exactly the critical points of the volume, seen as a functional on the space of sheared hyperbolic structures having a given total angles on all edges.

**Lemma 7.10.** Let \( \theta \in \Theta \) and let \( h \in \mathcal{H}_{sh}(\theta) \); then the restriction of \( dV \) to \( T_h \mathcal{H}_{sh}(\theta) \) is zero if and only if the shear of \( h \) at all interior edges is zero, that is, if and only if \( h \in \mathcal{H}_{si} \).

**Proof.** Lemma 7.7 shows that the volume as a functional on \( \mathcal{H}_{sh}(\theta) \) is critical when the shear is zero at all interior edges. Remark 7.9 shows that the local deformations are tangent to \( \mathcal{H}_{sh}(\theta) \).

Suppose that the shear at an edge \( e \) is non-zero, and consider the simplices containing \( e \), as in figure 3. It is possible to choose horospheres at the vertices of those simplices such that:

- at \( e_+ \) and \( e_- \), the choices are coherent at the triangles \((e, a^+_i, a^-_i)\), except for \( i = 0 \);
- at \( e_+ \) and \( e_- \), the distance between the two horospheres on each side of the triangle \((e, a^+_0, a^-_0)\) is equal to the shear at \( e \);
- for each \( i \neq 0 \), the two horospheres at the vertex \( a^-_i \cap a^+_i \) are coherent.

Consider a local deformation at \( e \). The only contributions to the variation of the volume comes from the \( a^-_i \) and the \( a^+_i \), since they are the only edges where the angles vary. Moreover they add up to zero except at \( a^+_0 \) and \( a^-_0 \), because of the choices of horospheres described above (they are coherent on both sides of the other edges). Finally, the contributions from \( a^-_0 \) and \( a^+_0 \) are non-zero and of the same sign.

So Proposition 7.6 shows that the shears have to be zero at all interior edges at critical points of \( V \).

**7.3 Consequences**

The results of the previous subsection now easily lead to interesting results on singular hyperbolic structures (i.e. those without shears).

**Corollary 7.11.** Let \( \theta_0 \in \Theta \) and \( g_0 \in \mathcal{H}_{si}(\theta_0) \). There exists a neighborhood \( U \) of \( \theta_0 \) in \( \Theta \) and a neighborhood \( V \) of \( g_0 \) in \( \mathcal{H}_{sm} \) such that, for any \( \theta \in U \), there exists a unique \( g \in \mathcal{H}_{si}(\theta) \cap V \).

**Proof.** By Corollary 2.6, \( V \) is a strictly concave function on \( \mathcal{H}_{sh} \). Moreover, the restriction of \( V \) to \( \mathcal{H}_{sh}(\theta_0) \) has a critical point at \( g_0 \). Therefore, for \( \theta \) close enough to \( \theta_0 \), \( V \) has a unique critical point of \( V \) on \( \mathcal{H}_{sh}(\theta) \) close to \( g_0 \); by Lemma 7.10 it is a point in \( \mathcal{H}_{si}(\theta) \).

**Corollary 7.12.** For each \( \theta \in \Theta \) such that \( \mathcal{H}_{sh}(\theta) \) is non-empty, \( \mathcal{H}_{sh}(\theta) \) is a submanifold of \( \mathcal{H}_{sh} \) with dimension equal to the number \( e_i \) of interior edges of the triangulation.

**Proof.** The space of ideal simplices in \( H^3 \) has dimension 2, so that the dimension of \( \mathcal{H}_{sh} \) is equal to twice the number of simplices in \( \Sigma \). Thus, by Lemma 5.11, \( \dim \mathcal{H}_{sh} = 2e_i + e_b - v \), where \( e_b \) and \( v \) are the number of boundary edges and of vertices of \( \Sigma \) respectively.
Specifying the dihedral angles and the excess angle at the interior edges adds $e_i + e_b$ constraints; they are not linearly independent, however, since they satisfy at least one linear condition for each vertex. The loss of dimensions due to those constraints is therefore at most $e_i + e_b - v$, so that, for $\theta \in \Theta$, $\dim \mathcal{H}_{\text{sl}}(\theta) \geq e_i$.

But Lemma 7.7 indicates that $dV$ depends only on the $e_i$ sheets at the interior edges, while the strict concavity of $V$ shows that the Jacobian of $dV$ is non-degenerate. Therefore, its restriction to $\mathcal{H}_{\text{sm}}(\theta)$ is also non-degenerate, so that $\dim \mathcal{H}_{\text{sh}}(\theta) \leq e_i$, and the result follows.

Finally, using the non-degeneracy of the volume functional also leads to an infinitesimal rigidity result.

**Corollary 7.13.** Let $h \in \mathcal{H}_{\text{sl}}$, and let $h \in T_h \mathcal{H}_{\text{sl}}$ be a first-order deformation of $h$. Then $h$ corresponds to a non-zero first-order deformation of either the dihedral angles of the boundary edges, or the excess angle at the interior edges. $\mathcal{H}_{\text{sl}}$ is, in the neighborhood of $h$, a manifold of dimension $e_i + e_b - v$.

**Proof.** This is a direct consequence of Corollary 7.12, which shows that $\mathcal{H}_{\text{sh}}$ is foliated by the $\mathcal{H}_{\text{sl}}(\theta)$, which are submanifolds of dimension $e_i$. We already know that $\mathcal{H}_{\text{sh}}$ is a manifold of dimension $2f$, and, by Lemma 5.11, $2f = 2e_i + e_b - v$, where $e_b$ is the number of boundary edges and $v$ is the number of vertices. But the dimension of the space $\Theta$ of possible angle assignments is at most $e_b + e_i - v$, since the dihedral angles at the boundary edges satisfy one linear condition for each vertex. Thus this dimension has to be exactly $e_b + e_i - v$, and each non-trivial deformation in $\mathcal{H}_{\text{sl}}$—which is transverse to the foliation of $\mathcal{H}_{\text{sh}}$ by the $\mathcal{H}_{\text{sh}}(\theta)$—induces a non-trivial deformation of some dihedral angle on the boundary or of some total angle on an interior edge.

Finally, an elementary dimension-counting argument, along with the previous corollary, shows that any infinitesimal deformation of the dihedral angles on the boundary—and of the angle excess around the interior edges—can be realized by some $\bar{h} \in T_h \mathcal{H}_{\text{sl}}$, if it satisfies the condition that, at each vertex, the sum of the exterior dihedral angles remains equal to $2\pi$ plus the sum of the angle excess at the interior edges. Restricting to the case of smooth hyperbolic structures, this can be formulated as follows. We call an admissible infinitesimal deformation of the dihedral angles such that the sum of the exterior angles of the edges at any vertex remains $2\pi$.

**Lemma 7.14.** Let $(M, h)$ be an ideal hyperbolic manifold. For any admissible infinitesimal variation $\theta$ of the dihedral angles on the boundary, there exists a (unique) infinitesimal deformation of $h$ in $\mathcal{H}_{\text{sl}}$ which induces the variation $\theta$.

**Proof.** By Lemma 5.4, there exists a finite cover $\overline{M}$ of $M$ so that the hyperbolic structure $\overline{h}$ lifted to $\overline{M}$ from $h$ admits an ideal triangulation, say $\Sigma$. Let $\bar{h}$ be the dihedral angles of $\overline{h}$. The previous corollary shows that there is no non-trivial deformation of $\bar{h}$ which does not change the dihedral angles, and that the dimension of $\mathcal{H}_{\text{sm}}$ in the neighborhood of $\overline{h}$ is $e_b - v$. Thus, for each admissible admissible infinitesimal deformation $\bar{\theta}$ of the dihedral angles of $\overline{h}$, there is a unique infinitesimal deformation $\bar{h}$ of $\overline{h}$ inducing $\bar{\theta}$. If now $\theta$ is an infinitesimal deformation of $\theta$, it lifts to an infinitesimal deformation $\bar{\theta}$ of the dihedral angles of $\overline{h}$, which is induced by a unique admissible infinitesimal deformation $\bar{h}$ of $\overline{h}$. Since $\bar{\theta}$ is equivariant, $\bar{h}$ is obviously equivariant (by uniqueness) and therefore defines an admissible infinitesimal deformation $h$ of $h$ inducing $\theta$, which is unique since $\overline{h}$ is.

Note that it is not necessary to suppose that the hyperbolic ideal manifold which we consider has a boundary which is triangulated, i.e. this lemma is also valid when some of the faces have more than 3 edges. In this case one can add edges to those faces, so as to obtain triangulation, and then choose the first-order variation of the dihedral angles at those new edges. In particular it is possible to choose first-order variations which "lose" the convexity of the boundary.

### 8 Dihedral angles

We consider here an ideal hyperbolic manifold $M$. The first point will be a compactness result for sequences of ideal structures on $M$ with a given triangulation of the boundary, when the dihedral angles converge. We will then describe in more details the situation with a fixed triangulation of $M$, then see what happens when one considers only a fixed cellulation $\sigma$ of the boundary, and finally how the affine structures corresponding to different triangulations of the boundary can be glued together.
8.1 Compactness of ideal manifolds

We consider a fixed triangulation $\sigma$ of $\partial M$; we will give in this subsection a compactness lemma for ideal structures with boundary combinatorics corresponding to $\sigma$, then use it in the next subsections to obtain a description of the possible dihedral angles.

**Lemma 8.1.** Let $(h_i)$ be a sequence of ideal structures on $M$ with boundary combinatorics given by $\sigma$. Suppose that the dihedral angles of the boundary vertices for the $h_i$ converge as $i \to \infty$ to limits in $(0, \pi)$. Then:

- either there is a non-elementary circuit in $\sigma$ on which the sum of the limit dihedral angles is $2\pi$;
- or there exists a sub-sequence of $(h_i)$ which converges to an ideal structure on $M$.

The intuitive idea behind this lemma is simple. Consider the induced metrics on one of the boundary components. For each $n$, it is a complete hyperbolic metric of finite area. So there is a subsequence which either converges to a hyperbolic metric of finite area, or has a closed geodesic whose length goes to 0. If the first case happens for all boundary components, then both the induced metric and the dihedral angles converge, so that the universal cover in $H^3$ of each boundary component converges, and thus the sequence of metrics on $M$ has to converge. In the second case, the short geodesic can either:

- be contractible in $M$, so that it corresponds to a non-elementary circuit on which the sum of the exterior dihedral angles goes to $2\pi$.
- be non-contractible in $M$, and then the sum of the dihedral angles of the edges it intersects has to tend to 0, since otherwise its lift in $\tilde{M}$ would have a very high “curvature per unit length”, which is impossible by a known argument.

In the real world, however, things are a little more complicated since one has to keep track of the relationship between the induced metrics and the triangulation; the proof will use a couple of definitions and propositions.

First, there is a natural notion of collapsing of vertices for the sequence $(h_i)$. We consider the triangulation $\sigma$ of $\partial M$, and its lift to a triangulation $\tilde{\sigma}$ of the universal cover of $\partial M$. Note that each hyperbolic structure $h_i$ on $M$ defines a hyperbolic structure on the universal cover $\tilde{M}$ of $M$, and thus a polyhedral embedding of $\partial \tilde{M}$ in $H^3$.

**Definition 8.2.** Let $S := \{s_1, \ldots, s_p\}$ be a subset, with non-empty complement and cardinal $p \geq 2$, of the set of vertices of $\sigma$ which are in $\partial M$. We say that $S$ 
\textbf{collapses} if there exists a subsequence of $(h_i)$, sequences $(\tilde{s}_{j,i})_{i \in \mathbb{N}}$, for each $j \in \{1, \ldots, p\}$, of lifts of the $s_j$ to $\partial \tilde{M}$, and a sequence of projective maps $\rho_i : H^3 \to B_0(1) \subset \mathbb{R}^3$ such that the following conditions are satisfied:

- Let $\Lambda_i \in \partial_\infty H^3$ be the limit set of the action of $\pi_1 M$ on corresponding to $h_i$. Then the sequence $(\rho_i(\Lambda_i))$ converges, in the Hausdorff topology, to a limit $\Lambda \subset S^2$.
- The $\rho_i(\tilde{s}_{j,i})$ converge, as $i \to \infty$, to a point $x_0$ in $S^2$, and $x_0 \notin \Lambda$.
- For any vertex $v$ of $\tilde{\sigma}$ which does not project to $S$, $\rho_i(v)$ remains outside some neighborhood of $x_0$.

Given such a collapsing set $S$, we will say that the collapse is:

- **finite** if, given $x_0$, the choice of the lift $\tilde{s}_{j,i}$ of $s_i$ is unique, as $i \to \infty$, for each $i \in \{1, \ldots, p\}$.
- **infinite** if, for some $j \in \{1, \ldots, p\}$ and for all $i \in \mathbb{N}$, there are two possibilities lifts $\tilde{s}_{j,i}$ and $\tilde{s}'_{j,i}$ of $s_j$, with $\tilde{s}_{j,i} \neq \tilde{s}'_{j,i}$, such that $\rho_i(\tilde{s}_{j,i}) \to x_0$ and $\rho_i(\tilde{s}'_{j,i}) \to x_0$.

Here $B_0(1)$ is the radius 1 ball centered at 0 in $\mathbb{R}^3$. Note that the terms “finite” or “infinite” do not apply to the set $S$ — which is always finite — but to the set of vertices of $\tilde{\sigma}$ which converge to the limit point $x_0$.

This notion of "collapse" is useful because it implies the existence of a closed path in $\sigma$ on which the sum of the dihedral angles goes either to $2\pi$ or to 0. We will see that this limit is $2\pi$ if the collapse is finite, and then the closed path is contractible in $M$; it is 0 if the collapse is infinite, and then the closed path is non-contractible in $M$.

**Proposition 8.3.** If a subset $S$ of vertices of $\sigma$ (with cardinal at least 2) has a finite collapse, then, maybe after taking a subsequence of $(h_i)$, there is a non-elementary circuit in the 1-skeleton of $\sigma$ on which the sum of the dihedral angles goes to $2\pi$.
Proof. The argument is the same as in section 4 (and that in [Sch98a, Sch01]): the polyhedron dual to $\partial M$ in de Sitter space, for the limit hyperbolic structure, has a face tangent to the boundary at infinity at the point where the collapse occurs, and the metric induced on this face is the metric of a hemisphere. Therefore the length of its boundary is $2\pi$, and this provides a non-elementary circuit in $\partial$ on which the sum of the dihedral angles converges to $2\pi$. □

Note that the non-elementary circuit on which the sum of the dihedral angles goes to $2\pi$ is simply the sequence of edges between a connected component of the lift of $S$ to $\partial M$ and its complement.

We now turn to the case where the "collapse" is infinite.

**Proposition 8.4.** Suppose that there exists a subset $S$ of the set of vertices of $\sigma$ in $\partial M$, which has an infinite collapses. Then there exists a closed curve in $\partial M$, which is not contractible in $M$, whose length for the induced metric goes to 0.

Proof. Let $s_{j_0} \in S$ be a vertex which is adjacent to a vertex $s$ of $\partial M$ which is not in $S$. Let $(\tilde{s}_i)_{i \in \mathbb{N}}$ be a sequence of lifts of $s$ to $\partial M$ such that, for each $i$, $\tilde{s}_i$ is adjacent to $\tilde{s}_{j_0,i}$. Then, as $i \to \infty$,

$$\rho_i(\tilde{s}_{j_0,i}) \to x_0,\quad \text{while,}
$$

maybe after taking a subsequence, $\rho_i(\tilde{s}_i) \to x_1$, where $x_1 \neq x_0$. For each $i \in \mathbb{N}$, we call $y_i$ the middle point of the segment $[\rho_i(\tilde{s}_{j_0,i}),\rho_i(\tilde{s}_i)]$. Then $y_i \to y_\infty$, where $y_\infty := (x_0 + x_1)/2$. Moreover, for each $i$:

$$1 - \|y_i\| = 1 - \left|\frac{\rho_i(\tilde{s}_{j_0,i}) - \rho_i(\tilde{s}_i)}{2}\right| \to 1 - \frac{|x_0 + x_1|}{2}.$$

Since the "collapse" is infinite, there exists an element $\gamma \in \pi_1 M$ such that $\rho_i(\tilde{\gamma}_{j_0,i}) \to x_0$. Let $\gamma_i := \rho_i \circ \gamma \circ \rho_i^{-1}$, so that $\gamma_i$ corresponds to the action of $\gamma$ on the projective model of $H^3$ given by $\rho_i$. $\gamma_i$ acts isometrically and without fixed point on $H^3$, so that $\gamma_i$ acts on $S^2$ by Möbius transformations. The fixed points of $\gamma_i$ are in $\rho_i(A)$, while $x_0 \notin \Lambda$. Since $\lim \rho_i(\tilde{s}_{j_0,i}) = x_0$, it follows that, as $i \to \infty$, the distance between $y_i$ and $\gamma_i y_i$ converges to 0.

For each $i \in \mathbb{N}$, let $c_i$ be a minimal path on $\rho_i(\partial M)$, for the Euclidean metric of $\mathbb{R}^3$, between $y_i$ and $\gamma_i y_i$. Then, as $i \to \infty$, $c_i$ remains at a distance from $S^2 = \partial H^3$ which is bounded from below. Otherwise, since $d(y_i,\gamma_i y_i) \to 0$ and $d(y_i, S^2)$ and $d(\gamma_i y_i, S^2)$ are bounded from below, it would follow by a convexity argument that $\rho_i(M)$ converges to a convex domain in a plane containing $x_0, x_1$, and this is clearly impossible.

Now the length of $c_i$ for the hyperbolic metric clearly goes to 0, since its Euclidean length goes to 0 while the Euclidean distance from $c_i$ to $S^2$ remains bounded from below. Since $c_i$ clearly corresponds to a curve which is non-contractible in $M$, the proposition follows. □

**Proposition 8.5.** Let $c$ be a closed curve in $\partial M$, which is not contractible in $M$. Suppose that there exists a family $(c_n)$ of curves homotopic to $c$, each $c_n$ being geodesic for $h_n$, such that the length of $c_n$ for $h_n$ converges to 0. Then the length of $c_n$ for $h\infty$ goes to 0.

Proof. Fix $n$, and consider the universal cover $\tilde{M}$ of $M$ as a subset of $H^3$. Each $c_n$ lifts to a geodesic in $(\partial \tilde{M}, h_n)$, with endpoints in $\partial_\infty H^3$. The result is obtained by applying to those lifted curves the following elementary statement of hyperbolic geometry (see [Thu97, EM86, BCO3]): there exists a constant $C > 0$ such that, if $\Omega \subset H^3$ is a convex set and $g$ is a complete geodesic in $\partial \Omega$ with endpoints on $\partial H^3$, then, for each segment $s$ of $g$ of length $l$, the total bending of $s$ is at most $C(l + 1)$. □

We now have to define the kind of metrics that appear on the boundary of an ideal hyperbolic manifold $M$. Here $S$ is a compact orientable surface of genus $g \geq 2$, with $N$ points $v_1, \cdots, v_N$ removed ($N \geq 1$), along with a triangulation $\sigma$ whose vertices are the $v_i$ — for instance, $S$ can be $\partial M$ with the vertices removed. We call $a_1, a_2, \cdots, a_k$ the edges of $\sigma$, and $\tau_1, \cdots, \tau_t$ its triangles.

**Definition 8.6.** A hyperbolic structure on $(S, \sigma)$ is a family of gluings between the adjacent triangles of $\sigma$, each being endowed with the metric of the ideal triangle in $H^2$. The set of hyperbolic structures on $(S, \sigma)$ will be denoted by $\mathcal{M}$.

Obviously a hyperbolic structure in this sense defines a hyperbolic metric on $S$, but it is in general not complete; indeed, there might be a "shift" at some of the edges, in the sense described below.

**Definition 8.7.** Let $g$ be a hyperbolic structure on $(S, \sigma)$, and let $E$ be an edge of $\sigma$. Choose an orientation of $E$, and let $T_+$ and $T_-$ be the triangles of $\sigma$ standing on the "right" and on the "left" of $E$ respectively. Let $u_+$ and $u_-$ be the orthogonal projections of $E$ of the vertices opposite to $E$ in $T_+$ and $T_-$ respectively. The shift $sh(E)$ of $g$ at $E$ is the oriented distance, along $E$, between $u_-$ and $u_+$. 34
Note that \( \text{sh}(E) \) does not depend on the orientation chosen for \( E \). This definition is somehow related to the notion of “shear” defined above for an ideal simplex at an edge, but it is better to use a different name to avoid confusions.

**Proposition 8.8.** Suppose that no set of vertices of \( \sigma \) collapses. Then, after taking a subsequence, the shifts of the edges of \( \sigma \) for the \( h_i \) converge.

**Proof.** Choose a "fundamental domain" in \( \tilde{\sigma} \), i.e. a connected subgraph \( \sigma_0 \subset \tilde{\sigma} \) such that each face in \( \sigma \) has a unique inverse image in \( \sigma_0 \) for the canonical projection \( \tilde{\sigma} \to \sigma \).

We consider again the polyhedral embeddings \( \phi_i : \partial M \to H^3 \) associated to the ideal hyperbolic structures \( h_i \) on \( M \). Since \( \sigma_0 \) has a finite number of edges, an elementary compactness argument shows that there exists a sequence \( (\rho_n) \) of conformal maps from \( H^3 \) to the radius 1 ball in \( \mathbb{R}^3 \) such that the images by \( \rho_i \circ \phi_i \) of the vertices of \( \sigma_0 \) converge in \( S^2 \). By applying projective transformations to the ball, we can moreover insure that not all vertices of \( \sigma_0 \) have the same limit.

Since by hypothesis there is no collapsing subset of vertices, there exists a constant \( C > 0 \) such that the edges of the image in \( S^2 \) of \( \sigma_0 \) by the limit have length between \( 1/C \) and \( C \). An elementary argument then shows that the distance on \( S^2 \) between the vertices of \( \sigma_0 \) and the orthogonal projections on the edges of the triangles of the opposite vertices remain bounded between \( 1/C' \) and \( C' \), where \( C' > 0 \) is another constant.

This in turns indicates that (after taking a subsequence) the shifts of \( \sigma \) converge. \( \square \)

**Proof of lemma 8.1.** By hypothesis the dihedral angles of the \( (h_i) \) converge. There can be no collapse of a subset of the set of vertices of \( \sigma \) by Proposition 8.3, Proposition 8.4 and Proposition 8.5. So, after taking a subsequence, the shifts at all edges converge to some limit by Proposition 8.8. Thus the polyhedral embeddings of \( \partial M \) in \( H^3 \) converge, and therefore the hyperbolic structures \( (h_i) \) also do so. \( \square \)

### 8.2 Dihedral angles with a given triangulation

We now have all the tools necessary to describe the dihedral angles of ideal manifolds with a given triangulation of the boundary. We consider a triangulation \( \Sigma \) of \( M \).

**Lemma 8.9.** Consider a triangulation \( \Sigma \) of the interior of \( M \). Let \( H^\Sigma_{\text{sm}} \) be the set of smooth hyperbolic structure obtained by gluing ideal simplices according to \( \Sigma \). Then:

1. the space of possible dihedral angles forms the interior of a convex polyhedron \( \Theta_\Sigma \) in \( \mathbb{R}^e \).
2. for any \( \theta \in \Theta_\Sigma \), there is a unique smooth hyperbolic structure on \( M \) with those dihedral angles; \( H^\Sigma_{\text{sm}} \) therefore inherits the affine structure of \( \Theta_\Sigma \).
3. on each point of the boundary of \( \Theta_\Sigma \), one of the following happens:
   
   (a) one of the simplices of \( \Sigma \) has a dihedral angle which goes to 0;
   (b) the dihedral angle of one of the boundary edges goes to 0 or to \( \pi \);
   (c) there is a non-elementary circuit in the 1-skeleton of \( \Sigma \) for which the sum of the dihedral angles goes to \( 2\pi \).
4. the volume \( V \) is a smooth, strictly concave function on \( \Theta_\Sigma \).

The proofs of points (1) and (2) are consequences of the previous section. Indeed the condition that the dihedral angles are given by \( \theta \) defines a family of linear conditions on the angles, so an affine subspace of the space of angle assignments of the simplices of \( \Sigma \). Adding the condition that the excess angle at each interior edge is \( 2\pi \) defines an affine subspace, and thus a smaller polyhedral in the space of angle assignments on the simplices of \( \Sigma \). Projecting to the space of possible dihedral angles on the boundary edges thus defines a convex polyhedron.

Point (2) follows from a deformation argument. Consider the map \( \Phi : H^\Sigma_{\text{sm}} \to \Theta_\Sigma \) sending an ideal hyperbolic structure to its dihedral angles. Corollary 7.11 shows that \( \Phi \) is locally injective (its differential is injective), while lemma 8.1 shows that \( \Phi \) is proper — it is therefore a covering. But \( H^\Sigma_{\text{sm}} \) is connected, while \( \Theta_\Sigma \) is contractible, so \( \Phi \) is a homeomorphism. Therefore each assignment of dihedral angles in \( \Theta_\Sigma \) is indeed realized by a smooth hyperbolic structure.
Then on $M$.

Lemma 8.11. Consider a cellulation $M$ ideal triangulation. Of course the point is that, by lemma 5.4, a finite cover of $M$ has an angle which goes to $0$ or to $\pi$.

This shows that $x$ is convex, and also that $f$ is strictly concave since $f$ is strictly concave.

**Proof.** Let $c : [0, 1] \to \Omega$ be the geodesic segment parametrized at constant speed such that $c(0) = x_0$ and $c(1) = x_1$. Since $\rho$ is linear, $\overline{c} = \rho \circ c$.

Moreover, since $f$ is strictly concave

$$\forall t \in (0, 1), \quad f \circ c(t) > tf \circ c(0) + (1 - t)f \circ c(1) \ .$$

Therefore, the definition of $\overline{f}$ shows that:

$$\forall t \in (0, 1), \quad \overline{f} \circ c(t) \geq f \circ c(t) > tf \circ c(0) + (1 - t)f \circ c(1) = \overline{f} \circ \overline{c}(0) + \overline{f} \circ \overline{c}(1) \ .$$

This shows that $\overline{f}$ is strictly concave.

8.3 Fixed triangulations of the boundary

Using the ideas and results of the previous section, we can now give a description of the set of dihedral angles which can be achieved on the ideal hyperbolic manifolds with a given cellulation of the boundary. Studying the possible degenerations and a global argument using the results of section 3 (on the fuchsian case) will then lead to theorem 8.17. So we now consider an ideal hyperbolic manifold $M$, which does not necessarily admit an ideal triangulation. Of course the point is that, by lemma 5.4, a finite cover of $M$ admits one.

**Lemma 8.11.** Consider a cellulation $\sigma$ of the boundary $\partial M$, and let $\mathcal{H}_{\text{sm}}^\sigma$ be the set of ideal hyperbolic structures on $M$ such that the boundary is triangulated according to $\sigma$. Then each connected component of $\mathcal{H}_{\text{sm}}^\sigma$ has a natural affine structure $A_\sigma$, obtained by gluing the $\mathcal{H}_{\text{sm}}^\Sigma$ — for $\Sigma$ a triangulation of the interior of $M$ inducing the triangulation $\sigma$ of the boundary — in the natural way on their intersections. $(\mathcal{H}_{\text{sm}}^\sigma, A_\sigma)$ is affinely equivalent to the disjoint union of the interiors of a finite set of convex polyhedra in $\mathbb{R}^3$, and $V$ is a concave function on each.

**Proof.** Let $\theta \in \Theta_\sigma$, and let $h \in \mathcal{H}_{\text{sm}}^\sigma(\theta)$. By lemma 7.14, the infinitesimal deformations of $h$ are parametrized by the admissible infinitesimal deformations of the dihedral angles at the exterior edges.

Let $\Theta_\Sigma$ be the set of dihedral angles that can be attained by deformation from $h$. By proposition 8.12, the boundary is made of angle assignments satisfying one of a number of possible affine equalities, so that $\Theta_\Sigma$ is the interior of a convex polyhedron in $\mathbb{R}^3$. Moreover, its boundary can be decomposed into the union of two components, $\partial \Theta_\Sigma = \partial_1 \Theta_\Sigma \cup \partial_2 \Theta_\Sigma$, where:

1. $\partial_1 \Theta_\Sigma$ is the set of boundary points where the sum of the dihedral angles on the edges of a non-elementary circuit goes to $2\pi$, or the dihedral angle at some boundary edge goes to $0$ or to $\pi$.

2. $\partial_2 \Theta_\Sigma$ is the set of boundary points where the previous condition does not apply, but one of the simplices has an angle which goes to $0$.
By lemma 3.11, \( \partial_i \Theta \) corresponds to the boundary of \( \mathcal{H}_{sm}^\sigma \). On the other hand, let \( \tilde{h} \) correspond to a point of \( \partial_i \Theta \). Lemma 7.14 shows that \( \tilde{h} \) can be deformed so that its dihedral angles go beyond \( \partial_i \Theta \). So there is another ideal triangulation \( \Sigma' \) of \( (M, \tilde{h}) \) such that \( \tilde{h} \in \mathcal{H}_{sm}^\sigma \). Since the affine structure on \( \mathcal{H}_{sm}^\sigma \) is defined in terms of the variations of the dihedral angles, it extends naturally to an affine structure on \( \mathcal{H}_{sm}^\sigma \). This shows that \( \mathcal{H}_{sm}^\sigma \) carries an affine structure \( \mathcal{A}_n \), obtained by gluing the affine structures on the \( \mathcal{H}_{sm}^\sigma \).

Moreover, at a point of \( \partial_i \Theta \cap \partial_i \Theta \), the faces of \( \mathcal{H}_{sm}^\sigma \) corresponding to \( \partial_i \Theta \) are the extensions of the faces of \( \mathcal{H}_{sm}^\sigma \), so that \( \mathcal{H}_{sm}^\sigma \) is locally convex for the affine structure. It is therefore affinely equivalent to the interior of a convex polyhedron in \( \mathbb{R}^3 \).

In addition, lemma 8.9 allows for a simple description of the possible boundary behavior; since the cases corresponding to the collapse of a simplex correspond to the boundary of the cells \( \mathcal{H}_{sm}^\sigma \) rather than to the boundary of \( \mathcal{H}_{sm}^\sigma \), the boundary of \( \mathcal{H}_{sm}^\sigma \) is characterized by one of the other possible limit cases.

**Proposition 8.12.** Let \( (h_n)_{n \in \mathbb{N}} \) be a sequence of elements of \( \mathcal{H}_{sm}^\sigma \) converging to a limit \( h \in \partial \mathcal{H}_{sm}^\sigma \). After taking a subsequence, one of the following occurs:

1. some boundary edge has a dihedral angle which goes to 0 or \( \pi \);
2. there exists a non-elementary circuit in \( \Sigma_{\partial M} \) on which the sum of the dihedral angles goes to \( 2\pi \).

### 8.4 Spaces of angle assignments

The results already obtained give us a good understanding of the local properties of the map sending an ideal hyperbolic manifold to the angle assignment on the boundary edges. To complete the proof of theorem 8.17, we need some global topological information on the space of angle assignments on the boundary satisfying the conditions of the theorem.

We need some additional notations. We will call \( (\partial_i M)_{1 \leq i \leq N} \) the connected components of \( \partial M \). For each \( n \in (\mathbb{N} \setminus \{0\})^N \), we call \( A_n \) the space of couples \((\Gamma, w)\), where \( \Gamma \) is the 1-skeleton of a cellulation of \( \partial M \) having \( n_i \) vertices in each \( \partial_i M \), and \( w : \Gamma \rightarrow (0, \pi) \) is a function satisfying the conditions of theorem 8.17; and we call \( M_n \) the space of ideal hyperbolic manifolds having \( n_i \) vertices in \( \partial_i M \), for each \( i \). \( \Phi_n \) will be the natural map sending an element of \( M_n \) to the corresponding dihedral angles assignment on a cellulation of \( \partial M \), which is in \( A_n \).

We will prove that if \( M \) has incompressible boundary, then, for each \( n \), \( A_n \) is connected. Thus \( \Phi_n \) is a covering. To prove that it is actually a homeomorphism, we will consider some special dihedral angle assignments for which the uniqueness follows from the Mostow rigidity.

**Lemma 8.13.** Suppose that \( M \) has incompressible boundary. Then, for each \( n \in (\mathbb{N} \setminus \{0\})^N \), \( A_n \) is connected.

This is the only point where we use the fact that \( M \) has incompressible boundary.

Recall that, for \( g \geq 2 \) and \( r \geq 1 \), \( A_{g,r} \) is the space of dihedral angles assignments for fuchsian manifolds of genus \( g \), with \( r \) vertices. From now on, we call \( g_i \) the genus of \( \partial_i M \). With the topological assumptions which we have made on \( M \), \( g_i \geq 2 \) for each \( i \).

**Remark 8.14.** Suppose that \( M \) has incompressible boundary. Then, for each \( n \in (\mathbb{N} \setminus \{0\})^N \), \( A_n \) is homeomorphic to \( \Pi_{i=1}^N A_{g_i,n_i} \).

**Proof.** The conditions in theorem 8.17, which define both \( A_n \) and the \( A_{g_i,n_i} \), describe each connected component of \( \partial M \) independently from the others. Since \( M \) has incompressible boundary, the curves in each connected component of \( \partial M \) which are contractible in \( M \) are those which are contractible in \( \partial M \); so the condition on the dihedral angles on \( \partial_i M \) are exactly those describing \( A_{g_i,n_i} \).

**Proposition 8.15.** For each \( g \geq 2 \) and each \( r \geq 1 \), \( A_{g,r} \) is homeomorphic to the space \( T_{g,r} \) of conformal structures on a surface of genus \( g \), with \( r \) marked points (up to isotopy).

**Proof.** Let \( \Sigma_g \) be a compact surface of genus \( g \), and let \( t \) be a conformal structure on \( \Sigma_g \). There is a unique hyperbolic metric on \( \Sigma_g \), say \( h \), in the conformal class defined by \( t \). Taking the warped product:

\[
M_g := (\Sigma_g \times \mathbb{R}, dt^2 + \cosh^2(t)h)
\]

determines a complete fuchsian hyperbolic manifold. If \( p_1, \ldots, p_r \) are points in \( \Sigma_g \), they define points \( p_1, \ldots, p_r, \pm \) in \( \partial_\infty M_g \), with \( p_{i,+} \) in the "upper" boundary and \( p_{i,-} \) in the "lower" boundary for each \( i \), and \( p_{i,\pm} \) exchanged with \( p_{i,-} \) by the isometric involution on \( M_g \).

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Let $M$ be the convex hull of the $p_{i,k}$. By remark 3.9, $M$ is an ideal hyperbolic manifold. Moreover, each fuchsian ideal hyperbolic manifold is obtained in this way, and thus is associated to a $\mathbf{t} \in \mathcal{T}_{g,r}$.

Finally, theorem 4.25 shows that $\mathcal{A}_{g,r}$ is homeomorphic to the space of fuchsian ideal hyperbolic manifolds with each boundary surface of genus $g$ with $r$ vertices, and the result follows.

Proof of lemma 8.13. For each $g \geq 2$ and $r \geq 1$, $\mathcal{T}_{g,r}$ is connected. So the result follows from the previous proposition.

### 8.5 Unique realization for some dihedral angle assignments

The content of the previous subsection is sufficient to ensure that the map $\Phi_n$ is a covering for each choice of $n$, and that the number of inverse images of all elements of $\mathcal{A}$ is the same. The next lemma states that this number is 1 for some cellulations of $M$ and choices of dihedral angles of the edges. Similar arguments were used earlier by D. Sullivan and by R. Brooks, in related contexts.

**Lemma 8.16.** Let $\sigma$ be a cellulation of $\partial M$ such that:
- the faces of $\sigma$ can be separated into two sets $F_-$ and $F_+$, with each edge bounding a face of $F_-$ and one of $F_+$.
- each vertex is adjacent to 4 faces.

Let $w : \sigma_1 \to (0, \pi)$ be the function assigning the value $\pi/2$ to each edge. Then $w$ satisfies the hypothesis of theorem 8.17, and there is at most one ideal hyperbolic manifolds with combinatorics given by $\sigma$ and dihedral angles given by $w$.

**Proof.** Let $g$ be an ideal hyperbolic metric on $M$ with boundary combinatorics $\sigma$ and all angles $\pi/2$. We consider 4 copies of $M$, say $M_1, M_2, M_3$ and $M_4$, each with the metric $g$. We identify the corresponding faces of the $M_i$ according to the following table, where a “+” on line $i$ and column $j$ means that each face of $M_i$ in $F_+$ is glued to the corresponding face on $M_j$, and the same for $-$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>+</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
<td>+</td>
<td>-</td>
<td></td>
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</tr>
<tr>
<td>4</td>
<td>+</td>
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</tbody>
</table>

Let $N$ be the result of those gluings. It is easy to check that:

1. the topology of $N$ is determined by the topology of $M$ and of $\sigma$, and does not depend on the ideal hyperbolic metric on $M$.
2. $N$, with the metric coming from the gluing of the $M_i$, is a complete, finite volume hyperbolic manifold.
3. two non-isometric ideal hyperbolic metrics on $M$ would result in two non-isometric hyperbolic metrics on $N$.

By Mostow rigidity there is at most one complete, finite volume hyperbolic metric on $N$. The result follows.

### 8.6 Proof of the main theorem

We can now prove theorem 8.17, concerning the description of dihedral angles of ideal hyperbolic manifolds.

**Theorem 8.17.** Suppose that $M$ has incompressible boundary. Then the set $\Theta_\sigma$ of dihedral angles of ideal hyperbolic structures whose boundary is cellulated according to $\sigma$ is given by the condition that:

1. on each elementary circuit in $\sigma$, the sum of the exterior dihedral angles is $2\pi$;
2. on each non-elementary circuit, the sum of the dihedral angles is strictly larger than $2\pi$.

Each dihedral angles assignment is obtained on a unique ideal hyperbolic manifold.
Proof. Let σ be a cellulation of ∂M. By lemma 8.11, there exists a number ν(σ) ∈ N such that each dihedral assignment on the edges of σ, satisfying the hypothesis of the theorem, is realized by exactly ν(σ) ideal hyperbolic manifolds with boundary combinatorics σ.

Let n ∈ (N \ {0})^N. By lemma 8.13, A_n is connected; therefore, for all cellulations σ of ∂M with n_i vertices in ∂_i M, the value of ν(σ) is the same, and is equal to a number ν(n) ∈ N.

Let σ be a cellulation of ∂M. Let σ̄ be the cellulation of ∂M obtained by “splitting” a vertex v of σ, i.e. replacing v by two vertices v_1 and v_2 connected by an edge e, and replacing two edges e_1 and e_2 containing v by two edges each, say e_11, e_12 and e_21, e_22. If v ∈ ∂_i M, let σ̄ = (n_1, n_2, · · · , n_i+1, · · · , n_N).

Each dihedral angle assignment θ to the edges of σ can be identified with a “degenerate” dihedral angle assignment σ̄ on σ̄ with a dihedral angle 0 at e, dihedral angles at e_1 and e_2 equal to half the dihedral angles at e, and the other dihedral angles equal to those of θ. If θ satisfies the condition of theorem 8.17, so does σ̄ (except of course for the fact that one of its angles is 0). Let (θ_n) be a sequence of dihedral angle assignments on the edges of σ̄ which converges to σ̄.

For each i ∈ {1, · · · , ν(π)}, there is a sequence of ideal hyperbolic metrics (g_i,n)_{n∈N} with dihedral angles equal to θ_n. As n → ∞, (g_i,n) converges to an ideal hyperbolic metric g_i on ∂M with boundary combinatorics given by σ, and dihedral angles by θ. So ν(π) ≥ ν(n) ≥ 1.

But it is easy to check that, for each n ∈ (N \ {0})^N, there exists some p ∈ (N \ {0})^N, with p_i ≥ n_i for all i ∈ {1, · · · , N}, such that there exists a cellulation of ∂M, with p_i vertices in each ∂_i M, to which lemma 8.16 applies.

So, for each n ∈ (N \ {0})^N, ν(n) = 1, which proves the theorem. □

It might be useful to note that the convexity of the boundary of M is not really necessary for many steps of the proof of this theorem; for instance the infinitesimal rigidity results still hold for at least some manifolds with non-convex boundary. This contrasts with most results concerning the induced metric on the boundary, where the convexity is crucial.

8.7 An affine structure on some Teichmüller spaces

A consequence of the previous considerations is that there is a natural piecewise affine structure on the space of ideal hyperbolic structures on M, coming from the parametrization by the dihedral angles (see also [Riv03]). Its definition is simple and depends directly only on the dihedral angles. It has “cone-like” singularities along the “cells” of codimension at least 2, corresponding to non-generic polyhedral structures of the boundary. The remarkable point, however, is that it also has a well-defined volume element. This, which I can not explain better than by doing an explicit computation, might indicate that this affine structure is somehow meaningful. The definition below should be understood with figure 4.

Definition 8.18. The affine structure A on H^c_{sm} is defined as follows.

• for each triangulation σ of the boundary, the restriction of A to H^c_{sm} is defined as A_{σ} above.

• if σ is a cellulation with one 4-gonal face f and the other faces triangles, then H^c_{sm} is a codimension 1 face of H_{sm} which bounds two maximal dimension faces H^c_{sm} and H^c_{sm}, where σ_1, σ_2 are triangulations obtained by refining σ by adding edges e_1 and e_2 respectively to the 4-gonal face of σ. Then H^c_{sm} and H^c_{sm} are glued by the map, from a neighborhood of σ in the extension of σ_1 corresponding to having a negative angle at e_1 to H^c_{sm}, which sends a configuration with angle −2u at e_1 and a, b, c, d at the 4 edges of f, to a configuration with angle 2u at e_2, and angles a − u, b − u, c − u, d − u at the edges of f.

Note that this definition makes sense since the transformation preserves the angle conditions defining the space of possible angle assignments. It is clear that the lengths conditions on elementary and non-elementary circuits are satisfied for one of the configurations if and only if they are satisfied for the other.

Lemma 8.19. A has a holonomy preserving a volume form.

Proof. To prove this one should check that the holonomy around all codimension two faces of H_{sm} preserve a volume form. There are two kinds of codimension two faces, those corresponding to a cellulation of ∂M with two 4-gonal faces, and those corresponding to the cellulations with one 5-gonal face. In the first case of the holonomy is trivial, so the only case to check is the second.

When a cellulation σ of ∂M has one 5-gonal face and its other faces are triangles, the corresponding cell H^c_{sm} of H_{sm} bounds five maximal dimension and five codimension one faces. The holonomy around H^c_{sm} is
computed by considering the diagram in figure 5, corresponding to the transformation of the angles as one goes through the five maximal dimension faces — the angles are marked on the dual graph, and the constant part of the angles on the five “exterior” edges are not marked.

The only part of the holonomy which is relevant for the volume is the one corresponding to the two “central” edges. The computation made on figure 5 shows that the holonomy for those two angles is given by the matrix:

\[
M := \frac{1}{32} \begin{pmatrix} 28 & -10 \\ 10 & 33 \end{pmatrix}
\]

But \(\det(M) = 1\), so that the holonomy around \(\mathcal{H}_{sm}^0\) preserves a volume form.

Remark 8.20. \((\mathcal{H}_{sm}, \mathcal{A})\) has locally convex boundary at all points which are in the closure of a face \(\mathcal{H}_{sm}^0\).

Proof. We have to prove that if \(\sigma\) is a triangulation of \(\partial M\) and \(F_0\) is a codimension 1 face of \(\mathcal{H}_{sm}^0\) which is in the boundary of \(\mathcal{H}_{sm}\), then, across each codimension 1 face of \(\mathcal{H}_{sm}^0\) which is not in \(\partial \mathcal{H}_{sm}\), the extension of \(F_0\) is either outside \(\mathcal{H}_{sm}\) or in its boundary.

Let \(\mathcal{H}_{sm}^0\) be a codimension 1 face of \(\mathcal{H}_{sm}^0\), so that \(\sigma_1\) is obtained from \(\sigma\) by removing an edge from \(\sigma\), and \(\sigma_1\) has exactly one face \(f\) with 4 edges. There are 3 cases to consider:
1. $F_0$ corresponds to a non-elementary circuit on which the sum of the dihedral angles is $2\pi$. Then the definition of $A$ shows directly that the extension of $F_0$ beyond $\mathcal{H}_{\text{SM}}^g$ remains in the boundary of $\mathcal{H}_{\text{SM}}^g$.

2. $F_0$ corresponds to an edge $e \not\in f$ with dihedral angle equal to $\pi$. It is then clear again from the definition that the extension of $F_0$ beyond $\mathcal{H}_{\text{SM}}^g$ remains in $\partial \mathcal{H}_{\text{SM}}^g$, since the transformation which occurs in the definition of $A$ does not change the angle at $e$.

3. $F_0$ corresponds to an edge $e \in f$ with dihedral angle equal to $\pi$. But then the definition of $A$ shows that, if one goes beyond $F_0$ on an affine line starting in $\mathcal{H}_{\text{SM}}^g$, the angle at $e$ is above $\pi$ after $\mathcal{H}_{\text{SM}}^g$, so that the extension of $F_0$ beyond $\mathcal{H}_{\text{SM}}^g$ is outside $\mathcal{H}_{\text{SM}}^g$.

An elementary consequence of the constructions which we have just seen — applied to the fuchsian case — is that there is a natural affine structure on the Teichmüller space of a surface of genus at least 2, with at least one marked point. The point is that fuchsian manifolds form an affine submanifold of the ideal hyperbolic manifolds (they corresponds to the case where the triangulations and dihedral angles are the same on the two components of the boundary). Moreover, if one remains in the category of fuchsian manifolds, there is no bent case (see remark 3.9), so that the affine structure defined on ideal manifolds spans the whole Teichmüller space. One thus obtain the:

**Theorem 8.21.** For each $g \geq 2$ and each $N \geq 1$, there is a natural unimodular piecewise affine structure $A_{g,N}$ on the Teichmüller $T_{g,N}$ space of the genus $g$ surface with $N$ marked points.

There are some natural questions that remain on this structure; for instance, how many cells it has, and whether the affine structure is globally convex.

## 9 Induced metrics

We will study in this section some properties of the metrics induced on the boundaries of ideal hyperbolic manifolds. They are finite area hyperbolic metrics on each connected component of the boundary of $M$ without its ideal points, and we will show that the infinitesimal deformations of the interior metric are parametrized by the infinitesimal deformations of the boundary metric; this is again a consequence of the Schläfli formula. In addition, we will prove a global result in the special case of fuchsian manifolds, then the induced metrics on the boundary are in one-to-one correspondence with the hyperbolic structures on the interior.

### 9.1 Finite area metrics on surfaces

We will use again here the notion of hyperbolic structure on a triangulated surface, as defined in the previous section. We consider again a compact, orientable surface $S$ of genus at least 2 with $N$ points $v_1, \cdots, v_N$ removed. The following additional definition is natural.

**Definition 9.1.** Let $g \in \mathcal{M}$ and let $v$ be a vertex of $\sigma$. The shift of $g$ at $v$ is the sum of the shifts of $g$ at the edges containing $v$. We will say that $g$ is complete if its shift is zero at all vertices. The set of complete structures will be denoted by $\mathcal{M}_c$.

Of course the notion of completeness defined here is the same as the usual, topological notion. Indeed if the shift of $g$ at a vertex $v$ is non-zero, it is possible to use this — and the fact that ideal triangles are exponentially thin near their vertices — to attain $v$ in a finite time, by “circling” around it to take opportunity of the shift. The reciprocal is not difficult to prove either.

Finally, knowing the shifts of a complete metric $g \in \mathcal{M}_c$ at all edges determines all the gluing along the edges; let $S_0$ be the set of maps from the set of edges of $\sigma$ to $\mathbb{R}$ such that, at each vertex, the sum of the adjacent edges is 0. We identify $S_0$ with $\mathbb{R}^{N-1}$. Then:

**Proposition 9.2.** The map $F$ from $\mathcal{M}_c$ to $S_0 = \mathbb{R}^{N-1}$ sending $g$ to $F(g)$ defined, for an edge $E$, by $F(g)(E) = sh(g)(E)$, is a bijection.

**Proof.** It is quite obvious that the hyperbolic structures on a given triangulated surface are determined by their shifts. On the other hand each shift function on the edges corresponds to a possible gluing (under the condition that the sum on the edges at any vertex is zero, by completeness).
9.2 The lengths of the edges of $\sigma$

Let $g \in M_c$. Each vertex $v_i$ of $\sigma$ has a neighborhood which is isometric to a neighborhood of a cusp in the quotient of $H^2$ by a parabolic isometry. For each such $v_i$, choose a horocycle $H_i$ centered at $v_i$. Then, for each edge $E$ of $\sigma$, going from a vertex $v_i$ to a vertex $v_j$, let $l(E)$ be the oriented length of $E$ between $H_i$ and $H_j$. The orientation is chosen so that $l(E)$ is negative when the horoballs bounded by $H_i$ and $H_j$ overlap. Clearly, replacing $H_i$ by another horocycle centered at $v_i$ changes the lengths of the edges containing $v_i$ as an end by the addition of a constant. So $l$ defines a function:

$$l : M_c \to \mathbb{R}^e/\mathbb{R}^n.$$

**Proposition 9.3.** $l$ is a bijection between $M_c$ and $\mathbb{R}^e/\mathbb{R}^n$.

The proof uses the following elementary property of ideal triangles in $H^2$, already used by several authors, in particular Penner.

**Sub-lemma 9.4.** Let $x_1, x_2, x_3, x_4$ be four distinct points on $S^1 = \partial_\infty H^2$, in this cyclic order. For each $i \in \{1, 2, 3, 4\}$, let $h_i$ be a horocycle centered at $v_i$, and, for $i \neq j$, let $l_{ij}$ be the distance between $h_i$ and $h_j$ along the geodesic going from $v_i$ to $v_j$ — which is negative if $h_i$ and $h_j$ overlap. Let $\pi_2$ and $\pi_4$ be the orthogonal projections on $(x_3, x_1)$ of $x_2$ and $x_4$ respectively, and let $\delta$ be the oriented distance between $\pi_2$ and $\pi_4$ on $(x_3, x_1)$. Then:

$$2\delta = l_{12} - l_{23} + l_{34} - l_{41}.$$

Figure 6: Ideal triangles (in the projective model of $H^2$)

**Proof.** It follows from figure 6, where numbers from 1 to 6 are attached to lengths of segments. Elementary properties of the ideal triangle show that:

$$l_{12} - l_{23} + l_{34} - l_{41} = (1 + 2) - (2 + 3) + (4 + 5) - (5 + 6) = 1 - 3 + 4 - 6 = (1 - 6) + (4 - 3) = 2\delta.$$

**Proof of Proposition 9.3.** Using Proposition 9.2, we are brought to proving that the map:

$$L : S_0 \simeq \mathbb{R}^{e-n} \to \mathbb{R}^e/\mathbb{R}^n$$

sending the $e$-uple of shifts of a metric to the lengths of its edges is a bijection. But the sub-lemma above shows that $L$ is linear, and explicitly describes its inverse. □
9.3 Rigidit y and the Šläfli formula

We now consider an ideal hyperbolic manifold \( M \); thanks to the previous paragraphs, we know that, if \( \partial M \) is triangulated, then its induced metric is determined by the lengths of the edges (defined up to the addition of a constant for each vertex). Of course, if \( \partial M \) is not triangulated — i.e. if some of the faces are polygons with more than three vertices — one can subdivide its cellulation to obtain a triangulation.

We will prove in this paragraph and the next that \( M \) is infinitesimally rigid, in the following sense. We call \( \Phi \) the map from \( \mathcal{H}_{\text{sm}} \) to \( \mathcal{M}_c \), sending \( g_0 \in \mathcal{H}_{\text{sm}} \) to the metric induced on its boundary. Then:

**Lemma 9.5.** For any \( g_0 \in \mathcal{H}_{\text{sm}} \), \( T_{g_0} \Phi \) is an isomorphism.

The proof will take two steps. We first treat the case when the boundary of \( M \) is triangulated, then the general case. The first case will rely on the Šläfli formula and Remark 8.10. Recall from Lemma 8.9 that, for any ideal hyperbolic manifold \( (M, g_0) \), the volume \( V \) is a strictly concave function on \( \mathcal{H}_{\text{sm}} \) in the neighborhood of \( g_0 \).

**Proposition 9.6.** Suppose that all faces of \( (M, g_0) \) are triangles, and that it is triangulable. Then \( T_{g_0} \Phi \) is an isomorphism.

**Proof.** We know by Corollary 7.13 that \( \dim \mathcal{H}_{\text{sm}} = e - v = \dim \mathcal{M}_c \). Moreover, the volume \( V(g) \) is a strictly concave function on \( \mathcal{H}_{\text{sm}} \) by Lemma 8.9.

Recall the Šläfli formula from section 1:

\[
dV = -\frac{1}{2} \sum_E l(E) d\theta(E),
\]

where the sum is over all edges, and \( \theta(E) \) is the interior dihedral angle of edge \( E \). The strict concavity of \( V \) with respect to the \( \theta(E) \) means that the Hessian of \( V \) with respect to those angles is negative definite, so that the differential of the lengths \( l(E) \) — with respect to the dihedral angles — is non-degenerate. Thus \( T_g l \) is injective, and the result follows.

Now let \( (M, g) \) be an ideal hyperbolic manifold with non-triangular boundary. By Lemma 5.4, \( M \) has a finite covering \( \tilde{M} \) which admits an ideal triangulation \( \Sigma \). \( \Sigma \) induces an ideal triangulation of \( \partial \tilde{M} \), and the argument above — the convexity of the volume and the Šläfli formula — still implies that, for each assignment of a real number \( r(e) \) to each edge \( e \) of \( \partial \tilde{M} \) (defined up to addition of a constant for each vertex) there is a unique infinitesimal deformation of \( \tilde{M} \) such that the length of \( e \) (defined with respect to a set of horospheres at the vertices) varies at the rate \( r(e) \). Therefore the argument given above shows that the infinitesimal deformations of \( \tilde{M} \) are uniquely determined by the induced variation of the boundary metric, and that each variation of the boundary metric is obtained.

This is in particular true for the variation of the boundary metric which are invariant under the action of \( \pi_1 M / \pi_1 \tilde{M} \) on \( \tilde{M} \), and the uniqueness implies that they are associated to infinitesimal deformations of \( M \) equivariant under the same action. So we obtain the proof of Lemma 9.5, which we can reformulate as:

**Lemma 9.7.** For each ideal hyperbolic manifold \( M \), and for each infinitesimal deformation \( \bar{h} \) of the boundary metric \( h \) on \( \partial M \), there is a unique infinitesimal deformation of \( M \) inducing \( \bar{h} \).

9.4 The fuchsian case

Lemma 9.7 provides only an infinitesimal deformation result for the induced metrics on the boundaries of ideal hyperbolic manifolds. It is then difficult to obtain a global existence and uniqueness results for the boundary case, in particular because we have no such infinitesimal rigidity result for bent hyperbolic manifolds, which necessarily enter the picture. But in the fuchsian case, the bent case is excluded by Remark 3.9, and a global result can be achieved.

**Theorem 9.8.** Let \( S \) be a surface of genus \( g \geq 2 \), and let \( N \geq 1 \). For each complete, finite area hyperbolic metric \( h \) on \( S \) with \( N \) cusps, there is a unique ideal fuchsian hyperbolic manifold \( M \) such that the induced metric on each component of the boundary is \( h \).

**Proof.** We will use a deformation proof, following the original approach of Aleksandrov [Ale58] for similar polyhedral question. We choose \( g \geq 2 \) and \( N \geq 1 \). We call \( T_{g,N} \) the Teichmüller space of marked conformal structures on the compact surface \( \Sigma_g \) of genus \( g \) with \( N \) marked points. There is a natural map \( \Phi_{g,N} \) from \( T_{g,N} \)
to itself, defined as follows. For \( h \in \mathcal{T}_{g,N} \), the conformal class on \( \Sigma_g \) (i.e. forgetting the marked points on \( \Sigma_g \)) contains a unique hyperbolic metric \( h_0 \) on \( \Sigma_g \); taking its universal cover defines a conformal map from \( \tilde{\Sigma}_g \) with the conformal structure in \( h \) to the upper hemisphere \( S^2_+ \). The marked points in \( h \) then define an equivariant set \( S \) of points in \( S^2_+ \). Taking the boundary of the convex hull of those points in \( \mathbb{R}^3 \) (and subtracting its intersection with the plane containing the equator) leads to a polyhedral surface, which is invariant under the natural action of \( \pi_1 \Sigma_g \) on \( H^3 \). The quotient is homeomorphic to \( \Sigma_g \), and has \( N \) vertices (which are the intersection points with \( S^2_+ \)); it carries a complete, finite area hyperbolic metric \( \overline{h} \) induced by the canonical metric on \( H^3 \). \( \Phi_{g,N}(h) \) is defined as this metric \( \overline{h} \), considered as an element of \( \mathcal{T}_{g,N} \).

The proof of the theorem is an immediate consequence of the following points.

1. \( \Phi_{g,N} \) is locally injective, i.e. its differential is an isomorphism at each point.
2. \( \Phi_{g,N} \) is proper.
3. \( \mathcal{T}_{g,N} \) is connected and simply connected.

Point (1) is just Lemma 9.5, while point (3) is well known (since we use the marked Teichmüller space). So we only have to prove point (2). It can be reformulated as a degeneration statement: if \((M_n)_{n \in \mathbb{N}}\) is a sequence of fuchsian ideal manifolds of genus \( n \) with \( N \) vertices, and if the sequence of induced metrics \((h_n)\) converges to a limit \( h \in \mathcal{T}_{g,N} \), then \((M_n)\) converges.

In other terms, we have to prove that if either the conformal structure on \( \partial E(M_n) \) degenerates or the marked points collapse, then the length of some closed geodesic on \( \partial M \) goes to 0.

In the case of a degeneration of the conformal structure on \( \partial E(M_n) \), consider the action of \( \pi_1 \partial M \) on the totally geodesic plane \( P_0 \subset H^3 \) which is fixed by the action of \( \pi_1 \partial M \). Clearly the action of \( \pi_1 \partial M \) on \( P_0 \) also degenerates, so there is a closed geodesic in the quotient whose length goes to infinity. Since the projection from \( \partial M \) on \( P_0 \) is contracting (as any projection on a convex set in \( H^3 \)) it means that the length of some closed geodesic in \( \partial M \) goes to infinity, so the sequence of induced metrics on \( \partial M \) can not converge.

9.5 Ideal polyhedra in \( H^3 \)

An elementary remark is that, for ideal polyhedra in \( H^3 \), the approach above gives a simple proof of the existence and uniqueness of ideal polyhedra having a given induced metric, see [Riv92, Riv96]. In particular this approach does not use the Cauchy method to get the global uniqueness, since it follows from the global deformation result.

Rather, the infinitesimal rigidity can be obtained as in Lemma 9.7 using the Schläfli formula and the results concerning the dihedral angles. Applying a deformation argument to conclude then only requires a compactness result — namely, that if a sequence of ideal polyhedra has induced metrics which converge, then it has a converging subsequence (modulo isometries). Such a compactness result is easy to obtain directly.

10 Circles patterns, circle packings

As mentioned in section 2, the polyhedral questions considered here can be formulated in terms of circle patterns in \( S^2 \). More precisely, let \( M \) be an ideal hyperbolic manifold. Consider its universal cover \( \tilde{M} \) as a subset of \( H^3 \). Its boundary \( \partial \tilde{M} \) is the disjoint union of a set of convex surfaces, and their boundary is the limit set \( \Lambda \) of the action of \( \pi_1 M \) on \( H^3 \).

Each face of \( \tilde{M} \) defines an oriented totally geodesic plane in \( H^3 \), with boundary at infinity an oriented circle in \( S^2 \setminus \Lambda \). If two faces \( F, F' \) of \( \partial M \) are adjacent, then the corresponding oriented circles \( C, C' \) intersect with angle equal to the exterior dihedral angle between \( F \) and \( F' \). Moreover, a simple convexity argument shows that the union of the interiors of the circles corresponding to the faces of \( \partial M \), along with the intersection points, is \( S^2 \setminus \Lambda \). So the results concerning the dihedral angles of ideal hyperbolic manifolds can be formulated in terms of circles patterns in the complement of \( \Lambda \), with given intersection angles.

Therefore, an ideal hyperbolic manifolds determines a circle pattern on \( \partial M \), in the following sense.

**Definition 10.1.** Consider \( \partial M \) endowed with a \( \mathbb{CP}^1 \)-structure, for instance coming from a complete hyperbolic metric on \( M \). A circle pattern on \( \partial M \) is a finite set of open disks \( C_1, \ldots, C_N \) in \( \partial M \) (for the given \( \mathbb{CP}^1 \)-structure), such that:

- the union of the closures of the disks is \( \partial M \).
For each $i \in \{1, \cdots, N\}$, there are at least 3 points in $C_i$ which are in one of the $C_j$, $j \neq i$, but not in any of the open disks bounded by the $C_k$.

Given a circle pattern, its incidence graph has one vertex for each circle, and it has an edge between two circles, $C_i$ and $C_j$, if and only if $C_i$ and $C_j$ intersect at two points which are not in the interior of the other circles. If a circle pattern in $\partial M$ comes from an ideal hyperbolic structure on $M$, then the incidence graph of the circle pattern is dual to the combinatorics of the boundary of the ideal hyperbolic structure.

This relationship between ideal hyperbolic manifolds and circle patterns means that the results concerning the dihedral angles should be compared to those concerning circle patterns, e.g. the rigidity results of [He99].

One can also consider circle packings in the more restrictive sense of sets of disks whose interiors are pairwise disjoint, with the complement made of disjoint, polygonal regions. Translating Theorem 8.17, we will find the following result, which reduces to the classical Koebe circle packing theorem as extended by Thurston (see [Koe36, Thu97]).

**Theorem 10.2.** Let $\Gamma$ be the 1-skeleton of a triangulation of $\partial M$. There is a unique couple $(g, c)$, where $g$ is a complete, convex co-compact hyperbolic metric on $M$, and $c$ is a circle packing on $\partial M$ (for the $CP^1$-structure defined on $\partial M$ by $g$) whose incidence graph is $\Gamma$.

It is helpful to use a trick due to Thurston [Thu97]. Consider such a circle packing, such that the complement of the circles is the disjoint union of "triangles" bounded by 3 circle arcs. Then add, for each connected component of the complement of the circles, a circle which is orthogonal to the 3 circles which it intersects. One then obtains a circle pattern intersecting at right angles. The same construction works in the other way. Consider a circle pattern intersecting at right angles, such that:

- the circles can be separated in two sets, the "white" and the "black" circles, such that each "black" circle only intersects transversally "white" circles, and conversely.
- each "black" circle intersects transversally exactly three "white" circles.

Then the set of "white" circles make up a circle packing.

Therefore, the proof Theorem 10.2 is reduced to finding circle pattern on $\partial M$, where the intersections between the circles are always at right angles. Moreover, for the circle patterns coming from circle packings by adding orthogonal circles, it is easy to check that all circuits are made of at least 4 edges, and strictly more than 4 unless they are elementary. Therefore, Theorem 8.17 applies, and Theorem 10.2 follows.

If one considers graphs which are the 1-skeleton of cellulations (more general than triangulation) the same construction works; one obtains circle packings with the added property that, for each intersticial region, there is a circle orthogonal to all the adjacent circles. The uniqueness, under this additional condition, is proved by noting that each such packing gives rise to a circle pattern with right angle intersections, and thus to an ideal hyperbolic manifold with boundary faces intersecting at right angle (whose uniqueness is known by Theorem 8.17).

Note that, seen in this light, the statements made here are related to different results on the rigidity of circle packings, see e.g. [Sch91].

In the fuchsian case, the results above can be stated as describing circle packings — or circle patterns with given angles — on surfaces of genus $g \geq 2$, see [Riv03, BS04].

11 Concluding remarks and questions

11.1 Manifolds with cusps

Most of what was done in this paper could presumably be extended from convex co-compact manifolds to manifolds with cusps: that is, we consider manifolds $M$ which remain of finite volume, with a boundary which is polyhedral, i.e. the union of a finite set of totally geodesic faces which intersect along edges, with all their vertices ideal; but we allow those manifolds to have some cusps, i.e. $M$ has finite volume but contains points arbitrarily far from the boundary.

The most delicate technical point to check is the content of section 4, concerning ideal triangulation. Proposition 5.5 extends to the case with cusps using the ideas of Epstein and Penner [EP88] in a manner which is closer to what they do in their paper, i.e. using the properties of the action of $\pi_1 M$ on the parabolic points.
The combinatorial arguments used in section 4 to obtain an ideal triangulation from a cellulation then work without any difference in the case with cusps.

Doing this in the special case of a manifold with one cusp, one could presumably recover some results of Thurston on circle packings on the torus (see [Thu97, section 13.7]). The dihedral angles, however, should not be restricted to be acute in this approach.

11.2 Hyperideal polyhedra

A remarkable point is that some of the properties described in the introduction survive when one considers polyhedra or surfaces with complete metrics of infinite area. In the polyhedral case this corresponds to hyperideal polyhedra; in the projective model of $H^3$, they can be described as polyhedra having some "usual" vertices in $H^3$ and some "hyperideal" vertices beyond infinity, but such that all edges meet the interior of $H^3$. An existence and uniqueness statement for the induced metric and the third fundamental form on those polyhedra can be found in [Sch08], while a recent work of Bao and Bonahon [BB02] describes the set of their dihedral angles — a problem which is related to the third fundamental form.

For smooth surfaces, it looks like a possible analog of hyperideal polyhedra is the class of convex surfaces in $H^3$ whose boundary at infinity is a circle. A statement concerning the induced metrics and the third fundamental forms of such surfaces can be found in [Sch08], but it deals only with the special case of surfaces of constant Gauss curvature. I also believe that one could prove a similar existence result for hyperbolic metrics on a manifold $M$ inducing a given metric, with constant curvature $K \in [-1, 0)$, on the boundary (see [Sch03]). A more general result, extending to metrics of non-constant curvature on the boundary, would be more demanding in terms of analytical techniques.

It would be interesting to understand whether those properties also remain valid for hyperbolic manifolds with boundaries, when the metric has a "boundary at finite distance" and also a "boundary at infinity", thus extending the notion of hyperideal polyhedron. One might consider a hyperbolic type "finite boundary" — which would have to be "hyperideal" in the sense that, for each end, all edges "converge" to a hyperideal vertex — or a smooth "finite boundary", maybe with the additional condition that it meets the boundary at infinity along a circle.

11.3 Complete manifolds of finite volume

Some of the tools used here could also have applications to the study of the deformations of complete hyperbolic manifolds of finite volume.

Those manifolds have a decomposition in cells, each isometric to an ideal polyhedron, by [EP88]. Thus the methods of section 5 might lead to ideal triangulations of finite covers, as in e.g. [NZ85].

One could then study the deformations obtained by changing the dihedral angles of the ideal triangles; the volume remains a concave functional on the space of angle assignments.

An important difference with the case of ideal hyperbolic manifolds, however, is that the shears along the singular edges is not the only obstruction to having a complete structure. The link of the vertices is now a torus — instead of a disk for ideal hyperbolic manifolds — which carries a similarity structure, with singularities at points corresponding to the singular edges; to have a complete structure, it is also necessary that the conformal part of the holonomy of each link vanishes. But this vanishing could also be a necessary and sufficient condition for the vanishing of the differential of the volume among deformations which do not change the total angle around each singular edge (at least it is sufficient).

11.4 Convex cores

The questions mentioned in the introduction, concerning the extension of Theorems 1.1 and 1.2 to cases where $\partial M$ is not smooth, have another natural setting: the case where $M$ is supposed to be the convex core of a complete, convex co-compact manifolds $N$. This is equivalent to supposing that $\partial M$ is convex and developpable, or that it is convex and has a hyperbolic (i.e. constant curvature $-1$) induced metric.

The question concerning Theorem 1.1 is whether, for any hyperbolic metric $h$ on $\partial M$, there is a unique hyperbolic metric $g$ on $M$ such that $\partial M$ is convex and developpable, with induced metric $h$. The existence part holds, as was proved by Labourie [Lab91] (another proof exists, using a degree argument and the known results from [EMS6] on a conjecture of Sullivan). The uniqueness, however, remains unknown.
When one considers the third fundamental forms of the boundary, one encounters a statement on the pleating measures of convex cores. Here again, satisfactory results on the existence part have been obtained recently by Bonahon, Otal [BO01] and Lecuire [Lec02], but the uniqueness is unknown.

The questions on the convex cores can rather naturally be extended to the "hyperbolic manifolds with horns" introduced in section 3. For instance, the question on the induced metric is:

**Question 11.1.** Let $h$ be a complete, finite area metric on $\partial M$ minus a finite number of points. Is there a unique metric $g$ on $M$ such that $(M, g)$ is a hyperbolic manifold with horns and that the induced metric on $\partial M$ is $h$?

### 11.5 Orbifolds, cone-manifolds

The results given here, in particular Theorem 8.17, can might also be used to construct hyperbolic orbifolds, or hyperbolic manifolds. This already happens in the fuchsian case, see [Thu97].

Other relation with cone-manifolds can be obtained, given an ideal hyperbolic manifold $M$, by gluing two copies of it along their common boundary. One obtains a finite volume, non-compact cone-manifold which is singular along a family of geodesics — corresponding to the edges of $M$ — each going between two points at infinity. Moreover, the convexity of $M$ means that the total angle around the singular geodesics is always strictly less than $2\pi$. The infinitesimal rigidity of ideal hyperbolic manifolds could therefore be a consequence of an infinitesimal rigidity result for such cone-manifolds, which could perhaps be proved using the ideas of [HK98]. Moreover, those non-compact cone-manifolds could themselves be of interest.

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### References


