A symplectic map between hyperbolic and complex Teichmüller theory

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Abstract

Let $S$ be a closed, orientable surface of genus at least 2. The cotangent bundle of the “hyperbolic” Teichmüller space of $S$ can be identified with the space $\mathbb{C}P$ of complex projective structures on $S$ through measured laminations, while the cotangent bundle of the “complex” Teichmüller space can be identified with $\mathbb{C}P$ through the Schwarzian derivative. We prove that the resulting map between the two cotangent spaces, although not smooth, is symplectic. The proof uses a variant of the renormalized volume defined for hyperbolic ends.

1 Introduction and main results

In all the paper $S$ is a closed, orientable surface of genus $g$ at least 2, $T$ is the Teichmüller space of $S$, $\mathbb{C}P$ the space of (equivalence classes of, see below) $\mathbb{C}P^1$-structures on $S$, and $\mathcal{ML}$ be the space of measured laminations on $S$.

1.1 The “hyperbolic” Teichmüller space

There are several quite distinct ways to define the Teichmüller space of $S$, e.g., the space of complex structures on $S$, or the space of all Fuchsian groups of genus $g$ (modulo conjugation), or the space of (equivalence classes of) appropriate Beltrami differentials. In this subsection we consider what can be called the “hyperbolic” Teichmüller space, defined as the space of hyperbolic metrics on $S$, considered up to isotopy. In this guise it is sometimes called the Fricke space of $S$. Here we denote this space by $T_H$ to remember its “hyperbolic” nature. This description emphasizes geometric properties of $T$, while some other properties, notably the complex structure on $T$, remain silent in it.

There is a natural identification between $T_H \times \mathcal{ML}$ and the cotangent bundle of $T_H$, which can be defined as follows. Let $l \in \mathcal{ML}$ be a measured lamination on $S$. For each hyperbolic metric $m \in T_H$ on $S$, let $L_m(l)$ be the length of $l$ for $m$. Thus $L_m(l)$ is a function on $T$, which is differentiable. For $m_0 \in T_H$, the differential of $m \mapsto L_m(l)$ at $m_0$ is a vector in $T_{m_0}^*T_H$, which we call $\delta(m,l)$. This defines a function $\delta : T_H \times \mathcal{ML} \rightarrow T^*T_H$, which is the identification we wish to use here. It is proved in section 2 (see Lemma 2.3) that $\delta$ is indeed one-to-one (this fact should be quite obvious to the specialists, a proof is included here for completeness). As a result of this identification the space $T_H \times \mathcal{ML}$ becomes a symplectic manifold. In the symplectic structure in question the differentials of the lengths of measured laminations in $\mathcal{ML}$ are “conjugate” to the earthquake vectors on the same measured laminations (see subsection 2.3).

Note that this identification between $T_H \times \mathcal{ML}$ and $T^*T_H$ is not identical with the better known identification, which goes through measured foliations and quadratic differentials, see e.g. [FLP91].

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Now let $\mathcal{CP}$ denote the space of (equivalence classes of, see below) $\mathbb{C}P^1$-structures (or complex projective structures) on $S$. Recall that a (complex) projective structure on $S$ is an atlas of charts from $S$ into $\mathbb{C}P^1$ such that all transition maps are Möbius transformations. Such a structure naturally yields a holonomy representation $hol : \pi_1(S) \to \mbox{PSL}(2, \mathbb{C})$, as well as an $hol(\pi_1(S))$-equivariant developing map $dev : S \to \mathbb{C}P^1$. Two complex projective structures are called equivalent if their corresponding holonomy representations are $\mbox{PSL}(2, \mathbb{C})$-conjugate.

There is a natural relation between complex projective structures on $S$ and complex structures along with holomorphic quadratic differentials on $S$. Thus, let $\sigma$ be a projective structure on $S$, and let $\sigma_0$ be the “Fuchsian” $\mathbb{C}P^1$-structure on $S$ obtained by the Fuchsian uniformization of the conformal structure underlying $\sigma$. Then the Schwartzian derivative of the complex map from $(S, \sigma_0)$ to $(S, \sigma)$ is a quadratic differential $q$ on $S$, holomorphic with respect to the conformal structure of both $\sigma, \sigma_0$, see e.g. [Dum08, McM00]. In this identification equivalent (i.e. with $\mbox{PSL}(2, \mathbb{C})$ conjugated holonomies) projective structures lead to the same holomorphic quadratic differentials. The converse identification, i.e. a map from the space of pairs $(c, q)$ of complex structures on $S$ and a holomorphic quadratic differential on $S$ to the space $\mathcal{CP}$ is also possible, via the Schwartzian differential equation. Thus, the identification between the space of pairs $(c, q)$ and $\mathcal{CP}$ is one-to-one.

Recall also that the space of couples $(c, q)$ where $c$ is a complex structure on $S$ and $q$ is a quadratic holomorphic differential on $(S, c)$ is naturally identified with the cotangent bundle of $T_C$, see e.g. [Ahl16]. So, we get the following natural map:

**Definition 1.2.** Let $\Phi_C : T^*T_C \to \mathcal{CP}$ be the map sending $(c, q) \in T^*T_C$ to the $\mathbb{C}P^1$-structure $\sigma$ such that $S(\sigma_0, \sigma) = q$. Let $\omega_C$ be the cotangent symplectic form on $T^*T_C$.

Contrary to $\Phi_H$, the map $\Phi_C$ is smooth, since both the conformal structure and the quadratic holomorphic differential determining a $\mathbb{C}P^1$-structure on a surface depend smoothly on the corresponding representation into $\mbox{PSL}(2, \mathbb{C})$.

Note that there is another way to associate a holomorphic quadratic differential to a complex projective structure on $S$, using as a “reference point” a complex projective structure given by the simultaneous uniformization (Bers slice) instead of the Fuchsian structure $\sigma_0$. This identification is not as canonical as the one above, as it depends on a chosen reference conformal structure needed for the simultaneous uniformization. It
turns out that the symplectic structure obtained in this way on \( CP \) is independent of the reference point and is the same as the one coming from the above construction using the Fuchsian projective structure \( \sigma_0 \). (This fact, while not obvious, is presumably well-known to the specialists, it can be proved e.g. using Proposition 8.9 in [KS08].) Most of what we say below is also applicable to this simultaneous uniformization way of identifying \( T^*T_C \) with \( CP \).

1.3 From the hyperbolic to the complex picture

The "hyperbolic" and the "complex" descriptions of Teichmüller space behave differently in some key aspects, and it is interesting to understand the relation between them. This has even been put forward as a key question by some researchers. In this paper we provide a simple relationship between the two pictures, using the symplectic forms \( \omega_H \) and \( \omega_C \) on \( CP \) coming from the cotangent symplectic forms on \( T^*T_H \) and \( T^*T_C \), respectively.

**Theorem 1.3.** The maps \( \Phi_C \) and \( \Phi_H \) give rise to the same symplectic structure on \( CP \), up to a factor of 2:

\[
(\Phi_C)^* \omega_C = 2(\Phi_H)^* \omega_H.
\]

Again, it should be kept in mind that the map \( \Phi_C^{-1} \circ \Phi_H \) is not smooth, so it is not even so clear what the statement of the theorem means. The precise statement is that the image by \( (\Phi_C)^{-1} \circ \Phi_H \) of the Liouville form of of \( 2\omega_H \) is the Liouville form of \( \omega_C \) plus the differential of a function. Below we shall give an alternative statement of the above theorem in terms of Lagrangian submanifolds.

Note also that another relation between the Schwarzian derivative and measured laminations, in a slightly different direction, is obtained by Dumas in [Dum07].

1.4 The character variety

There is a third way to define Teichmüller space, and the space of complex projective structures on \( S \) of \( S \); as a connected component of the space of equivalence classes of representations of \( \pi_1(M) \) in \( PSL(2, \mathbb{R}) \), resp. in \( PSL(2, \mathbb{C}) \). This viewpoint leads to another symplectic structure on \( CP \), see [Gol84], defined in terms of the cup-product of two 1-cohomology classes on \( S \) with values in the appropriate Lie algebra bundle over \( S \). Here we call \( \omega_G \) this symplectic form on \( CP \), which is \( \mathbb{C} \)-valued. It turns out that this symplectic form is also equal, up to a constant, to \( (\Phi_C)^* \omega_C \); this was proved by Kawai [Kaw96].

Note also that \( T^*T \) has yet other symplectic structures, some of them involving the Weil-Petersson symplectic structure on \( T \) and/or the complex structure on \( T \). We only consider here the symplectic structure on \( T^*T \) coming from the cotangent bundle structure, which is not related to the Weil-Petersson metric or symplectic structure.

1.5 Hyperbolic ends

The proof of Theorem 1.3 is based on the geometry of geometrically finite 3-dimensional hyperbolic ends. We define this notion here as follows.

**Definition 1.4.** A hyperbolic end is a 3-manifold \( M \), homeomorphic to \( S \times \mathbb{R}_{>0} \), where \( S \) is a closed surface of genus at least 2, endowed with a (non-complete) hyperbolic metric such that:

- the metric completion corresponds to \( S \times \mathbb{R}_{\geq 0} \),
- the metric \( g \) extends to a hyperbolic metric in a neighborhood of the boundary, in such a way that \( S \times \{0\} \) corresponds to a pleated surface,
- \( S \times \mathbb{R}_{>0} \) is concave in the neighborhood of this boundary.

Given such a hyperbolic end, we call \( \partial_0 M \) the "metric" boundary corresponding to \( S \times \{0\} \), and \( \partial_{\infty} M \) the boundary at infinity.
It is simpler to consider a quasifuchsian hyperbolic manifold \( N \). Then the complement of its convex core is the disjoint union of two hyperbolic ends. However, a hyperbolic end, as defined above, does not always extend to a quasifuchsian manifold. Note also that the hyperbolic ends as defined here are always geometrically finite, so our definition is more restrictive than others found elsewhere, and the longer name “geometrically finite hyperbolic end” would perhaps be more precise.

There are two natural ways to describe a hyperbolic end, either from the metric boundary or from the boundary at infinity, both of which are well-known. On the metric boundary side, \( \partial_0 M \) has an induced metric \( m \) which is hyperbolic, and is pleated along a measured lamination \( l \). It is well known that \( m \) and \( l \) uniquely determine \( M \), see e.g. [Dum08].

In addition, \( \partial_\infty M \) carries naturally a complex projective structure, \( \sigma \), because it is locally modeled on the boundary at infinity of \( H^3 \) and that hyperbolic isometries act at infinity by Möbius transformations. This complex projective structure has an underlying conformal structure, \( c \). Moreover the construction described above assigns to \( \partial_\infty M \) a quadratic holomorphic differential \( q \), which is none other than the Schwarzian derivative of the complex map from \( (S, \sigma_0) \) to \( (S, \sigma) \). It follows from Thurston’s original construction of the grafting map that \( \sigma = G_\tau(m) \).

### 1.6 Convex cores

Before we describe how the above hyperbolic ends can be of any use for the questions considered in this paper, let us consider what is perhaps a more familiar situation. Thus, consider a hyperbolic 3-manifold with boundary \( N \), which admits a convex co-compact hyperbolic metric. We call \( \mathcal{G}(N) \) the space of such convex co-compact hyperbolic metrics on \( N \). Let \( g \in \mathcal{G} \), then \( (N, g) \) contains a smallest non-empty subset \( K \) which is geodesically convex (any geodesic segment with endpoints in \( K \) is contained in \( K \)), its convex core, denoted here by \( CC(N) \). \( CC(N) \) is then homeomorphic to \( N \), its boundary is the disjoint union of closed pleated surfaces, each of which has an induced metric which is hyperbolic, and each is pleated along a measured geodesic lamination, see e.g. [EMS6]. So we obtain a map

\[
i' : \mathcal{G}(N) \to T_H(\partial N) \times \mathcal{ML}(\partial N).
\]

Composing \( i' \) with the identification \( \delta \) between \( T_H \times \mathcal{ML} \) and \( T^*T_H \), we obtain an injective map

\[
i : \mathcal{G}(N) \to T^*T_H(\partial N).
\]

**Theorem 1.5.** \( i(\mathcal{G}(N)) \) is a Lagrangian submanifold of \( \left(T^*T_H(\partial N), \omega_H\right) \).

As we have already discussed, that the map \( i \) is not smooth. The reason for considering convex cores in our context will become clear in the next two subsections.

### 1.7 Kleinian reciprocity

There is a direct relationship between the statement 1.5 and Theorem 1.3, in that Theorem 1.5 can be considered as a corollary of Theorem 1.3. This goes via the so-called “Kleinian reciprocity” of McMullen. Thus, consider a Kleinian manifold \( M \), and let \( \mathcal{G}(M) \) be the space of complete hyperbolic metrics on \( M \). Then each \( g \in \mathcal{G}(M) \) gives rise to a projective structure on all of the boundary components at infinity \( \partial_\infty M \). This gives an injective map \( j : \mathcal{G}(M) \to T^*T_C(\partial_\infty M) \). We then have the following statement:

**Theorem 1.6** (McMullen [McM00]). \( j(\mathcal{G}(M)) \) is a symplectic submanifold of \( \left(T^*T_C(\partial_\infty M), \omega_C\right) \).

This statement is quite analogous to 1.5, with the only difference being that the space of convex cores is replaced by the space of Kleinian manifolds, and the “hyperbolic” cotangent bundle at boundaries of the convex core is replaced by the “complex” one. This statement is proved in the appendix of [McM00] under the name of “Kleinian reciprocity”, and is an important technical statement allowing the author to prove the Kähler hyperbolicity of Teichmüller space.

Let us note that Theorem 1.5 is a direct consequence of Theorem 1.6 and of Theorem 1.3. This will become more clear below when we present another statement of 1.3. Below we will give a direct proof of Theorem 1.5, thus also giving a more direct proof of the Kleinian reciprocity result.

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Using the result of Kawai [Kaw95] already mentioned above, Theorem 1.6 is equivalent to the fact that the subspace of complex projective structures on $\partial N$ obtained from hyperbolic metrics on $N$ is a Lagrangian submanifold of $(CP(\partial N), \omega_G)$, a fact previously known to Kerckhoff through a different, topological argument in personal communication.

1.8 A Lagrangian translation of Theorem 1.3

In a similar vein to what we have done above, let us consider the space $\mathcal{G}(E)$ of hyperbolic ends $E$. Each such space gives a point in $T_H \times \mathcal{M} \mathcal{L}$ for its pleated surface boundary, and a point in $T^* T_C$ for its boundary at infinity. Thus, composing this with the map $\delta$ we get an injective map:

$$k : \mathcal{G}(E) \to T^* T_H(\partial_0 E) \times T^* T_C(\partial_\infty E).$$

Our main Theorem 1.3 can then be restated as follows:

**Theorem 1.7.** $k(\mathcal{G}(E))$ is a Lagrangian submanifold of $T^* T_H(\partial_0 E) \times T^* T_C(\partial_\infty E)$.

We will actually prove our main result in this version, which is clearly equivalent to Theorem 1.3.

1.9 Cone singularities

One interesting feature of the arguments used here is that they appear likely to extend to the setting of hyperbolic surfaces with cone singularities of angle less than $\pi$. One should then use hyperbolic ends with “particles”, i.e., cone singularities of angle less than $\pi$ going from the “interior” boundary to the boundary at infinity, as already done in [KS07] and to some extend in [KS08].

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2 The Schläfli formula and the dual volume

In this section we recall the Schläfli formula, first in the simple case of hyperbolic polyhedra, then in the more involved setting of convex cores of hyperbolic 3-manifolds (as extended by Bonahon). We then deduce from Bonahon’s Schläfli formula a “dual” formula for the first-order variation of the dual volume of the convex core.

2.1 The Schläfli formula for hyperbolic polyhedra

Let $P \subset H^3$ be a convex polyhedron. The Schläfli formula (see e.g. [Mil94]) describes the first-order variation of the volume of $P$, under a first-order deformation, in terms of the lengths and the first-order variations of the angles, as follows:

$$dV = \frac{1}{2} \sum \varepsilon L_e \, d\theta_e,$$

where the sum is over the edges of $P$, $L_e$ is the length of the edge $e$, and $\theta_e$ is its exterior dihedral angle.

There is also an interesting “dual” Schläfli formula. Let

$$V^* = V - \frac{1}{2} \sum \varepsilon L_e \theta_e,$$

be the dual volume of $P$, then, still under a first-order deformation of $P$,

$$dV = -\frac{1}{2} \sum \varepsilon \theta_e dL_e.$$

This follows from the Schläfli formula (1) by an elementary computation.
2.2 First-order variations of the volume of the convex core

The convex core of a quasifuchsian manifold is reminiscent in some ways of a polyhedron, but the edges and their exterior dihedral angles are replaced by a measured lamination describing the pleating of the boundary, see e.g. [Thu80, EM86].

Bonahon [Bon98a] has extended the Schl"afli formula to this setting as follows. Let $M$ be a convex co-compact hyperbolic manifold (for instance, a quasifuchsian manifold), let $\mu$ be the induced metric on the boundary of the convex core, and let $\lambda$ be its measured bending lamination. By a “first-order variation” of $M$ we mean a first-order variation of the representation of the fundamental group of $M$. Bonahon shows that the first-order variation of $\lambda$ under a first-order variation of $M$ is described by a transverse Hölder distribution $\lambda'$, and there is a well-defined notion of length of such transverse Hölder distributions. This leads to a version of the Schl"afli formula.

**Lemma 2.1** (The Bonahon-Schl"afli formula [Bon98a]). The first-order variation of the volume $V_C$ of the convex core of $M$, under a first-order variation of $M$, is given by

$$dV_C = \frac{1}{2} L_\mu(\lambda') .$$

2.3 The dual volume

Just as for polyhedra above, we define the dual volume of the convex core of $M$ as

$$V_C^* = V_C - \frac{1}{2} L_\mu(\lambda) .$$

**Lemma 2.2** (The dual Bonahon-Schl"afli formula). The first-order variation of $V^*$ under a first-order variation of $M$ is given by

$$dV_C^* = -\frac{1}{2} L'_\mu(\lambda) .$$

This formula has a very simple interpretation in terms of the geometry of Teichmüller space: up to the factor $-1/2$, $dV^*$ is equal to the Liouville form of the cotangent bundle $T^*T_H$. Note also that this formula can be understood in an elementary way, without reference to a transverse Hölder distribution: the measured lamination $\lambda$ is fixed, and only the hyperbolic metric $\mu$ varies. The proof we give here, however, is based on Lemma 2.1 and thus on the whole machinery developed in [Bon98a].

Theorem 1.5 is a direct consequence of Lemma 2.2: since $dV^*$ coincides with the Liouville form of $T^*T(\partial N)$ on $j(N)$, it follows immediately that $j(N)$ is Lagrangian for the symplectic form of $T^*T_H(\partial N)$.

**Proof of Lemma 2.2.** Thanks to Lemma 2.1 we only have to show a purely 2-dimensional statement, valid for any closed surface $S$ of genus at least 2: that the function

$$L : \mathcal{T} \times \mathcal{ML} \to \mathbb{R}$$

$$(\mu, \lambda) \mapsto L_\mu(\lambda)$$

is differentiable, with differential equal to

$$L_\mu(\lambda)' = L'_\mu(\lambda) + L_\mu(\lambda') . \quad (3)$$

Two special cases of this formula were proved by Bonahon: when $\mu$ is kept constant [Bon97] and when $\lambda$ is kept constant [Bon96].

To prove equation (3), suppose that $\mu_t, \lambda_t$ depend on a real parameter $t$ chosen so that the derivatives $\mu'_t, \lambda'_t$ exist for $t = 0$, with

$$\frac{d\mu_t}{dt} \bigg|_{t=0} = \mu', \quad \frac{d\lambda_t}{dt} \bigg|_{t=0} = \lambda'.$$
We can also suppose that \((m_t)\) is a smooth curve for the differentiable structure of Teichmüller space. We can then decompose as follows:

\[
L_{\mu_t}(\lambda_t) - L_{\mu_0}(\lambda_0) = \frac{L_{\mu_t}(\lambda_t) - L_{\mu_0}(\lambda_t)}{t} + \frac{L_{\mu_0}(\lambda_t) - L_{\mu_0}(\lambda_0)}{t}.
\]

The second term on the right-hand side converges to \(L_{\mu}(\lambda')\) by [Bon97] so we now concentrate on the first term.

To prove that the first term converges to \(L_{\mu}'(\lambda)\), it is sufficient to prove that \(L_{\mu}'(\lambda)\) depends continuously on \(\mu, \mu'\) and on \(\lambda\). This can be proved by a nice and simple argument, which was suggested to us by Francis Bonahon. \(\mu\) can be replaced by a representation of the fundamental group of \(S\) in \(PSL_2(\mathbb{C})\), as in [Bon96]. For fixed \(\lambda\), the function \(\mu \rightarrow L_{\mu}(\lambda)\) is then holomorphic in \(\mu\), and continuous in \(\lambda\). Since it is holomorphic, it is continuous with respect to \(\mu\) and to \(\mu'\), and the result follows.

\(\Box\)

### 2.4 A cotangent space interpretation

Here we sketch for completeness the argument showing that the map \(\delta : T_H \times ML \rightarrow T^*T_H\) defined in the introduction is a homeomorphism. This is equivalent to the following statement.

**Lemma 2.3.** Let \(m_0 \in T_H\) be a hyperbolic metric on \(S\). For each cotangent vector \(u \in T^*_m T_H\), there exists a unique \(l \in ML\) such that \(m \mapsto dL_m(l) = u\) at \(m_0\).

**Proof.** Wolpert [Wol83] discovered that the Weil-Petersson symplectic form on \(T_H\) has a remarkably simple form in Fenchel-Nielsen coordinates:

\[
\omega_{WP} = \sum_i dL_i \wedge d\theta_i,
\]

where the sum is over the simple closed curves in the complement of a pants decomposition of \(S\). A direct consequence is that, given a weighted multicurve \(w\) on \(S\), the dual for \(\omega_{WP}\) of the differential of the length \(L_w\) of \(w\) is equal to the infinitesimal fractional Dehn twist along \(w\).

This actually extends when \(w\) is replaced by a measured lamination \(\lambda\), with the infinitesimal fractional Dehn twist replaced by the earthquake vector along \(\lambda\), see [Wol85, SB01]. So the Weil-Petersson symplectic form provides a duality between the differential of the lengths of measured laminations and the earthquake vectors.

Moreover the earthquake vectors associated to the elements of \(ML\) cover \(T_m^* T_H\) for all \(m \in T_H\) (see [Ker83]), it follows that the differentials of the lengths of the measured laminations cover \(T_m^* T_H\).

\(\Box\)

Note that this argument extends directly to hyperbolic surfaces with cone singularities, when the cone angles are less than \(\pi\). In that case the fact that earthquake vectors still span the tangent to Teichmüller space follows from [BS06].

### 3 The renormalized volume

#### 3.1 Definition

We recall in this section, very briefly, the definition and one key property of the renormalized volume of a quasifuchsian – or more generally a geometrically finite – hyperbolic 3-manifold; more details can be found in e.g. [KS08]. The definition can be made as follows. Let \(M\) be a quasifuchsian manifold and let \(K\) be a compact subset which is geodesically convex (any geodesic segment with endpoints in \(K\) is contained in \(K\)), with smooth boundary.

**Definition 3.1.** We call

\[
W(K) = V(K) - \frac{1}{4} \int_{\partial K} H da,
\]

where \(H\) is the mean curvature of the boundary of \(K\).
Actually $K$ defines a metric $I^*$ on the boundary of $M$. For $\rho > 0$, let $S_\rho$ be the set of points at distance $\rho$ from $K$, then $(S_\rho)_{\rho>}$ is an equidistant foliation of $M \setminus K$. It is then possible to define a metric on $\partial M$ as

$$I^* := \lim_{\rho \to \infty} 2e^{-2\rho} I_\rho,$$

where $I_\rho$ is the induced metric on $S_\rho$. Then $I^*$ is in the conformal class at infinity of $M$, which we call $c_\infty$.

Not all choices of $I^*$ in $c_\infty$ can be obtained from some choice of $K$, but any choice of $I^* \in c_\infty$ does define a unique equidistant foliation of $M$ in the neighborhood of infinity. It is then still possible to define $W(I^*)$, although the foliation does not necessarily extend to all positive values of $\rho$, one way to see this is based on the fact that $W(I^*)$ can be obtained through the “usual” definition of the renormalized volume in terms of the asymptotic expansion of the volume bounded by one of the surfaces in the equidistant foliation (details can be found in [KS08]).

As a consequence, $W$ defines a function, still called $W$, which, to any metric $I^* \in c_\infty$, associates a real number $W(I^*)$.

**Lemma 3.2** (Krasnov [Kra00], Takhtajan, Teo [TT03], see also [TZ87]). Over the space of metrics $I^* \in c_\infty$ of fixed area, $W$ has a unique maximum, which is obtained when $I^*$ has constant curvature.

This, along with the Bers double uniformization theorem, defines a function $V_R : T(\partial M) \to \mathbb{R}$, sending a conformal structure on the boundary of $M$ to the maximum value of $W(I^*)$ when $I^*$ is in the fixed conformal class of metrics and is restricted to have area equal to $-2\pi \chi(\partial M)$. This number $V_R$ is called the renormalized volume of $M$.

### 3.2 The first variation of the renormalized volume

The first variation of the renormalized volume involves a kind of Schl"{a}fl"{i} formula, in which some terms appear that need to be defined. One such term is the second fundamental form at infinity $II^*$ associated to an equidistant foliation in a neighborhood of infinity, as in the previous subsection. The definition comes from the following lemma, taken from [KS08].

**Lemma 3.3.** Given an equidistant foliation as above, there is a unique bilinear symmetric 2-form $II^*$ on $\partial M$ such that, for $\rho \geq \rho_0$,

$$I_\rho = \frac{1}{2}(e^{2\rho} I^* + 2II^* + e^{-2\rho} III^*),$$

where $III^* = II^* I^{-1} II^*$, that is, $III^* = I^*(B^*, B^* \cdot )$ where $B^* : T\partial M \to T\partial M$ is the bundle morphism, self-adjoint for $I^*$, such that $II^* = I^*(B^*, \cdot )$.

The first variation of $W$ under a deformation of $M$ or of the equidistant foliation is given by another lemma from [KS08], which can be seen as a version “at infinity” of the Schl"{a}fl"{i} formula for hyperbolic manifolds with boundary found in [RS00, RS99].

**Lemma 3.4.** Under a first-order deformation of the hyperbolic metric on $M$ or of the equidistant foliation close to infinity, the first-order variation of $W$ is given by

$$dW = -\frac{1}{4} \int_{\partial M} \left< dII^* - \frac{H^*}{2} dI^*, I^* \right> da^*,$$

where $H^* := tr(B^*)$ and $da^*$ is the area form of $I^*$.

The “second fundamental form at infinity”, $II^*$, is actually quite similar to the usual second fundamental form of a surface. It satisfies the Codazzi equation

$$d\nabla^* II^* = 0,$$

where $\nabla^*$ is the Levi-Civita connection of $I^*$, as well as a modified form of the Gauss equation,

$$\text{tr}_{I^*}(B^*) = K^*,$$
where $K^*$ is the curvature of $I^*$. The proof can again be found in [KS08]. A direct consequence is that, if $I^*$ has constant curvature $-1$, the trace-less part $B_0^*$ of $B^*$ is the real part of a holomorphic quadratic differential on $\partial M$ for the complex structure of $I^*$. In addition, the first-order variation of $V_R$ follows from Lemma 3.4.

**Lemma 3.5.** In a first-order deformation of $M$,

$$dV_R = -\frac{1}{4} \int_{\partial M} (dI^*, B_0^*) da^*. $$

This statement is very close in spirit to Lemma 2.2, with the dual volume of the convex core replaced by the renormalized volume. The right-hand term is, up to the factor $-1/4$, the Liouville form on the cotangent bundle $T^*T_C(\partial M)$.

A simple proof of Theorem 1.6. We have just seen that $dV_R$ coincides with the Liouville form of $T^*T_C(\partial M)$ on $k(G)$. It follows that the symplectic form of $T^*T(\partial M)$ vanishes on $k(G(\partial M))$, which is precisely the statement of the theorem.

\[\square\]

### 4 The relative volume of hyperbolic ends

#### 4.1 Definition

We consider in this part yet another notion of volume, defined for (geometrically finite) hyperbolic ends rather than for hyperbolic manifolds. Here we consider a hyperbolic end $M$. The definition of the renormalized volume can be used in this setting, leading to the relative volume of the end. We will write that a geodesically convex subset $K \subset M$ is a collar if it is relatively compact and contains the metric boundary $\partial M$ of $M$ (possibly all geodesically convex relatively compact subsets of $M$ are collars, but it is not necessary to consider this question here). Then $\partial K \cap M$ is a locally convex surface in $M$.

The relative volume of $M$ is related both to the (dual) volume of the convex core and to the renormalized volume; it is defined as the renormalized volume, but starting from the metric boundary of the hyperbolic end. We follow the same path as for the renormalized volume and start from a collar $K \subset M$. We set

$$W(K) = V(K) - \frac{1}{4} \int_{\partial K} H da + \frac{1}{2} I_\mu(\lambda),$$

where $H$ is the mean curvature of the boundary of $K$, $\mu$ is the induced metric on the metric boundary of $M$, and $\lambda$ is its measured bending lamination.

As for the renormalized volume we define the metric at infinity as

$$I^* := \lim_{\rho \to \infty} 2e^{-2\rho} I_\rho,$$

where $I_\rho$ is the set of points at distance $\rho$ from $K$. The conformal structure of $I^*$ is equal to the canonical conformal structure at infinity $c_\infty$ of $M$.

Here again, $W$ only depends on $I^*$. Not all metrics in $c_\infty$ can be obtained from a compact subset of $E$, however all metrics do define an equidistant foliation close to infinity in $E$, and it is still possible to define $W(I^*)$ even when $I^*$ is not obtained from a convex subset of $M$. So $W$ defines a function, still called $W$, from the conformal class $c_\infty$ to $\mathbb{R}$.

**Lemma 4.1.** For fixed area of $I^*$, $W$ is maximal exactly when $I^*$ has constant curvature.

The proof follows directly from the arguments used in [KS08] so we leave the details to the reader.

**Definition 4.2.** The relative volume $V_R$ of $M$ is $W(I^*)$ when $I^*$ is the hyperbolic metric in the conformal class at infinity on $M$. 


4.2 The first variation of the relative volume

**Proposition 4.3.** Under a first-order variation of the hyperbolic end, the first-order variation of the relative volume is given by

\[
V'_R = \frac{1}{2} L_\mu'(\lambda) - \frac{1}{4} \int_{\partial E} (I'^*, H_\mu^*) d\sigma^* .
\]

The proof is based on the arguments described above, both for the first variation of the renormalized volume and for the first variation of the volume of the convex core. Some preliminary definitions are required.

**Definition 4.4.** A polyhedral collar in a hyperbolic end \( M \) is a collar \( K \subset M \) such that \( \partial K \cap M \) is a polyhedral surface.

**Lemma 4.5.** Let \( K \) be a polyhedral collar in \( M \), let \( L_e, \theta_e \) be the length and the exterior dihedral angle of edge \( e \) in \( \partial K \cap M \). In any deformation of \( E \), the first-order variation of the measured bending lamination on the metric boundary of \( M \) is given by a transverse Hölder distribution \( \lambda' \). The first-order variation of the volume of \( K \) is given by

\[
2V' = \sum_e L_e d\theta_e - L_\mu'(\lambda') .
\]

**Sketch of the proof.** This is very close in spirit to the main result of [Bon98a], with the difference that here we consider a compact domain bounded on one side by a pleated surface, on the other by a polyhedral surface. The argument of [Bon98a] can be followed line by line, keeping one surface polyhedral (of fixed combinatorics, say) while on the other boundary component the approximation arguments of [Bon98a] can be used.

**Corollary 4.6.** Let \( V^*(K) := V(K) + (1/2) L_\mu(\lambda) \), then, in any deformation of \( K \)

\[
2V' = \sum_e L_e d\theta_e + L_\mu'(\lambda) .
\]

**Sketch of the proof.** This follows from Lemma 4.5 exactly as Lemma 2.2 follows from Lemma 2.1.

It is possible to define the renormalized volume of the complement of a polyhedral collar in a hyperbolic end, in the same way as for quasifuchsian manifolds above. Let \( C \) be a closed polyhedral collar in the hyperbolic end \( M \), and let \( D \) be its complement. Let \( K' \) be a compact geodesically convex subset of \( M \) containing \( C \) in its interior, and let \( K := K' \cap D \). We define

\[
W(K) = V(K) - \frac{1}{4} \int_{D \cap \partial K} H da .
\]

In addition \( K \) defines a metric at infinity, \( I^* \), according to (4), and it is possible to show that \( K \) is uniquely determined by \( I^* \), so that \( W \) can be considered as a function of \( I^* \), a metric in the conformal class at infinity of \( M \) (in general, as explained in section 3.1, \( I^* \) only defines an equidistant foliation near infinity which might not extend all the way to \( K \)). The first-variation of \( W \) with respect to \( I^* \) shows (as in [KS08]) that \( W(I^*) \) is maximal, under the constraint that \( I^* \) has fixed area, if and only if \( I^* \) has constant curvature. We then define the renormalized volume \( V_R(D) \) as the value of this maximum.

**Lemma 4.7.** Under a first-order deformation of \( D \), the first-order variation of its renormalized volume is given by

\[
V'_R = -\frac{1}{4} \int_{D \cap \partial K} \left( I'^* - \frac{H^*}{2} I^* \right) da^* + \frac{1}{2} \sum_e L_e d\theta_e .
\]

Here \( L_e \) and \( \theta_e \) are the length and exterior dihedral angle of edge \( e \) of the (polyhedral) boundary of \( D \).

**Proof.** The proof can be obtained by following the argument used in [KS08], the fact that \( D \) is not complete and has a polyhedral boundary just adds some terms relative to this polyhedral boundary in the variations formulas.

**Proof of Proposition 4.3.** The statement follows directly from Corollary 4.6 applied to a polyhedral collar and from Lemma 4.7 applied to its complement, since the terms corresponding to the polyhedral boundary between the two cancel.
4.3 Proof of Theorem 1.3

Since hyperbolic ends are in one-to-one correspondence with $\mathbb{CP}^1$-structures, we can consider the relative volume $V_R$ as a function on $\mathbb{CP}$. Let $\beta_H$ (resp. $\beta_C$) be the Liouville form on $T^*T_H$ (resp. $T^*T_C$). Equation (5) means precisely that, for any first-order deformation of $E$,

$$\frac{dV_R}{2}(\Phi_H)_*\beta_H - \frac{1}{4}(\Phi_C)_*\beta_C,$$

and it follows that $2(\Phi_H)_*\omega_H = (\Phi_C)_*\omega_C$. This clearly proves Theorem 1.3.

References


