THE COST SHARE THEOREM

Let $Y(q(t))$ denote economic production mesured in units of currency where $q(t) = (q_1(t), q_2(t), q_3(t)) \in \mathbb{R}^3$. Let p_i be the price of q_i for $i = 1, 2, 3$. The *cost share* of the quantity q_i is defined as

$$c_i = \frac{q_i p_i}{Y} \tag{0.1}$$

We assume that Y is a Cobb-Douglas function of q, that is

$$Y = \prod_{i=1}^{3} q_i^{\alpha_i}. \tag{0.2}$$

Let $\sum_i \alpha_i = r$ and $\sum_i c_i = s$. If we assume that q_i are the costs of production of Y, then neoclassic equilibrium theory says

Theorem 1. *(The Cost Share Theorem)*

$$\alpha_i = c_i r/s. \tag{1.1}$$

Proof. Let $C(q) = \sum_{i=1}^{3} p_i q_i$ denote the cost of producing Y. At equilibrium, we have $Y = \max_{C(q)=\text{const}} Y(q)$. Therefore there is a $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} \nabla Y &= \lambda \nabla C \\ \alpha_i q_i^{\alpha_i - 1} \prod_{j \neq i} q_j^{\alpha_j} &= \lambda p_i,\ i = 1, 2, 3 \end{aligned}$$

Multiplying the last equation by q_i/Y, we obtain

$$\alpha_i = \lambda c_i$$

Summing the last equation over i yields $\lambda = r/s$ which gives (1.1). $\square$