GENERAL ECONOMIC PRODUCTION FUNCTIONS, PRICES, AND COST SHARE: THEORY AND APPLICATIONS

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ABSTRACT. We prove a general theorem about economic production functions under very mild hypotheses. We characterize economic motors and drags on the economy. We provide tools with which researchers in economics can empirically test hypotheses using economic production functions.

1. INTRODUCTION

This article provides a mathematically rigorous framework with which researchers can view empirical data and test economic hypotheses using economic production functions. This work is quite different from that of [21, 4, 13]. While these works treat the economy as if it were a business with inputs of labor and capital, and in the latter two cases energy, outputting GDP, we regard economic production as a measurable quantity. Thus it is a function of a large set, in fact all economic quantities. We wish to evaluate how changes in these quantities have effected economic production. Our only assumption on the structure of the economy is the existence of a well developed monetary system for the distribution of wealth. Our results are thus more general and can be applied not only to the previously cited works (since the inputs are measured quantities with prices), but also to a much larger set of both economic production functions and economies. We are concerned with characterizing economic motors or drags on the economy and measuring the effect of changes in these quantities and their prices on economic production. In particular, we would like to understand the structural changes an economy goes through when an economic motor transforms into an economic drag on the economy. A key motivation for this work is to understand the effects peak oil production will likely have on the economy.

We show

(1) The elasticity of a factor in the production function is related to the elasticity of its price divided by the cost share.
(2) The greater the importance of a quantity in economic production, the smaller the scarcity rent (in some sense).
(3) Economic motors are characterized by a shrinking cost share in an environment of economic growth, and an increasing cost share in an environment of economic contraction.

We provide tools for measuring the effect of quantities and their prices in the economic production function.

2. Measuring Economic Production

Units are a major problem in measuring economic production. If all economic production where apples, one could count the apples to evaluate economic production. But in an economy which produces apples, oranges, computers, software, and mathematical theorems among many other things, each unit one uses has its specific problems.

Different units have been studied by different authors to measure the economy. We feel each of the units have different advantages and disadvantages. Studying different units give different insights into how the economy functions.

2.1. Currency.

Currency. The major success of currency as a unit is that it is common to all things measurable (objects with a price), allowing us for example, to compare software production to apple production. The unit of currency does have drawbacks.

Money or currency is many things. A monetary system is a social contract for the distribution of goods produced by the society. One aspect of money is that it is a measure of value on measurable objects. The so called market compares the price of computer software and apples allowing us to compare the relative values of very diverse objects. The value system is not democratic, as persons controlling large relative amounts of money have more votes in determining the value of an object than a large number of people without money. For example, if a large part of the population is starving, one might expect the price of food to be high, but this is not necessarily the case if the people who are starving have no money. The distribution of money or wealth thus significantly effects the price of goods.

Another aspect of money is that as it is a man made construct, it can be manipulated so that there are distortions. Economists attempt to see through these distortions, for example by adjusting for inflation, using discount rates, and factoring in subsidies. It is nonetheless impossible to account for all distortions in a satisfactory manner. The success of the website Shadowstats attests to intensity of this debate with respect to inflation.

2.2. Energy.

Energy. Many authors have worked on Energy Return on Energy Invested, or EROEI

$$\text{EROEI} \stackrel{\text{def}}{=} E_o/E_i, \tag{2.1}$$

where E_i is the energy invested to obtain the output energy E_o [10, 8, 19, 24]. Rather than using currency, these authors use units of energy to measure economic activity. Energy as a unit, is known very well in all the physical sciences. It is impossible to manipulate. The drawback of energy is that one does not consider what the energy is used for, and how well it is used for that purpose, primary concerns for those wishing to understand the economy.

Energy balance equations are fundamental to understanding all sciences. We feel that this should be an important area of future research.

When measured with the unit of energy, the economy resembles one or more electric circuit(s) in parallel with a power supply(s). Each good can be traced back to the origin of the energy that permitted its manufacture. Energy, or more properly, exergy, is different from money in that it is exhaustible. Exergy cannot circulate forever, spent exergy never comes back. It needs to be produced from a power supply. To people measuring the economy with energy, exergy production is the most fundamental aspect of the economy. The IEA has recently made interactive Sankey diagrams available, a valuable tool for people studying energy and the economy.

2.3. **Useful Work I.** Ayres and Warr [3] have introduced the idea of useful work

$$U \stackrel{\text{def}}{=} eE, \tag{2.2}$$

where E is energy production and e is efficiency. Since efficiency is a dimensionless proportion, U has the units of energy so it is really just an aid to study the unit of energy rather than an entirely new unit. Essentially, U represents energy production less the energy lost in (non useful) heat. Note that $0 < e < 1$ so that e is bounded.

This variable has the advantage of adding a measure of technological progress to measures of energy production. It also has the advantage of attempting to measure what is actually done with the energy, an improvement over just measuring energy production alone.

We define Useful Work Return on Energy Investment in a similar way to EROEI:

$$\text{UWROEI} \stackrel{\text{def}}{=} U_o/E_i = eE_o/E_i. \tag{2.3}$$

We do not divide by e in the denominator as we assume that this is accounted for in the computation of EROEI.

2.4. **Useful Work II.** One can take the idea of useful work a step further by measuring precisely what we do with energy in the economy. For example one can measure vehicle kilometers traveled, ton kilometers of freight transported, person kilometers traveled, processor cycles cycled, or page views on the web. Such measures are direct measures of economic activity, but unfortunately one loses a uniform unit and it becomes difficult to compare different activities. One obtains measures of different parts of the economy with incompatible units. If the goal is to transport people or goods one can also talk about efficiency, though it is not as well defined as in Section 2.3. One can measure for example ton kilometers per kilowatt hour. The larger the quotient, the greater the efficiency. We note that in terms of vehicle kilometers traveled, U.S. and Western European economies have contracted since 2007.

Remark 2.1. *The theory of economic production functions developed here can be applied to variables of useful work II. It suffices to replace economic production by a variable of useful work II expressed in suitable monetary units.*

2.5. **Labor and Food.** Marx [16] noted the importance of hours of work in measuring value added. Again, different work requires different accounting methods. The amount of training and skill of workers varies. However, we can make meaningful averages based on population among other statistics. Population in turn can be estimated from food production. We believe that food production is the origin of labor specialization, and hence modern economies. In hunter gatherer societies all labor is devoted to food production. When a percentage of the population accepts to produce food for a larger population, then some laborers are available for other work.

Because food is an essential ingredient of labor, we may substitute food production for labor. With food and labor, we can do analysis similar to EROEI analysis done with energy. Knowing the percentage of the population engaged in food production tells us how much labor is available for producing other things. It is a measure of the diversity of the economy. In the United States, this represents about 1% of the population, leaving 99% of the population available for other work. In Burundi 93% of the population is in agriculture leaving a mere 7% available to do other things.

3. General Economic Production Functions

3.1. **Theory.**

3.1.1. *Definitions and Hypotheses.* Let $Y(t)$ be the economic production of an economy expressed in currency where t denotes time. We make no assumptions about the structure of the economy other than the existence of a well developed monetary system to determine the distribution of wealth. Let $\mathbf{q}(t) \in \mathbb{R}^d$ be the measurable quantities in the economy (quantities with a price). We will make use of the following assumptions:

H1:

$$Y(t) = Y(\mathbf{q}(t), t) \tag{3.1}$$

is a locally $C^1(\mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R})$ with $\mathbf{q}(t) = (q_1(t), \ldots q_d(t)) \in \mathbb{R}^d$. Prices are locally $C^1(\mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R})$ functions of quantities.

H2: Over short periods of time the economic production function depends more on $\mathbf{q}(t)$ than on the time variable, therefore, for short periods of time $Y(\mathbf{q}(t), t) \approx \tilde{Y}(\mathbf{q}(t))$, $\mathbf{q}(t) \in \mathbb{R}^d$.

H3: For short periods of time we have $\dfrac{\partial Y}{\partial q_i} \neq 0$ for $i = 1, \ldots, d$.

Remark 3.1.
(1) *The regularity assumptions are not essential to the theory developed below, their purpose is to simplify notation.*
(2) *In the real world, $d = d(t)$. Care must be taken when applying results to new quantities or disappearing quantities.*

H1 is not a strong assumption, as the very fact that GDP is measurable, means we measure certain quantities and use prices to evaluate the value added by the domestic economy.

H2 essentially says that we assume it takes time for the economy to change. Certain quantities are fungible, but we assume it takes a certain amount of time to switch from one item to the other.

H3 is not a strong assumption, as we are eliminating only quantities that have a small effect on the economy at any given time.

Let $p_i(t)$ be the cost per unit of $q_i(t)$ and let c_i be the *cost share* or *intensity* of q_i in the economy,

$$c_i(t) \stackrel{\text{def}}{=} p_i(t) q_i(t) / Y(t). \tag{3.2}$$

We make the following definitions.

Definition 3.1.
(1) *We will say that a currency is* adjusted for inflation with respect to the variable q_i *if* $\dfrac{\partial p_i}{\partial q_i} = 0$.
(2) *A quantity q_i is called a* super linear motor *of economic growth if* $\dfrac{\partial c_i}{\partial q_i} < 0$ *in a currency adjusted for inflation with respect to q_i.*

(3) *A quantity q_i is called a* drag *on economic growth if* $\dfrac{\partial c_i}{\partial q_i} > 1/q_i$ *in a currency adjusted for inflation with respect to q_i.*

We prove these definitions consistent in the appendix.

Remark 3.2. *The cost share of economic motors decreases in times of economic growth and increases in times of economic contraction.*

Definition 3.1 is an artificial construct to simplify the proof and statement of Theorem 3.1. However, during the 20'th century, adjusted for inflation in the standard way, the price of oil and other commodities were highly volatile around a constant average [9], while their cost shares decreased and the economy grew, making them strong candidates for being economic motors.

We will denote by $Y_{q_i}(u) = Y_i(\overline{\mathbf{q}}_i(t), t, u)$, with $\overline{\mathbf{q}}_i(t) \in \mathbb{R}^{d-1}$, consisting of q_j, $j \neq i$, the quantity

$$Y_{q_i}(u) \stackrel{\text{def}}{=} \int_0^u \frac{\partial Y}{\partial q_i} q_i'(s)\mathrm{d}s. \tag{3.3}$$

We denote by $Y_u(u)$ when the above integral is with respect to the last variable. We define the functions $p_{q_i}(u)$ and $c_{q_i}(u)$ similarly.

The function $Y(t)$ can also be seen as a function of the prices $Y(\mathbf{p}(t), t)$, $\mathbf{p} \in \mathbb{R}^d$. When the above derivatives and integrals are with respect to p_i, they will be denoted $Y_{p_i}(u)$, etc.

For any function $x(t)$ and $(t_0, t_1) \in \mathbb{R}_+^2$, we define the *index* of x:

$$I_x(t_0, t_1) \stackrel{\text{def}}{=} x(t_1)/x(t_0). \tag{3.4}$$

3.1.2. *Elasticity.* Elasticity, or how quantities scale in the economic production function is very important. Suppose $d = 1$. One can write $Y(t) = Cq^{\alpha(t)}$. If $\alpha(t) \equiv \alpha$, a constant, then the production function is homogeneous of degree α and we call α the elasticity or scaling factor. If $\alpha = 1$ Y is *linear* in q, if $\alpha < 1$, Y is *sublinear* in q, otherwise, Y is *superlinear* in q. Scaling factors are important in many sciences and mathematics. One looks for constant or average scaling empirically by normalizing quantities at a start date, taking logs and performing linear regression.

3.1.3. *Main Theorem.* We prove

Theorem 3.1. *Assume (H1), then*

(1) *If $\alpha_i(t)$ is the scaling factor of q_i, then $\alpha_i(t) - 1$ is the scaling factor of p_i/c_i.*
(2) *Assume that c_i is constant. Then sublinear scaling of Y in q_i occurs if and only if $p_i(q_i)$ is monotone decreasing, linear scaling implies price is independent of q_i, and superlinear scaling occurs if and only if $p_i(q_i)$ is monotone increasing.*
(3) *Super linear economic motors have superlinear scaling in Y, economic drags have sublinear scaling in Y.*
(4) *The greater* $\dfrac{\partial Y}{\partial q_i}$, *the smaller the scarcity rent in the in the sense of* (3.8) *(see discussion below).*

(5) *The quantity* $\dfrac{\partial Y}{\partial q_i}$ *is negative if and only if* q_i *is a drag on economic growth. If* q_i *is a motor of economic growth,* $\dfrac{\partial Y}{\partial q_i} > 0$.

(6) *The index of* $Y_{q_i}(u)$ *for any* $(t_1, t_2) \in \mathbb{R}^2_+$ *is given by*

$$I_{Y_{q_i}}(t_1, t_2) = I_{p_{q_i}}(t_1, t_2) I_{c_{q_i}}(t_2, t_1) I_{q_i}(t_1, t_2). \tag{3.5}$$

(7) *The index of* $Y_{p_i}(u)$ *for any* $(t_1, t_2) \in \mathbb{R}^2_+$ *is given by*

$$I_{Y_{p_i}}(t_1, t_2) = I_{q_{p_i}}(t_1, t_2) I_{c_{p_i}}(t_2, t_1) I_{p_i}(t_1, t_2). \tag{3.6}$$

Proof. From (3.2) one immediately obtains

$$p_i(t) = c_i(t) Y(t)/q_i(t). \tag{3.7}$$

Properties (1) and (2) can be read directly from (3.7). Property (3) is also clear from (3.7) since for an economic motor, a negative cost share derivative means the cost share decreases with constant price implying superlinear scaling in Y. A similar statement holds true in the case of an economic drag.

Taking the derivative of (3.7) one obtains

$$\frac{\partial p_i}{\partial q_i} = \frac{\partial c_i}{\partial q_i}\frac{Y}{q_i} + c_i \frac{\frac{\partial Y}{\partial q_i} q_i - Y}{q_i^2}. \tag{3.8}$$

The scarcity rent of a quantity varies inversely to its importance in the economic production function in the following sense. The more important a quantity in the economic production function, the greater the partial derivative of Y with respect to that quantity. But from (3.8), we see that the price is an increasing function of the partial derivative of Y with respect to q, or price decreases as quantity decreases, a negative scarcity rent [1]. This proves (4).

Solving (3.2) for Y, taking logs, and then the partial derivative with respect to q_i, one obtains

$$\frac{\partial Y}{\partial q_i} = Y \left(\frac{\frac{\partial p_i}{\partial q_i}}{p_i} - \frac{\frac{\partial c_i}{\partial q_i}}{c_i} + \frac{1}{q_i} \right). \tag{3.9}$$

The sign of the left hand side of (3.9) is the same as the sign in parentheses on the right hand side. Using a currency adjusted for inflation with respect to q_i the first term in parentheses is 0, this proves (5). To obtain (3.5), we multiply (3.9) by $\frac{\mathrm{d}q}{\mathrm{d}t}/Y$ and integrate from t_1 to t_2 and take the exponential of the resulting equation. The proof of (3.6) is similar. $\square$

Remark 3.3. (1) *Equation* (3.8) *should not be considered a precise model because different quantities in the equation move at different speeds. Prices move more quickly than does the reaction of the economy to price changes.*

[1] Of course, for an important quantity one would expect the derivative of the cost share to be strongly negative and dominate the positive term

(2) *Equation* (3.5) *can provide a method for measuring the relative contribution of a quantity in* $I_Y(t_1,t_2)$ *in cases where* $c_{q_i}(t)$ *or* $c_{p_i}(t)$ *can be estimated. See Section 4.2. Note that*

$$I_Y(t_1,t_2) = I_{Y_u} \prod_{j=1}^{d} I_{Y_{q_j}}(t_1,t_2). \tag{3.10}$$

Note also that the order in which $I_{Y_{q_j}}(t_1,t_2)$ *is computed in* (3.10) *can change it's value because this changes the values of the* $q_i(t)$, $i \neq j$ *in* (3.5).

(3) *In most cases one has*

$$I_{c_{q_i}}(t_1,t_0) \leq I_{c_i}(t_1,t_0). \tag{3.11}$$

In order for the inequality to be strict in (3.11)*, another independent quantity must increase. See Section 3.3.*

(4) *If useful work* $U = eE$ *is used as a variable, we can write* $Y(U(t)) = Y(e(t)E(t))$ (3.7) *becomes* $p_E = c_E Y(eE)/q_E$*. One sees that energy efficiency increases the price per unit of energy assuming* $c_E(t)$ *remains constant. We thus have a very simple explanation of the empirically observed Jevons paradox or the rebound effect* [12, 5].

(5) *The "cost share theorem" from neoclassic equilibrium theory* [4, Appendix A] *says that cost share is proportional to the scaling factors of the variables in the production function equation. Equation* (3.9) *suggests strongly that in a growing economy a large scaling factor should be associated with a shrinking cost share, thus the interaction between variables gives some variables larger scaling. We can think of many reasonable scenarios in which this theorem is not verified (see Section 3.3). We believe that the hypotheses from which the cost share theorem is derived are speculative and that empirical evidence should be inspected carefully before accepting this theorem.*

3.2. **Representative Variables and the Implicit Function Theorem.** One might be interested in using standard techniques to determine the nature of $Y(t)$. If one accepts assumptions (H2) and (H3), one can use the implicit function theorem to reduce the number of variables to a small set of representative variables locally. This is very familiar to investors who use economic indicators to gauge the state of the economy. By choosing overlapping time periods, empirical results will show whether or not the computed functions can be stitched together for longer periods of time.

3.3. **An Example.** We suppose a very limited economy produces 3 quantities: $q_1 = E$, $q_2 = G$, and $q_3 = F$. We assume prices are adjusted for inflation for the 3 quantities and we normalize all prices to one. The size of this economy is

$$Y(t) = \sum_{i=1}^{3} p_i q_i \tag{3.12}$$

$$= E + G + F. \tag{3.13}$$

Now suppose that E is a motor of economic growth in the following sense, when E grows 10% this produces a growth of 5% in both G and F in the next time period. Growth in G has no effect on E or F. However F is a drag on economic growth since a 10% growth in F causes a 5% percent contraction in E and G. We can name our quantities to make the example more realistic. We call E energy production, which permits us to produce more of G and F. Let us call G gold production and F fun production. Fun decreases growth in E and G because in fact many people do not like producing energy or mining for gold, so as soon as there is a fun event, they stop work to enjoy the fun which reduces production of E and G.

Let us assume that $t = n \in \mathbb{N}$ and that the initial conditions are $E_0 = G_0 = F_0 = 1$. Now assume that population growth would cause growth of 10% in each time period, but the interactions occur in the next time period. Thus $E_1 = G_1 = F_1 = 1.1$ but $E_2 = 1.21 - .055 = 1.155$ because of the fun interaction. We have $G_2 = 1.21 - .055 + .055 = 1.21$ and $F_2 = 1.21 + .055 = 1.265$. We see that the cost share c_E has dropped from 1/3 to 0.32, c_G is unchanged and c_F increases from 1/3 to .35, while $I_Y(0,2) = 1.21$. Computing the index of each quantity separately (assuming the other quantities constant at the t_0 value, that is, assuming the index is computed as if it was computed first in (3.10)) we find $I_{Y_E}(0,2) \approx 1.1$, $I_{Y_G}(0,2) = 1.07$, and $I_{Y_F}(0,2) \approx 1.04$. Note that $c_{q_1}(q_2(t_0), q_3(t_0), t_1) \approx 0.366 > c_E(t_1) = 0.32$ so that $I_{c_{q_1}}(t_1,t_0) < I_{c_E}(t_1,t_0)$. This is because E does not explain all the growth in Y. The individual scaling factors of E, G, and F are respectively .52, .35, and .19. Repeating the calculations with $E_0 = G_0/2 = F_0/2 = 1/2$, we obtain $E_1 = .55$, $G_1 = 1.1$, $F_1 = 1.1$, $E_2 = .575$, $G_2 = 1.21$, $F_2 = 1.26$. With these initial conditions, c_E decreases from .2 to .19, c_G is almost unchanged and c_F increases from .40 to .41. In this case $I_Y(0,2) = 1.22$ so that a smaller initial cost share of the economic motor produces greater overall growth. From these initial conditions, the individual scaling factors of E, G, and F are .5, .42, and .28 respectively.

Remark 3.4. *Not all quantities that drag on economic growth are fun.*

4. Applications

We wish to understand how economies can react to scarcity of essential items such as energy and food. We remark that Theorem 3.1 is consistent with the thesis of many authors that growth in energy and food production are primary drivers of growth in economic production [15, 23, 12, 17, 4, 6, 7, 13]. In times of economic growth, the cost share of food is in general either stable or decreasing [22], as is the price of energy, thus these quantities satisfy the characterization of economic motors.

Note that when prices are integrated into the example of Section 3.3, broad discussions are possible on measuring $Y(t)$. Suppose that the cost shares are as follows: $c_E(t_0) = 0.1$ and $c_G(t_0) = c_F(t_0) = 0.45$. Suppose that production of all quantities remain constant, but the price of energy doubles ($p_E(t_1) = 2 * p_E(t_0)$), thus $c_E(t_1) = .18$. If we use (3.12) to compute Y, we obtain $I_Y(t_0,t_1) = 1.1$. However if we adjust for inflation with respect to E before computing $Y(t)$, we find that P_G and P_F are halved so that $I_Y(t_0,t_1) = .55$. Of course one could interpolate any value between these two extreme values. In order to have an idea about which measurement is more appropriate, one needs to have an idea of how money is distributed and who is buying what with their money. Of course, it is quite unlikely that the price of energy should double and the other quantities remain the same. Agents react to price changes and quantities vary accordingly. We begin by trying to qualitatively understand secular cycles using Theorem 3.1.

4.1. **Qualitatively Understanding Secular Cycles.**

In [22], Turchin and Nefedov describe cycles that are common to many former civilizations. The cycle begins with a period of growth, in population and living standards lasting one or more hundreds of years. Then comes a period of stagflation in which population density approaches the carrying capacity of the land (one says increased population pressure) lasting on the order of half a century. During the stagflation period peasants leave the countryside for cities, the difference between the elite and the commoners increases, and the price of food rises relative to wages. Population ceases to grow in the working class. At first the elite are somewhat better off in the stagflation period because wages are low and they can employ a larger number of former peasants who have left the countryside. As the stagflation period progresses, the ratio elite to peasants rises (peasants have a higher mortality rate) creating competition among the elite. Social mobility increases, mostly downward as elites lose their status. The inter elite competition creates fissures which often lead to civil war and the final crisis stage lasting a few decades in which population decreases and the state breaks down. There follows an intercycle lasting on the order of 100 years before a new growth period ensues.

Several authors have observed that food production techniques permit the growth of civilizations by permitting work specialization, but degrade soil over time [18, 7]. During the growth phase of development, money flows create positive feedback loops with stable food prices relative to wages, thus food satisfies the characteristics of an economic motor during this period. With depleted soil, food production ceases to be an economic motor. Poor harvests become more common. Food prices rise, but in spite of higher food prices, many peasants are unable to make a living producing food and leave the countryside for the cities. This can partly be explained by Theorem 3.1 (4) which says that prices do not rise as fast as might be expected because economic growth stalls. Debt becomes a problem as do government finances as tax revenue falls at the same time that the government invests in infrastructure projects to increase food production. From Theorem 3.1, one would expect that hardship would hit the unproductive elite class first. This is not verified empirically. Hardship is initially born by the food producing peasants. We will not speculate on why this occurs, but we note that the phenomena decreases food prices as those that are hungry do not have money for food. Since the cost share of food for the elites is low and they represent the bulk of the wealth this decreases the cost share of food in the economy. From (3.7) we see that prices are not as high as they could be if money was more evenly distributed. Low wages become an economic motor for other parts of the economy as artisans proliferate and handcraft work increases.

4.2. **Measuring Effects.** The usefulness of equations (3.5) and (3.6) depend on estimating the quantities $I_{c_{q_i}}$ and $I_{p_{q_i}}$ or $I_{c_{p_i}}$ and $I_{q_{p_i}}$. Estimating these quantities for a modern economy is beyond the scope of this work, but in this section we explain what equations (3.5) and (3.6) can be used for.

Equation (3.5) should be used in the case of either a growing or contracting economy. Without loss of generality, one can assume that the average impact of other quantities on prices is 0. Therefore (3.5) should be used between two periods in which quantities differ, but the price (adjusted for inflation) is constant. In this case one can assume that $I_{p_{q_i}} = I_{p_i} = 1$. It remains to estimate $I_{c_{q_i}}$.

A simple case in which (3.5) can be applied directly is in the case in which Liebig's law of the minimum [25] holds. Liebig's law of the minimum holds in the case of a short term supply shock with the quantity q_i, in a restricted setting. In this case, we have $I_{c_{q_i}} = I_{c_i}$ and $I_{p_{q_i}} = I_{p_i}$ as the limiting factor in the economy is q_i and (3.5) reduces to $I_Y(t_0, t_1)$. For example if one had an accurate estimate of the amount of jet fuel produced in 2020, one could compute an upper bound on the number of kilometers flown by jets in that year. Computing I_{c_i} for different industries in the past in the case of supply shocks gives an indication of how to estimate $I_{c_{q_i}}$ in the general case. By beginning with variables in which the role of a quantity is very clear (as per Remark 2.1), such as the production of jet fuel with respect to kilometers flown by jets and then carefully evaluating the effect of changes in these sectors of the economy on the rest of the economy, one can slowly begin to estimate c_{q_i} on the economy as a whole.

One should use (3.6) in the stagflation period of the economy when prices of essential quantities increase but quantities are stagnant. Again, applying (3.6) to a modern economy is beyond the scope of this paper. It is simpler to apply it in a more restricted setting, for example to wage earners in a certain class and the effect of higher prices on their consumption. Hamilton [11] has analyzed the effects of an increased cost share of oil on wage earners in the U.S. Clearly if wages rise more slowly than energy prices and energy quantities are flat, the rest of the economy must shrink, either in price, quantity, or both relative to the wage earner.

4.3. **Peak Oil.** Empirical evidence points towards a current stagflation period with respect to oil production and the oil economy after a period of growth that lasted from 1857 to 2002. Oil price was volatile around a constant average throughout the growth period while quantities increased steadily. The cost share of oil for the consumer dropped from 7.3% in 1959 to 4.1% in 2002 [11]. The price of oil doubled from its average cost during the 20'th century between 2002 and 2005. Between 2005 and 2011, the price doubled again, but the increase in quantities slowed dramatically. In fact, without the notable exception of the U.S. (the worlds third largest producer), oil production is on a plateau since the end of 2004 [20]. Wages are flat and in many cases decreasing so that the cost share of oil is increasing for the average wage earner. The difference between rich and poor is increasing.

Theorem 3.1 (4) indicates that peak oil is more a low price problem than a high priced problem. The 5 major oil companies production peaked in 2004 [2]. Profits in the oil industry were down in 2013 as oil prices have not kept pace with increase in capital expenses and volumes are flat. The flow of money in the oil industry is changing. Low priced oil is being replaced by high priced oil (Aleklett reported marginal costs of oil production of $1 per barrel in an oil field in Dubai in 2005 [1] while Likvern [14] has estimated the marginal production cost of tight oil in the U.S. at between $70 and $80 per barrel) this results in money flowing to oil service companies rather to than the diverse industries that were supported with cheap oil. Clearly the structure of the economy is changing.

5. Conclusion

We believe the framework we have outlined should be of use to many researchers, both inside and outside the field of economics.

In view of predictions of oil production from Uppsala Global Energy Systems which are significantly lower than those of the EIA or the IEA and probably more reliable, we believe the most pressing economic problem is to resolve socio-economic problems in the absence of economic growth.

Appendix A

Let $Y(\mathbf{q}(t), t)$ be an economy with prices $p(t) \in \mathbb{R}^d$. We make Definition 3.1 (1) explicit by defining a currency adjusted for inflation with respect to q_i as follows. We choose t_0 and then define

$$\overline{p}_j(t) \stackrel{\text{def}}{=} \frac{I_{p_{q_i}}(t, t_0)}{p_i(t_0)} p_j(t) \tag{A.1}$$

for $j = 1, \ldots, d$. The currency $\overline{p}(t)$ is adjusted for inflation with respect to q_i. To see this note that $I_{p_i}(t_0, t)$ satisfies an equation similar to (3.5), thus $\overline{p}_i(t)$ will not depend on q_i. We now have:

Proposition A.1. *The definitions 3.1 are consistent.*

Proof. Since two currencies which adjust inflation with respect to q_i differ in prices by a constant factor and the units in cost share are currency divided by currency, the common factor in the numerator and denominator cancel and the cost shares (for all quantities q_j) are the same. □

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